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ON THE TRANSFORMATIONS OF THE LOGARITHMIC SERIES

NIKOLA NAIDENOV

In this paper we consider transformations of the series

$$l(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \text{ and } L(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$$

in the forms: (A) $l(x) = \sum_{n=1}^{\infty} \frac{A_n x^n}{1-\alpha_n x}$, (B) $L(z) = \sum_{n=0}^{\infty} \frac{B_n}{1-b_n z^2} \left(\frac{z}{1-\beta_n z^2}\right)^{4n+1}$ and (C) $l(x) = \sum_{n=1}^{\infty} \frac{C_n x^n}{(1-\gamma_1 x)\cdots(1-\gamma_n x)}$. Minimization of the coefficients in (A) and (B), under the restrictions $|\alpha_n|, |\beta_n| \le 1$, is explored numerically. The resulting hypothesis is that we can accelerate the convergence like a geometric progression. We prove that the unique lacunary series $l(x) = \sum_{i=0}^{\infty} \frac{A_i x^{2i+1}}{1-\alpha_i x}$ and $L(z) = \sum_{i=0}^{\infty} \frac{B_i z^{4i+1}}{1-b_i z^2}$ diverge for $x \ne 0$ and $z \ne 0$. Assuming $|\gamma_n| \le 1$ we prove lower and upper bounds for the optimal rate of convergence of (C). A similar upper bound for (A) is proved. Also, some new accelerated series for the logarithmic and other transcendental functions are obtained.

Keywords: Logarithm, Series acceleration, Recurrences, Rational approximation.

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1. INTRODUCTION

In this study we consider some rational transformations of the series

$$f(x) := a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \tag{1}$$

which is assumed to have radius of convergence equal to 1. Mainly, we restrict our attention to representations of $l(x) := \ln\Big(\dfrac{1}{1-x}\Big)$, i.e. with $a_n = \dfrac{1}{n}$, and other related functions of the forms

$$f(x) \approx \frac{A_1 x}{1-\alpha_1 x} + \frac{A_2 x^2}{1-\alpha_2 x} + \frac{A_3 x^3}{1-\alpha_3 x} + \cdots \tag{2}$$

and

$$f(x) \approx \frac{C_1 x}{1-\gamma_1 x} + \frac{C_2 x^2}{(1-\gamma_1 x)(1-\gamma_2 x)} + \frac{C_3 x^3}{(1-\gamma_1 x)(1-\gamma_2 x)(1-\gamma_3 x)} + \cdots . \tag{3}$$

The symbol "$\approx$" can be considered as coincidence of formal power series, or as asymptotic expansion for $x \to 0$. The goal is to obtain series that converge faster than the initial one and that coincide with the corresponding function in a neighborhood of $x = 0$. The form (2) is a sum of geometric series, while (3) is similar to a Newton series and having the same computational efficiency as (2) it allows much easier treatment.

Everywhere in this paper, if the area of validity of an equality involving series is not specified, then it can be considered as certain neighborhood of the origin or more specifically, the disk $\{w \in \mathbb{C} : |w| < |p_1|\}$, where p_1 is the closest to 0 non-zero singular point (sometimes 0 will be a removable singularity).

As there are extremely fast methods for computing the logarithmic function (see e.g. [4, Ch.1.3]), transformations (2) and (3) of (1) do not bring something new in this area. Actually, $l(x)$ serves as a model function in studying the possibilities of the forms like (2) and (3) for acceleration of power series. Such transformations can occur in calculating other transcendental functions like $Li_k(x)$ or the Euler digamma function. Another aim of the study is to point out to some interesting and difficult analytical problems which appear meanwhile.

Note that for the transformation of (1) in the form (2) (similarly for (3)) the convergence of the series does not matter. Given $\{a_n\}$, if we fix the series $\{\alpha_n\}$, then the numbers $\{A_n\}$ in (2) are obtained easily by the recursive formulas

$$A_1 \alpha_1^{n-1} + A_2 \alpha_2^{n-2} + \cdots + A_{n-1}\alpha_{n-1} + A_n = a_n. \tag{4}$$

Conversely, if we choose in advance $\{A_n\}$, then the numbers $\{\alpha_n\}$ are obtained by the same formulas, provided no division by zero is encountered. Formally, it is an easy task to rewrite the series (1) in the form (2) with coefficients $\{A_n\}$ that tend arbitrarily fast to 0. However, the requirement the series in (2) to converge to $f(x)$ in a neighborhood of $x = 0$ poses the restriction on the poles $\{1/\alpha_n\}$ to be distinct from zero, that is, the sequence of parameters $\{\alpha_n\}$ to be bounded.

What we have is a coding of the power series (1) by using twice as much parameters $\{A_n, \alpha_n\}$ (or $\{C_n, \gamma_n\}$). From this point of view we arrive at an extremal problem of optimizing over the extra parameters according to certain minimization criterion. We shall try to formulate simple criteria in order to decompose the minimization of the overall series $\{A_n\}$ ($\{C_n\}$) by greedy type algorithms, which determine the series step by step. Also, we study numerically other rational forms generalizing (2).

Finally, lacunary series are of great interest. We shall prove that the unique transformation of $l(x)$ in the type (2) with $A_{2k} = 0, k = 1, 2, \ldots$ is divergent. In contrast, it is easy to obtain lacunary representations of $l(x)$ in the type (3) that converge. Actually, the well known series for $\ln\Big(\dfrac{1+z}{1-z}\Big)$, $|z| < 1$, can be written as

$$l(x) = 2\Big(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots\Big) =: 2L(z), \quad z = \frac{x}{2-x}, \tag{5}$$

which is of type (3) with parameters sequences $\{\gamma_n\} = \big\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\big\}$ and $\{C_n\} = \Big\{\frac{1}{1}, 0, \frac{1}{3}\Big(\frac{1}{2}\Big)^2, 0, \frac{1}{5}\Big(\frac{1}{2}\Big)^4, 0, \ldots,\Big\}$. Also, this example shows that there is a choice of a bounded sequence $\{\gamma_n\}$ in (3) for $l(x)$, having rate of convergence of $\{C_n\}$ as a geometric series with ratio $\frac{1}{2}$.

The paper is organized as follows. In Section 2 some classical methods for accelerating series are applied to $l(x)$ and $L(z)$. In Section 3 we describe numerical experiments for optimization of the representations of $l(x)$ and $zL(z)$ in the form (2) and the lacunary form $f(x) \approx \sum_i \frac{B_i}{1-b_i x}\frac{x^{2i+1}}{(1-\beta_i x)^{k_i}}$. Using different algorithms we found parameter sequences such that $|\alpha_i|(|\beta_{i-1}|) \le 1$ and $|A_i|(|B_{i-1}|) \le q^{i-1}$, $i = 1, \ldots, i_1$ ($q < 1$). The above representation with $\beta_i = p$ is of particular interest. This special case is partially investigated for convergence in Section 4. As a result, the following theorem is proved there:

Theorem 1. *The unique lacunary representations*

$$a)\ l(x) \approx \sum_{i=0}^{\infty} \frac{A_i x^{2i+1}}{1-a_i x} \quad \text{and} \quad b)\ L(z) \approx \sum_{i=0}^{\infty} \frac{B_i z^{4i+1}}{1-b_i z^2}$$

are divergent for every nonzero value of the argument.

In Section 5 we consider the representation (3) for $l(x)$ and prove the following

Theorem 2. *Let $\{C_n\}$ and $\{\gamma_n\}$ be the parameters in (3) for $f(x) = l(x)$. Then, for every $\varepsilon \in (0, 1]$,*

a) there exists a choice of $\{\gamma_n\}$ such that $\gamma_n \in [0, 1]$ and the corresponding coefficients satisfy $|C_n| < M(4-\varepsilon)^{-n}$ for every $n \in \mathbb{N}$ with some $M = M(\varepsilon)$.

b) there is no choice of $\{\gamma_n\}$ such that $\gamma_n \in [0, 1]$ and $|C_n| < M(8+\varepsilon)^{-n}$ for every $n \in \mathbb{N}$ with some $M = M(\varepsilon)$.

As a consequence of this we obtain

Theorem 3. *Let $f(x) = l(x)$ and the parameters $\{\alpha_n\}_1^\infty$, $\{\gamma_n\}_1^\infty$ satisfy the restrictions $|\alpha_n|, |\gamma_n| \le 1$. Then for the sequences $\{A_n\}_1^\infty$ and $\{C_n\}_1^\infty$ determined by (2) and (3) correspondingly, there is no positive number M such that*

$$|A_n| \le M \cdot 31^{-n} \ \textit{for every}\ \ n \in \mathbb{N} \quad \text{or} \quad |C_n| \le M \cdot 25^{-n} \ \textit{for every}\ \ n \in \mathbb{N}.$$

Also, in this section some concrete series with periodic $\{\gamma_n\}$ are obtained and a comparison of the series (3) with continued fraction representation for $l(x)$ is done. Finally, in Section 6 we consider some accelerated series for other transcendental functions, including $Li_2(x)$ and $\psi(x)$.

We finish the introductory section with presenting another point of view. The form (2) can be considered as a power series with varying coefficients, i.e., $f(x) \approx \sum_{n=0}^{\infty} F_n(x).x^n$, where $\{F_n(x)\}$ are functions of a specific class (in (2), $F_n(x) = \frac{A_n}{1-\alpha_n x}$). Obviously, the simplest choice $F_n(x) = A_n + B_n x$ brings nothing for the acceleration of (1). The next natural choice actually is the complete linear fractional transformation $F_n(x) = \frac{A_n+B_n x}{C_n+D_n x}$. This form perhaps deserves more attention than (2) because of the following property, which is preserved by the form (3), but not by (2). Namely, if the first n poles $\{\gamma_i^{-1}\}_{i=1}^n$ in (3) interchange their order, then the residual (and the n-th partial sum) do not change. Similarly, in the above generalization of (2), we can change the order of two poles, with an appropriate change of the other parameters, so that the residual of the series remains the same. Indeed, let $S = \dfrac{\alpha_n + \beta_n x}{1-\gamma_n x} x^n + \dfrac{\alpha_{n+1} + \beta_{n+1} x}{1-\gamma_{n+1} x} x^{n+1}$ be the sum of two consecutive terms. Then, if $\gamma_{n+1} \neq 0$, we have the identity $S = \dfrac{\bar{\alpha}_n + \bar{\beta}_n x}{1-\gamma_{n+1} x} x^n + \dfrac{\bar{\alpha}_{n+1} + \bar{\beta}_{n+1} x}{1-\gamma_n x} x^{n+1}$, where $\bar{\alpha}_n = \alpha_n$, $\bar{\beta}_n = \alpha_{n+1} - \alpha_n \gamma_{n+1} + \frac{\beta_{n+1}}{\gamma_{n+1}}$, $\bar{\alpha}_{n+1} = \alpha_n \gamma_n + \beta_n - \frac{\beta_{n+1}}{\gamma_{n+1}}$ and $\bar{\beta}_{n+1} = \gamma_n \frac{\beta_{n+1}}{\gamma_{n+1}}$. In the exceptional case $\gamma_{n+1} = 0$ we have $S = \dfrac{\alpha_n + \alpha_{n+1} x}{1 - 0.x} x^n + \dfrac{\alpha_n \gamma_n + \beta_n}{1-\gamma_n x} x^{n+1} + \beta_{n+1} x^{n+2}$ and the last summand can be joined to the next term in the series.

2. SOME SIMPLE EXAMPLES

Let us consider the case $\alpha_n = 1, n = 1, 2, 3, \ldots$. Then it is easily verified that

$$l(x) = \frac{1}{1-x}\Big(x - \frac{x^2}{1.2} - \frac{x^3}{2.3} - \frac{x^4}{3.4} - \cdots\Big).$$

This is a Kummer type acceleration but also it can be explained as follows. $l(x)$ has a singularity at $x = 1$ which have logaritmic order divergence. Then $(1-x)l(x)$ is “more regular”, having at least finite limit when $x \to 1$. This explains why the later function has smaller Maclaurin series than $l(x)$. Following this line of reasoning, for every $r \in \mathbb{N}$, we can write the acceleration formula:

$$\frac{1}{r!}\Big(1 - \frac{1}{x}\Big)^r l(x) = P_{r-1}\Big(\frac{1}{x}\Big) + \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)\ldots(n+r)},$$

where $P_{r-1}(z)$ is a polynomial of degree $r-1$. The proof easily follows if we substitute in the infinite sum $\frac{1}{n(n+1)\ldots(n+r)} = \frac{(-1)^r}{r!}\Delta^r \frac{1}{n}$ by $\frac{1}{r!}\sum_{k=0}^{r}\binom{r}{k}\frac{(-1)^k}{n+k}$. For example, when $r = 2$ it follows that ([3, 1.513])

$$\Big(1-\frac{1}{x}\Big)^2 l(x) = \frac{1}{x} - \frac{3}{2} + 2\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)(n+2)}.$$

Since it is not easy to improve formula (5) for $l(x)$, by the end of this section we are going to accelerate $L(z)$. Similarly as above we get

$$\Big(\frac{1}{z} - z\Big)L(z) = 1 - 2z\Big(\frac{z}{1.3} + \frac{z^3}{3.5} + \frac{z^5}{5.7} + \frac{z^7}{7.9} + \cdots\Big)$$

and

$$\Big(\frac{1}{z} - z\Big)^2 L(z) = \frac{1}{z} - \frac{5}{3}z + 8\Big(\frac{z^3}{1.3.5} + \frac{z^5}{3.5.7} + \frac{z^7}{5.7.9} + \cdots\Big).$$

For another type acceleration let us consider the changes of the variables

$$L(z) = z\sum_{n=0}^{\infty} \frac{z^{2n}}{2n+1} = z\sum_{n=0}^{\infty} \frac{t^n}{2n+1} = z\sum_{n=0}^{\infty} C_n\Big(\frac{t}{1-pt}\Big)^n =: zf(\tau),$$

where $t = z^2$, $\tau = \dfrac{t}{1-pt}$ and p is a real parameter. We shall see that the best choice for p, when the sequence $\{C_n\}$ decreases in the fastest way, is $p = \frac{1}{2}$. Indeed, since the change $\tau = \frac{t}{1-pt}$ and its inverse $t = \frac{\tau}{1+p\tau}$ are regular in a neighborhood of the origin, the same is true for the function $f(\tau)$. The radius of convergence of $f(\tau)$ depends on its smallest singular point. For real τ we have

$$f(\tau) = \begin{cases} \frac{1}{2\sqrt{t}}\ln\frac{1+\sqrt{t}}{1-\sqrt{t}}, & \text{for } t \in (0,1) \\ \frac{1}{2\sqrt{-t}}\arctan\sqrt{-t}, & \text{for } t \in [-1,0)\ , \\ 1, & \text{for } t = 0 \end{cases} \quad t = \frac{\tau}{1+p\tau}.$$

It is quite clear from this expression that the singular points of any analytic continuation of $f(\tau)$ are $\tau = -\frac{1}{p}$ and $\tau = \frac{1}{1-p}$, when $t = 1$. (Note that $\tau = 0$ is a removable singular point.) Then the radius of convergence of f is $R(p) = \min\{\frac{1}{|p|}, \frac{1}{|1-p|}\}$ and it is easy to verify that $\max_{p\in\mathbb{R}} R(p) = R(1/2) = 2$. As a result we conclude that the optimal acceleration of $L(z)$ by this transformation gives coefficients $\{C_n\}$ that tend to 0 like a geometric series with ratio $\frac{1}{2}$. Next, with $p = \frac{1}{2}$, it is easy to check out the identity $f(\tau) + \tau(2+\tau)f'(\tau) = \frac{1+\tau/2}{1-\tau/2}$ from where we find the recurrence formula

$$(2n+1)C_n + (n-1)C_{n-1} = 2^{1-n}, \quad n = 1,2,3,\ldots \quad (C_0 = 1).$$

Thus, the transformed series starts as follows

$$L(z) = z\left[1 + \frac{1}{3}\tau + \frac{1}{2.5!!}\tau^2 + \frac{11}{4.7!!}\tau^3 + \frac{39}{8.9!!}\tau^4 + \frac{633}{16.11!!}\tau^5 + \cdots\right], \quad \tau = \frac{z^2}{1-z^2/2}.$$

An acceleration of the same order but with more explicit coefficients can be obtained using Euler transform applied in certain succession. If $F(x) = \sum_{n=0}^{\infty} a_n x^n$ then the Euler transform is defined by the identity $\dfrac{1}{1+t}F\Big(\dfrac{t}{1+t}\Big) = \sum\limits_{n=0}^{\infty}(\Delta^n a_0)t^n$, where $\Delta a_i = a_{i+1} - a_i$ and $\Delta^n a_i = \Delta(\Delta^{n-1}a_i)$. Sometimes by Euler transform it is understood the particular case for $x = -1$, i.e. when $t = -\frac{1}{2}$, which converts an alternating numerical series usually into a faster converging one. For $a_n = \frac{1}{2n+1}$, $n = 0, 1, 2, \ldots$ it is easy to find that $\Delta^n a_0 = (-1)^n \frac{(2n)!!}{(2n+1)!!}$. Then, the Euler transform leads to

$$L(z) = \frac{z}{1-z^2}\sum_{n=0}^{\infty}\frac{(2n)!!}{(2n+1)!!}(-y)^n, \quad y = \frac{z^2}{1-z^2}, \tag{6}$$

which is the well known series ([3, 1.515])

$$\frac{\ln(\sqrt{y}+\sqrt{1+y})}{\sqrt{1+y}} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!!}{(2n+1)!!}(\sqrt{y})^{2n+1}.$$

Note that the series (6) has approximately the same rate of convergence as (5), and if we apply the Euler transform to (6), then we return exactly at (5). Actually, the idempotence is a general property of the Euler transform after the change $y = -t$ (see [5]). The key observation for accelerating $L(z)$ in this way is that an application of the Euler transform from a larger index is more effective. So, leaving the first term in (5) unchanged and applying Euler transform to the residual we get

$$L(z) = z\left\{1 + \frac{z^2}{1-z^2}\Big[\frac{1}{3} - \frac{2!!}{5!!}y + \frac{4!!}{7!!}y^2 - \frac{6!!}{9!!}y^3 + - \cdots\Big]\right\}, \quad y = \frac{z^2}{1-z^2}.$$

Again leaving the first term in the square brackets and applying the Euler transform to the residual (with argument $-y$) we obtain

$$L(z) = z\left\{1 + \frac{1}{3}y - z^2 y\Big[\frac{2}{3.5} + \frac{2}{5.7}z^2 + \frac{2}{7.9}z^4 + \frac{2}{9.11}z^6 + \cdots\Big]\right\}.$$

Continuing in the same way we find

$$\begin{aligned}
L(z) &= z\left\{1 + \frac{1}{3}y - \frac{2}{3.5}yz^2 - 3y^2z^2\Big[\frac{2!!}{7!!} - \frac{4!!}{9!!}y + \frac{6!!}{11!!}y^2 - \cdots\Big]\right\}\\
&= z\left\{1 + \frac{1}{3}y - \frac{2!}{5!!}yz^2 - \frac{3!}{7!!}y^2z^2 + y^2z^4\Big[\frac{4!!}{5.7.9} + \frac{4!!}{7.9.11}z^2 + \frac{4!!}{9.11.13}z^4 + \cdots\Big]\right\}\\
&= z\left\{1 + \frac{1}{3}y - \frac{2!}{5!!}yz^2 - \frac{3!}{7!!}y^2z^2 + \frac{4!}{9!!}y^2z^4 + 5!!y^3z^4\Big[\frac{4!!}{11!!} - \frac{6!!}{13!!}y + \frac{8!!}{15!!}y^2 - \cdots\Big]\right\}
\end{aligned}$$

and so on to arrive at the series

$$L(z) = z\left\{1 + \frac{1!}{3!!}y - \frac{2!}{5!!}yz^2 - \frac{3!}{7!!}y^2z^2 + \frac{4!}{9!!}y^2z^4 + \frac{5!}{11!!}y^3z^4 - \frac{6!}{13!!}y^3z^6 - \cdots\right\}. \tag{7}$$

The same result can be obtained more directly. Namely, starting from (6) and using the identities $\frac{1}{1-z^2} = 1 + y$ and $y^2 = yz^2(1+y)$ we can transform $L(z)$ as follows:

$$
\begin{aligned}
L(z) &= z\left\{1+\frac{1}{3}y - y^2\Big[\frac{2!!}{5!!} - \frac{4!!}{7!!}y + \frac{6!!}{9!!}y^2 - \frac{8!!}{11!!}y^3 + \cdots\Big]\right\}\\
&= z\left\{1+\frac{1}{3}y - yz^2\Big[\frac{2!}{5!!} + \frac{3!}{7!!}y - \frac{3.4!!}{9!!}y^2 + \frac{3.6!!}{11!!}y^3 - \frac{3.8!!}{13!!}y^4 + - \cdots\Big]\right\}\\
&= z\left\{1+\frac{1}{3}y - \frac{2!}{5!!}yz^2 - \frac{3!}{7!!}y^2z^2 + 3y^3z^2\Big[\frac{4!!}{9!!} - \frac{6!!}{11!!}y + \frac{8!!}{13!!}y^2 - + \cdots\Big]\right\}
\end{aligned}
$$

$$=z\left\{1+\frac{1}{3}y-\frac{2!}{5!!}yz^2-\frac{3!}{7!!}y^2z^2+3y^2z^4\Big[\frac{4!!}{9!!}+\frac{5.4!!}{11!!}y-\frac{5.6!!}{13!!}y^2+\frac{5.8!!}{15!!}y^3-\cdots\Big]\right\}$$

$$=z\left\{1+\frac{1}{3}y-\frac{2!}{5!!}yz^2-\frac{3!}{7!!}y^2z^2+\frac{4!}{9!!}y^2z^4+\frac{5!}{11!!}y^3z^4-5!!y^4z^4\Big[\frac{6!!}{13!!}-\frac{8!!}{15!!}y+\cdots\Big]\right\}$$

$$=z\left\{1+\frac{1}{3}y-\frac{2!}{5!!}yz^2-\frac{3!}{7!!}y^2z^2+\frac{4!}{9!!}y^2z^4+\frac{5!}{11!!}y^3z^4-5!!y^3z^6\Big[\frac{6!!}{13!!}+\frac{7.6!!}{15!!}y-\cdots\Big]\right\}$$

and so on. Finally, let us remark that formula (7) is of type (3) with $z^2 = x$.

