ГОДИШНИК

НА

СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

КНИГА 2 — МЕХАНИКА

Том 86
1992

ANNUAIRE

DE

L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE

LIVRE 2 — MECANIQUE

Tome 86
1992

СОФИЯ, 1994, SOFIA
УНИВЕРСИТЕТСКО ИЗДАТЕЛСТВО „СВ. КЛИМЕНТ ОХРИДСКИ“
PRESSES UNIVERSITAIRES "ST. KLIMENT OHRIDSKI"

ISSN 0205–0808

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ON THE PERTURBATIONS OF A MECHANICAL SYSTEM FROM THE RIGID BODY DYNAMICS

OGNYAN CHRISTOV

Огнян Христов. О ПЕРТУРБАЦИЯХ ОДНОЙ МЕХАНИЧЕСКОЙ СИСТЕМЫ ИЗ ДИНАМИКИ ТВЕРДОГО ТЕЛА

В этой статье проверяются условия КАМ теории для одного интегрируемого случая в механической системе описывающей движения частицы, которая осцилирует в твердом теле с неподвижной точкой в отсуствие внешных сил.

Ognyan Christov. ON THE PERTURBATIONS OF A MECHANICAL SYSTEM FROM THE RIGID BODY DYNAMICS

In this paper the KAM–theory conditions are checked for an integrable case of a mechanical system describing the motion of a particle, oscillating in a rigid body with a fixed point without external forces.

1. INTRODUCTION

The question of integrability of Hamiltonian systems is one of the oldest problems of classical mechanics [1, 2]. Classical results due to Poincare and Bruns show that most of the Hamiltonian systems are not integrable. According to Poincare the main problem of dynamics is the study Hamiltonian systems which are close to integrable ones. The most powerful approach to such systems is the KAM–theory [3, 4, 5, 6]. Before giving a brief account of KAM–theory we remind the structure of the integrable Hamiltonian systems.

The phase space of the generic integrable Hamiltonian systems with n-degrees of freedom is foliated into invariant manifolds, the typical fibre being an n-dimensional torus on which the motion is quasiperiodic. A natural question is whether small perturbations destroy these tori. The KAM–theory gives conditions for the integrable systems which guarantee the survival of most of the invariant tori. The conditions are given in terms of the so-called action-angle variables J_1, J_2, ..., J_n; φ_1, φ_2, ..., φ_n. Without going into details we remind that the action-angle variables can be introduced for any integrable system locally near a fixed torus, and they have a property that $\boldsymbol{J} = (J_1, J_2, \ldots, J_n)$ maps a neighbourhood of a fixed torus on an open subset of $\mathbb{R}^n$. The functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are the co-ordinates on any of the nearby tori. Moreover, the first integrals become functions of the action variables J_1, J_2, ..., J_n. At last, to any fixed torus there corresponds an invariant torus on which the motion is quasiperiodic with frequencies $(\omega_1(\boldsymbol{J}), \ldots, \omega_n(\boldsymbol{J})) = (\partial H/\partial J_1, \ldots, \partial H/\partial J_n)$ (see [5] for details).

One condition, stated by Kolmogorov (see [3, 4] and [5], app. 8] and the cited literature) on the Hamiltonian of the integrable system that ensures the survival of most of the invariant tori under small perturbations, is that the frequency map

$$\boldsymbol{J} \longrightarrow (\omega_1(\boldsymbol{J}), \omega_2(\boldsymbol{J}), \ldots, \omega_n(\boldsymbol{J}))$$

should be non-degenerated. Analytically this means that the Hesseian

$$\det\left(\frac{\partial^2 H}{\partial J_j \partial J_k}\right), \quad j,\, k = 1, \ldots, n, \tag{1.1}$$

does not vanish. We should note that the measure of the surviving tori decreases with the increase of both perturbation and measure of the set, where the above Hesseian is too close to zero.

Another condition of this type, stated by V. Arnold and J. Moser (see [5, app. 8], [6]), is that of an isoenergetical non-degeneracy, which can be explained as follows. Fix an energy level $H_0 = h_0$. If the Hamiltonian H_0 is written in action variables, then define the following map F_{h_0} from the $(n-1)$ dimensional variety $H_0^{-1}(h_0)$ into the projective space $\mathbb{P}^{n-1}$:

$$F_{h_0} : \boldsymbol{J} \longrightarrow (\omega_1(\boldsymbol{J}) : \omega_2(\boldsymbol{J}) : \ldots : \omega_n(\boldsymbol{J})).$$

Then the system is isoenergetically non-degenerated if the map F_{h_0} is a homeomorphism. Analytically, the isoenergetical non-degeneracy is tantamount to non-vanishing of the determinant

$$\det\begin{pmatrix} \dfrac{\partial^2 H_0}{\partial \boldsymbol{J}^2} & \dfrac{\partial H_0}{\partial \boldsymbol{J}} \\ \dfrac{\partial H_0}{\partial \boldsymbol{J}} & 0 \end{pmatrix}. \tag{1.2}$$

The checking of the conditions (1.1) and (1.2) is a very difficult problem, however, there exist several methods for solving such problems.

Knorrer [7] found a method for checking the Kolmogorov's condition by reducing the number of degrees of freedom. Using this method he proved that for several systems, including the geodesic flow on the ellipsoid and K. Neumann's system, the Kolmogorov's condition is fulfilled almost everywhere.

In a recent paper Horozov [8] proved that for the system describing the spherical pendulum condition (1.1) is satisfied everywhere out of the bifurcation diagram of the energy–momentum map. The crucial role in [8] is played by certain algebraic curves and Abelian integrals on them. The condition (1.2) for the spherical pendulum is checked in [9].

The purpose of this paper is to check the KAM-theory conditions (1.1) and (1.2) for the following system.

A particle, attached to a spring, is oscillating in a rigid body with a fixed point O along a line that passes through the fixed point of the body. The motion of the particle is smooth. Without a loss of generality we assume that the fixed point O is an equilibrium position for the particle. We consider the particular case when the particle is oscillating along a principal inertia axes for the body (let this be the axes which inertia moment is denoted by C) and there are no external forces acting on the system. Then the equations of motion around the fixed point written in the body fixed co-ordinate system are (for the general case see [10])

$$
\begin{aligned}
&A\dot{\omega}_1 + (C - B)\omega_2\omega_3 = -2mr\dot{r}\omega_1 - mr^2\dot{\omega}_1 + mr^2\omega_2\omega_3, \\
&B\dot{\omega}_2 + (A - C)\omega_1\omega_3 = -2mr\dot{r}\omega_2 - mr^2\dot{\omega}_2 - mr^2\omega_1\omega_3, \\
&C\dot{\omega}_3 + (B - A)\omega_1\omega_2 = 0, \\
&\ddot{r} + r\left(\sigma/m - \omega_1^2 - \omega_2^2\right) = 0, \qquad (\dot{\ } = d/dt)
\end{aligned}
\tag{1.3}
$$

where ω_1, ω_2, ω_3 are the components of the angular velocity of the body, A, B, C — the components of the inertia tensor, r is the distance between the particle and the fixed point O, σ — the stiffness of the spring, and m — the mass of the particle.

The system (1.3) possesses the integrals.

$$H = \left\{\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) + m\left[\dot{r}^2 + r^2\left(\omega_1^2 + \omega_2^2\right)\right] + \sigma r^2\right\}/2 = H_0, \tag{1.4}$$

$$M^2 = \left(A + mr^2\right)^2\omega_1^2 + \left(B + mr^2\right)^2\omega_2^2 + C^2\omega_3^2 = M_0^2. \tag{1.5}$$

The system (1.3) is integrable when $A = B$, but we shall consider the simpler case $A = B = C$.

The paper is organized as follows. In Section 2 the system (1.3) is brought into more appropriate form. After that the action variables are introduced and the main results are formulated. The proofs are left for Section 3.

2. ACTION VARIABLES AND MAIN RESULTS

First we shall bring the sytem (1.3) into a more appropriate form. In order to do this we put

$$
\begin{aligned}
&z_1 = r, \quad z_2 = \dot{r}, \\
&M_1 = \left(A + mz_1^2\right)\omega_1, \quad M_2 = \left(B + mz_1^2\right)\omega_2, \quad M_3 = C\omega_3.
\end{aligned}
\tag{2.1}
$$

Then the system (1.3) reads

$$
\begin{aligned}
&\dot{M}_1 = M_2 M_3 \left(1/C - 1/\left(B + m z_1^2\right)\right), \\
&\dot{M}_2 = M_1 M_3 \left(1/\left(A + m z_1^2\right) - 1/C\right), \\
(2.2) \qquad &\dot{M}_3 = M_1 M_2 \left(1/\left(B + m z_1^2\right) - 1/\left(A + m z_1^2\right)\right), \\
&\dot{z}_1 = z_2, \\
&m\dot{z}_2 = m z_1 \left(M_1^2/\left(A + m z_1^2\right)^2 + M_2^2/\left(B + m z_1^2\right)^2 - \sigma/m\right).
\end{aligned}
$$

Now let $A = B = C$. If we consider the system (2.2) on the integral level $M_3 = M_{30}$, put

$$
I = M_1^2 + M_2^2, \quad \varphi = \operatorname{arc\,tg}(M_2/M_1)/(2M_{30}),
$$
$$
z = z_1, \quad p_z = m z_2,
$$

and after rescaling time and variables, we have

$$
\begin{aligned}
&\dot{I} = -\partial H/\partial \varphi = 0, \\
&\dot{\varphi} = \quad \partial H/\partial I = \left(1/(1+z^2) - 1\right)/2, \\
(2.3) \qquad &\dot{z} = \quad \partial H/\partial p_z = p_z, \\
&\dot{p}_z = -\partial H/\partial z = z\left(I/(1+z^2)^2 - s\right), \quad s > 0,
\end{aligned}
$$

where

$$
H = \left[p_z^2 + s z^2 + I(1/(1+z^2) - 1)\right]/2 = h.
$$

The first integrals of the system (2.3) are

$$
\begin{aligned}
&F = I = f, \\
&H = p_z^2 + s z^2 + f(1/(1+z^2) - 1) = 2h.
\end{aligned}
$$

The values of H and F, for which the real movement takes place, define the set

$$
\begin{aligned}
&U = U^{(1)} \cup U^{(2)}, \\
&U^{(1)} = \left\{(h,f),\ h \geqq 0,\ f \geqq 0\right\}, \\
&U^{(2)} = \left\{(h,f),\ f \geqq 0,\ h \leqq 0,\ h \geqq \sqrt{fs} - s/2 - f/2\right\}.
\end{aligned}
$$

In order to introduce the action-angle variables we need to exclude from U the critical values of the energy-momentum map (H, F). It is easy to calculate that these points are the boundaries of U, i.e. the points satisfying the equations

$$
f = 0, \quad h = 0, \quad h = \sqrt{fs} - s/2 - f/2.
$$

Denote by U_r the set of regular values of the energy-momentum map

$$
\begin{aligned}
&U_r = U_r^{(1)} \cup U_r^{(2)}, \quad \text{(Fig. 1)} \\
&U_r^{(1)} = \left\{(h,f),\ h > 0,\ f > 0\right\}, \\
&U_r^{(2)} = \left\{(h,f),\ f > 0,\ h < 0,\ h > \sqrt{fs} - s/2 - f/2\right\}.
\end{aligned}
$$

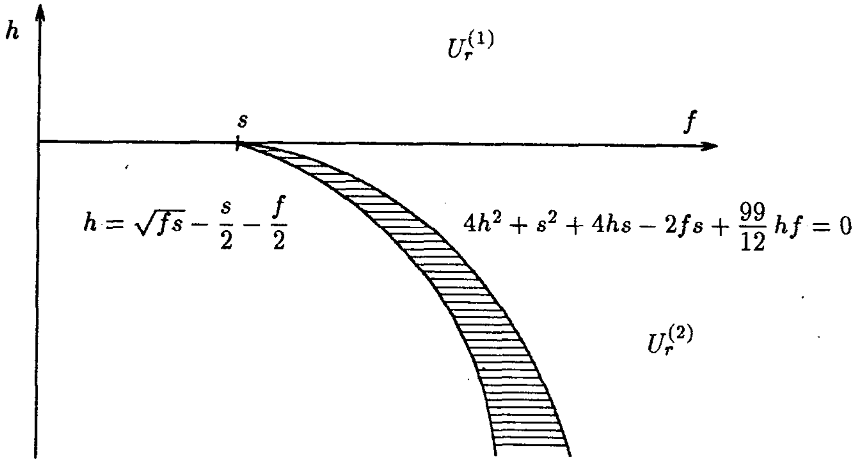

Fig. 1. The set of regular values of energy-moment mapping U_r.
The set V (shaded region)

For the points $(h, f) \in U_r$ the level surface, determined by the equations $H = h$, $F = f$, is a torus $T_{h,f}$. Choose a basis γ_1, γ_2 of the homology group $H_1(T_{h,f}, \mathbb{Z})$ with the following representatives. For γ_1 take the curve on $T_{h,f}$ defined by fixing z, p_z, I, and letting φ run through $[0, 2\pi]$. For γ_2 fix φ and let z, p_z make one circle on the curve given by the equation

$$p_z^2 + sz^2 + f(1/(1+z^2) - 1) = 2h.$$

Now, following [5], we can define the action co-ordinates J_1, J_2 by the formula

$$J_j = \int_{\gamma_j} \sigma, \quad j = 1, 2,$$

where σ is a canonical one-form $\sigma = I\,d\varphi + p_z\,dz$. Trivial computations give

(2.4) $$J_1 = 2\pi f,$$

(2.5) $$J_2 = \int_{\gamma_2} p_z\,dz = 2\int_{z_-}^{z_+} \sqrt{2h - sz^2 + fz^2/(1+z^2)}\,dz,$$

where $z_+ > z_-$ are the two roots of the equation

$$2h = sz^2 - fz^2/(1+z^2).$$

Remark. This is the construction for the action variable J_2 when we have one torus. Actually, when $(h, f) \in U_r^{(2)}$ for any fixed h, f there exist two tori. So, we define the action variables $J_2^{(1)}$; $J_2^{(2)}$ in a neighbourhood of these tori. Due to the symmetry of the curve, $J_2^{(1)} = J_2^{(2)}$, therefore in the following we shall consider anyone of them.

For later use it is convenient to make some changes of variables in the integral J_2. First we make a change $z_1 = z^2$.

$$J_2 = \frac{1}{2} \int\limits_{\gamma_2'} \frac{\sqrt{z_1(1+z_1)[(2h-sz_1)(1+z_1)+fz_1]}}{(1+z_1)z_1} \, dz_1 .$$

After that we put $1+z_1 = 1/z_2$.

$$J_2 = -\frac{1}{2} \int\limits_{\gamma_2''} \frac{\sqrt{(1-z_2)[-s+z_2(2h+s+f)-fz_2^2]}}{(1-z_2)z_2^2} \, dz_2 .$$

We write z and γ again instead of z_2 and γ_2'', respectively, for simplicity. Denote

$$y^2 = (1-z)[z(2h+f+s)-s-fz^2] \tag{2.6}$$

and by γ — the oval of the curve

$$\Gamma_{h,f} = \{(y,z) : y^2 = (1-z)[z(2h+f+s)-s-fz^2]\}.$$

Then we have

$$\psi(h,f) \overset{\text{def}}{=} J_2 = \int\limits_{\gamma} \frac{y\,dz}{(1-z)z^2} . \tag{2.7}$$

Denote by $\widetilde{H}(J_1, J_2)$ the Hamiltonian of the considered system in action-angle co-ordinates. We state the theorems, which are the aim of this paper.

Theorem 1. (i) For $(h,f) \in U_r^{(1)}$ the following determinant does not vanish:

$$\det\left(\frac{\partial^2 \widetilde{H}}{\partial J_j \partial J_k}\right) \neq 0, \quad j,\, k = 1,\, 2, \tag{2.8}$$

(ii) For $(h,f) \in U_r^{(2)}$ the above determinant does not vanish almost everywhere.

Theorem 2. (i) For $(h,f) \in U_r^{(1)}$ the following determinant does not vanish:

$$\det\begin{pmatrix} \dfrac{\partial^2 \widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial \widetilde{H}}{\partial J_1} \\ \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_2^2} & \dfrac{\partial \widetilde{H}}{\partial J_2} \\ \dfrac{\partial \widetilde{H}}{\partial J_1} & \dfrac{\partial \widetilde{H}}{\partial J_2} & 0 \end{pmatrix} \neq 0; \tag{2.9}$$

(ii) For $(h,f) \in V = \{f>0,\ h<0,\ 4h^2+s^2+f^2+4hs-2fs+99hf/12>0\}$ in $U_r^{(2)}$ the above determinant has at most two zeros.

We shall give the conditions (2.8) and (2.9) an explicit form in terms of Abelian integrals of the second kind. Using the expressions (2.4) and (2.7) for J_1 and J_2, we can determine $\widetilde{F}$ and $\widetilde{H}$ implicitly from the equations

$$J_1 = 2\pi \widetilde{F}, \quad J_2 = \psi(\widetilde{F}, \widetilde{H}).$$

Lemma 2.1 (Horozov [8]).

$$(2\pi)^2(\partial\psi/\partial h)^4 \det \begin{pmatrix} \dfrac{\partial^2 \widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} \\ \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_2^2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial h^2} & \dfrac{\partial^2 \psi}{\partial h \partial f} \\ \dfrac{\partial^2 \psi}{\partial h \partial f} & \dfrac{\partial^2 \psi}{\partial f^2} \end{pmatrix}.$$

Similarly, we have

Lemma 2.2.

$$(2\pi)\psi_h^3 \det \begin{pmatrix} \dfrac{\partial^2 \widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial \widetilde{H}}{\partial J_1} \\ \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_2^2} & \dfrac{\partial \widetilde{H}}{\partial J_2} \\ \dfrac{\partial \widetilde{H}}{\partial J_1} & \dfrac{\partial \widetilde{H}}{\partial J_2} & 0 \end{pmatrix} = \psi_{ff}.$$

It is easy to be seen that

$$\psi_h = \int\limits_\gamma \frac{dz}{zy} \neq 0$$

in U_r, because $z_+ > z_- > 0$ and $\int\limits_\gamma \frac{dz}{y} \neq 0$ since it is the period. So, instead of Theorem 1 and Theorem 2 we shall prove their equivalent.

Theorem 3. For $(h, f) \in U_r$ the following determinant does not vanish:

$$D = \det \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial h^2} & \dfrac{\partial^2 \psi}{\partial h \partial f} \\ \dfrac{\partial^2 \psi}{\partial h \partial f} & \dfrac{\partial^2 \psi}{\partial f^2} \end{pmatrix}.$$

Theorem 4. (i) For $(h, f) \in U_r^{(1)}$ the expression $D_1 = \psi_{ff}$ does not vanish;

(ii) For $(h, f) \in V = \{f > 0,\ h < 0,\ 4h^2 + s^2 + f^2 + 4hs - 2fs + 99hf/12 > 0\}$ in $U_r^{(2)}$ the above expression $D_1 = \psi_{ff}$ has at most two zeros.

Next we would like to show that the entries of D (and D_1) can be represented as elliptic integrals. If we differentiate (2.7) twice formally, we get the following expressions:

$$\frac{\partial^2 \psi}{\partial h^2} = \int\limits_\gamma \frac{(z-1)\,dz}{y^3},$$

(2.10)
$$\frac{\partial^2 \psi}{\partial h \partial f} = -\frac{1}{2}\int\limits_\gamma \frac{(1-z)^2\,dz}{y^3},$$

$$\frac{\partial^2 \psi}{\partial f^2} = -\frac{1}{4}\int\limits_\gamma \frac{(1-z)^3\,dz}{y^3}.$$

The differential forms containing y^{-3} have poles along γ. A standard way to get rid of the poles on the integration path is to consider $\Gamma_{h,f}$ as a complex curve. Topologically, it is a torus from which one point is removed [11]. If we deform the cycle γ into a cycle γ', homological to γ, on which the functions y and $z(1-z^2)$ have neither poles nor zeros, then by the Cauchy's theorem [11] the function $\psi(h,f)$ can be defined by the integral (2.7) on γ' instead of γ. After these notes it is clear that the derivatives (2.10) are well defined. We again denote γ' by γ.

3. PROOFS

First we need the functions

$$w_j = \int_\gamma \frac{z^j\,dz}{y^3}, \quad j = 0,\ 1,\ 2,\ \ldots \tag{3.1}$$

The next lemma gives a representation of D as a quadratic form in w_0, w_1.

Lemma 3.1. The determinant D has the representation

$$D = -h(w_1 - w_0)^2/(2f) - \left[(w_1 - w_0)(f - s - h) - hw_1\right]^2/(9f^2). \tag{3.2}$$

Proof. We have to express the derivatives (2.10) by the integrals w_0, w_1. Obviously, we have $\psi_{hh} = w_1 - w_0$. For ψ_{hf} and ψ_{ff} we have

$$\psi_{hf} = -(w_0 - 2w_1 + w_2)/2,$$

$$\psi_{ff} = -(w_0 - 3w_1 + 3w_2 - w_3)/4.$$

Now, we express integrals w_2 and w_3 via w_0 and w_1 in the following way:

$$w_2 = \int_\gamma \frac{z^2\,dz}{y^3} = \frac{1}{3f}\int_\gamma \frac{d[fz^3]}{y^3}$$

$$= \frac{1}{3f}\int_\gamma \frac{d[y^2 + z^2(2h + 2f + s) - z(2h + 2s + s) + s]}{y^3}$$

$$= \frac{1}{3f}\left\{2\int_\gamma \frac{dy}{y^2} + 2(2h + 2f + s)\int_\gamma \frac{z\,dz}{y^3} - (2h + 2s + f)\int_\gamma \frac{dz}{y^3}\right\}$$

$$= \left[2(2h + 2s + f)w_1 - (2h + 2s + f)w_0\right]/(3f),$$

i.e.

$$w_2 = \left[2(2h + 2s + f)w_1 - (2h + 2s + f)w_0\right]/(3f).$$

Similarly,

$$w_3 = \left[(2h + 2s + f)w_1 - 2sw_0\right]/f.$$

Consequently,

$$\psi_{ff} = -h(w_1 - w_0)/(2f)$$

and

$$\psi_{hf} = \left[(f - s - 2h)w_1 + (h + s - f)w_0\right]/(3f),$$

from where we obtain the representation (3.2). This completes the proof of Lemma 3.1.

Next we shall put the family of the elliptic curves $\Gamma_{h,f}$ into the normal form

$$\Gamma_p = \{(u,v) \in \mathbb{C}^2,\ v^2 = 2(u^3 - 3u + p)\} \tag{3.3}$$

by the translation $r = x + \delta$, where

$$\delta = (2h + 2f + s)/(3f), \tag{3.4}$$

and the rescaling $y = \alpha v$, $x = \beta u$, where

$$\beta = \sqrt{(2h + 2f + s)^2 - 3f(2h + 2s + f)}/(3f), \quad \alpha^2 = f\beta^3/2. \tag{3.5}$$

If we put

$$p(h,f) = \frac{f\delta^3 - (2h + 2f + s)\delta^2 + (2h + 2s + f)\delta - s}{2\alpha^2}, \tag{3.6}$$

we get (3.3). It was proven in [12] that $w_0 \neq 0$ in U_r. This allows us to introduce the function

$$\sigma = w_1/w_0. \tag{3.7}$$

Then in the variables u, v and p the integrals w_0 and w_1 become

$$w_0 = \frac{\beta}{\alpha^3} \int\limits_{\gamma(p)} \frac{du}{v^3}, \qquad w_1 = \frac{\beta}{\alpha^3} \int\limits_{\gamma(p)} \frac{(\beta u + \delta)\, du}{v^3}.$$

Here $\gamma(p)$ is the cycle homological to the oval of the curve Γ_p, defined for $p \in (-2, 2)$ and $v \neq 0$ on $\gamma(p)$.

Introduce the new functions

$$\theta_0(p) = \int\limits_{\gamma(p)} \frac{du}{v^3}, \quad \theta_1(p) = \int\limits_{\gamma(p)} \frac{u\, du}{v^3},$$

and their ratio

$$\rho(p) = \theta_1(p)/\theta_0(p).$$

In these notations we have

$$\sigma = \beta\rho + \delta. \tag{3.8}$$

The following result from [8] is crucial for the proof of the theorems.

Lemma 3.2 (Horozov, [8]). (i) The function $\rho(p)$ is strictly monotonous decreasing in the interval $[-2, 2]$;

(ii) $\rho(-2) = 7/5, \quad \rho(2) = 1$.

Proof of Theorem 3. It is seen from the representation (3.2) that if the entries in D are not simultaneously zero, the determinant D is negative in $U_r^{(1)}$ where $h > 0$. We shall show that they cannot be simultaneously zero. Suppose that $w_1 - w_0 = 0$. Then D takes the form

$$D = -(hw_1)^2/(9f^2) = -(hw_0)^2/(9f^2) \neq 0 \text{ in } U_r.$$

Now suppose that the second entry in D is zero, i.e.

$$(f-s-2h)w_1+(h+s-f)w_0=0.$$

The both coefficients vanish simultaneously on the boundary of U_r, so

$$D=-\frac{h^2w_0^2}{2f(f-s-2h)^2}\neq 0.$$

Let now $(h,f)\in U_r^{(2)}$. Here $h<0$, $f>s$, so the expressions in (3.2) have different signs. It is easy to calculate that D is

(3.9) $$D=\left(aw_1^2+2bw_1w_0+cw_0^2\right)/\left(18f^2\right)=w_0^2\left(a\sigma^2+2b\sigma+c\right)/\left(18f^2\right),$$

where $a=-9hf-2(f-s-2h)^2$, $b=9hf+2(f-s-2h)(h+s-f)$, $c=-9hf-2(h+s-f)^2$.

Consider the point in $U_r^{(2)}$, obtained when $h=-s/2$, $f=6s$. Then the expression $a\sigma^2+2b\sigma+c$ reads

$$T=-s^2\left(45\tilde{\sigma}^2+186\tilde{\sigma}+67/2\right),$$

where the wave over corresponding expressions means that they are evaluated in the point $h=-s/2$, $f=6s$. But $\tilde{\sigma}=\tilde{\beta}\tilde{\rho}+\tilde{\delta}$, where $\tilde{\beta}=(18)^{-1/2}$, $\tilde{\delta}=2/3$, and $\tilde{\rho}\in(1,7/5)$, since $h=-s/2$, $f=6s$ is in $U_r^{(2)}$. Obviously T is transformed in

$$T=-s^2\left(5\tilde{\rho}^2+82(2)^{1/2}\tilde{\rho}+355\right)/2.$$

Suppose that T (and therefore D) is zero in this point. It turns out, as it is easy to calculate that the two roots of the equation $T=0$ are negative and this is a contradiction with the admissible values for $\tilde{\rho}$. Therefore $\tilde{D}=D_{f=6s,\,h=-s/2}\neq 0$, i.e. D is not identically zero. Hence D is not zero almost everywhere in $U_r^{(2)}$. This completes the proof of the Theorem 3 and hence Theorem 1.

Proof of Theorem 4. First we transform D_1 in the following way

$$\begin{aligned}D_1&=-hw_0(w_1/w_0-1)/(12f)=-h\theta_0\beta(\beta\rho+\delta-1)/(12f\alpha^3)\\&=-\frac{h\beta^2}{12f\alpha^3}\theta_0\left(\rho-\frac{1-\delta}{\beta}\right).\end{aligned}$$

Denote

(3.10) $$Z=\frac{1-\delta}{\beta}.$$

First, we shall prove that $Z<1$ when $(h,f)\in U_r^{(1)}$. Let substitute in Z δ and β with their equals from (3.4) and (3.5). Then by direct computations it is seen that the inequality $Z<1$ is equivalent to $h>0$, i.e. $D_1\neq 0$ in $U_r^{(1)}$. This proves the first part of Theorem 4.

By the same way it is shown that $7/5>Z>1$ when $(h,f)\in V\cap U_r^{(2)}$, where by V the following domain is denoted:

$$V=\left\{(h,f):\ f>0,\ h<0,\ 4h^2+s^2+f^2+4hs-2fs+99hf/12>0\right\}.$$

Now let $\nu = 1/Z$. For any fixed $\nu \in [5/7, 1]$ the equation $\rho(p) - 1/\nu = 0$ has exactly one solution $p(\nu) \in [-2, 2]$ as Lemma 3.2 implies. This defines a function $\nu \to p(\nu)$, $\nu \in [5/7, 1]$, which is strictly increasing. Let $\nu_0 \in (5/7, 1)$, $(h, f) \in V \cap U_r^{(2)}$, and $p_0(\nu_0)$ is its correspondent via the equation $\rho(p_0) - 1/\nu_0 = 0$. Then from (3.7) it is seen that the preimage of ν_0 contains at most two points, when h is fixed, and from (3.6) — that the preimage of $p_0(\nu_0)$ contains at most six points. It is clear now that zeros of D_1 in $V \cap U_r^{(2)}$ can be at most two. This completes the proof of Theorem 4 and hence of Theorem 2.

Acknowledgement. I would like to thank E. Horozov for his interest to this work.

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Received 02.02.1993

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 86, 1992

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 86, 1992

ON THE ABSORPTION COEFFICIENT OF RANDOM DISPERSIONS

MIKHAIL KOLEV, KONSTANTIN MARKOV

Михаил Колев, Константин Марков. О КОЭФФИЦИЕНТЕ АБСОРБЦИИ СЛУЧАЙНОЙ ДИСПЕРСИИ СФЕР

Работа посвящена задаче определения эффективного коэффициента абсорбции сред случайной структуры. Вариационные оценки этого коэффициента, недавно предложенные авторами, вычислены явно для случайной суспенсии сфер до порядка „квадрата концентрации" и сравнены с оценками Талбота и Уиллиса. Оказывается, что оценки авторов ужe, однако они, в отличии от оценок Талбота и Уилиса, применимы лишь для концентрации сфер, не превошодящих 0.10.

Mikhail Kolev, Konstantin Markov. ON THE ABSORPTION COEFFICIENT OF RANDOM DISPERSIONS

The problem of predicting the effective absorption coefficient of random media is discussed. The variational estimates on this coefficient, recently derived by the authors, are explicitly evaluated for random dispersion of spheres to the order "square of concentration". A comparison with the bounds of Talbot and Willis is performed as well. It appears that the proposed bounds are more restrictive but, unlike those of Talbot and Willis, are only applicable for sphere concentrations that do not exceed 0.10.

1. INTRODUCTION

Consider the steady-state equation

$$\Delta\varphi(\mathbf{x}) - k^2(\mathbf{x})\varphi(\mathbf{x}) + K = 0 \tag{1.1}$$

that governs, at the expense of some simplifying assumptions, the concentration

$\varphi(\mathbf{x})$ of a diffusing species (say, irradiation defects), generated at the constant rate K, in a random absorbing (lossy) medium (see [1] for references and more details). The absorption coefficient $k^2(\mathbf{x})$ is a given random field, assumed positive, statistically homogeneous and isotropic. The problem is to evaluate the random field $\varphi(\mathbf{x})$, i.e. all its multipoint correlations, and, in particular, to find the mean defect concentration $\langle\varphi(\mathbf{x})\rangle$; the brackets $\langle\cdot\rangle$ hereafter denote ensemble averaging. The latter value allows to obtain the effective absorption coefficient (sink strength) k^{*2} of the medium, defined by the relation $k^{*2}\langle\varphi(\mathbf{x})\rangle = K$.

Recently the authors have proposed variational estimates on the coefficient k^{*2}, using the technique of truncated functional series and a procedure of Beran's type [2]. We shall recall now these bounds in the particular case of a two-phase medium. Having in mind the application to particulate media and dispersions of spheres in particular, we call one of the constituents, for definiteness sake, filler and denote its absorption coefficient by k_f^2 and its volume fraction — by $c_f = c$; the other constituent is called then matrix and its respective parameters are k_m^2 and $c_m = 1 - c$. Thus the random absorption field of the medium is

$$k^2(\mathbf{x}) = \begin{cases} k_m^2, & \text{if } \mathbf{x} \in \text{matrix}, \\ k_f^2, & \text{if } \mathbf{x} \in \text{filler}, \end{cases}$$

or

$$k^2(\mathbf{x}) = k_m^2 + [k^2]I_f(\mathbf{x}) = \langle k^2\rangle + [k^2]I_f'(\mathbf{x}), \tag{1.2}$$

where $[k^2] = k_f^2 - k_m^2$, $I_f(\mathbf{x})$ is the characteristic function of the region, occupied by the filler, and $I_f'(\mathbf{x}) = I_f(\mathbf{x}) - c$ is its fluctuating part.

The elementary (one-point) bounds on k^{*2} read

$$k_R^2 \leqq k^{*2} \leqq k_V^2, \tag{1.3}$$

where

$$k_V^2 = \langle k^2(\mathbf{x})\rangle = ck_f^2 + (1-c)k_m^2,$$

$$k_R^2 = \frac{1}{\langle\alpha^2(\mathbf{x})\rangle} = (c\alpha_f^2 + (1-c)\alpha_m^2)^{-1};$$

here $\alpha^2(\mathbf{x}) = 1/k^2(\mathbf{x})$ is the compliance field for the medium. The bounds (1.3) are the obvious counterparts of the well-known Voigt and Reuss estimates on the effective conductivity or elastic moduli of a heterogeneous medium.

The bounds on k^{*2}, announced in [3] and detailed in [4], are already three-point and thus they are always tighter than the elementary ones (1.3). The bounds have the form

$$k_R^2 \left\{ 1 - \frac{[\alpha^2]^2}{\langle\alpha^2\rangle^2} \frac{c(1-c)(I_2^\alpha)^2}{I_2^\alpha + \frac{[\alpha^2]}{\langle\alpha^2\rangle}(1-2c)I_3^\alpha} \right\}^{-1} \tag{1.4}$$

$$\leqq k^{*2} \leqq k_V^2 \left(1 - \frac{[k^2]^2}{\langle k^2\rangle^2} \frac{c(1-c)(I_2^k)^2}{I_2^k + \frac{[k^2]}{\langle k^2\rangle}(1-2c)I_3^k} \right).$$

The following four statistical parameters enter the bounds:

$$I_2^k = \frac{k_V^2}{M_2^k(0)} \int G_V(\mathbf{y}) M_2^k(\mathbf{y})\, d^3\mathbf{y}, \quad I_2^\alpha = -\frac{1}{M_2^\alpha(0)} \int \Delta G_R(\mathbf{y}) M_2^\alpha(\mathbf{y})\, d^3\mathbf{y}, \tag{1.5}$$

$$I_3^k = \frac{k_V^4}{M_3^k(0,0)} \int\int G_V(\mathbf{y}_1) G_V(\mathbf{y}_2) M_3^k(\mathbf{y}_1, \mathbf{y}_2)\, d^3\mathbf{y}_1\, d^3\mathbf{y}_2,$$
$$I_3^\alpha = \frac{1}{M_3^\alpha(0,0)} \int\int \Delta G_R(\mathbf{y}_1) \Delta G_R(\mathbf{y}_2) M_3^\alpha(\mathbf{y}_1, \mathbf{y}_2)\, d^3\mathbf{y}_1\, d^3\mathbf{y}_2. \tag{1.6}$$

Here $G_V(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}\exp(-k_V|\mathbf{x}|)$ is the Green function of the operator $\Delta - k_V^2$, i.e.

$$\Delta G_V(\mathbf{x}) - k_V^2 G_V(\mathbf{x}) + \delta(\mathbf{x}) = 0, \tag{1.7}$$

and similarly for $G_R(\mathbf{x})$ with k_V replaced by k_R;

$$M_2^k(\mathbf{y}) = \langle \delta k^2(0) \delta k^2(\mathbf{y}) \rangle, \quad M_3^k(\mathbf{y}_1, \mathbf{y}_2) = \langle \delta k^2(0) \delta k^2(\mathbf{y}_1) \delta k^2(\mathbf{y}_2) \rangle \tag{1.8}$$

are, respectively, the two- and three-point correlation functions for the field $k^2(\mathbf{x})$. The same functions for the compliance field $\alpha^2(\mathbf{x})$ are denoted by $M_2^\alpha(y)$ and $M_3^\alpha(\mathbf{y}_1, \mathbf{y}_2)$. Since the medium is two-phase, we have the well-known relations

$$M_2^k(0) = c(1-c)[k^2]^2, \quad M_3^k(0,0) = c(1-c)(1-2c)[k^2]^3, \tag{1.9}$$

and similarly for $M_2^\alpha(0)$ and $M_3^\alpha(0,0)$. Hereafter the integrals are over the whole space $\mathbb{R}^3$, if the integration domain is not explicitly indicated.

Note that in [4] it was shown, in particular, that the bounds (1.4) are third-order in the weakly-inhomogeneous case. Moreover, the explicit results, obtained in [4] for Miller's cellular media, indicate that the bounds remain useful even when the absorption capabilities of the constituents differ one hundred times.

In this paper we shall consider in detail the evaluation of the statistical parameters (1.5) and (1.6) for random dispersions of nonoverlapping spheres. In Section 2 we briefly summarize the needed in the sequel statistical description of random dispersions. In Section 3 we calculate the parameters I_2^k and I_2^α for the dispersion, given in (1.5), that depend on the two-point correlations. Similar calculations are performed in Section 4 for the parameters I_3^k and I_3^α, see (1.6), but unlike the "two-point" parameters, we are able to give analytical results correct to the asymptotic order c^2 only. In Section 5 we illustrate the performance of the bounds and compare them with those of Talbot and Willis [1].

2. STATISTICAL DESCRIPTION OF RANDOM DISPERSIONS

We consider a random dispersion of spheres, i.e. an unbounded matrix, containing an array of equal and nonoverlapping spherical inhomogeneities, each one of radius a. The medium is thus completely described by the system of random points $\{\mathbf{x}_\alpha\}$ — the centers of the spheres. The statistics of the system $\mathbf{x}_\alpha$ is conveniently represented by the multipoint distribution densities $f_p(\mathbf{y}_1, \ldots, \mathbf{y}_p)$, or probability density functions. They define the probability dP to simultaneously find a point of the random set $\{\mathbf{x}\}_\alpha$ per each of the infinitesimal volumes $\mathbf{y}_i < \mathbf{y} < \mathbf{y}_i + d\mathbf{y}_i$, $i = 1, \ldots, p$, to be

$$dP = f_p(\mathbf{y}_1, \ldots, \mathbf{y}_p)\, d^3\mathbf{y}_1 \ldots d^3\mathbf{y}_p. \tag{2.1}$$

We assume that the system $\{\mathbf{x}_\alpha\}$ is statistically isotropic and homogeneous; then, in particular, $f_1 = n$ and $f_p = f_p(\mathbf{y}_{2,1}, \ldots, \mathbf{y}_{p,1})$, where $\mathbf{y}_{j,i} = \mathbf{y}_j - \mathbf{y}_i$ and n denotes the number density, i.e. the mean number of points per unit volume.

Let us imagine now that by means of a certain manufacturing process we produce random point systems $\{\mathbf{x}\}_\alpha$ with different number densities n. The statistics of the system $\{\mathbf{x}\}_\alpha$ will then depend on n as a parameter, i.e. $f_p = f_p(\mathbf{Y}_p; n)$, $\mathbf{Y}_p = (\mathbf{y}_1, \ldots, \mathbf{y}_p)$. We shall assume, as usual, that $f_p \sim n^p$, i.e. f_p has the asymptotic order n^p at $n \to 0$, $p = 1, 2, \ldots$ In particular, for the two-point distribution density f_2, which most frequently appears in models and theoretical studies, we have

$$f_2(\mathbf{y}_1, \mathbf{y}_2) = n^2 g(r), \ g(r) = g_0(r) + O(n), \tag{2.2}$$

$r = |\mathbf{y}_2 - \mathbf{y}_1|$. (The point system $\{\mathbf{x}\}_\alpha$ hereafter will be assumed statistically isotropic as well.) Thus $g_0(r)$ is the zero-density limit of the radial distribution function $g(r)$ for the system $\{\mathbf{x}\}_\alpha$.

A convenient characteristics of the set of random points is the so-called random density field

$$\psi(\mathbf{x}) = \sum_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha). \tag{2.3}$$

This field was systematically used by Stratonovich [5] in the one-dimensional case when the role of $\mathbf{x}$ is played by the time. The random function $\psi(\mathbf{x})$ is uniquely defined by the random set $\mathbf{x}_\alpha$. The respective formulas [5] read:

$$\begin{aligned}
\langle \psi(\mathbf{y}) \rangle &= f_1(\mathbf{y}) = n, \\
\langle \psi(\mathbf{y}_1)\psi(\mathbf{y}_2) \rangle &= f_1(\mathbf{y}_1)\delta(\mathbf{y}_{1,2}) + f_2(\mathbf{y}_1, \mathbf{y}_2), \\
\langle \psi(\mathbf{y}_1)\psi(\mathbf{y}_2)\psi(\mathbf{y}_3) \rangle &= f_1(\mathbf{y}_1)\delta(\mathbf{y}_{1,2})\delta(\mathbf{y}_{1,3}) \\
&\quad + 3\left\{\delta(\mathbf{y}_{1,2}) f_2(\mathbf{y}_{1,3})\right\}_s + f_3(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3),
\end{aligned} \tag{2.4}$$

etc., where $\{\cdot\}_s$ means symmetrization with respect to all different combinations of indices in the braces.

The random absorption field (1.2) of the medium under study has a simple integral representation by means of the field $\psi(\mathbf{x})$, namely

$$k^2(\mathbf{x}) = \langle k^2 \rangle + [k^2] \int h(\mathbf{x} - \mathbf{y}) \psi'(\mathbf{y})\, d^3\mathbf{y}, \tag{2.5}$$

where $\psi'(\mathbf{y}) = \psi(\mathbf{y}) - n$ is the fluctuating part of the field $\psi(\mathbf{y})$, $h(\mathbf{y})$ is the characteristic function of a single sphere of radius a, located at the origin.

3. EVALUATION OF THE PARAMETERS I_2^k AND I_2^α

According to (1.2), (2.4) and (2.5), the parameter I_2^k for the dispersion has the form

$$I_2^k = \frac{k_V^2}{c(1-c)} \iiint G_V(\mathbf{y}) h(\mathbf{z}_1) h(\mathbf{y}-\mathbf{z}_2) \langle \psi'(\mathbf{z}_1)\psi'(\mathbf{z}_2)\rangle \, d^3\mathbf{y} d^3\mathbf{z}_1 d^3\mathbf{z}_2$$

(3.1)
$$= \frac{k_V^2}{c(1-c)} \iint h(\mathbf{z}_1)\chi_V(\mathbf{z}_2)\left[n\delta(\mathbf{z}_1-\mathbf{z}_2) - n^2 R(\mathbf{z}_1-\mathbf{z}_2)\right] d^3\mathbf{z}_1 d^3\mathbf{z}_2$$

$$= \frac{a_0 - a_1 c}{1-c}; \quad R(\mathbf{z}) = 1 - g(\mathbf{z}),$$

with the coefficients

(3.2)
$$a_0 = \frac{k_V^2}{V_a} \int h(\mathbf{z})\chi_V(\mathbf{z})\, d^3\mathbf{z},$$

(3.3)
$$a_1 = \frac{k_V^2}{V_a^2} \iint h(\mathbf{z}_1)\chi_V(\mathbf{z}_2) R(\mathbf{z}_1-\mathbf{z}_2)\, d^3\mathbf{z}_1 d^3\mathbf{z}_2.$$

Here $V_a = \frac{4}{3}\pi a^3$ and

(3.4)
$$\chi_V(\mathbf{z}) = (h * G_V)(\mathbf{z})$$

is the Helmholtz potential for a single sphere of radius a, located at the origin. Let us recall that it is the continuous and bounded everywhere solution of the Helmholtz equation $\Delta\chi_V - k_V^2\chi_V + h(\mathbf{z}) = 0$. A simple calculation yields

(3.5a)
$$\chi_V(\mathbf{z}) = \frac{1}{k_V^2} \begin{cases} A' \dfrac{a_V \sinh r_V}{r_V \sinh a_V} + 1, & r < a, \\ A'' \dfrac{a_V}{r_V} e^{a_V - r_V}, & r \geqq a, \end{cases}$$

(3.5b)
$$A' = -\frac{1+a_V}{a_V} e^{-a_V} \sinh a_V, \quad A'' = \frac{1}{a_V} e^{-a_V}(a_V \cosh a_V - \sinh a_V),$$

where $a_V = a k_V$ and $r_V = r k_V$ are dimensionless, $r = |\mathbf{z}|$.

Using (3.5), we find first the coefficient a_0:

(3.6)
$$a_0 = 1 - F_2(a_V),$$

where

(3.7)
$$F_2(x) = 3\frac{1+x}{x^3} e^{-x}(x\cosh x - \sinh x)$$

is the function that appeared when calculating the parameters I_2^k and I_2^α for cellular media with spherical shape of the cells, see [3,4].

For the coefficient a_1 we get in turn:

(3.8)
$$a_1 = \frac{k_V^2}{V_a^2} \int h(\mathbf{z}) P(\mathbf{z})\, d^3\mathbf{z},$$

where P denotes the convolution $P(\mathbf{z}) = (\chi_V * R)(\mathbf{z})$. Due to (1.7), the function P solves the equation

$$\Delta P - k_V^2 P + h * R = 0. \tag{3.9}$$

The assumption of nonoverlapping yields $g(\mathbf{z}) = 0$ and thus $R(\mathbf{z}) = 1 - g(\mathbf{z}) = 1$ at $|\mathbf{z}| \leq 2a$. That is why $(h * R)(\mathbf{z}) = V_a$ within the sphere $|\mathbf{z}| \leq a$ and the solution of eqn (3.9) within the same sphere has therefore the form

$$P(\mathbf{z}) = \frac{V_a}{k_V^2}\left(1 + B\frac{a_V \sinh r_V}{r_V \sinh a_V}\right), \quad r = |\mathbf{z}| < a. \tag{3.10}$$

The unknown constant B is found by means of the obvious relation

$$P(0) = \frac{V_a}{k_V^2}\left(1 + B\frac{a_V}{\sinh a_V}\right) = \int \chi_V(\mathbf{z}) R(\mathbf{z})\, d^3\mathbf{z},$$

or

$$B = \frac{\sinh a_V}{a_V}\left[\frac{k_V^2}{V_a}\int \chi_V(\mathbf{z})(1 - g(\mathbf{z}))\, d^3\mathbf{z} - 1\right]. \tag{3.11}$$

Simple calculations, using (3.5), yield eventually

$$a_1 = 1 - \frac{4a_V^2 e^{2a_V}}{(1 + a_V)^2} F_2^2(a_V) I, \tag{3.12}$$

where

$$I = \int_1^\infty s e^{-2a_V s} g(s)\, ds, \quad s = r/2a, \tag{3.13}$$

is the statistical parameter that appeared in Talbot and Willis' bounds on the effective absorption coefficient k^{*2} [1].

In the simplest two-point statistics — the so-called "well-stirred" case — one has $g(s) = 1$ at $s \geq 1$, so that

$$I = I^{ws}(a_V) = \frac{1 + 2a_V}{4a_V^2} e^{-2a_V}, \tag{3.14}$$

and thus

$$a_1 = 1 - \frac{1 + 2a_V}{(1 + a_V)^2} F_2^2(a_V). \tag{3.15}$$

Note that Talbot and Willis were able also to evaluate the parameter I in the case when the two-point statistics of the dispersion is governed by the well-known Percus-Yevick approximation:

$$I = I^{py}(a_V) = G(2a_V),$$

$$G(t) = \frac{tL(t)}{12c[L(t) + S(t)e^t]}, \quad L(t) = 12c\left[\left(1 + \frac{1}{2}c\right)t + 1 + 2c\right], \tag{3.16}$$

$$S(t) = (1 - c)^2 t^3 + 6c(1 - c)t^2 + 18c^2 t - 12c(1 + 2c).$$

A simple check shows that

$$I^{py}(a_V) = I^{ws}(a_V) + O(c),$$

as it should be.

Thus the needed statistical parameter I_2^k for the dispersion is

$$I_2^k = \frac{a_0 - a_1 c}{1 - c} = \varphi_2(a_V), \tag{3.17}$$

where a_0, a_1 are explicitly given in (3.6), (3.12) respectively. Hence φ_2 is a known function of the dimensionless parameter a_V, depending on the radial distribution function $g(r)$ for the dispersion through the statistical parameter I.

The evaluation of the second statistical parameter I_2^α, as given in (1.5), is now straightforward. Keeping in mind (1.7), we get immediately

$$I_2^\alpha = -\frac{1}{M_2^\alpha(0)} \int \Delta G_R(\mathbf{y}) M_2^\alpha(\mathbf{y})\, d^3\mathbf{y}$$

$$= 1 - \frac{k_R^2}{M_2^\alpha(0)} \int G_R(\mathbf{y}) M_2^\alpha(\mathbf{y})\, d^3\mathbf{y},$$

so that

$$I_2^\alpha = 1 - \varphi_2(a_R), \tag{3.18}$$

where $\varphi_2(a_R)$ is the function, defined in (3.17), in which a_V should be replaced everywhere by a_R.

4. EVALUATION OF THE PARAMETERS I_3^k AND I_3^α

Unlike I_2^k and I_2^α, we are able to evaluate the three-point parameters I_3^k and I_3^α to the order c^2 only. The reason is that the three-point probability density f_3 will enter the needed moments, so that the only way to obtain analytical results is to neglect it, assuming $f_3 \sim c^3$, see Section 2. Thus all formulae hereafter are correct to the order $O(c^2)$ only.

According to (1.9), (2.4) and (2.5), the parameter I_3^k for the dispersion has the form

$$I_3^k = \frac{k_V^4}{c(1-c)(1-2c)} \iiint \left[\int G_V(\mathbf{y}_1) h(\mathbf{y}_1 - \mathbf{z}_1)\, d^3\mathbf{y}_1\right]$$

$$\times \left[\int G_V(\mathbf{y}_2) h(\mathbf{y}_2 - \mathbf{z}_2)\, d^3\mathbf{y}_2\right] h(\mathbf{z}_3) \langle \psi'(\mathbf{z}_1)\psi'(\mathbf{z}_2)\psi'(\mathbf{z}_3)\rangle\, d^3\mathbf{z}_1 d^3\mathbf{z}_2 d^3\mathbf{z}_3$$

$$= \frac{k_V^4}{c(1-c)(1-2c)} \iiint \chi_V(\mathbf{z}_1)\chi_V(\mathbf{z}_2) h(\mathbf{z}_3) \tag{4.1}$$

$$\times \left[n\delta(\mathbf{z}_{1,2})\delta(\mathbf{z}_{1,3}) - n^2 3\{\delta(\mathbf{z}_{1,2})R_0(\mathbf{z}_{2,3})\}_s\right] d^3\mathbf{z}_1 d^3\mathbf{z}_2 d^3\mathbf{z}_3$$

$$= \frac{b_0 - b_1 c}{(1-c)(1-2c)}; \quad R_0(\mathbf{z}) = 1 - g_0(\mathbf{z}),$$

with the coefficients

$$b_0 = \frac{k_V^4}{V_a}\int h(\mathbf{z})\chi_V^2(\mathbf{z})\,d^3\mathbf{z}, \tag{4.2}$$

$$b_1 = 2J_1 + J_2, \tag{4.3}$$

$$J_1 = \frac{k_V^4}{V_a^2}\int\!\!\int h(\mathbf{z}_1)\chi_V(\mathbf{z}_1)\chi_V(\mathbf{z}_2)R_0(\mathbf{z}_1-\mathbf{z}_2)\,d^3\mathbf{z}_1 d^3\mathbf{z}_2, \tag{4.4}$$

$$J_2 = \frac{k_V^4}{V_a^2}\int\!\!\int h(\mathbf{z}_1)\chi_V^2(\mathbf{z}_2)R_0(\mathbf{z}_1-\mathbf{z}_2)\,d^3\mathbf{z}_1 d^3\mathbf{z}_2, \tag{4.5}$$

$g_0(\mathbf{z})$ is the zero-density limit of the radial distribution function $g(\mathbf{z})$ for the dispersion, see (2.2).

Using (3.5), we find first the coefficient b_0:

$$b_0 = 1 - 2F_2(a_V) + F_3(a_V), \tag{4.6}$$

where

$$F_3(x) = \frac{3}{2}\frac{(1+x)^2}{x^3}e^{-2x}(\sinh x\cosh x - x) \tag{4.7}$$

is the function that appeared when evaluating the three-point statistical parameters I_3^k and I_3^α for a cellular medium, see [3,4], and $F_2(x)$ is defined in (3.7).

Let us evaluate next the coefficient b_1 in (4.3). To this end we first recast the integral J_1 as

$$J_1 = \frac{k_V^4}{V_a^2}\int h(\mathbf{z})\chi_V(\mathbf{z})P_0(\mathbf{z})\,d^3\mathbf{z}, \tag{4.8}$$

where $P_0(\mathbf{z}) = (\chi_V * R_0)(\mathbf{z})$ is the convolution, similar to that used in Section 3. Keeping in mind (3.5), (3.10) and (3.11), we find straightforwardly that

$$J_1 = 1 - F_2(a_V) + \frac{3}{4}\frac{a_V\cosh a_V - \sinh a_V}{a_V^5(1+a_V)}e^{a_V}\left[F_3(a_V) - F_2(a_V)\right]I, \tag{4.9}$$

where I is the statistical parameter of Talbot and Willis, see (3.13), corresponding to the zero-density limit $g_0(\mathbf{z})$ of the radial distribution function.

In the particular case of a well-stirred dispersion we have, due to (3.14),

$$J_1 = 1 - F_2(a_V) + \frac{1+2a_V}{(1+a_V)^2}F_2(a_V)\left[F_3(a_V) - F_2(a_V)\right]. \tag{4.10}$$

The evaluation of the second integral J_2, entering the expression for the coefficient b_1, is more complicated. We first recast its definition (4.4) as

$$J_2 = \frac{k_V^4}{V_a^2}\int \chi_V^2(\mathbf{z})F_0(\mathbf{z})\,d^3\mathbf{z}, \tag{4.11}$$

where

$$F_0(\mathbf{z}) = \int h(\mathbf{z}-\mathbf{y})R_0(\mathbf{y})\,d^3\mathbf{y} = V_a - \int h(\mathbf{z}-\mathbf{y})g_0(\mathbf{y})\,d^3\mathbf{y}. \tag{4.12}$$

Let $h_A(\mathbf{y})$ be the characteristic function of a sphere of radius A located at the origin. Following [6], we denote

$$d_A h_A(\mathbf{y}) = h_{A+dA}(\mathbf{y}) - h_A(\mathbf{y}) = \begin{cases} 1, & \text{if } A < |\mathbf{y}| < A + dA, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily seen that

$$g_0(\mathbf{y}) = \int_{2a}^{\infty} g_0(A) d_A h_A(\mathbf{y}),$$

which is inserted into (4.12):

$$\begin{aligned} F_0(\mathbf{z}) &= V_a - \int_{2a}^{\infty} g_0(A) \left[\int h(\mathbf{z} - \mathbf{y}) \frac{h_{A+dA}(\mathbf{y}) - h_A(\mathbf{y})}{dA} d^3\mathbf{y} \right] dA \\ &= V_a - \int_{2a}^{\infty} g_0(A) \left[\frac{d}{dA} F^A(\mathbf{z}) \right] dA, \quad F^A(\mathbf{z}) = (h * h_A)(\mathbf{z}). \end{aligned} \tag{4.13}$$

We introduce, in turn, (4.13) into (4.11) and integrate by parts:

$$\begin{aligned} J_2 &= \frac{k_V^4}{V_a^2} \left\{ V_a \int \chi_V^2(\mathbf{z})\, d^3\mathbf{z} - \int_{2a}^{\infty} g_0(A) \left[\frac{d}{dA} \int \chi_V^2(\mathbf{z}) F^A(\mathbf{z})\, d^3\mathbf{z} \right] dA \right\} \\ &= \frac{k_V^4}{V_a^2} \left\{ \int_{2a}^{\infty} g_0'(A) \int \left[\chi_V^2(\mathbf{z}) F^A(\mathbf{z})\, d^3\mathbf{z} \right] dA + g_0(2a) \int \chi_V^2(\mathbf{z}) F^{2a}(\mathbf{z})\, d^3\mathbf{z} \right\}, \end{aligned} \tag{4.14}$$

having used the facts that $F_\infty(\mathbf{z}) = V_a$ and $g_0(\infty) = 1$.