3. SOME COMPUTER EXPERIMENTS

1. We start with the choice of the parameters $\{\alpha_n\}_{n=1}^{\infty}$ in the form (2) of $f(x) = l(x)$, suggested by the simplest greedy algorithm. Namely, we choose every next α_n such that $|A_{n+1}|$ to be minimal. Thus we arrive at a lacunary representation of $l(x)$. Let us explain the derivation of the first four coefficients. Clearly $A_1 = 1$ and the requirement $A_2 = 0$ leads, by (4), to the equation $A_1\alpha_1 + 0 = \frac{1}{2}$, i.e. to $\alpha_1 = \frac{1}{2}$. As a result of $A_2 = 0$, we have no control on A_3 and the relation $A_1\alpha_1^2 + A_2\alpha_2 + A_3 = \frac{1}{3}$ gives $A_3 = \frac{1}{12}$. Next, the choice $A_4 = 0$ is possible because the equation $A_1\alpha_1^3 + 0 + A_3\alpha_3 + 0 = \frac{1}{4}$ has a solution $\alpha_3 = \frac{3}{2}$. Continuing in this way we obtain

$\{\alpha_n\} = \{\frac{1}{2}, *, \frac{3}{2}, *, \frac{35}{12}, *, \frac{35077}{6324}, *, \frac{167344077283}{15930229780} = 10.504..., *, 19.899..., *, \ldots\}$;

$\{A_n\} = \{1, 0, \frac{1}{12}, 0, -\frac{1}{20}, 0, \frac{527}{4032}, 0, -\frac{1511407}{1214208}, 0, 42.385..., 0, -5174.4..., \ldots\}$,

where "*" means an arbitrary number. It is seen that the obtained series diverge rapidly and we shall prove this in the next section. An heuristic explanation is from the type of the recurrence relations (4). Once an $|\alpha_n|$ larger than 1 occurs, then larger and larger numbers will appear in (4), which most likely will draw $\{|A_n|\}$ to infinity. A similar behavior is observed in the following lacunary representation

$$L(z) \approx \frac{z}{1-\frac{1}{3}z^2} + \frac{\frac{4}{45}z^5}{1-\frac{25}{21}z^2} - \frac{\frac{4}{147}z^9}{1-\frac{1609}{693}z^2} + \frac{0.043699...\,z^{13}}{1-4.4448...\,z^2} - \frac{0.26698...\,z^{17}}{1-8.4284...\,z^2}\cdots.$$

2. In view of the above observations, in the next two examples we pose the requirement for the summands to be regular in the open unit disk, i.e. $|\alpha_{\mathbf{n}}| \leq \mathbf{1}$. We have that

$$
\begin{aligned}
l(x) \approx & \frac{x}{1-0.2\,x} + \frac{0.3\,x^2}{1-0.8\,x} + \frac{x^3/18.75}{1-\ x} - \frac{x^4/300}{1+0.4\,x} - \frac{x^5/101.35...}{1-0.8\,x} - \frac{x^6/694.44...}{1-\ x} \\
& - \frac{x^7/767.54...}{1-0.x} - \frac{x^8/2425.6...}{1+\ x} - \frac{x^9/9582.6...}{1-\alpha_9\,x} \pm \cdots ,
\end{aligned}
$$

where the coefficients satisfy $|A_n| \leq 3^{1-n}$, $n = 1, ..., 9$. For the method used for obtaining this series see the next example. Now we formulate the following

Hypothesis 1. There is a choice of $\{\alpha_n\} \subset \mathbb{R}$ with $|\alpha_n| \leq 1$ such that the coefficients in the form (2) of $l(x)$ satisfy $|A_n| \leq M\,q^n$ for some $M > 0$ and $q < 1$.

For the function L we found that

$$
\begin{aligned}
L(z) \approx & \frac{z}{1-\frac{51}{350}\,z^2} + \frac{z^3/5.329...}{1-\frac{1009}{1400}\,z^2} + \frac{z^5/22.96...}{1-\ z^2} - \frac{z^7/806.9...}{1+\frac{59}{100}\,z^2} - \frac{z^9/259.3...}{1-\ z^2} \\
& + \frac{z^{11}/1039.7...}{1-\frac{1567}{2100}\,z^2} - \frac{z^{13}/4274.2...}{1+\ z^2} + \frac{z^{15}/16697.9...}{1+\ z^2} - \frac{z^{17}/73749.6...}{1-\alpha_9\,z^2} ...,
\end{aligned}
\tag{8}
$$

where the n-th coefficient is less than 4^{1-n} for $n \leq 9$. The method is the following branch and bound algorithm. Fix an integer m and consider k nested cycles for $\alpha_n, n = 1, ..., k$ ranging from -1 to 1 with step $2/m$. The bound is $A_{n+1} \leq 4^{-n}$ and if this is not fulfilled, the corresponding cycle continues with the next iteration, avoiding going into deeper levels. The algorithm works successfully up to $k = 7$. For (8), a modification was used to justify the coefficients to $k = 8$.

3. Consider the following lacunary representation

$$
L(z) \approx \sum_{k=0}^{\infty} B_k \Big(\frac{z}{1-\beta_k z^2} \Big)^{4k+1}. \tag{9}
$$

The parameters in (9) are uniquely determined, with the first several of them given by:

$$
\begin{aligned}
& B_0 = 1, \quad \beta_0 = \frac{1}{3}; && B_1 = \frac{4}{45} = \frac{1}{11.25}, \quad \beta_1 = \frac{5}{21} = 0.23809...; \\
& B_2 = \frac{92}{63^2} = \frac{1}{43.14...}, && \beta_2 = \frac{163}{759} = 0.21475...; \\
& B_3 = \frac{22458728}{3015483471} = \frac{1}{134.2...}, && \beta_3 = \frac{4150546877}{20339185545} = 0.20406...; \\
& B_4 = \frac{1}{378.6...}, && \beta_4 = 0.197876...; \\
& B_5 = \frac{1}{1007.8...}, && \beta_5 = 0.193803...;
\end{aligned}
$$

Thus, we can formulate the following

Hypothesis 2. The representation (9) of $L(z)$ converges when z belongs to a certain disk centered at the origin.

Note that the analogous representation for $l(x)$ leads exactly to (5).

4. Let us consider the following combination of (2) and the above form:

$$L(z) \approx \frac{B_0}{1-b_0z^2}\left(\frac{z}{1-\beta_0z^2}\right)+\frac{B_1}{1-b_1z^2}\left(\frac{z}{1-\beta_1z^2}\right)^5+\frac{B_2}{1-b_2z^2}\left(\frac{z}{1-\beta_2z^2}\right)^9+\cdots. \tag{10}$$

In this form, keeping the lacunary property, we have a series of extra parameters in order to optimize the coefficients. Say if we choose $\{\beta_n\}$, the parameters $\{B_n\}$ and $\{b_n\}$ are uniquely determined, provided it does not appear division by zero. The following choice satisfy: $|\beta_n|, |b_n| \le 1$ for $n = 0, ..., 3$; $|B_n| \le 20^{-n}$ for $n = 0, ..., 4$; and provide a possibility for arbitrary small $|B_5|$ with $|\beta_4| \le 1$:
$\{\beta_n\}_0^3 = \{0.4254, 0.1427, 0.0238, 0.411\}$;
$\{b_n\}_0^3 = \{-0.092066..., 0.889557..., 0.925184..., -0.478074...\}$;
$\{B_n\}_0^4 = \{1, 1/20.1111..., -1/521.310..., 1/19118.7..., 1/161497.8...\}$.

The method is by considering the graphs of two consecutive B_n and B_{n+1} with respect to β_{n-1} and β_n in order to choose β_{n-1}. The graph of $B_n(\beta_{n-1})$ is a parabola and we introduce the notion *balanced choice* of the previous parameters if the graph intersects the abscissa for $\beta_{n-1} \in [-1, 1]$, that is if we can make $|B_n|$ arbitrarily small. But if we take $B_n = 0$ then B_{n+1} becomes undefined because of division by zero. This is clearly seen from the second graph of $B_{n+1}(\beta_{n-1}, \beta_n)$ which has infinite branches, at the places where $B_n = 0$. So, it is good to choose β_{n-1} close to these vertical asymptotes (the zeros of $B_n(\beta_{n-1})$) so that the corresponding section of the 3D graph (which is the planar graph for the next step) crosses the zero level. Actually, considering the 3D graphs is an auxiliary process, and we can avoid this. We can try several specific values of β_{n-1} close to the zeros of $B_n(\beta_{n-1})$ so that $|B_n|$ is small and the next graph of $B_{n+1}(\beta_n)$, $\beta_n \in [-1, 1]$ has zeros, i.e. the choice of β_{n-1} to be balanced. If, say, $B_n(\beta_{n-1})$ has two zeros in $[-1, 1]$, then it can happen to exist four appropriate areas for choosing β_{n-1}, on the both sides of the two zeros. An additional reasoning which helps the choice is the goal to keep the parameters $\{b_n\}$ in $[-1, 1]$. Then, the choice of β_{n-1} has to be such that $|b_{n-1}| \le 1$ and since the function $b_n(\beta_n) = A(\beta_{n-1})\beta_n + B(\beta_{n-1})$ is linear, it is easy to estimate in advance the range of b_n when $\beta_n \in [-1, 1]$.

A natural question is if there exists a balanced choice of $\{\beta_{n-1}\}$ for every $n \in \mathbb{N}$.

Revisiting example (9) considered as a particular case of (10) we make the following observations. The choice of $\{\beta_i\}_{i=0}^{n-1}$ is balanced up to $n = 30$, as the graphs of $B_{n+1}(\beta_n)$ (with specified previous $\{\beta_i\}$) have two roots in $[-1, 1]$ and the specific value for β_n in (9) is between the middle of them and the second root. It seems that the series $\{\beta_n\}$ has a limit around 0.17 and the ratios B_{n+1}/B_n belong to $(0, 1/2)$.

Other interesting choice is to specify β_n at the extreme point of the parabola $y = B_{n+1}(\beta_n)$. On the basis of calculations made up to $n = 30$, the situation appears to be very similar to the one described above, but now β_n is exactly the middle of the two roots of $B_{n+1}(\beta_n)$. Surprisingly, we observe that $b_n = \beta_n$ and seemingly this series tends to the same limit as above.

Clearly, there is much subjectivity in the approach described above, but it is not easy to avoid it. For example, if we use the least squares criterion $M_n = \lambda_n B_n^2 + (1-\lambda_n)B_{n+1}^2 \to \min$ then the subjectivity transfers to the choice of the λ-s. The function $M_n(\beta_{n-1}, \beta_n)$ usually has several local extrema and a decent optimization of the sequence $\{B_n\}_1^k$ needs considering of a tree of possibilities. Note that the attempt to manage the parameters by minimizing of the three term sums $\lambda_n B_n^2 + \mu_n B_{n+1}^2 + \nu_n B_{n+2}^2$ was not successful because of the complicatedness of this three variable function.

4. CONVERGENCE CONSIDERATIONS

Let us consider the representation (10) with equal parameters $\beta_n = p, n = 0, 1, 2, \ldots.$ This form is motivated as a simple generalization of the lacunary variant of (2) (for $zL(z), x = z^2$), which hopefully will converge for certain p. We start the study of the series $\{B_n(p)\}$ and $\{b_n(p)\}$ with some particular examples. For $p = 0$ we have the second lacunary example from 3.1, while for $p = \frac{1}{4}$ we have

$$\begin{aligned}\{B_n\} = \{&1, 1.0972...10^{-1}, 2.1442...10^{-2}, 4.5910...10^{-3}, 8.9862...10^{-4}, -7.5297...10^{-5},\\ &3.0626...10^{-3}, -3.8502...10^{-2}, 3.4662..., -1.2595...10^{3}, 1.6502...10^{6}, \ldots\};\\ \{b_n\} = \{&1/12, -0.15898..., -0.25539..., -0.20173..., 0.17256..., -6.7384..., -8.4293...,\\ &-17.216..., -33.759..., -64.545..., -122.64..., \ldots\}.\end{aligned}$$

The behavior of this sequence is typical: For common values of p, in the beginning $|B_n|$ decreases like a geometric series, later on the decreasing slows down and changes to increasing and finally we observe again a rapid divergence to ∞. Slightly before the turning of $\{B_n\}$ it is preceded by breaking the restriction $|b_n| \le 1$. Especially, for $p = 0.17$ the decreasing lasts up to $n = 336$, when $B_{336} = 9.1654...10^{-119}$, and after that again $|B_n|$ goes to ∞. A natural question is whether there exist real values of p for which $\{B_n(p)\}_0^\infty$ is bounded. However, a numerical search for such values encounters some difficulties. For example, the above number was obtained by using long arithmetics and a precision of 200 decimal digits was not sufficient.

Usually we get the limit behavior $b_n \approx A.q^n$ and $B_n \approx (-1)^n B.q^{n^2-\alpha n}$ with $q > 1$. While A, B and α in the above empirical formulas depend on p, it is interesting that $q \approx 1.894$ is an absolute constant. Indeed, assume that the above relations hold as asymptotic equivalences "$\sim$" and $|q| > 1$. By (10) with $\beta_n = p$ we obtain the following system for the coefficients B_n and b_n:

$$
\begin{aligned}
B_0 \sum_{i=0}^{2n} p^i b_0^{2n-i} + B_1 \sum_{i=0}^{2n-2} \binom{4+i}{i} p^i b_1^{2n-2-i} + B_2 \sum_{i=0}^{2n-4} \binom{8+i}{i} p^i b_2^{2n-4-i} + \cdots + B_n \;&=\; \frac{1}{4n+1}, \\
B_0 \sum_{i=0}^{2n+1} p^i b_0^{2n+1-i} + B_1 \sum_{i=0}^{2n-1} \binom{4+i}{i} p^i b_1^{2n-1-i} + B_2 \sum_{i=0}^{2n-3} \binom{8+i}{i} p^i b_2^{2n-3-i} + \cdots & \\
+ B_n(b_n + (4n+1)p) \;&=\; \frac{1}{4n+3}.
\end{aligned}
\tag{11}
$$

We observe that, for a sufficiently large n the back terms in (11) are significant, while the first terms are relatively small (we assume that $|p| < 1$). Also, the first summands $b_{n-j}^{2j+\delta}, \delta \in \{0, 1\}$ in the rear sums (for $i = 0$) are equivalent to the whole sums. For example, next to the last term in the left hand side of the first equation is $B_{n-1}(b_{n-1}^2 + (4n-3)pb_{n-1} + (2n-1)(4n-3)p^2) \sim B_{n-1}b_{n-1}^2$ for $n \to \infty$. That's why the terms containing p are negligible for $n \to \infty$ according to the assumption. Thus, for a sufficiently large n we come to the limit system

$$
\begin{aligned}
&B_n.1 + B_{n-1}.b_{n-1}^2 + B_{n-2}.b_{n-2}^4 + \cdots + B_0.b_0^{2n} = o\big(B_n\big), \\
&B_n.b_n + B_{n-1}.b_{n-1}^3 + B_{n-2}.b_{n-2}^5 + \cdots + B_0.b_0^{2n+1} = o\big(B_n\big).
\end{aligned}
$$

Substituting here the asymptotic relations for b_n and B_n we see that A^{2n} and $q^{\alpha n}$ are combined. Then, with $u = A^2 q^{\alpha}$, letting $n \to \infty$ we come to the following system

$$
\begin{aligned}
&1 - uq^{-1^2} + u^2.q^{-2^2} - u^3.q^{-3^2} + \cdots = 0, \\
&1 - uq^{-1.2} + u^2.q^{-2.3} - u^3.q^{-3.4} + \cdots = 0.
\end{aligned}
$$

We did not investigate this system for all real solutions, but considering truncated systems, which are algebraic, we found a series of real solutions that stabilizes to $(q, u) = (1.8947..., 6.1450...)$.

In order to understand better the behavior of the series $\{B_n\}$ and $\{b_n\}$ we consider first the truncated recurrence system

$$
\begin{aligned}
&B_n + B_{n-1}\Big[b_{n-1}^2 + \binom{4n-3}{1} pb_{n-1} + \binom{4n-2}{2} p^2\Big] = \frac{1}{4n+1} \\
&B_n\big(b_n + (4n+1)p\big) + B_{n-1}\Big[b_{n-1}^3 + \binom{4n-3}{1} pb_{n-1}^2 + \binom{4n-2}{2} p^2 b_{n-1} + \binom{4n-1}{3} p^3\Big] = \frac{1}{4n+3}.
\end{aligned}
\tag{12}
$$

and its specification (with $p = 0$)

$$
\begin{aligned}
B_n + B_{n-1}b_{n-1}^2 &= \frac{1}{4n+1} \\
B_n b_n + B_{n-1}b_{n-1}^3 &= \frac{1}{4n+3}.
\end{aligned}
\tag{13}
$$

The system (13) appears to have a similar behavior as (11) in qualitative sense, but in quantitative sense it is weaker. Depending on B_0 and b_0, in the general case the series fluctuates in the beginning and from some place on stabilizes to the asymptotic formulas $b_n \to \rho$, $|\rho| > 1$ and $B_n \sim C(-\rho^2)^n$. It is possible that the series terminates if some B_k vanishes and consequently $b_k = c/0$. We shall prove a "divergence criterion" which imply the asymptotic formulas (if they hold) for a concrete initial pair (B_0, b_0).

Consider first the case when $B_k = \varepsilon$ for a sufficiently small $|\varepsilon|$. Then, $b_k = c/\varepsilon$ and let us assume that $|c|$ is not very small, say $\varepsilon = o(c)$ for $\varepsilon(B_0, b_0) \to 0$ and $B_0 = const$. For the next terms, we find from (13)

$$
\begin{aligned}
B_{k+1} &= \frac{1}{4k+5} - \frac{c^2}{\varepsilon} = -\Big(\frac{c}{\varepsilon}\Big)^2 B_k \Big(1 + O(\varepsilon)\Big)\\
b_{k+1} &= \Big(\frac{1}{4k+7} - \frac{c^3}{\varepsilon^2}\Big)/B_{k+1} = b_k \Big(1 + O(\varepsilon)\Big).
\end{aligned}
$$

Similarly, $B_{k+j} = \Big(-\dfrac{c^2}{\varepsilon^2}\Big)^j B_k \Big(1 + O(\varepsilon)\Big)$ and $b_{k+j} = b_k \Big(1 + O(\varepsilon)\Big)$ for every fixed $j \in \mathbb{N}$. Thus, if B_k happens to be very close to 0, then the asymptotic formulas take place immediately after k with a large ρ.

Proposition 1. *Let $\{(B_n, b_n)\}$ satisfy (13) and for a fixed k the conditions $|b_{k-1}| \geq r > 1$ and $|B_{k-1} b_{k-1}^2| \geq \dfrac{Q}{4k+1}$ with $Q > 3$ hold true. Then*

$$
|b_k| \geq r - \frac{1+r}{Q-1} \quad \textit{and} \quad |B_k b_k^2| \geq \frac{r^2(Q-3)^2}{(4k+5)(Q-1)}.
$$

Proof. From (13) it follows $|B_k| \geq |B_{k-1} b_{k-1}^2| - \frac{1}{4k+1}$ and $B_k \frac{b_k}{b_{k-1}} + \Big(\frac{1}{4k+1} - B_k\Big) = \frac{1/b_{k-1}}{4k+3}$. Therefore, $|B_k| \geq \frac{Q-1}{4k+1}$ and $|B_k| \cdot \Big|\frac{b_k}{b_{k-1}} - 1\Big| \leq \frac{1}{4k+1} + \frac{1/r}{4k+3} \leq \frac{1+1/r}{4k+1}$. As a consequence we have $\Big|\frac{b_k}{b_{k-1}} - 1\Big| \leq \frac{1+1/r}{Q-1}$. Hence, $\frac{b_k}{b_{k-1}} \geq 1 - \frac{1+1/r}{Q-1}$, and since the latter number is positive ($Q > 3$), we obtain that

$$
|b_k| \geq |b_{k-1}| \Big(1 - \frac{1+1/r}{Q-1}\Big) \geq r\Big(1 - \frac{1+1/r}{Q-1}\Big) = r - \frac{1+r}{Q-1}
$$

and

$$
|B_k b_k^2| \geq \frac{Q-1}{4k+1} \Big(\frac{rQ - 2r - 1}{Q-1}\Big)^2 > \frac{r^2(Q-3)^2}{(4k+5)(Q-1)}.
$$

□

The following assertion (which is a divergence criterion) makes use of the fact that for a sufficiently large Q, the estimates from Proposition 1 essentially repeat recursively and imply that $\{B_n\}$ increases at least as a geometric sequence.

Corollary 1. *Let $\{(B_n, b_n)\}$ satisfy (13) and for a given k the estimates $|b_k| \geq q+\varepsilon$ and $|B_k b_k^2| \geq \frac{Q}{4k+5}$ hold true, where $q>1$, $\varepsilon>0$ and $Q>3$. If in addition $\varepsilon(Q-3) \geq \frac{q^2}{q-1}$, then for every $j \in \mathbb{N}_0$ we have*

$$|b_{k+j}| \geq q, \quad |B_{k+j} b_{k+j}^2| \geq \frac{(Q-3)q^{2j}}{4(k+j)+5} \quad \textit{and} \quad |B_{k+j+1}| \geq \frac{(Q-3)q^{2j}-1}{4(k+j)+5}.$$

Before proving Corollary 1, we will prove a technical lemma.

Lemma 1. *For given $q>1$, $\varepsilon>0$ and $Q>3$ let us define the sequences $\{\varepsilon_j\}_0^\infty$ and $\{Q_j\}_0^\infty$ by $\varepsilon_0=\varepsilon$, $(q+\varepsilon_{j+1}) = (q+\varepsilon_j) - \frac{1+(q+\varepsilon_j)}{Q_j-1}$ and $Q_0=Q$, $Q_{j+1} = (q+\varepsilon_j)^2 \frac{(Q_j-3)^2}{(Q_j-1)}$. If in addition $\varepsilon(Q-3) \geq \frac{q^2}{q-1}$, then for every $j \in \mathbb{N}_0$ the inequalities $\varepsilon_j \geq \frac{q^2}{(q-1)(Q_j-3)} > 0$ and $(Q_j-3) \geq (Q-3)q^{2j} > 0$ hold true.*

Proof. Clearly, it is enough to prove the assertion only for $j=1$, as for larger j it follows inductively. For brevity, set $\bar{Q} := Q-3$.

We start with the proof of inequality $(Q_1-3) \geq (Q-3)q^2$. It is equivalent to

$$(q+\varepsilon)^2 \bar{Q}^2 \geq (\bar{Q}q^2+3)(\bar{Q}+2).$$

In view of the additional assumption for ε, the above will follow from

$$q^2\bar{Q}^2 + \frac{2q^3}{q-1}\bar{Q} + \frac{q^4}{(q-1)^2} \geq (\bar{Q}q^2+3)(\bar{Q}+2),$$

which is $\dfrac{2q^3}{q-1}\bar{Q} + \dfrac{q^4}{(q-1)^2} \geq (2q^2+3)\bar{Q}+6$ and easily follows by termwise comparison of the summands in the left- and right-hand sides, taking into account that $q>1$.

The inequality $\varepsilon_1 \geq \dfrac{q^2}{(q-1)(Q_1-3)}$, by the definitions and the just proved $Q_1>3$, is equivalent to

$$\left[\varepsilon\Big(\frac{Q-2}{Q-1}\Big) - \frac{1+q}{Q-1}\right] \cdot \left[(q+\varepsilon)^2 \frac{(Q-3)^2}{Q-1} - 3\right] \geq \frac{q^2}{q-1}.$$

It is not difficult one to verify that the first factor in the left-hand side is positive, as it is positive for ε replaced with its lower bound $\frac{q^2}{(q-1)(Q-3)}$. Therefore, the above inequality will hold true if it is true with $\varepsilon = \frac{q^2}{(q-1)(Q-3)}$, which is

$$\left[\frac{q^2}{q-1} \cdot \frac{Q-2}{(Q-1)(Q-3)} - \frac{1+q}{Q-1}\right] \cdot \left[\Big(q + \frac{q^2}{(q-1)(Q-3)}\Big)^2 \frac{(Q-3)^2}{Q-1} - 3\right] \geq \frac{q^2}{q-1}.$$

The latter is equivalent to the inequality

$$\big[q^2+\bar{Q}\big] \cdot \big[q^2(q-1)^2\bar{Q}^2 + 2q^3(q-1)\bar{Q} + q^4 - 3(q-1)^2(\bar{Q}+2)\big] \geq q^2(q-1)^2(\bar{Q}+2)^2\bar{Q}$$

which after simplification takes the form $A\bar{Q}^2 + B\bar{Q} + C \geq 0$ with coefficients $A = (q-1)\big(2q^3 + (q-1)(q^4 - 4q^2 - 3)\big)$, $B = q^4 + 2(q-1)q^5 - (q-1)^2(7q^2+6)$ and $C = q^2(q^4 - 6(q-1)^2)$.

It is easy to verify that polynomials A, B and C are positive for $q > 1$. The positivity C follows from $q^4 - 6(q-1)^2 = (q-1)^4 + 4(q-1)^3 + 4(q-1) + 1 > 0$. To check that $A > 0$ we write $2q^3 + (q-1)(q^4 - 4q^2 - 3) = 2q^3 - 7(q-1) + (q-1)(q^2-2)^2$ and $2q^3 - 7(q-1) = 2(q-1)^3 + 6(q-1)^2 - (q-1) + 1 > 0$ since $q-1$ is majorized either by $(q-1)^2$ or by 1. Finally, to verify that $B > 0$ we rewrite it as $\big(q^4 - 6(q-1)^2\big) + q^2(q-1)\big(2q^3 - 7(q-1)\big)$, where the positivity of the both summands was already shown. The lemma is proved. $\square$

Proof of Corollary 1. Define the sequences $\{\varepsilon_j\}_0^\infty$ and $\{Q_j\}_0^\infty$ as in Lemma 1. The definitions are coherent with Proposition 1 so that (by induction) $|b_{k+j}| \geq q + \varepsilon_j$ and $|B_{k+j} b_{k+j}^2| \geq \frac{Q_j}{4(k+j)+5}$ for $j \geq 0$. The conditions of Proposition 1, $q + \varepsilon_j > 1$ and $Q_j > 3$, are ensured by Lemma 1 on the basis of the additional condition for ε and Q. Furthermore, the estimates from Lemma 1, $\varepsilon_j \geq 0$ and $(Q_j - 3) \geq (Q-3)q^{2j}$, imply the first two claimed estimates in the corollary. The third inequality is an elementary consequence from the second one and the first row of (13). $\square$

The next assertion claims that, essentially, the above lower estimates describe the asymptotical behavior of the series generated by (13).

Corollary 2. *Under the conditions of Corollary 1 the asymptotic relations $b_n \to \rho$, $|\rho| > 1$ and $B_n \sim C\,(-\rho^2)^n$ hold for $n \to \infty$, where $\rho = \rho(B_0, b_0)$ and $C = C(B_0, b_0)$.*

Proof. Denote $B_n b_n^2$ by M_n. Increasing if necessary the index k in Corollary 1, we may assume that $Q > 6$, hence $(4(k+j)+5)M_{k+j} > 3$ for every $j > 0$. Using (13) we obtain $B_{k+j+1} = -M_{k+j}\left(1 - \frac{1}{(4(k+j)+5)M_{k+j}}\right)$ and

$$b_{k+j+1} = \frac{-M_{k+j} b_{k+j} + \frac{1}{4(k+j)+7}}{B_{k+j+1}} = \frac{b_{k+j} - \frac{1}{(4(k+j)+7)M_{k+j}}}{1 - \frac{1}{(4(k+j)+5)M_{k+j}}} =: b_{k+j}\big(1 + \tau_{k+j}\big).$$

Then we have

$$|\tau_{k+j}| = \left| \frac{\frac{1}{(4(k+j)+5)M_{k+j}} - \frac{1}{(4(k+j)+7)b_{k+j}M_{k+j}}}{1 - \frac{1}{(4(k+j)+5)M_{k+j}}} \right| \leq \frac{\left(\frac{1}{4(k+j)+5} + \frac{q}{4(k+j)+7}\right)\frac{1}{|M_{k+j}|}}{1 - \frac{1}{(4(k+j)+5)|M_{k+j}|}}$$
$$\leq \frac{2}{(4(k+j)+5)|M_{k+j}|}\,\frac{1}{1-1/3} \leq \frac{3}{(Q-3)q^{2j}} \leq \frac{1/2}{q^{2j}}.$$

Therefore, for every natural $n \geq k$ and j the inequalities

$$\frac{b_{n+j}}{b_n} = \prod_{i=0}^{j-1}(1 + \tau_{n+i}) \geq \prod_{i=0}^{j-1}(1 - |\tau_{n+i}|) \geq \prod_{i=0}^{\infty}(1 - |\tau_{n+i}|) \geq 1 - \sum_{i=0}^{\infty}|\tau_{n+i}|$$

and

$$\frac{b_{n+j}}{b_n} = \prod_{i=0}^{j-1}(1+\tau_{n+i}) \le \prod_{i=0}^{j-1}(1+|\tau_{n+i}|) \le \prod_{i=0}^{\infty}(1+|\tau_{n+i}|) \le \exp\Big(\sum_{i=0}^{\infty}|\tau_{n+i}|\Big)$$

hold true. As a consequence, in view of the estimates for τ_m, we obtain

$$-\frac{(1/2)}{q^{2(n-k)}(1-q^{-2})} \le \frac{b_{n+j}}{b_n} - 1 \le \exp\Big(\frac{(1/2)}{q^{2(n-k)}(1-q^{-2})}\Big) - 1.$$

Since these bounds can be arbitrarily close to 0 for a sufficiently large n and $\{|b_m|\}$ is bounded (by the same inequalities with $n = k$), we conclude that the sequence $\{b_m\}$ is fundamental, and hence convergent to a limit ρ with $|\rho| \ge q > 1$.

In addition, letting j to infinity, we find the estimates

$$-\frac{1/2}{q^{2(n-k)}(1-q^{-2})} \le \frac{\rho}{b_n} - 1 \le \exp\Big(\frac{1/2}{q^{2(n-k)}(1-q^{-2})}\Big) - 1, \qquad n \ge k.$$

Or, we can simplify these to

$$\Big|\frac{\rho}{b_n} - 1\Big| \le \frac{c(q)}{q^{2(n-k)}}, \quad n \ge k,$$

where $c(q) = \max_{x\in[0,1]} \frac{1}{x}\Big[\exp\Big(\frac{x/2}{1-q^{-2}}\Big) - 1\Big]$. Then, with $b_n =: \rho/(1+\theta_n)$, the bound $|\theta_n| \le c(q)q^{2(k-n)}$ holds for $n \ge k$. Next, by the first equation in (13) it follows that

$$\frac{B_{n+1}}{B_n} = -b_n^2\Big(1 - \frac{1}{b_n^2(4n+5)B_n}\Big) = \frac{-\rho^2}{(1+\theta_n)^2}\Big(1 - \frac{1}{(4n+5)M_n}\Big).$$

Thus, for $n \ge k$ we have

$$B_n = B_k(-\rho^2)^{n-k}\prod_{j=0}^{n-k-1}\big(1+\theta_{k+j}\big)^{-2}\Big(1 - \frac{1}{(4(k+j)+5)M_{k+j}}\Big).$$

Finally, the estimates $|\theta_{k+j}| \le c(q)q^{-2j}$ and $\frac{1}{(4(k+j)+5)|M_{k+j}|} \le \frac{1}{(Q-3)q^{2j}}$ ensure the convergence of the infinite product $P := \prod_{j=0}^{\infty}\big(1+\theta_{k+j}\big)^{-2}\Big(1 - \frac{1}{(4(k+j)+5)M_{k+j}}\Big)$. (In view of Corollary 1 we have $B_n \ne 0$ and the all factors in P do not vanish.) Therefore, the partial product is asymptotically equivalent to its limit and we obtain $B_n \sim B_kP(-\rho^2)^{n-k} = C(-\rho^2)^n$. The proof is complete. □

Let us consider an example for application of Corollary 1. Let $B_0 = 1$ and $b_0 = 0$. By (13) we obtain: $B_1 = 1/5$, $b_1 = 5/7$; $B_2 = 4/441$, $b_2 = 153/77$; $B_3 = 0.04111...$, $b_3 = -0.10924...$; $B_4 = 0.05833...$, $b_4 = 0.90318...$; $B_5 = 3.4754...\times 10^{-5}$, $b_5 = 14.4152...$; $B_6 = 0.03277...$, $b_6 = -2.04617...$; $B_7 = -0.10275...$, $b_7 =$

$-3.04678...$; $B_8 = 0.98415...$, $b_8 = -2.92393...$; $B_9 = -8.38692...$, $b_9 = -2.93641...$; etc. It can be verified that the assumptions of Corollary 1 hold for $k = 7$ with $q = 2.8$, $\varepsilon = 0.2$ and $Q = 31$. Then, this particular sequence tends to infinity like a geometrical series and the lower bounds $|b_n| \geq 2.8$ and $|B_{n+1}| \geq \frac{28(2.8)^{2n-14}-1}{4n+5}$ hold for all $n \geq 7$.

Remark 1. It seems that there are bounded solutions of (13) even with $B_0 = 1$. We have not a strict proof but there is a particular candidate - the sequence with $B_0 = 1$ and $b_0 = b^*$, where $b^* \in (0.9512609, 0.9512610)$.

Let us turn our attention to the system (12). For $p \neq 0$ the usual limit behavior of the sequences defined by (12) is $B_n \sim C(p, b_0, B_0)(-4p^2)^n n![(n-1)!]^3$ and $b_n \sim -2pn^2$. A divergence criterion is given by the following

Proposition 2. *Let $\{(B_n, b_n)\}$ satisfy (12) and for certain $k \geq \frac{1}{2|p|}$ there holds $|B_k| \geq 1$. Then the sequence $\{|B_n|\}$ tends to infinity faster than any geometrical series.*

Proof. Let us set $L_n := b_n^2 + (4n+1)pb_n + \binom{4n+2}{2}p^2$. It is easily verified that $L_n \geq \frac{p^2}{4}(4n+1)(4n+3)$. Then by (12) and $|B_k| \geq 1$ we have

$$\begin{aligned} |B_{k+1}| = \; & \Big|\frac{1}{4k+5} - B_k L_k\Big| \geq \Big(L_k - \frac{1}{(4k+5)|B_k|}\Big)|B_k| \\ \geq \; & \Big(L_k - \frac{1}{(4k+5)}\Big)|B_k| \geq \Big(\frac{p^2}{4}(4k+1)(4k+3) - \frac{1}{(4k+5)}\Big)|B_k|. \end{aligned} \tag{14}$$

Now, the condition $k \geq \frac{1}{2|p|}$ imply that $(4k+1)^2 > \frac{4}{p^2}$ and by (14), $|B_{k+1}| > |B_k|$. It follows inductively that $|B_{n+1}| > |B_n| \geq 1$ for every $n \geq k$. Now, take an arbitrary $q > 1$. In view of the last inequality for B_n, we may assume that k is sufficiently large so that $\rho_k := \frac{p^2}{4}(4k+1)(4k+3) - \frac{1}{(4k+5)} \geq q$. Then (14) yields $|B_{k+1}| \geq q|B_k|$. Since ρ_k is increasing, we can prove by induction using (14) that $|B_{n+1}| \geq q|B_n|$ for every $n \geq k$. $\square$

Now we will prove that in the general case (except eventually for some special values of p) there is a choice of (B_0, b_0), such that the sequence $\{B_n\}$ is bounded. The basic observation is that the asymptotic formulas $B_n \sim \frac{6|\bar{y}|}{(4n)^3p^2}$ and $b_n \sim \bar{y}(4np)$ are compatible with the system (12) if $\bar{y} \approx -0.62654$ is the unique real solution of the equation $y^3 + y^2 + \frac{1}{2}\,y + \frac{1}{6} = 0$. The next assertion states the existence of such type solutions of (12).

Proposition 3. *For every nonzero real number p there exist $k \in \mathbb{N}_0$ and $B_k, b_k \in \mathbb{R}$ such that the sequences $\{B_n\}$ and $\{b_n\}$ determined by (12) for $n \geq k$ satisfy $B_n = O(n^{-3})$ and $b_n = O(n)$ as $n \to \infty$.*

For the proof of this proposition we need the following auxiliary result.

Lemma 2. *Let* (α,β) *belongs to the domain* $\mathcal{D}=\{(\alpha,\beta)\in\mathbb{R}^2:\ |\alpha|,|\beta|\le\frac{1}{10}\}$. *Then the equation* $y^3+y^2+\frac{1+\alpha}{2}\,y+\frac{1+\beta}{6}=0$ *has a unique real solution* $y(\alpha,\beta)\in(-0.75,-0.5]$ *which is a Lipschitz function in* $\mathcal{D}$. *Moreover, if* $(\alpha_i,\beta_i)\in\mathcal{D}$, $i=1,2$, *then* $|y(\alpha_2,\beta_2)-y(\alpha_1,\beta_1)|\le\frac{15}{8}|\alpha_2-\alpha_1|+\frac{5}{6}|\beta_2-\beta_1|$.

Proof. Let $f(y)=y^3+y^2+\frac{1+\alpha}{2}\,y+\frac{1+\beta}{6}$. Then $f'(y)=3y^2+2y+\frac{1+\alpha}{2}$ has a negative discriminant when $|\alpha|\le\frac{1}{10}$ and hence $f'(y)>0$ for every $y\in\mathbb{R}$. Consequently, for $(\alpha,\beta)\in\mathcal{D}$ the equation $f(y)=0$ has one real solution, which is denoted by $y(\alpha,\beta)$. Next, since $f(-\frac34)\le\big(-\frac34\big)^3+\big(-\frac34\big)^2+\frac{0.9}{2}\big(-\frac34\big)+\frac{1.1}{6}<0$ and $f(-\frac12)\ge\big(-\frac12\big)^3+\big(-\frac12\big)^2+\frac{1.1}{2}\big(-\frac12\big)+\frac{0.9}{6}=0$, then $y(\alpha,\beta)\in(-0.75,-0.5]$ provided $(\alpha,\beta)\in\mathcal{D}$.

For $(\alpha_0,\beta_0)=(0,0)$ and fixed $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in\mathcal{D}$ let us set $y_i:=y(\alpha_i,\beta_i)$ and $f_i(y):=y^3+y^2+\frac{1+\alpha_i}{2}\,y+\frac{1+\beta_i}{6}$, $i=0,1,2$. Then

$$\begin{aligned}
0 &= f_2(y_2)-f_1(y_1)=f_0(y_2)-f_0(y_1)+\frac{\alpha_2y_2}{2}+\frac{\beta_2}{6}-\frac{\alpha_1y_1}{2}-\frac{\beta_1}{6}\\
&= f_0'(\eta)(y_2-y_1)+\frac{\alpha_2}{2}(y_2-y_1)+\frac{\alpha_2-\alpha_1}{2}y_1+\frac{\beta_2-\beta_1}{6}
\end{aligned}$$

with some $\eta\in[y_1,y_2]$ (or $[y_2,y_1]$). Therefore,

$$|y_2-y_1|\le\Big(\frac{|\alpha_2-\alpha_1|}{2}|y_1|+\frac{|\beta_2-\beta_1|}{6}\Big)\Big/\Big|f_0'(\eta)+\frac{\alpha_2}{2}\Big|.$$

Using that $\eta\in[-\frac34,-\frac12]$ we obtain

$$f_0'(\eta)+\frac{\alpha_2}{2}=3\eta^2+2\eta+\frac{1+\alpha_2}{2}\ge3\eta^2+2\eta+0.45\ge0.2,$$

hence

$$|y_2-y_1|\le\Big(\frac{0.75}{2}|\alpha_2-\alpha_1|+\frac16|\beta_2-\beta_1|\Big)\times5=\frac{15}{8}|\alpha_2-\alpha_1|+\frac56|\beta_2-\beta_1|.$$

The lemma is proved. □

Proof of Proposition 3. Let us define the sequences $\{(b_{n,i},B_{n,i})\}_{n=k}^{\infty}$, for $i=0,1,2,\dots$ and $k\in\mathbb{N}$, which will be specified later, by the recurrence formulas:

$$b_{n,0}:=(4n+1)p\bar{y},\qquad B_{n,0}:=\frac{6|\bar{y}|}{(4n+1)^2(4n+5)p^2};$$

$$\begin{aligned}
b_{n,i}:\ B_{n,i-1}\Big[b_{n,i}^3+\tbinom{4n+1}{1}pb_{n,i}^2+\tbinom{4n+2}{2}p^2b_{n,i}+\tbinom{4n+3}{3}p^3\Big]&\\
=\frac{1}{4n+7}-B_{n+1,i-1}\big(b_{n+1,i-1}+(4n+5)p\big),&
\end{aligned}$$

$$B_{n,i}:\ B_{n,i}\Big[b_{n,i}^2+\tbinom{4n+1}{1}pb_{n,i}+\tbinom{4n+2}{2}p^2\Big]=\frac{1}{4n+5}-B_{n+1,i-1}.$$

Set $b_{n,i} := (1+\delta_{n,i})b_{n,i-1}$ and $B_{n,i} := (1+\Delta_{n,i})B_{n,i-1}$. We shall show that the relative distances $\delta_{n,i}$ and $\Delta_{n,i}$ decay (with i) like a geometrical series, from where it will follow that $(b_{n,i}, B_{n,i})$ converge to a certain limit as $i \to \infty$.

We estimate separately $\delta_{n,1}$ and $\Delta_{n,1}$. The ratio $X = X_n := b_{n,1}/((4n+1)p)$ is a solution of the equation $X^3+X^2+\frac{1}{2}\Big(1+\frac{1}{4n+1}\Big)X+\frac{1}{6}\Big(1+\frac{1}{4n+1}\Big)\Big(1+\frac{2}{4n+1}\Big) = A_0$, where $A_0 := \Big[\frac{1}{4n+7} - B_{n+1,0}\big(b_{n+1,0}+(4n+5)p\big)\Big]\Big/\Big[(4n+1)^3p^3B_{n,0}\Big]$. Let us set $\alpha = \frac{1}{4n+1}$ and $\beta = \beta_{n,1} = \frac{3}{4n+1} + \frac{2}{(4n+1)^2} - 6A_0$. In view of the definitions of $b_{n,0}$ and $B_{n,0}$ we have

$$\begin{aligned}|\beta| &\le \frac{3}{4n+1} + \frac{2}{(4n+1)^2} + \frac{4n+5}{4n+7}\cdot\frac{1}{(4n+1)|p\bar{y}|} + \frac{6(1+\bar{y})}{(4n+1)(4n+9)p^2}\\ &\le \frac{3+1.6/|p|}{4n+1} + \frac{2+2.25/p^2}{(4n+1)^2} =: \bar{\beta}_n.\end{aligned}$$

Now, choose k such that $\bar{\beta}_n \le \frac{1}{20}$ for $n \ge k$. Thus, $|\beta| \le \frac{1}{20}$ and $\frac{3}{4n+1} \le \frac{1}{20}$, i.e. $|\alpha| \le \frac{1}{60}$. Then, an application of Lemma 2 gives

$$|X-\bar{y}| = |y(\alpha,\beta)-y(0,0)| \le \frac{15}{8}\alpha + \frac{5}{6}|\beta| \le \frac{15/8}{60} + \frac{5/6}{20} \le 0.073\ .$$

Therefore, $X \in (-0.7, -0.553)$ and $|\delta_{n,1}| = \big|b_{n,1}/b_{n,0}-1\big| = \big|X/\bar{y}-1\big| < 0.118$.

Before estimating $\Delta_{n,1}$ we estimate

$$L_{n,1} = b_{n,1}^2+(4n+1)pb_{n,1}+\frac{1}{2}\Big(1+\frac{1}{4n+1}\Big)(4n+1)^2p^2 = (4n+1)^2p^2\Big(X^2+X+\frac{1+\alpha}{2}\Big).$$

Since $X \in (-0.7, -0.5)$ and $\alpha \in \big(0, \frac{1}{60}\big)$, then $L_{n,1}/(4n+1)^2p^2 \in (0.25, 0.3)$. This, in view of

$$B_{n,1} = \Big(\frac{1}{4n+5} - B_{n+1,0}\Big)\Big/L_{n,1} = B_{n,0}\Big(\frac{1}{6|\bar{y}|} - \frac{1/p^2}{(4n+5)(4n+9)}\Big)\Big/\Big[L_{n,1}/(4n+1)^2p^2\Big]$$

and $\frac{1/|p|}{4n+1} < 0.03$ (a consequence of $\bar{\beta}_n \le \frac{1}{20}$) implies that $B_{n,1}/B_{n,0} \in (0.883, 1.065)$ (the numerator belongs to $(0.265, 0.2661)$). Therefore, $|\Delta_{n,1}| \le 0.117$.