Let

$$\mu(\lambda, a_V) = \frac{k_V^4}{V_a^2} \int \chi_V^2(\mathbf{z}) F^{\lambda a}(\mathbf{z})\, d^3\mathbf{z}; \quad \lambda = \frac{A}{a} \geqq 2. \tag{4.15}$$

Simple algebra, based on the analytical form (3.5) of the Helmholtz potential $\chi_V(\mathbf{z})$, yields

$$\mu(\lambda, a_V) = 1 - 2F_2(a_V) + F_3(a_V) + F_4(a_V, \lambda) + F_5(a_V, \lambda), \tag{4.16}$$

where

$$\begin{aligned} F_4(x, y) &= \frac{3}{2} \frac{(x \cosh x - \sinh x)^2}{x^3} \left(e^{-2x} - e^{-2(y-1)x} \right), \\ F_5(x, y) &= \frac{3}{128} \left(\frac{x \cosh x - \sinh x}{x^3} \right)^2 \Big\{ \left[(12(y^3 + y^2 - y) + 52) x^3 \right. \\ &\quad \left. - 6(y+1)^2 x^2 + 6(y-1)x + 3 \right] e^{-2(y-1)x} \\ &\quad - \left[12(y-1)^2(y+1)x^3 - 6(y-1)^2 x^2 + 6(y+1)x + 3 \right] e^{-2(y+1)x} \\ &\quad + 24(y-1)^2(y+1)^2 x^4 \left[Ei(-2(y-1)x) - Ei(-2(y+1)x) \right] \Big\} \end{aligned} \tag{4.17}$$

and F_2 and F_3 are the functions, defined in (3.7) and (4.7), respectively. As usual

$$Ei(-t) = \int_{-\infty}^{-t} \frac{e^s}{s}\, ds$$

denotes the integral exponent.

Thus

$$J_2 = a\int_2^{\infty} g_0'(\lambda a)\mu(\lambda, a_V)\, d\lambda + g_0(2a)\mu(2, a_V). \tag{4.18}$$

In the particular case of a well-stirred dispersion

$$J_2 = b_0 + \frac{3}{128}\left(\frac{a_V\cosh a_V - \sinh a_V}{a_V^3}\right)^2$$
$$\times\Big\{\big[172a_V^3 - 54a_V^2 + 6a_V + 3\big]e^{-2a_V} - \big[36a_V^3 - 6a_V^2 + 18a_V + 3\big]e^{-6a_V} \tag{4.19}$$
$$+216a_V^4\big[Ei(-2a_V) - Ei(-6a_V)\big]\Big\}.$$

Eventually, the needed parameter I_3^k is

$$I_3^k = \frac{b_0 - b_1 c}{(1-c)(1-2c)} = \varphi_3(a_V), \tag{4.20}$$

where a_0, a_1 are explicitly given in (4.6), (4.3), etc., respectively. Hence φ_3 is a known function of the dimensionless parameter a_V, depending on the zero-density limit $g_0(r)$ of the radial distribution function for the dispersion through the integrals I and J_2, see (3.13) and (4.18) respectively.

The evaluation of the statistical parameter I_3^α is already easy. From its definition $(1.6)_2$ and eqn (1.7) (with k_V^2 replaced by k_R^2) we have

$$I_3^\alpha = 1 - \frac{2k_R^2}{M_3^\alpha(\mathbf{0},\mathbf{0})}\int G_R(\mathbf{y})M_3^\alpha(\mathbf{0},\mathbf{y})\, d^3\mathbf{y}$$
$$+ \frac{k_R^4}{M_3^\alpha(\mathbf{0},\mathbf{0})}\int\!\!\int G_R(\mathbf{y}_1)G_R(\mathbf{y}_2)M_3^\alpha(\mathbf{y}_1,\mathbf{y}_2)\, d^3\mathbf{y}_1 d^3\mathbf{y}_2. \tag{4.21}$$

But

$$M_3^\alpha(\mathbf{0},\mathbf{y}) = [\alpha]^3\left\langle I_f'^2(0)I_f'(\mathbf{y})\right\rangle = (1-2c)[\alpha]^3\left\langle I_f'(0)I_f'(\mathbf{y})\right\rangle = (1-2c)[\alpha]M_2^\alpha(\mathbf{y}),$$

since $I_f'^2(0) = (I_f(0) - c)^2 = (1-2c)I_f(0) + c^2$. (Note that $I_f^2(\mathbf{x}) = I_f(\mathbf{x})$.) Hence

$$\frac{k_R^2}{M_3^\alpha(\mathbf{0},\mathbf{0})}\int G_R(\mathbf{y})M_3^\alpha(\mathbf{0},\mathbf{y})\, d^3y = \frac{k_R^2}{M_2^\alpha(0)}\int G_R(\mathbf{y})M_2^\alpha(\mathbf{y})\, d^3\mathbf{y} = \varphi_2(a_R),$$

see (3.17), so that

$$I_3^\alpha = 1 - 2\varphi_2(a_R) + \varphi_3(a_R), \tag{4.22}$$

because the last term in the r.-h. side of (4.21) is immediately recognized as the function φ_3 from (4.20) in which a_V is to be replaced everywhere by a_R.

5. COMPARISON WITH THE BOUNDS OF TALBOT AND WILLIS

The results of Sections 3 and 4 allow us to evaluate the bounds (1.4) for the dispersion to the order c^2. Indeed the relations (3.17) and (3.18) give us the values of the two-point statistical parameters I_2^k and I_2^α for an arbitrary radial distribution function $g(r)$. In turn, eqns (4.20) and (4.22) provide the values of the three-point statistical parameters I_3^k and I_3^α, again for an arbitrary radial distribution function $g(r)$, but to the order c^2 only.

Let us recall that in [1] Talbot and Willis derived bounds on the effective absorption coefficient k^{*2} for a dispersion of spheres, using an original variational principle of Hashin–Shtrikman's type. Their bounds have the form

$$\text{(5.1a)} \qquad \frac{k^{*2}}{k_0^2} = 1 + \frac{\lambda\delta + \mu\gamma}{\alpha\delta + \beta\gamma}$$

with the coefficients

$$\alpha = \frac{3c(a_f \cosh a_f - \sinh a_f)}{(1-c)a_f^3}, \quad \beta = \frac{a^2 k_0^2}{k_f^2}\langle k^2 \rangle,$$

$$\gamma = e^{-ak_0}\left(\cosh a_f + \frac{k_0}{k_f}\sinh a_f\right) - \frac{3c(k_f^2 - k_0^2)(a_f\cosh a_f - \sinh a_f)\eta}{k_0^2 a_f^3 (1-c)} + \theta,$$

$$\text{(5.1b)} \qquad \eta = (1 + ak_0)e^{-ak_0} - \frac{12cI}{ak_0}(ak_0\cosh ak_0 - \sinh ak_0),$$

$$\theta = \frac{12cI}{ak_0}\left[\cosh ak_f \sinh ak_0 - \frac{k_0}{k_f}\sinh ak_f \cosh ak_0\right],$$

$$\delta = \frac{k_0^2}{k_f^2}(a_f^2 - a_m^2)\eta, \quad \lambda = \frac{3c(k_f^2 - k_0^2)(a_f\cosh a_f - \sinh a_f)}{(1-c)k_0^2 a_f^3},$$

$$\mu = c\frac{k_m^2}{k_f^2}(a_f^2 - (ak_0)^2) + (1-c)(a_m^2 - (ak_0)^2), \quad a_f = ak_f, \ a_m = ak_m.$$

(Our notations differ a bit from the original ones used in [1].)

Upon inserting $k_0 = \min(k_m, k_f)$ in (5.1) one obtains a lower bound on k^{*2} and, similarly, inserting $k_0 = \max(k_m, k_f)$ — an upper one. In (5.1b) I is the statistical parameter, defined in (3.13), which carries information about the two-point statistics of the dispersion. In this sense the bounds (5.1) are two-point and therefore should be expected to be less restrictive than ours (1.4) which are three-point.

It is to be pointed out, however, that Talbot and Willis' bounds (5.1) are useful for all values $c \in (0,1)$ of the sphere volume fraction while the bounds (1.4) have been calculated in the foregoing analysis only for dilute fractions — to the order c^2 — and thus may be expected to provide useful results for values of c not exceeding $0.10-0.15$. The numerical calculations confirm these expectations. The c^2-bounds (1.4) are closer to the exact values of k^{*2} and more restrictive than the Talbot and Willis estimates (5.1) only at sphere fractions c not exceeding 0.1. This is illustrated in Tables 1 and 2 for a well-stirred dispersion of spheres in the two cases $k_f^2/k_m^2 = 10$ and $k_f^2/k_m^2 = 0.1$ respectively (at $a_m = 1$). The exact values are found by means of the numerical procedure, developed in [7] which employs the techniques of the factorial functional series [8] and allows to obtain explicitly the full statistical solution of eqn (1.1) to the order c^2 for the dispersion and, in particular, the effective absorption coefficient k^{*2} to the same order. The results for other values of a_m ($a_m = 10$ and $a_m = 0.1$) are similar and therefore they are not shown here.

Acknowledgement. The support of this work by the Bulgarian Ministry of Science, Education and Culture under Grant No MM26-91 is gratefully acknowledged.

Table 1

c	TW-lower	KM-lower	exact	KM-upper	TW-upper
0.0	1	1	1	1	1
0.02	1.071	1.071	1.071	1.072	1.089
0.04	1.147	1.147	1.147	1.147	1.183
0.06	1.229	1.230	1.230	1.231	1.281
0.08	1.317	1.318	1.318	1.319	1.384
0.10	1.413	1.414	1.415	1.417	1.492

Estimates on the effective absorption coefficient k^{*2} for a well-stirred dispersion at $a_m = 1$, $k_f^2/k_m^2 = 10$: KM — the bounds (1.4); TW — the bounds (5.1) of Talbot and Willis

Table 2

c	TW-lower	KM-lower	exact	KM-upper	TW-upper
0.0	1	1	1	1	1
0.02	0.977	0.978	0.979	0.979	0.979
0.04	0.955	0.957	0.958	0.958	0.958
0.06	0.934	0.937	0.937	0.937	0.937
0.08	0.914	0.917	0.917	0.917	0.917
0.10	0.894	0.896	0.896	0.897	0.897

The same as in Table 1 at $k_f^2/k_m^2 = 0.1$

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Received 15.03.1993

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 86, 1992

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 86, 1992

ON THE TWO-SPHERE PROBLEM IN AN ABSORBING MEDIUM

MIKHAIL KOLEV

Михаил Колев. ОБ АБСОРБЦИОННОЙ ЗАДАЧЕ ДВУХ СФЕР

Работа посвящена решению абсорбционной задачи двух сфер, которая состоит в следующем. В неограниченной матрице содержатся две непересекающиеся сферы одинакового радиуса, в которых генерируются дефекты с постоянной скоростью. Дефекты поглощаются матрицей и сферами с различными коэффициентами абсорбции. Требуется определить стационарное распределение дефектов в среде. Эта задача возникает естественным образом при вычислении эффективного коэффициента абсорбции случайной суспенсии сфер. Предложенное аналитическое решение использует т. наз. метод двойных разложений, который удобен для численной реализации.

Mikhail Kolev. ON THE TWO-SPHERE PROBLEM IN AN ABSORBING MEDIUM

The paper is devoted to the two-sphere absorption problem. Namely, let two identical spheres be embedded into an unbounded matrix. Defects are created within the spheres at constant rate and are absorbed, with different absorption coefficients, by the matrix and the spheres. The steady-state defect distribution in the medium has to be found. This problem appears in a natural way when evaluating the effective absorption coefficient of a random dispersion of spheres. The herein proposed analytical solution employs the twin-expansions method and it is convenient for numerical implementation.

1. INTRODUCTION

Consider the equation

$$\Delta H^{(2)}(\mathbf{x};\mathbf{z}) - \{k_m^2 + [k^2](h(\mathbf{x}) + h(\mathbf{x}-\mathbf{z}))\} H^{(2)}(\mathbf{x};\mathbf{z}) - [k^2]\{h(\mathbf{x}) + h(\mathbf{x}-\mathbf{z})\} = 0. \tag{1}$$

Here $h(\mathbf{x})$ is the characteristic function of a sphere of radius a located at the origin, $[k^2] = k_f^2 - k_m^2$; the differentiation everywhere is with respect to $\mathbf{x}$, $\mathbf{z}$ plays the role of a parameter, $|\mathbf{z}| > 2a$. The solution $H^{(2)}(\mathbf{x};\mathbf{z})$, we are seeking, should be bounded and continuous everywhere in $\mathbb{R}^3$, and its normal derivative should be continuous on the surfaces $|\mathbf{x}| = a$ and $|\mathbf{z}-\mathbf{x}| = a$. The interpretation of eqn (1) is as follows. It describes the steady-state distribution $H^{(2)}(\mathbf{x};\mathbf{z})$ of a diffusing species (say, irradiation defects) generated within two non-overlapping spheres, embedded into an unbounded matrix, at the rate $-[k^2]$. The spheres are of radius a, located at the origin and at the point $\mathbf{z}$. The species is then absorbed by the spheres and by the matrix with different (and positive) absorption coefficients k_f^2 and k_m^2 respectively. Due to this interpretation, the problem (1) was called in [1,2] the two-sphere absorption problem. Let us recall that this problem appeared in a natural way in [1,2] when looking for the effective absorption coefficient of a random dispersion of spheres to the order c^2, where c is the volume fraction of the spheres. Hence, the solution of (1), especially in a form suitable for numerical implementation, is needed when evaluating the statistical characteristics of the diffusing species field in the random dispersion and, in particular, when calculating the effective absorption coefficient of the latter. The aim of this paper to describe such a solution of the problem (1), using the method of twin-expansions. It is to be pointed out that a similar method has been successfully employed in the respective two-spheres problems for heat conduction [3,4], elasticity [5], diffraction [6], etc.

2. TWIN-EXPANSION SOLUTION OF THE PROBLEM (1)

Let us introduce two Cartesian systems and two systems of spherical co-ordinates as shown in Fig. 1. Both spheres are of radius a, the origins of the systems are at the centres of the spheres, the χ-co-ordinate is common for them, and the distance between the centres, $|O_1O_2|$, is denoted by R, so that

$$x_1 = x_2, \quad y_1 = y_2, \quad z_1 = z_2 + R.$$

Then, obviously,

$$x_1 = r_1 \sin\theta_1 \cos\chi, \quad x_2 = r_2 \sin\theta_2 \cos\chi,$$

$$y_1 = r_1 \sin\theta_1 \sin\chi, \quad y_2 = r_2 \sin\theta_2 \sin\chi,$$

$$z_1 = r_1 \cos\theta_1, \quad z_2 = -r_2 \cos\theta_2,$$

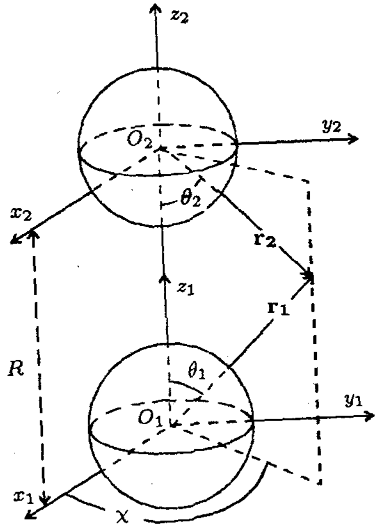

Fig. 1

where $0 \leqq r_1, r_2 < \infty$, $0 \leqq \theta_1, \theta_2 \leqq \pi$, $0 \leqq \chi < 2\pi$.

We need, first of all, a convenient form of the relation between the spherical wave functions given in the two spherical co-ordinate systems. We start with the identity [6]:

$$Z_n^{(1)}(k_m r_2) P_n(\cos\theta_2) = (-1)^n \sum_{s=0}^{\infty} Q_{oson}(k_m R, \pi) j_s(k_m r_1) P_s(\cos\theta_1), \tag{2}$$

where

$$Z_n^{(1)}(k_m r_2) = \sqrt{\frac{\pi}{2k_m r_2}} H^{(1)}_{n+\frac{1}{2}}(k_m r_2), \quad j_s(k_m r_1) = \sqrt{\frac{\pi}{2k_m r_1}} J_{s+\frac{1}{2}}(k_m r_1) \tag{3}$$

are the spherical Bessel functions,

$$H_n^{(1)}(z) = J_m(z) + iN_m(z)$$

denotes as usual the Hankel function of the first kind,

$$Q_{oson}(k_m R, \pi) = \frac{2i^{s-n}}{N_{os}} \sum_{\sigma=|s-n|}^{s+n} i^{\sigma} b_{\sigma}^{(sono)} Z_{\sigma}^{(1)}(k_m R) P_{\sigma}(-1), \quad N_{os} = \frac{2}{2s+1}, \tag{4}$$

$P_s(x)$ are the Legendre polynomials, and

$$b_n^{(n_1 m_1 n_2 m_2)} = (-1)^{m_2} \sqrt{\frac{(n_1+m_1)!(n_2+m_2)!(n-m_1+m_2)!}{(n_1-m_1)!(n_2-m_2)!(n+m_1-m_2)!}}$$

$$\times (n_1 n_2 oo|no)(n_1 n_2 m_1, -m_2|n, m_1 - m_2),$$

where $(n_1 n_2 m_1 m_2|n, m_1 + m_2)$ are the Clebsh–Gordan coefficients.

According to the fact that $P_\sigma(-1) = (-1)^\sigma$, we have from (2):

$$\sqrt{\frac{\pi}{2k_m r_2}} H^{(1)}_{n+\frac{1}{2}}(k_m r_2) P_n(\cos\theta_2) = (-1)^n \sqrt{\frac{\pi}{2k_m r_1}} \sqrt{\frac{\pi}{2k_m R}} \sum_{s=0}^{\infty} J_{s+\frac{1}{2}}(k_m r_1)$$

(5)

$$\times P_s(\cos\theta_1) \frac{2i^{s-n}}{N_{0s}} \sum_{\sigma=|s-n|}^{s+n} (-i)^\sigma b_\sigma^{(s0n0)} H^{(1)}_{\sigma+\frac{1}{2}}(k_m R).$$

Using the well-known relations between the Bessel functions:

(6) $$I_{s+\frac{1}{2}}(x) = i^{-(s+\frac{1}{2})} J_{s+\frac{1}{2}}(ix), \quad K_{s+\frac{1}{2}}(x) = \frac{\pi i}{2} i^{(s+\frac{1}{2})} H^{(1)}_{s+\frac{1}{2}}(ix),$$

we find from (5) the formula

$$\frac{1}{\sqrt{k_m r_2}} K_{n+\frac{1}{2}}(k_m r_2) P_n(\cos\theta_2) = (-1)^n \sqrt{\frac{2\pi}{k_m R}} \frac{1}{\sqrt{k_m r_1}}$$

(7)

$$\times \sum_{s=0}^{\infty} \frac{(-1)^s}{N_{0s}} I_{s+\frac{1}{2}}(k_m r_1) P_s(\cos\theta_1) \Big[\sum_{\sigma=|s-n|}^{s+n} (-1)^\sigma b_\sigma^{(s0n0)} K_{\sigma+\frac{1}{2}}(k_m R) \Big]$$

between the spherical wave functions in the two co-ordinate systems (Fig. 1), which is just the form suitable for our purposes.

We look for the solution $H^{(2)}(\mathbf{x};\mathbf{z})$ of the eqn (1), that is independent of the co-ordinate φ, in the following form:

— inside the sphere "i" as

(8a) $$H^{(2)}_{(i)} = -\frac{[k^2]}{k_f^2} + \sum_{n=0}^{\infty} A_n \sqrt{\frac{a}{r_i}} I_{n+\frac{1}{2}}(k_f r_i) P_n(\cos\theta_i), \quad i = 1, 2;$$

— outside the spheres as

$$H^{(2)}_{(out)} = \sum_{n=0}^{\infty} \Big\{ C_n^{(1)} \sqrt{\frac{a}{r_1}} K_{n+\frac{1}{2}}(k_m r_1) P_n(\cos\theta_1)$$

(8b)

$$+ C_n^{(2)} \sqrt{\frac{a}{r_2}} K_{n+\frac{1}{2}}(k_m r_2) P_n(\cos\theta_2) \Big\}.$$

Due to the obvious symmetry of the problem under study we have $C_n = C_n^{(1)} = C_n^{(2)}$, $n = 0, 1, \ldots$ Thus

$$H^{(2)}_{(out)} = \sum_{n=0}^{\infty} C_n \Big\{ \sqrt{\frac{a}{r_1}} K_{n+\frac{1}{2}}(k_m r_1) P_n(\cos\theta_1)$$

(9)

$$+ \sqrt{\frac{a}{r_2}} K_{n+\frac{1}{2}}(k_m r_2) P_n(\cos\theta_2) \Big\}.$$

The coefficients A_n and C_n are to be found from the above mentioned boundary conditions, namely, the continuity of the field $H^{(2)}(\mathbf{x};\mathbf{z})$ and its normal derivative across each spherical interface (with respect to $\mathbf{x}$; recall that $\mathbf{z}$ plays the role of a parameter).

According to (7), we recast the solution (8b) of eqn (1) outside the spheres as

$$H^{(2)}_{(out)} = \sqrt{\frac{a}{r_1}}\sum_{n=0}^{\infty} P_n(\cos\theta_1)\Big\{C_n K_{n+\frac{1}{2}}(k_m r_1) + \sqrt{\frac{2\pi}{k_m R}}\frac{1}{N_{on}} I_{n+\frac{1}{2}}(k_m r_1) \tag{10}$$
$$\times \sum_{s=0}^{\infty}(-1)^{n+s}C_s\Big[\sum_{\sigma=|s-n|}^{s+n}(-1)^{\sigma} b_{\sigma}^{(sono)} K_{\sigma+\frac{1}{2}}(k_m R)\Big]\Big\}.$$

Making use of the above mentioned boundary conditions, the orthogonality of the Legendre polynomials and the fact that $b^{(sooo)} = 1$, $s = 0, 1, \ldots$, see [6], we find the following relations between the unknown coefficients

$$A_0 I_{\frac{1}{2}}(a_f) - \frac{[k^2]}{k_f^2} = C_0 K_{\frac{1}{2}}(a_m) + \frac{1}{N_{oo}} I_{\frac{1}{2}}(a_m)\sqrt{\frac{2\pi}{k_m R}}\left[\sum_{s=0}^{\infty} C_s K_{s+\frac{1}{2}}(k_m R)\right], \tag{11a}$$

$$A_n I_{n+\frac{1}{2}}(a_f) = C_n K_{n+\frac{1}{2}}(a_m) + \frac{1}{N_{on}} I_{n+\frac{1}{2}}(a_m)\sqrt{\frac{2\pi}{k_m R}} \tag{11b}$$
$$\times\left[\sum_{s=0}^{\infty} C_s\Big(\sum_{\sigma=|s-n|}^{s+n}(-1)^{n+s+\sigma} b_{\sigma}^{(sono)} K_{\sigma+\frac{1}{2}}(k_m R)\Big)\right], \quad n = 1, 2, \ldots,$$

$$A_0[2a_f I'_{\frac{1}{2}}(a_f) - I_{\frac{1}{2}}(a_f)] = C_0\left[2a_m K'_{\frac{1}{2}}(a_m) - K_{\frac{1}{2}}(a_m)\right] \tag{12a}$$
$$+ \frac{1}{N_{oo}}\sqrt{\frac{2\pi}{k_m R}}[2a_m I'_{\frac{1}{2}}(a_m) - I_{\frac{1}{2}}(a_m)]\sum_{s=0}^{\infty} C_s K_{s+\frac{1}{2}}(k_m R),$$

$$A_n[2a_f I'_{n+\frac{1}{2}}(a_f) - I_{n+\frac{1}{2}}(a_f)] = C_n\left[2a_m K'_{n+\frac{1}{2}}(a_m) - K_{n+\frac{1}{2}}(a_m)\right]$$
$$+\frac{1}{N_{on}}\sqrt{\frac{2\pi}{k_m R}}\left[2a_m I'_{n+\frac{1}{2}}(a_m) - I_{n+\frac{1}{2}}(a_m)\right] \tag{12b}$$
$$\times\left[\sum_{s=0}^{\infty} C_s\Big(\sum_{\sigma=|s-n|}^{s+n}(-1)^{n+s+\sigma} b_{\sigma}^{(sono)} K_{\sigma+\frac{1}{2}}(k_m R)\Big)\right], \quad n = 1, 2, \ldots$$

Simple manipulations, employing the well-known properties

$$I'_s(x) = \frac{1}{2}[I_{s+1}(x) + I_{s-1}(x)], \quad K'_s(x) = -\frac{1}{2}[K_{s+1}(x) + K_{s-1}(x)]$$

of the modified Bessel functions, allow to exclude A_n from eqns (11), so that (12) yields the needed equations for the coefficients C_n:

(13)
$$C_0 U_0 + V_0 \sum_{s=0}^{\infty} C_s K_{s+\frac{1}{2}}(k_m R) = -2\frac{[k^2]}{k_f^2} a_f I_{\frac{3}{2}}(a_f),$$

$$C_n U_n + V_n \sum_{s=0}^{\infty} C_s \left[\sum_{\sigma=|s-n|}^{s+n} (-1)^{n+s+\sigma} b_\sigma^{(sono)} K_{\sigma+\frac{1}{2}}(k_m R) \right] = 0,$$

$n = 1, 2, \dots$, where

(14a)
$$U_n = a_f K_{n+\frac{1}{2}}(a_m)\left[I_{n-\frac{1}{2}}(a_f) + I_{n+\frac{3}{2}}(a_f)\right] + a_m I_{n+\frac{1}{2}}(a_f)\left[K_{n-\frac{1}{2}}(a_m) + K_{n+\frac{3}{2}}(a_m)\right],$$

(14b)
$$V_n = \frac{1}{N_{on}}\sqrt{\frac{2\pi}{k_m R}}\Big\{a_f I_{n+\frac{1}{2}}(a_m)\left[I_{n-\frac{1}{2}}(a_f) + I_{n+\frac{3}{2}}(a_f)\right] - a_m I_{n+\frac{1}{2}}(a_f)\left[I_{n-\frac{1}{2}}(a_m) + I_{n+\frac{3}{2}}(a_m)\right]\Big\},$$

and $a_f = ak_f$, $a_m = ak_m$ are dimensionless parameters.

For the coefficients A_n we get in turn:

(15a)
$$A_0 = \frac{1}{I_{\frac{1}{2}}(a_f)}\Big\{\frac{[k^2]}{k_f^2} + C_0 K_{\frac{1}{2}}(a_m) + \frac{1}{2} I_{\frac{1}{2}}(a_m)\sqrt{\frac{2\pi}{k_m R}} \sum_{s=0}^{\infty} C_s K_{s+\frac{1}{2}}(k_m R)\Big\},$$

(15b)
$$A_n = \frac{1}{I_{n+\frac{1}{2}}(a_f)}\Big\{C_n K_{n+\frac{1}{2}}(a_m) + \frac{1}{N_{on}} I_{n+\frac{1}{2}}(a_m)\sqrt{\frac{2\pi}{k_m R}} \times \Big[\sum_{s=0}^{\infty} C_s \Big(\sum_{\sigma=|s-n|}^{s+n} (-1)^{n+s+\sigma} b_\sigma^{(sono)} K_{\sigma+\frac{1}{2}}(k_m R)\Big)\Big]\Big\}, \quad n = 1, 2, \dots$$

In this way the two-sphere problem (1) is reduced to the solution of the infinite system of linear equations (13) for any given separation distance R between the spheres.

The natural numerical procedure to solve the problem (1) is the method of truncation. Namely, assuming $C_n = 0$ at $n > N$ in (13), we get a linear system of $N + 1$ equations for the first $N + 1$ coefficients C_n, $n = 0, 1, \dots$ Solving the latter, we find the approximate values $C_n^{(N)}$ of these coefficients. Then, at $N \to \infty$, the approximations $C_n^{(N)}$ will converge to the exact values C_n, as we shall argue in the next section.

Due to the exponential decay of the modified Bessel functions $K_{n+\frac{1}{2}}(x)$, $n = 0, 1, \dots$ the series solution developed in the present section converges very rapidly when the spheres are well apart. For instance, to obtain the values of the coefficients A_n, C_n and the field $H^{(2)}(\mathbf{x};\mathbf{z})$ with three decimal digits, it suffices to take $N = 10$ if $R/a \geqq 3$. However, as the spheres approach each other, more equations should be kept in the truncated system, e.g. at $2.1 < a/R < 3$ we should take $N = 20$ in order to have the same three decimal digits correct.

3. JUSTIFICATION OF THE TRUNCATION METHOD

To justify the above used truncation method a bit more detailed investigation of the infinite system (13) is needed. With this aim in view we recast it in the form

(16)
$$C_0 + \frac{U_0}{V_0}\sum_{s=0}^{\infty} C_s K_{s+\frac{1}{2}}(k_m R) = -2\frac{[k^2]}{k_f^2}\frac{a_f I_{\frac{3}{2}}(a_f)}{U_0},$$
$$C_n + \frac{V_n}{U_n}\sum_{s=0}^{\infty} C_s\left[\sum_{\sigma=|s-n|}^{s+n}(-1)^{n+s+\sigma} b_\sigma^{(sono)} K_{\sigma+\frac{1}{2}}(k_m R)\right] = 0,$$

$n = 1, 2, \ldots$ In turn, using the relations (6) and the definition (4) of Q_{oson}, we rewrite eqn (16) as

(17)
$$C_n + \sum_{s=0}^{\infty} d_{sn} C_s = f_n, \quad n = 0, 1, \ldots$$

where

(18a)
$$d_{sn} = \frac{\pi}{2} i^{3s+n+2} Q_{onos}(ik_m R, \pi)\frac{W_n}{U_n}, \quad s, n = 0, 1, \ldots$$

(18b)
$$f_n = \begin{cases} -2\frac{[k^2]}{k_f^2}\frac{a_f I_{\frac{3}{2}}(a_f)}{U_0}, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases}$$

(18c)
$$W_n = N_{on}\sqrt{\frac{k_m R}{2}} V_n.$$

Let us replace in the system (17) the unknowns C_n by X_n:

(19)
$$C_n = I_{n+\frac{1}{2}}(a_m) X_n.$$

The infinite set of equations (17), when written with respect to the new unknowns X_n, becomes

(20)
$$X_n + \sum_{s=0}^{\infty} D_{sn} X_s = g_n, \quad n = 0, 1, \ldots$$

where

(21)
$$D_{sn} = d_{sn}\frac{I_{s+\frac{1}{2}}(a_m)}{I_{n+\frac{1}{2}}(a_m)}, \quad g_n = \frac{f_n}{I_{n+\frac{1}{2}}(a_m)}.$$

The asymptotical behaviour of the coefficients D_{sn}, for large values of s, n, can be easily deduced by means of the Debay formulae [7, p. 25]:

$$(22) \qquad K_\lambda(x) \sim \sqrt{\frac{\pi}{2\lambda}}\left(\frac{2\lambda}{ex}\right)^\lambda, \quad I_\lambda(x) \sim \frac{1}{\sqrt{2\pi\lambda}}\left(\frac{2\lambda}{ex}\right)^{-\lambda} \quad \text{for } \lambda \gg x.$$

Introducing (22) into the definitions (14a) and (18c) of the coefficients U_n and W_n we find, after simple algebra, that

$$(23) \qquad \left|\frac{W_n}{U_n}\right| < c_1(2n+3)\left(\frac{ea_m}{2n+1}\right)^{2n+1}, \quad n = 0, 1, \ldots$$

with a certain constant c_1.

For bounding the function Q_{onos} we use the result of Ivanov [6]:

$$(24) \qquad |Q_{onos}| < \frac{c_2 s(2s+2n+1)^{n+s}}{(k_m R)^{3/2}(ek_m R)^{n+s+\frac{1}{2}}},$$

where c_2 is another constant.

We next introduce (22), (23) and (24) into (21) and take into account (18a):

$$(25) \qquad |D_{sn}| < c\frac{s}{2s+1}\left(\frac{2n+2s+1}{2n+1}\right)^n\left(\frac{2n+2s+1}{2s+1}\right)^s\left(\frac{a}{R}\right)^{n+s+1},$$

$s, n = 0, 1, \ldots$

Since $R > 2a$ (the spheres are nonoverlapping), $a/R < \dfrac{1}{2}$ and thus

$$(26a) \qquad \sum_{s,n=0}^{\infty} |D_{sn}|^2 < \infty.$$

Let us note that obviously

$$(26b) \qquad \sum_{n=0}^{\infty} |g_n|^2 < \infty.$$

Hence the system (20) can be recast as

$$(27) \qquad \mathbf{X} + \mathbf{D}\cdot\mathbf{X} = \mathbf{G},$$

where $\mathbf{X} = \{X_n\}$, $\mathbf{G} = \{g_n\}$ and $\mathbf{D} = \{D_{sn}\}$, $s, n = 0, 1, \ldots,$ Due to (26), eqn (27) is an equation in the Hilbert space ℓ^2 with a compact operator $\mathbf{D}$, so that the Fredholm alternative holds [8, Ch. 13]. Therefore, in particular, the system (27) will have a unique solution for any $\mathbf{G} \in \ell^2$ if the homogeneous system (27) possesses a unique (trivial) solution. But the latter is obviously the case for our problem, since $\mathbf{G} = \mathbf{0}$ corresponds to $[k^2] = 0$, i.e. to a homogeneous equation (1). (The homogeneous equation (1) has a trivial solution only, since the absorption coefficients k_f^2 and k_m^2 are positive.) The uniqueness and existence theorem for infinite system (13) is thus proved. Its obvious corollary is then the needed justification of the truncation method of Section 2: due to (26a), the solutions $C_n^{(N)}$ of the truncated systems (13) will indeed converge in ℓ^2 to the solution of (13) at $N \to \infty$.

4. CONCLUDING REMARKS

In this note we have presented and justified a method of effective numerical solution of the two-sphere problem (1). When combined with the general results of [2] concerning diffusion of defects in a random dispersion, it allows to obtain, in particular, the effective absorption coefficient k^{*2} of the dispersion and the two-point correlation function of the defect fields to the order c^2. More details and the respective numerical results are given in [2].

Acknowledgement. The support of this work by the Bulgarian Ministry of Science, Education and Culture under Grant No MM26-91 is gratefully acknowledged.

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Received 20.03.1993

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 86, 1992

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 86, 1992

NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS IV. THE EULERIAN DYNAMICAL EQUATIONS

GEORGI CHOBANOV, IVAN CHOBANOV

While the problem of oscillation of a heavy rigid body about a fixed axis had been solved correctly by Huygens, and while a more satisfactory method containing the germ of several later principles had been created by James Bernoulli in 1703, in 1750 it could not be said that the general motion of a rigid body was understood at all. Even for motion about a fixed axis, the reaction of the body upon its support could not be calculated, and no method for determining the behaviour of a spinning top was known.

Euler's "first principles" changed the scene overnight ... in the paper *Découverte d'un nouveau principe de mécanique*, written 1750 and published 1752, where these principles are published, Euler obtained the general *equations of motion of a rigid body* about its center of gravity. He applied the "first principles" to the elements of mass making up the body, at the same time replacing the acceleration of the element by its expression in terms of the *angular velocity vector*, which makes its first appearance here. Taking moments about the center of gravity then yields, after some reduction, the differential equations of motion known as "Euler's equations" for a rigid body, subject to assigned torque about its center of mass. In the process arise naturally the six components of what is now called the "tensor of inertia".

C. Truesdell: *A Program Towards Rediscovering the Rational Mechanics of the Age of Reason*

Георги Чобанов, Иван Чобанов. ДИНАМИЧЕСКИЕ АКСИОМЫ НЬЮТОНА И ЭЙЛЕРА. IV. ДИНАМИЧЕСКИЕ УРАВНЕНИЯ ЭЙЛЕРА

В этой четвертой части серии статьей [1–3], посвещенные динамическим аксиомам Ньютона и Эйлера, особое внимание уделено динамическим уравнениям Эйлера, управляющие движения всяких твердых тел, как свободных,так и подчиненных произвольным конечным и инфинитезимальным связям. В частности, анализированы уравнения движения твердых прутьев. Работа содержит подробный анализ динамической философии Даламбера и Лагранжа, рассматривающая "le corps proposé comme l'assemblage d'une infinité de corpuscules ou points massifs unis ensamble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances", который показывает, что подобная гипотеза приводит к противоречию с второй аксиомой Ньютона, а именно, что "mutationem motus proportionalem esse vi motrici impressae et fieri secundum lineam rectam, qua vis illa imprimitur".

Georgi Chobanov, Ivan Chobanov. NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS. IV. THE EULERIAN DYNAMICAL EQUATIONS

In this fourth part of the series of articles [1–3], dedicated to the Newtonian and Eulerian dynamical axioms, special stress is put on Euler's dynamical equations governing the motion of any rigid body both free and subjected to arbitrary finite and infinitesimal constraints. In particular, the equations of motion of rigid rods are discussed. The paper contains a detailed analysis of D'Alembert's and Lagrange's dynamical philosophy, regarding "le corps proposé comme l'assemblage d'une infinité de corpuscules ou points massifs unis ensemble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances", which displays clearly that such a hypothesis leads to a contradiction with Newton's second dynamical law, namely "mutationem motus proportionalem esse vi motrici impressae et fieri secundum lineam rectam, qua vis illa imprimatur".

This paper is the natural sequel of a series of articles [1–3], published in this *Annual* some time ago and concerning the logical status of the Newtonian and Eulerian dynamical axioms (the laws, or principles, or postulates, or hypotheses, etc. of momentum and of moment of momentum for mass-points and rigid bodies respectively) in the edifice of analytical mechanics and their connection with Hilbert's Sixth Problem for the axiomatic consolidation of its logical foundations. As almost any second year student has it at his finger's ends, though not every author of dynamical treatises is aware of the fact, the whole of rigid body dynamics is based on, and is developed from, the following two assumptions, or suppositions, or conjectures, or maxims, or tenets, etceteras, formulated by Euler as early as 1750 and nowadays bearing his name:

Ax 1 E (*First Eulerian dynamical axiom* or *principle of momentum of a rigid body*). There exists such a rigid system of reference Σ that, all derivatives being taken with respect to Σ, for any rigid body S and for any system of forces $\underset{\rightarrow}{\Phi}$, *acting* on S, the derivative with respect to the *time* of the momentum of S equals the basis of $\underset{\rightarrow}{\Phi}$.

Df 1 E. Any system of reference, satisfying Ax 1 E, is called *inertial according to Euler.*

Ax 2 E (*Second Eulerian dynamical axiom* or *principle of moment of momentum* (*kinetical moment*) *of a rigid body*). Σ being an inertial according to Euler system of reference and all derivatives being taken with respect to Σ, for any rigid body S and for any system of forces $\underset{\rightarrow}{\Phi}$, *acting* on S, the derivative with respect to the *time* of the moment of momentum of S equals the moment of $\underset{\rightarrow}{\Phi}$, both moments being taken with respect to the origin of Σ.

Before proceeding further, let us make a most important remark that is a matter of principle, since it concerns the logical status of Ax 1 E and Ax 2 E in the system of rigid body dynamics. Euler's dynamical axioms Ax 1 E and Ax 2 E involve a set of terms specific for analytical mechanics and proclaim certain relations between the mechanical entities these terms nominate. The terms themselves are: system of reference, rigid system of reference, origin of a system of reference, derivative (of a vector function) with respect to a system of reference, momentum of a rigid body, moment of momentum (kinetical moment) of a rigid body, system of forces, basis of a system of forces, moment of a system of forces (with respect to a given pole), acting, and time.

Now all the above terms, with the explicit exception of the last two, are capable to a strict mathematical definition — at least as strict as the term "integral" in analysis. The meaning of this statement reduces to the following mathematical fact. As it is well-known [4], a real standard vector space V is defined axiomatically as a set, in which four operations (addition in V, multiplication of the real numbers with the elements of V, scalar multiplication of the elements of V, and vector multiplication in V) are defined, satisfying 15 (3, 4, 5, and 3, respectively, for any of the operations listed above) specific axioms. Now V once granted, all the terms Ax 1 E and Ax 2 E include, with the exceptions of *acting* and *time*, are potentially and actually definitable by means of the algebraical and analytical apparatus in V. It goes without saying that the effective reproduction of the mentioned definitions is out of question here: the reader, taking an interest in this matter, may be referred to the corresponding literary sources. The cold fact remains that using the algebra and analysis in V as mathematical tools and the elements of V as mathematical building materials all the notions the Eulerian dynamical axioms Ax 1 E and Ax 2 E include, with the emphasized exception of *acting* and *time*, may be given specific mathematical definitions satisfying the most severe logical standards of Twentieth Century's mathematics. As regards the notions *acting* and *time*, their logical status in analytical mechanics is identical with that of the notions *point*, *line*, and *plane* in Euclidean geometry.

In other words, if all the terms the Eulerian dynamical axioms Ax 1 E and Ax 2 E include were capable of explicit mathematical definitions, then these statements would be (true of false) dynamical theorems. Now the fact that Ax 1 E and Ax 2 E involve terms incapable of such definitions reduces these statements to dynamical axioms which, in their turn, define (along with other, as yet unstated, dynamical axioms) the terms *acting* and *time* implicitly. In such a manner, Ax 1 E and Ax 2 E are mathematical predicates that are neither provable nor disprovable, just like the fifth postulate in geometry or the mathematical induction in arithmetic. Putting it in another way, one has every right to accept Ax 1 E and Ax 2 E or to reject them. In the first case one arrives at the Eulerian rigid body dynamics; as regards the second case, one is faced with one's own problems.

The Eulerian dynamical axioms are nowadays universally accepted — at least as universally as Euclidean geometry. Analytical dynamics may be then, if not defined, at least rather adequately described, as the mathematics of equilibria and motions of mass-points and rigid bodies, and of the forces, which generate these equilibria and motions and are generated by them. In the same manner, the Eulerian rigid body dynamics may be described as the set of mathematical corollaries derived from Ax 1 E and Ax 2 E. In other words, one may look upon the Eulerian dynamical axioms as questions: if Ax 1 E and Ax 2 E, then what? The answer is one and only: then modern analytical dynamics.

After these general and hence somewhat vague memoranda let us now proceed to the mathematical formalization of Ax 1 E and Ax 2 E.

First and foremost, let $Oxyz$ be an inertial according to Euler orthonormal right-hand orientated Cartesian system of reference (the existence of one at least such a system is ensured by Ax 1 E) with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of the axes Ox, Oy, Oz, respectively. In other words,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = 1, \quad \boldsymbol{i}\boldsymbol{j} = 0, \quad \boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}. \tag{1}$$

Second, let $\Omega\xi\eta\zeta$ be an orthonormal right-hand orientated Cartesian system of reference, invariably connected with the rigid body S (the existence of one at least such a system is ensured by the very definition of the rigid body concept) with unit vectors $\bar{\xi}^0$, $\bar{\eta}^0$, $\bar{\zeta}^0$ of the axes $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$, respectively. In other words,

$$\left(\bar{\xi}^0\right)^2 = \left(\bar{\eta}^0\right)^2 = 1, \quad \bar{\xi}^0\bar{\eta}^0 = 0, \quad \bar{\zeta}^0 = \bar{\xi}^0 \times \bar{\eta}^0. \tag{2}$$

Let the *cosine-directors* $a_{\mu\nu}$ (μ, $\nu = 1, 2, 3$) of $Oxyz$ and $\Omega\xi\eta\zeta$ be defined by

$$\begin{cases} \boldsymbol{i} = a_{11}\bar{\xi}^0 + a_{12}\bar{\eta}^0 + a_{13}\bar{\zeta}^0, \\ \boldsymbol{j} = a_{21}\bar{\xi}^0 + a_{22}\bar{\eta}^0 + a_{23}\bar{\zeta}^0, \\ \boldsymbol{k} = a_{31}\bar{\xi}^0 + a_{32}\bar{\eta}^0 + a_{33}\bar{\zeta}^0. \end{cases} \tag{3}$$

Then (1)–(3) imply

$$\begin{cases} \bar{\xi}^0 = a_{11}\boldsymbol{i} + a_{21}\boldsymbol{j} + a_{31}\boldsymbol{k}, \\ \bar{\eta}^0 = a_{12}\boldsymbol{i} + a_{22}\boldsymbol{j} + a_{32}\boldsymbol{k}, \\ \bar{\zeta}^0 = a_{13}\boldsymbol{i} + a_{23}\boldsymbol{j} + a_{33}\boldsymbol{k}. \end{cases} \tag{4}$$

Let P be any point and

$$\boldsymbol{r} = \boldsymbol{OP}, \quad \boldsymbol{r}_\Omega = \overline{O\Omega}, \quad \bar{\rho} = \overline{\Omega P}. \tag{5}$$

Then the identity $\boldsymbol{OP} = \overline{O\Omega} + \overline{\Omega P}$ implies

$$\boldsymbol{r} = \boldsymbol{r}_\Omega + \bar{\rho}. \tag{6}$$

If

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \tag{7}$$

$$\boldsymbol{r}_\Omega = x_\Omega\boldsymbol{i} + y_\Omega\boldsymbol{j} + z_\Omega\boldsymbol{k}, \tag{8}$$

$$\bar{\rho} = \xi\bar{\xi}^0 + \eta\bar{\eta}^0 + \zeta\bar{\zeta}^0, \tag{9}$$

then (6) and (3), (4) imply

$$\begin{cases} x = x_\Omega + a_{11}\xi + a_{12}\eta + a_{13}\zeta, \\ y = y_\Omega + a_{21}\xi + a_{22}\eta + a_{23}\zeta, \\ z = z_\Omega + a_{31}\xi + a_{32}\eta + a_{33}\zeta \end{cases} \tag{10}$$

and inversely

$$(11) \qquad \begin{cases} \xi = a_{11}(x - x_\Omega) + a_{21}(y - y_\Omega) + a_{31}(z - z_\Omega), \\ \eta = a_{12}(x - x_\Omega) + a_{22}(y - y_\Omega) + a_{32}(z - z_\Omega), \\ \zeta = a_{13}(x - x_\Omega) + a_{23}(y - y_\Omega) + a_{33}(z - z_\Omega). \end{cases}$$

If

$$(12) \qquad \boldsymbol{k} \times \bar{\zeta}^0 \neq \boldsymbol{o},$$

then let the elementary angle θ and the orientated angles ψ and φ be defined as follows:

$$(13) \qquad \cos\theta = \boldsymbol{k}\bar{\zeta}^0 \qquad (0 < \theta < \pi),$$

$$(14) \qquad \sin\psi = \boldsymbol{j}\bar{\gamma}^0, \quad \cos\psi = \boldsymbol{i}\bar{\gamma}^0 \quad (0 \leqq \psi < 2\pi),$$

$$(15) \qquad \sin\varphi = -\bar{\eta}^0\bar{\gamma}^0, \quad \cos\varphi = \bar{\xi}^0\bar{\gamma}^0 \quad (0 \leqq \varphi < 2\pi),$$

provided

$$(16) \qquad \bar{\gamma}^0 = \frac{\boldsymbol{k} \times \bar{\zeta}^0}{\sin\theta}.$$

Then ψ, φ, θ are called the *Eulerian angles* of the systems of reference $Oxyz$ and $\Omega\xi\eta\zeta$, and

$$(17) \qquad \begin{cases} a_{11} = \cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta, \\ a_{12} = -\cos\psi\sin\varphi - \sin\psi\cos\varphi\cos\theta, \\ a_{13} = \sin\psi\sin\theta, \\ a_{21} = \sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta, \\ a_{22} = -\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta, \\ a_{23} = -\cos\psi\sin\theta, \\ a_{31} = \sin\varphi\sin\theta, \\ a_{32} = \cos\varphi\sin\theta, \\ a_{33} = \cos\theta, \end{cases}$$

i.e.

$$(18) \qquad a_{\mu\nu} = a_{\mu\nu}(\psi, \varphi, \theta) \qquad (\mu, \nu = 1, 2, 3)$$

are completely determined functions of ψ, φ, θ.

The first and the second derivatives of the scalar functions with respect to the *time* t are traditionally denoted in analytical mechanics by means of one and two dots, respectively, placed over the corresponding symbols representing those functions. As regards the vector functions, the dots and the symbols $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$ are reserved for their derivatives with respect to the system of reference $Oxyz$ only, their derivatives with respect to $\Omega\xi\eta\zeta$ being denoted by the symbols $\frac{\delta}{\delta t}$ and $\frac{\delta^2}{\delta^2 t}$.

Thus, if

$$(19) \qquad \boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$

and

(20) $$a = a_\xi \bar{\xi}^0 + a_\eta \bar{\eta}^0 + a_\zeta \bar{\zeta}^0,$$

then the derivatives of a with respect to the time and with regard to $Oxyz$ and $\Omega\xi\eta\zeta$ are

(21) $$\frac{da}{dt} = \dot{a} = \dot{a}_x i + \dot{a}_y j + \dot{a}_z k$$

and

(22) $$\frac{\delta a}{\delta t} = \dot{a}_\xi \bar{\xi}^0 + \dot{a}_\eta \bar{\eta}^0 + \dot{a}_\zeta \bar{\zeta}^0,$$

respectively. The derivatives (21) and (22) are sometimes qualified by the use of the adjectives *absolute* and *relative*, respectively.

The *instantaneous angular velocity*

(23) $$\bar{\omega} = \frac{1}{2}\left(\bar{\xi}^0 \times \dot{\bar{\xi}}^0 + \bar{\eta}^0 \times \dot{\bar{\eta}}^0 + \bar{\zeta}^0 \times \dot{\bar{\zeta}}^0\right)$$

of $\Omega\xi\eta\zeta$ with respect to $Oxyz$ is defined as the only solution of the system of vector equations

(24) $$\bar{\omega} \times \bar{\xi}^0 = \dot{\bar{\xi}}^0, \quad \bar{\omega} \times \bar{\eta}^0 = \dot{\bar{\eta}}^0, \quad \bar{\omega} \times \bar{\zeta}^0 = \dot{\bar{\zeta}}^0.$$

If

(25) $$\bar{\omega} = \omega_\xi \bar{\xi}^0 + \omega_\eta \bar{\eta}^0 + \omega_\zeta \bar{\zeta}^0,$$

then the relations

(26) $$\begin{cases} \omega_\xi = \dot{\psi} \sin\theta + \dot{\theta}\cos\varphi, \\ \omega_\eta = \dot{\psi} \sin\theta - \dot{\theta}\sin\varphi, \\ \omega_\zeta = \dot{\psi}\cos\theta + \dot{\varphi} \end{cases}$$

are called the *Eulerian kinematical equations.*

If (19)–(22), then

(27) $$\frac{da}{dt} = \frac{\delta a}{\delta t} + \bar{\omega} \times a,$$

whence

(28) $$\frac{d\bar{\omega}}{dt} = \frac{\delta\bar{\omega}}{\delta t}.$$

Now (28) and (25), (22) imply

(29) $$\dot{\bar{\omega}} = \dot{\omega}_\xi \bar{\xi}^0 + \dot{\omega}_\eta \bar{\eta}^0 + \dot{\omega}_\zeta \bar{\zeta}^0.$$

By definition the point P belongs to the rigid body S if, and only if,

(30) $$\frac{\delta\bar{\rho}}{\delta t} = o \qquad (\forall t),$$

in other words, iff

$$(31) \qquad \dot{\xi} = \dot{\eta} = \dot{\zeta} = 0 \qquad (\forall t)$$

provided (9) by virtue of (22). Since (6) and (27) imply

$$(32) \qquad \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}_\Omega}{dt} + \frac{\delta\bar{\rho}}{\delta t} + \bar{\omega} \times \bar{\rho},$$

the definition (30) and

$$(33) \qquad \boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}, \quad \boldsymbol{v}_\Omega = \frac{d\boldsymbol{r}_\Omega}{dt}$$

imply that

$$(34) \qquad \boldsymbol{v} = \boldsymbol{v}_\Omega + \bar{\omega} \times \bar{\rho} \qquad (\forall t)$$

is a necessary and sufficient condition for P in order to belong to the rigid body S.

The set V_S of all points P belonging to a rigid body S, i.e. of all (9) with (30) or, just the same, with (31), constitutes a real standard vector space. Now the very definition of the rigid body concept presupposes that a function

$$(35) \qquad \varkappa : V_S \longrightarrow [0, \infty)$$

is defined, such that the integral

$$(36) \qquad m = \int\limits_{V_S} \varkappa(\bar{\rho})\, d\mu$$

exists; $\varkappa(\bar{\rho})$ is called the *density* of S at the point P, and m is called the *mass* of S. The density, as well as the mass of a rigid body play a fundamental role in both analytical statics and analytical dynamics. In mechanics of rigid bodies with *constant mass* it is supposed that

$$(37) \qquad \frac{d\varkappa(\bar{\rho})}{dt} = 0 \qquad (\forall \bar{\rho} \in V_S, \forall t)$$

— a condition that will be hypothesized in the sequel.

The traditional notation

$$(38) \qquad dm = \varkappa(\bar{\rho})\, d\mu,$$

as convenient as incorrect, is frequently used, dm being called an *elementary mass* of S; besides, V_S is usually omitted in the record of the integral (36), being implied by the context. Using this convention and the notation (38), the definition (36) may be written in the following popular though somewhat enigmatic form:

$$(39) \qquad m = \int dm.$$

Now (36)–(39) imply

$$(40) \qquad \frac{dm}{dt} = 0 \qquad (\forall t).$$

Before proceeding further, it is *nec plus ultra* necessary to say in this place some words about the integral (36) and about some other important dynamical integrals which will appear immediately below. The point is that, in the present state of affairs at least — videlicet, at such a logical level of exposition as the present one, no specification may be made as regards the mathematical nature of the process of integration in (36): in order to fix the ideas one may purely and simply suppose that the integral in (36) and elsewhere is taken *im Riemannschen Sinne*, $d\mu$ denoting infinitesimal volume (if S is a 3-dimensional rigid body), or infinitesimal area (if S is 2-dimensional), or infinitesimal length (if S is 1-dimensional). As regards any further information, it is imbedded in the very definition of the notion of a dynamical rigid body.

Being at this stage forced into accepting those as vague as to seem void of sense explanations, let us proceed to the definition of the *mass–centre* G of the rigid body S. It is introduced traditionally by means of the relation

(41) $$\bar{\rho}_G = \frac{1}{m}\int \bar{\rho}\, dm$$

provided

(42) $$\bar{\rho}_G = \overline{\Omega G},$$

the integral being taken over V_S. Along with (30) and (40) the definition (41) implies

(43) $$\frac{\delta \bar{\rho}_G}{\delta t} = 0 \qquad (\forall t),$$

i.e. the mass-centre of a rigid body S belongs to S.

If by definition

(44) $$\boldsymbol{r}_G = \boldsymbol{OG},$$

then the identity $\boldsymbol{OG} = \overline{O\Omega} + \overline{\Omega G}$, together with (5) and (42), implies

(45) $$\boldsymbol{r}_G = \boldsymbol{r}_\Omega + \bar{\rho}_G,$$

and (45), (41), (6), (39) imply

(46) $$\boldsymbol{r}_G = \frac{1}{m}\int \boldsymbol{r}\, dm,$$

the integral being taken over V_S provided (6).

The identities (6) and (45) imply

(47) $$\boldsymbol{r} = \boldsymbol{r}_G - \bar{\rho}_G + \bar{\rho}.$$

If by definition

(48) $$\boldsymbol{v}_G = \frac{d\boldsymbol{r}_G}{dt},$$

then (45), (33), (34), (43) imply

(49) $$\boldsymbol{v}_G = \boldsymbol{v}_\Omega + \bar{\omega} \times \bar{\rho}_G,$$

and (49), (34) imply

(50) $$v = v_G - \bar{\omega} \times \bar{\rho}_G + \bar{\omega} \times \bar{\rho}.$$

On the other hand, (46), (48), (40), (33) imply

(51) $$v_G = \frac{1}{m} \int v \, dm.$$

By definition the integrals

(52) $$K = \int v \, dm$$

and

(53) $$L = \int r \times v \, dm,$$

taken over V_S provided (6), are called the *momentum* and the *moment of momentum* (*kinetical moment*), respectively, of the rigid body S with regard to $Oxyz$.

We shall now subject the quantities K and L to certain identical transformations. First of all, we observe that (52) and (51) imply

(54) $$K = m v_G.$$

On the other hand, (53) and (47), (50) imply

(55) $$L = \int (r_G - \bar{\rho}_G + \bar{\rho}) \times (v_G - \bar{\omega} \times \bar{\rho}_G + \bar{\omega} \times \bar{\rho}) \, dm$$

and (55), (39) imply

(56) $$L = m r_G \times v_G + L_G,$$

where by definition

(57) $$L_G = \int \bar{\rho} \times (\bar{\omega} \times \bar{\rho}) \, dm - m \bar{\rho}_G \times (\bar{\omega} \times \bar{\rho}_G).$$

If by definition

(58) $$I_{\xi\xi} = \int (\eta^2 + \zeta^2) \, dm, \quad I_{\eta\eta} = \int (\zeta^2 + \xi^2) \, dm, \quad I_{\zeta\zeta} = \int (\xi^2 + \eta^2) \, dm,$$

(59) $$I_{\eta\zeta} = \int \eta\zeta \, dm, \quad I_{\zeta\xi} = \int \zeta\xi \, dm, \quad I_{\xi\eta} = \int \xi\eta \, dm,$$

(60) $$J_{\xi\xi} = m \left(\eta_G^2 + \zeta_G^2\right), \quad J_{\eta\eta} = m \left(\zeta_G^2 + \xi_G^2\right), \quad J_{\zeta\zeta} = m \left(\xi_G^2 + \eta_G^2\right),$$

(61) $$J_{\eta\zeta} = m \eta_G \zeta_G, \quad J_{\zeta\xi} = m \zeta_G \xi_G, \quad J_{\xi\eta} = m \xi_G \eta_G,$$

(62) $$A = I_{\xi\xi} - J_{\xi\xi}, \quad B = I_{\eta\eta} - J_{\eta\eta}, \quad C = I_{\zeta\zeta} - J_{\zeta\zeta},$$

(63) $$D = I_{\eta\zeta} - J_{\eta\zeta}, \quad E = I_{\zeta\xi} - J_{\zeta\xi}, \quad F = I_{\xi\eta} - J_{\xi\eta}$$

provided (9) and

$$\bar{\rho}_G = \xi_G \bar{\xi}^0 + \eta_G \bar{\eta}^0 + \zeta_G \bar{\zeta}^0, \tag{64}$$

then the quantities A, B, C are called the moments of inertia and the quantities D, E, F are called the moments of deviation of the rigid body S with respect to $\Omega\xi\eta\zeta$.

If now

$$\boldsymbol{L}_G = L_{G\xi}\bar{\xi}^0 + L_{G\eta}\bar{\eta}^0 + L_{G\zeta}\bar{\zeta}^0, \tag{65}$$

then (57) implies

$$L_{G\xi} = \int \bar{\xi}^0 \times \bar{\rho}.\bar{\omega} \times \bar{\rho}\, dm - m\bar{\xi}^0 \times \bar{\rho}_G.\bar{\omega} \times \bar{\rho}_G, \tag{66}$$

and (66), (25), (9), (64) imply

$$L_{G\xi} = \int \left(\bar{\rho}^2\omega_\xi - (\bar{\rho}\bar{\omega})\xi\right)\, dm - m\left(\bar{\rho}_G^2\omega_\xi - (\bar{\rho}_G\bar{\omega})\xi_G\right), \tag{67}$$

whence

$$L_{G\xi} = A\omega_\xi - F\omega_\eta - E\omega_\zeta \tag{68}$$

by virtue of (58)–(63). Similarly,

$$L_{G\eta} = B\omega_\eta - D\omega_\zeta - F\omega_\xi, \tag{69}$$

$$L_{G\zeta} = C\omega_\zeta - E\omega_\xi - D\omega_\eta. \tag{70}$$

Now (68)–(70) and (65) imply

$$\begin{aligned}\boldsymbol{L}_G = (A\omega_\xi - F\omega_\eta - E\omega_\zeta)\bar{\xi}^0 + (B\omega_\eta - D\omega_\zeta - F\omega_\xi)\bar{\eta}^0 \\ + (C\omega_\zeta - E\omega_\xi - D\omega_\eta)\bar{\zeta}^0.\end{aligned} \tag{71}$$

On the other hand, (27) implies

$$\dot{\boldsymbol{L}}_G = \frac{\delta}{\delta t}\boldsymbol{L}_G + \bar{\omega} \times \boldsymbol{L}_G \tag{72}$$

and (72), (71), (29) imply

$$\begin{aligned}\dot{\boldsymbol{L}}_G = &\left(A\dot{\omega}_\xi - (B-C)\omega_\eta\omega_\zeta - D(\omega_\eta^2 - \omega_\zeta^2) - E(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) - F(\dot{\omega}_\eta - \omega_\zeta\omega_\xi)\right)\bar{\xi}^0 \\ &+ \left(B\dot{\omega}_\eta - (C-A)\omega_\zeta\omega_\xi - E(\omega_\zeta^2 - \omega_\xi^2) - F(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) - D(\dot{\omega}_\zeta - \omega_\xi\omega_\eta)\right)\bar{\eta}^0 \\ &+ \left(C\dot{\omega}_\zeta - (A-B)\omega_\xi\omega_\eta - F(\omega_\xi^2 - \omega_\eta^2) - D(\dot{\omega}_\eta + \omega_\zeta\omega_\xi) - E(\dot{\omega}_\xi - \omega_\eta\omega_\zeta)\right)\bar{\zeta}^0.\end{aligned} \tag{73}$$

Up to here all considerations have been purely kinematical, in the sense that no forces have ever appeared. Let us now suppose that the rigid body S is subjected to the action of the system of forces

$$\overrightarrow{F}_\sigma = (\boldsymbol{F}_\sigma, \boldsymbol{M}_\sigma) \qquad (\sigma = 1, \ldots, s), \tag{74}$$

all moments $\boldsymbol{M}_\sigma$ ($\sigma = 1, \ldots, s$) being taken with respect to O. Let by definition

$$\boldsymbol{F}^* = \sum_{\sigma=1}^{s} \boldsymbol{F}_\sigma \tag{75}$$

and

$$M^* = \sum_{\sigma=1}^{s} M_\sigma \tag{76}$$

be the basis and the moment, respectively, of the system (74), the latter being obviously taken with regard to O. If now one calls to mind that the system of reference $Oxyz$ is inertial according to Euler by hypothesis, then one may write down the Eulerian dynamical axioms Ax 1 E and Ax 2 E in the form

$$\dot{K} = F^* \tag{76'}$$

and

$$\dot{L} = M^*, \tag{77}$$

respectively.

If by definition

$$w_G = \frac{d^2 r_G}{dt^2}, \tag{78}$$

then (76′), (54), (40), (48) imply

$$m w_G = F^*. \tag{79}$$

Now (69), (78),

$$r_G = x_G i + y_G j + z_G k, \tag{80}$$

$$F^* = F_x^* i + F_y^* j + F_z^* k \tag{81}$$

imply

$$m\ddot{x}_G = F_x^*, \quad m\ddot{y}_G = F_y^*, \quad m\ddot{z}_G = F_z^*. \tag{82}$$

Let

$$M_G^* = M^* + F^* \times r_G \tag{83}$$

be the moment of the system of forces (74) with respect to the mass-centre G of the rigid body S. Then (83) and (79) imply

$$M^* = M_G^* + m r_G \times w_G, \tag{84}$$

and (84), (77), (56) imply

$$\dot{L}_G = M_G^*. \tag{85}$$

If by definition

$$M_G^* = M_{G\xi}^* \bar{\xi}^0 + M_{G\eta}^* \bar{\eta}^0 + M_{G\zeta}^* \bar{\zeta}^0, \tag{86}$$

then (85), (73) imply

(87)
$$\begin{cases} A\dot{\omega}_\xi - (B - C)\omega_\eta\omega_\zeta - D(\omega_\eta^2 - \omega_\zeta^2) - E(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) - F(\dot{\omega}_\eta - \omega_\zeta\omega_\xi) = M_{G\xi}^*, \\ B\dot{\omega}_\eta - (C - A)\omega_\zeta\omega_\xi - E(\omega_\zeta^2 - \omega_\xi^2) - F(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) - D(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) = M_{G\eta}^*, \\ C\dot{\omega}_\zeta - (A - B)\omega_\xi\omega_\eta - F(\omega_\xi^2 - \omega_\eta^2) - D(\dot{\omega}_\eta + \omega_\zeta\omega_\xi) - E(\dot{\omega}_\xi - \omega_\eta\omega_\zeta) = M_{G\zeta}^*. \end{cases}$$

The relations (82) are called *Euler's dynamical equations for the motion of the mass-centre of a rigid body*, and the relations (87) are called *Euler's dynamical equations for the motion of a rigid body around its mass-centre.* At that, the equations (82) represent a mathematically developed equivalent of the first Eulerian dynamical axiom Ax 1 E, while the equations (87) are a mathematical reflexion, by means of the moments of inertia and of the moments of deviation of a rigid body, of the second Eulerian dynamical axiom Ax 2 E.

Let us for a while come to a standstill here and let our thought dwell upon the equations (82), (87). The latter supply us with a system of 6 conditions for the motion of any rigid body concerning the quantities they involve. And which quantities do they involve? Along with the moments of inertia A, B, C and the moments of deviation D, E, F (which are known quantitative characteristics for any particular rigid body S), the equations (82), (87) include the canonic parameters

(88) $$x_\Omega,\ y_\Omega,\ z_\Omega,\ \psi,\ \varphi,\ \theta$$

of S and the components

(89) $$F_x^*,\ F_y^*,\ F_z^*,\ M_{G\xi}^*,\ M_{G\eta}^*,\ M_{G\zeta}^*$$

of the basis (75) and of the moments (83) with respect to G of the system of forces (74) acting on S. At that, according to a convention, sanctified by the centuries-old experience and tradition of analytical mechanics, it is hypothesized that

(90) $$\boldsymbol{F}^* = \boldsymbol{F}^*(x_\Omega, y_\Omega, z_\Omega, \psi, \varphi, \theta; \dot{x}_\Omega, \dot{y}_\Omega, \dot{z}_\Omega, \dot{\psi}, \dot{\varphi}, \dot{\theta}; t)$$

and

(91) $$\boldsymbol{M}_G^* = \boldsymbol{M}_G^*(x_\Omega, y_\Omega, z_\Omega, \psi, \varphi, \theta; \dot{x}_\Omega, \dot{y}_\Omega, \dot{z}_\Omega, \dot{\psi}, \dot{\varphi}, \dot{\theta}; t)$$

are certain determined functions of the canonic parameters (88), of their derivatives

(92) $$\dot{x}_\Omega,\ \dot{y}_\Omega\ \dot{z}_\Omega,\ \dot{\psi},\ \dot{\varphi},\ \dot{\theta}$$

with respect to the time t, and possibly of t itself, whence the same supposition is valid for the components (89) of (90) and (91). In such a manner, the Eulerian dynamical equations (82), (87) represent a system of 6 differential relations of second order with respect to the time t for the 6 canonic parameters (88) of the rigid body S.

All that would be ***nuda veritas*** under the assumption that all the parameters (88) of the rigid body are ***mutually independent***, alias that any of them could vary, along with its derivative with respect to the time t, completely independently from the variations of the rest of these parameters and of their derivatives with respect to the time. Is that always the case?

If all canonic parameters (88) of a rigid body S are mutually independent, then it is said that S is a *free rigid body*, the term implying that no restrictions are imposed on the thinkable (or possible, or feasible, or imaginary, or potential, or virtual) positions of S in space and on the velocities of its points. In the case of a free rigid body S a classical for analytical mechanics hypothesis presupposes (or demands, or exacts, or requires, or insists on, or announces, or promulgates, or declares, or proclaims) that all the forces (74), acting on S, are *active forces*, in other words, all of them are given (or known, or familiar, or prescribed, or specified) functions of (88), of (92), and possibly of t. In such a way, in the case of a free rigid body the Eulerian dynamical equations (82), (87) represent a wholly determined system of 6 *genuine*, or *pure* differential equations of second order with respect to the time t for the 6 unknown functions (88) of the time t. If now *initial values* of these functions and of their derivatives (92) with respect to t are given (i.e. admissible values of (88) and (92) for any particular moment τ of t, say $t = 0$), then the dynamical problem concerning the motion of this free rigid body S presents itself in the capacity of a perfectly correct mathematical problem with one and only solution (provided certain conditions are satisfied concerning the right-hand sides of the equations (82), (87), i.e. if some requirements affecting the analytical nature of the functions (90), (91) are fulfilled).

The situation is shifted in a trice if some of the forces (74) are unknown and, in the same time, not all of the canonic parameters (88) are mutually independent. Millenial physical experience, engineering praxis, and sound mechanical common sense display that the idyllic picture of free rigid body motions ceases to interpret adequately the dynamical realities which have surprisingly engendered nightmarish problems for all the mechanicians from the Seventeenth till Twenty-First Century. Of course, the interplay of mathematical discovery and physical experience is a dangerous game, and we by no means venture to imitate D'Alembert's unfortunate improvisations on this theme and variations, giving good reason for Truesdell's statement that "in attempting to connect physical experience with mathematics, he heaped folly on folly ... one searches for the little solid matter as a sparrow pecks out a few nutricious seeds from a dungheap — a task not altogether savory" [5]. In the same time, especially in "physikalischen Disziplinen, in denen schon heute die Mathematik eine hervorragende Rolle spielt: dies sind in erster Linie die Wahrscheinlichkeitsrechnung und die Mechanik" [6], this interaction or, should we say, heuristic symbiosis, is *un fait accompli* that no one may disregard without disturbing all sense of reality:

"... mathematics, however abstract and however precise, is a science of *experience*, for experience is not confined to the gross senses: Also the human mind can experience, and we need not be so naive as to see in an oscilloscope an instrument more precise than the brain of a man.

That rational mechanics grew out of practical mechanics and co-operated with it, if not always gracefully, to produce applied mechanics and mechanical engineering, is obvious. In writing the first treatise on rational mechanics [7] Newton established its standard of mathematical rigor as precisely that of geometry. Not always has this standard been maintained, but today as in 1687 it remains the ideal. Newton's comparison with geometry is enlightening, for geometry, too, grew from physical experience. To those who scoff at geometry for its precise calculations when all measurements are liable to error, the geometer for millenia has replied: Geometry is *mental*, not *instrumental*. The scoffers have always been with us and remain today; not only does the ultimate practical and physical value of geometry need no defense before scientists, but also no-one who has known a geometer needs reminder that practical and physical usefulness seldom has supplied or suppressed a single equation in the progress of geometrical research.

The analogy to geometry is a good one. That rational mechanics speaks not only of space and time but also of mass, force, and energy does not make it any the less precise. Since it deals with a greater number of physical concepts than does geometry, its applications to physical problems may be expected to be more frequent and more far-reaching, but physical applications are not its objective.

But does not rational mechanics deal with quantities of physical experience? Indeed it does; so does geometry, for lengths, surfaces, and volumes are equally related to physical experience. The geometer may visualize a surface in terms of a twisted strip of paper, as in mechanics one may think of a force as a push with the hand, but whatever these motivations, the symbols in the equations of geometry and mechanics are precisely defined mathematical quantities. Origin in broader experience may make mechanics more interesting, but it need not make it any less rigorous" [8, p. 335–336].

Could it be said more clearly and more simply? There are other places in [8], dedicated to the interplay of mathematics and physics, that one plainly cannot leave out not mentioned. Reminding Daniel Bernoulli's words "there is no philosophy which is not founded upon knowledge of the phenomena, but to get any profit from this knowledge it is absolutely necessary to be a mathematician" and Huygens' motto "from experience and from reason", Truesdell speculates:

"What was, then, the method? Rational mechanics was a science of *experience*, but no more than geometry was it *experimental*. While some great mechanical experiments were done in the Age of Reason, they had only occasional bearing on the growth of the theories we now regard as classical. Experiment and theory result from different kinds of reaction to experience. If, ideally, they should complement and check one another, yet even today, with all our superior knowledge not only of facts but also of scientific method, it is difficult enough to relate them, why should it has been easier 300 years ago? It was not. A factual view of the history of mechanics must concede that rational mechanics and experimental mechanics, both arising from human beings' intelligent reaction to mechanical experience, grew up separately.

Not only private, individual experimental researches were performed in the eighteenth century; there were also large, cooperative projects. As today, they cost more than real science, and they attracted administrators. But the effect of all this expense on what we now consider the achievement of the period was nil. The method used in the great researches was entirely mathematical, but the result was not what would now be called pure mathematics. *Experience* was the guide; *experience*, physical experience and the experience of accumulated previous theory. If we are to seek a word for what was done, it would not be physics and it would not be pure mathematics; least of all would it be applied mathematics. It would be ***rational mechanics*** ...

Without *experience*, there would be no rational mechanics, but I should mislead you if I claimed that experiment, either now or 200 years ago, had greatly influenced those who study rational mechanics. In this connection experiment, like alcohol, is a stimulant to be taken with caution. To consult the oracle of a fine vintage at decent intervals exhilirates, but excess of the common stock brings stupor" [*ibid.*, p. 135–136, 357].

One of the most primitive, most fundamental, and, together with that — no wonder *ergo propter hoc* — most complicated mathematical formalization of physical experience in rigid body analytical dynamics focalizes in the idea of — no matter accidental or intentional — restrictions imposed on the possible (virtual, potential) positions of rigid bodies in space. At first sight the underlying idea looks as simple as to seem obvious; as simple as to seem obvious are the mathematical means, too, by the aid of which, until this very day, mechanicians are trying to formalize mathematically this same idea — the height of perfection of their efforts inevitably calling to one's mind Mark Twain's observation that *for any problem there is a solution that is simple, obvious, and wrong.* In point of fact, the simplicity of the idea is spurious to such a degree that anyone who ventures to get into the swing of the work unavoidably wanders through the intricacies of a true Labyrinth with no Ariadne at its mouth.

Calling *ficus ficus, ligonem ligonem*, we are obliged to fathom the fact that the physical cause underlying any restriction in the positions of a rigid body in space is rooted in that attribute of matter which is described by the categorical though somewhat enigmatical term *impenetrability.* This property, characterized also by the substantives *impermeability* and *imperviousness*, is available in the very commencement of the notorious dynamical *Traité* [9] of D'Alembert — in its first sentence, to all intents and purposes, see the section *Définitions et Notions préliminaires* (p. 1) — as an inseparable part of the author's definition of the rigid body concept:

"Si deux portions d'étendüe semblables & égales entr'elles sont *impénétrables*, c'est-à-dire, si elles ne peuvent être imaginées unis & confondües l'une avec l'autre, de maniére qu'elles ne fassent qu'une même portion d'étendüe moindre que la somme des deux, chacune de ces portions d'étendüe sera ce qu'on appelle un *Corps.* L'impénétrabilité est la propriété principale par laquelle nous distinguons les Corps des parties de l'espace indéfini, où nous imaginons qu'ils sont placés."

The most natural question, coming to the mind of the reader of this *définition*, is how does its author use "la propriété principale par laquelle nous distinguons les Corps" described as *impenétrabilité* in order to achive his object so modestly proclaimed in the *Préface* of the *Traité*:

"Je me suis proposé dans cet Ouvrage de satisfaire à ce double objet, de reculer les limites de la Méchanique, & d'en applanir l'abord; & mon but principal a été de remplir en quelque sorte un de ces objets par l'autre, c'est-à-dire, nonseulement de deduire les Principes de la Méchanique des notions les plus claires, mais de les appliquer aussi à de nouveaux usages; de faire voir tout à la fois, & l'inutilité de plusieurs Principes qu'on avoit employés jusqu'ici dans la Méchanique, & l'avantage qu'on peut tirer de la combinaison des autres pour le progrès de cette Science; en un mot, d'étendre les Principes en les réduisant."

Strange to say, the straightforward answer of this question is: not at all, not the least bit, never a whit. In spite of his promise "de déduire les Principes de la Méchanique des notions les plus claires" and "de les appliquer aussi à de nouveaux usage", D'Alembert never, nowhere, and in no wise uses "la notion" *impenetrabilité* to this end. The principles in question are formulated in the very beginning of the

Premiere Partie. Loix générales du mouvement et de l'équilible des Corps of [9], where one reads:

"On peut réduire tous les Principes de la Méchanique a trois, la force d'inertie, le mouvement composé, & l'équilibre. Au moins j'espere faire voir par ce Traité, que toute cette science peut être déduite de ces trois Principes. Je traiterai de chacun d'eux en particulier dans chacun des Chapitres suivans."

Well, Sir! One reads *Chapitre Premier. De la force d'inertie, et des propriétés du mouvement qui en résultent*, and one does not come across the word *impénétrabilité* at all. Afterwards one reads *Chapitre II. Du Mouvement composé*, and one does not encounter this word there too. Ultimately, one turns the pages of the book over *Chapitre III. Du Mouvement detruit ou changé par des obstacles*, and one does not run into *impénétrabilité* again. It is true that in the last chapter one reads:

"Un Corps qui se meut, peut rencontrer des obstacles qui altérent, ou même qui anéantissent tout–à–fait son Mouvement; ces derniers sont, ou invincibles par eux–mêmes, ou n'ont précisément de resistance, que ce qu'il en faut pour détruire le Mouvement imprimé au Corps.

Un obstacle invincible peut être tel, qu'il ne permette au Corps aucun Mouvement, comme quand un Corps tire une verge droit attachée a un point fixe; ou l'obstacle pourroit être de telle nature, qu'il n'empêchât pas le Corps de se mouvoir dans une autre direction que celle qu'il a, comme quand un Corps rencontre un plan inébranlable" (p. 31).

It is also true that the very idea of "obstacles" is inextricably bound up with the "propriété principale impénétrabilité". At last, it is true that a bit further down D'Alembert writes:

"Dela il s'ensuit, qu'un Corps sans ressort qui vient choquer perpendiculairement un plan immobile & *impénétrable*, doit s'arrêter après ce choc, & rester en repos. Car il est visible que si ce Corps a du Mouvement après la rencontre du plan, ce ne peut être qu'en arriére, & dans la direction de la perpendiculaire" (p. 32, our italics).

At the same time it is also true that the adjective "impénétrable" is used, in the last passage, sporadically, haphazardly, and contrastively — assigned to a subject exterior to the rigid body, the motion of which is studied rather than to this latter body itself. The same applies to other cases when the term "impénétrable" is used, for instance in 30. paragraph of the Traité, where the word "impénétrable" is used as a synonym for the word "invincible".

The reason for this state of affairs is a quite simple one: the quality "impénétrable" is an attribute to rigid bodies, whereas the *Traité de Dynamique* of D'Alembert has nothing to do with such matters: at the best it could be accepted as a writing dedicated to mass-point dynamics (if at all), as the ceaseless usage of the term "vitesse" at once displays, which becomes meaningless when assigned to rigid bodies. As regards the bodies themselves, D'Alembert is the originator of the conception, shortly afterwards adopted and developed further by his younger contemporary De la Grange — an outlook that was fated to play an extremely unenviable role in the supervening history of rational mechanics.

Entre parenthèses: In spite of the solemn promise of its author, promulgated in its title (namely, to "give a general principle for discovering the motions of several rigid bodies acting one upon another in an arbitrary manner"), the *Traité de Dynamique* of D'Alembert does not provide the reader with mathematical means for solving even one and only dynamical problem concerning a sole rigid body. Faced with this situation, one is at a loss for what reason has this *Traité* gained its "immortal" fame? The answer of this quite justifiable question is given by Truesdell, though his words refer to Leonardo rather than to D'Alembert:

"To learn the source, we recall the method of the Renaissance: Self-advertising ... In his skill of speech and his self-promotion he was a true son of the Renaissance. Like the humanists, with much adroitness but little solid achievement he blew himself into renown for all times"[8, p. 80–81].

In D'Alembert's case it would be appropriate to recall Vasari's words apropos of Leonardo: "Even though he talked much more about his works than he actually achieved, his name and fame will never be extinguished." We close the brackets.

In such a manner, a sound physical idea has been compromised mathematically in the *Traité de Dynamique* of D'Alembert. Without entering into details, we confine ourselves to the statement that it is discredited also in the natural logical extension [10] of D'Alembert's illogical dynamical philosophy. As a matter of fact, the collapse of the idea reaches in [10] such apocalyptic scales that evokes memories of biblical sinister omens for the original sin. *Nuda veritas* is that the reader of [10] is missing the forest for the trees. Directly contrary to Lagrange's overweening advertisements (namely that he proposes "des formules générales, dont le simple développement donne tous les équations nécessaires pour la solution de chaque problème ... la manière dont j'ai tache de remplir cet objet ne laissera rien à desirer ... Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou méchaniques, mais seulement des opérations algébriques, assujetties à une marche régulière et uniforme"), the bulk of formulae one bumps up against in the *Méchanique Analitique* is entirely helpless when faced with the problem of the dynamical behaviour of a single rigid body subjected to any mechanical constraints: the cold fact is that the blazing upper strata of Lagrange's dynamical performances is as high as the movements (if any) of discrete systems of a finite number of masspoints. Truesdell's observations apropos of [7]: "Newton *gives no evidence of being able to set up differential equations of motion for mechanical systems* ... the cold fact is, the equations are not in Newton's book ... In Newton's *Principia* occur no equations of motion for systems of more than two free mass-points or more than one constrained mass-point; Newton's theories of fluids are largely false; and the spinning top, the bent spring, lie altogether outside Newton's range" [8, p. 92–93], may be paraphrased apropos of [10] in the following manner: Lagrange gives no evidence of being able to set up differential equations of motion for mechanical systems including rigid bodies; the cold fact is the equations are not in Lagrange's book. In Lagrange's *Méchanique Analitique* occur no equations of motion for systems of more than a finite number of mass-points; Lagrange's theories of rigid bodies are largely false; and the spinning top, the billiard ball, lie altogether outside Lagrange's range.

The long and the short of the whole span of Lagrange's mechanical philosophy, of his statical and dynamical *Weltanschauung*, may be incarnated in a sole phrase of his *Traité*:

"... considérons un système de corps, disposés les uns par rapport aux autres comme on voudra et animé par des forces accélératrices quelconques.

Soit m la masse de l'un quelconque de ces corps, *regardé comme un point*" [11, p. 264; our italics].

This mechanical ideology of Lagrange's has ripened into the manhood a long time before he settled down to composing his *Méchanique Analitique* — as a matter of fact, not later than 1772 when he wrote his articles [12] wherein one reads:

"... si l'on imagine un système d'un nombre indéfini de corps considérés comme des points et liés ensemble de manière que leur distances mutuelles restent toujours les mêmes ..." (p. 579).

Alibi:

"En général, si l'on a un système d'autant de corps qu'on voudra, disposés de manière qu'ils soient forcés de conserver toujours les mêmes distances tant entre eux qu'à l'égard d'un point donné ..." (p. 587).

Alibi again:

"Je considere le corps propose comme l'assemblage d'une infinité de corpuscules ou points massifs unis ensemble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances" (p. 590).

In such a manner, there can be no mistaking Lagrange's words: his *corps* and *systèmes de corps* are special kinds of finite systems of discrete mass-points rather than rigid bodies in the genuine sense of the word. This circumstance has not been left unheeded, not mentioned, and untraversed. It did not escape Euler's attention. Apropos of [12] he wrote in [13] with undubitable while latent irony:

"But when I tried with greatest avidity to follow in detail his extremely profound thoughts, truly I could not get myself to go through all his calculations. Even the first lemma so deterred me that on account of my blindness I could not hope in any way to check through all the analytic devices he used" (quoted according to [8], p. 260).

Considerably later, in 1853 to be more precise, J. Bertrand made some critical remarks in this connection in the third edition of [10] *publiée par* himself. *Voilà* two of them, quoted after [11]:

"Le mot *corps* désigne ici un point matériel" (p. 11);

"Le mot *corps*, ici comme plus haut, désigne un point matériel" (p. 32).

In our days Noll, for instance, brought to the fore, from general considerations, the untenability of the efforts to regard "le corps proposé comme l'assemblage d'un infinité de corpuscules ou points massifs unis ensemble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances":

"Many textbooks on theoretical mechanics dismiss continuous bodies with the remark that they can be regarded as the limiting case of a particle system with an increasing number of particles. They cannot. The erroneous belief that they can had the unfortunate effect that no serious attempt was made for a long period to put classical continuum mechanics on a rigorous axiomatic basis" [14, p. 266].

Though a home truth, these statements at first sight appear to be ill-founded, since they are not substanciated by a mathematical proof. *Incredibile dictu*, as far as our knowledge goes, nobody has as yet answered mathematically the following question, fundamental for the whole of Lagrangean dynamical tradition:

Possibility Problem. Can rigid bodies be regarded as the limiting case of a particle system with an increasing number of particles?

Lagrange's answer is *yes*. Noll's answer is *no*. Let us cast our eyes about some other stands. *Voilà* a *Traité* [15] that out and out belongs to the mechanical classics. The feather in the author's cap, as regards the rigid body notion, consists in the following "extremely profound thoughts" in the words of Euler, *absit invidia verbo*:

"*Un corps solide est un ensemble de points matériels invariablement liés entre eux.* — Lorsqu'une force est appliquée à l'un de ces points, on dit qu'elle est appliquée au corps. Le corps solide ainsi défini est une abstraction. Tous les corps de la nature se déforment sous l'action des forces qui leur sont appliquées; mais les corps appelés communement solides subissent des déformations très petits, qui peuvent être négligees dans une première approximation" (t. I, p. 123–124).

In other words, the *Possibility Problem* is answered in the affirmative by Appell too. Skipping more than half a century, let us peek into a dynamical treatise [16] of comparatively recent time, its author promising in his *Introduction* "to give a compact, consistent, and reasonably complete account of the subject *as it now stands*" (p. VII, our italics). How does he define the rigid body concept?

Bona venia vestra, he does not define it at all. In the index of the book this term does not appear independently or, should we say, single-handed, unaided, off its own bat. Indeed, the text one finds there reads:

"Rigid body, motion in two dimensions, 111–113, 204; in space, 205–207. *See also* Euler's equations, spinning top, rolling sphere, rolling penny, rolling ellipsoid" (p. 640).

In order to find a description if not a definition of the notion of rigid body in [16] one must search "for the little solid matter as a sparrow pecks out a few nutricious seeds from a dungheep — a task not altogether savory", if it is permitted to use here Truesdell's words apropos of D'Alembert. While Chapter I of the book, entitled *Motion of a particle*, is dedicated to mass-point dynamics, the term "rigid body" comes into view for the first time in [16] in the beginning of Chapter II, headed *Dynamical systems*. Therein one reads:

"In the preceding chapter we considered the dynamics of a single particle. It might seem natural, following the historical order of development, to discuss next the theory of the motion of a single rigid body; this is in fact the order usually followed in a first study of rigid Dynamics. Our approach will however be somewhat different. In Analytical Dynamics we proceed directly from the single particle to the general dynamical system. The single rigid body is of course a special case of a dynamical system and indeed one that we shall frequently find useful as a special illustration" (p. 20).

In such a way, the reader of [16] comes to know at the same breath the following truths as great as to seem divine revelations:

1. In Analytical Dynamics it is proceeded directly from the single particle to the general dynamical system.

2. The single rigid body is a special case of a dynamical system.

3. The latter statement needs no proof: it is "of course" true.

4. The single rigid body is frequently useful as a special illustration.

5. Ergo: the single rigid body represents no interest *in se*, that is to say as *ein Ding an sich.*

6. An indirectly implied corollary: any attributes ascribed to rigid bodies must be derived from attributes of general dynamical systems of single particles.

7. Ergo: constraints imposed on rigid bodies must be implied by constraints imposed on single particles.

We shall see now how does the author of this Treatise (having bidden fair, we recall, "to give a compact, consistent, and reasonably complete account of the subject [of analytical dynamics] as it now stands") materialize this new kind of mathematical induction — his limiting process $1 \to \infty$. *Qui habet aures audiendi, audiat*:

"The idea of a rigid body in the classical dynamics is a collection of particles set in a rigid and imponderable frame. Similarly we shall think of the general dynamical system as a collection of particles acted on by given forces and controlled by various kinds of constraints" (*ibid.*).

In such a manner, Pars answers the *Possibility Problem* also in the affirmative — in a most categorical manner at that. There is a point, however, that ought not be left unnoticed.

All those yes-answers and no-answers (or should we say can-answers and cannot-answers) are, alas, no mathematical answers at all. Quite much the reverse: those replies are sooner reflexions of inner convictions, of professional habits, of intellectual indolence, if you will, and in this respect they are not a jot more reliable than the possible responses of the question, say, which faith is more preferable — the Christian or the Mohammedan. The only way a mathematician can solve a *Possibility Problem* is to solve an *Existence Problem* — to prove that the object, the possibility of which is investigated, exists in actual fact.

It will remain an enigma of enigmas *in saecula saeculorum* why, in the course of more than two clear centuries, the idea flashed through nobody's mind that Lagrange's mental picture of "le corps ... comme l'assemblage d'un infinité de corpuscules ou points massifs unis ensemble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances" must be unconditionally submitted to a mathematical proof or disproof, in the same manner as it must be proved, or disproved, that there exist natural numbers x, y, z and $n > 2$ for which $x^n+y^n = z^n$ holds. Some mathematicians believe that such numbers exist, others disbelieve it — but, with the nasty exception of a swarm of illiterate idiots, there was a sole mathematician worthy of the name in the last four centuries, who stated he knew there exist no such numbers, and he knew it since he found a proof. However, he did not leave us such a proof, and Gauss, for instance, thought that Fermat misled himself; that is why the negation of $x^n + y^n = z^n$ is qualified by modern mathematicians as a hypothesis rather than a theorem.

If "un corps solide est un ensemble de points matériels invariablement liés entre eux", then the question quite naturally arises: what does connect them in such a manner? Since we claim to be mechanicians rather than fakirs, we accept that the only factors determining the mechanical behaviour of mass-points are forces. In such a way, the *Possibility Problem* formulated above may be re-redacted in the following manner:

Existence Problem. S being a system of mass-points, do there exist forces acting on them and conserving invariant in the course of the time the mutual distances between these mass-points?

Solution. In order to accomplish a *reductio ad absurdum* let us suppose that this question is answered in the affirmative. Since the number n of the points of S is indeterminate, it may be supposed, without a loss of generality, that $n = 2$. Let P_ν be the points of S with masses m_ν, respectively, and let $\boldsymbol{r}_\nu = \boldsymbol{OP}_\nu$ $(\nu = 1, 2)$, O denoting the origin of an inertial according to Newton system of reference $Oxyz$. Let $\boldsymbol{v}_\nu = \dfrac{d\boldsymbol{r}_\nu}{dt}$ $(\nu = 1, 2)$, the derivatives being taken with respect to $Oxyz$. At last, let $\boldsymbol{F}_\nu$ be the forces acting on P_ν $(\nu = 1, 2)$, respectively, in accordance with the supposition made above that such forces exist. Then, by virtue of Newton's dynamical axiom,

$$\text{(93)} \qquad \frac{d}{dt}(m_\nu \boldsymbol{v}_\nu) = \boldsymbol{F}_\nu \qquad (\nu = 1, 2),$$

the derivatives being taken with respect to $Oxyz$.

By hypothesis the forces $\boldsymbol{F}_\nu$ $(\nu = 1, 2)$ are such that

$$\text{(94)} \qquad \frac{d}{dt}(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2 = 0 \qquad (\forall t)$$

or, just the same,

$$\text{(95)} \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{v}_1 - \boldsymbol{v}_2) = 0 \qquad (\forall t).$$

Let τ be a particular moment of the time t and let

$$\text{(96)} \qquad \boldsymbol{r}_{\nu\tau} = \boldsymbol{r}_\nu(\tau), \quad \boldsymbol{v}_{\nu\tau} = \boldsymbol{v}_\nu(\tau) \qquad (\nu = 1, 2)$$

be the initial positions and the initial velocities, respectively — in other words, the *initial conditions* — of the dynamical problem under consideration. Since the relation (95) holds for any t, it is valid for $t = \tau$ too:

$$\text{(97)} \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{v}_1 - \boldsymbol{v}_2) = 0 \qquad (t = \tau),$$

and (96), (97) imply

$$\text{(98)} \qquad (\boldsymbol{r}_{1\tau} - \boldsymbol{r}_{2\tau})(\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}) = 0.$$

Now the equation (98) is an *absurdity*, since it represents a restriction imposed on the initial conditions (96) of the system S, due to the hypothesis that there exist forces $\boldsymbol{F}_\nu$ $(\nu = 1, 2)$ for which (93) with (94) hold: it is a principle of principles in rational mechanics that the initial conditions of a mechanical system are independent of the forces acting on it, and this principle is rooted in the very essence of the theory of ordinary differential equations, according to which the initial conditions of a system of differential equations are wholly arbitrary, independent of the particular functions available there. The absurdity (98) traverses the hypothesis in question and gives a negative answer of the question posed in the *Existence Problem. Quod erat demonstrandum.*

Scholium 1. A colleague and, strange enough, a good friend of ours, when for the first time faced with the absurdity (98), ejaculated: Now the same is true for any two points of any rigid body! At first sight this is a most well-founded doubt. This is only seemingly, however.

Let S be a rigid body and P_ν ($\nu = 1, 2$) be any two of its points. Under the above notations the very definition of the rigid body concept implies the relation (95) and, following the chain of the above argumentation, leads ultimately to the conclusion (98). For a rigid body, however, the relation (98) is no restriction at all imposed on the initial position of the body in space and on its initial velocities. In other words, (98) puts no restraints on the initial values

$$x_\Omega(\tau),\ y_\Omega(\tau),\ z_\Omega(\tau),\ \psi(\tau),\ \varphi(\tau),\ \theta(\tau) \tag{99}$$

of the canonical parameters (88) of S and on the initial values

$$\dot{x}_\Omega(\tau),\ \dot{y}_\Omega(\tau),\ \dot{z}_\Omega(\tau),\ \dot{\psi}(\tau),\ \dot{\varphi}(\tau),\ \dot{\theta}(\tau) \tag{100}$$

of their derivatives (92). As a matter of fact, in the rigid body case the relation (98) is reduced to the identity

$$0 = 0, \tag{101}$$

as it is at once seen by a scalar multiplication with $\boldsymbol{r}_1 - \boldsymbol{r}_2$ of the necessary and sufficient condition

$$\boldsymbol{v}_1 - \boldsymbol{v}_2 = \bar{\omega} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad (\forall t) \tag{102}$$

in order that the points P_1 and P_2 belong to S. *Sapienti sat.*

Mais revenons a nos moutons! In other words, let us return to Euler's dynamical equations (82), (87), where no specification is made as yet as regards the mechanical nature of the forces (74). As it has been underlined, the cases of a free rigid body, when all the canonic parameters (88) are mutually independent and when all the forces (74) are known beforehand as given data in the conditions of the particular dynamical problem under consideration, are as *rara avis in terris* as a honest politician; it has been emphasized also that the physical cause underlying any restriction in the position of a rigid body in space is rooted in the impenetrability of matter resulting in the phenomenon of mutual contact between bodies. The latter is a fact *homo sapiens* has been on closer acquaintance with from his very childhood in the literal as well as the metaphorical sense of the word — to such an extent as to feel it by intuition. In real fact, all the motions the same *homo* observes in nature are movements of non-free bodies, he himself being perpetually coerced to set his feet on earth.

Now that one comes to think of it, one realizes to his or her amazement that there is not a single motion in this God's earth accomplished on account of "pure" forces, that is to say without the interference of reactions due to surrounding environment. Even the free fall of ponderous bodies thrown in the air is influenced by the resistance of the medium, affecting sometimes the projectile motions to such a degree as to plunge artillerists into dispair. In reality, the only "pure" motions observable in our universe in the days of Galileo and Newton have been the planet movements; these, however, have been "polluted", first, by the Earth's own motion, and, second, by their non-observability as movements in the proper sense of the word (as, for instance, the fall of a meteor): for the naked eye, as well as for the aided by any instrument whichever, the planet motion is a series of discrete positions of the luminary rather than a continuous process in the course of time.

Let us make a parenthesis for a brief lyrical digression. Let us fancy the epoch of Galileo and Newton, at daybreak of dynamics, when no dynamical law has been as yet grasped by human mind, but hints of such one were already felt in the air. The acceleration concept has been shaped by now, the outlines of the force concept have picked out in the dark (let alone in the statical case), some kind of a mutual relation between them was already suspected, and yet nobody came to know it. If life begins *ab ovo*, then dynamics begins *ab corpusculo*: identifying, as Galileo and Newton did, bodies with mass–points, we know today that any dynamical phenomenon, observable in their days, has been governed by the law

$$mw = \boldsymbol{P} + \boldsymbol{R}, \tag{103}$$

$\boldsymbol{P}$ denoting the *innate force* of the body (in other words, the gravitational effects as established on the Earth's surface), and $\boldsymbol{R}$ — the reactions of the constraints (resistance including) imposed on the body. Now while $\boldsymbol{P}$ is a completely determined mechanical entity (at least as far as a particular geographic point is concerned), $\boldsymbol{R}$ on the contrary escapes a direct observation and measurement like a ghost. However, $\boldsymbol{R}$ being unknown and the equation (103) itself being buried in the impenetrable future, how could one hope to unearth it in broad daylight?

The only chance one has at his disposal is the case

$$\boldsymbol{R} = o. \tag{104}$$

Such "pure" motions are proposed by planets, by planets only, and by nothing save planets. Newton grasped this chance — *his* chance — with both hands. The result is immortality, as far as stars are immortal, since his law governs stellar motions:

Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.

Now the universality of this discovery of Newton's lies in the fact that, although discovered in the special case (104) of (103), it is not only applicable — moreover, it is a *conditio sine qua non* — for the motion of any corpuscular body subjected to any constraints imposed on it, generating any reactions the Human Mind and Mother Nature may devise. This inference is one of the most daring, true though incredible, inductional hypothesis in all the history of science, with wholly nonforecastable after-effects.

Summing up, one could quite justifiably state that no rational dynamics could be created if stellar motions were un-get-at-able to observation and measurements — if, for instance, the average earth temperature was some degrees higher, so that no stars could be seen on account of clouds. *Finis* of the lyrical digression.

All those meditations are much more philosophical than mathematical by nature, and we apologize to the reader begging his pardon. And yet, the character of the mathematical phenomenon described by the enigmatic expression *geometrical constraints imposed on rigid bodies* cannot be grasped rightly without these verbal explanations. Since, *summa summarum*, all this has a bearing on one of the most fundamental concepts in rigid dymanics.

Squaring accounts as regards the heuristic origins of the notion, we must perceive that, although technically feasible by means of an infinite variety of contrivances, all restrictions on the positions of a rigid body in space, described in the mechanical literary sources by means of phrases like "the body is constrained", or "compelled", or "coerced". or "forced", or "imposed", etceteras repeatedly used, reduce, when all is said and done, to a most simple mathematical device: those are *geometrical constraints* imposed on certain points of the rigid bodies. However, since a logical *anguis in herba latet* here, and the witchcraft of the words may play a practical joke on the uninitiated, converting sound intentions into a germ of regrettable misunderstandings, it is of paramount importance to nip in the bud any chance for any misconception by taking special pains for explaining the exact meaning of those synonymous terms.

Here is a point that must become crystal clear for anybody who has made up his mind to work professionally rather than dilettantish in analytical dynamics: in spite of the fact that the combination of words *geometrical constraint* has infiltrated the whole span of mechanical language, it is by no means a mathematical term — it is a concise expression of most knotty, most catchy, and most mazy mathematical situations that badly need a formal specification in any particular case. All of those particular cases reduce to the essentiality that specific mathematical hypothesis of one kind or another must be announced in the very conditions of the dynamical problem under consideration, concerning the mechanical behaviour of one or more points of the rigid body or rigid bodies. The corresponding point or points are promulgated, or proclaimed, or declared *points of contact* between the rigid body and the geometrical constraint in question. The importance of this notion may be emphysizèd by the maxim *no point of contact — no dynamical problem* concerning non-free rigid bodies, in the genuine mathematical sense of the word.

There are three geometrical entities in space, and there are also three geometrical entities invariably connected with a rigid body S, that can be juxtaposed in such mutual relations among each other as to restrict the possible positions of S in space, and these entities are points, lines, and surfaces. The relations in question reduce to one of the following combinations:

A fixed point of S is constrained to coincide with a given point in space, or to describe a given curve line in space, or to lie on a given surface in space.

Or a fixed curve line in S is constrained to pass through a given point in space, or to intersect a given curve line in space, or to touch a given curve line in space, or to touch a given surface in space.

Or a fixed surface in S is constrained to pass through a given point in space, or to touch a given curve line in space, or to touch a given surface in space.

(In all those cases the term *in space* means *external* for the rigid body S; at that, the special points, lines, and surfaces may be both *scleronomic* and *rheonomic*, that is to say fixed in space, or variable in position, or in shape in the course of time, respectively.)

Whenever any of these 10 cases is at hand in a dynamical problem (separately or in combination with others), it is said that *a geometrical constraint is imposed* on the rigid body. It is immediately seen that in such a case a singular point comes out into the open, namely the particular point common for both the geometrical entity fixed in the rigid body S and for the geometrical entity in space, playing the part of a geometrical constraint. This namely point is called the *point of contact* of S with the geometrical constraint in question.

The cardinal significance of the notion *point of contact* for rigid mechanics is predeterminated by the following dynamical axiom, reflecting age-old practical experience.

Ax 3 E. Any geometrical constraint imposed on a rigid body S generates a force acting on S, the directrix of which is passing through the point of contact of S with the geometrical constraint.

Df 2 E. The force of Ax 3 E is called the *reaction* of the geometrical constraint.

Scholium 2. The term *reaction* is fabricated as an antipode, or at least in contrast, to the term *action*, by means of which the forces indicated in the conditions of the dynamical (as well as statical) problem are described. Another terminology exploits the terms *active forces* and *passive forces*, respectively. At that, *active* are by definition those forces that are completely determined in the conditions of the statical or dynamical problem for any position and any motion of the rigid body S, that is to say for any admissible values of the canonic parameters (88) of S, of their velocities (92), and possibly of the time t, whereas nothing else is known for the *passive* forces save what Ax 3 E sermonizes, namely that they are acting on S and that their directrices are running through the corresponding points of contact with the geometrical constraints generating those reactions.

The latter statement necessitates some specification. Let A be the point of contact of the rigid body S with a geometrical constraint γ and let

$$\overrightarrow{R} = (\boldsymbol{R}, \boldsymbol{N}) \tag{105}$$

be the reaction of γ, its moment $\boldsymbol{N}$ being taken with respect to O. As it is well-known, the equation of the directrix d of (105) is

$$\boldsymbol{r} \times \boldsymbol{R} = \boldsymbol{N}, \tag{106}$$

$\boldsymbol{r} = \boldsymbol{OP}$ denoting the fluent radius-vector of any point P of d. If by definition $\boldsymbol{r}_A = \boldsymbol{OA}$, then (106) implies

$$\boldsymbol{r}_A \times \boldsymbol{R} = \boldsymbol{N} \tag{107}$$

by virtue of Ax 3 E.

Scholium 3. As a matter of fact, Ax 3 E states 3 things:

1. The existence of the force $\overrightarrow{R}$.
2. $\overrightarrow{R}$ is acting on S.
3. $\boldsymbol{N}$ is known as far as $\boldsymbol{r}_A$ and $\boldsymbol{R}$ are known.

In other words, any constraint imposed on a rigid body S introduces a new force in the right-hand sides of the equations (82), (87), governing the motion of S. Besides, any such constraint introduces 3 new unknown quantities in the mathematical problem to be solved, namely the components of $\boldsymbol{R}$ according to

$$\boldsymbol{R} = R_x\boldsymbol{i} + R_y\boldsymbol{j} + R_z\boldsymbol{k} \tag{108}$$

in view of (107).

Let, in a particular dynamical problem, S be under the action of the active forces

$$\overrightarrow{F}_\mu = (\boldsymbol{F}_\mu, \boldsymbol{M}_\mu) \qquad (\mu = 1, \ldots, m) \tag{109}$$

and let by definition

$$\boldsymbol{F} = \sum_{\mu=1}^{m} \boldsymbol{F}_\mu, \quad \boldsymbol{M} = \sum_{\mu=1}^{m} \boldsymbol{M}_\mu. \tag{110}$$

Let n geometrical constraints be imposed on S, generating passive forces

$$\overrightarrow{R}_\nu = (\boldsymbol{R}_\nu, \boldsymbol{N}_\nu) \qquad (\nu = 1, \ldots, n), \tag{111}$$

and let by definition

$$\boldsymbol{R} = \sum_{\nu=1}^{n} \boldsymbol{R}_\nu, \quad \boldsymbol{N} = \sum_{\nu=1}^{n} \boldsymbol{N}_\nu. \tag{112}$$

(Naturally, all moments $\boldsymbol{M}_\mu$ and $\boldsymbol{N}_\nu$ ($\mu = 1, \ldots, m$; $\nu = 1, \ldots, n$) in (109) and (111) are taken with respect to O.) Besides, let

$$\boldsymbol{M}_G = \boldsymbol{M} + \boldsymbol{F} \times \boldsymbol{r}_G, \quad \boldsymbol{N}_G = \boldsymbol{N} + \boldsymbol{R} \times \boldsymbol{r}_G \tag{113}$$

be the moments of the system of forces (109) and (111), respectively, with regard to the mass-centre G of S. Under these hypothesis, the Eulerian dynamical equations (82), (87) take the form

$$m\ddot{x}_G = F_x + R_x, \quad m\ddot{y}_G = F_y + R_y, \quad m\ddot{z}_G = F_z + R_z, \tag{114}$$

$$\begin{cases} A\dot{\omega}_\xi - (B - C)\omega_\eta\omega_\zeta - D(\omega_\eta^2 - \omega_\zeta^2) \\ \qquad - E(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) - F(\dot{\omega}_\eta - \omega_\zeta\omega_\xi) = M_{G\xi} + N_{G\xi}, \\ B\dot{\omega}_\eta - (C - A)\omega_\zeta\omega_\xi - E(\omega_\zeta^2 - \omega_\xi^2) \\ \qquad - F(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) - D(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) = M_{G\eta} + N_{G\eta}, \\ C\dot{\omega}_\zeta - (A - B)\omega_\xi\omega_\eta - F(\omega_\xi^2 - \omega_\eta^2) \\ \qquad - D(\dot{\omega}_\eta + \omega_\zeta\omega_\xi) - E(\dot{\omega}_\xi - \omega_\eta\omega_\zeta) = M_{G\zeta} + N_{G\zeta}, \end{cases} \tag{115}$$

provided by definition

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}, \tag{116}$$

$$\boldsymbol{M}_G = M_{G\xi}\bar{\xi}^0 + M_{G\eta}\bar{\eta}^0 + M_{G\zeta}\bar{\zeta}^0, \tag{117}$$

$$\boldsymbol{N}_G = N_{G\xi}\bar{\xi}^0 + N_{G\eta}\bar{\eta}^0 + N_{G\zeta}\bar{\zeta}^0. \tag{118}$$

Scholium 4. Even a cursory analysis of the mathematical formalism describing a geometrical constraint of the kinds enumerated above at once displays that any such constraint imposes one, two, or at most three analytic restrictions on the canonic parameters (88) of the rigid body S. In the case of n constraints this circumstance diminishes the number of the unknown functions

$$x_\Omega(t),\ y_\Omega(t),\ z_\Omega(t),\ \psi(t),\ \varphi(t),\ \theta(t) \tag{119}$$

of the time t, the determination of which as a solution of the system of differential equations (114), (115) is required, by at least n and at most $3n$ units. On the other hand, the reactions (111) introduce $3n$ new unknowns. In such manner, any problem of rigid dynamics is reduced to a system of 6 ordinary differential equations (114), (115) of second order with respect to the time t of a heterogenously mixed type: a part of the unknown quantities are some of the functions (119) and they are at hand in (114), (115) analytically, that is to say together with their first and second derivatives with respect to t; another part are the $3n$ unknown components of the reactions (111), provided

$$\boldsymbol{R}_\nu = R_{\nu x}\boldsymbol{i} + R_{\nu y}\boldsymbol{j} + R_{\nu z}\boldsymbol{k} \qquad (\nu = 1, \ldots, n), \tag{119'}$$

and they are at hand in (114), (115) algebraically, as linear unknown quantities, in point of fact.

Scholium 5. The first query arising when a problem of rigid dynamics is put for discussion is the question, whether the system (114), (115) of differential equations is consistent, i.e. whether it does or does not possess a solution. In other words, this is the *Existence Problem* for the dynamical problem under consideration or, in view of the physical interpretation of the mathematical circumstances, the *Possibility Problem* for the motion of the rigid body under the conditions this dynamical problem announces.

On account of the mathematical complications the existence problem gives rise to, it is an object of a particular investigation we shall soon turn back to. For the time being we shall confine us to the remark that most authors of mechanical writings leave the existence problem out in the cold in the most flagrant manner: not only they do not proceed to its solution, but even do not make mention of the existence of the existence problem.

Scholium 6. We shall bring our exposition to an end with a note concerning the application of the Eulerian dynamical equations (114), (115) to that special kind of rigid bodies, which are known under the name of *rigid rods*.

A rigid rod L is a rigid body the density (35) of which has the eccentricity to be zero everywhere save along a straight line l, called the *directrix* of L. Let us connect with L invariably an orthonormal right-hand orientated Cartesian system of reference $\Omega\xi\eta\zeta$ in the following manner: the axis $\Omega\xi$ coincides with the directrix l; the axis $\Omega\zeta$ is parallel to the line of intersection of the plane Oxy with the plane through Ω perpendicular to $\Omega\xi$ (supposing those two planes non-parallel); the unit vectors $\bar{\xi}^0$ and $\bar{\zeta}^0$ of the axes $\Omega\xi$ and $\Omega\zeta$, respectively, once defined, the axis $\Omega\eta$ is determined by its unit vector $\bar{\eta}^0 = \bar{\zeta}^0 \times \bar{\xi}^0$. The axis Oz being obviously perpendicular to the axis $\Omega\zeta$, the definition (13) implies

$$\theta = \frac{\pi}{2}, \tag{120}$$

and (120), (17) imply

$$(121)\qquad \begin{cases} a_{11} = \cos\psi\cos\varphi, & a_{12} = -\cos\psi\sin\varphi, & a_{13} = \sin\psi, \\ a_{21} = \sin\psi\cos\varphi, & a_{22} = -\sin\psi\sin\varphi, & a_{23} = -\cos\psi, \\ a_{31} = \sin\varphi, & a_{32} = \cos\varphi, & a_{33} = 0. \end{cases}$$

Besides, (120) and (26) imply

$$(122)\qquad \omega_\xi = \dot\psi\sin\varphi, \quad \omega_\eta = \dot\psi\cos\varphi, \quad \omega_\zeta = \dot\varphi.$$

By virtue of the condition (120) a dynamical problem concerning a rigid rod is presumably overdetermined. Indeed, the canonic parameters of L are now 5 in number, namely

$$(123)\qquad x_\Omega,\ y_\Omega,\ z_\Omega,\ \psi,\ \varphi,$$

whereas there are 6 equations (114), (115) for their determination: in the case of a free rigid rod they outnumber the unknown quantities (123). This contradiction is, however, only an ostensible one.

Let us take a closer view of the situation. The definition of a rigid rod implies

$$(124)\qquad \varkappa(\bar\rho) = 0$$

for

$$(125)\qquad \eta \neq 0$$

or

$$(126)\qquad \zeta \neq 0,$$

whence, formally at least,

$$(127)\qquad dm = \varkappa(\bar\rho)\, d\xi d\eta d\zeta$$

implies

$$(128)\qquad dm = \varkappa(\xi)\, d\xi$$

provided (9). Now (128), (124)–(126), and (41) imply

$$(129)\qquad \bar\rho_G = \frac{1}{m}\int \xi\varkappa(\xi)\, d\xi \bar\xi^0,$$

i.e.

$$(130)\qquad \eta_G = \zeta_G = 0$$

provided (64).

On the other hand, (128), (124)–(126) and (58), (59) imply

$$(131)\qquad I_{\xi\xi} = 0, \quad I_{\eta\eta} = I_{\zeta\zeta} = \int \xi^2\varkappa(\xi)\, d\xi, \quad I_{\eta\zeta} = I_{\zeta\xi} = I_{\xi\eta} = 0$$

and (130), (60), (61) imply

$$(132)\qquad J_{\xi\xi} = 0, \quad J_{\eta\eta} = J_{\zeta\zeta} = m\xi_G^2, \quad J_{\eta\zeta} = J_{\zeta\xi} = J_{\xi\eta} = 0.$$

Now (131), (132), (62), (63) imply

$$(133)\qquad A = 0, \quad B = C = I, \quad D = E = F = 0,$$

provided by definition

$$I = \int \xi^2 \varkappa(\xi)\, d\xi - m\xi_G^2, \tag{134}$$

and (133), (115) imply

$$\begin{cases} 0 = M_{G\xi} + N_{G\xi}, \\ I(\dot\omega_\eta - \omega_\zeta\omega_\xi) = M_{G\eta} + N_{G\eta}, \\ I(\dot\omega_\zeta + \omega_\xi\omega_\eta) = M_{G\zeta} + N_{G\zeta}. \end{cases} \tag{135}$$

We are faced now with a most interesting and instructive phenomenon — a danger hanging like the sword of Damocles over the head of everyone working in rational mechanics. Let us first suppose that the rigid rod is free; then

$$N_{G\xi} = 0 \tag{136}$$

and the first equation (135) implies

$$M_{G\xi} = 0. \tag{137}$$

In other words, (137) is a necessary condition for a free rigid rod dynamical problem to be consistent, videlicet to possess a solution or, using a mechanical language, in order that the rigid body could move. Now is this *conditio sine qua non* satisfied indeed?

This is a question God Almighty cannot answer.

A Mister Someone with a more physical than mathematical mental constitution would at once exclaim: Nonsense! You bet (137) is true!

What are his motives?

His mental picture of a rigid rod is suggested by his everyday experience. He cannot imagine a spade, or a mattock, or an ax working save when hands are holding its shank, in other words, save when the forces acting on the instrument are applied on its handle. And the meaning of the term "applied" in this context is: when the directrices of the forces intersect the directrix of the rod.