Our goal is to prove by induction that $|\delta_{n,i}|, |\Delta_{n,i}| \le 7^{-i}$. The above estimates prove this assertion for $i = 1$, and we assume that $i \ge 2$ by the end of the proof. Next, with $Y_i = Y_{n,i} := b_{n,i}/(4n+1)p$, we have

$$Y_i^3 + Y_i^2 + \frac{1}{2}\Big(1+\frac{1}{4n+1}\Big)Y_i + \frac{1}{6}\Big(1+\frac{1}{4n+1}\Big)\Big(1+\frac{2}{4n+1}\Big) = A_{i-1},$$

where $A_{i-1} := \Big[\frac{1}{4n+7} - B_{n+1,i-1}\big(b_{n+1,i-1}+(4n+5)p\big)\Big]\Big/\Big[(4n+1)^3p^3B_{n,i-1}\Big]$. As above, we set $\alpha = \frac{1}{4n+1}$ and $\beta = \beta_{n,i} = \frac{3}{4n+1} + \frac{2}{(4n+1)^2} - 6A_{i-1}$. Then

$$\begin{aligned}\beta_{n,i} - \beta_{n,i-1} = &\frac{-6/p^2}{(4n+1)^3}\left[\frac{1/p}{4n+7}\Big(\frac{1}{B_{n,i-1}} - \frac{1}{B_{n,i-2}}\Big)\right. \\ &\left. -\Big(\frac{b_{n+1,i-1}}{p} + 4n+5\Big)\frac{B_{n+1,i-1}}{B_{n,i-1}} + \Big(\frac{b_{n+1,i-2}}{p} + 4n+5\Big)\frac{B_{n+1,i-2}}{B_{n,i-2}}\right].\end{aligned}$$

Consequently,

$$|\beta_{n,i} - \beta_{n,i-1}| \le \frac{6/p^2}{(4n+1)^3}\left[\frac{\big(1/|p|\big)|K|}{(4n+7)B_{n,i-2}} + \frac{|b_{n+1,i-2}/p+4n+5|B_{n+1,i-2}|M|}{B_{n,i-2}}\right], \quad (15)$$

where $K = 1 - \frac{B_{n,i-2}}{B_{n,i-1}}$ and $M = 1 - \frac{b_{n+1,i-1}/p+4n+5}{b_{n+1,i-2}/p+4n+5}\cdot\frac{B_{n+1,i-1}}{B_{n+1,i-2}}\cdot\frac{B_{n,i-2}}{B_{n,i-1}}$. By induction, the following estimate for K holds true:

$$|K| = \Big|1 - \frac{1}{1+\Delta_{n,i-1}}\Big| = \frac{|\Delta_{n,i-1}|}{1+\Delta_{n,i-1}} \le \frac{49}{48}\cdot 7^{1-i}.$$

Here we have used $|\Delta_{n,i-1}| \le \frac{1}{49}$ for $i \ge 3$ and $|\Delta_{n,1}| \le 0.117$ for $i = 2$. Now, we estimate the factor M. By induction,

$$\frac{B_{n+1,i-1}}{B_{n+1,i-2}}\cdot\frac{B_{n,i-2}}{B_{n,i-1}} = \frac{1+\Delta_{n+1,i-1}}{1+\Delta_{n,i-1}} \in \Big[\frac{1-7^{1-i}}{1+7^{1-i}}, \frac{1+7^{1-i}}{1-7^{1-i}}\Big]$$

and let

$$\frac{b_{n+1,i-1}/p + 4n+5}{b_{n+1,i-2}/p + 4n+5} = 1 + \frac{b_{n+1,i-1} - b_{n+1,i-2}}{b_{n+1,i-2} + (4n+5)p} =: 1 + \varepsilon\,.$$

Then $|\varepsilon| = |\delta_{n+1,i-1}|\Big/\Big|1 + \frac{(4n+5)p}{b_{n+1,i-2}}\Big|$ and we need to estimate the denominator. For $i = 2$, by definition it is $|1 + 1/\bar{y}| \approx 0.5961$ (hence $|\varepsilon| < 0.2$), while for $i \ge 3$ we have (see above)

$$Y^{-1}_{n+1,i-2} = \frac{(4n+5)p}{b_{n+1,i-2}} = \frac{b_{n+1,i-3}}{b_{n+1,i-2}}\,\frac{b_{n+1,i-4}}{b_{n+1,i-3}}\cdots\frac{b_{n+1,1}}{b_{n+1,2}}\,\frac{(4n+5)p}{b_{n+1,1}} = \Big[X\prod_{j=2}^{i-2}(1+\delta_{n+1,j})\Big]^{-1}.$$

By induction we conclude that

$$Y^{-1}_{n+1,i-2} \in \Big(\Big[X\prod_{j=2}^{\infty}(1-7^{-j})\Big]^{-1}, \Big[X\prod_{j=2}^{\infty}(1+7^{-j})\Big]^{-1}\Big) \subset \big(-1.853, -1.395\big),$$

where we have used $X = X_{n+1} \in (-0.7, -0.553)$ and $\prod_{j=2}^{\infty}(1+7^{-j}) \approx 1.02388$, $\prod_{j=2}^{\infty}(1-7^{-j}) \approx 0.97626$. Hence, $|\varepsilon| \le 2.54\times 7^{1-i}$.

It follows from the above estimates that

$$M \in \Big[1 - \frac{1+7^{1-i}}{1-7^{1-i}}\Big(1 + 2.54\times 7^{1-i}\Big), 1 - \frac{1-7^{1-i}}{1+7^{1-i}}\Big(1 - 2.54\times 7^{1-i}\Big)\Big].$$

As a consequence, $|M| \le \frac{4.54+2.54\times 7^{1-i}}{1-7^{1-i}} 7^{1-i} \le 5.72\times 7^{1-i} \quad (i\ge 2)$.

The remaining factors in (15) we estimate by the induction. For $j=0,1$ we have

$$\frac{B_{n+j,i-2}}{B_{n+j,0}} = (1+\Delta_{n+j,1})\cdots(1+\Delta_{n+j,i-2}) \in \begin{cases}(0.883, 1.065), & \text{for } \ i=3\\ (0.862, 1.091), & \text{for } \ i>3\end{cases}.$$

From $Y^{-1}_{n+1,i-2} = \frac{(4n+5)p}{b_{n+1,i-2}} \in \big(-1.853, -1.395\big)$ we infer $(b_{n+1,i-2}/p+4n+5)/(4n+5) \in (0.283, 0.461)$, and the latter inclusion holds for $i=2$ as well. Then (15) implies

$$\begin{aligned}&|\beta_{n,i}-\beta_{n,i-1}| \le \\ &\frac{6/p^2}{(4n+1)^3\times 0.862 B_{n,0}}\Big[\frac{1/|p|}{(4n+7)}\times\frac{49}{48}\times 7^{1-i}+0.461(4n+5)\times 1.091 B_{n+1,0}\times 5.72\times 7^{1-i}\Big]\\ &\le \frac{7^{1-i}}{0.862(4n+1)}\Big[\frac{1}{|p\bar{y}|}\cdot\frac{49}{48}+\frac{2.877\times 6}{(4n+9)p^2}\Big] \le \Big[\frac{13.24/|p|}{4n+1}+\frac{140.2/p^2}{(4n+1)^2}\Big]\times 7^{-i}.\end{aligned}$$

Hence, from $\frac{1/|p|}{4n+1} < 0.03$ we get $|\beta_{n,i}-\beta_{n,i-1}| \le 0.524\times 7^{-i} \ \ (i\ge 2)$.

Inductively, similar inequalities hold for all $\{\beta_{n,j}\}_{j=2}^{i-1}$ and we conclude that

$$|\beta_{n,i}| \le |\beta_{n,1}| + \sum_{j=2}^{i} |\beta_{n,j}-\beta_{n,j-1}| < \frac{0.524\times 7^{-2}}{1-7^{-1}} + \frac{1}{20} < \frac{1}{10}.$$

Therefore, we can apply Lemma 2 to find for $i\ge 2$

$$|Y_i - Y_{i-1}| = |y(\alpha,\beta_{n,i}) - y(\alpha,\beta_{n,i-1})| \le \frac{15}{8}\times 0 + \frac{5}{6}|\beta_{n,i}-\beta_{n,i-1}| < 0.437\times 7^{-i}\,.$$

Also, Lemma 2 gives the estimate $Y_{i-1}\in(-0.75,-0.5]$ and we obtain

$$|\delta_{n,i}| = \big|Y_i/Y_{i-1}-1\big| = \big|Y_i - Y_{i-1}\big|/|Y_{i-1}| \le 0.874\times 7^{-i}, \quad i\ge 2.$$

In order to estimate $\Delta_{n,i}$ for $i\ge 2$ we use the identity

$$\Delta_{n,i} = \frac{B_{n,i}}{B_{n,i-1}} - 1 = \frac{1/(4n+5) - B_{n+1,i-1}}{L_{n,i}}\,\frac{L_{n,i-1}}{1/(4n+5) - B_{n+1,i-2}} - 1\,,$$

where $L_{n,i} := b^2_{n,i} + \binom{4n+1}{1} p b_{n,i} + \binom{4n+2}{2}p^2$. Clearly,

$$\begin{aligned}|L_{n,i}-L_{n,i-1}| &= |b_{n,i}-b_{n,i-1}|\times|b_{n,i}+b_{n,i-1}+(4n+1)p|\\ &= |b_{n,i-1}\,\delta_{n,i}|\times(4n+1)|p|\times|Y_i+Y_{i-1}+1|.\end{aligned}$$

Recalling that $Y_0 = \bar{y}$ and $Y_1 = X \in (-0.7, -0.553)$ we can refine the estimate $Y_i \in (-0.75,-0.5]$ by $|Y_i| \le |X| + |Y_1 - Y_2| + \cdots + |Y_{i-1}-Y_i| < 0.7 + \frac{0.437\times 7^{-2}}{1-7^{-1}} < 0.711$, i.e. $Y_i\in(-0.711,-0.5]$. The same holds for Y_{i-1} as well. Hence, $|Y_i+Y_{i-1}+1| < 0.422$ and we get

$$\Big|\frac{L_{n,i}}{L_{n,i-1}} - 1\Big| \le 0.369\times 7^{-i}\,\frac{|b_{n,i-1}(4n+1)p|}{|b^2_{n,i-1}+(4n+1)pb_{n,i-1}+\binom{4n+2}{2}p^2|} < \frac{0.369\times 7^{-i}|z|}{z^2+z+1/2},$$

where $z = \frac{b_{n,i-1}}{(4n+1)p} = Y_{i-1}$. The last expression we estimate by

$$\frac{|z|}{z^2 + z + 1/2} = \frac{1}{|z| - 1 + \frac{1}{2|z|}} =: \frac{1}{g(|z|)} \le \frac{1}{\min_{t>0}\, g(t)} = \sqrt{2} + 1\,.$$

Therefore, $\left|\frac{L_{n,i}}{L_{n,i-1}} - 1\right| < 0.891 \times 7^{-i}$. Next, let

$$N := \frac{1/(4n+5) - B_{n+1,i-1}}{1/(4n+5) - B_{n+1,i-2}} = 1 - \frac{B_{n+1,i-1} - B_{n+1,i-2}}{1/(4n+5) - B_{n+1,i-2}} = 1 - \frac{B_{n+1,i-2}\,\Delta_{n+1,i-1}}{1/(4n+5) - B_{n+1,i-2}}.$$

We found above $\frac{B_{n+1,i-2}}{B_{n+1,0}} \in (0.862, 1.091)$, hence $(4n+5)B_{n+1,i-2} \le \frac{1.091\times 6|\bar{y}|}{(4n+5)(4n+9)p^2}$ $< 4.102 \times 0.03^2 < 0.0037$. As a consequence, $\frac{B_{n+1,i-2}}{1/(4n+5)-B_{n+1,i-2}} < 0.004$. Therefore, $|1 - N| < 0.004 \times 7^{1-i} = 0.028 \times 7^{-i}$ and, finally,

$$|\Delta_{n,i}| = \left|\frac{N}{L_{n,i}/L_{n,i-1}} - 1\right| \le \frac{(0.028 + 0.891)7^{-i}}{1 - 0.891 \times 7^{-i}} \le 0.937 \times 7^{-i}, \quad i \ge 2.$$

The claimed estimates $|\delta_{n,i}|, |\Delta_{n,i}| \le 7^{-i}$ are proved.
These estimates imply that the sequences

$$b_{n,i} = b_{n,0} \prod_{j=1}^{i} (1 + \delta_{n,j})\,, \quad B_{n,i} = B_{n,0} \prod_{j=1}^{i} (1 + \Delta_{n,j})$$

converge as $i \to \infty$ to certain limits b_n^* and B_n^*, $n \ge k$. By the definitions of $\{(b_{n,i}, B_{n,i})\}_{n=k}^{\infty}$ it follows that the limit sequences satisfy (12). In addition, $b_n^*/b_{n,0}$ and $B_n^*/B_{n,0}$ are bounded, i.e. $b_n^* = O(np)$ and $B_n^* = O(1/n^3p^2)$. $\square$

Thus we proved the existence of a solution $\{(B_n, b_n)\}$ of (12) with bounded B_n and $b_n = O(n)$ starting from a certain index $k(p)$. It is easily seen that (12) considered as a system for (B_{n-1}, b_{n-1}) is solvable in $\mathbb{R}^2$ provided $\frac{1}{4n+1} - B_n \ne 0$. Then, with the exception of some very special values for p, we can complete the obtained bounded sequence to the starting values $(B_0^*(p), b_0^*(p))$. For example, when $p = \frac{1}{2}$, the condition $\beta_k \le \frac{1}{20}$ is fulfilled for $k = 32$ and the values $(B_{32}^*, b_{32}^*) = (6.7280929... \times 10^{-6}, -40.023137...)$ allow to complete uniquely the sequence up to $(B_0^*, b_0^*) = (0.28687201..., 0.34268557...)$ $(b_n^* < 0, n \ge 1)$. Note that in contrast to the backward calculations, which are stable, in order to get the above values for (B_{32}^*, b_{32}^*) starting from (B_0^*, b_0^*), the latter have to be given with at least 100 decimal digits.

Remark 2. In the special case $B_0 = 1$, $b_0 = \frac{1}{3} - p$, which is of interest for us (see (11)), it seems that there is no real p which determines a bounded sequence $\{B_n\}$ satisfying (12). This claim is based on exhaustive computer experiments. For example: when $|p| \ge \frac{5}{4}$, $|B_1| = |\frac{1}{5} - [(\frac{1}{3} - p)^2 + p(\frac{1}{3} - p) + p^2]| > 1$ and by

Proposition 2, $|B_n| \to \infty$; when $p \in [0.04277, 0.0428]$, the graph of $\log|B_{23}(p)|$ is clearly positive and Proposition 2 implies $|B_n| \to \infty$ in this case, too.

In view of the above results it is reasonable to consider lacunary transformations of $L(z)$ depending on two parameters. We formulate the following

Hypothesis 3. There is a choice of the real parameters p and q such that the representation

$$L(z) \approx qt + \sum_{n=0}^{\infty} \frac{B_n t^{4n+1}}{1 - b_n z^2}, \quad t = \frac{z}{1 - pz^2}$$

has coefficients satisfying $B_n = O(\rho^n)$ for some $\rho > 0$.

There is even some reason to expect the validity of Hypothesis 3 for $p = 0$, i.e. when $t = z$. This is because B_n is a rational function of q, whose numerator is an odd degree polynomial, and hence for every n there are values of q producing arbitrarily small B_n.

Of course, the magnitude of b_n is also important for the convergence of the series. If $\{B_n\}$ is bounded, but $\{b_n\}$ is not, then still it is enough b_n to be negative for $n \geq n_0$ and the convergence will hold in a real neighborhood of $z = 0$.

For the system (11) we will consider theoretically only the case $p = 0$. We can prove the following divergence criterion.

Proposition 4. *Assume that the sequences $\{B_n\}_0^\infty$ and $\{b_n\}_0^\infty$ satisfy the system*

$$\begin{array}{rcl} B_n + B_{n-1}b_{n-1}^2 + B_{n-2}b_{n-2}^4 + \cdots + B_0 b_0^{2n} & = & d_n^{(0)} \\ B_n b_n + B_{n-1}b_{n-1}^3 + B_{n-2}b_{n-2}^5 + \cdots + B_0 b_0^{2n+1} & = & d_n^{(1)}, \end{array} \tag{16}$$

where $|d_n^{(j)}| \leq 1$ for $j = 0, 1$. Let us denote $Y_n := b_n/b_{n-1}$, $Z_n := -B_n/(B_{n-1}b_{n-1}^2)$, $X_n := Y_n Z_n$ and $(\tilde{Z}^*, \tilde{X}^*) := (0.30834705, 0.58425448)$. Then the conditions

$$\begin{aligned} &|Z_{k-i} - \tilde{Z}^*|, \ |X_{k-i} - \tilde{X}^*| \leq r, \quad i = 0,1,2,3\,; \\ &|Z_{k-4} - \tilde{Z}^*|, \ |X_{k-4} - \tilde{X}^*| \leq 5 \times 10^{-5}\,; \\ &\Big(1 + \sum_{i=0}^{k-6} \big|B_i b_i^{2(k+1-i)+j}\big|\Big) \Big/ \big|B_k b_k^{2+j}\big| \leq 10^{-8}, \quad j = 0,1\,; \\ &|b_{k-1}| = \max_{i \leq k-1} |b_i| \geq 1, \end{aligned} \tag{17}$$

for a given $k \geq 5$ and $r = 10^{-6}$ imply that $|B_n|$ tends to infinity faster than any geometrical series and $|b_n| \to \infty$.

Proof. We first change the variables and introduce vector notations. It is not difficult to verify that (16) is equivalent to

$$\begin{array}{rcl} 1 - 1/Z_n + 1/(Z_n Z_{n-1} Y_{n-1}^2) - 1/(Z_n Z_{n-1} Z_{n-2} Y_{n-1}^2 Y_{n-2}^4) + \cdots & = & d_n^{(0)}/B_n \\ 1 - 1/(Z_n Y_n) + 1/(Z_n Z_{n-1} Y_n Y_{n-1}^3) - 1/(Z_n Z_{n-1} Z_{n-2} Y_n Y_{n-1}^3 Y_{n-2}^5) + \cdots & = & d_n^{(1)}/B_n b_n, \end{array}$$

which is

$$
\begin{aligned}
Z_n = &\; 1 - \frac{Z_{n-1}}{X_{n-1}^2} + \frac{Z_{n-1}Z_{n-2}^3}{X_{n-1}^2 X_{n-2}^4} - \frac{Z_{n-1}Z_{n-2}^3 Z_{n-3}^5}{X_{n-1}^2 X_{n-2}^4 X_{n-3}^6} + \cdots - \frac{d_n^{(0)}}{B_{n-1} b_{n-1}^2} \\
X_n = &\; 1 - \frac{Z_{n-1}^2}{X_{n-1}^3} + \frac{Z_{n-1}^2 Z_{n-2}^4}{X_{n-1}^3 X_{n-2}^5} - \frac{Z_{n-1}^2 Z_{n-2}^4 Z_{n-3}^6}{X_{n-1}^3 X_{n-2}^5 X_{n-3}^7} + \cdots - \frac{d_n^{(1)}}{B_{n-1} b_{n-1}^3},
\end{aligned}
\tag{18}
$$

where the sums are expanded to Z_1 and X_1, respectively. In this way we reduce the problem to the proof that the stationary point near $(\tilde{Z}^*, \tilde{X}^*)$ is stable. Indeed, then it will follow that, for $n \to \infty$, $b_n \approx A.(Y^*)^n$, where $Y^* \approx \tilde{X}^*/\tilde{Z}^* \approx 1.8948$ and $|B_{n+1}/B_n| \approx \tilde{Z}^* b_n^2 \to \infty$.

To prove the stability of the stationary point (Z^*, X^*) of (18) we use the approach based on fixed point theorems (see e.g. [8] and the references therein). Note that, formally, we will not use the existence of (Z^*, X^*).

In what follows we care mainly for the impact of the first four summands in (18) (excluding 1) while the remainder we estimate with less precision. So, let us denote $\overline{V}_n := (Z_n, X_n, Z_{n-1}, X_{n-1}, Z_{n-2}, X_{n-2}, Z_{n-3}, X_{n-3})^T$, then (18) becomes $\overline{V}_n = \bar{f}(\overline{V}_{n-1}) + \bar{\theta}_n$, or more precisely,

$$
\left|
\begin{array}{l}
\overline{V}_n(1) = \varphi(\overline{V}_{n-1}) + \epsilon_n \\
\overline{V}_n(2) = \psi(\overline{V}_{n-1}) + \delta_n \\
\overline{V}_n(i) = \overline{V}_{n-1}(i-2), \qquad i = 3, \ldots, 8,
\end{array}
\right.
$$

where

$$
\varphi(z_1, x_1, \ldots, z_4, x_4) := \tilde{a}^* + \sum_{i=1}^{4} (-1)^i z_1 \ldots z_i^{2i-1} / (x_1^2 \ldots x_i^{2i}),
$$

$$
\psi(z_1, x_1, \ldots, z_4, x_4) := \tilde{b}^* + \sum_{i=1}^{4} (-1)^i z_1^2 \ldots z_i^{2i} / (x_1^3 \ldots x_i^{2i+1}).
$$

Here,

$$
\tilde{a}^* = 1 + \sum_{i=5}^{\infty} (-1)^i (\tilde{Z}^*)^{i^2} / (\tilde{X}^*)^{i(i+1)} \quad \text{and} \quad \tilde{b}^* = 1 + \sum_{i=5}^{\infty} (-1)^i (\tilde{Z}^*)^{i(i+1)} / (\tilde{X}^*)^{i(i+2)}
$$

are approximations of the remainders of the sums in (18) when (Z_n, X_n) approaches the stationary point (Z^*, X^*). Assuming convergence of $\{(Z_n, X_n)\}$, the residuals ϵ_n and δ_n will become very small, but do not tend exactly to 0, because of the difference between $(\tilde{Z}^*, \tilde{X}^*)$ and (Z^*, X^*).

Our next step is to prove that conditions (17), but with $k = n$ and $r = 2 \times 10^{-5}$, imply the representation

$$
(\overline{V}_{n+1} - \tilde{V}^*) = \tilde{J}.(\overline{V}_n - \tilde{V}^*) + \bar{\varepsilon}_n, \tag{19}
$$

where $\tilde V^* = (\tilde Z^*, \tilde X^*, \dots, \tilde Z^*, \tilde X^*) \in \mathbb{R}^8$, $\tilde J$ is a given approximation of the Jacobi matrix $\tilde J^* = \frac{D(\bar f)}{D(\overline V)}$ calculated at $\tilde V^*$ and $||\overline\varepsilon_n||_\infty \le \varepsilon := 10^{-7}$.