Since the latter in our case is the axis $\Omega\xi$, Mister Someone presupposes that the directices

$$\boldsymbol{r} \times \boldsymbol{F}_\mu = \boldsymbol{M}_\mu \qquad (\mu = 1, \ldots, m) \tag{138}$$

of the forces (109) intersect $\Omega\xi$, the equation of which is

$$\boldsymbol{r} \times \bar\xi^0 = \boldsymbol{r}_\Omega \times \bar\xi^0 \tag{139}$$

or

$$\bar\rho \times \bar\xi^0 = \boldsymbol{o} \tag{140}$$

in view of (6). On the other hand, (64) and (130) imply

$$\bar\rho_G = \xi_G \bar\xi^0, \tag{141}$$

and (138), (6), (45) imply

$$(\bar\rho + \boldsymbol{r}_G - \bar\rho_G) \times \boldsymbol{F}_\mu = \boldsymbol{M}_\mu \qquad (\mu = 1, \ldots, m), \tag{142}$$

whence

$$(\bar\rho + \boldsymbol{r}_G - \bar\rho_G) \times \boldsymbol{F} = \boldsymbol{M} \tag{143}$$

by virtue of (110) or, just the same,

$$\bar{\rho} \times \boldsymbol{F} = \boldsymbol{M}_G + \xi_G \bar{\xi}^0 \times \boldsymbol{F} \tag{144}$$

in view of (141) and

$$\boldsymbol{M}_G = \boldsymbol{M} + \boldsymbol{F} \times \boldsymbol{r}_G. \tag{145}$$

Since the equations (140) and (144) are, by a physical hypothesis, consistent, the relation

$$\bar{\rho} = \lambda \bar{\xi}^0 \tag{146}$$

with an appropriate λ according to (140) and (144) imply

$$\lambda \bar{\xi}^0 \times \boldsymbol{F} = \boldsymbol{M}_G + \xi_G \bar{\xi}^0 \times \boldsymbol{F}, \tag{147}$$

whence

$$\bar{\xi}^0 \boldsymbol{M}_G = 0, \tag{148}$$

i.e. (137) by virtue of (117).

In such a manner, the necessary condition (137) for a free rigid rod dynamical problem to be consistent is a corollary from the hypothesis that the directrices of all active forces applied on the rod intersect the directrix of the rod, i.e. the line along which its density is different from zero. But this hypothesis does not follow from the hitherto formulated definition of the rigid rod concept consisting in the only requirement (124) for (125) or (126): it is a new aspect of this notion that has been just now substantiated physically and mathematically and that must necessarily take part in the definition of this concept.

In such a manner we arrive at the following newly improved formulation:

A rigid rod is a rigid body the density of which is zero anywhere save along a straight line (its directrix), where its density is such that the integral (39) is non-zero; moreover, if an active force is acting on a rigid rod, its directrix intersects (or coincides with) the directrix of the rod.

This definition accepted, (137) implies that the first equation (135) becomes (136). The relation (136), however, is by no means an obligatory one. The meaning of this statement is that the condition (136) is both beyond proof and beyond disproof. Now we are faced with the same logical perplexities as in the case of the necessary condition (137). This dilemma is settled in the same way as in the preceding case. In other words, it is supposed that the only points of contact of a rigid rod with any geometrical constraint, imposed on it, must be lying on its directrix.

Summing up, we may now state that (137) and (136) are presumptive necessary conditions for any problem of rigid rod dynamics. *Praemonitus et praemunitus* with this new clause, one has now every right to state that in the case of a rigid rod the first equation (115) turns out to become an identity of the kind (101) (provided the system of reference $\Omega\xi\eta\zeta$, invariably connected with the rigid body, is chosen in such a manner that (120) and (124) provided (125) or (126) hold). As a result, in the case of rigid rod dynamics one has at his disposal exactly 5 equations of motion, namely (114) and

$$I(\dot{\omega}_\eta - \omega_\zeta \omega_\xi) = M_{G\eta} + N_{G\eta}, \quad I(\dot{\omega}_\zeta + \omega_\xi \omega_\eta) = M_{G\zeta} + N_{G\zeta}, \tag{149}$$

while the number of the unknown quantities in the dynamical problem is not lesser than 5.

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Received 8.04.1993

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 86, 1992

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 86, 1992

NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS
V. PREHISTORY OF MECHANICAL CONSTRAINTS

GEORGI CHOBANOV, IVAN CHOBANOV

Not only the public press but also the precis of the paedagogues has taught us to gulp superlatives as daily narcotics. The professor who forgets to class, rate, and rank his subject as the first and finest something will fail to find it mentioned in the students' terminal examinations. The busy modern calls for culture in predigested quintessence pills, packaged in abridged paperbacks, explained by folksy prefaces in pellet-paragraphs of sugared baby-talk, lullabies to smugness. Suggestion to a general audience that historical facts must precede if not replace historical enthusiasm may expect only oblivion, the palm of dullness.

C. Truesdell: *The Mechanics of Leonardo da Vinci*

Георги Чобанов, Иван Чобанов. ДИНАМИЧЕСКИЕ АКСИОМЫ НЬЮТОНА И ЭЙЛЕРА. V. ПРЕДИСТОРИЯ МЕХАНИЧЕСКИХ СВЯЗЕЙ

Это есть пятая часть серии статьей, посвещенные динамических аксиом Ньютона и Эйлера; она естественное продолжение и развитие последней из них [17], в которой подробно дискутированы физические мотивировки понятия о механических связах, наложенных систем масовых точек и твердых тел. В работе приведены многочисленные автентические данные в связи с зачатием и преждевременном рождении понятия связи в ранной истории рациональной механики, причем особое внимание уделено сочинений *Discorsi e Dimostrazioni Matematiche Intorno à Due Nuove Scienze* Галилея, *Philosophiae Naturalis Principia Mathematica* Ньютона и *Traité de Dynamique* Даламбера.

Georgi Chobanov, Ivan Chobanov. NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS. V. PREHISTORY OF MECHANICAL CONSTRAINTS

This is the fifth part of a series of articles dedicated to the Newtonian and Eulerian dynamical axioms; it is the natural continuation and development of the last of them [17], where the physical motivations of the notion of mechanical constraints imposed on mass-point and rigid body systems are discussed at length. Numerous authentical data are adduced in connection with the conception and premature birth of the constraint concept in the early history of rational mechanics, special stress being laid on Galileo's *Discorsi e Dimostrazioni Matematiche Intorno à Due Nuove Scienze*, Newton's *Philosophiae Naturalis Principia Mathematica*, and D'Alembert's *Traité de Dynamique*, where the germs of the constraint notion may be traced.

This fifth part of a series of articles dedicated to the Newtonian and Eulerian dynamical axioms is a natural continuation and development of the last of them [17], where the physical motivations of the notion mechanical constraints imposed on mass-points and rigid bodies are discussed at length and which is published in this volume of the *Annual*; that is why, in order to avoid reiterations, the literature quoted in this paper and in [17] has a unified numeration.

The motivations of the engineering praxis are as old as human race too. The *vectis*, *axis in peritrochio*, *trochlea seu polispastus*, *cochlea* and *cuneus* (that is to say the lever, axis and wheel, pulley, screw, and wedge, respectively) have been utilized already by the ancient to the fullest extent. In the *Auctoris praefacio ad lectorem* of his *Principia* [7] Newton states:

"Pars haec mechanicae a veteribus in potentiis quinque ad artes manuales spectantibus exculta fuit, qui gravitatem (cum potentia manualis non sit) vix aliter quam in ponderibus per potentis illas movendis considerarunt."

As Krilov observes in his Russian version [18] of *Principia*, the term *potentia* is used here in two ways: the first time as a synonym of *machina*, and the second time as a synonym of *power*. As an illustration he adduces the following excerpt from Maclaurin's book [19]:

"It is distinguished by Sir I. Newton into *practical* and *rational* mechanics; the former treats of the *mechanical powers* viz. the *lever*, the *axis* and *wheel*, the *pulley*, the *wedge* and the *screw* to which the *inclined* plane is to be added and of the various combinations together. Rational mechanics comprehends the whole theory of motion and shews when the *powers* of *forces* are given how to determine the motions that are produced by them ... in tracing the powers that operate in nature from the phenomena we proceed by analysis and deducing the phenomena from the powers or causes that produce them we proceed by synthesis."

The close relations of early mechanicians with engineering experience is reflected in an excellent manner in Galileo's *Dialoghi delle nuove scienze* [20], *Giornata prima* of which begins with the following inferences of Salviati and Sagredo:

SAL. Largo campo di filosofare a gl'intelletti specolativi parmi che porga la frequente pratica del famoso arsenale di voi, Signori Veneziani, ed in particolare in quella parte che mecanica si dominanda; atteso che quivi ogni sorte di strumento e di machina vien continuamente posta in opera da numero grande d'artefici, tra i quali, e per l'osservazioni fatte dai loro antecessori, e per quelle che di propria avvertenza vanno continuamente per se stessi facendo, è forza che ve ne siano de i pertissimi e di finissimo discorso.

SAGR. V. S. non s'inganna punto: ed io, come per natura curioso, frequento per mio diporte la visita di questo luogo e la pratica di questi che noi, per certa preminenza che tengono sopra 'l resto della maestranza, domandiamo proti; la conferenza de i quali mi ha piu volte aiutato nell'investigazione della ragione di effetti non solo maravigliosi, ma reconditi ancora e quasi inopinabili. E vero che tal volta anco mi ha messo in confusione ed in disperazione di poter penetrare comme possa seguire quello che, lontano da ogni mio concetto, mi dimostra il senso esser vero ... [21, II, p. 81].

The question now quite reasonably arises: are there in Galileo's mechanical writings solutions of dynamical problems concerning motions of constrained mass-points or rigid bodies? Before answering this question we may point out that it is by no means groundless, since motions of mass-points along inclined (to the vertical) lines or circumferences are *par excellence* constrained motions, and such problems are abundant in *Giornata terza* of *Discorsi*. As a matter of fact, all theorems, propositions, corollaries, problems, and scholiums of section *De motu naturaliter accelerato* of *Giornata terza*, as well as all concomitant commentaries of Salviati, Sagredo, and Simplicio, beginning with *Theorema III*, *Propositio III*, are concerned with *motus naturalis* along inclined lines. On that ground, formally at least, one may expect that germs leastwise of constraint dynamics may be found in Galileo's works.

Alas, those are blighted hopes, and the reason is a quite simple one. In spite of all traditional physical folklore there is no dynamics at all in anything Galileo has written on mechanics. In vain will remain all Lagrange's efforts to render *quae sunt Caesaris Deo et quae sunt Dei Caesari*:

"La Dynamique est la science des forces accélératrices ou retardatrices et des mouvements variés qu'elles doivent produire. Cette science est due entièrement aux modernes, et Galilée est celui qui en a jeté les premiers fondements. Avant lui on n'avait considéré les forces qui agissent sur les corps que dans l'état d'équilibre; et quoiqu'on ne pût attribuer l'accélération des corps pesants et le mouvement curviligne des projectiles qu'à l'action constante de la gravité, personne n'avait encore réussi à déterminer les lois de ces phénomènes journaliers, d'après une cause si simple. Galilée a fait le premier ce pas important et a ouvert par là une carrière nouvelle et immense à l'avancement de la Méchanique. Cette découverte est exposée et developpée dans l'Ouvrage intitulé: *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*, lequel parut, pour la première fois, à Leyde, en 1638. Elle ne procura pas à Galilée, de son vivant, autant de célébrité que celles qu'il avait faites dans le ciel; mais elle fait aujourd'hui la partie la plus solide et la plus reelle de la gloire de ce grand homme" [11, p. 237].

This brilliant appraisal of Galileo's mechanical performances has been multiplied in the course of two clear centuries in a myriad physical and mechanical text-books, treatises, monographs, articles, journals, newspapers, etceteras in precise conformity with the renowned verse of Vergilius *Fama mobilitate viget viresque adquirit eundo*. As a result of the loathsome aptitude of human mind to idolatry the image of Galileo as the founder of non-peripatetic dynamics is rooted in public spiritedness as tight as the image of Marx as the founder of non-capitalistic economics. Fortunately, the first case is not this far sinister.

The quoted excerpt from [11] may be written only by someone who has not read *Discorsi*; or by someone who has read it carelessly; or by someone who has read it attentively and has understood nothing. Otherwise one cannot explain why the adjective *dynamical* is ascribed to purely *kinematical* investigations.

For the whole of the content of *Giornata terza & quarta* is kinematics, only kinematics, and nothing save kinematics.

This is a point that needs a close attention. In order to make things transparent and to leave no room for gratuitous and subjective interpretations, let us drink one sip or two out of the very spring:

"Quae in motu aequabili contingunt accidentia, in praecedenti libro considerata sunt: modo de motu accelerato pertractandum.

Et primo, definitionem ei, quo utitur natura, apprime congruentem investigare atque explicare convenit. Quamvis enim aliquam lationis speciem ex arbitrio confingere, et consequentes eius passiones contemplari, non sit inconveniens ..., tamen, quandoquidem quadam accelerationis specie gravium descendentium utitur natura, eorundem speculari passiones decrevimus, si eam, quam allaturi sumus de nostro motu accelerato definitionem, cum essentia motus naturaliter accelerati congruere contigerit" [21, II, p. 254].

In such a manner, we come to know that Galileo:

1. Proceeds to study motions with impermanent velocity.

2. Realizes the possibility of an infinite variety of such motions.

3. Is interested in that special kind of accelerated motions Nature makes use of.

4. Does not know the definition of the *motus naturaliter acceleratus*.

How does Galileo solve the last problem? *Ipse dixit*:

"Postremo, ad investigationem motus naturaliter accelerati nos quasi manu duxit animadversio consuetudinis atque instituti ipsiusmet naturae in ceteris suis operibus omnibus, in quibus exercendis uti consuevit mediis primis, simplicissimis, facillimis. Neminem enim esse arbitror qui credat, natatum aut polatum simpliciori aut faciliori modo exerceri posse, quam eo ipso, quo piscem et aves instinctu naturali utuntur" [*ibid.*].

Ergo, Galileo:

5. Intends to discover the definition of naturally accelerated motion by observing (and possibly measuring) natural motions.

6. Proclaims a philosophical principle Nature obeys unquestingly: Simplicity.

What does, however, *simplicity* mean in the special case of natural motions? *Verba magistri*:

"Dum igitur lapidem, ex sublimi a quiete descendentem, nova deinceps velocitatis acquirere incrementa animadverto, cur talia additamenta, simplicissima atque omnibus magis obvia ratione, fieri non credam? Quod si attente inspiciamus, nullum additamentum, nullum incrementum, magis simplex inveniemus, quam illud, quod semper eodem modo superaddit ... Et sic a recta ratione absonum nequaquam esse videtur, si accipiamus, intentionem velocitatis fieri iuxta temporis extensionem; ex quo definitio motus, de quo acturi sumus, talis accipi potest: Motum aequabiliter, seu uniformiter, acceleratum dico illum, qui, a quiete recedens, temporibus aequalibus aequalia celeritatis momenta sibi superaddit" [*ibid.*, p. 254–255].

In such a manner, following a most natural, logical, and methodical course of thought, Galileo:

7. Arrives at the modern definition of uniformly accelerated motion.
8. Declares the latter the law of free fall.

Moreover, the author puts the last two items to the test of experimental control:

"In un regolo, o vogliàn dir corrente, di legno, lungo circa 12 braccia, e largo per un verso mezo braccio e per l'altro 3 dita, si era in questa minor larghezza incavato un canaletto, poco piu largo d'un dito; tiratolo drittissimo, e, per averlo ben pulito e liscio, incollativi dentro una carta pecora zannata e lustrata al possibile, si faceva in esso scendere una palla di bronzo durissimo, ben rotondata e pulita; constituito che si era il detto regolo pendente, elevando sopra il piano orizontale una delle sue estremitá un braccio o due ad arbitrio, si lasciava (come dico) scendere per il detto canale la palla, notando, nel modo che appresso dirò, il tempo che consumava nello scorrerlo tutto, replicando il medesimo atto molte volte per assicurarsi bene della quantitá del tempo, nel quale non si trovavà mai differenza né anco della decima parte d'una battila di polso. Fatta e stabilita precisamente tale operazione, facemmo scender la medesima palla solamente per la quarta parte della lunghezza di esso canale; e misurato il tempo della sua scesa, si trovava sempre puntualissimamente esser la metá dell'altro: e facendo poi l'esperienze di altre parti, esaminando ora il tempo di tutta la lunghezza col tempo della metá, o con quello delli duo terzi o de i 3/4, o in conclusione con qualunque altra divisione, per esperienze ben cento volte replicate sempre s'incontrava, gli spazii passati esser tra di loro come i quadrati de i tempi, e questo in tutte le inclinazioni del piano, cioè del canale nel quale si faceva scender la palla; dove osservamo ancora, i tempi delle scese per diverse inclinazioni mantener esquisitamenre tra di loro quella proporzione che piú a basso troveremo essergli assegnata e dimostrata dall'Autore. Quanto poi alla misura del tempo, si teneva una gran secchia piena d'acqua, attaccata in alto, la quale per un sottil cannellino, sal da togli nel fondo, versava un sottil filo d'acqua, che s'andava ricevendo con un piccol bicchiero per tutto 'l tempo che la palla scendeva nel canale e nelle sue parti: le particelle poi dell'acqua, in tal guisa raccolte, s'andavano di volta in volta con esattissima bilancia pesando, dandoci le differenze e proporzioni de i pesi loro le differenze e proporzioni de i tempi; e questo con tal giustezza, che, come ho detto, tali operazioni, molte e molte volte replicata, giá mai non differivano d'un notabil momento" [*ibid.*, p. 274–275].

Once the definition of *motus naturaliter acceleratus* established, all that follows in *Giornata terza & quarta* of *Movimenti Locali* are kinematical exercises of free fall, sliding along a line inclined towards the horizon, and flying of projectiles. At that, the moving objects are points rather than bodies; moreover, those are geometrical — by no means mechanical — points that are moving on the pages of Galileo's *Discorsi*. The meaning of this statement is that the mass of the moving point is completely irrelevant to Galileo's meditations and calculations — there is no relationship between moving object and moving cause. As a mere child could say, the mass concept is void of sense if estranged from forces, and there are no forces at all in Galileo's mechanical studies — at least no such ones that would be found congenial today.

And yet, even only kinematically, no one can justly deny that there are constrained motions in Galileo's *Discorsi*. There may only be divergences of views on the degree of originality of the contributions of this notionalist to the field of rational mechanics. To Galileo's worshippers Truesdell's standpoint may seem blasphemy:

"Historians of letters had meanwhile created the myth of the 'Renaissance'. According to this myth, in the Middle Ages man hibernated beneath a pall of scholastic repetitions, borrowed from Aristotle and enforced by the Church; the Renaissance, casting all this aside, opened its eyes and discovered man and the world by personal sensation ...

... such historians of science ... found several of Galileo's ideas, more or less, in Leonardo's notes ... published in facsimile in the years 1881 to 1936 ... Previously historians had believed that Galileo thought these things out of 'genius' applied to thin air ... 'Discovery' of Leonardo transferred the point of application of this same theory a century backwards. He, too, had the same material to work with: 'genius' and thin air, and the remaining problem for this group of historians was only to see how Leonardo's ideas got to Galileo, thus making the latter a true grandson, if not son, of the Renaissance" [8, p. 25, 27].

To make matters worse, strange characters emerge from days long gone by:

"...the main kinematical properties of uniformly accelerated motions, still attributed to Galileo by the physics texts, were discovered and proved by scholars of Merton College — William Heytesbury, Richard Swineshead, and John of Dumbleton — between 1328 and 1350. Their work distinguished *kinematics*, the geometry of motion, from *dynamics*, the theory of the causes of motion. Their approach was mathematical. They succeeded in formulating a fairly clear concept of instantaneous speed, which means that they foreshadowed the concepts of function and derivative, and they proved that the space traversed by a uniformly accelerated motion in a given time is the same as that traversed by a uniform motion whose speed is the mean of the greatest and the least speeds in the accelerated motion. In principle, the qualities of Greek physics were replaced, at least for motions, by the numerical quantities that have ruled Western science ever since. This work was quickly diffused into France, Italy, and other parts of Europe. Almost immediately, Giovanni da Casale and Nicole Oresme found how to represent the results by geometrical graphs, introducing the connection between geometry and the physical world that became a second characteristic habit of Western thought — a habit so deep-seated that it is known to every carpenter and passes unremarked only in certain highly specialized professions ...

Clagett [22] has cited much evidence to show that these ideas, which originated in England and France in the early fourteenth century, were discussed back and forth in periods of varying activity and inactivity in France, the Empire, and Italy in the latter half of the same century and were taught in Italian universities in the next one, at the end of which a flood of printed books opened the subject to everyone — everyone who could understand Latin and mathematics" [*ibid.*, p. 30–31].

In the light of this information Galileo's pretensions in the introductory words of *De motu locali* seem a bit overdone:

"De subiecto vetustissimo novissimam promovemus scientiam. Motu nil forte antiquis in natura, et circa eum volumina nec pauca nec parva a philosophis conscripta reperiuntur; symptomatum tamen, quae complura et scitu digna insunt in eo, adhuc inobservata, necdum indemonstrata, comperio. Leviora quaedam adnotantur, ut, gratia exempli, naturalem motum gravium descendentium continue accelerari; verum, iuxta quam proportionem eius fiat acceleratio, proditum hucusque non est: nullus enim, quod sciem, demonstravit, spacia a mobili descendente ex quiete peracta in temporibus aequalibus, eam inter se retinere rationem, quam habent numeri impares ab unitate consequentes" [21, II, p. 247].

The first place in *Discorsi*, where inclined plane comes into view, is *Theorema III, Propositio III* of *De motu naturaliter accelerato* in *Giornata terza*, namely:

"Si super plano inclinato atque in perpendiculo, quorum eadem sit altitudo, feratur ex quiete idem mobile, tempora lationum erunt inter se ut plani ipsius et perpendiculi longitudines" [*ibid.*, p. 282].

Inclined planes are repeatedly used in well-nigh all the following theorems, propositions, corollaries, problems, scholia, and commentaries of Salviati, Sagredo, and Simplicio of *Giornata terza* of *Discorsi*. As a matter of fact, the whole content of this part of *Due scienze* consists of exercises on theme and variations point kinematics of uniformy accelerated motions.

It is quite immaterial to us whether Galileo's statements in the said propositions are true or false: the cold fact is, there is point kinematics of constrained motions in his book. This applies especially to an important problem — that of *lationem omnium velocissimam*, which later became the starting point of Johann Bernoulli's *Problemata novum, ad cuius solutionem geometrici invitantur*, as well as a stimulus for variational calculus. Formulated by Galileo in the form of a *Scholium*, it reads:

"Ex his quae demonstrata sunt, colligi posse videtur, lationem omnium velocissimam ex termino ad terminum non per brevissimam lineam, nempe per rectam, sed per circuli portionem, fieri" [*ibid.*, p. 333].

In such a way, Galileo:

9. Includes the circumference in the family of geometrical constraints.

10. Formulates a minimalization problem concerning constrained motions (for the first time in the history of mechanics, as far as our knowledge goes).

The importance of the last event is not diminished by the fact that Galileo's solution was wrong: another contingency was purely and simply impossible in his days. As Truesdell says:

" Now a mathematician has a matchless advantage over general scientists, historians, and exponents of other professions: He can be wrong. *A fortiori*, he can also be right. There are errors in Euclid, and, to within a set certainly of measure zero on the ordinary human scale, what Euclid proved to be true in ancient Greece is true even in the colossal, unprecedented, nucleospacial, totally welfared today. In the advance through the physical, social, historical, and other sciences, the demarcation between truth and falsehood grows vaguer, until in some areas truth can be rezoned as falsehood and falsehood enshrined into truth by consensus of "acknowledged experts and autorities" or even popular vote. One professor discussing the doctrines of Karl Marx may label them as grave errors; a second, equally qualified, may present them as problematic, partly true and partly not so; while a third, living in a different part of the world, may proclaim them as the quintessence of human knowledge. In the mathematical science as taught by the colleagues of these same three social scientists, there is no disagreement as to what is true and what is not. A mistake made by a mathematician, even a great one, is not a "difference of point of view" or "another interpretation of the data" or a "dictate of a conflicting ideology", it is a mistake. The greatest of all mathematicians, those who have discovered the greatest quantities of mathematical truths, are also those who have published the greatest number of lacunary proofs, insufficiently qualified assertions, and flat mistakes.

The mistakes made by a great mathematician are of two kinds: first, trivial slips that anyone can correct, and second, titanic failures reflecting the scale of the struggle which the great mathematician waged. Failures of this latter kind are often as important as successes, for they give rise to major discoveries by other mathematicians. One error of a great mathematician has often done more for science than a hundred little theorems proved by lesser men" [8, p. 140].

While Galileo did not have at his disposal the tool for solving dynamical problems involving constrained mass-points, Newton did. That is why it is interesting to the utmost degree (at least as far as our topic is concerned) to see what did he actually accomplish by its aid.

To this end it is sufficient to take a look at Newton's mechanical archives, his *Principia* [7]. The realization of the fact that this book is a treatise on point dynamics, not in the least on rigid dynamics, is as old as Euler:

"... while Newton had used the word 'body' vaguely and in at least three different meanings, Euler realized that the statements of Newton are generally correct only when applied to masses concentrated at isolated points ..." [8, p. 107].

Therefore, we must discover how far Newton has penetrated into the field of constrained mass-point dynamics. Already a mere glance at the contents of *Principia* at once displays that the only place of the work, where constrained motions may be treated, is *Sect. X: De Motu Corporum in Superficiebus datis, deq; Funipendulorum Motu reciproco* of *Liber Primus, De Motu Corporum.* It begins with *Prop. XLVI. Prob. XXXII*, namely:

"Posita cujuscunq; generis vi centripeta, datoq; tum virium centro tum plano quocunq; in quo corpus revolvitur, et concessis Figurarum curvilinearum quadraturis: requiritur motus corporis de loco dato data cum velocitate secundum Rectam in Plano illo datum egressi" [7, p. 145].

This is a constrained mass-point dynamical problem *par excellence*: it proposes to find the motion of a mass-point constrained to remain on a given plane and subjected to the action of an arbitrary central force, the pole of which is lying outside the plane.

Newton's Problem XXXII is extraordinary interesting with a view to our topic, namely the nascency of the idea of a mechanical constraint imposed on a mass-point or a rigid body. We shall therefore follow the train of thoughts of *Principia*'s author exposed in his solution of this problem. At that, with an eye to a greater clearness, we shall quote the English version of the work in Motte's translation rather than the original Latin text:

"Let S be the centre of force, SC the least distance of that centre from the given plane, P a body issuing from the place P in the direction of the right line PZ, Q the same body revolving in its curve, and PQR the curve itself which is required to be found, described in that given plane. Join CQ, QS, and if in QS we take SV proportional to the centripetal force with which the body is attracted towards the centre S, and draw VT parallel to CQ, and meeting SC in T; then will the force SV be resolved into two (by Cor. II of the Laws of Motion), the force ST, and the force TV; of which ST attracting the body in the direction of a line perpendicular to the plane, does not at all change its motion in that plane. But the action of the other force TV, coinciding with the position of the plane itself, attracts the body directly towards the given point C in that plane; and therefore causes the body to move in the plane in the same manner as if the force ST were taken away, and the body were to revolve in free space about the centre C by means of the force TV alone. But there being given the centripetal force TV with which the body Q revolves in free space about the given centre C, there is given (by Prop. XLII) the curve PQR which the body describes; the place Q, in which the body will be found at any given time; and, lastly, the velocity of the body in that place Q. And conversely, Q. E. I." [23, vol. T, p. 148–149].

In this solution two places of the book are quoted in the capacity of arguments: Corollary II of the introductory *Axiomata sive Leges Motus* and Proposition XLII. Problem XXIX. The corresponding texts of [23] read as follows:

"And hence is explained the composition of any one direct force AD, out of any two oblique forces AC and CD; and, on the contrary, the resolution of any one direct force AD into two oblique forces AC and CD: which composition and resolution are abundantly confirmed from mechanics" (p. 15).

"The law of centripetal force being given, it is required to find the motion of a body setting out from a given place, with a given velocity, in the direction of a given right line" (p. 133).

As it is immediately clear, Newton reduces his Problem XXXII to Problem XXIX. The fact itself is irrelevant to our concern, since we are interested in Newton's idea of a constraint imposed on a mass-point rather than in particular dynamical problems whichever concerning such constraints. That is why we shall present Newton's arguments in a modern form that will help us to expose the roots of the matter.

Using Newton's notations, let by definition $\boldsymbol{r} = \boldsymbol{SQ}$, $\boldsymbol{n} = \boldsymbol{SC}$, where it is supposed $\boldsymbol{n} \neq \boldsymbol{o}$, so that the unit vector $\boldsymbol{n}^0 = \frac{1}{n}\boldsymbol{n}$ exists. Under these notations, the equation of the plane π (that is to say CPQ is)

$$(1) \qquad \boldsymbol{r}\boldsymbol{n} = \nu$$

with an appropriate ν. On the other hand, the motion of the mass-point Q is governed by the equation

$$(2) \qquad m\ddot{\boldsymbol{r}} = \boldsymbol{F} + \boldsymbol{R},$$

dots denoting, as traditionally in analytical dynamics, derivatives with respect to the time t, m — the mass of Q, $\boldsymbol{F} = \boldsymbol{VS}$ — the "centripetal force", acting on Q, and $\boldsymbol{R}$ — the reaction of the plane π on Q. Besides, let by definition $\bar{\rho} = CQ$, whence $\boldsymbol{r} = \boldsymbol{n} + \bar{\rho}$, and, since π is constant (ergo $\dot{\boldsymbol{n}} = \boldsymbol{o}$),

$$(3) \qquad \ddot{\boldsymbol{r}} = \ddot{\bar{\rho}}.$$

Now (2), (3) imply

$$(4) \qquad m\ddot{\bar{\rho}} = \boldsymbol{F} + \boldsymbol{R}.$$

Moreover, since obviously

$$(5) \qquad \boldsymbol{n}^2 = \nu$$

(the point C lying in the plane π), the relations (1), (5) imply

$$(6) \qquad \bar{\rho}\boldsymbol{n} = 0.$$

In such a manner, the problem of the *constrained* motion of the mass-point Q is reduced to that of the motion of the *free* mass-point Q under the action of the forces $\boldsymbol{F} + \boldsymbol{R}$.

As regards $\boldsymbol{F}$, Newton's decomposition $\boldsymbol{F} = \boldsymbol{VT} + \boldsymbol{TS}$ implies

$$(7) \qquad \boldsymbol{F} = F\bar{\rho}^0 + N\boldsymbol{n}^0,$$

$\bar{\rho}^0$ denoting the unit vector of $\bar{\rho}$, and F and N — the projections of $\boldsymbol{F}$ on $\bar{\rho}^0$ and $\boldsymbol{n}^0$, respectively. In such a manner, the relations (4), (7) imply

$$(8) \qquad m\ddot{\bar{\rho}} = F\bar{\rho}^0 + N\boldsymbol{n}^0 + \boldsymbol{R}.$$

Now, reducing Problem XXXII to Problem XXIX, Newton presupposes

$$(9) \qquad N\boldsymbol{n}^0 + \boldsymbol{R} = 0.$$

Why?

As regards the reaction $\boldsymbol{R}$ we know nothing save that it is acting on the mass-point Q, and this condition is satisfied by writing equation (4). Now Newton assumes on the sly that the plane π is *smooth*, in other words, that

$$(10) \qquad \boldsymbol{R} = R\boldsymbol{n}^0$$

with an appropriate R. Then (9), (10) imply

$$(11) \qquad N + R = 0.$$

As regards the equation of motion of Q as a free mass-point under the action of the central force $F\bar{\rho}^0$, namely

$$(12) \qquad m\ddot{\bar{\rho}} = F\bar{\rho}^0,$$

which is a corollary from (8) and (9), we shall not discuss the problem to what extend Newton could attack it by the aid of the mathematical artillery he had at his disposal in those times. (In the history of mechanics the solution of (12) is connected with the name of Binet, 1786–1856). As Truesdell says, "it is not the function of the historian to guess what Newton might have done of could have done" [8, p. 92]. The cold fact is that under the hypothesis (10) for a smooth plane π the condition (11) is necessary and sufficient for the plane motion of the mass-point Q.

We shall systematize our observations in connection with Newton's Problem XXXII in the form of several scholia.

Scholium 1. Newton's treatment does not make use of any system of reference.

Scholium 2. The reasons of the antecedent inference are rooted already in the formulation of *Lex II*, namely:

"Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur " [7, p. 12], where not a word is said about a system of reference with respect to which *mutationem motus* is calculated.

Scholium 3. This is a most regrettable circumstance since the validity of *Lex II* is not unconditional: *Lex II* holds only for the so-called inertial systems of reference.

Scholium 4. Newton makes no mention of any force acting on Q save $\boldsymbol{VS}$; in particular he does not even allude to the reaction $\boldsymbol{R}$.

Scholium 5. Therefore he does not require explicitly (10) expressing the smoothness of π.

Scholium 6. In the case of a non-smooth constraint π the whole of Newton's construction collapses.

Scholium 7. Newton's supposition "of which $\boldsymbol{ST}$ attracting the body in the direction of a line perpendicular to that plane, does not at all change its motion in that plane" may be physically well-founded, but mathematically it is entirely groundless: a mathematical conclusion about forces and motions is legitimate if, and only if, it is derived from the equations of motion.

Scholium 8. Newton does not at all submit for discussion the question for the possibility of the geometrical constraint imposed on the mass-point Q: he considers this question apriorily settled.

Scholium 9. The foregoing conclusion stands in a causal connection with the *existence problem* in rational mechanics.

Scholium 10. All preceding ascertainments are by no means reprimands: *Impossibilium nulla est obligatio.* Juxtaposed with his epoch, Newton's performances seem superhuman. History in general, however, history of science, in particular, accepts no condolences. Our aim is to ascertain how the notion of mechanical constraint is conceived, born, and bred; and this goal cannot be achieved without the works of classics of mechanics, with all their merits and demerits. *Quod erat explicandum.*

It is pointless to expose the remaining propositions of section X of *Principia.* All of them concern particular motions of mass-points along given curve lines or surfaces; the treatment of any of them is imbued with the spirit of the age. It is true that the ratio of the mechanical content of *Principia* to its mythical fame is negligibly small. As Truesdell emphasizes, "except for certain simple if important special problems, Newton *gives no evidence of being able to set up differential equations of motion for mechanical systems* ... in Newton's *Principia* occur no equations of motion for systems of more than two free mass-points or more than one constrained mass-point" [8, p. 92–93]. At the same time, *quod sciamus*, this is the first mechanical work where constrained motions are considered in an almost modern way — in any case, by the aid of the momentum axiom.

Our goal has by no means been to propose a systematic historical investigation on the mechanical constraint notion. It is a hard nut to crack for a historian of rational mechanics who has made it his set purpose to track out the shady affair of development of dynamics of constrained systems. The hardships are due not only to the fact, properly explained in [17], that almost all motions Leonardo, Galileo, Tartaglia, and their successors had the chance to observe and, although extremely rarely, to measure, have been "impure" — that is to say, attended with the aftereffects of reactions now here of constraints and now there of resisting media; moreover, embarrassments come into being owing to the propensity of authors of mechanical writings to explain in fluent phrases, readily, ardently, and rather life-like to their readers somethings obscure to the authors in question themselves. Aggraviating the situation, the atmosphere becomes electrified by the fact that the mechanical constraint concept is, mathematically speaking, as yet in its historical phase; one cannot — as one can in other fields of mathematics — put one's finger on a certain line of a certain page in a certain book and pronounce the sacramental abracadabra: *this* is a mechanical constraint imposed on a rigid body.

Our occupations with related literary sources have confronted us with a non-incurious phenomenon. If one is apt to have faith in Truesdell's assessments of complicated mechanical situations — as we readily aknowledge we are — then one may find congenial the following opinion of this eminent author apropos of the early story of constrained mechanical systems:

"D'Alembert was the first to give a general rule for obtaining equations of motion of constrained systems. After decomposing the motion into two parts, one being 'natural' and the other due to the presence of the constraints, he asserts that the forces corresponding to the accelerations due to the constraints form a system in static equilibrium. Thus his principle is a development of one of the ideas of James Bernoulli's great paper of 1703; it is still closer to the principle stated even more obscurely by Daniel Bernoulli in his treatment of the hanging cord in 1732–1733 (published 1740). Like the older assertions of Descartes and Leibniz, it is a statement about the system as a whole, not about its parts, and it is insufficient to solve the general problems of dynamics; D'Alembert tacitly invoked other principles as well, but he got results; moreover, he was the first to derive a partial differential equation as the statement of a law of motion, the particular case being that of a heavy hanging cord" [8, p. 113].

Malum nullum est sine aliquo bono. When we for the first time were faced with these acknowledgements of D'Alembert's mechanical performances, we flat and plain could not co-ordinate them with the scientific image his mechanical writings have shaped in our consciousness. Using *vera rerum vocabula*, one cannot set at naught the fact that — more than half a century after *Principia* — D'Alembert declares in everyone's hearing in his *Traité* [9]:

"... j'ai, pour ainsi dire, détourné la vûe de dessus les *causes motrices*, pour n'envisager uniquement que le Mouvement qu'elles produisent; que j'aie entiérement proscrit les forces inhérentes au Corps en Mouvement, êtres obscurs & Méta-physiques, qui ne sont capables que de répandre les ténèbres sur une Science claire par elle-même" (p. XVI).

Moreover, as Truesdell *ibidem* makes out, "while Euler was soon to become the champion of Newton's approach to mechanics, D'Alembert started a new and opposed way of thinking. If the motion is known, he observed, then what we call forces are merely manifestations which may be calculated from it" (p. 113):

"Pourquoi donc aurions-nous recours à ce Principe dont tout le monde fait usage aujourd'hui, que la force accélératrice ou retardatrice est proportionelle à l'Elément de la vitesse; principe appuyé sur cet unique axiôme vague & obscur, que l'effet est proportionnel à sa cause. Nous n'examinerons point si ce Principe est de verité nécessaire; nous avouerons seulement que les prouves qu'on en a données jusqu'ici, ne nous paroissent pas fort convaincantes: nous ne l'adopterons pas non plus, avec quelques Geométres, comme de verité purement contingente, ce qui rumeroit la certitude de la Méchanique, & la réduiroit à n'être plus qu'une Science expérimentale: nous nous contenterons d'observer, que vrai ou doutreux, claire ou obscure, il est inutile à la Méchanique, & que per conséquent il doit en être banni [9, p. XI–XII].

Amicus Socrates, amicus Plato, sed magis amica veritas. In spite of our peerless veneration to Truesdell's erudition, independence of thought, and uprightness of judgements, let us penetrate the roots of matter of the problem of constrained systems of mass-points, in order to acquire an uninfluenced opinion on D'Alembert's dynamical performances. To this end, let us first see how the land lies as regards some indispensable definitions.

From here further let $Oxyz$ denote an inertial orthonormal right-hand orientated Cartesian system of reference with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of the axes Ox, Oy, Oz, respectively, and let all derivatives of vector functions be taken with respect to $Oxyz$.

A mass-point P is said to be *free* if it may, according to the conditions of the particular dynamical problem under consideration, take any position in space and move with any velocity. If P is free, $\boldsymbol{r} = \boldsymbol{OP}$, and

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \tag{13}$$

then $\boldsymbol{r}$ and

$$\boldsymbol{v} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \tag{14}$$

may accept any conceivable values.

A mass-point P is said to be *non-free*, if it is not free. Instead of "non-free" the adjective *constrained* is often used. According to both definitions of *free* and *non-free* mass-points, P is non-free if some restrictions on the admissible values of $\boldsymbol{r}$ or $\boldsymbol{v}$ are imposed by the conditions of the particular dynamical problem under consideration.

There are two, and two only, *modi operandi*, sanctioned by the age-old mechanical tradition, to make a mass-point P constrained, and both are described immediately below.

The first one consists in the hypothesis that P is compelled, by the very conditions of the particular dynamical problem under consideration, to remain on a given surface

$$f(x, y, z, t) = 0, \tag{15}$$

"given" meaning "completely determined" by the said conditions of the problem. At that, it is supposed that the relation

$$\operatorname{grad} f \neq o \tag{16}$$

holds provided by definition

$$\operatorname{grad} f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}. \tag{17}$$

The second one consists in the hypothesis that P is compelled, by the very conditions of the particular dynamical problem under consideration, to remain on a given curve line

$$f_\nu(x, y, z, t) = 0 \qquad (\nu = 1,\, 2), \tag{18}$$

"given" meaning "completely determined" by the said conditions of the problem. At that, it is supposed that the relation

$$\operatorname{grad} f_1 \times \operatorname{grad} f_2 \neq o \tag{19}$$

holds provided by definition

$$\operatorname{grad} f_\nu = \frac{\partial f_\nu}{\partial x}\boldsymbol{i} + \frac{\partial f_\nu}{\partial y}\boldsymbol{j} + \frac{\partial f_\nu}{\partial z}\boldsymbol{k} \qquad (\nu = 1,\, 2). \tag{20}$$

Both the surface (15) and the line (18) are called *geometrical constraints* imposed on the mass-point. If a geometrical constraint is independent of the time t, it is called *scleronomic*; otherwise it is called *rheonomic*.

According to a dynamical axiom, any geometrical constraint, imposed on a mass-point P, generates a force $\boldsymbol{R}$ acting on P. It is called the *reaction* of the geometrical constraint and, along with other forces acting on P, it predestinates the mechanical behaviour of P.

The meaning of the last statement is as follows. Let $\boldsymbol{F}$ be the resultant of all active forces acting on P. This means that $\boldsymbol{F}$ is the sum of all forces acting on P in accordance with the conditions of the particular dynamical problem under consideration. In other words, $\boldsymbol{F}$ is a vector quantity, wholly determined by the said conditions for any position $\boldsymbol{r}$ of P, for any velocity $\boldsymbol{v}$ of P, and for any moment t. This implies that $\boldsymbol{F}$ belongs to the *data* of the dynamical problem concerned, being a completely determined function

$$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v}, t) \tag{21}$$

of $\boldsymbol{r}$, $\boldsymbol{v}$ and t. In such a manner, the term *active forces* is a synonym of the terms *given*, or *known*, or *determined* by the conditions of the dynamical problem. On the contrary, nothing is known about the reactions $\boldsymbol{R}$ of the geometrical constraints imposed on the mass-point P save that they are *acting* on P, the meaning of the last term being specified immediately. Therefore, in contrast to the term *active forces*, the reactions of the constraints are called also *passive forces*.

Acting means that the motion of P is governed by the equation

$$\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F} + \boldsymbol{R}, \tag{22}$$

m denoting the mass of P and $m\boldsymbol{v}$ being by definition the *momentum* of P with respect to $Oxyz$. In such a manner, (22) is a mathematically formalized expression of Newton's *Lex II* already quoted above. Now there is much to be said about the quasi-differential equation (22).

Let us turn back to our constraints (15) and (18). Under certain hypotheses concerning the analytic nature of the left-hand sides of (15) and (18), let us suppose that:

1. If (15), then there exist certain functions

$$x = x(q_1, q_2, t), \quad y = y(q_1, q_2, t), \quad z = z(q_1, q_2, t) \tag{23}$$

of certain arguments q_1, q_2, satisfying (15) identically, i.e.

$$f(x(q_1, q_2, t),\ y(q_1, q_2, t),\ z(q_1, q_2, t),\ t) = 0 \tag{24}$$

for any values of q_1 and q_2 in their definitional domain. Therefore, no restrictions are imposed on the "velocities" of q_1 and q_2, i.e. on their derivatives $\dot{q}_1$ and $\dot{q}_2$ with respect to the time t.

2. If (18), then there exist certain functions

$$x = x(q, t), \quad y = y(q, t), \quad z = z(q, t) \tag{25}$$

of a certain argument q, satisfying (18) identically, i.e.

$$f_\nu(x(q, t),\ y(q, t),\ z(q, t),\ t) = 0 \qquad (\nu = 1,\ 2) \tag{26}$$

for any value of q in its definitional domain. Therefore, no restrictions are imposed on the "velocity" of q, i.e. on its derivative $\dot{q}$ with respect to the time t.

In both cases (15) and (18) there exists a number l ($1 \leqq l \leqq 2$) and l quantities

$$q_\lambda \qquad (\lambda = 1, \ldots, l), \tag{27}$$

mutually independent, together with their velocities

$$\dot{q}_\lambda \qquad (\lambda = 1, \ldots, l), \tag{28}$$

such that any position of the mass-point P consistent with the geometrical constraints imposed on P is uniquely determined by (27). Under these notations the number l is called the *amount of the degrees of freedom* (or simply *degrees of freedom*) of P, and (27) are called the *independent parameters* (or simply *parameters*) of P; sometimes (27) are called the *generalized co-ordinates*, and (28) — the *generalized velocities* of P.

Introducing the acceleration $\boldsymbol{w} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}$ and supposing the mass m of P invariable in the course of the time t, one may write down (22) in the form

$$m\boldsymbol{w} = \boldsymbol{F} + \boldsymbol{R}. \tag{29}$$

The definition of $\boldsymbol{w}$ and (13), (14) imply

$$\boldsymbol{w} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}. \tag{30}$$

Let by definition

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}, \tag{31}$$

$$\boldsymbol{R} = R_x\boldsymbol{i} + R_y\boldsymbol{j} + R_z\boldsymbol{k}. \tag{32}$$

Now (30)–(32) imply that the equation (29) is equivalent with the system of equations

$$(33)\qquad m\ddot{x} = F_x + R_x, \quad m\ddot{y} = F_y + R_y, \quad m\ddot{z} = F_z + R_z.$$

With a view to generality, the efficiency of which will become clear later, let us work with l instead of 2 in the case (23) and of 1 in the case (25). In other words, let us compute the left-hand sides of (33) at an arbitrary l. Then we obviously obtain

$$(34)\qquad \dot{x} = \sum_{\lambda=1}^{l} \frac{\partial x}{\partial q_\lambda} \dot{q}_\lambda + \frac{\partial x}{\partial t},$$

$$(35)\qquad \ddot{x} = \sum_{\lambda=1}^{l} \frac{\partial x}{\partial q_\lambda} \ddot{q}_\lambda + \sum_{\lambda=1}^{l}\sum_{\mu=1}^{l} \frac{\partial^2 x}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda \dot{q}_\mu + 2\sum_{\lambda=1}^{l} \frac{\partial^2 x}{\partial q_\lambda \partial t} \dot{q}_\lambda + \frac{\partial^2 x}{\partial t^2},$$

provided

$$(36)\qquad \frac{\partial^2 x}{\partial q_\lambda \partial q_\mu} = \frac{\partial^2 x}{\partial q_\mu \partial q_\lambda}, \qquad \frac{\partial^2 x}{\partial q_\lambda \partial t} = \frac{\partial^2 x}{\partial t \partial q_\lambda}$$

(λ, $\mu = 1, \ldots, l$), and two similar expressions for $\ddot{y}$ and $\ddot{z}$. Let us lay a special emphasis upon the fact that all coefficients of the quantities $\ddot{q}_\lambda$, $\dot{q}_\lambda \dot{q}_\mu$, $\dot{q}_\lambda$ (λ, $\mu = 1, \ldots, l$), as well as the free member $\frac{\partial^2 x}{\partial t^2}$ in (35) are completely determined functions of the parameters (27) of the mass-point, since by hypothesis the functions (23), as well as (25) of (27) are wholly certain.

On the other hand, as already underlined, the active forces (21) are entirely determined functions of $\boldsymbol{r}$, $\boldsymbol{v}$, and t. Now, with a view to (13), (14), (23), (25), (34) (and similar for $\dot{y}$ and $\dot{z}$), one arrives at the conclusion that (21) may be written in the form

$$(37)\qquad \boldsymbol{F} = \boldsymbol{F}(q_1, \ldots, q_l; \dot{q}_1, \ldots, \dot{q}_l; t),$$

where the right-hand side is a completely determined function of the parameters (27), of the velocities (28), and possibly of the time t. Considering the decomposition (31), one may now quite lawfully state that the same holds for the projections F_x, F_y, F_z of $\boldsymbol{F}$ on the axes Ox, Oy, Oz, respectively.

Summing up, we may now state that the mechanical behaviour of the non-free mass-point P subjected to the geometrical constraint (15) or (18) is governed by the following system of differential-algebraic relations, qualified above as "quasi-differential equations":

$$(38)\qquad \sum_{\lambda=1}^{l} X_\lambda \ddot{q}_\lambda = X + R_x, \quad \sum_{\lambda=1}^{l} Y_\lambda \ddot{q}_\lambda = Y + R_y, \quad \sum_{\lambda=1}^{l} Z_\lambda \ddot{q}_\lambda = Z + R_z,$$

where by definition

$$(39)\qquad X_\lambda = \frac{\partial x}{\partial q_\lambda}, \quad Y_\lambda = \frac{\partial y}{\partial q_\lambda}, \quad Z_\lambda = \frac{\partial z}{\partial q_\lambda} \qquad (\lambda = 1, \ldots, l),$$

$$
(40) \qquad \begin{cases} X = F_x - \sum_{\lambda=1}^{l} \sum_{\mu=1}^{l} \frac{\partial^2 x}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda \dot{q}_\mu - 2 \sum_{\lambda=1}^{l} \frac{\partial^2 x}{\partial q_\lambda \partial t} \dot{q}_\lambda - \frac{\partial^2 x}{\partial t^2}, \\ Y = F_y - \sum_{\lambda=1}^{l} \sum_{\mu=1}^{l} \frac{\partial^2 y}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda \dot{q}_\mu - 2 \sum_{\lambda=1}^{l} \frac{\partial^2 y}{\partial q_\lambda \partial t} \dot{q}_\lambda - \frac{\partial^2 y}{\partial t^2}, \\ Z = F_z - \sum_{\lambda=1}^{l} \sum_{\mu=1}^{l} \frac{\partial^2 z}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda \dot{q}_\mu - 2 \sum_{\lambda=1}^{l} \frac{\partial^2 z}{\partial q_\lambda \partial t} \dot{q}_\lambda - \frac{\partial^2 z}{\partial t^2}. \end{cases}
$$

As it is immediately seen from (39), (40) and (37), (31), the equations (38) of motion of P involve unknown quantities of two entirely different kinds:

1. *Infinitesimal* unknowns, that is to say the parameters (27) of P together with their first and second derivatives with respect to the time t (the latters being present linearly).

2. *Finitesimal* unknowns, that is to say the projections of the reaction $\boldsymbol{R}$ of the constraint acting on P (the latters also being present linearly).

This circumstance is the reason calling the equations (38) *differential-algebraic* and *quasi-differential.*

The equations (38) represent the most adequate formal-mathematical expression of the dynamical problem under consideration. Therefore they deserve a special attention.

As any mathematical problem, the system of equations (38) engenders two challenges:

1. Existence?

2. Uniqueness?

It stands to reason, it would be an extravagant luxury to answer the second question before the first one is answered in the affirmative: it could be compared to taking down finger-prints of a ghost. And yet, the course of the solutions of mathematical problems is traditionally topsyturvied. Habitually first and foremost, disregarding the existence-problem, a provisional solution is sought by the problem solver, and only afterwards it is proved, commonly by means of an immediate check-up, that this potential solution is an actual one. At that, as a rule, the existence-problem is mathematically incomparably harder to solve than the uniqueness-problem.

Horribile visu, horribile dictu, horribile auditi — horresco referens: in rational mechanics, in general, and in rigid dynamics, in particular, the existence problem does not exist at all. Or, more correctly, it exists like the ozone-hole: everybody knows and nobody cares. Evidence? — Any treatise on analytical mechanics you like: the choice is yours. To express this statement in concrete form by indicating one particular from among countless amount of dynamical textbooks, books of problems, treatises, monographs, and articles would mean to do unjustice to the author of the selected work, converting him into a scapegoat for a widespread sin. And yet, under this reservation, we shall quote a practical example — solely in order not to be upbraided with groundless idle talk. As regards the pitiable absence of mind of ancient and modern mechanicians in connection with the existence-problem, it is in a full agreement with Seneca's observation *Quae fuerant vitia, mores sunt.*

The scapegoat in question is the Treatise [16] on rigid dynamics, published comparatively recently. Turning over its pages at random, we arrive at the problems of motion of a rod in a rotating plane (p. 119), rolling penny (p. 120–122), sleeping top (p. 152–156), sphere on turntable (p. 207–209), sphere on a rotating inclined plane (p. 209–211), sphere rolling on a fixed surface (p. 211–213), and so on, and so forth, etcetera. (As regards examples from other literary sources, *nomen illis legio.*) Are in the solutions of all those problems in [16] answers of the existence-question? Not one jot! Not a whit! By no means! There is even not the least hint for such a thing. Incredible? — Incredible. Fact? — Fact. If somebody dares contest this statement, then there is a sole possible answer: *Hic Rhodus, hic salta.* That is to say, hic Pars' Treatise, *hic* points a finger at an existence proof.

Saeculi vitia, non hominis. The cause for this state of affairs in analytical dynamics is rooted in its dual nationality: down to the present day it is simultaneously a citizen both of United Kingdom's Mathematics and of United States' Physics. At least such is the mental disposition of most who work in this domain, in spite of the danger to fall between two stools. Indeed, mechanics is occupied studing motions, and motion is something that exists — isn't it? Then why worrying about such a nonsense as existence-problem?

Maybe. Maybe not. Do you remember the nursery rhymes:

> "For the want of a nail the shoe was lost,
> For the want of a shoe the horse was lost,
> For the want of a horse the rider was lost,
> For the want of a rider the battle was lost,
> For the want of a battle the kingdom was lost —
> And all for the want of a horseshoe nail."

Let us now make an *en gros* assessment of the situation. Our dynamical problem of the mechanical behaviour of the mass-point P, submitted to the geometrical constraint (15) or (18) and to the action of active forces with resultant (21), consists, first, in determining (if such exist) the parameters (27) of P as functions

(41) $$q_\lambda = q_\lambda(t) \qquad (\lambda = 1, \ldots, l)$$

of the time t, P starting from a fixed though wholly arbitrary initial position

(42) $$q_{\lambda_0} = q_\lambda(0) \qquad (\lambda = 1, \ldots, l)$$

with a fixed though completely arbitrary initial velocity

(43) $$\dot{q}_{\lambda_0} = \dot{q}_\lambda(0) \qquad (\lambda = 1, \ldots, l);$$

and, second, in determining (if such exists) the reaction $\boldsymbol{R}$ of the corresponding geometrical constraint, that is to say, the projections

(44) $$P_x,\ P_y,\ P_z$$

of $\boldsymbol{R}$ on the axes Ox, Oy, Oz of $Oxyz$, respectively, according to (32). Consequently, the unknown quantities of the mathematical problem under consideration are 5 in number in the case of constraint (15) (since then $l = 2$) and 4 in number in the case of constraint (18) (since then $l = 1$). Since the number of the equations (38) with (39), (40) we have at our disposal for the determination of those $l + 3 > 3$ unknown quantities (41), (42) is exactly 3, our dynamical problem is, in the general case at least, mathematically indeterminate.

This conclusion is as two-faced as Ianus. Its favourable face is that one may hope that the existence-problem might be answered in the affirmative; its unfavourable face is that the answer might be as arbitrary as to seem meaningless. Both expectations are vindicated by reality.

There is one, and one only, way to make a constrained mass-point dynamical problem completely determined mathematically, and it consists in the hypothesis that the corresponding geometrical constraint is *smooth*. Physically the concept of *smoothness* is reduced to the idea that the corresponding surface or the corresponding curve line is *polished* like a mirror. The same physical idea suggests that the constraint generates *no friction*. Mathematically *smoothness* means that the reaction of the constraint is *normal* to the latter, i.e.

(45) $$\boldsymbol{R} \times \operatorname{grad} f = 0$$

in the case (15) and

(46) $$\boldsymbol{R}.\operatorname{grad} f_1 \times \operatorname{grad} f_2 = 0$$

in the case (18).

Indeed, (45) and (16) imply that there exists a scalar μ with

(47) $$\boldsymbol{R} = \mu \operatorname{grad} f.$$

Now (47), (17), and (32) imply that the equations (38) take the form

(48) $$\begin{cases} \sum_{\lambda=1}^{l} X_\lambda \ddot{q}_\lambda = X + \mu \dfrac{\partial f}{\partial x}, \\ \sum_{\lambda=1}^{l} Y_\lambda \ddot{q}_\lambda = Y + \mu \dfrac{\partial f}{\partial y}, \\ \sum_{\lambda=1}^{l} Z_\lambda \ddot{q}_\lambda = Z + \mu \dfrac{\partial f}{\partial z}. \end{cases}$$

Similarly, (46) and (19) imply that there exist scalars μ_1 and μ_2 with

(49) $$\boldsymbol{R} = \mu_1 \operatorname{grad} f_1 + \mu_2 \operatorname{grad} f_2.$$

Now (49), (20), and (32) imply that the equations (38) take the form

(50) $$\begin{cases} \sum_{\lambda=1}^{l} X_\lambda \ddot{q}_\lambda = X + \mu_1 \dfrac{\partial f_1}{\partial x} + \mu_2 \dfrac{\partial f_2}{\partial x}, \\ \sum_{\lambda=1}^{l} Y_\lambda \ddot{q}_\lambda = Y + \mu_1 \dfrac{\partial f_1}{\partial y} + \mu_2 \dfrac{\partial f_2}{\partial y}, \\ \sum_{\lambda=1}^{l} Z_\lambda \ddot{q}_\lambda = Z + \mu_1 \dfrac{\partial f_1}{\partial z} + \mu_2 \dfrac{\partial f_2}{\partial z}. \end{cases}$$

In both cases (48) and (50) the number of the unknown quantities equals the number of the equations available for their determination, namely 3: in the case (15) the unknowns are q_1, q_2 and μ, and in the case (18) they are q, μ_1 and μ_2. *Q. E. D.*

Naturally, in both cases a horseshoe is still wanting: the solution of the existence-problem.

The situation around a single mass-point being, in such a manner, settled on principle, let us now turn back to D'Alembert's "Principe général pour trouver le Mouvement de plusieurs Corps qui agissent les uns sur les autres, d'une manière quelconque". Let us carry ourselves mentally in the age when he wrote his *Traité*. Truesdell might be helpful again:

"... a large part of the literature of mechanics for sixty years following the *Principia* searches various principles with a view to finding the equations of motion for the systems Newton had studied and for other systems nowadays thought of as governed by the 'Newtonian' equations" [8, p. 92–93].

Now all mathematicians of many decades after *Principia* passionately strove for disclosing the mysteries mystifying the motions of the most enigmatic of all mechanical systems called *rigid bodies*. Most of them, D'Alembert in the first place, chose the most natural, most obvious, and most wrong way: the idea that a rigid body is an aggregate of mass-points, constrained in such a manner that their mutual distances remain invariable. The rise and fall of this idea is reflected in the *Traité de Dynamique* and *Méchanique Analitique*. But let us not go so far. Let us first formulate the basic notions of a system of constrained mass-points.

Let Σ be such a system, i.e. a set of n mass-points P_ν with masses m_ν and radius-vectors $\boldsymbol{r}_\nu = \boldsymbol{OP}_\nu$ $(\nu = 1, \ldots, n)$. Some of the points of Σ may be free, some may be constrained to remain on certain surfaces, and some may be compelled to slide on certain curve lines. If one applies to any of these points the arguments used in the case of a single mass-point adduced above, one sees at once that for any of them there exists a number, at least 1 and at most 3, of mutually independent parameters determining its admissible by the corresponding geometrical constraints positions in space; let (27) be those parameters for all the points of Σ arranged in a definite order, say according to the increasing number ν of the point P_ν.

Besides, let $\boldsymbol{F}_\nu$ and $\boldsymbol{R}_\nu$ be the resultants respectively of the active forces and the reaction of the constraint imposed on the ν-th point of Σ, and let $\boldsymbol{w}_\nu = \ddot{\boldsymbol{r}}_\nu$ $(\nu = 1, \ldots, n)$ be its acceleration with respect to $Oxyz$. Then, obviously, according to Newton's *Lex II*, the motion of P_ν will be governed by the equation

(51) $$m_\nu \boldsymbol{w}_\nu = \boldsymbol{F}_\nu + \boldsymbol{R}_\nu \qquad (\nu = 1, \ldots, n).$$

The dynamical problem we are faced with in such a manner, concerning the mechanical behaviour of Σ, consists in solving the system of equations (51) under entirely arbitrary, though fixed, initial conditions (42), (43), i.e. in discovering such functions (41) and such linear unknown quantities

(52) $$R_{\nu x},\ R_{\nu y},\ R_{\nu z} \qquad (\nu = 1, \ldots, n)$$

provided

(53) $$\boldsymbol{R}_\nu = R_{\nu x}\boldsymbol{i} + R_{\nu y}\boldsymbol{j} + R_{\nu z}\boldsymbol{k} \qquad (\nu = 1, \ldots, n),$$

that satisfy (51) identically, taking (42), (43) into account.

Does D'Alembert solve this problem in [9]? Is he "the first to give a general rule for obtaining equations of motion of constrained systems" as Truesdell generously states? Do we find in [9] the system (51) or at least some semblance, some similarity, some likeness of it?

Certainly not. Nothing of the kind. Never a whit. *Traité de Dynamique* is as far from (51) as Stahl from Lavoisier.

Let us lay special emphasis on the fact of extraordinary importance on principle that to describe mathematically the mechanical behaviour of the system Σ means to determine the dynamical demeanour of any mass-point entering into the composition of Σ. This means to know the functions

$$\text{(54)} \qquad \boldsymbol{r}_\nu = \boldsymbol{r}_\nu(t) \qquad (\nu = 1, \ldots, n)$$

if the initial conditions

$$\text{(55)} \qquad \boldsymbol{r}_{\nu 0} = \boldsymbol{r}_\nu(0), \qquad \boldsymbol{v}_{\nu 0} = \boldsymbol{v}_\nu(0) \qquad (\nu = 1, \ldots, n)$$

are prescribed provided $\boldsymbol{v}_\nu = \dot{\boldsymbol{r}}_\nu$, as well as $\boldsymbol{R}_\nu$ for any $\nu = 1, \ldots, n$. Now in D'Alembert's *Traité* there is not the slightest trace of a solution of this problem even in its most elementary case $n = 2$.

Extending our analysis in connection with the system (51), let us note that, in contrast to the case of a single mass-point P, when the active force $\boldsymbol{F}$ acting on P may, according to (21), depend only on the position $\boldsymbol{r}$ and the velocity $\boldsymbol{v}$ of P itself, in the case of a system Σ of mass-points the active force $\boldsymbol{F}_\nu$ acting on P_ν may depend on the positions $\boldsymbol{r}_\mu$ and the velocities $\boldsymbol{v}_\mu$ of all the points P_μ ($\mu = 1, \ldots, n$) of Σ. In other words, in the general case it is supposed that the active forces $\boldsymbol{F}_\nu$ are completely determined functions

$$\text{(56)} \qquad \boldsymbol{F}_\nu = \boldsymbol{F}_\nu(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n; \boldsymbol{v}_1, \ldots \boldsymbol{v}_n; t)$$

of all $\boldsymbol{r}_\nu$, $\boldsymbol{v}_\nu$ ($\nu = 1, \ldots, n$) and possibly of the time t. In such a manner, although the solution of the system (51) requires the determination of (54) provided (55), and of (52) provided (53) for any particular $\nu = 1, \ldots, n$, the integration of the system (51) of quasi-differential equations cannot be accomplished separately for any particular ν, since (51) represents a system of interdependent relations.

We proceed now to one of the greatest mistifications in all the history of rational mechanics. *Chapitre Premier. Exposition du Principe of Second Partie. Principe général pour trouver le Mouvement de plusieurs Corps qui agissent les uns sur les autres d'une maniére quelconque, avec plusieurs applications de ce Principe* of [9] begins with the following declaration:

"Les Corps n'agissent les uns sur les autres que de troi maniéres différentes qui nous soient connus: ou par impulsion immédiate, comme dans le choc ordinaire, ou par le moyen de quelque Corps interposé entr'eux, & auquel ils sont attachés, ou enfin, par une vertu d'attraction réciproque, comme sont dans le systême Newtonien le Soleil & les Planetes. Les effets de cette derniére espece d'action ayant été suffisamment examinés, je me bornerai à traiter ici du Mouvement des Corps qui se choquent d'une maniére quelconque, ou de ceux qui se tirent par des fils ou des verges inflexibles. Je m'arrêterai d'autant plus volontiers sur ce sujet, que les plus grands Géométres ne nous ont donné jusqu'à présent qu'un très petit nombre de Problêmes de ce genre, & que j'espere, par la Méthode générale qui je vais donner, mettre tous ceux qui sont au fait du calcul & des Principes de la Mécanique, en état de résoudre les plus difficiles Problêmes de cette espece" (p. 49–50).

A *Définition* follows:

"J'appelerai dans la suite *Mouvement* d'un Corps, la vitesse de ce même Corps considerée en ayant égard à sa direction, & par *quantité de Mouvement*, j'entendrai à l'ordinaire le produit de la masse par la vitesse" (p. 50).

The formulation of *Problême general* reads:

"*Soi donné un systême de Corps disposés les uns par rapport aux autres d'une maniére quelconque; et supposons qu'on imprime à chacun de ces Corps un Mouvement particulier, qu'il ne puisse suivre à cause de l'action des autres Corps, trouver le Mouvement que chaque Corps doit prendre*" (*ibid.*).

Before proceeding to D'Alembert's "Solution" let us fix our eyes on the formulation of *Problême general.* First of all, it sticks out a mile that D'Alembert's *Corps* are, as it is, purely and simply *mass-points* and *no bodies* at all: the formulation of *Problême general* attaches *Mouvement* to *chaque Corps*, that is to say *vitesse* according to D'Alembert's "Définition", and velocity is a mechanical entity that becomes wholly meaningless when attached to rigid bodies — it is meaningful only when localized to points. The second circumstance that cannot slip anybody's attention is that D'Alembert's *Problême general* is, when all is said and done, a purely kinematical proposition with not an atom of dynamics. Now one is at a loss how could D'Alembert, on the basis of a purely kinematical *Principe general*, redeem his promise made with such an aplomb in the title of the *Seconde Partie* of the work, namely to "trouver le Mouvemens de plusieurs Corps qui agissent les uns sur les autres d'une maniére quelconque"? Be that as it may, let us proceed, after these remarks, to D'Alembert's *Solution*:

"Soient A, B, C, & c. les Corps qui composent le systême, & supposons qu'on leur ait imprimé les Mouvemens *a*, *b*, *c*, & c. qu'ils soient forces, a cause de leurs action mutuelle, de changer dans les Mouvemens a, b, c, & c. Il est clair qu'un peut regarder le Mouvement *a* imprimé au Corps A comme composé du Mouvement a qu'il a pris, & d'un autre Mouvement α; qu'on peut de même regarder les Mouvemens *b*, *c*, & c. comme composés des Mouvemens b, β; c, $\varkappa$; & c. d'ou il s'ensuit que le Mouvement des Corps A, B, C, & c. entr'eux auroit été le même, si au lieu de leur donner les impulsions *a*, *b*, *c*, on leur êut donné à la fois les doubles impulsions a, α; b, β; c, $\varkappa$, etc. Or par la supposition, les Corps A, B, C, & c. ont pris d'eux-mêmes les Mouvemens a, b, c; etc. Donc les Mouvemens α, β, $\varkappa$ & c. doivent être tels qu'ils ne dérangent rien dans les Mouvemens a, b, c, etc. c'est-à-dire, que si les Corps n'avoient reçu que les Mouvemens α, β, $\varkappa$ & c. ces Mouvemens auroient dû se détruire mutuellement, & le systême demeurer en repos.

Delà résulte le Principe suivant, pour trouver le Mouvement de plusieurs Corps qui agissent les unes sur les autres. *Décomposés les Mouvemens a*, *b*, *c* & c. *imprimés a chaque Corps*, *chacun en deux autres* a, α; b, β; c, $\varkappa$; & c. *qui soient tels, que si l'on n'eût imprimé aux Corps que les Mouvemens* a, b, c, & c. *ils eussent pû conserver ces Mouvemens sans se nuire réciproquement*; *et que si on ne leur eût* *imprimé que les Mouvemens* α, β, $\varkappa$, & c. *le système fut demeuré en repos*; il est clair que a, b, c seront les Mouvemens que ces Corps prendront en vertu de leur action. Ce *Q. F. Trouver*" (*ibid.*, p. 50–51).

This "Solution" of D'Alembert's provides the occasion for quite a lot of commentaries, all of them curious, instructive, and beneficial. We shall, however, spare them for the time being, postponing a detailed discussion of the preceding text for immediate future. For the time being we shall restrict our attention on the sequels this *Principe général* of D'Alembert has had in the subsequent development of rigid dynamics.

Disregarding the *Eigenwerte* D'Alembert himself placed on his principle in the *Préface* of [9] and in its application to various problems of mechanics in this very work, let us first note that some decades later the same principle has been rediscovered and brought back to life by Lagrange, D'Alembert's true spiritual son. Meanwhile, let us read the commentary of the Russian translator of [9], made immediately after the principle is announced in the book:

"В настоящем n° Даламбером формулируется то правило, которое ныне называется „принципом Даламбера". Как видно, этот „принцип" выглядит у его автора совсем не так, как он излагается ныне в учебниках. Форма, близкая к современной, придана была принципу Даламбера Лагранжем в его „Аналитической механике".

Даламбер дал изложение своего „принципа" и в „Энциклопедии", в статье „Dynamique" (Динамика). Приведем здесь это изложение буквально ..." [24, c. 333–334].

The author of this quite equitable finding takes into consideration several somethings, the first of which is the singing praise to the skies of D'Alembert's *Principe général* in *Section Première. Sur les différents principes de la dynamique* of *Second Partie, La Dynamique* of [10] by Lagrange, who has been 7 years old when D'Alembert published his *Principe* and, as regards the penetrating into the roots of matter, did not fledge much since.

For the time being at least we shall wind up our exposition by a mathematical *coup de grâce, in arenam cum aequalibus descendi.*

Let us rewrite (51) in the form

$$m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu = \boldsymbol{o} \qquad (\nu = 1, \ldots, n), \tag{57}$$

and let regard the formal expressions

$$A_\lambda = \sum_{\nu=1}^{n} (m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu) \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \qquad (\lambda = 1, \ldots, l) \tag{58}$$

and

$$\delta A = \sum_{\lambda=1}^{l} A_\lambda \delta q_\lambda, \tag{59}$$

δq_λ denoting arbitrary infinitesimal variations of q_λ ($\lambda = 1, \ldots, l$), respectively, not necessarily co-ordinated with the dynamical equations (51), the $\boldsymbol{r}_\nu$ ($\nu = 1, \ldots, n$)

in (58) being subordinated to the geometrical constraints imposed on the system Σ of mass-points. The following computations are traditional. The identities

$$\boldsymbol{v}_\nu = \sum_{\lambda=1}^{l} \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \dot{q}_\lambda + \frac{\partial \boldsymbol{r}_\nu}{\partial t} \qquad (\nu = 1, \ldots, n) \tag{60}$$

imply

$$\frac{\partial \boldsymbol{v}_\nu}{\partial \dot{q}_\lambda} = \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \qquad (\lambda = 1, \ldots, l;\ \nu = 1, \ldots, n) \tag{61}$$

and

$$\frac{\partial \boldsymbol{v}_\nu}{\partial q_\mu} = \sum_{\lambda=1}^{l} \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\mu \partial q_\lambda} \dot{q}_\lambda + \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\mu \partial t} \tag{62}$$

$(\mu = 1, \ldots, l;\ \nu = 1, \ldots, n)$. Now (62) and

$$\frac{d}{dt} \frac{\partial \boldsymbol{r}_\nu}{\partial q_\mu} = \sum_{\lambda=1}^{l} \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda + \frac{\partial^2 \boldsymbol{r}_\nu}{\partial t \partial q_\mu} \tag{63}$$

$(\mu = 1, \ldots, l;\ \nu = 1, \ldots, n)$ imply

$$\frac{d}{dt} \frac{\partial \boldsymbol{r}_\nu}{\partial q_\mu} = \frac{\partial \boldsymbol{v}_\nu}{\partial q_\mu} \qquad (\mu = 1, \ldots, l;\ \nu = 1, \ldots, n) \tag{64}$$

provided

$$\frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\lambda \partial q_\mu} = \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\mu \partial q_\lambda}, \quad \frac{\partial^2 \boldsymbol{r}_\nu}{\partial t \partial q_\mu} = \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\mu \partial t} \tag{65}$$

$(\lambda,\ \mu = 1, \ldots, l;\ \nu = 1, \ldots, n)$. If by definition

$$Q_\lambda^{(a)} = \sum_{\nu=1}^{n} \boldsymbol{F}_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda}, \quad Q_\lambda^{(p)} = \sum_{\nu=1}^{n} \boldsymbol{R}_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \tag{66}$$

$(\lambda = 1, \ldots, l)$ and

$$T = \frac{1}{2} \sum_{\nu=1}^{n} m_\nu v_\nu^2, \tag{67}$$

then (58), (61), (64), (66), (67) imply

$$A_\lambda = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda^{(a)} - Q_\lambda^{(p)} \qquad (\lambda = 1, \ldots, l). \tag{68}$$

If the constraints imposed on the system Σ of mass-points are *smooth*, then the second definition (66) implies

$$Q_\lambda^{(p)} = 0 \qquad (\lambda = 1, \ldots, l) \tag{69}$$

and (68), (69), (58) imply

$$\sum_{\nu=1}^{n} (m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu) \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda^{(a)} \tag{70}$$

($\lambda = 1, \ldots, l$), whence it is immediately seen that Newton's *Lex II* (57) applied on Σ automatically leads to Lagrange's dynamical equations

$$(71) \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda^{(a)} = 0 \qquad (\lambda = 1, \ldots, l).$$

As a matter of fact, the *fundamental identities of Lagrangean formalism* (70) at once display that the left-hand sides of Lagrange's dynamical equations (71) are, purely and simply, linear combinations of the projections of the left-hand sides of Newton's *Lex II* (57) on axes, defined by the directions

$$(72) \qquad \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \qquad (\lambda = 1, \ldots, l;\ \nu = 1, \ldots, n)$$

co-ordinated with the geometrical constraints imposed on Σ.

If by definition

$$(73) \qquad \delta \boldsymbol{r}_\nu = \sum_{\lambda=1}^{l} \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \delta q_\lambda \qquad (\nu = 1, \ldots, n),$$

then (58), (59) imply

$$(74) \qquad \delta A = \sum_{\nu=1}^{n} (m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu),$$

and the relation $\delta A = 0$, i.e.

$$(75) \qquad \sum_{\nu=1}^{n} (m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu) \delta \boldsymbol{r}_\nu = 0,$$

is usally accepted in the traditional dynamical literature as a modern mathematical expression of D'Alembert's original *Principe général.* For the time being at least we shall refrain from commentaries as to the degree of adequacy of such an interpretation, in accordance with *Davus sum, non Oedipus* of Terentius.

As far as our experience goes, the composing of the true history of the theory of mechanical constraints is as yet postponed *ad Calendas Graecas.* As already emphasized and as maybe it becomes transparent from our exposition, this is a back-breaking task. Neither shall we dare penetrate imprudently the vast white fields of this *terra incognita.* One thing is certain: before one sets one's foot in its Arcadia, one must cross the rocky mountains of Lagrangean formalism.

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Received 8.04.1993

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 86, 1992

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 86, 1992

NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS VI. INDUCTIO PER ENUMERATIONEM SIMPLICEM

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Opium facit dormire, quare est in eo virtus dormitiva
Molière: *Le malade imaginaire*

Георги Чобанов, Иван Чобанов. ДИНАМИЧЕСКИЕ АКСИОМЫ НЬЮТОНА И ЭЙЛЕРА. VI. INDUCTIO PER ENUMERATIONEM SIMPLICEM

Это есть шестая часть серии статьей, посвещенные динамических аксиом Ньютона и Эйлера; она естественное продолжение и развитие последней из них [25], в которой дан предисторический эскиз возникновения и первоначального формирования идеи о механических связях, налагаемых твердым телам. Настоящая работа содержит подробный анализ сегодняшнего положения дел в этой области; специальное внимание уделено динамическому трактату [15] Аппеля, принадлежащему в настоящее время механической классики, а также его статьи [27], где исследована природа механических связей. Главный вывод авторов касательно математического описания связы может быть выражен формулой через *inductio per enumerationem simplicem* к *definitio per enumerationem simplicem.*

Georgi Chobanov, Ivan Chobanov. NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS. VI. INDUCTIO PER ENUMERATIONEM SIMPLICEM

This is the sixth part of a series of articles dedicated to the Newtonian and Eulerian dynamical axioms; it is the natural continuation and development of the last of them [25], where a prehistorical sketch is given of the origination and first shaping of the idea of mechanical constraints imposed on rigid bodies. The present paper contains a detailed analysis of the state of affairs in the domain nowadays, a special attention being paid to Appell's dynamical treatise [15], now pertaining to the mechanical classic, as well as to his article [27] where the nature of mechanical constraints is examined. The main inference of the authors concerning the mathe-

matical description of the constraint concept may be expressed by the slogan via *inductio per enumerationem simplicem* towards *definitio per enumerationem simplicem*.

Being the sixth part of a series of studies dedicated to various aspects of *Newtonian and Eulerian dynamical axioms*, the present paper is the natural continuation of the last of them [25], published in this very volume of the Annual; that is why the quoted literature in the present article has a unified numeration with that of [25].

As it is well-known, the Eulerian dynamical equations [17; (114), (115)] representing a mathematically formalized version of Eulerian dynamical axioms (the laws, or principles, or postulates, or hypotheses, etc. of momentum and of moment of momentum of a rigid body) become completely meaningless unless the nature of the mechanical constraints imposed on the body is specified. Since there is still a discrepancy between the physical ideas reflected in the naive conception of a mechanical constraint imposed on a rigid body, say, and the mathematical devices by means of which these physical ideas are formalized; since the said mathematical apparatus is undergoing a process of perfection as yet; since, at last, any scientific concept is perceived best in its historical development — in view of all these considerations a brief and unpretentious information has been adduced in [25] concerning the prehistory of the idea of such constraints.

As it has been underlined in this latter part, any attempt at composing a genuine history of the kinetical (statical as well as dynamical) concept of mechanical constraints imposed on a mechanical system is, for the time being at least, bound up with insurmountable difficulties. Due to that, there is not the slightest trace of such an attempt in the present paper. If, here and there, disperced at sixes and sevens, some historical records may be found here, their presence is due only to a trend towards a better substanciation.

In the spirit of these reservations, a long–drawn–out–interval of time in the history of rational mechanics will be left out: as a matter of fact, the period between D'Alembert [9] and Appell [15]. There are two almost exigent reasons to do so. First and foremost, one hardly could in sober earnest sustain that there have happened, in this space of time, some important developments that have contributed in a degree, worthy of mention, to the mathematical clarification and finalization of the mechanical constraint concept. As we shall soon see, Appell's *Traité* [15] is exactly as much in captivity of Lagrange's mechanical ideology [10] as Lagrange himself had "fallen under the personal influence of D'Alembert" [8, p. 248]. Could one explain otherwise the presence of such statical apparitions in [15]:

"*Principes généraux relatifs aux ensembles de points matériels.* Si l'ensemble est formé de points libres et indépendants les uns des autres, on peut répéter pour chacun d'eux ce que nous avons dit sur le point matériel complètement libre. Pour que l'équilibre existe, il faut et il suffit que la resultante des forces qui agissent sur chaque point soit nulle. Cette condition n'est plus nécessaire si l'ensemble est soumis a des liaisons définies géométriquement ou exprimées par des équations entre les coordonées des points. C'est qui arrive, par exemple, si l'un des points est assujetti à rester sur une surface, ou encore, si la distance de deux points de l'ensemble est constante. Relativement à ses ensembles, nous poserons les deux principes suivants:

1°. *Si un ensemble est en équilibre sous l'action d'un système de forces, l'équilibre sera conservé si, sans changer les forces, on introduit de nouvelles liaisons.*

2°. *Si un ensemble est en équilibre sous l'action d'un système de forces* (*A*), *l'équilibre sera conservé, si l'on ajoute ou supprime à* (*A*) *un système* (*B*) *qui maintient l'ensemble en équilibre*" [I, p. 122–123]?

Or the following logical *bijou*:

"*Principe de solidification.* Nous avons étudié, jusqu'à présent, les conditions de l'équilibre d'un corps solide, c'est-à-dire d'un système de forme invariable. Imaginons un système matériel dont les différentes parties sont liées les unes aux autres d'une certaine façon, mais non d'une façon invariable: le système est alors déformable. Nous pourrons, pour tous ces systèmes, énoncer la proposition suivante, qu'on appelle quelquefois *principe de solidification* et qui est un cas particulier du premier principe énoncé [above].

Quand un systéme deformable est en équilibre, les forces extérieures (c'est–à–dire les forces autres que les réactions mutuelles des différentes partis) *qui lui sont appliquées satisfont aux conditions d'équilibre des forces appliquées a un corps solide.* En effet, le système, étant en équilibre, y restera évidemment si l'on relie les points matériels les uns aux autres d'une manière invariable, c'est–à–dire si l'on solidifie le système. Les forces extérieures doivent se fair équilibre sur le corps solide ainsi constitué; elles satisfont donc aux six équations générales de l'équilibre. Ces conditions, nécéssaires, ne sont pas, en general, suffisantes" [*ibid.*, p. 165]?

The hitherto quoted excerpts from the *Traité* [15] reflect the statical philosophy of its author; as regards his dynamical *Weltanschauung*, it becomes transparent from the following place, *exempli gratia*:

"On regarde un système matériel quelconque, formé de corps solides, liquides, gazeux, comme composé d'un tres grande nombre de points matériels assujettis à certain liaisons. Une corps solide, par example, est un ensemble de points assujettis à rester à des distances invariables les uns des autres.

Les théorèmes généraux s'obtiennent en supposant qu'on ait écrit les équations du mouvement de ces différents points matériels et qu'on en fasse des combinaisons" [II, p. 70].

All these three passages from [15] have a common characteristic: all of them concern *liaisons* imposed on the mechanical system in question. The reader may expect to come to know, what does by the way this term mean. If so, then those are give–up–all–hope–expectations: the term *liaison* is explained in [15] mathematically as irreproachably as the term *mésalliance*. That is another topic though. For the time being the important point is that, as regards the logical levels of exposition concerning the mechanical constraint concept at least, the *niveau différence* between [15] and [10] is ignorably small.

This first. Second, the same reversibility exists between [15] and a vast horde of modern mechanical literary youngsters — textbooks, treatises, as well as books of problems, monographs, or articles. In order not to be baseless, let us mention one and only of them, namely [16]. So much for that now, however: later we shall harp it on the same string. There is a point, however, that must be settled here and now.

All that has been quoted from [15] has been written about 1896. It is true that *littera scripta manet.* It is also true that *littera occidit, spiritus autem vivificat.* At last, it is not the lesser true that the genuine credo of a professional historian of science must be *verbatim et litteratim.* Now in commenting ancient written sources there is always a danger of prochronistic deviations — that is to say, to interpret terms wrongly, ascribing modern meanings to words they did not possess in times long over and done with. Is there such a danger in our case?

Well yes, as well as no.

This — somewhat enigmatic to be sure — answer stands *vis-à-vis* a two-faced problem: the ethos, and the letter.

Let us not set at naught *Anno Domini* 1896, when Appell's *Traité* was first published. It is a date Cantor's *Mannigfaltigkeitslehre* was violently controverted as yet; the integral concept was still in a process of fermentation, in Lebesgue's wood above all things; and Hilbert's *Grundlagen der Geometrie* — that were to topsyturvy in a fortnight the mathematical way of thinking all the world over — were still drowsing *in cunabula*; in a word, the logical spirit of Twentieth Century's Mathematics was as yet cooped up in the tight frames of Nineteenth's as the jinnee in the bottle. All this as regards the yes-answer. It is as infantile to lay claim to *obligatio impossibilium* as to cry for the moon.

As regards the no-answer, we must take into account several considerations, the first of which is that — as regards the *liaisons*-concept at least — no such changes have set in rigid dynamics since 1896 as to be seen with a naked eye. In the beginning of this century a clever man, Voss, has written with deep regret:

"Die Erscheinung, dass die Resultate mathematischer Lehrgebäude von grundlegender Wichtigkeit oft eine lange Zeit hindurch ihrer strengen wissenschaftlichen Begründung vorausgeeilt sind, hat sich in weit höherem Grade bei der *Mechanik*, wie bei der Arithmetik oder der Infinitesimalrechnung wiederholt. Man kann den Standpunkt, welchen die systematische Entwicklung der Mechanik in ihrer gegenwärtigen Gestalt einnimmt, etwa mit dem der Infinitesimalrechnung vor *Cauchy* vergleichen, auf den sich fast wörtlich die Bemerkungen von *Hertz* in seiner Einleitung zur Mechanik anwenden lassen ... siehe die Bemerkungen von *Hertz*, Mechanik, p. 8, über das bei der Exposition der Grundlagen der Mechanik häufig hervortretende Bestreben, über die Schwierigkeiten und Verlegenheiten in denselben möglichst bald hinaus und zu konkreten Beispielen zu kommen" [26, Erster Teilband, S. 8–9].

Today, December 17, 1992, anybody can calmly countersign this standpoint of Voss and sleep the sleep of the just, unmolested that his bill might be protested within a century of this date. *C'est la vie mécanique* ...

The second of the considerations mentioned above is that — as regards *liaisons* at least — we shall quote the same author, only grown considerably wiser during the thirty years gone by. The literary source we bear in mind is [27], and it is a very interesting scientific document in several aspects indeed.

As its title implies, [27] is concerned with "des équations de la dynamique"; what the title of [27] does not imply is that these "équations de la dynamique" are offsprings of the author of this article *lui-même*. Those are the famous ***dynamical equations of Appell*** (or *of Gibbs–Appell*, as they are sometimes called) — so famous that we cannot desist from adducing some independent appraisals.

Pars, for instance, states:

"... *the Gibbs–Appell equations* ... were first discovered by Willard Gibbs in 1879, and studied in detail by Appell twenty years later ... The Gibbs–Appell equations provide what is probably the simplest and most comprehensive form of the equations of motion so far discovered. They are of superlatively simple form, they apply with equal facility to holonomic and to non-holonomic systems alike, and quasi-co-ordinates may be used freely" [16, p. 201–202].

This is a generally shared view. For instance, in the Mathematical Encyclopaedia [28, p. 301–302] one reads:

"*Аппеля уравнения* — обыкновенные дифференциальные уравнения описывающие движения как голономных, так и не голономных систем, установленные П. Аппелем [29, 30]. Иногда называются уравнениями Гиббса–Аппеля, так как для голономных систем ранее их установил Дж. У. Гиббс [31] ... Аппеля уравнения являются наиболее общими уравнениями движения механических систем."

Ipse dixit:

"Les équations que nous avons en vue se rapportent donc à la mécanique classique d'aujourd'hui; elles s'appliquent, comme on le verra, quelle que soit la nature des liaisons, pourvu que les liaisons soient realisées de telle façon que l'équation générale de la dynamique soit exacte" [27, p. 1–2].

The meaning of the last supposition is revealed on p. 10–11 of the article:

"Écrivons l'équation générale de la dynamique, telle qu'elle résult du principe de d'Alembert combiné avec le théorème du travail virtuel. Nous emploierons, dans tout ce qui suit, pour désigner les dérivées par rapport au temps, la notation des accents de Lagrange. L'équation générale de la dynamique est alors ..."

Meanwhile, Appell proceeds:

"On verra que, pour obtenir ces equations, nous sommes obligés de calculer l'énergie d'accélération du système $S = \frac{1}{2}\sum mJ^2$, c'est-a-dire d'aller au second ordre de dérivation par rapport au temps. Si l'on veut s'en tenir au premier ordre de dérivation, comme Lagrange, on est conduit a des équations assez compliquées qui généralisent celles de Lagrange [32–33], qu'on a appelées équations de Lagrange–Euler: cette méthode a été étudiée d'abord par Volterra en 1898 [34 — 38]; on pourra aussi consulter des mémoires de Tzenoff [39] et de Hamel [40]. Nous donnerons des applications à de questions de mécanique rationelle. Mais nous esperons que ces équations pourront aussi être utilisées par les physiciens dans des cas ou les équations de Lagrange et les équations canoniques d'Hamilton qui s'en déduisent ne sont plus applicables" [27, p. 2].

As regards this article of Appell's we declare our earnest intention to split hairs, at least in reference to some of its parts: he, who sows the wind, shall reap the whirlwind, and old sins cast long shadows, as the saying goes; and it proceeds: God's mills grind slowly, but they grind superfine. There is wind sowing in [27], and there are old sins there, no matter that the author is hiding himself under the umbrella of one or two great names, Poincaré's and especially Gauss', quoting the latter apropos of Appell's magister Lagrange:

"Le principe des vitesses virtuells transforme, comme on sait, tout problème de statique en une question de mathématiques pures, et, par le principe de d'Alembert, la dynamique est, à son tour, ramenée a la statique. Il result de là qu'aucun principe fondamental de l'équilibre et du mouvement ne peut être essentiellement distinct de ceux que nous venons de citer et que l'on pourra toujours, quel qu'il soit, le regarder comme leur consequence plus ou moint immediate.

On ne doit pas conclure que tout théorème nouveau soit, pour cela, sans mérite. Il sera, au contraire, toujours intéressant et instructif d'étudier les lois de la nature sous un nouveau point de vue, soit que l'on parvienne ainsi à traiter plus simplement telle ou telle question particulière ou que l'on obtienne seulement une plus grande precision dans les énoncés.

Le grand géomètre, qui a si brillamment fait reposer la science du mouvement sur le principe des vitesses virtuelles, n'a pas dédaigné de perfectionner et de généraliser le principe de Maupertuis, relatif a la *moindre action*, et l'on sait que ce principe est employé souvent par les géomètres d'une manière tres advantageuse" (see *Journal de Crelle*, tome IV).

A most symptomatic for the mechanical philosophy of the author of [27], along with the manifestation of the above ideology, is his dynamical credo, revealed in the very inceptive sentence of the article:

"Il faut tout d'abord prendre ici le mot *dynamique* dans son sens ancien, dans le sens de Galilée, de Newton, de Lagrange, de d'Alembert, de Carnot, de Lavoisier, de Mayer."

Do you see the name of Euler in this register of maestri of *la dynamique*? We certainly do not. A chance oversight, maybe? By no means. The fact is a result of a traditional, systematical, and most intentional scientific policy. The *Index bibliographique* of the article [27] includes 49 items (much more, in reality, since some of them, say No 39, involve more than one titles); as regards Euler's name, however, it has not a word to throw at a dog. Why is in this respect Appell as dumb as a fish — as silent as a grave? He certainly is not a regular oyster. He has found in [27] place enough for lyrical digressions — even for scientific poetry too fair to be sane, like H. Poincaré's:

"Peut-être devrons-nous construire toute une mécanique nouvelle que nous ne faisons qu'entrevoir, où, l'inertie croissant avec la vitesse, la vitesse de la lumièr deviendrait un obstacle infranchissable. La mécanique vulgaire, plus simple, resterait une première approximation puisqu'elle serait vraie pour les vitesses qui ne seraient pas très grandes, de sorte qu'on retrouverait encore l'ancienne dynamique sous la nouvelle. Nous n'aurions pas à regretter d'avoir cru aux principes, et meme, comme les vitesses trop grandes pour les anciennes formules ne seraient jamais qu'exceptionelles, le plus sûr dans la pratique serait encore de faire comme si l'on continuait à y croire. Ils sont si utiles qu'il faudrait leur conserver une place. Vouloir les exclure tout à fait, ce serait se priver d'une arme prècieuse. Je me hâte de dire, pour terminer, que nous n'en sommes pas là, et que rien ne prouve qu'il ne sortiront pas de là victorieux et intact"(p. 1, see *La valeur de la Science*, p. 231).

The absent-mindedness of P. Appell towards L. Euler apropos of *systèmes dynamiques non holonomes* is by no manner of means due to lack of good upbringing, or of want of place, or in default of bond — it is by no means accidental, fortuitous, and twopenny-halfpenny. This is a selective absent-mindedness. An idealist would say that it is a Freudean forgetfulness. A cynic certainly would qualify it as an unfair competition.

It is true that Euler never wrote a single line dedicated expressly to non-holonomic dynamics; it is true that Euler never solved even a most simple of all the non-holonomic problems; it is even true that he never suspected the existence of such a branch of dynamics and he never heard the term "non-holonomic" itself — the first study [41] in this domain has been published more than half a century after Euler's death. All this is true. At the same time it is also true that already in 1750 Euler discovered in his work [42] (see § 22, 40–58) the one and only system of differential equations describing adequately, authentically, and autoritatively the mechanical behaviour of any rigid body, submitted to the action of any active forces and subjected to any mechanical constraints whatever — including the non-holonomic case in a most natural way and as a most trivial particular case. Moreover, it turns out that any other kind of differential equations of motion of non-holonomic dynamical systems (including Appell's) as yet proposed represent only necessary and by no means sufficient conditions for the motion of the body, being only corollaries from Euler's dynamical equations and therefore being unable to solve ultimately a single non-holonomic dynamical problem (leaving unanswered the cardinal question of existence of a solution, as well as the crucial problem of the motive causes of the non-holonomic dynamical phenomenon under consideration, that is to say, the question, which are the reactions of the non-holonomic constraints). To cap it all one must add to the calamities of Lagrangean dynamical tradition in non-holonomic dynamics two great misfortunes. First, all Lagrange's versions of non-holonomic differential equations are adopted under the hypothesis that the constraints are ideal, and the Lagrangeans have no modus operandi to make sure of the mathematical reliability of this hypothesis which may be verified only by means of Euler's equations. Second, in the most frequent case of non-ideal non-holonomic constraints those Lagrange's versions become wholly unworkable, and the only way to solve the non-holonomic dynamical problem is to apply namely Euler's equations. But those are other topics we shall return later on; for the time being they have been mentioned in passing only in order to become crystal-clear that the absence of Euler's name, say, in the *Литература* of neither more nor less than 515 quoted authors in the monograph [43] especially dedicated to non-holonomic dynamics is a fact attesting at least a professional ignorance, putting it politely.

The only consolation Euler might find in — a Dutch comfort though, maybe — is that Newton is, in this respect, *ejusdem farinae.*

After these introductory explanations let us dot the i's and cross the t's of that part of the article [27] which concerns itself with the *Nature des liaisons.* The first paragraph entitled *Systèmes essentiellement holonomes ou essentiellement non holonomes*; *ordre d'un système non holonome* begins with the following explications:

"Imaginons un système matériel, a k degrés de liberté, formé de n points de masse m_μ ($\mu = 1, 2, \ldots, n$) ayant pour coordonnées rectangulaires x_μ, y_μ, z_μ dans un triedre d'axes orientés, animés, par rapport aux axes considérés comme fixes dans la mécanique classique, d'un mouvement de translation rectiligne et uniforme; les déplacements, les vitesses, les accélérations que nous considérerons sont des déplacements, des vitesses, des accélérations par rapport à ce trièdre" (p. 4).

With a view to vantage references, we shall organize our remarks in the form of several scholia.

Scholium 1. The mechanical systems Appell intends investigating (unless the reader hears to the contrary) represent sets of a finite number n of discrete mass-points.

Scholium 2. Appell's description does not exclude the case $n = 1$.

Scholium 3. The *triedre d'axes orientés* described by Appell so loquaciously and so indefinitely at the same time (namely, *animés d'un mouvement de translation rectiligne et uniforme par rapport aux axes considérés* [?!] *comme fixes dans la mécanique classique*) is, when all is said and done, purely and simply an *inertial* according to Newton (right-hand orientated) orthonormal Cartesian system of reference $Oxyz$, i.e. such that Newton's *Lex II*

$$\frac{d}{dt}(mv) = \boldsymbol{F} \tag{1}$$

holds for any mass-point P, the mass of which is m and which is acted on by forces with the resultant $\boldsymbol{F}$, provided $\boldsymbol{r} = \boldsymbol{OP}$ and

$$v \doteq \frac{dr}{dt}, \tag{2}$$

all derivatives being taken with respect to $Oxyz$. Except for being circumlocutory and, as a result, obscure, Appell's description is physical rather than mathematical: it speaks about *axes considérés comme fixes dans la mécanique classique*, hinting (without the explicit use of the term *space*, to tell the truth) at a purely physico-philosophical idea — as deep seated as short witted — of a kind of *absoluter Raum*, a broken puppet from mechanics' childhood the grown up mathematicians have junked long ago.

Scholium 4. Another physical remnant that has slipped through Appell's fingers is the adjective *matériel* — that much *persona gratissima* in analytical mechanics as Old Harry in church. This term is a string vibrating psychologically dangerous overtones. It cherishes the vain hopes in circles earning their bread and butter from mechanics that the objects rational mechanics studies have something to do with certain entities in the real world. They have not. The erroneous belief that they have has damaged immensely rational mechanics in the course of its whole history from Aristotle to Einstein and has muddled the sound connection of mathematical mechanics with the other two faces of mechanical triunity — physical mechanics and engineering mechanics.

Scholium 5. The quoted above excerpt from [27] includes an expression that is a virtual logical dalayed action bomb for the whole following exposition, namely *système à k degrés de liberté*. Why? Because the definition of the notion *degrés de* *liberté* requires, in the capacity of a *conditio sine qua non*, the precursory definition of the notion *liaisons imposées a un système mécanique*, and no definition of the term *liaison* may be found in [27] preceding page 4 of the article.

Scholium 6. How come? That seems a pretty how-d'ye-do. The section is entitled *Nature des liaisons* and the reader is rightfully expecting to learn what does a *liaison* mean and which attributes pertain to its *nature*. Maybe the author of [27] presupposes that the reader is presumably familiar with the *liaison*-concept? Where from? Appell gives no answer. To what extent? Silence again. If the degree of the reader's knowledge is inconsiderable, he won't be able to penetrate the author's exposition. If it is too high, the said exposition would be needless.

Scholium 7. A quite natural supposition is that the definition of the *liaison*-concept is to be searched for in Appell's *Traité* [15]. It is not a bad idea. Let us part for a while from [27] and peep into this treatise. At that, according to the above supposition, we presume that we know nothing about *liaisons* and that the genuine mathematical definition of this notion is in store for us in [15].

This mental experiment has been accomplished at the cost of considerable time, attention, and patience. Independent of the celebrity of this world-famous book, gone through countless editions and translations into many languages, from a constructional point of view it is a pell-mell achievably only by the French genius applied on such a slithery material like mechanics. Since there is no explicit definition of the *liaison*-concept in it; since the work is lacking in an index of subjects; and since its author rambles from subject to subject like a grasshopper, or rather hurries up and down the same theme like a shuttle — for all those reasons we have been compelled, in order to accomplish our task scrupulously, to reread through the magnifying glass of the prospected definition the 548 + 538 printed pages of the first two volumes of [15] on the lookout for *liaisons* with the avidity of gold-diggers. *Voilá* what remained in our cradle rocker after all the auriferous gravel has been panned out.

As well as the scientific philosophy of a chemist is reduced to the idea of atoms, the scientific phylosophy of a mechanician of Appell's phylum is reduced to the idea of *point matériel*. With a view to the fundamental importance of this notion, let us see how it is introduced in [15]:

"A fin de commencer par le problème le plus simple, on étudie d'abord le mouvement d'un portion de matière assez petite pour qu'on puisse, sans erreur sensible, déterminer sa position comme celle d'un point géométrique. Une telle portion de matière s'appelle un *point matériel*. On considère ensuit les corps comme formés par la réunion d'un tres grand nombre de points matériels" (I, p. 78).

The only congenial commentary this "définition" is worthy of is Louis Carroll's quatrain from his *Alice*:

"Twas brillig, and the slithy loves
Did gyre and gimble in the wabe:
All mimsy were the borogoves,
And the mome raths outgrabe."

For the first time a significant, symptomatic, and even (as immediate future is to prove) crucial adjective is attached to the notion point (matériel) in *Chapitre V. Équilibre d'un point; équilibre d'un corps solide*, vol. I of [15], where one reads:

"Pour qu'un point libre M soit en équilibre, il faut et il suffit que ..." (p. 115).

We are not interested now what, as a matter of fact, the necessary and sufficient condition in question is; we do not even take an interest in the most curious fact that no definition of the term *équilibre* precedes this mathematical criterion for the mathematical category "equilibrium"; what attracts our attention now in the above excerpt is the adjective *libre*. Its use is by no means accidental.

Indeed, not only is the considered paragraph entitles *Point libre*, but also two following paragraphs are entitled *Point mobile sans frottement sur une surface fixe* and *Point mobile sans frottement sur une courbe fixe*, respectively. Since in the book there is no definition of the notion *point libre*, the only reasonable conclusion a reasonable reader could educe from those texts of Appell's (provided such a thing is possible) is that a *point matériel* is *libre* if it is not compelled to be *mobile sans frottement sur une surface fixe* or *sur une courbe fixe*. In such a manner, the notion of *point libre* is defined through its demerits rather than through its merits, in other words, by means of what it is not rather than what it is. Besides, there are two more problems:

1. What about *points matériels* compelled to be *mobile sur une surface fixe* or *sur une courbe fixe*, respectively, *avec frottement*?

2. Is this description of *point libre*, based on *statical* considerations, usable under *dynamical circumstances*?

The solutions of both these problems are lying on the mechanical conscience of the author of [15].

As far as our control goes, the term *liaison* comes forward, explicitly at least, for the first time in the following excerpt from [15]:

"La méthode générale que nous emploierons consiste à regarder les corps comme libres, en introduisant comme inconnues auxiliaires les réactions provenant des liaisons qui leur sont imposées, réactions que l'on nomme *forces de liaison*" (*ibid.*, p. 145).

Alas, this cryptic legend does not a whit lend one a helping hand to come to know what does actually a *liaison* mean, if one does not know it already. The immediately following paragraphs of the work are dedicated to particular though important examples of equilibria in special cases, like *corps ayant un point fixe*, *corps ayant un axe fixe*, *corps tournant autour d'un axe et glissant le long de l'axe* and *corps s'appuyant sur un plan fixe*. Thence the term *liaison* escapes at all the memory of the author of [15], in order to come across his mind not until *Chapitre VII. Principe des vitesses virtuelles*, where the reader comes to know that "nous exposerons une démonstration classique [of this principle] qui repose sur l'analyse des diverses sortes de *liaisons* simples" (*ibid.*, p. 226, our italics).

This promise is very hopeful. Looking on the bright side, if not of life, then at least of [15], the optimistic reader is in anticipation of at least three things:

1. To learn ultimately what does by Jove actually a *liaison* mean.

2. To become acquainted with *diverses sortes de liaisons*.

3. Moreover, to penetrate deeper into the *liaison*-concept by means of an appropriate *analyse* of this notion.

(The fourth possible expectation, namely:

4. To attend at une *démonstration classique du principe des vitesses virtuelles*, the reader must postpone *ad calendas Graecas*.)

Unfortunately, the following exposition of [15] blights all those hopes in *pulvis et umbra*.

Indeed, after the "définition" of the terms *déplacement virtuel*, *travail virtuel*, and *vitesse virtuelle* (I, p. 226), the logical level of which equals that of the implications *canis a non canendo* and *lucus a non lucendo*, the first text containing the term *liaison* reads:

"... imaginons un système de points assujettis à des liaisons sans frottement. Divisons les forces appliquées aux différents points en deux classes: *les forces de liaison* qui proviennent des liaisons imposées au système, et *les forces directement appliquées* ou *forces données* que l'on fait agir sur le système; le principe des vitesses virtuelles s'énonce alors de la façon suivante:

La condition nécéssaire et suffisante de l'équilibre d'un système est que, pour tout déplacement virtuel de ce système, compatible avec les liaisons, la somme des travaux virtuels des forces directement appliquées soit nulle" (*ibid.*, p. 227).

If the principle of virtual velocities is quoted here, the reason is not concealed in its importance: it is a mechanical anachronism, remains of peripatetic antiquity, wreckage of the Great Dynamical Catastrophe called Lagrangean Tradition. Its falsity, its spuriousness, its phoniness are seen from miles away with a naked eye. Its inclusion in modern mechanical textbooks — moreover, its supplying with counterfeit "proves", "demonstrations", and "deductions" in these books — unavoidably provokes one to exclaim together with Cicero: *Mirabile videtur, quod non rideat haruspex, cum haruspicem viderit; hoc mirabilius, quod vos risum tenere possitis.* Immediately following the formulation of the *principe des vitesses virtuelles*, quoted above, the special cases are treated of *point sur une surface* (p. 228–229) and *point sur une courbe* (p. 230–231). Now a mass-point, moving on a smooth surface or along a smooth curve line *inertially* (that is to say, under the action of no *forces directement appliquées*, in other words, movable only by *les forces de liaison*) is a counter-example *par excellence* of the *principe des vitesses virtuelles* that destroys it without leaving a trace. If the principle of virtual velocities is quoted here, we repeat, the only reason is that its formulation and discussion is in the consecutive text of [15], where the term *liaison* is visible.

The next text with this same characteristic is:

"... un solide libre ... est formé d'un grand nombre de points matériels assujettis à rester à des distances invariables les uns des autres: ce sont la les liaisons imposées au système. Dans ce nouveau cas, les seuls déplacements possibles, compatibles avec les liaisons, sont ceux pour lesquels la forme du solide reste invariable" (*ibid.*, p. 231).

These profound thoughts are followed by the text:

"Que le corps soit en équilibre ou non, la somme des travaux des forces de liaisons, qui sont ici les actions mutuelles des points du système, est nulle pour tout déplacement compatible avec les liaisons ... Si les déplacements virtuels imprimés aux deux points sont compatibles avec la liaison imposée aux deux points de rester à *une distance invariable*, r reste constant, δr est nul et la somme des travaux des forces de liaisons est nulle" (*ibid.*, p. 232–233).

Immediately afterwards Appell formulates *le lemme suivant*:

"Qu'un système de points matériels soit en équilibre ou non, pour tout déplacement virtuel compatible avec les liaisons, la somme des travaux virtuels des forces dues à ces liaisons est nulle, en supposant essentiellement qu'il n'y a pas de frottement.

Il suffit évidemment d'établir ce lemme pour chacune des liaisons du système et, pour cela, nous passerons en revue les diverses sortes de liaisons. Nous les diviserons en deux catégories:

1°. Liaisons des corps du système avec des corps fixes.

2°. Liaisons des corps du système entre eux". (*ibid.*, p. 233).

We shall not particularize the subsequent meditations. Their mathematical value is below zero: at the best they may be qualified as physical casuistry of a mathematically vicious practice. What we are interested in is the *liaison*-concept as it is shaped in [15]. If the hitherto adduced patterns of Appell's way of thinking and writing are still insufficient in this connection, let the following ones replenish the shortage with fresh samples:

"Les liaisons réalisées dans les machines sont les combinaisons des précédentes. Ainsi il est aisé de fair rentrer dans les liaisons examinées ci-dessus les liaisons réalisées à l'aide de fils ou de chaînes.

Imaginons, par exemple, que deux points M et M_1 du système sont liés l'un à l'autre par une chaîne C inextensible, tendue dans une partie de sa longueur sur une surface S sur laquelle elle peut glisser sans frottement, cette surface S étant d'ailleurs fixe ou mobile. Cette liaison est une combinaison des précédentes; les chaînons sont des corps solides; chacun d'eux est articulé au suivant en un point ou suivant un axe; ceux qui sont en contact avec la surface glissent sans frottement sur une surface S. L'un des deux points, M_1 par exemple, pourrait, de plus, être lié invariablement à la surface S: ce serait encore une liaison précédemment examinée. Ce genre de liaisons comprend en particulier les liaisons effectuées à l'aide de poulies" (*ibid.*, p. 236–237).

All these particular cases settled in the described manner at that place, following a process of *inductio per enumerationem simplicem*, Appell arrives at a *conception générale des liaisons sans frottement*:

"Nous venons de voir que, pour les liaisons les plus simples et leurs combinaisons, la somme des travaux virtuels des forces de liaison est nulle, pour tout deplacement virtuel compatible avec les liaisons, du moment qu'il n'y a pas de frottement. Pour des liaisons d'une nature plus compliquées, par example des liaisons qui sont exprimées par des équations, ou prend la propriété précédente comme la definition même de l'absence de frottement; les liaisons sont sans frottement si, pour tout déplacement compatible avec les liaisons, la somme des travaux des forces de liaisons est nulle" (*ibid.*, p. 237).

We shall not discuss the qualities, the advisability, or even the very reasonableness of this "définition". Two points must be, however, unconditionally emphasized.

First, it makes use of the notion *déplacement virtuel compatible avec les liaisons*, which is void of mathematical roots of matter in the frames of the treatise [15].

Second, it would define the notion *liaisons sans frottement* if the notion *liaison* was defined preliminary; and it is in no way.

The following paragraph is dedicated to a pseudodemonstration of the *principe des vitesses virtuelles*, namely:

"*Pour que le système soit en èquilibre dans un certain position, il faut et il suffit que, si l'on imprime au système un déplacement virtuel quelconque compatible avec les liaisons, la somme des travaux virtuels des forces directement appliquées soit nulle*" (*ibid.*, p. 237).

After his hymerical proof of this unveracious mathematical proposition, the author of [15] takes liberties with a mechanical caprice, namely *liaisons effectuées à l'aide de corps sans masse*:

"Il arrive quelquefois que, dans un système en mouvement ou en équilibre, il se trouve des corps dont on néglige la masse par rapport aux autres corps du système et qu'on regard comme ayant une masse nulle. On traduit cette hypothèse en exprimant que les forces appliquées à un corps sans masse se font équilibre ... Par exemple, si deux points matériels M et M_1 sont liés l'un à l'autre par *une tige rigide et sans masse*, les actions de la tige sur les deux points sont deux forces F et F' egales et directement opposées" (*ibid.*, p. 240).

An occasional gleam of mathematical professionalism as regards the *liaison*-concept may be spotted in the following text (*ibid.*, p. 249):

"Soit donné un système forme de n points

$$M_1(x_1, y_1, z_1),\ M_2(x_2, y_2, z_2),\ \ldots,\ M_n(x_n, y_n, z_n)$$

soumis à des liaisons qui s'expriment par des relations entre leurs coordonnées

$$(1)\qquad \begin{cases} f_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0, \\ f_2(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0, \\ \ldots\ldots \\ f_h(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0." \end{cases}$$

Anything that follows till the very end of *Chapitre VII* are routine mathematical manipulations and applications. As regards *Chapitre VIII*, it is dedicated to *Notions sur le frottement*. Against the background of the hotchpotches around the general *liaison*-concept its content may be skipped with a *Graecum est, non legitur*.

Scholium 8. All foregoing observations have been made on the basis of *Deuxième partie*: *Statique* of vol. I of [15]. All of them run upon the *liaison*-concept, the mathematical zenith of which attained in [15] being the text quoted above from page 249. Now we are on the horns of a dilemma: to proceed further, repeating in the dynamical case all that we have already done in the statical one; or to suspend discussions with a *Sapienti sat* or *Intelligenti pauca*. Both solutions have their good points and their drawbacks.

We have a preference for the *aurea mediocritas* — which is maybe the silliest decision. We shall at once point at the mathematical climax Appell has attained in dynamics in connection with the *liaison*-concept. It is formulated on p. 319–320 of vol. II of [15], where one may read:

"Soit un système de n points $m_1, m_2, \ldots, m_n$ de coordonnées $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$, assujettis à des liaisons données, realisées sans frottement; ces liaisons peuvent d'ailleurs dépendre du temps ... Supposons, ce qui n'est pas toujours possible, les liaisons exprimées par des équations finies, entre les coordonées des points et le temps,

$$(2) \qquad \begin{cases} f_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n, t) = 0, \\ f_2(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n, t) = 0, \\ \quad \cdots\cdots \\ f_h(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n, t) = 0. \end{cases}$$

On concoit comment les liaisons peuvent dépendre du temps; c'est ce qui arrive, par exemple, quand un point du système est assujetti à glisser sur une surface ou sur une courbe animée d'un mouvement connu.

Comme pour le cas de l'équilibre, il faut supposer le nombre des équations de liaison inférieur à $3n$; si ce nombre était $3n$, le mouvement du système serait déterminé. Nous poserons encore

$$h = 3n - k.$$

Imprimons au système un déplacement virtuel $\delta x_1, \delta y_1, \delta z_1, \ldots, \delta x_n, \delta y_n, \delta z_n$, compatible avec les liaisons qui ont lieu a l'instant t. Ce déplacement devra se faire de facon que les équations (2), dans lesquelles t possède la valeur numerique qui correspond a l'instant considéré, soient satisfaites; on aura donc les relations entre les différentielles δx_ν, δy_ν, δz_ν, en différentiant les équations de liaisons ou t sera considéré comme une constante. On a de cette façon

$$(3) \qquad \begin{cases} \dfrac{\partial f_1}{\partial x_1}\,\delta x_1 + \dfrac{\partial f_1}{\partial y_1}\,\delta y_1 + \dfrac{\partial f_1}{\partial z_1}\,\delta z_1 + \cdots + \dfrac{\partial f_1}{\partial z_n}\,\delta z_n = 0, \\ \quad \cdots\cdots \\ \dfrac{\partial f_h}{\partial x_1}\,\delta x_1 + \dfrac{\partial f_h}{\partial y_1}\,\delta y_1 + \dfrac{\partial f_h}{\partial z_1}\,\delta z_1 + \cdots + \dfrac{\partial f_h}{\partial z_n}\,\delta z_n = 0. \end{cases}$$

... Les équations (3) montrent que parmi les $3n$ variations δx_ν, δy_ν, δz_ν, il y en a k d'arbitraires; les h autres s'exprimeront linéairement en fonction des k premières au moyen de ces équations ..."