Before doing this, we adopt the convention (for this proof only) that $||\cdot|| := ||\cdot||_\infty$ and $\alpha \approx \beta$ will mean that β is the rounded value of α to the corresponding decimal digit. For example $\alpha \approx -1.230 \times 10^{-5}$ means $|\alpha + 1.23 \times 10^{-5}| \le \frac12 \times 10^{-8}$.

The Jacobian $D(\varphi, \psi, v_1, ..., v_6)/D(v_1, ..., v_8)$, $\bar v = (z_1, ..., x_4)$, at the point $\tilde V^*$, is calculated to be $\tilde J^* \approx \tilde J := (g_1, g_2, e_1, e_2, e_3, e_4, e_5, e_6)^T$, where

$$g_1 = (-2.2431, 2.3676, 2.0593, -1.4491, -0.2532, 0.1604, 0.0071, -0.0043),$$

$$g_2 = (-2.6966, 2.1347, 0.7911, -0.5219, -0.0451, 0.0278, 0.0006, -0.0004),$$

and e_j is the unit row vector in $\mathbb{R}^8$ whose j-th component equals 1.

It is important that the spectral radius $\rho(\tilde J) \approx 0.2539$ is less than 1. This means that the iterations of (19) will remain bounded provided the perturbation is sufficiently small. Now we start with the estimation of $\overline\varepsilon_n$.
We have $\overline V_{n+1} = \bar f(\overline V_n) + \bar\theta_{n+1}$ and set $\overline\varepsilon^1 := \bar\theta_{n+1} = (\epsilon_{n+1}, \delta_{n+1}, 0, ..., 0)^T$.

Next we justify the approximation $\bar f(\tilde V^*) \approx \tilde V^*$ by introducing $\overline\varepsilon^2 = \bar f(\tilde V^*) - \tilde V^*$, hence $(\overline V_{n+1} - \tilde V^*) = \bar f(\overline V_n) - \bar f(\tilde V^*) + \overline\varepsilon^1 + \overline\varepsilon^2$.

Applying Taylor's formula to the second order around $\tilde V^*$ we get

$$\bar f(\overline V_n) = \bar f(\tilde V^*) + \tilde J^*.(\overline V_n - \tilde V^*) + \frac12 \bar Q(\overline V_n - \tilde V^*),$$

where $\bar Q(\overline V)$ is a vector whose components are quadratic forms of $\overline V$ and more precisely $\bar Q_1(\overline V) = \sum_{i,j} \frac{\partial^2\varphi}{\partial v_i \partial v_j}(\bar\eta_1) V_i V_j$, $\bar\eta_1 \in [\tilde V^*, \overline V_n]$, $\bar Q_2(\overline V) = \sum_{i,j} \frac{\partial^2\psi}{\partial v_i \partial v_j}(\bar\eta_2) V_i V_j$, $\bar\eta_2 \in [\tilde V^*, \overline V_n]$ and $\bar Q_i(\overline V) = 0$ for $i = 3, \dots, 8$. We denote $\frac12 \bar Q(\overline V_n - \tilde V^*)$ by $\overline\varepsilon^3$ and then $(\overline V_{n+1} - \tilde V^*) = \tilde J^*.(\overline V_n - \tilde V^*) + \sum_{i=1}^3 \overline\varepsilon^i$.

Finally, the Jacobian $\tilde J^*$ is calculated approximately, hence with $\Delta\tilde J := \tilde J^* - \tilde J$ and $\overline\varepsilon^4 := \Delta\tilde J.(\overline V_n - \tilde V^*)$ we arrive at

$$(\overline V_{n+1} - \tilde V^*) = \tilde J.(\overline V_n - \tilde V^*) + \sum_{i=1}^4 \overline\varepsilon^i.$$

The estimation of $\overline\varepsilon^4$ is easy: we have

$$||\overline\varepsilon^4|| \le ||\Delta\tilde J||.||\overline V_n - \tilde V^*|| \le 8 \cdot \frac12 \times 10^{-4} r = 8 \times 10^{-9}.$$

In order to estimate $||\overline\varepsilon^3|| = \dfrac12 \max\limits_{i=1,2} |\bar Q_i(\overline V_n - \tilde V^*)| \le \max\limits_{i=1,2} ||\bar Q_i|| \cdot r^2/2$ we use the obvious inequality: $||\sum\limits_{i,j} a_{i\,j} v_i v_j||_\infty := \max\limits_{||\bar v||=1} \left|\sum\limits_{i,j} a_{i\,j} v_i v_j\right| \le \sum\limits_{i,j} |a_{i\,j}|$.

The quadratic form $\bar{Q}_1$ has coefficients $\frac{\partial^2\varphi}{\partial v_i \partial v_j}(\bar{\eta}_1) = \sum_{k=1}^4 (-1)^k \frac{\partial^2\varphi_k}{\partial v_i \partial v_j}(\bar{\eta}_1)$, where $\varphi_k(\bar{v}) := \varphi_k(z_1, x_1, ..., z_4, x_4) = z_1 ... z_k^{2k-1}/(x_1^2 ... x_k^{2k})$ (for $k \le 4$). Since φ_k (and ψ_k below) has the form $z_1^{\alpha_1} ... z_k^{\alpha_k} x_1^{-\beta_1} ... x_k^{-\beta_k}$, $\alpha_i, \beta_i \in \mathbb{N}$ we can use the general estimate

$$\begin{aligned}\sum_{i,j} \Big| \frac{\partial^2\varphi_k}{\partial v_i \partial v_j}(\bar{\eta}_1) \Big| &\le 2\, z_+^{\sum \alpha_i} x_-^{-\sum \beta_i} \Bigg[\frac{\sum_{i<j} \alpha_i\alpha_j + \sum_i \frac{\alpha_i(\alpha_i - 1)}{2}}{z_+^2} \\ &\qquad + \frac{\sum_{i<j} \beta_i\beta_j + \sum_i \frac{\beta_i(\beta_i+1)}{2}}{x_-^2} + \frac{\sum_{i,j} \alpha_i\beta_j}{z_+ x_-} \Bigg] \\ &= z_+^{\sum \alpha_i} x_-^{-\sum \beta_i} \Big[\Big(\frac{\sum \alpha_i}{z_+} + \frac{\sum \beta_i}{x_-} \Big)^2 - \frac{\sum \alpha_i}{z_+^2} + \frac{\sum \beta_i}{x_-^2} \Big],\end{aligned}$$

where $z_+ (x_-)$ is an upper(a lower) bound of the odd(even) components of $\bar{\eta}_1$. Then, from $\bar{\eta}_1 \in [\tilde{V}^*, \overline{V}_n]$ and $||\overline{V}_n - \tilde{V}^*|| \le r$ it follows that $||\bar{\eta}_1 - \tilde{V}^*|| \le r$. Hence, we can take $z_+ = 0.3084 > \tilde{Z}^* + r$ and $x_- = 0.5842 < \tilde{X}^* - r$. So, for $k = 1, 2, 3, 4$ we have
$\bar{\alpha} = (1)$, $\bar{\beta} = (2) \Rightarrow \sum_{i,j} \Big| \frac{\partial^2\varphi_1}{\partial v_i \partial v_j}(\bar{\eta}_1) \Big| \le \frac{z_+}{x_-^2} \Big[\frac{6}{x_-^2} + \frac{4}{z_+ x_-} \Big] < 36$;
$\bar{\alpha} = (1, 3)$, $\bar{\beta} = (2, 4) \Rightarrow \sum_{i,j} \Big| \frac{\partial^2\varphi_2}{\partial v_i \partial v_j}(\bar{\eta}_1) \Big| \le \frac{z_+^4}{x_-^6} \Big[\Big(\frac{4}{z_+} + \frac{6}{x_-} \Big)^2 - \frac{4}{z_+^2} + \frac{6}{x_-^2} \Big] < 117.4$;
$\bar{\alpha} = (1, 3, 5)$, $\bar{\beta} = (2, 4, 6) \Rightarrow \sum_{i,j} \Big| \frac{\partial^2\varphi_3}{\partial v_i \partial v_j}(\bar{\eta}_1) \Big| \le \frac{z_+^9}{x_-^{12}} \Big[\Big(\frac{9}{z_+} + \frac{12}{x_-} \Big)^2 - \frac{9}{z_+^2} + \frac{12}{x_-^2} \Big] < 39$;
$\bar{\alpha} = (1, 3, 5, 7)$, $\bar{\beta} = (2, 4, 6, 8) \Rightarrow \sum_{i,j} \Big| \frac{\partial^2\varphi_4}{\partial v_i \partial v_j}(\bar{\eta}_1) \Big| \le \frac{z_+^{16}}{x_-^{20}} \Big[\Big(\frac{16}{z_+} + \frac{20}{x_-} \Big)^2 - \frac{16}{z_+^2} + \frac{20}{x_-^2} \Big] < 2.3$.

As a consequence, $||\bar{Q}_1|| \le \sum_{i,j} \Big| \frac{\partial^2\varphi}{\partial v_i \partial v_j}(\bar{\eta}_1) \Big| \le 195$.

Analogously, $\bar{Q}_2$ has coefficients $\frac{\partial^2\psi}{\partial v_i \partial v_j}(\bar{\eta}_2) = \sum_{k=1}^4 (-1)^k \frac{\partial^2\psi_k}{\partial v_i \partial v_j}(\bar{\eta}_2)$, where $\psi_k(\bar{v}) := \psi_k(z_1, x_1, ..., z_4, x_4) = z_1^2 ... z_k^{2k}/(x_1^3 ... x_k^{2k+1})$ and hence $||\bar{Q}_2||$ is estimated by the sum of

$$\sum_{i,j} \Big| \frac{\partial^2\psi_1}{\partial v_i \partial v_j}(\bar{\eta}_2) \Big| < 59;\ \sum_{i,j} \Big| \frac{\partial^2\psi_2}{\partial v_i \partial v_j}(\bar{\eta}_2) \Big| < 68;\ \sum_{i,j} \Big| \frac{\partial^2\psi_3}{\partial v_i \partial v_j}(\bar{\eta}_2) \Big| < 10;\ \sum_{i,j} \Big| \frac{\partial^2\psi_4}{\partial v_i \partial v_j}(\bar{\eta}_2) \Big| < 1.$$

Thus, $||\bar{Q}_2|| < 138$ and therefore $||\bar{\varepsilon}^3|| \le \max\big(||\bar{Q}_1||, ||\bar{Q}_2||\big) r^2/2 < 4 \times 10^{-8}$.

Next, we have

$$\begin{aligned}\bar{\varepsilon}^2 &= \big(\varphi(\tilde{V}^*) - \tilde{Z}^*, \psi(\tilde{V}^*) - \tilde{X}^*, 0, ..., 0 \big)^T \\ &= \Big(\sum_{i=0}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i^2}}{(\tilde{X}^*)^{i(i+1)}} - \tilde{Z}^*, \sum_{i=0}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i(i+1)}}{(\tilde{X}^*)^{i(i+2)}} - \tilde{X}^*, 0, ..., 0 \Big)^T \\ &\approx \big(-8.0 \times 10^{-9}, -8.1 \times 10^{-9}, 0, ..., 0 \big)^T,\end{aligned}$$

and the Leibnitz type series are easy to estimate, yielding $||\bar{\varepsilon}^2|| < 10^{-8}$.

For an estimate of $\overline{\varepsilon}^1 = (\epsilon_{n+1}, \delta_{n+1}, 0, ..., 0)^T$ we write

$$
\begin{aligned}
\epsilon_{n+1} &= -\frac{Z_n Z_{n-1}^3 Z_{n-2}^5 Z_{n-3}^7 Z_{n-4}^9}{X_n^2 X_{n-1}^4 X_{n-2}^6 X_{n-3}^8 X_{n-4}^{10}} + - \cdots \text{ (to } X_1) - \frac{d_{n+1}^{(0)}}{B_n b_n^2} - \sum_{i=5}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i^2}}{(\tilde{X}^*)^{i(i+1)}} \\
&= \Big[-\varphi_5(Z_n, \ldots, X_{n-4}) + \frac{(\tilde{Z}^*)^{25}}{(\tilde{X}^*)^{30}} \Big] + \Big[\cdots \Big] - \sum_{i=6}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i^2}}{(\tilde{X}^*)^{i(i+1)}} \\
&=: A + B + C.
\end{aligned}
$$

$$
|A| = \big|\varphi_5(Z_n, ..., X_{n-4}) - \frac{(\tilde{Z}^*)^{25}}{(\tilde{X}^*)^{30}}\big| \le \max\Big\{ \big|\frac{(z_+)^{25}}{(x_-)^{30}} - \frac{(\tilde{Z}^*)^{25}}{(\tilde{X}^*)^{30}}\big|, \big|\frac{(z_-)^{25}}{(x_+)^{30}} - \frac{(\tilde{Z}^*)^{25}}{(\tilde{X}^*)^{30}}\big| \Big\},
$$

where z_+ and x_- are as above while $z_- := 0.30829$ and $x_+ := 0.58431$. Note that $\tilde{Z}^* \pm 5 \times 10^{-5} \in [z_-, z_+]$ and $\tilde{X}^* \pm 5 \times 10^{-5} \in [x_-, x_+]$, so in view of (17), $\{Z_{n-i}\}_{i=0}^4$ and $\{X_{n-i}\}_{i=0}^4$ belong to these intervals, too. Thus, we find $|A| < 1.3 \times 10^{-8}$.

For the estimate of $|B|$ we return to the initial variables and use (17):

$$
|B| \le \Big(\sum_{i=0}^{n-6} \big|B_i b_i^{2(n+1-i)}\big| + |d_{n+1}^{(0)}| \Big) \Big/ |B_n b_n^2| < 10^{-8}.
$$

We also easily find $|C| < (\tilde{Z}^*)^{36}/(\tilde{X}^*)^{42} < (z_+)^{36}/(x_-)^{42} < 3 \times 10^{-9}$. As a result we have $|\epsilon_{n+1}| < 3 \times 10^{-8}$. In a very similar way we estimate

$$
\begin{aligned}
\delta_{n+1} &= -\frac{Z_n^2 Z_{n-1}^4 Z_{n-2}^6 Z_{n-3}^8 Z_{n-4}^{10}}{X_n^3 X_{n-1}^5 X_{n-2}^7 X_{n-3}^9 X_{n-4}^{11}} + - \cdots \text{ (to } X_1) + \frac{d_{n+1}^{(1)}}{B_n b_n^3} - \sum_{i=5}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i(i+1)}}{(\tilde{X}^*)^{i(i+2)}} \\
&= \Big[-\psi_5(Z_n, \ldots, X_{n-4}) + \frac{(\tilde{Z}^*)^{30}}{(\tilde{X}^*)^{35}} \Big] + \Big[\cdots \Big] - \sum_{i=6}^{\infty} (-1)^i \frac{(\tilde{Z}^*)^{i(i+1)}}{(\tilde{X}^*)^{i(i+2)}} \\
&=: A_1 + B_1 + C_1,
\end{aligned}
$$

whence $|\delta_{n+1}| \le |A_1| + |B_1| + |C_1| < 7 \times 10^{-10} + 10^{-8} + 6 \times 10^{-11} < 2 \times 10^{-8}$.

Thus we obtain $||\overline{\varepsilon}^1|| < 3 \times 10^{-8}$ and hence $||\overline{\varepsilon}_n|| \le \sum_{i=1}^4 ||\overline{\varepsilon}^i|| < 9 \times 10^{-8} < \varepsilon$. The relation (19) is proved under the corresponding conditions.

Now we will prove the following

Claim. Let conditions (17) are fulfilled with $r = 2 \times 10^{-5}$ and assume that $|Z_{k+i} - \tilde{Z}^*|$, $|X_{k+i} - \tilde{X}^*| \le r$, $i = 1, \ldots, 6$. Then $||\overline{V}_n - \tilde{V}^*|| \le r$ $\forall n \ge k$, i.e. the above inequalities hold for all $i \ge 0$.

First note that from (17) with any positive $r \le 2 \times 10^{-5}$ they follow all but the first two similar inequalities with $k+1$ in place of k. Indeed, the relations $|Z_{k+1-i} - \tilde{Z}^*|$, $|X_{k+1-i} - \tilde{X}^*| \le r$, $i = 1, 2, 3$ are contained in (17) and the inequalities $|Z_{k+1-4} - \tilde{Z}^*|$, $|X_{k+1-4} - \tilde{X}^*| \le 5 \times 10^{-5}$ are obvious consequences.

The last condition follows from $|b_k| = |Y_k b_{k-1}| > \frac{x_-}{z_+}|b_{k-1}| > 1.894|b_{k-1}|$. It remains to estimate for $j = 0, 1$

$$
\begin{aligned}
R^{(j)} &:= \Big(1 + \sum_{i=0}^{k-5} \big|B_i b_i^{2(k+2-i)+j}\big|\Big)\Big/\big|B_{k+1} b_{k+1}^{2+j}\big| \\
&\le \Big(1 + b_{k-1}^{2+j} \sum_{i=0}^{k-6} \big|B_i b_i^{2(k+1-i)}\big| + \big|B_{k-5} b_{k-5}^{14+j}\big|\Big)\Big/\big|B_k b_k^2\big| \cdot \Big|\frac{B_k b_k^2}{B_{k+1} b_{k+1}^{2+j}}\Big| \\
&\le b_{k-1}^{2+j}\Big(1 + \sum_{i=0}^{k-6} \big|B_i b_i^{2(k+1-i)}\big| + \Big(\frac{b_{k-5}}{b_{k-1}}\Big)^{2+j}\big|B_{k-5} b_{k-5}^{12}\big|\Big)\Big/\big|B_k b_k^2\big| \cdot \Big|\frac{1}{Z_{k+1} b_{k+1}^{2+j}}\Big| \\
&\le \frac{1}{\big|Z_{k+1}(Y_{k+1}Y_k)^{2+j}\big|}\Big[10^{-8} + \big(Y_{k-1}...Y_{k-4}\big)^{-2-j}|\varphi_5(Z_k, ..., X_{k-4})|\Big] \\
&\le \frac{Z_{k+1} Z_k^2}{\big|X_{k+1}^2 X_k^2 (Y_{k+1}Y_k)^j\big|}\big[10^{-8} + (x_-/z_+)^{-8-4j} z_+^{25}/x_-^{30}\big],
\end{aligned}
$$

where we have used $Y_n = X_n/Z_n$ and $Z_{k-i} < z_+$, $X_{k-i} > x_-$ for $i = 0, \dots, 4$. Now, by (19) we conclude that $||\overline{V}_{k+1} - \tilde{V}^*|| \le ||\tilde{J}||r + \varepsilon < 8.54r + \varepsilon < 2 \times 10^{-4}$, which implies $Z_{k+1} < 0.3086$, $X_{k+1} > 0.5840$ and $Y_{k+1} > 1$. Therefore, in view of $Y_k > 1$, we get $R^{(j)} < 0.2522\big[10^{-8} + z_+^{33}/x_-^{38}\big] < 10^{-8}$, $j = 0, 1$.

From this and the conditions of the claim we conclude that (17) and (19) are fulfilled for $n = k, \dots, k+6$. Therefore we have

$$
\begin{aligned}
||\overline{V}_{k+7} - \tilde{V}^*|| &= \big\|\tilde{J}.(\overline{V}_{k+6} - \tilde{V}^*) + \overline{\varepsilon}_{k+6}\big\| = \cdots = \Big\|\tilde{J}^7.(\overline{V}_k - \tilde{V}^*) + \sum_{i=0}^{6} \tilde{J}^i \overline{\varepsilon}_{k+6-i}\Big\| \\
&\le \big\|\tilde{J}^7\big\|.r + \sum_{i=0}^{6} \big\|\tilde{J}^i\big\|.\varepsilon\,.
\end{aligned}
$$

We calculated $\|\tilde{J}\| \approx 8.53$, $\|\tilde{J}^2\| \approx 11.13$, $\|\tilde{J}^3\| \approx 11.13$, $\|\tilde{J}^4\| \approx 11.13$, $\|\tilde{J}^5\| \approx 11.13$, $\|\tilde{J}^6\| \approx 2.38$ and $\|\tilde{J}^7\| \approx 0.274$. Thus we obtain

$$
||\overline{V}_{k+7} - \tilde{V}^*|| < 0.275r + 56.5\varepsilon < 1.2 \times 10^{-5} < r\,,
$$

and the claim follows by induction.

To accomplish the proof of Proposition 4 it remains to show that conditions (17) with $r = 10^{-6}$ imply the conditions of the claim. Indeed, it follows from the claim that $b_n/b_{n-1} = Y_n > x_-/z_+ > 1.894$ for $n \ge k$, and hence $|B_n/B_{n-1}| = Z_n b_{n-1}^2 > (z_-) b_{n-1}^2 \to \infty$.

Let (17) be fulfilled with $r = 10^{-6}$ and for a $j \in \{1, \dots, 6\}$, $\|\overline{V}_{k+i} - \tilde{V}^*\| \le 2 \times 10^{-5}, i = 1, \dots, j-1$. Then (19) holds for $n = k, \dots, k+j-1$ and

$$
\begin{aligned}
\|\overline{V}_{k+j} - \tilde{V}^*\| &= \big\|\tilde{J}^j.(\overline{V}_k - \tilde{V}^*) + \tilde{J}^{j-1}.\overline{\varepsilon}_k + \cdots + \tilde{J}^0.\overline{\varepsilon}_{k+j-1}\big\| \\
&\le 11.14\, r + 54.1\, \varepsilon < 2 \times 10^{-5}.
\end{aligned}
$$

This observation inductively implies the conditions of the claim and completes the proof of the stability of (Z^*, X^*), i.e. the proof of Proposition 4. $\square$

Proof of Theorem 1. We apply Proposition 4 to the series a) for $l(x)$ with $B_n = A_n$, $b_n = a_n$ and $d_n^{(j)} = \frac{1}{2n+1+j}$, $j = 0, 1$. It is calculated that for $k = 13$ the conditions of the proposition are fulfilled. Namely,
$\{Z_n - \tilde{Z}^*\}_{n=9}^{13} \approx \{1 \times 10^{-6}, 9 \times 10^{-8}, -6 \times 10^{-8}, -1 \times 10^{-8}, -1 \times 10^{-9}\}$;
$\{X_n - \tilde{X}^*\}_{n=9}^{13} \approx \{2 \times 10^{-6}, 7 \times 10^{-8}, -9 \times 10^{-8}, -1 \times 10^{-8}, 7 \times 10^{-9}\}$;
$\left(1 + \sum_{i=0}^{k-6} \left|A_i a_i^{2(k+1-i)+j}\right|\right) \Big/ \left|A_k a_k^{2+j}\right| \approx 3 \times 10^{-9}, 6 \times 10^{-11}$ for $j = 0, 1$;
$a_{k-1} = \max_{i \le k-1} |a_i| \approx 1744.92 \ge 1$.

Then, by Proposition 4, $|A_n|$ tends to ∞ faster than any geometrical series, while a_n in the denominator behave as $C.(Y^*)^n$. Therefore, the common term in the sum of a) does not tend to 0 (unless for $x = 0$) and the series diverges.

A very similar argument holds for the series b). Now, $d_n^{(j)} = \frac{1}{4n+1+2j}$, $j = 0, 1$ and again for $k = 13$ the conditions of Proposition 4 are fulfilled:
$\{Z_n - \tilde{Z}^*\}_{n=9}^{13} \approx \{1 \times 10^{-6}, 5 \times 10^{-7}, -3 \times 10^{-8}, -3 \times 10^{-8}, -5 \times 10^{-9}\}$;
$\{X_n - \tilde{X}^*\}_{n=9}^{13} \approx \{3 \times 10^{-6}, 7 \times 10^{-7}, -7 \times 10^{-8}, -4 \times 10^{-8}, 2 \times 10^{-9}\}$;
$\left(1 + \sum_{i=0}^{k-6} \left|B_i b_i^{2(k+1-i)+j}\right|\right) \Big/ \left|B_k b_k^{2+j}\right| \approx 3 \times 10^{-9}, 6 \times 10^{-11}$ for $j = 0, 1$;
$b_{k-1} = \max_{i \le k-1} |b_i| \approx 1399.65 \ge 1$.

Then, Proposition 4 implies the divergence of b) for $z \neq 0$. Theorem 1 is proved. $\square$

5. BOUNDS FOR THE RATE OF CONVERGENCE OF (2) AND (3).

We first note an useful formula connecting the coefficients in the representation

$$\frac{A_0}{1 - \alpha_0 x} + \frac{A_1 x}{(1 - \alpha_0 x)(1 - \alpha_1 x)} + \frac{A_2 x^2}{(1 - \alpha_0 x)(1 - \alpha_1 x)(1 - \alpha_2 x)} + \cdots \approx \sum_{n=0}^{\infty} a_n x^n.$$

Namely,

$$\begin{aligned}
&A_0 = a_0; \quad A_1 = a_1 - \alpha_0 a_0; \\
&A_2 = a_2 - (\alpha_0 + \alpha_1) a_1 + (\alpha_0 \alpha_1) a_0; \\
&A_3 = a_3 - (\alpha_0 + \alpha_1 + \alpha_2) a_2 + (\alpha_0 \alpha_1 + \alpha_1 \alpha_2 + \alpha_2 \alpha_0) a_1 - (\alpha_0 \alpha_1 \alpha_2) a_0; \qquad (20) \\
&A_4 = a_4 - \sigma_1(\alpha_0, ..., \alpha_3) a_3 + \sigma_2(\alpha_0, ..., \alpha_3) a_2 - \sigma_3(\alpha_0, ..., \alpha_3) a_1 + \sigma_4(\alpha_0, ..., \alpha_3) a_0; \\
&\quad \vdots
\end{aligned}$$

where $\sigma_k(\alpha_0, \dots, \alpha_n) = \sum_{0 \le i_1 < \cdots < i_k \le n} \alpha_{i_1} \dots \alpha_{i_k}$. Formulas (20) easily follow by induction. Indeed, the relations for A_0 and A_1 are easily verified. Let the formula holds for A_k with a fixed $k \geq 1$ and arbitrary parameters $\{\alpha_i\}$ and $\{a_i\}$. Then, removing the denominator $1-\alpha_0 x$, subtracting A_0 and dividing by x we obtain

$$\frac{A_1}{(1-\alpha_1 x)} + \frac{A_2 x}{(1-\alpha_1 x)(1-\alpha_2 x)} + \cdots \approx \sum_{n=0}^{\infty} a_{n+1} x^n - \alpha_0 \sum_{n=0}^{\infty} a_n x^n.$$

So, by the induction and the linearity, for the coefficient of $x^k / \prod_{i=1}^{k+1} (1- \alpha_i x)$ we find

$$\begin{aligned} A_{k+1} = & \big[a_{k+1} - \sigma_1(\alpha_1, ..., \alpha_k) a_k + \sigma_2(\alpha_1, ..., \alpha_k) a_{k-1} - \cdots + (-1)^k \sigma_k(\alpha_1, ..., \alpha_k) a_1\big] \\ & - \alpha_0 \big[a_k - \sigma_1(\alpha_1, ..., \alpha_k) a_{k-1} + \cdots + (-1)^k \sigma_k(\alpha_1, ..., \alpha_k) a_0\big] \\ = & a_{k+1} - \sigma_1(\alpha_0, ..., \alpha_k) a_k + \big(\sigma_2(\alpha_1, ..., \alpha_k) + \alpha_0 \sigma_1(\alpha_1, ..., \alpha_k)\big) a_{k-1} - + \cdots \\ & + (-1)^k \big(\sigma_k(\alpha_1, ..., \alpha_k) + \alpha_0 \sigma_{k-1}(\alpha_1, ..., \alpha_k)\big) a_1 + (-1)^{k+1} \alpha_0 \alpha_1 ... \alpha_k \cdot a_0 , \end{aligned}$$

as for $k = 1$ the middle terms in the brackets do not appear. For $k \geq 2$ the induction step follows by the properties of the combinatorial sums $\{\sigma_i\}$.

In the particular case $a_0 = \alpha_0 = 0$ and $a_n = \frac{1}{n}$, $n \geq 1$, we arrive at the formula

$$l(x) \approx \frac{C_1 x}{1-\gamma_1 x} + \frac{C_2 x^2}{(1-\gamma_1 x)(1-\gamma_2 x)} + \frac{C_3 x^3}{(1-\gamma_1 x)(1-\gamma_2 x)(1-\gamma_3 x)} + \cdots ,$$

where

$$\begin{aligned} C_k &= \frac{1}{k} - \sigma_1(\gamma_1, ..., \gamma_{k-1}) \frac{1}{k-1} + \sigma_2(\gamma_1, ..., \gamma_{k-1}) \frac{1}{k-2} - \cdots + (-1)^{k-1} (\gamma_1 \cdots \gamma_{k-1}) \\ &= \int_0^1 (x-\gamma_1) ... (x-\gamma_{k-1})\, d\, x. \end{aligned} \tag{21}$$

Let us consider some concrete representations of $l(x)$ with periodic $\{\gamma_i\}$. As was mentioned before, the choice $\{\gamma_i\}_1^\infty = \{\frac{1}{2}\}_1^\infty$ leads to the series (5). Let now $\{\gamma_i\}_1^\infty = \{0, a, b, a, b, ...\}$. We take $\gamma_1 = 0$ in order to write the series in the form

$$l(x) = D_0 x + (B_1 + D_1 x) u + (B_2 + D_2 x) u^2 + (B_3 + D_3 x) u^3 + \cdots , \tag{22}$$

where $u = \dfrac{x^2}{(1-ax)(1-bx)}$. Then $D_0 = 1$ while for $n \geq 1$, from (21) and

$$(B_n + D_n x) u^n = \frac{B_n x^{2n}}{(1-0.x)(1-ax)^n (1-bx)^{n-1}} + \frac{(bB_n + D_n) x^{2n+1}}{(1-0.x)(1-ax)^n (1-bx)^n},$$

it follows that $B_n = \int_0^1 x \big[(x-a)(x-b)\big]^{n-1} d\, x$, $D_n + bB_n = \int_0^1 x (x-a)^n (x-b)^{n-1} d\, x$, hence $D_n = \int_0^1 x^2 \big[(x-a)(x-b)\big]^{n-1} d\, x - sB_n$, where $s = a + b$.

Introducing $A_n = \int_0^1 \big[(x-a)(x-b)\big]^{n-1} d\, x$, $C_n = \int_0^1 x^2 \big[(x-a)(x-b)\big]^{n-1} d\, x$ and $a' = 1-a$, $b' = 1-b$, one can easily verify the recurrence relations

$$\begin{aligned} B_n &= \frac{(a'b')^n - (ab)^n}{2n} + \frac{s}{2} A_n; & C_n &= \frac{(a'b')^n + (n+1) s B_n - ab A_n}{2n+1}; \\ D_n &= C_n - s B_n; & A_{n+1} &= D_n + ab A_n. \end{aligned}$$

To minimize the asymptotics of the coefficients in (22) we choose $(x-a)(x-b) = x^2 - x + \frac{1}{8}$, which is the Chebyshev polynomial of the first kind $T_2^*(x)$, transformed to the interval $[0,1]$. Recall that the polynomials $T_k^*(x) = x^k + \cdots$ associated with the interval $[\alpha,\beta]$ provide the minimal uniform norm on $[\alpha,\beta]$ amongst all polynomials of the form $x^k + \sum_{i=0}^{k-1} a_i x^i$ and the value of this minimal norm is $\frac{(\beta-\alpha)^k}{2^{2k-1}}$ ([4, Ch.2.2.3]). In the our case, $\|T_2^*\|_{C[0,1]} = -T_2^*(1/2) = \frac{1}{8}$ which yields that the above integrals are asymptotic to $\frac{const}{\sqrt{n}}\Big(-\frac{1}{8}\Big)^n$ for $n \to \infty$. For this choice we have $s = a + b = 1$ and the recurrence relations simplify to

$$A_{n+1} = \frac{1}{2n+1}\Big[\frac{1}{8^n} - \frac{n}{4}A_n\Big];\quad B_n = \frac{1}{2}A_n;\quad D_n = A_{n+1} - \frac{1}{8}A_n.$$

It is convenient to substitute $A_n = \Big(\frac{1}{8}\Big)^{n-1}\alpha_n$, where $\alpha_{n+1} = \dfrac{1-2n\alpha_n}{2n+1}$. Then, the rate of convergence of this special case of (22), considered as a series of type (3), i.e. with each summand counted twice, is like a geometrical series with ratio $\frac{\sqrt{u}}{2\sqrt{2}} \sim \frac{x}{2\sqrt{2}}$ $(x \to 0)$. In addition we can rewrite the series in a lacunary form. Namely, by the recurrence formulas and $D_0 = A_1 = 1$ we get

$$\begin{aligned} l(x) =&\ A_1 x + \Big(\frac{1}{2}A_1 + \big(A_2 - \frac{1}{8}A_1\big)x\Big)u + \Big(\frac{1}{2}A_2 + \big(A_3 - \frac{1}{8}A_2\big)x\Big)u^2 + \cdots \\ =&\ \Big[x + \Big(\frac{1}{2} - \frac{x}{8}\Big)u\Big]\Big[A_1 + A_2 u + A_3 u^2 + \cdots\Big], \end{aligned} \tag{23}$$

where A_n satisfies the above formulas and $u = \frac{x^2}{1-x+x^2/8}$. (The second factor is lacunary, considered as a series of the form (3).)

Remark 3. Although the series (23) converges faster than (5), and is lacunary as well, it is still less effective. This is because the coefficients are more complicated. Indeed, let us count only multiplications and divisions as the most costly arithmetic operations with equal cost. Then every next term in (5) needs two operations ($z^{2n+1} = z^{2n-1} \times z^2$ and $z^{2n+1}/(2n+1)$), while every next term in (23) needs three operations ($\alpha_{n+1} = (1+\alpha_n)/(2n+1) - \alpha_n$, $(u/8)^n = (u/8)^{n-1} \times (u/8)$ and $\alpha_{n+1} \times (u/8)^n$). Actually, even the example below hardly improves (5).

Remark 4. We see that the first factor in (23) vanishes for $x = 0$ and $x = 2$. In fact, $l(0) = 0$ but $l(2) = \log(-1) \neq 0$. Recall our adoption that when the region of validity of some identity is not specified, it is certain neighborhood of 0. In particular, (23) converges for $|x| < 1$ and represents $l(x)$ in the open unit disc. On the other hand, a continuation of (23) for x outside the unit disc is questionable because of $l(1) = \infty$.

Consider now "periodic" representations

$$l(x) = b_0 x + c_0 x^2 + \sum_{n=1}^{\infty}(a_n + b_n x + c_n x^2)v^n,\quad v = \frac{x^3}{(1-ax)(1-bx)(1-cx)},$$

i.e. of the form (3) with $\{\gamma_i\}_1^\infty = \{0, 0, a, b, c, a, b, c, ...\}$. Transforming (3) in the above form, in view of (21), we find the following integral formulas for $n \geq 1$:

$$
\begin{aligned}
a_n &= \int_0^1 t^2 \big[(t-a)(t-b)(t-c)\big]^{n-1}\, dt; \\
b_n &= \int_0^1 t^2 (t-a-b-c)\big[(t-a)(t-b)(t-c)\big]^{n-1}\, dt; \\
c_n &= \int_0^1 t^2 \big(t^2-(a+b+c)t+ab+bc+ca\big)\big[(t-a)(t-b)(t-c)\big]^{n-1}\, dt.
\end{aligned}
$$

For simplicity let us assume that the points a, b and c are symmetrically placed in $[0,1]$, or more precisely let $a+b=1$ and $c=1/2$. Then, with the notations $I_n := \int_0^1 [P(t)]^n dt$ and $J_n := \int_0^1 t[P(t)]^n dt$ where $P(t)= (t-a)(t-b)(t-c)$, we easily get $I_{2m-1}=0$ and $J_{2m}=\frac{1}{2}I_{2m}$. Next, using $P(t)= \big(\frac{1}{3}P'(t)+\frac{4ab-1}{6}\big)\big(t-\frac{1}{2}\big)$ we calculate

$$
\begin{aligned}
I_n =& \frac{n/2}{3n+1}\Big[(4ab-1)\Big(J_{n-1}-\frac{1}{2}I_{n-1}\Big)+\frac{1-(-1)^n}{n}\Big(\frac{ab}{2}\Big)^n\Big]; \\
J_n =& \frac{n/2}{3n+2}\Big[\frac{1}{n}I_n+(2ab-\tfrac{1}{2})\Big(J_{n-1}-\frac{2ab+1}{3}I_{n-1}\Big) \\
&\qquad + \Big(\frac{ab}{2}\Big)^n\Big(\frac{1}{n}+(4ab-1)\frac{1-(-1)^n}{3n}\Big)\Big]; \\
a_n =& J_{n-1}-\frac{2ab+1}{6}I_{n-1}+\frac{1-(-1)^n}{3n}\Big(\frac{ab}{2}\Big)^n; \\
b_n =& I_n-\big(ab+\frac{1}{2}\big)J_{n-1}+\frac{ab}{2}I_{n-1}; \\
c_n =& J_n+\frac{ab}{2}J_{n-1}.
\end{aligned}
$$

In particular, a nice formula is obtained if we take $a=0$ and $b=1$. Then

$$
J_n = \begin{cases} -\frac{1}{12}\frac{n}{3n+2}J_{n-1} & \text{for odd } \ n>0 \\ -\frac{1}{4}\frac{n}{3n+1}J_{n-1} & \text{for even } \ n>0\,, \end{cases}
$$

$$
a_n+b_nx+c_nx^2 = \begin{cases} \big(\frac{2}{3}-\frac{x}{2}\big)J_{n-1}+x^2J_n & \text{for odd } \ n \\ \big(1-\frac{x}{2}\big)J_{n-1}+(x^2+2x)J_n & \text{for even } \ n\,. \end{cases}
$$

The starting value is $J_0=\frac{1}{2}$ and even the second formula holds for $n=0$ with $J_{-1}:=0$. Therefore, with $v=\frac{x^3}{(1-x/2)(1-x)}$,

$$
\begin{aligned}
l(x) \;=\; & \Big(x+\frac{x^2}{2}+\Big(\frac{1}{3}-\frac{x}{4}\Big)v\Big)\Big[1+\frac{1}{48}\frac{2!v^2}{5.7}+\frac{1}{48^2}\frac{4!v^4}{5.7.11.13}+\frac{1}{48^3}\frac{6!v^6}{5.7.11.13.17.19}+\cdots\Big] \\
& -\frac{1}{24}\Big(x^2+\Big(1-\frac{x}{2}\Big)v\Big)\Big[\frac{1!v}{5}+\frac{1}{48}\frac{3!v^3}{5.7.11}+\frac{1}{48^2}\frac{5!v^5}{5.7.11.13.17}+\cdots\Big]. \qquad (24)
\end{aligned}
$$

The both parts of (24), considered together and as a series of type (3), converge like a geometric series with ratio $\sqrt[6]{\frac{1}{48\cdot 9}}x \approx \frac{x}{2.75}$ $(x \to 0)$.

Proof of Theorem 2. To prove a), we denote by $0 < x_1 < x_2 < \cdots < x_k < 1$ the zeros of the Chebyshev polynomial $T_k^*(x)$ for the interval $[0,1]$ and take the periodic sequence $\{\gamma_i\}_1^\infty = \{x_1, \ldots, x_k, x_1, \ldots, x_k, \ldots\}$. Then, by (21) we have

$$|C_{n+1}| \le \int_0^1 |T_k^*(t)|^m \, |(t-\gamma_{km+1})...(t-\gamma_n)| d\,t \le \left(2^{-2k+1}\right)^m < 2^{(-2k+1)(n/k-1)},$$

where $n = k\,m + r$, $\;r \in \{0, \ldots, k-1\}$. Hence,

$$\limsup_{n\to\infty} |C_{n+1}|^{1/n} \le 4^{-1+1/2k} =: q(k).$$

Therefore, with a fixed $\varepsilon \in (0,1]$, the inequality $q(k) < \frac{1}{4-\varepsilon}$ holds true, provided k is sufficiently large. With such a k, the sequence $\{\gamma_i\}$ above satisfies a).

To prove b) we shall exploit some properties of the Legendre polynomials $P_n(x) = \frac{1}{2^n n!}[(x^2-1)^n]^{(n)}$. Let $p_n(x) = P_n(2x-1)$ be the normalized Legendre polynomials for the interval $[0,1]$. It follows from $\int_{-1}^1 P_n(x)Q_m(x)d\,x = 0$ for every polynomial $Q_m(x)$ of degree $m < n$ that $\int_0^1 p_n(x)q_m(x)d\,x = 0$ provided $\deg(q_m) = m < n$. In addition, $\int_{-1}^1 P_n^2(x)d\,x = \frac{2}{2n+1}$ implies $\int_0^1 p_n^2(x)d\,x = \frac{1}{2n+1}$.

Now consider a representation of $l(x)$ in the form (3) with $\gamma_i \in [0,1]$. By (21) $C_{n+1} = \int_0^1 \o_n(t)d\,t$, where $\o_n(x) := \prod_{i=1}^n (x-\gamma_i)$. Given a fixed $n \in \mathbb{N}$, let us represent $p_n(x)$ by the Newton interpolation formula at the points $\{\gamma_i\}_{i=n+1}^{2n+1}$, namely

$$\begin{aligned} p_n(x) &= \sum_{k=0}^n p_n[\gamma_{n+1}, ..., \gamma_{n+k+1}](x-\gamma_{n+1})...(x-\gamma_{n+k}) \\ &= \sum_{k=0}^n \frac{p_n^{(k)}(\eta_k)}{k!}(x-\gamma_{n+1})\ldots(x-\gamma_{n+k}), \end{aligned}$$

where $\eta_k \in [0,1]$, $\;k = 0, \ldots, n$. Now we will use the relation $P_n(x) = P_n^{\left(\frac{1}{2}\right)}(x)$ and the following properties (see [7, Ch.4.7,7.33]) of the ultraspherical polynomials $P_n^{(\lambda)}(x)$ (note that here λ represents a parameter, and not derivative order): $\frac{d}{dx}\left\{P_m^{(\lambda)}(x)\right\} = 2\lambda P_{m-1}^{(\lambda+1)}(x)$; $P_m^{(\lambda)}(1) = \binom{m+2\lambda-1}{m}$ and $\max\limits_{-1\le x\le 1}|P_m^{(\lambda)}(x)| = P_m^{(\lambda)}(1)$, for $\lambda > 0$. Then, with $\lambda = k + \frac{1}{2}$, we have $P_n^{(k)}(x) = (2k-1)!!P_{n-k}^{(\lambda)}(x)$ and hence, for $k = 0, \ldots, n$,

$$|p_n^{(k)}(\eta_k)| = 2^k |P_n^{(k)}(2\eta_k - 1)| \le 2^k \|P_n^{(k)}\|_{C[-1,1]} = 2^k P_n^{(k)}(1) = \frac{(n+k)!}{k!(n-k)!}.$$

In particular, for $k=n$ we find that the leading coefficient of $p_n(x)$ is $q_n = \binom{2n}{n}$. Thus, from the above representation of $p_n(x)$ and (21) we obtain

$$I_n := \int_0^1 \o_n(x) p_n(x) d\,x = \sum_{k=0}^{n} \frac{p_n^{(k)}(\eta_k)}{k!} C_{n+k+1}. \tag{25}$$

On the other hand

$$I_n = q_n^{-1} \int_0^1 \big((q_n \o_n(x) - p_n(x)) + p_n(x)\big) p_n(x) d\,x = q_n^{-1} \int_0^1 p_n^2(x) d\,x = \frac{(n!)^2}{(2n+1)!}.$$

Therefore, for at least one summand in (25) we have that

$$\begin{aligned} \frac{(n!)^2}{(n+1)(2n+1)!} &\le \frac{p_n^{(k)}(\eta_k)}{k!} C_{n+k+1} \\ &\le \frac{(n+k)!}{(k!)^2 (n-k)!} |C_{n+k+1}| = \tbinom{n+k}{2k} \tbinom{2k}{k} |C_{n+k+1}| < 2^{n+3k} |C_{n+k+1}|, \end{aligned}$$

i.e. $|C_{n+k+1}| > 2^{-n-3k} \cdot \frac{(n!)^2}{(n+1)(2n+1)!} = \frac{2^{-n-3k}}{(n+1)(2n+1)\binom{2n}{n}} > \frac{1}{(n+1)(2n+1)} \left(\frac{1}{8}\right)^{n+k}$. As a consequence, the inequalities $|C_j| \le \frac{M}{(8+\varepsilon)^j} \quad \forall j \in \mathbb{N}$ can not hold true for any positive M and ε. Hence part b) and the theorem are proved. □

Proof of Theorem 3. First we shall prove the assertion for $\{C_n\}$. We can apply the same reasoning as in the proof of part b) of Theorem 2 to the estimation

$$\frac{(n!)^2}{(n+1)(2n+1)!} \le \frac{p_n^{(k)}(\eta_k)}{k!} C_{n+k+1}, \tag{26}$$

for some $k \in \{0, \dots, n\}$, but now the restriction is $|\eta_k| \le 1$. So, we need of an upper bound for $|p_n^{(k)}(\eta_k)|$. In view of the monotonicity of $|P_n^{(k)}(x)|$ for $|x| \ge 1$ we have

$$\begin{aligned} |p_n^{(k)}(\eta_k)| &= 2^k |P_n^{(k)}(2\eta_k - 1)| \le 2^k |P_n^{(k)}(-3)| = 2^k |P_n^{(k)}(3)| \\ &= 2^k Q_{n,k} (3^2 - x_{1,k}^2)(3^2 - x_{2,k}^2) \dots \le 2^k Q_{n,k}\, 3^{n-k}, \end{aligned}$$

where $Q_{n,k}$ and $\{\pm x_{i,k}\}$ are the leading coefficient and the zeros of $P_n^{(k)}$, respectively. From the definition of $P_n(x)$ (by the Rodrigues' formula) we find $Q_{n,k} = \frac{(2n)!}{2^n n! (n-k)!}$ from where $|p_n^{(k)}(\eta_k)| \le \frac{(2n)!}{n!(n-k)!} \left(\frac{3}{2}\right)^{n-k}$. Hence, taking modulus in (26), we can write

$$\begin{aligned} &(n+1)(2n+1)|C_{n+k+1}| \ge \frac{(n!)^2}{(2n)!} \cdot \frac{k!}{|p_n^{(k)}(\eta_k)|} \ge \left[\binom{2n}{n}^2 \binom{n}{k} \left(\frac{3}{2}\right)^{n-k}\right]^{-1} \\ &> \left[4^{2n} \binom{n}{k} 1^k \left(\frac{3a}{2}\right)^{n-k} a^{k-n}\right]^{-1} > \left[4^{2n} \left(1 + \frac{3a}{2}\right)^n a^{k-n}\right]^{-1} = \left[\left(\frac{16}{a} + 24\right)^n a^k\right]^{-1}. \end{aligned}$$

Now, if we choose $a = 12 + \sqrt{160} < 25$, that is, the positive root of the equation $z = 16/z + 24$, then for any fixed $M > 0$ we obtain $|C_{n+k+1}| > \frac{1}{(n+1)(2n+1)} a^{-n-k} > M \cdot 25^{-(n+k+1)}$, provided n is sufficiently large. The assertion for $\{C_n\}$ is proved.