In such a manner, the system of equations (2) may be adopted (with a grain of salt) in the capacity of a definition of *liaisons* imposed on a system of n discrete mass-points, and the system of equations (3) may be taken up (again for what it is worth) as a definition of *virtual displacements* compatible with such *liaisons* imposed on such a system.

Unfortunately, the range of action of those definitions and constructions is negligibly small. Before writing down the system of equations (2) Appell proposes to "supposons, ce qui n'est pas toujours possible ..."; as a matter of fact, this supposition is impossible for the most part — in any case, every time when a rigid body is concerned. In other words, in the rigid case all the above considerations become as illusive as, *exempli causa*, panacea, philosophorum lapis, or phlogiston. In the rigid case all those considerations purely and simply become completely void of sense. And yet, it is the rigid case, namely, where the overwhelming majority of applications is done. As an ephemeral illustration, let us cite the *Remarque* of Appell immediately following the excerpt quoted above:

"Il n'est pas toujours possible d'exprimer les liaisons par des équations finies telles que (2) entre les coordonnées. Par exemple, si une surface S est assujettie à rouler et à pivoter sur une surface fixe Σ, on exprime cette liaison en écrivant que la vitesse du point de S au contact avec Σ est nulle, ce qui ne donne pas une équation finie" (p. 322–323).

Now, *iurare iovem lapidem, quid hoc ad Iphicli boves*? What does the surface S have in common with a finite system of n discrete mass-points? The question is purely rhetorical, of course.

As regards the illegitimate applications of the above definitions, formulations, and constructions upon rigid bodies, any text-book and book of problems on analytical dynamics dealing with Lagrange's dynamical equations *verbatim et litteratim* is swarming with such a breed born on the wrong side of the blanket.

Scholium 9. In such a way the program announced in the beginning of *Scholium 7* may be considered settled. If the essence of this program is rooted in the question, whether Appell's *Traité* [15] contains a strict and irreproachable mathematical definition of the *liaison*-concept, then the answer is a most categorical *NO*.

Scholium 10. In such a manner, the excerpt from the article [27], cited immediately before *Scholium 1*, is a false start or, just the same, a logical *circulus vitiosus*. The vitiosity of the mathematical procedure involved is traditionally qualified by the phrase *idem per idem* or *definitio per idem*; by analogy with the terms *circulus in demonstrando* or *circulus in probando* it could be called also *circulus in definando*.

The phenomenon is no news in mathematics. The precedents are legion in order to be exemplified here; one of them is, however, too congenial to be passed over in silence: the notion of measure and the concept of integral.

Scholium 11. In the absence of a clear-cut definition of the *liaison*-concept the statements of Appell cited immediately below *sunt verba et voces, praetereaque nihil* — absolutely arbitrary formulations that could be true, and could be untrue, and heaven only knows where does the dividing line between verity and falsehood lie:

"Pour obtenir le déplacement virtuel le plus général du système [matériel, formé de n points de mass m_μ ($\mu = 1, 2, \ldots, n$)] compatible avec les liaisons existant à l'instant t, il suffit de faire varier k paramètres $q_1, q_2, \ldots, q_k$, convenablement choisis, de quantités arbitraires infiniment petites $\delta q_1, \delta q_2, \ldots, \delta q_k$. On a alors pour le déplacement virtuel du point m_μ

$$(1)\qquad \begin{cases} \delta x_\mu = a_{\mu,1}\delta q_1 + a_{\mu,2}\delta q_2 + \cdots + a_{\mu,k}\delta q_k, \\ \delta y_\mu = b_{\mu,1}\delta q_1 + b_{\mu,2}\delta q_2 + \cdots + b_{\mu,k}\delta q_k, \\ \delta z_\mu = c_{\mu,1}\delta q_1 + c_{\mu,2}\delta q_2 + \cdots + c_{\mu,k}\delta q_k, \end{cases}$$

et pour le déplacement réel du même point pendant le temps dt

$$(2)\qquad \begin{cases} dx_\mu = a_{\mu,1}dq_1 + a_{\mu,2}dq_2 + \cdots + a_{\mu,k}dq_k + a_\mu dt, \\ dy_\mu = b_{\mu,1}dq_1 + b_{\mu,2}dq_2 + \cdots + b_{\mu,k}dq_k + b_\mu dt, \\ dz_\mu = c_{\mu,1}dq_1 + c_{\mu,2}dq_2 + \cdots + c_{\mu,k}dq_k + c_\mu dt. \end{cases}$$

Dans ces équations les coefficients $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$, a_μ, b_μ, c_μ ($\mu = 1, 2, \ldots, n$; $\nu = 1, 2, \ldots, k$) sont quelconques; ils dépendent uniquement de la position du système a l'instant t et du temps t; la constitution de ces coefficients ne joue aucun rôle dans le cas général" (p. 4).

Scholium 12. In order to proceed further, we are compelled now to play the hypocrite: we must dissimulate that we understand Appell's arguments. In this respect we are not the only pebbles on the beach: the students of this *Professeur à la Faculté des Sciences* (as well as the students of any professor on analytical dynamics all over the world who is presenting his subject according to the canons of the Lagrangean mechanical tradition) have also been constrained to feign understanding of the enforced material in order to take no risk of failing in the examinations.

Under these conditions we come to know that:

"D'après la terminologie de Hertz, un système est dit *holonome*, quand les liaisons qui lui sont imposées s'expriment par des relations en termes finis entre les coordonnées déterminant les positions des divers corps dont il est compose; dans ce cas, on peut choisir pour $q_1, q_2, \ldots, q_k$ des variables dont les valeurs numériques, a l'instant t, déterminent la position du système; les quantités $q_1, q_2, \ldots, q_k$ sont alors les coordonnées du système holonome, dont la position est déterminée par le point figuratif ayant pour coordonnées rectangulaires $q_1, q_2, \ldots, q_k$ dans l'espace à k dimensions; les coordonnées x_μ, y_μ, z_μ sont des fonctions de $q_1, q_2, \ldots, q_k$ et du temps t exprimables en termes finis et les second membres des équations (2) sont les différentielles totales de fonctions de $q_1, q_2, \ldots, q_k$ et t. Les équations du mouvement prennent alors la forme donnee par Lagrange. Il peut arriver, au contraire, que les liaisons entre certains corps du système s'expriment par des relations différentielles *non intégrables* entre les coordonnées dont dépendent les positions de ces corps; c'est ce qui arrive, par exemple, si un solide du système est terminé par une surface ou une ligne assujettie à rouler sans glisser sur une surface fixe ou sur la surface d'un autre solide du système; cette liaison s'exprime en effet, dans le premier cas en écrivant que la vitesse du point matériel au contact est *nulle*, et, dans le deuxième, que les vitesses des deux points matériels au contact sont les mêmes. D'apres Hertz, on dit que le système n'est pas holonome dans ce cas; même si l'on supposer que les $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$ peuvent être exprimés à l'aide des seuls variables $q_1, q_2, \ldots, q_k, t$, les seconds membres des formules (2) ne sont par supposés des différentielles exactes" (p. 4-5).

Scholium 13. It is symptomatic that, though by chance, the number of this scholium sounds almost as fatally as the content of the text it contains:

"Dans ce qui précède, nous avons considéré avec Hertz les systèmes eux-mêmes; pour les distinguer nous dirons qu'ils sont *essentiellement holonomes* ou *essentiellement non holonomes*. On peut aussi définir la nature d'un système pour un certain choix des paramètres; à cet égard on peut définir *l'ordre d'un système non holonome, pour un choix de paramètres*. Il y a alors deux éléments à rapprocher, le système matériel et le choix des paramètres. On dira qu'un système est holonome, pour un certain choix $q_1, q_2, \ldots, q_k$ de paramètres, si les équations de Lagrange s'appliquent a tous les paramètres. On appellera ordre, pour un certain choix de paramètres $q_1, q_2, \ldots, q_k$, d'un système non holonome, le nombre des paramètres auxquels les équations de Lagrange ne s'appliquent pas ...

D'après cela, un système qui est, pour un certain choix de paramètres, *non holonome d'ordre zéro* est *holonome.*

L'ordre peut rester le même ou changer quand on remplace le système des paramètres $q_1, q_2, \ldots, q_k$ par un autre ...

On voit que l'ordre d'un système non holonome est défini par rapport a un certain choix des paramètres et qu'en faisant varier ce choix on peut faire varier *l'ordre*; mais il existe neanmoins un ordre essentiel attaché à chaque système, c'est le *minimum* ω des ordres obtenus en faisant varier d'une facon quelconque le choix des paramètres. Par exemple, un système essentiellement holonome est un système non holonome d'ordre essentiel zero" (p. 5–7).

For a *connaisseur* of the real state of affairs in rigid dynamics this verbiage rings at least as whimsical as the already cited quatrain of Carroll's, which we shall quote again, this time in French:

"Il brilgur les toves lubricéilleux
Se gyrent en vrillant dans le guave
Enmimes sont les gouge bosqueux
Et le mômerade horsgrave."

It is an hopeless task to enlighten the mind of a Lagrangean mechanician by bringing to light his almost fanatical superstitions as to convert a religious neurasthenic to the dogmata of modern physics. Lagrangean mechanicians put as much confidence in old wifes' tales about "les corps comme formés par la reunion d'un très grand nombre de points matériels, c'est à dire portions de matière essez petites pour qu'on puisse, sans erreur sensible, déterminer ses positions comme celles des points géométriques" as simple-hearted infant children in Arabian Nights fairy-tales. And yet, some commentaries in connection with the last fragment from [27] are purely and simply inevitable. At that, in order to make a long story short, we shall once more cast a glance into Truesdell's *Essays* [8]:

"At the end of the [eighteenth] century there was a dismaying tendency to turn away from the basic problems, both in mechanics and in pure analysis. Directly contrary to the great tradition set by James Bernoulli and Euler, this formalism grew rapidly in the French school and is reflected in the *Méchanique Analitique.* Much of the misjudgement that historians and physicists have passed upon the work of the eighteenth century comes from unwillingness to look behind and around the *Méchanique Analitique* to the great works of Euler and Bernoullis which are left unmentioned. As its title implies, the *Méchanique Analitique* is not a treatise on rational mechanics, but rather a monograph on one method of deriving differential equations of motion, mainly in the special branch now called, after it, *analytical mechanics* ... While it contains interesting historical parentheses, the presentation of mechanics is strictly algebraic, with no explanation of concepts, no illustrations either by diagrams or by developed examples, and no attempt to justify any limit process by rigorous mathematics" (p. 134, 173).

No explanation of concepts, no illustration either by diagrams or by developed examples, and no attempt to justify any limit process by rigorous mathematics one may find in Appell's *Sur une forme générale des équations de la dynamique* too. There is, however, a characteristic feature of this article that Truesdell has overlooked in his portrait of Lagrange the Mechanician, and it is the *absolute absence* *of existence arguments*. As regards Lagrange himself, this non-attendance of the existence problem may be apologized by *Saeculi vitia, non hominis*. As regards [27], however, no vindication save *Mea culpa, mea maxima culpa* may be accepted.

The article [27] is published in 1925, that is to say a quarter of a century after Hilbert's *Grundlagen* [44] and *Probleme* [6]. Born in 1855, Appell was in the fullness of his mental powers when these titanic works entered the depository of human knowledge. Now the existence problem is *Problem Number One* of the whole of Hilbert's mathematical philosophy. Obviously, Appell's mathematical blood proved perfect immunity against axiomatic infections.

Existence problem — what does it mean in mechanics? In order to answer this question we shall put a counter-question: *which mechanics*? There is no such thing as universal mechanics — there is a mathematical mechanics, there is a physical mechanics, and there is an engineering mechanics, at last. For an engineer something exists if he can operate it. For a physicist something exists if he can experiment it. For a mathematician something exists if he can demonstrate it. Since we are interested in mathematical mechanics (which is a synonym of rational mechanics), for us an *existence problem* means a *problem of proof*.

Appell is speaking about *liaisons*. Moreover, he describes them. In this respect he completely satisfies Fontenelle's epigram: "Mathematicians are like lovers ... Grant a mathematician the least principle, and he will draw from it a consequence, which you must also grant him, and from this consequence another ..." Like a lover fancies the eyes, the hair, the bosom, and other attributes of his beloved, Appell fancies the smoothness, the holonomeness, the order of his *liaisons*, virtual displacements compatible with the latters, and what not yet. All the same, both Appell and the lover overlook the cardinal problem: Does she love me? (the lover); Do *liaisons* exist? (Appell).

What does *existence* in rational mechanics mean?

The answer of this question is twofold, since there are two kinds of objects in rational mechanics:

1. Non-specific objects.
2. Specific objects.

Restricting ourselves to rigid dynamics we could state that *non-specific* objects of this domain of rational mechanics are those mechanical entities that live a self-dependent life, detached from the concept of force. Such are, for instance, all paraphernalia of kinematics, including geometrical constraints *in se*, as *Dinge an sich*, imposed on kinematical rigid bodies (that is to say, rigid bodies considered as purely geometrical objects devoid of density and completely insubordinated to any forces). On the contrary, *specific objects* of rigid dynamics are those mechanical entities, the very definition of which becomes meaningless in the absence of the force concept.

As mathematical notions, both non-specific and specific objects of rigid mechanics require, with a view to the logical legalization of their definitions, existence proofs. The difference between the first and the second categories consists in the fact that while no forces are needed to prove existence in the first case, in the second one no existence proof is thinkable without the essential use of the force concept.

Since this is a problem that will be analysed in details elsewhere, we confine for the time being our exposition to these brief indications.

Scholium 14. The last remarks apropos of [27] concern the following fragment of the article:

"Les deux jeux préférés des enfants, la toupie et le cerceau, fournissent les exemples de systèmes essentiellement holonomes ou essentiellement non holonomes. Pour le montrer, définissons d'abord les six coordonnées d'un corps solide entièrement libre (système essentiellement holonome). Soient trois axes rectangulaires fixes $O\xi\eta\zeta$; appelons ξ, η, ζ les coordonnées du centre de gravité G du corps solide par rapport a ces axes; θ, φ, ψ les angles d'Euler d'un système d'axes rectangulaires $Gxyz$ liés au corps avec des axes de directions fixes $Gx_1y_1z_1$ parallèles aux axes fixes. Ces six coordonnées ξ, η, ζ, θ, φ, ψ définissent la position d'un corps solide libre. Les coordonnées d'un point quelconque du corps sont des fonctions de ces six coordonnées. Si l'on impose des liaisons au solide, cela revient, suivant les cas, à établir certaines relations en termes finis entre les six coordonnées ou encore à établir certaines relations différentielles du premier ordre non intégrable: le nombre des degrés de liberté est alors diminué" (p. 7).

This fragment from [27] represents the only mathematically sane text of Appell's article up to this place. But it is not Appell's — it is Euler's, for whose name Appell's mind was a complete blanc when, in the beginning of his article, he explained that he accepts "ici le mot *dynamique* dans son sens ancient, dans le sens de Galilée, de Newton, de Lagrange, de d'Alembert, de Carnot, de Lavoisier, de Mayer" (see our commentary immediately above Scholium 1). All constructions described above are Euler's inventions: even Appell — volens nolens or nolens volens — is coerced to call θ, φ, ψ "les angles d'Euler". Nowadays all these constructions may seem trivial: Lavoisier's chemical ideology also seems trivial *nowadays*; but anyone having the slightest idea of the tragicomical or comitragical history of chemistry realizes that Lavoisier's chemical philosophy is a Promethean gift to mankind. As Truesdell says, Euler "put most of mechanics into its modern form; from his books and papers, if indirectly, we have all learned the subject, and *his way of doing things is so clear and natural as to seem obvious.* In fact, it was he who *made mechanics simple and easy, and for the straightforward it is unnecessary to give references.* In return, the scientist of today who consults Euler's later writings will find them *perfectly modern*, while other works of that period require efforts and some historical generosity to be appreciated" [8, p. 106, our italics].

Facit indignatio versum: is the above excerpt from [27] not an *argumentum ad ignorantiam*? Anyway, it is a pound to a penny that the mathematical procedure just now described and thence frivolously applied to the *toupie* and *cerceau* has nothing to do with the mathematical procedure that led to the differential systems (1) and (2) from p. 4 of [27]: it would be the height of mathematical effrontery to maintain that a *toupie* or a *cerceau* represents "un système matériel ... formé de n points de masse m_μ ($\mu = 1, 2, \ldots, n$)", in spite of the fact that such a mathematical imprudence, such a mechanical insolence, such a dynamical impertinence is exhibited as many as 40 years after [27] in the treatise [16] on analytical dynamics, where a collection of typical rigid bodies (a spinning top, p. 113; a rigid rod, p. 119;

a rolling penny, p. 120; a sphere, p. 207; an ellipsoid, p. 224; etc.), are substituted by the counterfeit of "a collection of particles set in a rigid and imponderable frame" (p. 20). This treatise [16] will not be mentioned in the sequel: the game is not worth the candle.

As regards Appell's article [27], its further content is rather interesting with a view to the degree of freedom of mare's nest a phony mechanical idea may result in. Instead of exploring and exploiting the concept underlying that fragment of [27], where Appell speaks about "les angles d'Euler", in the section *Réalisation des liaisons. Asservissement* of the article its author adduces arguments in connection with the *liaisons* which give a good grounds to repeat Truesdell's words apropos of D'Alembert, namely "in attempting to connect physical experience with mathematics, he heaped folly on folly" [45, p. 12]. But why ask the Bishop when the Pope is around:

"Dans ce qui précède, les liaisons sont considérées à un point de vue purement analytique, indépendent de la manière particulièr dont elles sont réasisées ... Or, peut-on faire abstraction de la manière dont une liaison est réalisée? La question a fait l'objet de nombreuses études. Voici quelques considérations générales empruntées à Beghin ... et à Delassus ... Une liaison L d'un système Σ peut être réalisée avec ou sans le secours d'un système auxiliaire Σ_1. Dans le premier cas, la réalisation de la liaison est dite *parfaite*; dans le second cas, la réalisation de la liaison est encore *parfaite*, si l'introduction du système auxiliare Σ_1 n'apporte aucune restriction aux déplacements virtuels du système Σ qui restent alors tous les déplacements compatibles avec la liaison L; mais elle est *imparfaite*, si l'introduction du système Σ_1 apporte des restrictions aux déplacements virtuels du system Σ" (p. 9).

And *alibi*:

"Mais il faut faire remarquer que, même si l'on se borne aux liaisons parfaites, il existe une catégorie importante de mécanismes dans lesquels les liaisons se trouvent réalisées par des méthodes différentes de celles qui permittent l'application pure et simple de l'équation générale de la dynamique: dans ces liaisons spéciales, on ne peut fair abstraction du mode de réalisation et se contenter de leur expressions analytique. Ces liaisons sont celles que l'on obtient par *asservissement*; nous dirons qu'il y a *asservissement* lorsque les liaisons correspondantes, au lieu d'être réalisées d'une façon en quelque sorte passive, par contact de deux solides qui glissent ou roulent l'un sur l'autre à titre d'exemple, le sont pas l'utilisation appropriée de forces quelconques (forces électromagnétiques, pression de fluides, forces produites par un être animé, etc.). De ces liaisons d'asservissement, il resulte des forces de liaison que M. Beghin ... appelle de *deuxième espece* et dont le travail virtuel est généralement différent de zero, même si le déplacement est compatible avec la liaison. Il est entendu que nous liaisons ce genre de liaisons de côté, renvoyant pour ce cas à la thèse de M. Beghin qui utilise la forme générale d'équations que nous indiquons" (p. 10).

Now, if a reader of the article [27] puts in a claim on penetrative understanding of those texts, then he produces his autocertificates either for hypocrisy or for self-deceit: *tertium non datur*. Unfortunately, a reader of the second category who swallowed that bait — hook, line, and sinker — has been the author of the paper [46], as inconvincing as the Immaculate Conception; fortunately, nevertheless, he never returned to that subject.

Scholium 15. Today, Anno Domini 1993, the situation around the liaison-concept is, on principle at least, as mirthless as 70 years ago.

Causa causarum for this *status quo ad praesens* is, one and only, the Lagrangean dynamical tradition conceived by D'Alembert and fanatically supported by the overwhelming majority of modern mechanicians. In such a sense, it would not be iniquitous to state that, as regards the logical crisis rigid dynamics has lapsed into, it is a *causa sui: volenti non fit iniuria*. As sure as death, if Euler could see a Lagrangeanist of today, his words would be *Nescio vos*. And yet, there is a hope: rational mechanics is worthy of *Augustinus Sanctus' encomium patiens, quia aeternus*.

Any analysis whatever of any features whichever of Lagrangean dynamical tradition lie entirely outside the frames of present brief sketch. Summing up our observations, we intend to fix the reader's attention on some cardinal points connected with the *liaison*-concept.

Every unswayed examination of any attempt at a mathematically consistent definition of the notion mechanical constraint imposed on a rigid body is predestinated to establish the total collapse of any such a try. Moreover, any future efforts in this direction are doomed to fail. The grounds for that prophecy — which is no prophecy at all, but only an earth-born, earthbound, and earthly minded plausible inference of mechanical experience of long standing — are quite plain and utterly simple ones: the constraint concept is insusceptible of a strict mathematical definition, since it is a *non-mathematical notion* in the proper sense of the word.

This is a situation one must fathom if one has the intention of working professionally in rigid dynamics instead of imitating third-rate laic parrotries of amateurish mimicries of dilettante forgeries.

In order to attain this perspicacious insight, one must come back mentally to his mechanical childhood — in other words, to the time of his first steps in rigid dynamics. Let us imagine such a backward journey to the first dynamical problems, including rigid bodies we have solved, or at least we have considered solved.

Inasmuch as we are interested in constraints imposed on rigid bodies, the free bodies must be excluded. As regards the non-free bodies, a mere look in the *Table des matiéres* of vol. II of [15] will refresh our memory. The first and the second sections of *Chapitre XIX* are entitled *Mouvement d'un corps solide autour d'un axe fixe* and *Mouvement d'un solide parallèlement à un plan fixe*, respectively, and *Chapitre XX* itself is entitled *Mouvement d'un solide autour d'un point fixe*. These are classical dynamical problems, ergo classical mathematical problems. We have solved them in our student youth on the supposition (not always, and maybe not at all, explicitly expressed) that such motions exist. As a matter of fact, neither we nor somebody else ever raised this question. It was considered answered by Mother Nature itself: didn't we see any moment such motions performed in plain sight of everybody? Why make mountains out of molehills?

In this same time, when we missed the very idea to pose the question of existence of motions of one kind or another, in other branches of mathematics we have been instructed in then the existence problem has grown up to such dimensions and proportions as to shut out the whole horizon. In Euclidean geometry, for instance, the problem of existence of more than one parallel proved to be a hard nut to crack for the strongest mathematical teeth in the course of two clear millenia. In arithmetic the non-existence of a rational measure for the diagonal of the square blighted the hopes of Pythagorean philosophical school. Again in geometry the non-existence of certain solutions (by means of ruler and compasses) of three famous problems of antiquity preoccupied the most brilliant mathematical minds for more than twenty centuries. Other examples? *Nomen illis legio.* As a matter of fact, the efforts to solve various existence problems in the course of long periods of time ultimately led to the creation of dozens of most important domains of modern mathematics. In dynamics solely no existence problem about motions is not merely solved, but even submitted. Why?

Also sprach Zarathustra. Lagrange, we mean — Lagrange "under the personal influence of D'Alembert" [8, p. 248]. And all neophytes since *Méchanique Analitique*, Hamilton in the first place. Gauss too, *mirabile miserabileque dictu.*

Leaving for the time being the existence problem aside, let us fix our attention on the modes of making a free rigid body constrained. First of all, let us announce in everyone's hearing that all considerations in [15], concerning *mouvement d'un corps solide autour d'un axe fixe* and *mouvement d'une solide parallelement à un plan fixe*, are dynamically absolutely illusionary ones. Indeed, one of the aims and purposes of dynamics is to discover the causes leading to one dynamical phenomenon or another. This means revealing of the forces producing the dynamical happening. Now, what makes the rigid body rotate around a fixed axis or move parallel to a fixed plane? God Almighty? Pars gives no evidence in this connection. He even goes so far as to forget to ask such a question. He simply hypothesizes that these bodies perform obediently this or that motion. Do they indeed? Is such a procedure mathematically possible? Is Appell's hypothesis consistent with the dynamical principles? Ultimately, are those motions conformable with Euler's dynamical equations [17, (114), (115)] with appropriate reactions of the constraints? Apropos of all these questions there is a dead silence in [15]. (It is a gospel truth that these questions cannot be answered. In the case of a *mouvement d'un corps solide autour d'un axe fixe* the contact between the rigid body and the axis is accomplished along a line, and in the case of a *mouvement d'un solide parallélement à un plan fixe* this contact is carried into effect upon a surface; in both cases rigid dynamics is as ignorant of the nature of the reactions of the constraints as, for instance, Euclidean geometry is know-nothing about the temperature, the colour, and the sound of an equilateral triangle. The competence of rigid dynamics is tied to point-contacts only. If in the first case two different points of the rigid body are compelled to coincide with two fixed points of the axis, then rigid dynamics is quite sure what does this imply: the fixed points generate passive forces acting on the rigid body with directrices running through those points. If in the second case three non-collinear points of the rigid body are compelled to rest on a fixed plane, then rigid dynamics is also entirely certain what does this imply: the plane generates three passive forces acting on the rigid body with directrices running through those three points of the body. All these conclusions are based on Ax 3 E, formulated in [17]. The application of Ax 3 E, however, presupposes the availability of one or several absolutely strictly defined points of contact. As regards extravagant, extraordinary, and exotic "constraints" violating this *conditio sine qua non*, in such cases rigid dynamics is as helpless as a tortoise on its back.)

After these parentheses let us consider somewhat closer that case in [15] which arouses no such remonstrances, namely *mouvement d'un solide autour d'un point fixe*. If this fixed point is O, then Ax 3 E of [17] implies that it generates a reaction $\vec{R}$ acting on the rigid body S with a directrix running through O. On the other hand, if P is any point of S different from O, then P obviously is compelled to rest on a sphere with centre O and radius $a = OP$. Let now P_ν be three points of S with $OP_\nu = a$ $(\nu = 1, 2, 3)$ and $\boldsymbol{OP}_1 \times \boldsymbol{OP}_2 . \boldsymbol{OP}_3 \neq 0$. Then it is obvious that, instead supposing O fixed, one could consider the same motion of S hypothesizing that P_ν $(\nu = 1, 2, 3)$ are compelled to remain on a sphere with centre O and radius a. And yet, though geometrically the same, this condition is mechanically a quite different one. Indeed, Ax 3 E implies that now, instead of a single reaction $\vec{R}$ through O, three reactions $\vec{R}_\nu$ are acting on S through P_ν $(\nu = 1, 2, 3)$, respectively. Mathematically this is a quite different problem, the unknown reactions introducing 9 new unknown quantities instead of the 3 components of $\vec{R}$, namely the 3 times 3 unknown components of $\vec{R}_\nu$ $(\nu = 1, 2, 3)$. In such a manner, even though geometrically the same, dynamically we are faced with a quite different problem. As a matter of fact, there is even an infinite variety of such problems, since the radius a and the points P_ν $(\nu = 1, 2, 3)$ may be chosen in infinitely many ways. Moreover, one could cast away the condition that P_ν $(\nu = 1, 2, 3)$ must remain on the same sphere: any of those points may remain on a sphere of its own; it is essential only those points to be non-complanar with O.

Nonsense? Maybe, but nonsense non stop. For almost any geometrical constraint imposed on a rigid body there exist ways that constraint to be substituted by others, geometrically equivalent and mechanically different nevertheless. These instances suggest that in dealing with *liaisons* imposed on rigid bodies no cautiousness can be surplus. On the contrary, a mathematician must approach the *liaisons* with the wariness of one stepping up to a rattlesnake: *anguis in herba*.

One thing is surer than sure for the present: the approach to the *liaison*-concept must be inductive rather than deductive: the deductive attemps exhibited their emasculation in the course of three clear centuries. Before reaching a satisfactory generalization of the formulations, one must pan off enormous amounts of gravel in order to attain to genuine dynamical nuggets. There is one and one only way to an adequate mathematical description of the constraint concept: via *inductio per enumerationem simplicem* towards *definitio per enumerationem simplicem*.

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Received 8.04.1993

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 86, 1992

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 86, 1992

КАЧЕНИЕ ШАРА ПО АБСОЛЮТНО ШЕРОХОВАТОМУ ТОРУ

СОНЯ ДЕНЕВА

Соня Денева. КАЧЕНИЕ ШАРА ПО АБСОЛЮТНО ШЕРОХОВАТОМУ ТОРУ

В работе затронуты некоторые аспекты неголономной задачи о качении шара по абсолютно шероховатой поверхности под действием силы тяжести.

Sonia Deneva. PRIVATE MOVEMENTS OF ROLLING SPHERE ON ABSOLUTELY ROUGH TORE

In this paper is considered some aspects of the classical unholonomic problem about rolling sphere on absolutely rough surface under the action of weight.

В работе [1] рассмотрены некоторые частные движения катящего шара по абсолютно шероховатому тору, когда точка соприкосновения шара описывает параллель. В одном из этих случаев угол нутации подвижного триэдра шара остается постоянным во время движения. В настоящей работе исследуется более общий случай этой задачи, при котором движение подвижного триэдра шара определяется по методу Дарбу. Для этой цели, как искомые функции времени рассматриваются компоненты вектора угловой скорости на неподвижные оси координат.

Введем следующие обозначения: G — центр масс шара, $G\xi\eta\zeta$ и $Oxyz$ — соответственно подвижная, связанная с движущимся телом с началом в его центре масс G, и неподвижная системы координат, $n^0(n_x, n_y, n_z)$ — единичный вектор внешней нормали в точке P соприкосновения шара с поверхностью качения, ω — угловая скорость тела, $R(R_x, R_y, R_z)$ — радиус-вектор произвольной точки тора на неподвижные оси, m и a — масса шара и его радиус, a_{ij} — директорные косинусы триэдра $G\xi\eta\zeta$ относительно $Oxyz$, α и β — параметры поверхности тора, v_G и w_G — скорость и ускорение центра шара, φ, ψ, θ — углы Эйлера подвижного триэдра $G\xi\eta\zeta$.

Так как исследуется движение однородного шара, каждая система $G\xi\eta\zeta$, которая неизменно связанна с шаром, является системой его главных осей инерции. Для определенности можно принять, что положение $G\xi\eta\zeta$ по отношении $Oxyz$ в начальном моменте движения задано, т. е. углы φ_0, ψ_0, θ_0 — фиксированы.

Из условия, что шар катится по тору без скольжения, имеем неголономную связь

$$v_G = a(\omega \times n^0) \tag{1}$$

или из проекций на $Oxyz$ получаем

$$\begin{aligned} \dot{x}_G &= a(\omega_y n_z - \omega_z n_y), \\ \dot{y}_G &= a(\omega_z n_x - \omega_x n_z), \\ \dot{z}_G &= a(\omega_x n_y - \omega_y n_x). \end{aligned} \tag{2}$$

Для уравнений тора будем иметь [1]

$$\begin{aligned} R_x &= (R_1 + R_2 \sin\beta)\cos\alpha, \\ R_y &= (R_1 + R_2 \sin\beta)\sin\alpha, \\ R_z &= R_2 \sin\beta. \end{aligned} \tag{3}$$

Из (3) находим нормальный вектор n^0:

$$n^0 = \cos\alpha\sin\beta i + \sin\alpha\sin\beta j + \cos\beta k, \tag{4}$$

где i, j, k — орты системы $Oxyz$.

Принимая ввиду, что рассматривается движение по параллели на торе, т. е. $\beta = \text{const}$, из (2), (3) и (4) находим зависимости

$$\omega_x \sin\alpha - \omega_y \cos\alpha = 0, \tag{5}$$

$$\dot{\alpha}\cos\alpha\,\frac{R_1 + (R_2 + a)\sin\beta}{a} = \omega_z \cos\alpha\sin\beta - \omega_x\cos\beta. \tag{6}$$

Уравнения движения шара находим из уравнений Аппеля в квазикоординатах

$$\frac{\partial S}{\partial \ddot{\pi}_j} = Q_j, \tag{7}$$

где S — энергия ускорений тела, Q_j — обобщенные силы. Для квазикоординатах выберем проекции угловой скорости шара

$$\dot{\pi}_1 = \omega_x, \quad \dot{\pi}_2 = \omega_y, \quad \dot{\pi}_3 = \omega_z. \tag{8}$$

По теореме Кенига энергия ускорений S имеет следующий вид:

$$S = \frac{A}{2}\left(\dot{\omega}_x^2 + \dot{\omega}_y^2 + \dot{\omega}_z^2\right) + \frac{m}{2} w_G^2 + \cdots, \tag{9}$$

где $A = \frac{2}{5} ma^2$ (A, B, C — главные моменты инерции). Согласно (1) находим

$$w_G^2 = a^2 \left[\dot{\omega}^2 - (\dot{\omega}.n^0)^2\right] - 2a^2 (\dot{\omega}.\dot{n}^0)(\omega.n^0) + \cdots \tag{10}$$

Здесь повсюду многоточием обозначены члены, которые не содержат $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$. Заменяем (10) в (9) и получаем согласно (4)

$$\begin{aligned} S = &\frac{7}{10} ma^2 (\dot{\omega}_x^2 + \dot{\omega}_y^2 + \dot{\omega}_z^2) \\ &- \frac{m}{2} a^2 (\dot{\omega}_x \cos\alpha \sin\beta + \dot{\omega}_y \sin\alpha \sin\beta + \dot{\omega}_z \cos\beta)^2 \\ &- ma^2 (\omega.n^0) \dot{\alpha} \sin\beta (-\sin\alpha \dot{\omega}_x + \cos\alpha \dot{\omega}_y) + \cdots \end{aligned} \tag{11}$$

Обобщенные силы Q_j определяются по формулам

$$Q_j = \sum_{\nu=1}^{N} a_{\nu j}.F_\nu, \tag{12}$$

где F_ν — силы тяжести и a_{ij} находим из формул для скоростей

$$v_\nu = \sum_{j=1}^{3} a_{\nu j} \dot{\pi}_j + a_\nu = a_{\nu 1}\omega_x + a_{\nu 2}\omega_y + a_{\nu 3}\omega_z + a_\nu. \tag{13}$$

С другой стороны, согласно (1) скорости v_ν определяются из зависимости

$$v_\nu = a(\omega \times n^0) + \omega \times \rho_\nu, \tag{14}$$

где $\rho_\nu = x_\nu i + y_\nu j + z_\nu k$ — радиус-вектор произвольной точки шара.

Из сравнений (13) и (14) находим

$$\begin{aligned} a_{\nu 1} &= \sin\alpha \sin\beta k - \cos\beta j - z_\nu j + y_\nu k, \\ a_{\nu 2} &= \cos\beta i - \sin\beta \cos\alpha k + z_\nu i - x_\nu k, \\ a_{\nu 3} &= \sin\beta(\cos\alpha j - \sin\alpha i) + x_\nu j - y_\nu i. \end{aligned} \tag{15}$$

Из (12) и (15) имеем

$$\begin{aligned} Q_1 &= -mga \sin\alpha \sin\beta, \\ Q_2 &= mga \cos\alpha \sin\beta, \\ Q_3 &= 0. \end{aligned} \tag{16}$$

Здесь имеется ввиду, что

$$\sum_{\nu=1}^{N} m_\nu x_\nu = \sum_{\nu=1}^{N} m_\nu y_\nu = \sum_{\nu=1}^{N} m_\nu z_\nu = 0.$$

Заменяем (11) и (16) в (7) и получаем

$$\frac{7}{5} a\dot{\omega}_x - a \cos\alpha \sin\beta (\dot{\omega}_x \cos\alpha \sin\beta + \dot{\omega}_y \sin\alpha \sin\beta + \dot{\omega}_z \cos\beta) \tag{17}$$

$$+ a\dot{\alpha}\sin\beta\sin\alpha(\omega.n^0) + g\sin\alpha\sin\beta = 0,$$

$$\frac{7}{5}\,a\dot{\omega}_y - a\sin\alpha\sin\beta(\dot{\omega}_x\cos\alpha\sin\beta + \dot{\omega}_y\sin\alpha\sin\beta + \dot{\omega}_z\cos\beta)$$

$$- a\dot{\alpha}\sin\beta\cos\alpha(\omega.n^0) - g\cos\alpha\sin\beta = 0,$$

$$\frac{7}{5}\,a\dot{\omega}_z - a\cos\beta(\dot{\omega}_x\cos\alpha\sin\beta + \dot{\omega}_y\sin\alpha\sin\beta + \dot{\omega}_z\cos\beta) = 0.$$

Уравнения (17) допускают первые интегралы кинетического момента и кинетической энергии, которые найдем непосредственно. Пусть P — точка касания шара и тора. Кинетичный момент имеет следующий вид

$$K_P = A\omega + m(GP \times v_G)$$

или согласно (1) находим

(18) $$K_P = \frac{7}{5}\,ma^2\omega - ma^2(\omega.n^0)n^0.$$

Из уравнения кинетического момента имеем

$$\frac{dK_P}{dt} = PG \times (-mgk),$$

откуда получаем интеграл кинетического момента для точки

(19) $$K_P.k = C_1 = \mathrm{const}.$$

Из (18) и (19) находим

(20) $$\omega_x\cos\alpha\sin\beta\cos\beta + \omega_y\sin\alpha\sin\beta\cos\beta - \omega_z(0{,}4 + \sin^2\beta) = C_1',$$

где C_1' — константа.

Интеграл кинетической энергии следует из теоремы изменения кинетической энергии T

(21) $$dT = mg.dr_G + N.dr_P,$$

где N — реакция тора на шар.

Для дифференциалов в (21) имеем

$$dr_P = v_P.dt,$$

так как шар катится без скольжения. Соответственно

$$g.dr_G = -gdz_G = 0$$

из-за движения по параллели. Тогда из (21) находим, что

(22) $$T = h = \mathrm{const}.$$

Из теоремы Кенига для кинетической энергии имеем

$$T = \frac{A}{2}\,\omega^2 + \frac{m}{2}\,v_G^2$$

или согласно (1) получаем

(23) $$T = \frac{7}{10}ma^2(\omega_x^2 + \omega_y^2 + \omega_z^2) - \frac{ma^2}{2}(\omega_x\cos\alpha\sin\beta + \omega_y\sin\alpha\sin\beta + \omega_z\cos\beta)^2.$$

Из (22) и (23) находим

(24) $7\left(\omega_x^2+\omega_y^2+\omega_z^2\right)-5(\omega_x\cos\alpha\sin\beta+\omega_y\sin\alpha\sin\beta+\omega_z\cos\beta)^2=h_1=\mathrm{const}.$

Из (5), (6) и (20) составляем систему уравнений относительно компонентов угловой скорости

$$
\begin{aligned}
&\omega_x\sin\alpha-\omega_y\cos\alpha=0,\\
(25)\qquad &\omega_y\cos\beta-\omega_z\cos\alpha\sin\beta=-\dot\alpha\cos\alpha\,\frac{R_1+(R_2+a)\sin\beta}{a},\\
&\omega_x\cos\alpha\sin\beta\cos\beta+\omega_y\sin\alpha\sin\beta\cos\beta-\omega_z(0{,}4+\sin^2\beta)=C_1'.
\end{aligned}
$$

Без ограничения общности, предположим, что в начальном моменте

(26) $$\alpha_0=0.$$

Из (25) в этом случае получаем зависимости

(27) $$\omega_{y_0}=0,$$

(28) $$\dot\alpha_0=\frac{a}{R_1+(R_2+a)\sin\beta}\,[\omega_{z_0}\sin\beta-\omega_{x_0}\cos\beta],$$

(29) $$C_1'=\omega_{x_0}\sin\beta\cos\beta-\omega_{z_0}(0{,}4+\sin^2\beta).$$

Начальное условие (27) является необходимым условием для угловой скорости шара при движении по параллели.

Решая линейную систему (25) относительно ω_x, ω_y, ω_z, получаем

$$
\begin{aligned}
&\omega_x=-\frac{2{,}5\cos\alpha}{\cos\beta}\left[C_1'\sin\beta+\frac{R_1+(R_2+a)}{a}\,\dot\alpha(0{,}4+\sin^2\beta)\right],\\
(30)\qquad &\omega_y=-\frac{2{,}5\sin\alpha}{\cos\beta}\left[C_1'\sin\beta+\frac{R_1+(R_2+a)}{a}\,\dot\alpha(0{,}4+\sin^2\beta)\right],\\
&\omega_z=-2{,}5\left[C_1'+\dot\alpha\sin\beta\frac{R_1+(R_2+a)\sin\beta}{a}\right].
\end{aligned}
$$

После замены (30) в (24) получается алгебрическое уравнение относительно $\dot\alpha$, из которого следует, что $\dot\alpha$ — константа для рассматриваемого движения, т. е. имеем

(31) $$\dot\alpha=\dot\alpha_0=\frac{a}{R_1+(R_2+a)\sin\beta}[\omega_{z_0}\sin\beta-\omega_{x_0}\cos\beta].$$

Заменяя (29) и (31) в (30), находим

(32) $$\omega_x=\omega_{x_0}\cos\alpha,\quad \omega_y=\omega_{x_0}\sin\alpha,\quad \omega_z=\omega_{z_0},$$

т. е. ω_x и ω_y являются периодическими функциями времени. Третье из уравнений (17) удовлетворяется тождественно согласно (32), а первые два уравнения редуцируются на следующую зависимость

(33) $$\frac{7}{5}\,a\omega_{x_0}-a\sin\beta(\omega_{x_0}\sin\beta+\omega_{z_0}\cos\beta)\dot\alpha-g\sin\beta=0.$$

Заменяя (31) в (33), получаем

$$\omega_{z_0}^2 \sin^2\beta\cos\beta + \omega_{x_0}^2 \cos\beta(0{,}4 + \cos^2\beta) - 2\sin\beta(0{,}2+\cos^2\beta)\omega_{x_0}\omega_{z_0} + \frac{g}{a^2}\sin\beta[R_1 + (R_2+a)\sin\beta] = 0. \tag{34}$$

Уравнение (34) показывает, что компоненты начальной угловой скорости зависимы при движении по параллели.

Из (34) получим

$$\omega_{z_0} = \frac{(0{,}2+\cos^2\beta)\omega_{x_0} \pm \sqrt{0{,}04\omega_{x_0}^2 - \dfrac{g}{a^2}\sin\beta\cos\beta R}}{\sin\beta\cos\beta}, \tag{35}$$

где с R обозначили величину

$$R = R_1 + (R_2+a)\sin\beta. \tag{36}$$

Очевидно, чтобы движение было реальным, начальная угловая скорость должна удовлетворять ограничению

$$\omega_{x_0} \geqq \frac{5}{a}\sqrt{gR\sin\beta\cos\beta}. \tag{37}$$

Другое ограничение для ω_{x_0} получим из условия, что проекция реакции N должна быть направлена по внешней нормали тора, т. е.

$$N.n^0 > 0. \tag{38}$$

Из уравнения движения центра шара имеем

$$N.n^0 = mg\cos\beta - m\dot\alpha^2\sin\beta R.$$

Условие (38) получает следующий вид:

$$|\dot\alpha| < \sqrt{\frac{g}{R}\operatorname{cotg}\beta}, \tag{39}$$

где R определено из (36).

Из (31) и (35) находим

$$\dot\alpha = \frac{a}{R\cos\beta}\left(0{,}2\omega_{x_0} \pm \sqrt{0{,}04\omega_{x_0}^2 - \frac{g}{a^2}R\sin\beta\cos\beta}\right). \tag{40}$$

Из (39) и (40) получаем неравенство

$$0{,}2\omega_{x_0} \pm \sqrt{0{,}04\omega_{x_0}^2 - \frac{g}{a^2}R\sin\beta\cos\beta} < \frac{\cos\beta}{a}.\sqrt{gR\operatorname{cotg}\beta}. \tag{41}$$

Неравенство (41) имеет различные решения в зависимости от знака радикала в (35). При знаке плюс (41) возможно только для угла β, удовлетворяющий условию $0 < \beta < \dfrac{\pi}{4}$.

Для ω_{x_0} получится ограничение

$$\omega_{x_0} < \frac{2{,}5}{a\cos\beta}\sqrt{gR\operatorname{cotg}\beta}\,.$$

Когда знак перед радикалом в (35) — минус, возможны два случая:

а) При $0 < \beta < \frac{\pi}{4}$ имеем условие

$$\omega_{x_0} < \frac{5\cos\beta}{a}\sqrt{gR\operatorname{cotg}\beta}\,.$$

б) При $\frac{\pi}{4} \leqq \beta < \frac{\pi}{2}$ — имеется ограничение

$$\omega_{x_0} > \frac{2{,}5}{a\cos\beta}\sqrt{gR\operatorname{cotg}\beta} \geqq \frac{5}{a}\sqrt{gR\sin\beta\cos\beta}\,.$$

Так доказана следующая

Теорема. Качение без скольжения однородного шара по абсолютно шероховатому тору, когда точка касания описывает параллель на торе, возможно только тогда, когда начальные угловые скорости шара и скорость его точки касания по параллели тора удовлетворяют следующие условия:

1) $\frac{5}{a}\sqrt{gR\sin\beta\cos\beta} \leqq \omega_{x_0} < \frac{2{,}5}{a\cos\beta}\sqrt{gR\operatorname{cotg}\beta}$,

$$\omega_{z_0} = \frac{(0{,}2+\cos^2\beta)\omega_{x_0} + \sqrt{0{,}04\omega_{x_0}^2 - \frac{g}{a^2}R\sin\beta\cos\beta}}{\sin\beta\cos\beta},$$

$$\dot{\alpha} = \frac{a}{R\cos\beta}\left(0{,}2\omega_{x_0} + \sqrt{0{,}04\omega_{x_0}^2 - \frac{g}{a^2}R\sin\beta\cos\beta}\right),$$

только для угла параллели β в интервале $\left(0, \frac{\pi}{4}\right)$.

2) $\frac{5}{a}\sqrt{gR\sin\beta\cos\beta} \leqq \omega_{x_0} < \frac{5\cos\beta}{a}\sqrt{gR\operatorname{cotg}\beta}$,

$$\omega_{z_0} = \frac{(0{,}2+\cos^2\beta)\omega_{x_0} - \sqrt{0{,}04\omega_{x_0}^2 - \frac{g}{a^2}R\sin\beta\cos\beta}}{\sin\beta\cos\beta},$$

$$\dot{\alpha} = \frac{a}{R\cos\beta}\left(0{,}2\omega_{x_0} - \sqrt{0{,}04\omega_{x_0}^2 - \frac{g}{a^2}R\sin\beta\cos\beta}\right)$$

для угла β в интервале $\left(0, \frac{\pi}{4}\right)$,

3) $\omega_{x_0} > \dfrac{2{,}5}{a\cos\beta}\sqrt{gR\operatorname{cotg}\beta}$,

$$\omega_{z_0} = \frac{(0{,}2+\cos^2\beta)\omega_{x_0} - \sqrt{0{,}04\omega_{x_0}^2 - \dfrac{g}{a^2}R\sin\beta\cos\beta}}{\sin\beta\cos\beta},$$

$$\dot\alpha = \frac{a}{R\cos\beta}\left(0{,}2\omega_{x_0} - \sqrt{0{,}04\omega_{x_0}^2 - \frac{g}{a^2}R\sin\beta\cos\beta}\right),$$

когда β удовлетворяет условию $\dfrac{\pi}{4} \leqq \beta < \dfrac{\pi}{2}$.

Здесь повсюду R определена из (36). Во всех этих случаях $\omega_{y_0} = 0$. Угловые скорости ω_x, ω_y — периодические функции времени, которые заданы из (32); ω_z — постоянная, а точка касания шара описывает параллель по тору с постоянной скоростью, задана выражением

$$v = (R_1 + R_2\sin\beta)\dot\alpha.$$

Интересно отметить, что движение по параллели однородного шара при частном движении с постоянным углом нутации невозможно по внешней стороне тора. Когда переменные углы Эйлера подвижного триэдра — это возможно, конечно со сделаными ограничениями для начальной скорости.

Наконец, рассмотрим сферическое движение шара около его центра, т. е. найдем углы Эйлера подвижного триэдра $G\xi\eta\zeta$ в функции времени. Приложим метод Дарбу — Риккати для решения этой проблемы. Согласно этому методу, рассматриваем уравнение Риккати

$$\frac{d\lambda}{dt} = \frac{i\omega_x + \omega_y}{2}\lambda^2 + i\omega_z\lambda + \frac{i\omega_x - \omega_y}{2}, \tag{42}$$

где неизвестная величина λ определяется из

$$\lambda = -i\operatorname{cotg}\frac{\theta}{2}e^{i\psi}. \tag{43}$$

Здесь ψ, θ — углы Эйлера триэдра $G\xi\eta\zeta$ — неизвестные функции времени. Заменяем в (42) величины ω_x, ω_y, ω_z, которые для рассматриваемого движения определены из (32), т. е. имеем

$$2\frac{d\lambda}{dt} = i\omega_{x_0}e^{-i\alpha}\lambda^2 + 2i\omega_{z_0}\lambda + i\omega_{x_0}e^{i\alpha}. \tag{44}$$

Уравнение (44) допускает частное решение

$$\lambda_0 = ke^{i\alpha}, \tag{45}$$

где k — постоянная, которая согласно (44) удовлетворяет уравнению

$$\omega_{x_0}k^2 + 2(\omega_{z_0} - \dot\alpha)k + \omega_{x_0} = 0. \tag{46}$$

Согласно (35) и (40) легко увидеть, что уравнение (46) при $0 < \beta \leqq \pi/4$ имеет реальные отрицательные корни. Для определенности остановимся на этом случае, т. е. когда k — вещественное число.

Частное решение (45) соответствует случаю $\theta = \text{const}$, $\psi = \alpha - \dfrac{\pi}{2}$, который мы уже рассмотрели в работе [1]. В настоящем исследовании этот случай мы должны отбросить, так как зависимость (35), в этом случае, не имеет реального решения. В действительности, с помощью формул Эйлера для ω_x, ω_y, ω_z таже зависимость здесь получает следующий вид:

$$\sin^2\beta\cos\beta\left[\frac{a}{R_1+R_2\sin\beta}\sin(\theta_0+\beta)+\cos\theta_0\right]^2\dot\psi^2$$
$$+\cos\beta(0{,}4+\cos^2\beta)\sin^2\theta_0\dot\varphi^2$$
$$+2\sin\beta(0{,}2+\cos^2\beta)\left[\cos\theta_0+\frac{a}{R_1+R_2\sin\beta}\sin(\theta_0+\beta)\right]\sin\theta_0\dot\varphi^2+\frac{g}{a^2}R\sin\beta=0. \tag{47}$$

Очевидно $\dot\varphi$ из (47) не имеет реальных значений.

Чтобы найти общее решение (44) предположим, что

$$\lambda = ke^{i\alpha}+u, \tag{48}$$

где u — новая неизвестная функция. Заменяем (48) в (44) и получим уравнение Бернули

$$\frac{du}{dt}=\frac{i}{2}\omega_{x_0}e^{-i\alpha}u^2+i(\omega_{x_0}k+\omega_{z_0})u. \tag{49}$$

Общее решение (49) имеет вид

$$u(t)=\frac{1}{D(t)}\left[(C_1-iC_2)e^{i(k\omega_{x_0}+\omega_{z_0})t}+\frac{i\omega_{x_0}}{2}\frac{e^{i\alpha}}{k\omega_{x_0}+\omega_{z_0}-\dot\alpha}\right], \tag{50}$$

где $D(t)$ — величина

$$D(t)=C_1^2+C_2^2+\frac{1}{4}\omega_{x_0}^2\frac{1}{(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)^2}$$
$$+\frac{\omega_{x_0}\left[C_1\sin(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)t-C_2\cos(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)t\right]}{k\omega_{x_0}+\omega_{z_0}-\dot\alpha} \tag{51}$$

и C_1, C_2 — произвольные действительные постоянные.

Здесь принято ввиду, что согласно (26) и (31) величина α является функцией

$$\alpha(t)=\dot\alpha t. \tag{52}$$

Согласно (43), (48), (50) и (52), находим следующие зависимости для углов Эйлера θ и ψ

$$\begin{aligned}\operatorname{cotg}\frac{\theta}{2}\sin\psi &= k\cos(\dot\alpha t)+\frac{1}{D(t)}\Big[C_1\cos(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)t\\&+C_2\sin(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)t-\frac{\omega_{x_0}\sin(\dot\alpha t)}{2(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)}\Big],\\ \operatorname{cotg}\frac{\theta}{2}\cos\psi &= -k\sin(\dot\alpha t)+\frac{1}{D(t)}\Big[C_2\cos(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)t\\&-C_1\sin(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)t-\frac{\omega_{x_0}\cos(\dot\alpha t)}{2(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)}\Big],\end{aligned} \tag{53}$$

где $D(t)$ определена из (51) и k — из (46).

Постоянные C_1 и C_2 определяются из начальных условий, применимы для (53). Предполагая, что начальные углы θ_0 и ψ_0 — заданы, получим систему

$$
(54)\quad
\begin{aligned}
C_1 &= \left(\operatorname{cotg}\frac{\theta_0}{2}\sin\psi_0 - k\right)\left[C_1^2 + C_2^2 - \frac{\omega_{x_0}}{k\omega_{x_0}+\omega_{z_0}-\dot\alpha}C_2 \right.\\
&\quad \left. + \frac{\omega_{x_0}^2}{4(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)^2}\right],\\
C_2 &= \operatorname{cotg}\frac{\theta_0}{2}\cos\psi_0\left[C_1^2 + C_2^2 - \frac{\omega_{x_0}}{k\omega_{x_0}+\omega_{z_0}-\dot\alpha}C_2 + \frac{\omega_{x_0}^2}{4(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)^2}\right]\\
&\quad + \frac{\omega_{x_0}}{2(k\omega_{x_0}+\omega_{z_0}-\dot\alpha)}.
\end{aligned}
$$

Нетрудно увидеть, что система (54) редуцируется на одно квадратное уравнение относительно одной из неизвестных. Система (53) определяет углы ψ и θ как функции времени. Третий из углов Эйлера находим из кинематических формул Эйлера, т. е. согласно (32)

$$\varphi(t) = \varphi_0 + \int_0^t \frac{\omega_{z_0} - \dot\psi}{\cos\theta}\,dt.$$

Очевидно то, что согласно (53) $\dot\psi$ и $\cos\theta$ являются знакомыми функциями времени.

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Поступила 12.04.1993 г.

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 86, 1992

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 86, 1992

ХИДРОДИНАМИЧНИ ВЗАИМОДЕЙСТВИЯ НА ТВЪРДИ ИЛИ ФЛУИДНИ ЧАСТИЦИ

ЗАПРЯН ЗАПРЯНОВ

Запрян Запрянов. ХИДРОДИНАМИЧЕСКОЕ ВЗАИМОДЕЙСТВИЕ ТВЕРДЫХ ИЛИ ЖИДКИХ ЧАСТИЦ

В начале работы сделан обзор исследований гидродинамического взаимодействия твердых и жидких частиц при малых числах Рейнольдса. Рассматриваются тоже системы, в которых микроскопический масштаб длины во много раз превышает размер молекул (так что система имеет макроскопические свойства) и одновременно с этим, во много раз меньше характерной длины макроскопического образца. При таких условиях, неоднородная среда (смесь) может рассматриваться в макроскопическом масштабе как континуум, который характеризуется „эффективными" свойствами.

В работе дискутируется тоже количественное определение осреднения по „ансамблю реализации", т. е. осреднение относительно большого числа систем, которые являются идентичными в макроскопических детайлях, но различаются в микроскопическом масштабе. Рассмотреные результаты (из-за математической аналогии) для эффективной вязкости, легко переносятся и на коэффициент теплопроводимости, коэффициент электропроводимости и коэффициент диффузии. Поступательные и вращательные движения частиц и их гидродинамическое взаимодействие описываются через несколько внутренних тензоров, которые зависят только от форм и размеров частиц.

Получено „точное" решение задачи обтекания двух сферических пузырей потокам Пуазелля. Исследуется влияние поверхностно-активных веществ на структуру течения.

Zapryan Zapryanov. HYDRODYNAMIC INTERACTION OF RIGID OR FLUID PARTICLES

The hydrodynamic interaction of rigid or fluid particles at small Reynolds numbers is reviewed. The paper focuses on the many instances in which the "microscopic" lenght scale (eg.

the average domain size) is much larger than the molecular dimensions (so that the domain possess macroscopic properties) but much smaller than the characteristic length of the macroscopic sample. In such circumstances the heterogeneous material can be viewed as a continuum on the macroscopic scale and macroscopic or "effective" properties can be ascribed to it.

A quantitative definition of an ensemble average for particulate systems is discused. The ensemble average refers to an average over a collection of a large number of systems which are identical in their macroscopic details but are different in their microscopic details. For reasons of mathematicle analogy, the general results given here for the effective viscosity translate into equivalent results for the thermal conductivity, electrical conductivity, dielectric constant and diffusion coeficient. The translational and rotational particle motions are marked to be governed by several intrinsic tensors that depend only upon the size and shape of the particle.

Bipolar coordinates are employed to obtain "exact" solution of the slow, Poisenille flow past two spherical bubbles. The influence of surface active agents on the flow is investigated.

1. ХИДРОДИНАМИЧНИ ВЗАИМОДЕЙСТВИЯ И МОДЕЛИРАНЕ НА СУСПЕНЗИИ И ЕМУЛСИИ

При движение на капки, мехури или твърди частици във вискозен флуид съществуват два вида хидродинамични взаимодействия — директно, когато всяка частица (твърда или флуидна) си взаимодействува със заобикалящия флуид, и косвено, когато частиците си взаимодействуват помежду си посредством флуида между тях. Резултатите от изследванията на хидродинамичните взаимодействия се прилагат много често при моделирането на суспензии и емулсии. Използувайки например директното взаимодействие на твърди или флуидни частици с вискозен флуид, са получени важни формули за „ефективния“ вискозитет на разредените суспензии и емулсии.

Пръв Айнщайн [1, 2] през 1906 и 1911 г. пресметна ефективния вискозитет на разредени суспензии на твърди сферични частици (дисперсна фаза) в нютонова течност (непрекъсната фаза):

$$\mu^* = \mu_0\left(1 + \frac{5}{2}\varphi\right), \tag{1.1}$$

където μ_0 е вискозитетът на непрекъснатата фаза, φ — отношението на обема на диспергираните частици към обема на суспензията, а μ^* — нейният ефективен вискозитет.

За да изведе тази формула, Айнщайн най-напред намира хидродинамичното взаимодействие на твърда сферична частица с градиентно вискозно течение. Извършените експериментални измервания на ефективния вискозитет на разредени суспензии [3, 4] показват, че коефициентът пред φ приема стойности от 1,5 до 5.

През 1922 г. Джефри [5] изследва суспензии, в които за дисперсна фаза са взети елипсовидни частици. В зависимост от елипсовидността ε (където $\varepsilon = \dfrac{a-b}{a}$ за продълговат и $\varepsilon = \dfrac{b-a}{a}$ за сплеснат елипсоид) на частиците той получава за ефективния вискозитет на суспензията формулата

$$\mu^* = \mu_0(1 + \nu\varphi), \tag{1.2}$$

където ν зависи от ε, т. е. от геометрията на елипсоида.

Ефективният вискозитет на силно разредени емулсии (сферични капки от една течност в друга течност) е пресметнат от Тейлър [6, 7, 8]:

$$\frac{\mu^*}{\mu_0} = 1 + \frac{5}{2} \cdot \frac{\hat{\mu} + \frac{5}{2}\mu}{\hat{\mu} + \mu}\varphi. \tag{1.3}$$

Тук μ и $\hat{\mu}$ са вискозитетите съответно на носещата и дисперсната фаза. Когато $\frac{\hat{\mu}}{\mu} \to \infty$, от формулата на Тейлър се получава като частен случай формулата на Айнщайн (1.1), а при $\frac{\hat{\mu}}{\mu} \to 0$, т. е. когато флуидните частици са мехури

$$\mu^* = \mu_0(1 + \varphi). \tag{1.4}$$

Формулата (1.3) е изведена при предположение, че $\varphi \ll 1$, и експерименталните изследвания показват, че тя дава добри резултати при $\varphi \ll 0{,}02$. Когато обемната концентрация не отговаря на това условие, трябва да се отчита и косвеното хидродинамично взаимодействие между флуидните частици.

Смолуховски [9] първи теоретично изследва седиментацията на частици в неограничена течност. За тази цел създава т. нар. „метод на отражението", съгласно който граничните условия върху отделните частици се удовлетворяват последователно едно след друго. Физически „методът на отражението" се интерпретира с помощта на предположението, че някакво начално смущение се отразява от границите на частиците, като ефектът при следващото отражение отслабва. При първото отражение се предполага, че в суспензията се движи само една от n-те сферични частици и граничното условие за полепване се удовлетворява само върху нея. В резултат на това възниква смутено течение, което влияе на всяка от останалите $n-1$ частици. При второто отражение граничното условие се удовлетворява само върху една от тях и т. н. Поради линейността на уравненията на Стокс и на граничните условия първото приближение на решението за i-тата частица се получава, като се съберат „първите отражения" на всички $n-1$ частици. Тъй като общото съпротивление, което изпитват частиците, се разпределя върху всяка от тях, то i-тата частица изпитва по-малко съпротивление, отколкото, ако е сама в течността. Сходимостта на „метода на отражението" не е доказана, въпреки че се използва от много изследователи. Прилагайки го, Смолуховски [9] получава в първо приближение за съпротивлението, което изпитва първата частица, формулата

$$F_1 = 6\pi\mu a U \left[1 - \frac{3}{4}\sum_{i=2}^{n}\frac{1}{r_i}\left(1 + \frac{x_i^2}{r_i^2}\right)\right],$$

където r_i е разстоянието между частиците 1 и i, x_i — проекцията на това разстояние върху оста Oz, U — скоростта на едновременно падащите частици и a — големината на радиуса им. Тъй като знакът на израза пред сумата е "$-$", то при по-голям брой n на сферите съпротивлението, което изпитва всяка от тях, ще бъде малко. Това означава, че при по-голям брой на сферичните частици скоростта им на падане ще бъде по-голяма.

За да извърши пресмятанията, Смолуховски взема центровете на частиците да съвпадат с върховете на кубическа решетка. Нека U_0 е скоростта на седиментация на коя да е от n-те частици в безкрайна течност, когато не си взаимодействува с другите частици, а l — разстоянието между две произволни частици. Тогава скоростта U на седиментация на коя да е от n-те частици, когато те падат заедно, е равна на

$$U = \frac{U_0}{1 + 2{,}6(a/l)}. \tag{1.5}$$

В съвкупност от частици, образуващи кубическа решетка, обемната концентрация на частиците е равна на $\frac{4}{3}\pi\left(\frac{a}{l}\right)^3$. Следователно $\frac{a}{l} = 0{,}62\varphi^{1/3}$ и формулата (1.5) добива вида

$$U = \frac{U_0}{1 + k\varphi^{1/3}}, \tag{1.6}$$

където $k \approx 1{,}6$.

Предполагайки, че частиците могат с една и съща вероятност да заемат всички положения около коя да е от тях, Бюргерс [10] получава същата формула (1.6), но при $k = 6{,}88$.

Интересни резултати за скоростта на седиментация на частици в суспензия получава Кинч [11]. Той разглежда подробно случаите на две и три сферични частици, като прави количествени оценки и сравнения с резултати на други изследователи. Според него в резултатите на Бюргерс влиянието на концентрацията φ е завишено, защото не са отчетени ефектите от по-висок порядък.

Интересен метод за решаване на задачата за обтичане на сферични частици, разположени във върховете на правоъгълна решетка, предлага Хасимото [12]. Правоъгълната периодична структура се състои от последователно повтарящи се правоъгълни паралелепипеди, определени от три независими вектора $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$. Ако се избере координатна система с начало в един от центровете (възлите) на частиците, радиус-векторът $\vec{r}_n$ на центъра на коя да е от останалите частици ще се определя от равенството

$$\vec{r}_n = n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3,$$

където $n_1, n_2, n_3 = 0, \pm1, \pm2, \pm3, \ldots$

Хасимото предлага оригинална идея — частиците да се заменят с концентрирани (точкови) сили, действуващи в центровете им. При моделирането на течението той модифицира уравненията на Стокс, добавяйки нов член, който описва сили на реакции, действуващи във всички възли на решетката. Така се получават уравненията

$$\mu\Delta\overrightarrow{V} = \nabla + \overrightarrow{F}\left(\sum_{n_1}\sum_{n_2}\sum_{n_3}\delta\left(\overrightarrow{r} - \overrightarrow{r}_n\right)\right), \qquad \nabla.\overrightarrow{V} = 0, \tag{1.7}$$

където $\overrightarrow{r}$ е радиус-векторът на произволна точка от течността и функцията на Дирак се задава по обикновения начин

1) $\int\limits_\tau \delta\left(\overrightarrow{r} - \overrightarrow{r}_n\right) d\tau = \begin{cases} 1 & \text{при } \overrightarrow{r}_n \in \tau \\ 0 & \text{при } \overrightarrow{r}_n \notin \tau; \end{cases}$

2) $\delta\left(\overrightarrow{r} - \overrightarrow{r}_n\right) = 0$ при $\bar{r} \neq \bar{r}_n$

за произволен обем τ.

За да намери решенията на уравненията (1.7), Хасимото търси скоростта и налягането във вид на фуриеров ред с тройно разлагане

$$\overrightarrow{V} = \sum_{n_1}\sum_{n_2}\sum_{n_3}\overrightarrow{V}_n \exp\left[-2\pi i\left(\overrightarrow{k} + \overrightarrow{r}\right)\right],$$

$$-\nabla p = \sum_{n_1}\sum_{n_2}\sum_{n_3}\overrightarrow{P}_n \exp\left[-2\pi i\left(\overrightarrow{k} + \overrightarrow{r}\right)\right].$$

Тук $\overrightarrow{k}$ може да се разглежда като радиус-вектор на взаимна решетка с нови базисни вектори $\overrightarrow{b}_1$, $\overrightarrow{b}_2$, $\overrightarrow{b}_3$, за които са изпълнени равенствата

$$\overrightarrow{k} = n_1\overrightarrow{b}_1 + n_2\overrightarrow{b}_2 + n_3\overrightarrow{b}_3, \qquad \overrightarrow{k}\,.\,\overrightarrow{a}_i = n_i,$$

където $i = 1, 2, 3$.

Като се заменят изразите за $\overrightarrow{V}$ и ∇p в уравненията (1.7), се получава система уравнения относно компонентите на векторите $\overrightarrow{V}_n$ и $\overrightarrow{P}_n$, които могат да се намерят във функция на неизвестната сила $\overrightarrow{F}$. За да определи $\overrightarrow{F}$, Хасимото поставя изискването средната скорост върху повърхността на сферичните частици да бъде равна на нула, т. е.

$$\langle\overrightarrow{V}\rangle = \frac{1}{4\pi a^2}\iint\limits_{r=a}\overrightarrow{V}\,dS,$$

и получава

$$F = \frac{6\pi\mu aU}{1 - 1{,}7601\varphi^{1/3}} + O(a^3), \tag{1.8}$$

където φ е обемната концентрация на частиците, а U — скоростта на филтрация на течението. Тъй като (1.8) може да се запише и във вида

$$F = 6\pi\mu aU\left(1 + 1{,}7601\varphi^{1/3}\right) + O\left(\varphi^{2/3}\right),$$

за скоростта на седиментация на частици, образуващи кубическа решетка, се получава

$$\frac{U}{U_0} = \frac{1}{1 + 1{,}7601\varphi^{1/3}}\,.$$

Ще обърнем внимание, че този резултат може да се използува само за силно разредени суспензии.

При определяне на вискозитета на концентрирани суспензии е необходимо да се отчита не само хидродинамичното взаимодействие между частиците, но и тяхната ротация, взаимни удари, образуване на дублети и по-сложни конфигурации. Най-голямата трудност при моделирането на по-кондензираните суспензии е, че случайната им структура не може да бъде описана посредством прост модел.

В много експериментални и теоретични изследвания зависимостта между ефективния вискозитет и обемната концентрация на частиците се представя чрез следния степенен ред:

$$\frac{\mu^*}{\mu} = 1 + k_1\varphi + k_2\varphi^2 \pm k_3\varphi^3 \pm \cdots$$

За първата константа k_1 се приема стойността, получена от Айнщайн — $k_1 = \frac{5}{2}$. Като използуват метода на отражението, в който се предполага, че допълнителното течение около втората сфера се компенсира от смущенията в течението около първата сфера, Гут и Симха [13] получават за втората константа $k_2 = 14{,}1$. Изключвайки ефекта на заемания от частиците обем, Сайто [14] получава $k_2 = 12{,}6$. По-добро отчитане на ефектите на взаимодействие получава Ванд [15], като отчита образуването на дублетите. За обемни концентрации, по-големи от 0,15 до 0,20, отрязването на степенния ред след члена с φ^2 води до грешки над 10%. Включването на члена с φ^3 дава възможност да се разшири валидността на степенния ред до $\varphi \approx 0{,}40$. Мони [16] предлага да се използува формулата

$$\frac{\mu^*}{\mu_0} = \exp\left[2{,}5(1 - a_2\varphi)\right],$$

където константата a_2 се определя от експеримента. Като изхожда от модела, основаващ се на кубично наредени частици, и от факта, че при допиращи се сферични частици относителната вискозност трябва да клони към безкрайност, Мони предлага за a_2 стойности, отговарящи на условието $1{,}35 < a_2 < 1{,}91$.

Крайният размер на частиците при по-концентрирани суспензии пречи на централната частица да взаимодействува с по-отдалечените. Затова ефектът на екраниране редуцира резултата, получен при заместване на частиците с материални точки. Като отчита това, Симха [17] получава за разредени суспензии формулата

$$\frac{\mu^*}{\mu} = 1 + 2{,}56\left(1 + \frac{25\varphi}{4f^3}\right),$$

където $1 < f < 2$.

В [18] се обръща внимание на важността на размера на частиците. За частици с диаметър, по-малък от 1 до 10 микрона, започват да действуват колоидно-химичните сили, които са причина за ненютоновото поведение на суспензиите. В резултат на това относителната вискозност расте с намаляване размерите на частиците и намалява с растенето на интензитета на градиентното течение. За частици с диаметър, по-голям от 1 до 10 микрона, увеличаването на диаметъра на частиците и въртенето им водят до допълнително увеличаване на дисипацията на енергията на относителния вискозитет.

Съществуват два начина за теоретично описание на даден материал в зависимост от линейния мащаб, в който се извършват изследванията:

а) макроскопичен, в който материалът се разглежда като сложна, но хомогенна непрекъсната среда;

б) микроскопичен, в който материалът се предполага, че е съставен от дискретни частици, които се движат транслационно и ротационно, деформират се и взаимодействуват помежду си.

Суспензиите са дисперсни среди, представляващи смес от две хомогенни среди. При течните суспензии едната от средите (фазите) е течна, а другата — частици (включвания), които могат да бъдат твърди, течни или газообразни.

Много от макроскопичните свойства на дисперсните среди могат да се предсказват теоретично. Такива са например т. нар. преносни свойства. Те характеризират способността на дисперсната система като цяло да пренася топлина, момент на количество движение и др. под влияние на градиентите на тези величини в разглежданата среда.

Ще отбележим, че въпреки големия брой статии, посветени на изследването на дисперсните среди, досега са предложени строги неевристични теории само за разредените суспензии. Изчерпателни сведения за изследванията до 1974 г. са дадени в обзорите на Бренер [19, 20, 21] и Кокс и Бренер [22].

Заменяйки приближено влиянието на съседните частици върху произволно взета частица с действието на концентрирани („точкови“) сили, действуващи в техните центрове, Хасимото [12] получава следната формула за редукцията на скоростта при седиментация:

$$\frac{\Delta U}{U_0} = 1{,}76\varphi^{1/3}, \qquad (1.9)$$

където $\Delta U = U_0 - U$, U_0 е установената (терминалната) скорост на единична частица, движеща се под действието на силата на тежестта ѝ, а U — средната скорост на седиментация на частиците от суспензията.

Теоретичните резултати за пресмятане на ефективния вискозитет на суспензии на твърди сферични частици, на твърди сфероидални частици и на сферични флуидни частици се отнасят за случая, когато взаимодействието между частиците може да се пренебрегне, т. е. при силно разредени суспензии, при които дисперсната фаза заема само няколко процента от обема на суспензията. Практиката обаче изисква създаването и на модели, отнасящи се за суспензии с по-големи концентрации. Такъв е „клетъчният модел“, предложен от Симха [23], а след това модифициран от Хапел [24, 25], Кувабара [26] и др. Основно предположение на „клетъчния модел“ е, че действието на дадена частица от суспензията върху останалите частици на течението може да се локализира вътре в клетка, съдържаща частицата и имаща външна граница с подходящо избрано гранично условие върху нея. Когато външната граница на клетката е сфера, нейният радиус се взема равен на $a\varphi^{-1/3}$. Коефициентът на пропорционалност във формулата на Кувабара [26]

$$\frac{\Delta U}{U_0} = 1{,}62\varphi^{1/3} \tag{1.10}$$

се различава от съответния коефициент във формулата на Гал–Ор и Васло [27]

$$\frac{\Delta U}{U_0} = 1{,}5\varphi^{1/3}. \tag{1.11}$$

При седиментация на капки в течност Гал–Ор и Васло [27] получават следната по-обща формула:

$$\frac{\Delta U}{U_0} = \frac{1 + \dfrac{3}{2}\lambda}{1+\lambda}\varphi^{1/3}, \qquad \lambda = \frac{\hat{\mu}}{\mu}.$$

Ще отбележим, че за друга външна граница на клетката (цилиндрична, кубична и др.) или други гранични условия върху нея коефициентът на пропорционалност при $\varphi^{-1/3}$ променя големината си. Това показва, че значителните опростявания на пресмятанията при „клетъчния модел“ стават за сметка на точността на резултатите.

Бринкмън [28] приема, че влиянието на всички частици върху дадена частица може да се моделира с влиянието, което изпитва разглежданата частица при движението ѝ в пореста среда с феноменологични коефициенти, зависещи от обемната концентрация φ на суспензията.

През 1947 г. чрез този подход Бринкмън получава формула за усреднeното въздействие, което оказва дадена частица, преминаваща през слой от много неподвижни частици. Идеите на този подход получават по-нататъшно развитие в работите на Том [29], Лундрен [30], Чилдрес [31], Хаулес [32] и на Буевич и сътрудници [33]–[36].

В работите от това направление основният резултат при определяне скоростта на седиментация е

$$\frac{\Delta U}{U_0} = \frac{3\sqrt{2}}{2}\varphi^{1/2}. \tag{1.12}$$

Тази формула е обобщена през 1977 г. от Сиркар [37] за емулсия, в която флуидните частици имат различни радиуси.

Важно предположение при прилагането на този подход е, че поведението на дадена микрочастица се влияе макроскопично при обтичането й от течението в порестата среда, състояща се също от микрочастици. Това е недостатък на този подход, защото излиза, че може да намерим поведението на дадена молекула в течение на течност, състояща се от подобни молекули, като решим уравненията, описващи обтичането на тази молекула от същото течение.

Ще отбележим, че използването на методите на самосъгласуваното поле дава възможност въпреки направената по-горе забележка да се моделират суспензии с обемна концентрация на частиците 30–40%.

В т. нар. статистически подход се предполага, че суспензията е статистически хомогенна среда, т. е. че частиците се движат свободно и центровете им могат да заемат статистически произволно място в носещата течност. Това означава, че многобройните частици на суспензията са разположени хаотично (неподредено) в носещата течност и има смисъл само статистическо описание на разположението им. Изследванията в това направление започват от Бюргерс [10]. Той използва динамично определение на вискозността посредством напрежението, вместо да се основава на дисипацията на енергията, както прави Айнщайн.

Изследвайки седиментацията на случайно разпределение на частиците в суспензия, Бюргерс [10] и Пюн и Фиксмън [38] получават, че средното относително изменение на скоростта на падане е пропорционално на φ, което се различава значително от формулите (1.9)–(1.13), получени при предположение, че частиците са подредени регулярно или че се използва „клетъчният модел“. За скоростта на седиментация на суспензия от еднакви твърди частици Батчелор [39, 40] получава формулата

$$\frac{\Delta U}{U_0} = 6{,}55\varphi. \tag{1.13}$$

Вашолдър [41] обобщава изследванията на Батчелор за случая на суспензия (емулсия) от еднакви сферични капки, но допуска грешка, която е коригирана в работата на Хабер и Хетсрони [42].

В [43] Сафмън прави пълен анализ на разликите на степените на концентрацията φ във формулите (1.9)–(1.13).

Друг метод за изследване на суспензиите предлагат през 1959 г. Ландау и Лифшиц [44]. На микро- (молекулно) равнище суспензиите имат хетерогенен състав и нестационарно движение на молекулите. За да се получат макроскопични характеристики на суспензията, е необходимо да се използва някакъв вид усредняване. Като използват обемно усредняване, Ландау и Лифшиц показват, че обемното усреднено напрежение в суспензия, съдържаща много частици, е равно на напрежението, усреднено в голям обем на една единствена тестова (пробна) частица от суспензията, намираща се вътре в него. Това означава, че средното в даден обем напрежение на суспензията може да се пресметне, без да се знаят напреженията вътре в частиците.

Конститутивното уравнение на суспензията и при този метод се получава, като се запише зависимостта между усреднения тензор на напреженията и усреднения тензор на скоростта на деформацията. За съжаление обаче при този подход (както и при подхода на Айнщайн) се появяват интеграли, които не са абсолютно сходящи.

Петерсън и Фиксмън [45] използуват метода на Ландау и Лифшиц, за да пресметнат хюйгенсовия коефициент K_H, появяващ се в разлагането на ефективната вискозност

$$\mu = \mu_0 \left[1 + \frac{5}{2}\varphi + K_H \left(\frac{5}{2}\right)^2 \varphi^2 + \cdots \right].$$

При тези изследвания те за първи път осъзнават трудностите, свързани с липсата на абсолютна сходимост на интеграли, и предлагат начин за тяхното преодоляване. Поради не съвсем ясното изложение обаче работата [45] не получава нужното признание в научната литература.

Систематичен подход за пълно преодоляване на трудностите, свързани с появата на неабсолютно сходящи интеграли при пресмятане на ефективния коефициент на вискозността на суспензиите, дава Батчелор [39, 40] [46, 47]. Като обобщава подхода на Ландау и Лифшиц, той разработва мощни методи в теорията на суспензиите. В случая на сферични частици Батчелор и Грин [46, 47] извършват всестранен анализ на взаимодействието на частиците, взети по двойки, и на плътността на функцията на вероятностното разпределение на разстоянията между сферите. За пресмятане на ефективния коефициент на вискозност те предлагат начин, посредством който се избягват трудностите, свързани с появата на неабсолютно сходящи интеграли. С точност до втори порядък $O(\varphi^2)$ те получават известната формула

$$\mu^* = \mu_0 \left(1 + \frac{5}{2}\varphi + 7{,}6\varphi^2 \right).$$

Освен за точното пресмятане на хюйгенсовия коефициент K_H подходът на Батчелор [48] е използуван и за решаване на други проблеми в теорията на суспензиите — седиментация [40], намиране на обемните макроскопични свойства на суспензии със силно удължени частици [49], хидродинамично взаимодействие на браунови частици [50] и др.

Без преувеличение може да се твърди, че прогресът в изследването на суспензиите след 1970 г. е пряко свързан с последователното използуване на подхода на т. нар. „усредняване по ансамбъл“ или „ансамблово усредняване“, използувано най-напред в теорията на суспензиите от Хашин [51] и Батчелор [39].

Друго направление в реологията на дисперсните системи са изследванията, свързани с ненютоновите свойства на течащите суспензии — зависимост на вискозитета от тензора на скоростта на деформацията, ефектите на релаксация и др., които възникват дори при малки стойности на обемната концентрация. Тези свойства на суспензиите са изследвани от Джефри [5] — за елипсоиди, Олдрид [52, 53] — за течни капки, Годард и Милер [54] и Роски [55] — за еластични сфери, Батчелор [56] — за удължени твърди частици. При тези изследвания линейният мащаб на движението на суспензията е много по-голям от размера на частиците. Затова суспензията може да се разглежда като непрекъсната среда, чиито макроскопични свойства се получават посредством ансамблово усредняване на съответните микроскопични величини. След получаване на детайлното поле около всяка частица е възможно получаването на реологично уравнение на състоянието, т. е. на функционална зависимост между напрежението и съответните физични величини. Този подход създава сериозни трудности, защото намирането на картината на течението около всяка частица е почти невъзможно. Затова за предпочитане е получаването на по-обща зависимост между напрежението и деформацията — например уравнението на Ривлин—Ериксън [57], уравнението на Олдрид [58], уравнението на Ханд [59] и др.

Досега взаимодействието между реологията на суспензиите и феноменологичните теории е твърде слабо, но може да се предполага, че в бъдеще ще се увеличи. Известни надежди за това дават работите на Гордън и Шоуолтър [60] и Ханд [61]. Сравнявайки някои феноменологични уравнения със зависимостите между напрежението и деформацията, Ханд показва, че усредненото напрежение на разредени суспензии на твърди елипсоиди, еластични сфери и течни капки удовлетворява предложеното от него феноменологично конститутивно уравнение.

Изследванията на Тейлър [6, 7] за моделиране на суспензии (емулсии) с течни капки са продължени от Шоуолтър, Чафи и Бренер [62] и Франкъл и Акривос [63]. Като използуват метода на регулярните разлагания, Франкъл и Акривос решават хидродинамичната задача за обтичане на деформируема флуидна частица от градиентен поток и чрез подхода на Батчелор [39] пресмятат усредненото напрежение на емулсията и ефективния ѝ вискозитет.

Резултатите на Франкъл и Акривос [63] са разширени от Барти–Бисел и Акривос [64] при отчитане деформацията на капките от по-висок ред. Направеният от тях анализ на реологията на разредени емулсии от деформируеми флуидни частици им позволява да получат две различни множества от уравнения — уравнение, свързващо напрежението и скоростта на деформацията посредством израз, който описва големината на анизотропията, и множество от диференциални уравнения, моделиращи изменението на анизотропията като функция на времето и скоростта на деформацията. Ще отбележим, че всички членове, участвуващи във феноменологичното уравнение на Ханд [61], възникват по естествен начин и в разгледания проблем от Франкъл и Акривос [63]. При това разредените емулсии трябва да се разглеждат като „псевдо-анизотропичен“ флуид, защото тяхната анизотропия се дължи на движението. От това следва, че тяхното конститутивно уравнение трябва да се редуцира към ненютоновия „прост флуид“.

Подобно е положението и с разредените суспензии от еластични сфери, изследвани от Годард и Милер [54] — те също са „псевдо-анизотропни“, защото анизотропията им се появява само при движение на суспензията.

Уравнението, получено от Ханд [59] през 1961 г. за суспензия с елипсовидни твърди частици, през 1970 г. е получено от Батчелор [39] посредством по-общ и по-ефективен подход. Чрез този подход средното (макроскопичното) напрежение се пресмята като „усреднено по ансамбъла на конфигурациите“ микроскопично напрежение върху репрезентативен обем V, съдържащ N частици. Така се получава

$$T_{ij} = -p\delta_{ij} + 2\mu e_{ij} + \frac{4\pi\mu}{V} \sum S_{ij},$$

където сумирането обхваща всички частици и S_{ij} е напрежението, възникнало от наличието им в суспензията.

В повечето от публикуваните досега изследвания върху суспензиите инерционните сили се пренебрегват, т. е. използуват се уравненията на Стокс. Посредством метода за срастване на асимптотичните разлагания Лин, Пери и Шоуолтър [65] въз основа на уравненията на Навие—Стокс при малки числа на Рейнолдс решават приближено задачата за обтичане на суспендирана във вискозен флуид сфера и получават картина на течението, която не е симетрична относно частицата поради действието на инерционните сили. Това води до анизотропия в течаща разредена суспензия от твърди сферични частици в условията на различно от нула число на Рейнолдс. Затова при моделирането на такъв вид течащи суспензии е необходимо да се използува обобщено уравнение на Ханд, което да включва тензор с по-висок от втори ранг. Направеният анализ и полученото разпределение на скоростта и налягането в [65] позволяват на авторите да пресметнат влиянието на инерционните сили върху реологията на суспензии от такъв вид. За ефективния вискозитет те получават формулата

$$\frac{\mu^*}{\mu} = 1 + \frac{5}{2}\varphi + 1{,}34\mathrm{Re}^{3/2}\varphi,$$

където Re е числото на Рейнолдс.

Друго ефективно преносно свойство, което притежават суспензиите, е топлопроводността. Първото изследване на влиянието на взаимодействието на частиците в суспензиите върху изменението на коефициента на топлопроводността е направено от Рейли [66] през 1892 г. Това изменение той търси във вид на ред по степените на параметъра $\frac{a}{d}$, където a е радиусът на сферичните частици, а d — средното разстояние между тях. За да определи влиянието на околните частици върху интензивността на силовия дипол, характеризиращ тестовата частица, Рейли предполага, че градиентът на силата в центъра на тази частица е равен на нейния температурен градиент плюс сумата от температурните полета, индуцирани от останалите сферични частици. При пресмятането на тази безкрайна сума той стига до интеграл, който не е абсолютно сходящ. Преодоляването на тази трудност е обект на изследванията на Левин [67], Зазовски и Бренер [68], Маккинзи и Макфердан [69] и др. Както Рейли, така и споменатите автори правят обаче идеализираното (грубото) предположение, че частиците в суспензията са регулярно подредени. Ефективната топлинна и електрическа проводимост на суспензии, в които частиците не са подредени, а са хаотично разположени, е изследвана от Джефри [70, 71]. За преодоляване на трудностите, свързани с получаването на интеграли, които не са абсолютно сходящи, той използува подхода на Батчелор. Друг метод за пресмятане на ефективните (преносни) свойства на суспензии с взаимодействуващи случайно разположени частици е предложен от О'Брейн [72].

Наред с монодисперсните смеси в много химически технологии се използуват и полидисперсните смеси. Тяхното изучаване започва с работите на Том [29], Буевич и Марков [35] и Лундрен [30].

През 1976 г. Батчелор [73] изследва движението на полидисперсна суспензия, като отчита хидродинамичното взаимодействие на частиците при брауновото им движение. Обобщавайки метода на изследване на седиментацията на частиците на монодисперсна суспензия [40], Батчелор [74] пресмята допълнителната скорост на частицата от вида i, дължаща се на наличието на частицата от вида j, посредством функциите, характеризиращи тяхната мобилност. Той намира плътността на вероятността $f_{ij}(\overrightarrow{r})$ на относителното разположение на i-тата и j-тата частица, като решава диференциално уравнение от вида на уравнението на Фокер – Планк. Това уравнение описва ефектите на относителното движение на тези частици, дължащо се на гравитацията, на силата на тяхното взаимодействие и на брауновата дифузия.

Усреднените скорости на двата вида частици, отбелязани с индекси i и j, се получават от общите формули:

$$\langle U_i \rangle = U_i^{(0)} (1 + S_{ii}\varphi_i + S_{ij}\varphi_j)\,,$$

$$\langle U_j \rangle = U_j^{(0)} (1 + S_{ji}\varphi_i + S_{jj}\varphi_j)\,.$$

Тук S_{ii} и S_{jj} описват хидродинамичното взаимодействие между частици от един и същ вид; S_{ij} зависят от $\lambda = \dfrac{a_j}{a_i}$ и $\gamma = \dfrac{\rho_j - \rho}{\rho_i - \rho}$, където ρ е плътността на течната среда.

Получените в [74] формули, отчитащи влиянието на хидродинамичното взаимодействие между две частици в разглежданата статистически хомогенна, разредена полидисперсна суспензия, са изследвани числено от Батчелор и Уин [75]. За суспензия, при която двата вида частици се характеризират с $\lambda = \dfrac{1}{4}$ и $\gamma = 1$, за средните скорости Батчелор и Уин получават

$$\langle U_i \rangle = U_i^{(0)} (1 - 6{,}55\varphi_i - 3{,}83\varphi_j)\,,$$

$$\langle U_j \rangle = U_j^{(0)} (1 - 24{,}32\varphi_i - 6{,}55\varphi_j) .$$

Използвайки подхода на Батчелор, Филибойс [76] разглежда суспензия, която не е хомогенна във вертикално направление, и получава формула за средната скорост на седиментация на сферичните частици. Ако суспензията е хомогенна, като частен случай от тази формула се получава формулата (9.1.38) на Батчелор. Квалифициран обзор на изследванията по седиментация на неколоидални частици в суспензии е направен от Акривос [77] през 1985 г.

Интересът към изследването на хидродинамичното взаимодействие на браунови частици напоследък нарасна много. Това се дължи на голямото му влияние в разтвори с големи макромолекули, където отчитането му води до значителни отклонения в сравнение с разглеждането на всяка браунова частица поотделно. От хидродинамична гледна точка от голямо значение е влиянието на брауновото движение върху усреднените характеристики на суспензиите — напрежение, вискозитет, топлопроводност и др.

Известно е, че ориентацията на несферичните частици (например елипсовидни частици) в разредена суспензия зависи от вида на течението, в което тя участвува. За разлика от флуидното течение, което „подрежда" в определен ред частиците от суспензията, брауновото движение води до случайното им разполагане. Взаимодействието на тези два ефекта се описва от уравнението на Фокер – Планк. В частност коефициентът k_1 във формулата $\mu^* = \mu(1 + k_1\varphi)$ за ефективния вискозитет зависи от формата на частиците и вида на течението, в което те участвуват. Батчелор [78] показа, че брауновото движение при разредена суспензия на сферични частици не влияе на коефициента пред първата степен на φ във формулата за ефективния вискозитет. Той установи, че за такива суспензии при чисто деформационно течение хюйгенсовият коефициент е равен на 6,2 вместо на 7,6, когато не се отчита влиянието на брауновото движение. Използвайки метода на т. нар. „точкова апроксимация", Фелдерхот [79] също изследва влиянието на хидродинамичното взаимодействие при браунови частици в суспензии и получава резултати, които са много близки до резултатите на Батчелор.

Първото системно изследване на реологията на разредени суспензии от еднакви, достатъчно малки частици с произволна форма, за които брауновите двоици са от значение, е извършено от Жисикус [80] през 1962 г. Влиянието на брауновото движение върху реологичните свойства на суспензии с несферични частици е изследвано и от Хинч и Лил [81] и Лил и Хинч [82]. Влиянието на брауновото въртене на еднакви частици с произволна форма в разредена суспензия върху реологията ѝ е изследвано от Ралисън [83]. Като се основава на подхода на Батчелор, Ралисън извежда конститутивните уравнения, описващи нестационарното течение, индуцирано от браунови двоици, действуващи върху разредената суспензия на еднакви частици с произволна форма. Пълен обзор на изследванията до 1981 г., свързани с взаимодействието между частиците при брауновото им движение в суспензиите, е направен от Ръсел [84]. Стабилността (устойчивостта) на суспензиите е от голямо значение за много технологични процеси. Под стабилност на коя да е система, включително и на суспензиите, се разбира способността ѝ да запазва непроменено състоянието си в целия обем. Нестабилността на грубо дисперсните системи се дължи главно на значителната скорост на утаяване на частиците им под действието на гравитацията. От голямо значение е процесът на непосредствено слепване (обединяване) на частиците при ударите между тях. Този процес се нарича коагулация или флокулация.

При удар на две частици между тях действуват сили на привличане и сили на отблъскване. Докато първите са обикновено от вандерваалсов тип, вторите се приема, че се дължат на взаимодействието на заредените повърхности на частиците. Когато преобладават силите на привличане, стабилността е малка и, обратно, нарастването на силите на отблъскване повишава стабилността на суспензията. При коагулация се образуват двойни частици, които при среща с единична дават тройни, а при среща с двойни — агрегати от четири частици и т. н. През 1917 г. Смолуховски [85] изчислява изменението на броя на частиците с времето независимо от големината им. Предполагайки, че коалисценцията между две флуидни частици настъпва веднага щом разстоянието между тях стане много по-малко от радиуса на по-голямата от тях, Хоскинг и Джонс [86] пресмятат скоростта на образуване на агрегати при процеса на седиментация на частиците. Скоростта на коагулация на суспензии, извършващи просто градиентно движение, е пресметната от Картис и Хоскинг [87] и Ван де Вен и Мейсън [88], а при чисто деформационно движение — от Зихнер и Шоуолтър [89]. За разлика от тези изследвания през 1983 г. Фик и Шоуолтър [90] изследват и влиянието на слабо брауново движение на частиците върху скоростта на коагулация както при просто градиентно течение, така и при чисто деформационно течение. Скоростта на образуване на дублети при липса на конвективна скорост на движение на частиците е изследвана от Дерягин и Мюлер [91]. През 1984 г. Шоуолтър [92] прави обзор на изследванията по устойчивост и коагулация на колоиди в градиентни течения. Нов метод за изследване скоростта на коагулация в разредена суспензия от малки частици предлагат Уин и Батчелор [93]. Този метод включва определянето на плътността на вероятността от диференциалното уравнение в „граничния слой“, където са сравними по големина силите, привеждащи суспензията в движение, и вандерваалсовите сили на привличане на частиците.

Наред с другите въпроси в обзорите на Джефри и Акривос [94] и на Херченски и Пиенковска [95] се разглежда и проблемът за използването на вариационни принципи за намиране на долна и горна граница на изменението на скаларните параметри (ефективен вискозитет и ефективен коефициент на топлопроводността), характеризиращи преносните свойства на суспензиите. Заслужава да се спомене работата на Хашин [51] за възможните математически граници, между които се намира ефективният вискозитет на изотропна суспензия на частици с произволна форма. Ако вискозитетът на средата вътре в тях е равен на $\hat{\mu}$, а повърхностното напрежение на междуфазовите граници е равно на нула, за долната и горната граница Хашин съответно получава

$$\mu + \frac{\varphi\mu}{\frac{2}{5}(1-\varphi) + \mu(\hat{\mu}-\mu)}, \qquad \hat{\mu} + \frac{(1-\varphi)\hat{\mu}}{\frac{2}{5}\varphi + \hat{\mu}(\mu-\hat{\mu})}.$$

В [96] Уолпол изказва съображения, позволяващи да се предположи, че сферичната форма на частиците в изотропна суспензия е екстремен случай, така че ако една от границите на Хашин се реализира, това е със сигурност суспензия със сферични частици. Границите, в които може да се изменя ефективния вискозитет на изотропна, не обезателно разредена суспензия със сферични флуидни частици с вътрешен вискозитет $\hat{\mu}$, са пресметнати от Келер, Рубенфелд и Molиньо [97].

Интересно е да се отбележи, че долната граница на ефективния вискозитет при малки стойности на обемната концентрация на суспензиите със сферични частици съвпада с формулата (9.1.1) на Айнщайн. Коментар на резултатите, отнасящи се до долната и горната граница на ефективния вискозитет на суспензиите, *може да се намери в обзора на Хашин* [51] и монографията на Беран [98].

Като се използува известната аналогия между механиката на твърдото деформируемо тяло и механиката на флуидите, граници на Хашин и Стрикмън [99] за еластични модули на композитните материали могат да се получат от границите за ефективния вискозитет на нютонови суспензии. Келер, Рубенфелд и Молиньо [97] намират горната и долната граница на измененията на ефективния вискозитет за суспензии, в които частиците са подредени. Общият случай за намиране границите на изменение на суспензии, в които частиците са разположени хаотично, досега не е решен в научната литература.

В обзорната статия на Голдсмит и Мейсън [100] се анализират изследванията на много автори, които разглеждат кръвта като суспензия, състояща се от кръвна плазма и еритроцити. В тези и много други изследвания еритроцитите се моделират като твърди частици — [101], [102], [103]; капки — [104], [105], [106]; еластични частици — [107], [108], [109], или капсули — [110], [111], [112].

Първите опити за моделиране на нехомогенни дисперсни среди са направени въз основа на хипотезата за взаимнопроникващи многоскоростни континууми [113]. Впоследствие към дисперсните системи започват да се прилагат методите на кинетичната теория на газовете [114]. Широко разпространение получават изследванията, в които резултатите от кинетичната теория на газовете се прилагат непосредствено за описание само на отделни параметри на дисперсните среди [115]. В цикъл от изследвания Струмински и сътрудниците му [116] прилагат също кинетичната теория на газовите смеси за моделиране на разредени нехомогенни дисперсни среди. През 1972 г. Садър и Ли [117] дават статистическо описание на случайно движение на сферични частици в концентрирана суспензия и получават някои резултати за нейния ефективен вискозитет. Те показват, че:

1. Функцията, характеризираща разпределението на частиците в суспензията, не зависи от техните линейни и ъглови скорости, а единствено от положението им.

2. Ефективният коефициент на суспензията не зависи от константата S, характеризираща интензивността на градиентното течение.

3. Ефективният коефициент може да се запише във вид на ред по степените на обемната концентрация φ на частиците, като n-тата степен на φ в реда отговаря на хидродинамичното взаимодействие, включващо n частици.

4. За да се изчислят коефициентите в развитието на ефективния вискозитет в ред по степените на φ, трябва да се познава скоростта на флуида около повърхността на сферичната частица, движеща се заедно със своите съседни частици в градиентно течение.

Тъй като точното определяне на тази скорост, когато си взаимодействуват повече от две частици, е практически невъзможно, Садър и Ли използват метода на „тегловните остатъци“ за получаване на приближено решение за скоростното поле при кубично разпределение на частиците. Разпределението на разстоянието между частиците те намират, като използват приближен статистически модел.

В по-концентрирани суспензии стават важни не само взаимодействията между две частици, но и взаимодействията между три, четири и т. н. частици. При много големи концентрации се стига до уплътнени слоеве от частици, където взаимодействието на много частици доминира над движението на частиците. Определянето на ефективния вискозитет на силно концентрирани суспензии следователно е сложна задача, защото изисква: 1) пресмятане скоростта на флуида на дадена конфигурация с отчитане взаимодействието на много частици и 2) статистически анализ на вероятностите на положението на частиците и техните скорости.

Христов и Марков [119] въз основа на изследванията на Огура [118] прилагат друг статистически подход.

Първите теоретични изследвания на вискозитета на концентрирани суспензии (Симха [23], Хапел [25] и Кинч [11]) са описани подробно в монографията на Хапел и Бренер [120]. Асимптотичното поведение на зависимостта между ефективния вискозитет и концентрацията на дадена суспензия със сферични частици е изучено от Франкъл и Акривос [121]. За целта те изследват подробно вискозното течение на носещия флуид на суспензията в малък процеп между две близко разположени сферични частици. Предполага се, че частиците образуват кубична решетка, чиито ребра са насочени успоредно на главните оси на тензора на напрежението.

2. УСРЕДНЯВАНЕ И ТЕНЗОР НА СЪПРОТИВЛЕНИЕТО И МОБИЛНОСТТА НА ДВИЖЕЩИ СЕ ЧАСТИЦИ В СУСПЕНЗИИ

В механиката на хетерогенните среди се изследват макроскопичните свойства на суспензиите, двуфазните течения и композитните материали по зададени свойства на техните компоненти. Трудностите, които възникват при математическото моделиране на суспензиите, произтичат от това, че техните макроскопични свойства зависят не само от макроскопичната структура на компонентите им, но и от тяхното хидродинамично взаимодействие и процесите, протичащи в околността на отделните включвания (частици). Тъй като относителното положение и ориентацията на отделните частици в различните части на дисперсните системи се изменят случайно, най-подходящо е макроскопичните им свойства да се описват статистически. Широкото навлизане на методите на теорията на вероятностите при изучаване на различните видове суспензии е трайна съвременна тенденция.

В механиката на флуидите се използват няколко различни начина за усредняване с цел да се получат уравнения, които не съдържат детайли от разглежданите течения, а описват само основни техни закономерности. Усредняване на хидродинамичните и газодинамични величини и съответните уравнения се извършва в теорията на турбулентността, кинетичната теория на газовете, динамиката на многофазните среди и др. Известни са следните видове усреднявания — усредняване относно времето, относно обем от разглежданата среда, относно „ансамбъл от реализации", асимптотично усредняване и др.

Под „ансамбъл" се разбира множество от голям брой системи, които са различни относно някои техни микроскопични детайли, но са идентични в макроскопичен смисъл. В механиката на хетерогенните среди под „реализация" на дадена суспензия, съдържаща N сферични частици, се разбира моментната конфигурация на системата от N-те частици, определена от разположението на радиус-векторите на центровете им. „Ансамбъл на реализациите" се нарича съвкупността от всички физически допустими конфигурации, които се приемат за равноправни. При усредняването по „ансамбъл от реализации" радиус-векторите на центровете на частиците се разглеждат като независими.

Понятията „ансамбъл" и усредняване относно „ансамбъл" са въведени в термодинамиката от Гибс [122].

Целесъобразността от използването на усредняване по „ансамбъл от реализации" при математическото моделиране на хетерогенните среди е подчертавана през последните години от много учени [123, 124].

За да бъде полезно и удобно за прилагане, усредняването трябва да удовлетворява следните условия:

1) $\langle F+G\rangle = \langle F\rangle + \langle G\rangle$;

2) $\langle\langle F\rangle.G\rangle = \langle F\rangle.\langle G\rangle$;

3) $\langle C\rangle = C \quad (C = \text{const})$;

4) $\left\langle \dfrac{\partial F}{\partial t} \right\rangle = \dfrac{\partial}{\partial t}\langle F\rangle$;

5) $\left\langle \dfrac{\partial F}{\partial x} \right\rangle = \dfrac{\partial}{\partial t}\langle F\rangle$.

Тук средните стойности на функциите F и G са означени с ъглови скоби. Първите три условия се наричат правила на Рейнолдс, четвъртото — правило на Лайбниц и петото — правило на Гаус.

Известно е, че в статистическата механика цялата информация за вероятността за динамичното състояние на частиците, т. е. за тяхното разположение и скорост, се съдържа във функцията на плътността на вероятностите. Всички макроскопични свойства на системата от частици се изразяват като интегрални моменти от тази функция. Аналогично е положението и при суспензиите.

Да разгледаме течна суспензия с обем V, която съдържа N твърди сферични частици с радиус a. При движението на суспензията частиците се движат една спрямо друга случайно по сложни траектории, които могат да се опишат само статистически. Да означим с $\vec{r}_N$ радиус-векторите на центровете на частиците, имащи общо начало в точката $\vec{r}_0$, и с C_N — множеството от радиус-векторите $\vec{r}_N$ на центровете на N-те сфери при коя да е тяхна реализация. Множеството C_N ще наричаме конфигурация на N-те сфери. Ако $f(C_N)$ е функцията на плътността на вероятностите, тогава вероятността за едновременно разполагане на центровете на сферичните частици в диференциалните обеми δv_1 около $\vec{r}_0 + \vec{r}_1$, δv_2 около $\vec{r}_0 + \vec{r}_2$, ... и δv_n около $\vec{r}_0 + \vec{r}_n$ е равна на

$$f(C_N)\,\delta C_N = f(r_1, r_2, \ldots, r_n)\,\delta v_1 \delta v_2 \ldots \delta v_N.$$

Главната идея е законите за запазване и реологичните зависимости, които са в сила при някакво произволно положение (реализация) на N-те частици (т. е. при произволна конфигурация C_N на частиците), да се усредняват по ансамбъла на всевъзможните конфигурации на радиус-векторите на частиците в суспензията. Този ансамбъл се описва напълно от плътността на вероятността $f(C_N)$ на конфигурациите на N-те частици в $3N$-мерното фазово пространство на компонентите на радиус-векторите им $\vec{r}_1$, $\vec{r}_2$, ..., $\vec{r}_N$. Поради хомогенността на разположението на частиците в суспензията функцията $f(C_N)$ не зависи от радиус-вектора $\vec{r}_0$, който е начална точка за множеството на конфигурациите C_N. Зависимостта на конфигурацията C_N от $\vec{r}_0$ трябва да се задава само когато C_N се описва експлицитно.

Ако $\vec{r}_0$ е радиус-векторът на центъра на допълнителната (пробна) частица, може да се разглежда плътността на условната вероятност на конфигурациите, при които центърът на допълнително зададената частица не се променя, т. е. $\vec{r}_0 = \overrightarrow{\text{const}}$. Тази условна вероятност се бележи с $f(C_N \mid r_0)$.

Тъй като N-те частици в обема са неразличими, функциите $f(C_N)$ и $f(C_N|r_0)$ могат да се нормират посредством равенствата

$$\underbrace{\iint\ldots\int}_{3N} f(C_N)\,dC_n = \underbrace{\iint\ldots\int}_{3N} f(C_N|r_0)\,dC_n = N!.$$

Тук всеки от $3N$-мерните интеграли се пресмята по всевъзможните положения на радиус-векторите на центровете на N-те частици, намиращи се в обема V, т. е. прието е означението

$$dC_N = \delta v_1 \delta v_2 \ldots \delta v_N = d^3r_1 d^3r_2 \ldots d^3r_N = dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 \ldots dx_N dy_N dz_N.$$

Плътността на вероятността на „конфигурациите" на радиус-вектора на една сфера се дава с равенствата

$$f(C_1) = f(\vec{r}_0 + \vec{r}_1) = \frac{N}{V} = n,$$

където n е средният брой сфери в единица обем. Освен това връзката между условната и безусловната вероятност има следния вид:

$$f(C_N|\vec{r}_0) = f(\vec{r}_0 + \vec{r}_k)\ldots f\left[C_{N-1} \mid (r + r_k)\right] = nf\left[C_{N-1} \mid (r + r_k)\right],$$

където $\vec{r}_0 + \vec{r}_k$ е радиус-векторът на центъра на произволна сферична частица от конфигурацията C_N. Верни са и равенствата

$$f(C_N \mid \vec{r}_0) = f(\vec{r} + \vec{r}_k \mid \vec{r}_0).f\left[C_{N-1} \mid (\vec{r}_0, \vec{r}_0 + \vec{r}_k)\right].$$

Ако центровете на частиците са отдалечени на разстояния, много по-големи от a, може да приемем, че вероятностите на разположение на центровете на две частици, разстоянието между които е много по-голямо от a, са независими. Тогава, ако всяка от сферичните частици на конфигурацията C_N се намира на разстояние от частицата с център в точката $\vec{r}_0$, което е много по-голямо от a, то

$$f(C_N|r_0) \approx f(C_N).$$

Ако е известна вероятностната плътност $f(C_N)$ на реализациите на конфигурациите C_N на радиус-векторите $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ в суспензията, то под „усреднени стойности по ансамбъл на конфигурациите C_N" на макроскопичните величини (скорост, напрéжение и скорост на деформация) разбираме

$$\langle\vec{v}(r)\rangle = \frac{1}{N!}\int \vec{v}(r|C_N)f(C_N)\,dC_N,$$

$$\langle T(r)\rangle = \frac{1}{N!}\int T(r|C_N)f(C_N)\,dC_N$$

и

$$\langle E(r)\rangle = \frac{1}{N!}\int E(r|C_N)f(C_N)\,dC_N.$$

В подинтегралните изрази $\vec{v}(r|C_N)$, $T(r|C_N)$ и $E(r|C_N)$ са стойностите на разглежданите величини (скорост, напрежение и скорост на деформацията) в точката с радиус-вектор $\vec{r}$, когато конфигурацията на центровете има вида C_N. Тези функции трябва да са дефинирани за радиус-вектора $\vec{r}$, отнасящ се за точки, лежащи както вън, така и вътре в частицата.

Усреднението на функцията $H(r_0|C_N)$ по ансамбъл на конфигурациите на радиус-векторите на центровете на частиците в една суспензия, което е свързано с пробна (фиксирана) частица с център в точката $\vec{r}_0$, се дефинира, като се използува плътността на условната вероятност $f(C_N|r_0)$ чрез равенството

$$\langle H(\vec{r}_0)\rangle = \frac{1}{N!}\int H(r_0|C_N)f(C_N|r_0)\,dC_N.$$

Стойността на физичните величини вътре или вън от частиците на суспензията може да се специализира посредством т. нар. характеристична структурна функция

$$\theta(\vec{r}|C_N) = \theta(\vec{r}, \vec{r}_1, \vec{r}_2, \vec{r}_2, \ldots, \vec{r}_N) = 1 - \sum_{i=1}^{N} \eta(\vec{a} - |\vec{r} - \vec{r}_i|),$$

където $\eta(x)$ е стъпаловидната функция на Хевисайд

$$\eta(x) = \begin{cases} 1 & \text{при } x \geqq 0 \\ 0 & \text{при } x < 0 \end{cases}$$

и $\vec{r}_i(t)$ са радиус-векторите на центровете на частиците. Характеристичната функция е равна на нула за точки вътре в частиците $(|\vec{r} - \vec{r}_i| \leqq a)$ и единица — за точки вън от частиците $(|\vec{r} - \vec{r}_i| > a)$.

Поради независимостта на функциите на плътността на вероятностите $f_N(C_N)$ от пространствената променлива r операторът за „ансамблово усредняване" е комутативен относно диференцирането по r, т. е.

$$\left\langle \frac{\partial G}{\partial r} \right\rangle = \frac{\partial}{\partial r}\langle G\rangle.$$

Освен това, като вземем предвид, че макроскопичният времемен мащаб τ превишава много микроскопичния времемен мащаб T, получаваме, че с точност $O(\tau/T)$ операторът на „ансамбловото усредняване" е комутативен и с диференцирането относно времето:

$$\left\langle \frac{\partial G}{\partial t} \right\rangle = \frac{\partial}{\partial t}\langle G\rangle.$$

Понякога се използува „ансамблово усредняване" отделно за всяка от двете фази съгласно формулите

$$\langle \theta G\rangle = \int \theta G f_N(t, C_N)\,dC_N,$$

$$\langle(1-\theta)G\rangle = \int (1-\theta) G f_N(t, C_N)\, dC_N.$$

В сила е комутативното свойство на оператора на „фазовото ансамблово усредняване“ относно диференцирането както по отношение на пространствената променлива r, така и по отношение на времето t.

За определяне на хидродинамичното взаимодействие между частиците е необходимо да се дефинират т. нар. тензор на съпротивлението и тензор на мобилността (подвижността). Когато са дадени силите $\overrightarrow{F}$ и моментите им $\overrightarrow{M}$, които действат върху частиците от суспензията, и се търси релативното движение на последните във флуида, казва се, че се разглежда проблемът за мобилността на частиците. Обратно, когато е известно релативното движение на частиците и се търсят съпротивлението $\overrightarrow{F}$ и моментът $\overrightarrow{M}$, които изпитват частиците при движението им във флуида, казва се, че се разглежда проблемът за съпротивлението.

Да означим с $\overrightarrow{U}$ и $\overrightarrow{\omega}$ транслационната и ротационната скорост на дадена частица в неподвижен флуид и с U^∞ и Ω^∞ — транслационната и ротационната скорост на флуида в безкрайност. Тогава скоростта $\overrightarrow{v}^0(\overrightarrow{r})$ на смутеното движение на флуида върху повърхността S_p на частицата е

$$\overrightarrow{v}^0(\overrightarrow{r}) = \overrightarrow{U} - \overrightarrow{U}^\infty + \left(\overrightarrow{\omega} - \Omega^\infty\right) \times \overrightarrow{r} - E^\infty.\overrightarrow{r}, \tag{2.1}$$

където E^∞ е тензорът на скоростта на деформацията на несмутения в безкрайност поток. Ако частицата се движеше само с транслационна скорост $U - U^\infty$, поради линейността на уравненията на Стокс връзката между съпротивлението $\overrightarrow{F}$ и скоростта $U - U^\infty$ е

$$\overrightarrow{F} = \mu A.\left(\overrightarrow{U}^\infty - \overrightarrow{U}\right), \tag{2.2}$$

където A е тензор от втори ранг, зависещ от формата на движещата се частица, а μ — вискозитетът на флуида.

Когато скоростта на смутения поток $\overrightarrow{v}^0(\overrightarrow{r})$ на повърхността на частицата има вида (2.1), равенството (2.2) се обобщава чрез следната линейна зависимост между моментите и параметрите на течението:

$$\begin{pmatrix} \overrightarrow{F} \\ \overrightarrow{M} \\ S \end{pmatrix} = \mu \begin{pmatrix} A & \widetilde{B} & \widetilde{G} \\ B & C & \widetilde{H} \\ G & H & T \end{pmatrix} \begin{pmatrix} U^\infty - U \\ \Omega^\infty - \omega \\ E^\infty \end{pmatrix}. \tag{2.3}$$

В това равенство A, B и C са тензори от втори ранг, G и H — тензори от трети ранг, T е тензор от четвърти ранг, а величината S се нарича стокслет на течението. Квадратната матрица в равенството (2.3) се нарича матрица на съпротивлението. От правилата за умножение на елементите на матрицата в (2.3) следват правилата за получаване на вътрешните тензорни произведения, т. е.

$$
\begin{aligned}
&F_i = \mu A_{ij}\left(U_j^\infty - U_j\right) + \mu \widetilde{B}_{ij}\left(\Omega_j^\infty - \omega_j\right) + \mu \widetilde{G}_{ijk} E_{jk}^\infty,\\
&S_{ij} = \mu G_{ijk}\left(U_k^\infty - U_k\right) + \mu H_{ijk}\left(\Omega_k^\infty - \omega_k\right) + \mu T_{ijkl} E_{kl}^\infty.
\end{aligned} \tag{2.4}
$$

При проблема за мобилността на частиците може да се запише равенството

$$
\begin{pmatrix} \vec{U}^\infty - \vec{U} \\ \Omega^\infty - \omega \\ \mu^{-1} S \end{pmatrix} = \begin{pmatrix} a & \widetilde{b} & \widetilde{g} \\ b & c & \widetilde{h} \\ g & h & m \end{pmatrix} \begin{pmatrix} \mu^{-1} F \\ \mu^{-1} M \\ E^\infty \end{pmatrix}. \tag{2.5}
$$

В това равенство a, b, c са тензори от втори ранг, g, h — тензори от трети ранг, а m е тензор от четвърти ранг. Квадратната матрица в (2.5) се нарича матрица на мобилността.