In order to prove the impossibility of the bounds for $\{A_n\}$ in the theorem, let us assume that for some $M > 0$, $q < 1$ and $\{\alpha_n\}$ such that $|\alpha_n| \le 1$, the estimates $|A_n| \le Mq^n$, $n = 1, 2, \dots$ hold true. Then for $|x| < 1$ we have

$$l(x^2) = l(x) + l(-x) = \sum_{m=1}^{\infty} \frac{2A_{2m-1}\alpha_{2m-1}x^{2m}}{1 - \alpha_{2m-1}^2 x^2} + \frac{2A_{2m}x^{2m}}{1 - \alpha_{2m}^2 x^2}.$$

That is, a representation $l(u) = \displaystyle\sum_{n=1}^{\infty} \frac{B_n u^{\lfloor (n+1)/2 \rfloor}}{1 - \beta_n u}$ holds true for $|u| < 1$, where $\beta_n \in [0, 1]$ and $|B_n| \le 2Mq^n$. We shall show that this series can be written in the form (3) with parameters $\{\gamma_i\} = \{\beta_1, \beta_2, 0, \beta_3, \beta_4, 0, \beta_5, \beta_6, \dots\}$. This will follow from the possibility of the representations

$$\frac{u^m}{1 - \beta_n u} = \sum_{j=1}^{k} \frac{a_{j,n} u^j}{(1 - \gamma_1 u) \dots (1 - \gamma_j u)} =: \sum_{j=1}^{k} a_{j,n} c_j(u), \tag{27}$$

where $m = \lfloor \frac{n+1}{2} \rfloor$ and $k = n + m - 1 = \lfloor \frac{3n-1}{2} \rfloor$, i.e $\gamma_k = \beta_n$. Note that $\{\gamma_i\}_1^k$ contains $m - 1$ zeros. Then, with $v = 1/u$ we have $c_j(u) = \frac{1}{(v-\gamma_1)\dots(v-\gamma_j)}$ and (for $v \ne 0, \gamma_1, \dots, \gamma_k$) equality (27) is equivalent to

$$g_{n-1}(v) := (v - \beta_1) \dots (v - \beta_{n-1}) = \sum_{i=0}^{k-1} a_{k-i,n} (v - \gamma_k) \dots (v - \gamma_{k-i+1}),$$

where the indices of $\{\gamma_j\}$ decrease, so the first product in the sum is assumed equal to 1. Thus, we have a representation of $g_{n-1}(x)$ by the Newton interpolating formula at the points $\gamma_k, \dots, \gamma_1$, hence ($k \ge n$) the representation exists and $a_{j,n} = g_{n-1}[\gamma_k, \dots, \gamma_j]$, $j = 1, \dots, k$. As a consequence, $a_{k-i,n} = \frac{g_{n-1}^{(i)}(\xi_i)}{i!}$, where $\xi_i \in [0, 1]$ since $\{\gamma_j\} \subset [0, 1]$. The last equality implies that $a_{1,n} = \dots = a_{m-1,n} = 0$ and

$$|a_{k-i,n}| = \binom{n-1}{i} |(\xi_i - x_1^{(i)}) \dots (\xi_i - x_{n-1-i}^{(i)})| \le \binom{n-1}{i}, \quad i = 0, \dots, n-1,$$

where $\{x_j^{(i)}\}$ are the zeros of $g_{n-1}^{(i)}(x)$. Using these estimates and (27) we obtain

$$\begin{aligned} l(u) &= \sum_{n=1}^{\infty} \frac{B_n u^m}{1 - \beta_n u} = \sum_{n=1}^{\infty} B_n \sum_{\frac{n}{2} \le j \le \frac{3n-1}{2}} a_{j,n} c_j(u) = \sum_{j=1}^{\infty} c_j(u) \sum_{\frac{2j+1}{3} \le n \le 2j} B_n a_{j,n} \\ &=: \sum_{j=1}^{\infty} C_j c_j(u). \end{aligned}$$

Thus we have written $l(u)$ in the form (3) with coefficients that satisfy

$$|C_j| \le \sum_{\frac{2j+1}{3} \le n \le 2j} 2Mq^n \binom{n-1}{k-j} \le 2M \sum_{\frac{2j+1}{3} \le n \le 2j} q^n \binom{2j-1}{\lfloor \frac{3n-1}{2} \rfloor - j}.$$

The numbers $l_n(j) := \lfloor \frac{3n-1}{2} \rfloor - j,\ n = \lceil \frac{2j+1}{3} \rceil, \dots, 2j$ belong to $\{0, \dots, 2j-1\}$ and are distinct. So, in view of $\frac{3n}{2} > \lfloor \frac{3n-1}{2} \rfloor = l_n(j) + j$,

$$|C_j| < 2M \sum_{l=0}^{2j-1} q^{\frac{2}{3}(l+j)} \binom{2j-1}{l} = 2Mq^{\frac{2}{3}j} \big[1 + q^{2/3}\big]^{2j-1} = M_1 \big[q^{1/3} + q\big]^{2j}.$$

Therefore, the assumption $q \le \frac{1}{31}$ leads to $|C_j| < M_1 \left(\frac{1}{8}\right)^j$ which is a contradiction to Theorem 2 since $\gamma_j \in [0, 1]$. Theorem 3 is proved. $\square$

Now we shall make a comparison between the form (3) and the method of continued fractions (see e.g. [1, Ch.4] for the used results). The similarity of the two approaches is obvious – in both cases the n-th partial sum of the Maclaurin series is recovered. We mean the usual representation of a function by a continued fraction

$$f(z) = b_0 + \frac{a_1 z}{b_1 +}\ \frac{a_2 z}{b_2 +}\ \frac{a_3 z}{b_3 +}\ \cdots =: b_0 + \mathbf{K}_{i=1}^{\infty}(a_i z / b_i). \tag{28}$$

But there are also some essential differences. Only seemingly the form (28) depends on two sequences $\{a_i\}$ and $\{b_i\}$. In fact, a nonsingular continued fraction (28), i.e. with $a_i, b_i \ne 0,\ i \ge 1$, elementary can be transformed into an equivalent form, say with $a_i = 1$ or with $b_i = 1$. For example, the fraction

$$\log(1+z) = \frac{z}{1 +}\ \frac{1^2 z}{2 +}\ \frac{1^2 z}{3 +}\ \frac{2^2 z}{4 +}\ \frac{2^2 z}{5 +}\ \cdots + \frac{n^2 z}{2n +}\ \frac{n^2 z}{2n+1 +}\ \cdots \tag{29}$$

is transformed (by dividing the numerator and the denominator of the $2n$-th and $2n+1$-th terms to $n\sqrt{z}$) into

$$\begin{aligned} log(1+z) \quad = \quad & \frac{\sqrt{z}}{(1/\sqrt{z}) +}\ \frac{1}{(2/\sqrt{z}) +}\ \frac{1}{(3/1\sqrt{z}) +}\ \frac{2/1}{(2/\sqrt{z}) +}\ \frac{1}{(5/2\sqrt{z}) +}\ \cdots \\ & + \frac{n/(n-1)}{(2/\sqrt{z}) +}\ \frac{1}{((2n+1)/n\sqrt{z}) +}\ \cdots \end{aligned}$$

This form has the advantage to (29) that it is close to a continued fraction $F = \mathbf{K}_{i=1}^{\infty}(1/b_i)$ (with unit numerators). The convergence of such a fraction is very easy to realize in view of the Seidel's theorem which states that when the elements $\{b_i\}_1^{\infty}$ are positive, then F is convergent iff the series $\sum_1^{\infty} b_i$ is divergent. Moreover, for "relatively large" elements (say $|b_i| \ge 3, i \ge n_0$) the fraction converges approximately like $[b_1 b_2 \dots b_n]^{-2}$. In the case of $\log(1+z)$, and equivalently of $l(x)$, this rule gives an approximate rate of convergence like $[(2/\sqrt{z})^n]^{-2} = (z/4)^n\ (z \to 0)$.

The above arguments can be done precise using the formulas

$$\mathbf{K}_{i=1}^{\infty}(a_i/b_i) = \frac{a_1}{b_1} + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{B_i B_{i-1}} \prod_{j=1}^{i} a_j,$$

where $A_n/B_n = b_0 + \mathbf{K}_{i=1}^{n}(a_i/b_i)$ is the n-th convergent of the fraction, and

$$\begin{aligned} A_i &= b_i A_{i-1} + a_i A_{i-2}, \quad A_0 = b_0,\ A_{-1} = 1; \\ B_i &= b_i B_{i-1} + a_i B_{i-2}, \quad B_0 = 1,\ B_{-1} = 0. \end{aligned}$$

In particular, for a fraction with $a_i = 1$ the remainder is

$$R_n := \mathbf{K}_{i=1}^{\infty}(1/b_i) - \mathbf{K}_{i=1}^{n}(1/b_i) = \sum_{i=n+1}^{\infty} \frac{(-1)^{i-1}}{B_i B_{i-1}}.$$

Then, for "relatively large" $|b_i|$, the relation $B_i = b_i B_{i-1} + B_{i-2}$ usually implies $B_i \approx b_i B_{i-1}$ and $B_i \to \infty$ for $i \to \infty$, which in turn yields the approximate rule $|R_n| \approx 1/|B_n B_{n+1}| \approx B_n^{-2}$. It has to be mentioned however, that there are some special cases for the data $\{b_i\}$ when the principal asymptotic behaviour of $\{B_i\}$ as a solution of the above three term recurrence relation is suppressed and the magnitude of the sequence is not the usual one. This corresponds to a fraction of value $1/0 = \infty$ and, in our case of interest, for $l(x)$ with $|x| < 1$, such situations do not appear.

Now, the question is which is the right correspondence for comparing the two methods for accelerating power series? We argue that the most natural way is to compare the n-th partial sum of (3) with the n-th convergent of (28). The calculation of both approximations can be organized in different ways, say backward, and the formal counting of the the cost of arithmetic operations then gives the same result ($2n$). Indeed, the coefficients in the continued fraction for $l(x)$ are much simpler, but (as we have seen) taking $\{\gamma_i\}$ at the zeros of $T_k(x)$ and grouping summands we obtain rational parameters in the series, too.

Let us summarize the above comments. Both methods transform a series with rate of convergence like z^n into a series (sequence) converging approximately as $(z/4)^n$. This accelerating factor $(1/4)^n$ appears often in the continued fraction expansions, for example in

$$L(z) = \operatorname{arcth}(z) = \frac{z}{1-}\ \frac{1^2 z^2}{3-}\ \frac{2^2 z^2}{5-}\ \cdots\ -\frac{n^2 z^2}{2n+1-}\ \cdots\ .$$

Thus, in many cases the both methods have approximately the same efficiency.

6. ACCELERATION OF SERIES FOR OTHER FUNCTIONS

First we consider the function

$$f = f_\alpha(x) := \frac{1}{\alpha} + \frac{x}{1+\alpha} + \frac{x^2}{2+\alpha} + \frac{x^3}{3+\alpha} + \cdots,$$

which contains $l(x)$ and $L(\sqrt{x})$ as particular cases. We describe some transformations of f_α allowing its effective computation. Let us change the variable by $t = \frac{x}{1-x/2}$. An explicit formula for the coefficients in $f = \sum_{n=0}^\infty a_n t^n =: S_0(t)$ can be written using the Euler transform, but it is not convenient for computation and estimation of $\{a_n\}$. More important is the recursive rule, which follows from the differential equation

$$\frac{df}{dt} = -\frac{2\alpha}{t(2+t)} f + \frac{1}{t(1-t/2)},$$

a consequence of $f = \frac{1}{x^\alpha}\int_0^x \frac{z^{\alpha-1}}{1-z}dz = \frac{(1+t/2)^\alpha}{t^\alpha}\int_0^{\frac{t}{1+t/2}} \frac{z^{\alpha-1}}{1-z}dz$. We have

$$a_{n+1} = \Big(\frac{1}{2^n} - \frac{n}{2}a_n\Big)\Big/(n+1+\alpha), \quad a_0 = \frac{1}{\alpha}. \tag{30}$$

When $\alpha > 0$, it is easily seen from (30) that $a_n \in (0, 2^{1-n}),\ n > 0$, hence the transformation gives an acceleration of $f_\alpha(x)$, for $x \to 0$, like 2^{-n}.

An interesting consequence is obtained when we replace the coefficients in $f = \sum_{n=0}^\infty a_n t^n$ from (30), namely

$$\Big(1+\frac{t}{2}\Big) f_\alpha\big(\frac{t}{1+t/2}\big) + \frac{1}{\alpha} = 2 f_\alpha\big(\frac{t}{2}\big) + \frac{1+\alpha}{2t^\alpha}\int_0^t z^\alpha f_\alpha\big(\frac{z}{1+z/2}\big)dz, \quad \alpha > -1.$$

By differentiating the identity $\bar f_\alpha(x) = \sum_{n=1}^\infty a_n(\alpha) t^n$, where $\bar f_\alpha := f_\alpha - \frac{1}{\alpha}$, with respect to α at $\alpha = 0$, we obtain another interesting result. Note that $\bar f_0 = l(x)$ and the above transformation leads to (5): $\{a_n(0)\}_1^\infty = \Big\{\frac{1}{1}, 0, \frac{1}{3}\Big(\frac{1}{2}\Big)^2, 0, \frac{1}{5}\Big(\frac{1}{2}\Big)^4, 0, \dots\Big\}$. Also, by (30) it follows $(n+1)a'_{n+1}(0) + a_{n+1}(0) = -na'_n(0)/2$ and one easily represents $\{na'_n(0)2^n\}$ as certain sums. With $z = t/2 = \frac{x}{2-x}$, this gives

$$\begin{aligned} Li_2(x) &= \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \cdots \\ &= 2\Big[\frac{1}{1}\Big(\frac{z^1}{1} - \frac{z^2}{2}\Big) + \Big(\frac{1}{1}+\frac{1}{3}\Big)\Big(\frac{z^3}{3} - \frac{z^4}{4}\Big) + \Big(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}\Big)\Big(\frac{z^5}{5} - \frac{z^6}{6}\Big) + \cdots\Big], \end{aligned}$$

and using that $\arctan^2(z) = \big(\frac{1}{1}\big)z^2 - \frac{1}{2}\big(\frac{1}{1}+\frac{1}{3}\big)z^4 + \frac{1}{3}\big(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}\big)z^6 - + \cdots$ we arrive at

$$Li_2(x) = 2\Big[\frac{1}{1}\frac{z^1}{1} + \Big(\frac{1}{1}+\frac{1}{3}\Big)\frac{z^3}{3} + \Big(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}\Big)\frac{z^5}{5} + \cdots\Big] - L^2(z). \tag{31}$$

The explicit formula $f_\alpha(x) = \sum_{n=0}^\infty \frac{n!}{(\alpha)_{n+1}}\frac{(-x)^n}{(1-x)^{n+1}} =: S_1(x)$, which follows from the Euler transform and $(\alpha)_k := \alpha(\alpha+1)\dots(\alpha+k-1)$, is also of certain interest. This identity gives an acceleration for f if $x < 0$ and especially when $x \approx -1$. Note that the larger is α, the smaller are the terms in the series. Therefore, it makes sense to shift the parameter to the right according to the formula $f_\alpha(x) := \sum_{i=0}^{k-1} \frac{x^i}{i+\alpha} + x^k f_{\alpha+k}(x)$.

Similarly to $l(x)$, $f_\alpha(x)$ has an analytic continuation in $\mathcal{D}_f = \mathbb{C} \setminus [1, \infty)$. From now on $f_\alpha(z)$ will mean this continuation of the series $f_\alpha(z)$. Then, let us justify the domains where the above identities take place. Note that some series representation $S(z)$ of $f_\alpha(z)$ coincide with the function in this connected component of the intersection of the definition domains, which contains $z = 0$. Since the domain of convergence of $S_1(z)$ is $\left|\frac{-z}{1-z}\right| < 1$ and a part of the boundary, depending on α, then $S_1(z)$ represents $f_\alpha(z)$ in the half-plane $Re(z) < 1/2$. Similarly, the domain of convergence of $S_0(\frac{z}{1-z/2})$ is included in $\mathcal{D}_f$, then this series can be used for calculation of $f_\alpha(z)$ in the half-plane $Re(z) < 1$. The remaining part of $\mathcal{D}_f$ can be covered by the following two formulas which are consequences from the relation

$$\alpha f_\alpha(z) = F(1, \alpha; 1 + \alpha; z). \tag{32}$$

For the properties of the hypergeometric function $F(a, b; c; z)$ see [2] and the multiple labels below refer to this book. Now, applying the identity 2.1(17) (which is 2.9(34)) we obtain

$$f_\alpha(z) = \frac{1}{z} f_{1-\alpha}(z^{-1}) + \frac{\pi(-z)^{-\alpha}}{\sin \pi\alpha},$$

where $y^\beta := e^{\beta \log_0(y)}$. According to the above note this relation holds for $z \in \mathbb{C} \setminus [0, \infty)$ and $\alpha \notin \mathbb{Z}$. (When α is an integer, then $f_\alpha(z)$ reduces to $l(z)$ and its analytic continuation is clear.) Another easy consequence of this formula is that when the variable z crosses the segment $(1, +\infty)$ at z_0 in positive direction, then the value of $f_\alpha(z)$ jumps by $2\pi i z_0^{-\alpha}$.

The next transformation changes the argument to $1 - z$ and is very useful for $z \approx 1$. Notice however that f_α belongs to the set of the so-called degenerate cases of the hypergeometric function and many known identities can not be used directly but after a limit passage. Thus, from 2.9(33), applied for $F(1, \alpha; 1 + \alpha + \varepsilon; z)$ with $\varepsilon \to 0$, or directly by 2.3(2) with $l = 0$, we get

$$\begin{aligned} f_\alpha(z) = \sum_{n=1}^{\infty} \Big(\frac{1}{1} - \frac{1}{\alpha} + \frac{1}{2} - \frac{1}{\alpha+1} + \cdots + \frac{1}{n} - \frac{1}{\alpha+n-1}\Big) \frac{(\alpha)_n}{n!} (1-z)^n \\ -\big(\psi(\alpha) + C + \log(1-z)\big) \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} (1-z)^n, \end{aligned} \tag{33}$$

where $\psi(\alpha)$ is the digamma function and C is the Euler-Mascheroni constant. The relation (33) holds in the domain $\{|z-1| < 1\} \setminus [1, 2)$.

Some other consequences of (32) are:

$$f_\alpha(x) = \int_0^1 \frac{t^{\alpha-1}}{1-xt} dt, \quad Re(\alpha) > 0, \ x \notin [1, \infty),$$

which follows from the Euler integral 2.1.3;

$$f_\alpha(x) = \frac{1}{\alpha -} \; \frac{\alpha^2 x}{\alpha+1 -} \; \frac{1^2 x}{\alpha+2 -} \; \frac{(\alpha+1)^2 x}{\alpha+3 \;\; -} \; \frac{2^2 x}{\alpha+4 -} \; \frac{(\alpha+2)^2 x}{\alpha+5 \;\; -} \; \cdots \; ,$$

see 2.5.4; next, the forth equality at the definition of u_1 in 2.9 gives

$$f_\alpha(x) = (1-x)^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \frac{y^n}{\alpha+n}, \quad y = \frac{-x}{1-x};$$

and again there, from the second equality, by a limit pass with respect to any parameter of $F(a,b;c;x)$, it follows

$$l(x) f_\alpha(x) = \sum_{n=1}^{\infty} \left(h_n + \frac{1}{\alpha} + \frac{1}{\alpha+1} + \cdots + \frac{1}{\alpha+n-1} \right) \frac{x^n}{\alpha+n}.$$

Another interesting identity is obtained from the relation $f_\nu(z) = \Phi(z,1,\nu)$. Namely, the formula 1.11(9) (which holds for $m=1$ as well), in view of 1.10(11), gives

$$f_\nu(z) = z^{-\nu} \left\{ -\sum_{n=1}^{\infty} \frac{B_n(\nu)}{n} \cdot \frac{(\log z)^n}{n!} + \left[\psi(1) - \psi(\nu) - \log\log\frac{1}{z} \right] \right\}, \qquad (34)$$

where $B_n(\nu)$ are the Bernoulli polynomials and $|\log z| < 2\pi$.