Ще отбележим, че ако се изключи от разглеждане стокслетът S, между двете споменати матрици съществува зависимост. Наистина от

$$
\begin{aligned}
\begin{pmatrix} \vec{F} \\ \vec{T} \end{pmatrix} &= \mu \begin{pmatrix} A & \widetilde{B} \\ B & C \end{pmatrix} \begin{pmatrix} \vec{U}^\infty - \vec{U} \\ \vec{\Omega} - \vec{\omega} \end{pmatrix} + \mu \begin{pmatrix} \widetilde{G} \\ \widetilde{H} \end{pmatrix} (E^\infty) \\
&= \mu \begin{pmatrix} A & \widetilde{B} \\ B & C \end{pmatrix} \left[\begin{pmatrix} a & \widetilde{b} \\ b & c \end{pmatrix} \begin{pmatrix} \mu^{-1}\vec{F} \\ \mu^{-1}\vec{M} \end{pmatrix} + \begin{pmatrix} \widetilde{g} \\ \widetilde{h} \end{pmatrix} (E^\infty) \right] + \mu \begin{pmatrix} \widetilde{G} \\ \widetilde{H} \end{pmatrix} E^\infty
\end{aligned}
$$

следват равенствата

$$
\begin{pmatrix} A & \widetilde{B} \\ B & C \end{pmatrix} \begin{pmatrix} a & \widetilde{b} \\ b & c \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}, \qquad \begin{pmatrix} A & \widetilde{B} \\ B & C \end{pmatrix} \begin{pmatrix} \widetilde{g} \\ \widetilde{h} \end{pmatrix} + \begin{pmatrix} \widetilde{G} \\ \widetilde{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Това означава, че

$$
\begin{pmatrix} a & \widetilde{b} \\ b & c \end{pmatrix} = \begin{pmatrix} A & \widetilde{B} \\ B & C \end{pmatrix}^{-1}, \qquad \begin{pmatrix} \widetilde{g} \\ \widetilde{h} \end{pmatrix} = -\begin{pmatrix} A & \widetilde{B} \\ B & C \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{G} \\ \widetilde{H} \end{pmatrix}.
$$

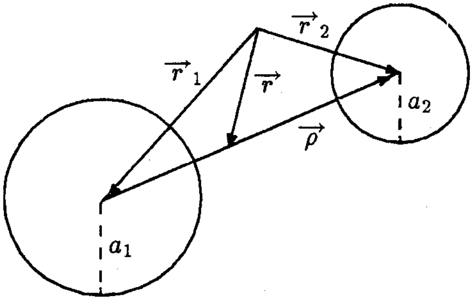
След известни преобразувания получаваме

$$
\begin{aligned}
&a = \left(A - B^t C^{-1} B\right)^{-1},\\
&b = -C^{-1} B \left(A - B^t C^{-1} B\right)^{-1},\\
&c = \left(C - B A^{-1} B^t\right)^{-1}.
\end{aligned} \tag{2.6}
$$

Да разгледаме сега две сферични частици, които под действие на външни сили и техните моменти се движат транслационно и ротационно във вискозен флуид. Ако на частиците не действуват моменти, то поради линейността на уравненията на Стокс за скоростта на двете частици можем да запишем

$$
\vec{U}_i = \omega_{i1} \vec{F}_1 + \omega_{i2} \vec{F}_2 \quad (i = 1, 2). \tag{2.7}
$$

Стойностите на компонентите на тензора ω_{ij}, характеризиращ влиянието на j-тата частица върху i-тата, зависят от разстоянието между тях



Фиг. 1

(фиг. 1). Като вземем предвид, че относителното движение на двете сферични частици и движението на техния център на масите се дефинират чрез равенствата

$$\frac{d\vec{S}}{dt} = \vec{U}_2 - \vec{U}_1, \qquad \frac{d\vec{r}}{dt} = \frac{\vec{U}_1 + \vec{U}_2}{2}, \tag{2.8}$$

можем да запишем формулите

$$\frac{d\vec{s}}{dt} = (\omega_{21} - \omega_{11})\vec{F}_1 + (\omega_{22} - \omega_{12})\vec{F}_2, \qquad \frac{d\vec{r}_c}{dt} = \frac{\omega_{21} + \omega_{11}}{2}\vec{F}_1 + \frac{\omega_{22} + \omega_{12}}{2}\vec{F}_2. \tag{2.9}$$

От физични съображения е ясно, че когато разстоянието между частиците расте неограничено, хидродинамичното взаимодействие между тях трябва да изчезва, т. е. при $i \neq j$

$$\lim_{r \to \infty} \omega_{ij} = 0, \tag{2.10}$$

а при $j = i$

$$\lim_{r \to \infty} \omega_{ii} = \frac{\delta}{6\pi\mu a_i}. \tag{2.11}$$

От получените резултати за динамиката на две сферични частици следва, че за тензора на мобилността ω_{ij} може да се запише формулата

$$\omega_{ij} = \frac{1}{3\pi\mu(a_1 + a_2)} \left[A_{ij}(\rho) \frac{\vec{\rho}\vec{\rho}}{\rho^2} + B_{ij}(\rho) \left(\delta - \frac{\vec{\rho}\vec{\rho}}{\rho^2} \right) \right], \tag{2.12}$$

като при $i \neq j$ съгласно (2.10) и (2.11)

$$\lim_{\rho \to \infty} A_{ij}(\rho) = \lim_{\rho \to \infty} B_{ij}(\rho) = 0, \tag{2.13}$$

а при $j = i$

$$\lim_{\rho \to \infty} A_{ii}(\rho) = \lim_{\rho \to \infty} B_{ii}(\rho) = 1. \tag{2.14}$$

От физични съображения следва, че ако радиусите на двете частици са равни ($a_1 = a_2 = a$), то $\omega_{11} = \omega_{22}$ и $\omega_{12} = \omega_{21}$. Ако $\vec{F}_1 = \vec{F}$ и $\vec{F}_2 = 0$, то при големи разстояния между частиците първата ще се движи със скорост $\vec{U}_1 = \dfrac{\vec{F}}{6\pi\mu a_1}$ и ще генерира около себе си флуидно течение със скорост

$$\vec{v}_1 = \frac{3a}{\rho}\left(1 + \frac{a^2}{3\rho^2}\right)\vec{U}_1 + \frac{3a}{4\rho}\left(1 - \frac{a^2}{\rho^2}\right)\frac{\vec{\rho}.\vec{U}_1\vec{\rho}}{\rho^2}. \tag{2.15}$$

Съгласно пъвия закон на Факсен скоростта $\vec{v}_1$ на смутеното течение около първата частица индуцира върху втората частица движение със скорост

$$\vec{U}_2 = \vec{v}_1(\vec{\rho}) + \frac{a_2^2}{6}\nabla^2\vec{v}_1(\vec{\rho}). \tag{2.16}$$

Ще отбележим също, че скоростта $\vec{v}_1(\rho)$ действува на втората частица като силов дипол с интензитет

$$S_2 = \frac{20}{3}\pi\mu a_2^3\left[l(\vec{r}) + \frac{1}{10}a_2^2\nabla^2 l(\vec{r})\right], \tag{2.17}$$

където $l(\vec{r})$ е тензорът на скоростта на деформацията на течението, генерирано от $\vec{v}_1(\vec{\rho})$. От друга страна, силовият дипол, имащ интензитет S_2, генерира течение около втората частица със скорост

$$\vec{v}_2 = S_2 : \nabla I. \tag{2.18}$$

Това течение променя скоростта $\vec{U}_1$ на първата частица и за нея се получава

$$\vec{U}_1 = \frac{\vec{F}}{6\pi\mu a_1} + \vec{v}_2(\vec{\rho}). \tag{2.19}$$

Оценките за A_{ij} и B_{ij} $(i, j = 1, 2)$ при големи стойности на ρ се намират, като се сравнят равенствата (2.16), (2.19) и (2.12). Окончателните резултати за коефициентите A_{ij} и B_{ij} при големи стойности на $\xi = \dfrac{2\rho}{a_1 + a_2}$ имат вида

$$A_{11} = 1 - \frac{60\eta^3}{(1+\eta)^4\xi^4} + O\left(\xi^{-6}\right), \qquad A_{12} = \frac{3}{2\xi} - \frac{2(1+\eta^2)}{(1+\eta)^2\xi^3} + O\left(\xi^{-7}\right),$$

$$B_{11} = 1 + O\left(\xi^{-6}\right), \qquad B_{22} = \frac{3}{4\xi} + \frac{1+\eta^2}{(1+\eta)^2\xi^3} + O\left(\xi^{-7}\right),$$

където $\eta = \dfrac{a_2}{a_1}$. Когато $\eta \to 0$, влиянието на втората частица върху първата изчезва.

3. ВЛИЯНИЕ НА ПАВ ВЪРХУ ХИДРОДИНАМИЧНОТО ВЗАИМОДЕЙСТВИЕ ПРИ ОБТИЧАНЕ НА ДВА МЕХУРА ОТ ГРАДИЕНТЕН ВИСКОЗЕН ПОТОК

Изменението на повърхностното напрежение σ се определя от разликата на температурата в различните точки на междуфазовата граница и наличието на повърхностно-активни вещества (ПАВ) върху нея. Често тези два фактора действуват едновременно. Ще разгледаме само случая, когато повърхностното напрежение се изменя поради наличието на ПАВ на междуфазовите граници.

Изменението на повърхностното напрежение води до деформиране на сферичната форма на движещите се във вискозно течение флуидни частици. При достатъчно малки концентрации на ПАВ и големи стойности на повърхностното напрежение обаче може да се предположи, че движещите се флуидни частици имат сферична форма. Ще предполагаме, че:

1. Отклонението на концентрацията Γ' на адсорбираното ПАВ на границата от равновесната ѝ стойност Γ_0 е малко в сравнение с Γ_0, т. е. $\Gamma = \Gamma_0 + \Gamma'$, $\Gamma' \ll \Gamma_0$.

2. Повърхностното напрежение е функция само на концентрацията на ПАВ върху повърхността. Това предположение е валидно само за много малка концентрация в обема.

Движението на повърхностите на флуидните частици се стреми да увеличи повърхностната концентрация Γ в задната им част. Този ефект се неутрализира чрез повърхностна дифузия и чрез обмен на ПАВ между границата и обемния разтвор. Обменът може да бъде извършен чрез бавна дифузия към обема на разтвора или чрез бавна адсорбция — десорбция. Тези три фактора — повърхностна дифузия, бавна дифузия в обема и бавна адсорбция — са разгледани отделно от Левич [125]. Нюман [126] намира, че трите ефекта могат да бъдат разгледани едновременно без допълнителни предположения. Относителната важност на тези три ефекта се отразява на получения коефициент на забавяне.

Да разгледаме два мехура, които се обтичат от несвиваем вискозен флуид при малки числа на Рейнолдс, като движението се описва от ососиметричен неограничен градиентен поток с параболичен профил на скоростта. Освен това в непрекъснатата фаза има ПАВ, които се привличат от фазовата граница и следователно се изменя повърхностното напрежение. ПАВ са разпределени неравномерно по повърхността. За решаването на тази задача са използувани работата на Стимсън и Джефри [127], които разглеждат две твърди частици, движещи се равномерно по посока на оста, свързваща центровете им, в неограничен вискозен флуид, и работата на Калицова – Запрянов [128] за две твърди сферични частици, обтичани от градиентен поток. Ще използуваме хидродинамичния модел, предложен в работата на Нюман [126] за флуидна частица, падаща със скорост U във вискозен флуид при малки числа на Рейнолдс и при наличието на ПАВ.

Да предположим, че два мехура се обтичат от ососиметричен градиентен поток с параболичен профил на скоростта при малки числа на Рейнолдс при наличието на ПАВ. Уравнението на Навие – Стокс за стационарни движения на несвиваеми вискозни флуиди е

$$\left(\overrightarrow{v}.\nabla\right)\overrightarrow{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\overrightarrow{v},$$

където ν е кинематичният коефициент на вискозност, а ρ — плътността на флуида. При бавни движения, каквито се разглеждат тук, се пренебрегват конвективните членове, които са малки, и уравнението в приближение на Стокс има вида

$$\nu\Delta\overrightarrow{v} = \frac{1}{\rho}\nabla p. \tag{3.1}$$

Уравнението на непрекъснатостта е

$$\nabla.\overrightarrow{v} = 0.$$

Кинематичните условия на границата между две течни фази са:

$$\text{а) } v_t^{(1)} = v_t^{(2)},$$

т. е. няма хлъзгане между частиците на двете фази (тук v_t е тангенциалната компонента на скоростта);

$$\text{б) } v_n^{(1)} = v_n^{(2)} = 0, \tag{3.2}$$

където v_n е нормалната компонента на скоростта. Това е условие за непротичане, т. е. двете фази не се смесват.

Поради действието на ПАВ върху междуфазовата граница се появява тангенциална сила, която е равна на

$$P_t = \operatorname{grad}_S \sigma = \frac{\partial\sigma}{\partial\Gamma} \operatorname{grad}_S \Gamma$$

и е насочена по допирателната към повърхността. Тук grad_S е повърхностният градиент. Предполагаме, че σ е функция само на Γ, т. е. $\sigma = \sigma(\Gamma)$.

Следователно динамичните условия, които се поставят на границата между две течни фази при наличие на ПАВ, са:

$$\text{а) } P_{nn} - P'_{nn} = p_\sigma,$$

където $p_\sigma = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ е капилярното налягане (тук R_1 и R_2 са радиусите на кривината на повърхността), а P_{nn} — нормалната компонента на тензора на напреженията;

$$\text{б) } P_{tt} - P'_{tt} = p_t, \tag{3.3}$$

където P_{tt} е тангенциалната компонента на тензора на напреженията, а p_t е тангенциалната сила на повърхността.

За да се намерят повърхностните сили p_σ и p_t, е необходимо да се намери разпределението на ПАВ по повърхността. Това разпределение се обуславя от няколко фактора. ПАВ на повърхността се увлича от движението на течността и възниква конвективен поток на ПАВ по повърхността:

$$j_{\text{пов}} = \Gamma v_t,$$

където v_t е тангенциалната скорост на течността на границата в резултат от разликите в концентрацията на ПАВ. На повърхността възниква дифузионен поток на ПАВ:

$$j_{\text{диф}} = -D_S \operatorname{grad}_S \Gamma,$$

където D_S е коефициентът на повърхностна дифузия.

Ако ПАВ е разтворимо в течността, то във всяка точка от повърхността може да се извърши преход на веществото от повърхностната фаза в обемния разтвор или обратно. Нека означим с j_n потока от ПАВ от единица повърхност към обемната фаза или обратно. Тогава законът за запазване на ПАВ има вида

$$j_n = \operatorname{div}_S \left(j_{\text{пов}} + j_{\text{диф}} \right),$$

или

$$j_n = \operatorname{div}_S (\Gamma v_t) - \operatorname{div}_S (D_S \operatorname{grad}_S \Gamma). \tag{3.4}$$

Тук div_S е повърхностната дивергенция.

За решението на задачата е необходимо съвместното решаване на уравнението на движение, закона за запазване на ПАВ и граничните условия.

Ако въведем цилиндрична координатна система (r, φ, z) с център, лежащ на отсечката, свързваща центровете на двете сфери, и ос Oz, съдържаща центровете им, то разглеждайки градиентен ососиметричен неограничен поток с параболичен профил на скоростта, граничното условие в безкрайност е

$$v_r = 0, \qquad v_z = U_\infty = Kr^2,$$

където K е константа. Тъй като разглеждаме ососиметрична задача, то $v_\varphi = 0$, т. е. задачата не зависи от φ.

Да въведем бисферична координатна система (ξ, η, φ), чиято връзка с цилиндричната координатна система се дава от равенствата

$$r = c\,\frac{\sin\xi}{\operatorname{ch}\eta - \cos\xi}, \qquad z = c\,\frac{\operatorname{sh}\eta}{\operatorname{ch}\eta - \cos\xi},$$

където $c > 0$, $0 \leqq \xi \leqq \pi$, $-\infty < \eta < \infty$.

Нека сферата с радиус a лежи в отрицателното полупространство $z < 0$, а сферата с радиус b лежи в положителното полупространство $z > 0$. Тогава в бисферични координати за сферата с радиус a имаме

$$\eta = \eta_1 < 0 \qquad \left(a = c|\operatorname{csch}\eta_1| = \frac{c}{|\operatorname{sh}\eta_1|} \right),$$

а за сферата с радиус b —

$$\eta = \eta_2 > 0 \qquad \left(b = c|\operatorname{csch}\eta_2| = \frac{c}{|\operatorname{sh}\eta_2|} \right).$$

Тъй като задачата е ососиметрична, можем да въведем функция на тока Ψ:

$$v_\xi = -\frac{1}{H_2 H_3}\,\frac{\partial\Psi}{\partial\eta}, \qquad v_\eta = \frac{1}{H_1 H_3}\,\frac{\partial\Psi}{\partial\xi}, \tag{3.5}$$

където $H_1 = H_2 = \dfrac{c}{\operatorname{ch}\eta - \cos\xi}$, $H_3 = \dfrac{c\sin\xi}{\operatorname{ch}\eta - \cos\xi}$ са метричните коефициенти на Ламе.

От зависимостта $v_z = -\dfrac{1}{r}\dfrac{\partial\Psi}{\partial r}$ следва, че условието в безкрайност има вида

$$\Psi = -Kc^4\frac{\sin^4\xi}{4(\operatorname{ch}\eta - \cos\xi)^4}, \quad \xi^2 + \eta^2 \to 0. \tag{3.6}$$

От условието за непротичане (3.2) получаваме

$$\Psi = 0 \quad \text{при} \quad \eta = \eta_1,\ \eta = \eta_2. \tag{3.7}$$

Освен това изискваме $\Psi = 0$ при $\xi = 0$, $\xi = \pi$.

В криволинейна координатна система тангенциалната компонента на тензора на напреженията е

$$P_{12} = P_{21} = \mu^*\left\{\frac{1}{H_2}\frac{\partial v_1}{\partial q_2} + \frac{1}{H_1}\frac{\partial v_2}{\partial q_1} - \frac{v_1}{H_1H_2}\frac{\partial H_1}{\partial q_2} - \frac{v_2}{H_1H_2}\frac{\partial H_2}{\partial q_1}\right\}.$$

Следователно, използувайки (3.5) в бисферични координати, получаваме

$$P_{\xi\eta} = \mu^*\frac{(\operatorname{ch}\eta - \cos\xi)^3}{c^3\sin\xi}\frac{\partial^2\Psi}{\partial\xi^2} - \frac{(\operatorname{ch}\eta - \cos\xi)^3}{c^3\sin\xi}\frac{\partial^2\Psi}{\partial\eta^2}$$
$$+\frac{(\operatorname{ch}\eta - \cos\xi)^2}{c^3\sin\xi}\cdot\frac{2\sin^2\xi + 1 - \cos\xi\operatorname{ch}\eta}{\sin\xi}\frac{\partial\Psi}{\partial\xi} - \frac{(\operatorname{ch}\eta - \cos\xi)^2}{c^3\sin\xi}\cdot 3\operatorname{sh}\eta\frac{\partial\Psi}{\partial\eta},$$

където μ^* е динамичният коефициент на вискозност на флуида.

Нека $\Gamma = \Gamma^0 + \Gamma'$, където Γ^0 е равновесната стойност на повърхностната концентрация на ПАВ, а Γ' — отклонението от нея. Ще предполагаме, че Γ' е много по-малко от Γ^0. Имаме представянето

$$\sigma = \sigma_0 + \frac{\partial\sigma}{\partial\Gamma}(\Gamma - \Gamma^0),$$

където $\dfrac{\partial\sigma}{\partial\Gamma}$ е константа или $\sigma = \sigma_0 + \dfrac{\partial\sigma}{\partial\Gamma}\Gamma'$.

Условието (3.3) е

$$P_{\xi\eta} = \operatorname{grad}_S\sigma \quad \text{при} \quad \eta = \eta_1,\ \eta = \eta_2$$

и го записваме във вида

$$\mu^*\left\{\frac{(\operatorname{ch}\eta - \cos\xi)^3}{c^3\sin\xi}\frac{\partial^2\Psi}{\partial\xi^2} - \frac{(\operatorname{ch}\eta - \cos\xi)^3}{c^3\sin\xi}\frac{\partial^2\Psi}{\partial\eta^2}\right. \tag{3.8}$$
$$\left.+\frac{(\operatorname{ch}\eta - \cos\xi)^2}{c^3\sin\xi}\frac{2\sin^2\xi + 1 - \cos\xi\operatorname{ch}\eta}{\sin\xi}\frac{\partial\Psi}{\partial\xi} - \frac{(\operatorname{ch}\eta - \cos\xi)^2}{c^3\sin\xi}\cdot 3\operatorname{sh}\eta\frac{\partial\Psi}{\partial\eta}\right\}$$
$$= \pm\frac{(\operatorname{ch}\eta - \cos\xi)}{c}\frac{\partial\sigma}{\partial\Gamma}\cdot\frac{\partial\Gamma'}{\partial\xi}, \qquad \eta = \eta_1,\ \eta = \eta_2.$$

Тук „+“ е за $\eta = \eta_1$, а „−“ е за $\eta = \eta_2$.

От векторното уравнение (3.1) елиминираме налягането p и използувайки връзките (3.5), получаваме уравнението

$$D^4(\Psi) = 0, \tag{3.9}$$

където $D^4 = D^2(D^2)$ и

$$D = \frac{H_3}{H_1 H_2}\left\{\frac{\partial}{\partial q_1}\left(\frac{H_2}{H_3 H_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_1}{H_2 H_3}\frac{\partial}{\partial q_2}\right)\right\}$$

или в бисферични координати

$$D^2 = \frac{\operatorname{ch}\eta - \mu}{c^2}\left\{\frac{\partial}{\partial \eta}\left[(\operatorname{ch}\eta - \mu)\frac{\partial}{\partial \eta}\right] + (1-\mu^2)\frac{\partial}{\partial \mu}(\operatorname{ch}\eta - \mu)\frac{\partial}{\partial \mu}\right\},$$

където $\mu = \cos\xi$.

Да разгледаме уравнението за съхранение на ПАВ (3.4). Големината на потока от ПАВ от повърхността към обемната фаза j_n се определя от по-бавния от двата процеса адсорбция — десорбция или пренасяне на молекулите на ПАВ от обема към повърхността.

Следвайки Левич, ще изразим j_n чрез дебелината на дифузионния слой на Нернст. Имаме връзката

$$j_n = D(C_b - C_\delta)/\delta,$$

където C_b е концентрацията на ПАВ в обемния разтвор, C_δ е концентрацията в разтвора близо до границата, а δ — дебелината на дифузионния слой на Нернст. Последното уравнение можем да запишем във вида

$$j_n = (D/\delta)(\Gamma^0 - \Gamma_\delta)/(\partial\Gamma/\partial C)_{\text{eq}}, \tag{3.10}$$

където $(\partial\Gamma/\partial C)_{\text{eq}}$ е константа.

Сега, разглеждайки механизма на адсорбция, можем да запишем следната линейна връзка:

$$j_n = -\alpha(\Gamma - \Gamma_\delta). \tag{3.11}$$

Елиминираме Γ_δ от (3.10) и (3.11) и получаваме

$$j_n = -\alpha D\Gamma'/\left[D + \alpha\delta(\partial\Gamma/\partial C)_{\text{eq}}\right]. \tag{3.12}$$

Разглеждаме първия член в дясната страна на (3.4):

$$\begin{aligned}\operatorname{div}_S(\Gamma v_t) &= \operatorname{div}_S\left[(\Gamma^0 + \Gamma')v_t\right] = (\Gamma^0 + \Gamma')\operatorname{div}_S v_t + v_t \operatorname{grad}_s(\Gamma^0 + \Gamma') \\ &= \Gamma^0 \operatorname{div}_S v_t + \operatorname{div}_S(\Gamma' v_t) + v_t \operatorname{grad}_S \Gamma^0 \approx \Gamma^0 \operatorname{div}_S v_t,\end{aligned}$$

защото Γ' е малко и v_t е малко поради бавното движение, следователно $\Gamma' v_t$ е малка величина от втори порядък и я пренебрегваме, а освен това Γ^0 е константа.

Вторият член в дясната страна на (3.4) е

$$\operatorname{div}_S(D_S \operatorname{grad}_S \Gamma) = D_S \operatorname{div}_S \operatorname{grad}_S \Gamma'.$$

Тогава (1.4) добива вида

$$j_n = \Gamma^0 \operatorname{div}_S v_t - D_S \operatorname{div}_S \operatorname{grad}_S \Gamma'. \tag{3.13}$$

Комбинираме уравненията (3.12) и (3.13) и получаваме уравнение за Γ':

$$-\alpha D\Gamma'/\left[D+\alpha\delta(\partial\Gamma/\partial C)_{\text{eq}}\right] = \Gamma^0 \operatorname{div}_S v_t - D_s \operatorname{div}_S \operatorname{grad}_S \Gamma'$$

или, ако положим

$$K_i = -\alpha D/\left[D+\alpha\delta(\partial\Gamma_i/\partial C)_{\text{eq}}\right], \quad i = 1,\ 2,$$

то можем да запишем уравнение за изменението на повърхностната концентрация Γ'_i върху всяка от частиците $\eta = \eta_i$:

(3.14) $$K_i\Gamma'_i = \Gamma^0_i \operatorname{div}_S v^i_t - D_S \operatorname{div}_S \operatorname{grad}_S \Gamma'_i, \quad i = 1,\ 2.$$

За да обезразмерим уравненията, да вземем за характерен линеен размер C, а за характерна скорост $U = \dfrac{\mu^*}{\rho C}$. Размерността на функцията на тока Ψ е UC^2. Разделяме двете страни на (3.6) на UC^2 и получаваме

(3.15) $$\Psi = -\frac{1}{4}\,\frac{\sin^4\xi}{(\operatorname{ch}\eta - \cos\xi)^4}, \quad \xi^2+\eta^2 \to 0$$

(тук предполагаме, че $\dfrac{KC}{U} = 1$). Делим двете страни на (3.7) също на UC^2 и получаваме

(3.16) $$\Psi = 0 \quad \text{при} \quad \eta = \eta_1,\ \eta = \eta_2.$$

За да обезразмерим (3.8), умножаваме двете страни по $\dfrac{C}{\sigma_0}$ и получаваме

(3.17) $$Ca\left\{\frac{(\operatorname{ch}\eta-\cos\xi)^3}{\sin\xi}\,\frac{\partial^2\Psi}{\partial\xi^2} - \frac{(\operatorname{ch}\eta-\cos\xi)^3}{\sin\xi}\,\frac{\partial^2\Psi}{\partial\eta^2}\right.$$
$$\left.+\frac{(\operatorname{ch}\eta-\cos\xi)^2}{\sin\xi}\,\frac{2\sin^2\xi+1-\cos\xi\operatorname{ch}\eta}{\sin\xi}\,\frac{\partial\Psi}{\partial\xi} - \frac{(\operatorname{ch}\eta-\cos\xi)^2}{\sin\xi}\cdot 3\operatorname{sh}\eta\,\frac{\partial\Psi}{\partial\eta}\right\}$$
$$= \pm(\operatorname{ch}\eta-\cos\xi)\,\frac{\partial\sigma^\delta}{\partial\Gamma}\,\frac{\partial\Gamma'}{\partial\xi}, \quad \eta=\eta_1,\ \eta=\eta_2.$$

Тук $C_a = \dfrac{\mu^* U}{\sigma_0}$ се нарича капилярно число, където σ_0 е равновесната стойност на повърхностното напрежение и

$$\sigma^\delta = \frac{\sigma}{\sigma_0} = 1 + \frac{\partial\sigma^\delta}{\partial\Gamma}\Gamma'.$$

Умножаваме уравнението (3.9) по $\dfrac{C^2}{U}$ и получаваме

(3.18) $$D^4(\Psi) = 0,$$

където

$$D^2 = (\operatorname{ch}\eta-\mu)\left\{\frac{\partial}{\partial\eta}\left[(\operatorname{ch}\eta-\mu)\frac{\partial}{\partial\eta}\right] + (1-\mu^2)\frac{\partial}{\partial\mu}(\operatorname{ch}\eta-\mu)\frac{\partial}{\partial\mu}\right\}, \qquad \mu=\cos\xi.$$

Умножаваме уравнението (3.14) по $\frac{C}{U\Gamma_x}$ и получаваме безразмерното уравнение

$$K_i\Gamma_i' = \Gamma_i^0 \operatorname{div}_S v_t^i - D_S \operatorname{div}_S \operatorname{grad}_S \Gamma_i', \quad i = 1, 2, \tag{3.19}$$

където

$$\operatorname{div}_S v_t^i = \frac{(\operatorname{ch}\eta_i - \cos\xi)^2}{\sin\xi} \frac{\partial}{\partial\xi}\left(v_t^i \frac{\sin\xi}{\operatorname{ch}\eta_i - \cos\xi}\right),$$

$$\operatorname{div}_S \operatorname{grad}_S \Gamma_i' = \frac{(\operatorname{ch}\eta_i - \cos\xi)^2}{\sin\xi} \frac{\partial}{\partial\xi}\left(\sin\xi \frac{\partial\Gamma_i'}{\partial\xi}\right).$$

Тук сме приели, че $K = 1$, $C = 1$.

Нека сега да въведем функцията $\overline{\Psi}$, която е свързана с Ψ чрез следното уравнение:

$$\overline{\Psi} = \Psi + \frac{1}{4}\frac{\sin^4\xi}{(\operatorname{ch}\eta - \cos\xi)^4}, \tag{3.20}$$

и клони към 0 в безкрайност (вж. 3.15). Тогава търсим точно решение на задачата във вида

$$\overline{\Psi} = (\operatorname{ch}\eta - \mu)^{-\frac{3}{2}} \sum_{n=1}^{\infty} U_n(\eta)V_n(\mu), \tag{3.21}$$

където

$$V_n(\mu) = P_{n-1}(\mu) - P_{n+1}(\mu), \tag{3.22}$$

(3.23)

$$U_n(\eta) = A_n \operatorname{ch}\left(n - \frac{1}{2}\right)\eta + B_n \operatorname{ch}\left(n + \frac{1}{2}\right) + C_n \operatorname{ch}\left(n - \frac{3}{2}\right)\eta + D_n \operatorname{ch}\left(n + \frac{3}{2}\right)\eta.$$

Тук $P_n(\mu)$ е полиномът на Льожандър от степен n. Като вземем предвид (3.20), от (3.16) получаваме

$$\overline{\Psi} = \frac{1}{4}\frac{\sin^4\xi}{(\operatorname{ch}\eta - \cos\xi)^4}, \quad \eta = \eta_1, \ \eta = \eta_2, \tag{3.24}$$

или

$$\sum_{n=1}^{\infty} U_n(\eta)V_n(\mu) = \frac{1}{4}\frac{\sin^4\xi}{(\operatorname{ch}\eta - \cos\xi)^{5/2}}, \quad \eta = \eta_1, \ \eta = \eta_2. \tag{3.25}$$

След дълги, но стандартни пресмятания, включващи използването на общото решение и граничните условия, се получава безкрайна система от линейни уравнения за коефициентите A_n, B_n, C_n, D_n, E_n^1 и E_n^2. Тази система се решава числено. Като пресметнем приближено неизвестните коефициенти, изчисляваме съпротивлението, което изпитват двете частици, от формулите

$$F_1 = -2\sqrt{2}\pi \sum_{n=1}^{\infty}(2n+1)(A_n - B_n + C_n - D_n) \quad \text{за } \eta = \eta_1 < 0,$$

$$F_2 = -2\sqrt{2}\pi \sum_{n=1}^{\infty}(2n+1)(A_n + B_n + C_n + D_n) \quad \text{за} \ \eta = \eta_2 > 0.$$

Получените числени резултати показват, че възпрепятстването на мобилността на междуфазовите граници на частиците нараства (чрез механизма на повърхностната дифузия на ПАВ), когато радиусите на частиците растат и разстоянието между тях намалява.

Настоящите изследвания са извършени в изпълнение на договор № 72/91, сключен с Министерството на образованието и науката.

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Получена 04.02.1993

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 86, 1992

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 86, 1992

ДИСПЕРСИОННО УРАВНЕНИЕ ПРИ ВЪЛНОВ РЕЖИМ НА ИЗТЪНЯВАНЕ НА ТЪНКИ ТЕЧНИ ФИЛМИ С ОТЧИТАНЕ НА ЕЛАСТИЧНИТЕ СВОЙСТВА НА МЕЖДУФАЗОВИТЕ ГРАНИЦИ

ЗАПРЯН ЗАПРЯНОВ

Запрян Запрянов. ДИСПЕРСИОННОЕ УРАВНЕНИЕ ПРИ ВОЛНОВОМ РЕЖИМЕ ДВИЖЕНИЯ ПЕННЫХ И ЭМУЛЬСИОННЫХ ПЛЕНОК

Адсорбирование поверхностно-активных веществ (ПАВ) модифицирует поверхностную вязкость и эластичность, и поэтому играет важную роль в стабилизации пен и эмульсий. Настоящая работа исследует волновые движения жидкости при утончении жидких пленок. Изучается влияние эластичных свойств межфазовых (газо–жидких или жидко–жидких) поверхностей на волновом режиме движения эмульсионных и пенных пленок. Получено дисперсионное уравнение волнового движения жидкости в пленках.

Zapryan Zapryanov. DISPERSION EQUATION OF WAVE REGIME FOR FILM DRAINAGE WITH ELASTIC PROPERTIES OF INTERFACE BOUNDARIES

Adsorbed species play an important role in foam stabilization, emulsion stability, and suspension polymerization by modifying interfacial viscosity and elasticity. The present paper deals with the wave motion in tinning films. The influence of elastic properties on the gas–liquid (and (liquid–liquid) film interfaces is investigated. The dispersion equation for wave motion of the film interfaces is derived.

Важна роля в поведението на някои дисперсни системи (пени, емулсии и др.) играят тънките течни слоеве между частиците, изграждащи системата. В много случаи свойствата на дисперсната среда се определят от времето на изтъняване на тънките течни филми между частиците.

Макар, пропорционално взето, обемната част на хидродинамичния филм да съдържа много повече флуид в сравнение с флуида, намиращ се върху границите му, процесът на неговото изтъняване се влияе съществено от реологичните свойства на междуфазовите му повърхности.

Когато флуидът във филма съдържа повърхностно активни вещества (ПАВ), които поради характера си се стремят „да отседнат“ на повърхността му, междуфазовият слой по границите изменя своите композиционни свойства значително в сравнение със свойствата на обемната фаза. Това довежда в някои случаи до силни отклонения от свойствата на нютоновите междуфазови повърхнини. По такъв начин процесът на изтъняване и разрушаване на такива филми започва да се влияе силно от адсорбираните на повърхността еластични характеристики и тяхното ненютоново поведение.

Както е известно, през 1913 г. Бусинеск [1] предложи двумерен аналог на тримерните уравнения на Навие—Стокс и с това даде описание на поведението на нютоновите междуфазови повърхности.

Конститутивните уравнения за изотропичните вискозно-еластични междуфазови повърхнини се извеждат аналогично на уравненията за нютоновите междуфазови повърхнини.

Ламб [2] и Годрич [3] показват как изучаването на капилярните вълни може да послужи като удобно средство за изучаване на реологичните свойства на междуфазовите граници, например за определяне на повърхностното напрежение, повърхностния вискозитет, еластичните характеристики и др. Капилярни вълни в неизтъняващи емулсионни филми са изследвани още от Люкасен и др. — [4] в случая на неразтворими ПАВ, Гумерман и Хомси — [5] за случая на сферични филми от чисти флуиди, и Джейн и Рукенщейн — [6] за едностранни филми с разтворими ПАВ.

В докторската си дисертация на тема „Извод на междуфазовите свойства от експерименти с капилярни вълни“ Мауер [7] изследва еластичните свойства на междуфазовите повърхности. Въз основа на модела на на Войт в теоретичната част от своите изследвания той разглежда ненютонови повърхности на неизтъняващи течни филми. Сравняването на теоретичните и експерименталните резултати му дава възможност да определи еластичните характеристики на междуфазовите повърхности. В споменатия модел Войт [8] дава най-простите конститутивни уравнения за вискозно-еластични междуфазови повърхнини като подходяща комбинация на конститутивните уравнения за чисто вискозни и чисто еластични междуфазови повърхнини. Олдроид [9] също предлага конститутивни уравнения на вискозно-еластични повърхнини, но те са по-сложни. Напоследък Гарднер и др. [10] предложиха конститутивни уравнения за сложни вискозно-еластични повърхности на „запаметяване“.

В тази статия ще изследваме вълновия режим за изтъняване на тънки течни филми с отчитане на еластичните свойства на междуфазовите граници. Граничните условия на междуфазовите граници на изтъняващия филм се вземат съгласно конститутивните уравнения на Войт за вискозно-еластични междуфазови повърхнини. Основните предположения са:

1) Реологичното поведение на обемните фази считаме нютоново, докато реологичното поведение на границите на филма ще предполагаме ненютоново. За да комбинираме вискозните и еластичните напрежения, ще изхождаме от конститутивните уравнения на Войт [8].

2) Изтъняването на филма и вълновото движение във филма считаме за независими [11, 12].

3) Ще считаме, че ПАВ са разтворени само в дисперсната среда (филма) и че обменът на ПАВ между филма и неговата повърхност е дифузно контролируем.

Като изберем за характерни величини общоприетите в тази област, ще получим следните безразмерни величини, които характеризират разглеждания проблем:

$$W_e = \frac{\nu\mu}{R\sigma_0}, \quad D_s^* = \frac{D_s R}{\nu}, \quad S_e = \frac{\nu}{D},$$

$$a = \frac{\hat{\mu}}{\mu}, \quad \mu_s^* = \frac{\mu_s}{\mu R}, \quad \Lambda^* = \frac{\Lambda}{\frac{\mu^2}{\rho R}}, \quad G^* = \frac{G}{\frac{\mu^2}{\rho R}}.$$

Нови параметри тук са Λ и G — повърхностните коефициенти на еластичността (Λ е разтягащият коефициент на еластичността, а G — тангенциалният).

Във връзка с изучаването на вълновия режим на движение на флуидите през 1890 г. лорд Келвин [13], като използува уравненията за движение на вискозни флуиди и гранични условия, включващи повърхностното напрежение, получава известното дисперсно уравнение

$$\omega^2 = gk + \sigma\frac{k^3}{\rho},$$

където ω е честотата, k — вълновото число, g — земното ускорение, ρ — плътността и σ — повърхностното напрежение. Той извършва изследванията си за капилярни вълни въз основа на нестационарните уравнения на Стокс, които описват бавните движения. През 1932 г. Ламб [2] обобщава резултатите на Келвин, като използува нов подход, станал след това класически в изследванията на вълновите движения в хидродинамиката, но също включва в граничните условия само повърхностното напрежение като характеристика на междуфазовите граници. Много автори, които са изследвали вълнови движения на вискозни флуиди, са се придържали към постановката и анализа, дадени от Ламб. През 1943 г. Уейгхард [14] включва в граничното условие на междуфазовата граница на вълните освен повърхностното напрежение още и повърхностния вискозитет, както той е бил въведен от Бусинеск. Известен опит за включване в граничните условия на междуфазовата граница на вълните и на еластичните характеристики на флуидните повърхнини наред с повърхностния вискозитет и повърхностното напрежение прави през 1951 г. Дорещейн [15]. През 1941 г. Левич [16] премахва от своите изследвания повърхностния вискозитет и поставя ограничителното условие за неподвижност на междуфазовата граница в хоризонтално направление не само при неразтворимите монослоеве (филми), но и за разтворимите. През 1964 г. Нансен и Ман [17] предлагат теория на вълновите движения при наличие на безкрайна междуфазова граница, като намират връзка между обемната вискозност, повърхностния вискозитет и еластичните модули на границата. Аналогични изследвания правят и Ван ден Темпл и Ван ден Рит [18]. Интересен, но по-сложен модел за изследване на вълновите движения е предложен от Ринтар, Израел и Васан [19].

През 1979 г. Иванов и др. [12] разглеждат вълнов режим на движение на филми с оглед изследването на устойчивостта на тънки течни филми с нютонови междуфазови граници.

В настоящата статия въз основа на простия модел на Войт е изследван вълновият режим на движение на радиално ограничените течни филми, като в граничните условия на междуфазовите граници са отчетени освен повърхностното напрежение и повърхностния вискозитет още и еластичните характеристики на междуфазовите повърхности на филма, т. е. изследван е вълновият режим на движение на тънки течни филми с ненютонови междуфазови граници. Изведено е обобщено дисперсно уравнение за вълновото движение на пенните и емулсионните филми.

Вследствие на термични флуктуации и други причини повърхностите на водните системи най-често са с вълнообразен характер. Профилът на такава повърхност във всеки момент може да се представи като суперпозиция на безкраен брой вълни с различни дължини и амплитуди. Положението на всяка вълна в тънък течен слой се определя до голяма степен от съотношението на два фактора — локалното капилярно налягане, което се стреми да анулира амплитудата на вълната, и отрицателното разклинящо налягане, което се стреми да увеличи нейната амплитуда. При малки дебелини на течния филм преобладава действието на разклинящото налягане, а при големи — капилярното налягане. При определени дебелини на филма двата ефекта се уравновесяват.

Ако двете междуфазови граници на филма са нагънати симетрично, той ще се скъса, когато амплитудата на вълната стане равна на половината от дебелината му, т. е. на h. Тази дебелина, при която става скъсване на филма, се нарича критична дебелина на скъсване — h_{cr}.

При движението на флуида адсорбиралият монослой на ПАВ върху границата на филма се свива и разпуща, т. е. проявява своето еластично поведение, което от своя страна се отразява на характера на вълновото движение.

За всяка вълна с вълново число k съществува дебелина h_0 на филма, при която се извършва преход от колебания около равновесното положение към преминаване към непрекъснато увеличаваща се амплитуда на вълната. През 1962 г. Шелудко [20] предложи следната формула за величината h_0:

$$h_0 = \left(\frac{3\pi^2 K}{16\sigma k^2}\right)^{1/4}, \tag{1}$$

където K е константата на Вандерваалс, σ — повърхностното напрежение и $\pi = -\dfrac{K}{h^3}$ е разклинящото налягане.

Иванов и др. [12] показаха, че когато отклонението на вълната ζ от равновесното положение на филма h е малко, общите уравнения за движение на флуида във филма могат да се разделят на две системи — първата, описваща изтъняването на цилиндричния плоскопаралелен филм, и втората — вълновия режим на движението му. Съгласно Врей [21] и Иванов и др. [22] симетричните смущения са най-съществени, когато се разглежда въпросът за устойчивостта на филма. При разглеждането на вълновия режим на движение във филма ние ще имаме предвид филми с дебелини, близки до h_{cr}, така че честотата ω на вълната да не зависи явно от времето. Това позволява да разглеждаме независимо вълнообразното движение от изтъняването на филма.

За вертикалното ζ и хоризонталното ξ отклонение на повърхнината на филма от равновесното му положение ще имаме

$$v_z = v_\zeta = \frac{\partial \zeta}{\partial t} \quad \text{при} \quad z = h, \tag{2}$$

$$v_r = U = \frac{\partial \xi}{\partial t} \quad \text{при} \quad z = h. \tag{3}$$

Тогава дебелината H на филма ще бъде

$$H = 2(h + \zeta).$$

Във връзка с направените по-горе предположения ще използваме следното гранично условие за ζ:

$$\zeta = 0 \quad \text{при} \quad r = 1. \tag{4}$$

Уравненията на движение в дисперсната фаза имат вида

$$\frac{\partial v_r}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right), \tag{5}$$

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right), \tag{6}$$

$$\frac{\partial}{\partial r}(r v_r) + \frac{\partial}{\partial z}(r v_z) = 0. \tag{7}$$

Решението за вълновия режим на движение ще търсим във вида

$$\begin{aligned} v_r &= v_r^{(0)} + v_r^{(1)}, \\ v_z &= v_z^{(0)} + v_z^{(1)}. \end{aligned} \tag{8}$$

Тук $v_r^{(0)}$ и $v_z^{(0)}$ ще считаме компоненти на движението на потенциално идеално флуидно течение и следователно съществува потенциална функция φ, такава че

$$v_r^{(0)} = -\frac{1}{r}\frac{\partial\varphi}{\partial r}, \qquad v_z^{(0)} = -\frac{1}{r}\frac{\partial\varphi}{\partial z}. \tag{9}$$

Като заместим (8) в (7), получаваме, че $v_r^{(1)}$ и $v_z^{(1)}$ удовлетворяват уравнението (7). Полагаме

$$v_r^{(1)} = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad v_z^{(1)} = \frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{10}$$

Тъй като по дефиниция $v_r^{(0)}$ и $v_z^{(0)}$ удовлетворяват уравнението на идеален флуид, то функцията ψ трябва да удовлетворява уравненията

$$\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial t} - \nu\Delta\psi\right) = 0, \quad \frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial t} - \nu\Delta\psi\right) = 0.$$

От тези две уравнения обаче следва, че функцията ψ трябва да удовлетворява уравнението

$$\frac{\partial\psi}{\partial t} = \nu\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right). \tag{11}$$

Предполагаме, че функциите φ и ψ имат вида

$$\varphi = \left[Ae^{-k_0(z-h)} + A_1e^{k_0(z-h)}\right] J_0(k_0r)e^{\omega t},$$

$$\psi = \left[Be^{-k(z-h)} + B_1e^{k(z-h)}\right] rJ_1(k_0r)e^{\omega t},$$

където $J_0(k_0r)$ и $J_1(k_0r)$ са функции на Бесел съответно от нулев и първи ред, а A, A_1, B и B_1 са константи. Тъй като при $z \to \infty$, $\vec{v} = 0$, то $A_1 = 0$ и $B_1 = 0$. Следователно

$$\varphi = Ae^{-k_0(z-h)}J_0(k_0r)e^{\omega t}, \qquad \psi = Be^{-k(z-h)}rJ_1(k_0r)e^{\omega t}, \tag{12}$$

където

$$k = k_0\sqrt{1 + \frac{\omega}{\nu k_0^2}} \approx \left(1 + \frac{\omega}{2\nu k_0^2}\right). \tag{13}$$

Компонентите на скоростта $\hat{v}_r$ и $\hat{v}_z$ могат да бъдат записани така:

$$\hat{v}_z = -\frac{\partial\varphi}{\partial z} + \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad \hat{v}_r = -\frac{\partial\varphi}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial z}. \tag{14}$$

Следователно

$$\hat{v}_z = \left[Ae^{-k_0(z-h)} + Be^{-k(z-h)}\right] k_0 J_0(k_0 r)e^{\omega t},$$
(15)
$$\hat{v}_r = \left[Ak_0 e^{-k(z-h)} + Bke^{-k(z-h)}\right] J_1(k_0 r)e^{\omega t}.$$

Тогава за хоризонталното и вертикалното отклонение ще имаме

$$\xi = \frac{Ak_0 + Bk}{\omega} J_1(k_0 r)e^{\omega t},$$

$$\zeta = \frac{Ak_0 + Bk_0}{\omega} J_0(k_0 r)e^{\omega t}.$$

По този начин за амплитудите на отклоненията ξ и ζ ще имаме съответно

$$\xi_0 = \frac{Ak_0 + Bk}{\omega}, \quad \zeta_0 = \frac{Ak_0 + Bk_0}{\omega}.$$

И така, засега нашето решение съдържа три неизвестни константи A, B и ω. Кинематичните гранични условия върху границата на филма имат вида

(16) $$\hat{v}_z = v_z,\ \hat{v}_r = v_r \quad \text{при} \quad z = h + \zeta.$$

При $\zeta \ll h$ за разклинящото налягане Π имаме

$$\Pi(2h + 3\zeta) \approx \Pi(2h) + \zeta\,\frac{d\Pi}{dh},$$

а капилярното налягане, дължащо се на кривината на профила на вълната, е равно на $\sigma^f \Delta_r \zeta$, където

$$\sigma^f = \sigma^0 + \frac{1}{2}\int_h^\infty \Pi\, dh$$

е повърхностното напрежение на филма [12].

Тогава за връзката между нормалните напрежения на границата на филма ще имаме

(17) $$P_{zz} - \hat{P}_{zz} = \sigma^f \Delta_r \zeta + \zeta\,\frac{d\Pi}{dh}$$

или

(18) $$P_{zz} - \hat{P}_{zz} = -q\zeta_0 J_0(k_0 r)e^{\omega t},$$

където

$$P_{zz} = -p + 2\mu\frac{\partial v_z}{\partial z}, \quad \hat{P}_{zz} = -\hat{p} + 2\hat{\mu}\frac{\partial \hat{v}_z}{\partial z}, \quad q = k_0^2\sigma^f - 2\Pi'.$$

Тъй като

$$\left(\hat{p}^{(0)}\right)_{z=h} = -\rho\frac{\partial\varphi}{\partial t}\bigg|_{z=h} = -\rho A\omega J_0(k_0 r)e^{\omega t},$$

то

$$\hat{P}_{zz}\Big|_{z=h} = -\hat{p}\Big|_{z=h} + 2\hat{\mu}\,\frac{\partial \hat{v}_z}{\partial z}\Big|_{z=h} = \left[\omega\rho A - 2\hat{\mu}(Ak_0^2 + Bk_0k)\right] J_0(k_0r)e^{\omega t}.$$

От друга страна,

$$P_{zz}\Big|_{z=h} = -p\Big|_{z=h} + 2\mu\,\frac{\partial v_z}{\partial z}\Big|_{z=h}.$$

За краткост на изложението ще положим

(19) $$p = \beta J_0(k_0r)e^{\omega t},$$

където β е неизвестен коефициент. От уравненията

(20) $$\mu\,\frac{\partial^2 v_r}{\partial z^2} = \frac{\partial p}{\partial r} \quad \text{и} \quad \frac{\partial p}{\partial z} = 0$$

следва

$$\mu\,\frac{\partial v_r}{\partial z} = \frac{\partial p}{\partial r}\,z + C_1(r,t).$$

От $\left(\dfrac{\partial v_r}{\partial z}\right)_{z=0} = 0$ следва, че $C_1(r,t) \equiv 0$.

Като интегрираме по z още веднъж, намираме

$$v_r = \frac{1}{2\mu}\,\frac{\partial p}{\partial r}\,z^2 + C_2(r).$$

От това равенство при $z = h$ получаваме

$$U(r,t) = \frac{1}{2\mu}\,\frac{\partial p}{\partial r}\,h^2 + C_2(r,t).$$

Следователно

(21) $$v_r = \frac{1}{2\mu}\,\frac{\partial p}{\partial r}(z^2 - h^2) + U(r,t).$$

Но

$$\frac{\partial p}{\partial r} = -\beta k_0 J_1(k_0r)e^{\omega t}.$$

Тогава

$$v_r = -\frac{1}{2\mu}(z^2 - h^2)\beta k_0 J_1(k_0r)e^{\omega t} + U(r,t).$$

От уравнението на непрекъснатостта, записано във вида

$$\frac{1}{r}\,\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0$$

и интегрирано по z от 0 до z, намираме

(22) $$v_z = -\frac{1}{2\mu}\left(\frac{z^3}{3} - zh^2\right)\beta k_0^2 e^{\omega t} J_0(k_0r) + \frac{z}{r}\,\frac{d}{dr}(rU).$$

От условието

(23) $$(v_z)_{z=h} = (\hat{v}_z)_{z=h}$$

следва, че $U(r,t)$ удовлетворява диференциалното уравнение

$$\frac{1}{3\mu} h^3 \beta k_0^2 e^{\omega t} J_0(k_0 r) + h\left[\frac{\partial U}{\partial r} + \frac{U}{r}\right] = (A+B) k_0 J_0(k_0 r) e^{\omega t}.$$

Като положим

$$U(r,t) = E.J_1(k_0 r) e^{\omega t},$$

където E е неизвестна константа, лесно получаваме, че

$$\nabla_r U = k_0 E J_0(k_0 r) e^{\omega t}.$$

От уравнението за $U(r,t)$ намираме

(24) $$E = \frac{A+B}{h} - \frac{1}{3\mu} h^2 k_0 \beta.$$

Следователно

(25) $$v_z = -\frac{1}{2\mu}\left(\frac{z^3}{3} - zh^2\right)\beta k_0^2 J_0(k_0 r) e^{\omega t} + z k_0 E J_0(k_0 r) e^{\omega t}.$$

От второто кинематично условие

$$(v_r)_{z=h} = (\hat{v}_r)_{z=h}$$

намираме

(26) $$A k_0 + B k = E.$$

Като използуваме (24) и граничното условие (18) за нормалните напрежения, получаваме

(27) $$\left(1 + \frac{2}{3} k_0^2 h^2\right)\beta = \frac{2\mu k_0}{h}(A+B) - \omega\rho A + 2\hat{\mu}(A k_0^2 + B k_0 k) + q\,\frac{A k_0 + B k_0}{\omega}.$$

От уравнението

(28) $$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial z^2}\right)$$

и полагането

$$C = \alpha\, \mathrm{sh}(lz) J_0(k_0 r) e^{\omega t}$$

намираме съотношението

(29) $$l^2 = k_0^2 + \frac{\omega}{D}.$$

От закона за запазване на веществото на ПАВ върху границата $z = h$ получаваме

(30) $$\Gamma_0 k_0 E = \alpha\left[D_s\left(\frac{\partial\Gamma}{\partial C}\right)^0 k_0^2\, \mathrm{ch}(lh) + Dl\, \mathrm{sh}(lh)\right].$$

От граничното условие за тангенциалните напрежения

$$(31)\qquad \frac{\partial v_r}{\partial z} - a\frac{\partial \hat{v}_r}{\partial z} = \frac{1}{W_l}\left(\frac{\partial \sigma}{\partial C}\right)^0 \frac{\partial C}{\partial r} + \mu_s^* \frac{\partial}{\partial r}(\nabla_r v_r) + (\Lambda^* + G^*)\frac{\partial}{\partial r}(\nabla_r \xi)$$

намираме

$$(32)\qquad \begin{aligned} &\hat{\mu}\left[Ak_0^2 + Bk^2\right] - \mu\left[\frac{h}{k}k_0\beta + hk_0^2E - \frac{h^3k_0}{3\mu}\beta\right] \\ &\quad = -\left(\frac{\partial \sigma}{\partial C}\right)^0 \alpha k_0 \operatorname{ch}(lh) - \mu_s^* k_0^2 E - (\Lambda^* + G^*)k_0^2\frac{kA + Bk}{\omega}\cdot \end{aligned}$$

Ако изразим константите E, β и α от уравненията (24), (26), (30) чрез константите A и B, ще получим две линейни хомогенни уравнения за A и B.

Тогава дисперсионното уравнение на търсения вълнов режим на движение ще се получи от анулирането на съответната детерминанта в системата за A и B. По този начин намираме, че търсеното уравнение за пенни филми има вида

$$(33)\qquad \frac{qh^2\left(1 + \dfrac{h^2k_0^2}{3}\right)}{\omega - \dfrac{h^3}{3\mu}k_0^2} + \frac{\Gamma_0\left(\dfrac{\partial \sigma}{\partial C}\right)^0 \operatorname{sh}(lh)}{D_s\left(\dfrac{\partial \Gamma}{\partial C}\right)^0 k_0^2 \operatorname{sh}(lh) + Dl\operatorname{ch}(lh)} = \mu_s^* + \frac{\Lambda^* + G^*}{\omega} + \mu h.$$

В случая за емулсионни филми дисперсионното уравнение има аналогичен вид.

В анализа, който извършихме дотук, предполагахме, че дебелината на филма h е постоянна, т. е. не отчитахме влиянието на изтъняването върху вълновото движение и обратно. Те обаче могат да се свържат чрез подходяща квазистационарна процедура, а именно, тъй като ζ би зависила от t само чрез h, можем да запишем [23]

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \zeta}{\partial h}\frac{dh}{dt},$$

или

$$\omega.\zeta = -V\frac{\partial \zeta}{\partial h}\cdot$$

След като интегрираме това уравнение, получаваме

$$(34)\qquad \ln\frac{\zeta(h)}{\zeta(h_t)} = -\int_{h_t}^{h}\frac{\omega}{V}\,dh.$$

Тук h_t е дебелината на прехода към неустойчивост. За да пресметнем h_t, трябва да използуваме

$$(35)\qquad q(h_t) = \sigma^f k_0^2 - \left.\frac{d\Pi}{dh}\right|_{h_t} = 0.$$

Така получаваме, че уравнение (34), а следователно и h_{cr} няма да зависят от всеки фактор, който променя V и ω в равна степен.

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Received 04.02.1993