Finally, we consider the digamma function, because it is closely connected with $f_\alpha(z)$. Indeed, if in place of the divergent series $f_\alpha(1) = \sum_{n=0}^{\infty} \frac{1}{\alpha+n}$ we take $\bar\psi(\alpha) := \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\alpha+n} \right)$, then $\bar\psi(\alpha) = \psi(\alpha) + C + \frac{1}{\alpha}$. As effective methods for calculation of $\psi(\alpha)$ (and $\bar\psi(\alpha)$) one can use (33) or (34). Also, the formula 1.7(30):

$$\psi(a+z) = \psi(a) + \frac{z}{a} - \frac{1}{2}\frac{z(z-1)}{a(a+1)} + \frac{1}{3}\frac{z(z-1)(z-2)}{a(a+1)(a+2)} - + \cdots$$

can serve for this purpose. Namely, assume that $x = O(1)$ and the value $\psi(x)$ is needed with accuracy 27^{-k}. Then, with $a = k$ and $z = x + k$ take $2k$ summands of the formula. The terms at that place are approximately $(k!)^3/(3k)!$ and decay as $const/3^n$. So, eventually taking several additional summands we stop when the last one becomes less than the required accuracy. Also, the shift formulas 1.7(9): $\psi(k) = h_{k-1} - C$ and 1.7(10): $\psi(x+2k) = \psi(x) + \sum_{j=0}^{2k-1} \frac{1}{x+j}$ are needed for the calculation, and they require $3k$ additional divisions.

We refer to [6] for more recent methods for computation of $\psi(z)$ (and $\Gamma(z)$).

Actually, the series $\psi(\alpha)$ easily can be transformed into a series that converges like $1/n!$, but the problem is that there appear infinitely many unknown constants. For example, such a rearrangement is given by the following formula of type (3)

$$\bar\psi(\alpha) = \frac{c_1 \alpha}{1+\alpha} + \frac{c_2 \alpha^2}{(1+\alpha)(2+\alpha)} + \frac{c_3 \alpha^3}{(1+\alpha)(2+\alpha)(3+\alpha)} + \cdots \, ,$$

where $c_1 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $c_2 = 2\sum_{n=2}^{\infty} \frac{n-1}{n^3} = 0.8857...$, $c_3 = 3\sum_{n=3}^{\infty} \frac{(n-1)(n-2)}{n^4} = 0.6102...$, $c_4 = 4\sum_{n=4}^{\infty} \frac{(n-1)(n-2)(n-3)}{n^5} = 0.4663...$, etc. The above series is a consequence of the more general relation

$$\frac{\alpha}{x_1(\alpha+x_1)} + \frac{\alpha}{x_2(\alpha+x_2)} + \frac{\alpha}{x_3(\alpha+x_3)} + \cdots = \frac{c_1\alpha}{\alpha+x_1} + \frac{c_2\alpha^2}{(\alpha+x_1)(\alpha+x_2)} + \cdots,$$

with $\frac{c_1}{x_1} = \sum_{n=1}^{\infty} \frac{1}{x_n^2}$, $\frac{c_2}{x_2} = \sum_{n=2}^{\infty} \frac{x_n - x_1}{x_n^3}$, $\frac{c_3}{x_3} = \sum_{n=3}^{\infty} \frac{(x_n-x_1)(x_n-x_2)}{x_n^4}$, ..., provided the series are convergent. If we set $x_{k+1} = x_{k+2} = \cdots = \infty$, then the relation becomes a polynomial identity, which is not difficult to verify.

Other interesting series are obtained by expanding $\bar{\psi}(\alpha)$ on rational terms containing $\alpha^{(n)} := \alpha(\alpha-1)...(\alpha-n+1)$, for example

$$\begin{aligned}
\bar{\psi}(\alpha) &= 2\left\{\frac{1}{1}\cdot\frac{\alpha}{\alpha+1} + \frac{1}{2}\cdot\frac{\alpha(\alpha-1)}{(\alpha+1)(\alpha+2)} + \frac{1}{3}\cdot\frac{\alpha(\alpha-1)(\alpha-2)}{(\alpha+1)(\alpha+2)(\alpha+3)} + \cdots\right\} \\
&= \sum_{k=1}^{\infty} \frac{\alpha(\alpha^2-1^2)...(\alpha^2-(k-1)^2)}{(1+\alpha)_{2k}} \cdot \frac{(8k-3)\alpha + k(10k-3)}{(2k-1)(2k)} \\
&= \alpha\Big(\frac{1}{\alpha+1} + \frac{1}{2.1}\Big) - \frac{\alpha(\alpha-1)}{2.3}\Big(\frac{1}{\alpha+2} + \frac{1}{2.2}\Big) + \frac{\alpha^{(3)}}{(3)_3}\Big(\frac{1}{\alpha+3} + \frac{1}{2.3}\Big) \\
&\quad - \frac{\alpha^{(4)}}{(4)_4}\Big(\frac{1}{\alpha+4} + \frac{1}{2.4}\Big) + - \ \cdots .
\end{aligned}$$

Note that the last two series converge like a geometrical series with ratio $\frac{1}{4}$.

We shall prove in details only the first identity. We start by proving the formula

$$\frac{\alpha}{\alpha+k} = \sum_{j=1}^{k} c_j(k)\frac{\alpha^{(j)}}{(\alpha+1)_j}, \text{ where } c_j(k) = 2j\frac{(k-1)^{(j-1)}}{(k+1)_j}, \ k = 1, 2, 3, \ldots \tag{35}$$

To prove the existence of such a representation with certain coefficients we remove the denominators and divide by α arriving to an equality between polynomials of degree $k-1$. Now, choosing the coefficients $\{c_j\}_1^k$ successively by substituting $\alpha = 1, \ldots, k$, the equality follows by the uniqueness of the interpolating polynomial. In order to verify the formula for the coefficients we multiply the identity by $(\alpha+1)_j$ and obtain

$$\frac{\alpha}{\alpha+k}(\alpha+1)_j = \alpha P_{j-1}(\alpha) + (\alpha+1)_j \sum_{i=j+1}^{k} c_i(k)\frac{\alpha^{(i)}}{(\alpha+1)_i},$$

where $P_{j-1}(\alpha)$ is a polynomial of degree $j-1$. Rewriting the last equality as

$$\frac{(1+\alpha)_j - (1-k)_j}{\alpha+k} + \frac{(1-k)_j}{\alpha+k} = P_{j-1}(\alpha) + (\alpha+1)_j \sum_{i=j+1}^{k} c_i(k)\frac{(\alpha-1)^{(i-1)}}{(\alpha+1)_i}$$

and noticing that the second term on the right-hand side vanishes for $\alpha = 1, \dots, j$, we conclude that $P_{j-1}(\alpha) - \frac{(1+\alpha)_j - (1-k)_j}{\alpha+k}$ is the interpolating polynomial for $\frac{(1-k)_j}{\alpha+k}$. In particular the leading coefficient equals

$$c_1(k) + \dots + c_j(k) - 1 = \frac{(1-k)_j}{\alpha+k}\big[1, 2, \dots, j\big] = \frac{(k-1)^{(j)}}{(k+1)_j},$$

which easily implies the formula for $c_j(k)$.

Now, we substitute $\frac{\alpha}{\alpha+k}$ from (35) into $\bar{\psi}(\alpha) = \sum_{k=1}^{\infty} \frac{\alpha}{k(\alpha+k)}$ and rearrange the summation with respect to the basis $\left\{\frac{\alpha^{(j)}}{(\alpha+1)_j}\right\}$, which is admissible since the double sum has positive terms. Then for the coefficients we get

$$\sum_{k=j}^{\infty} \frac{c_j(k)}{k} = 2j \sum_{k=j}^{\infty} \frac{(k-1)^{(j-1)}}{(k)_{j+1}} = \frac{2j!}{(j)_{j+1}} F(j, j; 2j-1; 1) = \frac{2}{j},$$

where we used that $F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. The first formula for $\bar{\psi}(\alpha)$ is proved.

For the proof of the second relation one can use the identity

$$\frac{\alpha}{\alpha+n} = \sum_{j=1}^{\lceil n/2 \rceil} \frac{(\alpha+j-1)^{(2j-1)}}{(\alpha+1)_{2j}} \big(e_j(n)\alpha + d_j(n)\big), \quad n = 1, 2, 3, \dots,$$

where $e_j(1)\alpha + d_j(1) = \alpha + 1$ and

$$e_j(n)\alpha + d_j(n) = \frac{n^{(2j-1)}}{(n-j)_{2j+1}} \big((4j-1)(n\alpha - j^2) - j(5j-2)(\alpha - n)\big), \quad n = 2, 3, \dots,$$

while the third one is a consequence of

$$\frac{\alpha}{\alpha+n} = \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{(n+1)_j} \cdot \alpha^{(j)} + \frac{(-1)^{n-1}}{(n+1)_{n-1}} \cdot \frac{\alpha^{(n)}}{\alpha+n}.$$

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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DEFINABILITY ISSUES IN THE ω-TURING DEGREES

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We show the equivalence of the first order definabilities of the jump of the least element and of the jump operator in the upper semilattice of the ω-Turing degrees.

Keywords: Turing reducibility, ω-Turing degrees, degree structures, definability, jump.

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1. INTRODUCTION

The investigation of the reducibilities between sequences of sets of natural numbers is initiated by Soskov. In the work [8] he introduces the ω-*enumeration* reducibility $\leq_\omega$, which compares the informational content of sequences of sets in a way that generalizes the Selman characterizing theorem for the enumeration reducibility[1]. As a preorder, the reducibility $\leq_\omega$ induces a degree structure – the structure $\mathcal{D}_\omega$ of the ω-enumeration degrees. Again in [8] it is given a definition of a jump operation $'$ over the ω-enumeration degrees. In [9] Soskov and Ganchev continue the studying of the structure $\mathcal{D}_\omega$. They derive that $\mathcal{D}_\omega$ is a proper extension of the structure $\mathcal{D}_e$ of the enumeration degrees whose group $\mathrm{Aut}(\mathcal{D}'_\omega)$ of the jump preserving automorphisms is isomorphic to the automorphism group $\mathrm{Aut}(\mathcal{D}_e)$ of $\mathcal{D}_e$. Recently Ganchev and Sariev show that in $\mathcal{D}_\omega$ the jump operation is first-order definable in the language of the structure order. In this way each automorphism of $\mathcal{D}_\omega$ is jump preserving, so the structures of enumeration and ω-enumeration degrees have isomorphic automorphism groups: $\mathrm{Aut}(\mathcal{D}_e)\cong \mathrm{Aut}(\mathcal{D}_\omega)$.

[1] The Selman Theorem states that $A \leq_e B \iff (\forall X \subseteq \omega)[B \leq_{c.e.} X \to A \leq_{c.e.} X]$

The ω-Turing reducibility $\leq_{T,\omega}$ arises as a 'Turing' analogue of $\leq_\omega$ and just like the ω-enumeration reducibility compares the informational content of the sequences of sets of natural numbers. In this computational framework the informational content of a sequence is uniquely determined by the set of the Turing degrees of the sets that *code* the sequence. We say that a set codes a sequence iff *uniformly* in k, it can compute the k-th element of the considered sequence in its k-th Turing jump:

$$X \subseteq \omega \text{ codes } \{A_k\}_{k<\omega} \iff A_k \leq_T X^{(k)} \text{ uniformly in } k.$$

Having this, we shall say that the sequence $\mathcal{A}$ is ω-Turing reducible to the sequence $\mathcal{B}$ iff each set that codes $\mathcal{B}$ also codes $\mathcal{A}$:

$$\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff (\forall X \subseteq \omega)[X \text{ codes } \mathcal{B} \Rightarrow X \text{ codes } \mathcal{A}].$$

This reducibility is introduced in [6], where its basic properties are explored. The relation $\leq_{T,\omega}$ is a preorder on the set of the sequences of sets of natural numbers and in the standard way induces a degree structure – the upper semi-lattice $\mathcal{D}_{T,\omega}$ of the ω-Turing degrees.

Again in [6] is defined a jump operation on sequences, which induces a corresponding jump operation in the degree structure. Namely the jump $\mathcal{A}'$ of the sequence $\mathcal{A}$ is defined in such a way that:

$$X \text{ codes } \mathcal{A}' \iff (\exists Y)[X \equiv_T Y' \;\&\; Y \text{ codes } \mathcal{A}].$$

How $\mathcal{D}_{T,\omega}$ can be seen as an extension of the structure $\mathcal{D}_T$ of the Turing degrees? By the uniform properties of the Turing jump, it is well known that for all $A, X \subseteq \omega$:

$$A \leq_T X \iff A^{(k)} \leq_T X^{(k)} \text{ uniformly in } k.$$

Thus, the informational content of the set A, described in the Turing universe by the set of the degrees of the sets that decides A, is the same as the content of the sequence $\{A^{(k)}\}_{k<\omega}$ in the context of the ω-Turing reducibility. This observation allows us to define a very natural embedding of the Turing degrees into the ω-Turing:

$$\deg_T(A) \longmapsto \deg_{T,\omega}(\{A^{(k)}\}_{k<\omega}).$$

This embedding preserves the order, the least upper bound operation and even the jump. In this way we may assume the Turing degrees as a proper substructure of $\mathcal{D}_{T,\omega}$. But there are much more strong connections between the both structures. In [6] it is shown that $\mathcal{D}_T$ is definable in $\mathcal{D}_{T,\omega}$ by a first-order formula in the language of the structure order and the jump operation. Also it is proved that the group $\mathrm{Aut}(\mathcal{D}_T)$ of the automorphisms of the Turing degrees is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(\mathcal{D}_{T,\omega})$ of $\mathcal{D}_{T,\omega}$ – namely to the subgroup $\mathrm{Aut}(\mathcal{D}'_{T,\omega})$ of the jump preserving automorphisms of the ω-Turing degrees.

The purpose of this paper is to show that in order to prove that the jump operator is first-order definable in the ω-Turing degrees it is sufficient to prove that the jump $\mathbf{0}_{T,\omega}{}'$ of the least element is definable by a first-order formula in the language of the structure order. We also show that the definability of $\mathbf{0}_{T,\omega}{}'$ implies the definability of $\mathcal{D}_T$ only in the language of the structure order.

2. PRELIMINARIES

2.1. BASIC NOTIONS

We shall denote the set of natural numbers by ω. If not stated otherwise, a, b, $c, \ldots$ shall stand for natural numbers, A, B, $C, \ldots$ for sets of natural numbers, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}, \ldots$ for degrees and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}, \ldots$ for sequences of sets of natural numbers. We shall further follow the following convention: whenever a sequence is denoted by a calligraphic Latin letter, then we shall use the Roman style of the same Latin letter, indexed with a natural number, say k, to denote the k-th element of the sequence (we always start counting from 0). Thus, if not stated otherwise, $\mathcal{A} = \{A_k\}_{k<\omega}$, $\mathcal{B} = \{B_k\}_{k<\omega}$, $\mathcal{C} = \{C_k\}_{k<\omega}$, etc. We shall denote the set of all sequences (*of length* ω) of sets of natural numbers by $\mathcal{S}_\omega$.

As usual $A \oplus B$ shall stand for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$. By A^+ we shall denote the set $A \oplus (\omega \setminus A)$.

We assume that the reader is familiar with the notion of Turing reducibility, $\leq_T$, and with the structure of the Turing degrees $\mathcal{D}_T$ (for a survey of basic results on the Turing degree structure we refer the reader to [2, 3, 4].

The relation $\leq_T$ is a preorder on the powerset 2^ω of the natural numbers and induces a nontrivial equivalence relation $\equiv_T$. The equivalence classes under $\equiv_T$ are called Tuirng degrees. The Tuirng degree which contains the set A is denoted by $\deg_T(A)$. The set of all Turing degrees is denoted by $\mathbf{D}_T$. The Tuirng reducibility between sets induces a partial order $\leq_T$ on $\mathbf{D}_T$ by

$$\deg_T(A) \leq_T \deg_T(B) \iff A \leq_T B.$$

We denote by $\mathcal{D}_T$ the partially ordered set $(\mathbf{D}_T, \leq_T)$. The least element of $\mathcal{D}_T$ is the Turing degree $\mathbf{0}_T$ of $\emptyset$. Also, the degree of $A \oplus B$ is the least upper bound of the degrees of A and B. Therefore $\mathcal{D}_T$ is an upper semi-lattice with least element.

The (Turing) jump A' of $A \subseteq \omega$ is defined as the halting problem for machines with an oracle A,

$$A' = \{e \mid \text{the } e\text{-th Turing machine with oracle } A \text{ halts on input } e\}.$$

The jump operation preserves the Turing reducibility, so we can define $\deg_T(A)' = \deg_T(A')$. Since $A <_T A'$, then we have $\mathbf{a} <_T \mathbf{a}'$ for every Turing degree $\mathbf{a}$. The jump operator is uniform, i.e. there exists a recursive function j such that for every sets A and B, if $A \leq_T B$ via the Turing operator with index e, then $A' \leq_T B'$ via the operator with index $j(e)$.

The ω-Turing reducibility and the corresponding degree structure $\mathcal{D}_{T,\omega}$ are introduced by Sariev and Ganchev in [6]. An equivalent, but more approachable definition in the terms of the uniform Turing reducibility is derived again in the same paper. Here we shall present only on the latter definition. According to it, the sequence $\mathcal{A}$ is ω-Turing reducible to the sequence $\mathcal{B}$, denoted by $\mathcal{A} \leq_{T,\omega} \mathcal{B}$, iff for every $n < \omega$,

$$A_n \leq_T P_n(\mathcal{B}) \text{ uniformly in } n.$$

Here, for each $\mathcal{X} \in \mathcal{S}_\omega$, $\mathcal{P}(\mathcal{X})$ is the so-called *jump sequence* of $\mathcal{X}$ and it is defined as the sequence $\{P_k(\mathcal{X})\}_{k<\omega}$ such that: $P_0(\mathcal{X}) = X_0$ and for each $k < \omega$, $P_{k+1}(\mathcal{X}) = (P_k(\mathcal{X}))' \oplus X_{k+1}$.

Clearly $\leq_{T,\omega}$ is a reflexive and transitive relation, and the relation $\equiv_{T,\omega}$ defined by

$$\mathcal{A} \equiv_{T,\omega} \mathcal{B} \iff \mathcal{A} \leq_{T,\omega} \mathcal{B} \text{ and } \mathcal{B} \leq_{T,\omega} \mathcal{A}$$

is an equivalence relation. The equivalence classes under this relation are called ω-Turing degrees. In particular the equivalence class $\deg_{T,\omega}(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_{T,\omega} \mathcal{B}\}$ is called the ω-Tuirng degree of $\mathcal{A}$. The relation $\leq_{T,\omega}$ defined by

$$\mathbf{a} \leq_{T,\omega} \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b} (\mathcal{A} \leq_{T,\omega} \mathcal{B})$$

is a partial order on the set of all ω-Turing degrees $\mathbf{D}_{T,\omega}$. By $\mathcal{D}_{T,\omega}$ we shall denote the structure $(\mathbf{D}_{T,\omega}, \leq_{T,\omega})$. The ω-Turing degree $\mathbf{0}_{T,\omega}$ of the sequence $\emptyset_\omega = \{\emptyset\}_{k<\omega}$ is the least element in $\mathcal{D}_{T,\omega}$. Further, the ω-Turing degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \deg_{T,\omega}(\mathcal{A})$ and $\mathbf{b} = \deg_{T,\omega}(\mathcal{B})$. Thus $\mathcal{D}_{T,\omega}$ is an upper semi-lattice with least element.

It is not difficult to notice that each sequence and its jump sequence belong to the same ω-Turing degree, i.e. for all $\mathcal{A} \in \mathcal{S}_\omega$,

$$\mathcal{A} \equiv_{T,\omega} \mathcal{P}(\mathcal{A}). \tag{2.1}$$

In this way, $\mathcal{P}(\mathcal{A})$ is an equivalent to $\mathcal{A}$ sequence, whose members are monotone with respect to $\leq_T$ and each its member decides the halting problems of the previous members.

Given a set $A \subseteq \omega$, denote by $A \uparrow \omega$ the sequence $(A, \emptyset, \emptyset, \ldots, \emptyset, \ldots)$. The definition of $\leq_{T,\omega}$ and the uniformity of the jump operation imply that for all sets of natural numbers A and B,

$$A \uparrow \omega \leq_{T,\omega} B \uparrow \omega \iff A \leq_T B. \tag{2.2}$$

The latter equivalence means that the mapping $\kappa : \mathbf{D}_T \to \mathbf{D}_{T,\omega}$, defined by

$$\kappa(\deg_T(X)) = \deg_{T,\omega}(X \uparrow \omega),$$

is an embedding of $\mathcal{D}_T$ into $\mathcal{D}_{T,\omega}$. Further, the so defined embedding κ preserves the order, the least element and the binary least upper bound operation.

We shall refer to κ as *the natural embedding* of the Turing degrees into the ω-Turing degrees. The range of κ shall be denoted by $\mathbf{D}_1$ and shall be called *the natural copy* of the Turing degrees.

The following theorem makes the connection between the original definition of the ω-Turing reducibility and this one we took here.

Theorem 1 *Let $\mathbf{a} \in \mathbf{D}_{T,\omega}$ be a ω-Turing degree and $\mathbf{C} \subseteq \mathbf{D}_{T,\omega}$ be at most countable set of ω-Turing degrees. Let for each $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} \not\leq_{T,\omega} \mathbf{a}$. Then there exists $\mathbf{f} \in \mathbf{D}_1$ such that $\mathbf{a} \leq_{T,\omega} \mathbf{f}$ and for each $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} \not\leq_{T,\omega} \mathbf{f}$.*

A full proof[2] of this result can be found in the PhD thesis of the first author, [5].

From the above property easily follows that each ω-Turing degree is uniquely determined by the set of the degrees in $\mathbf{D}_1$, which bound it,

$$\mathbf{a} \leq_{T,\omega} \mathbf{b} \iff (\forall \mathbf{x} \in \mathbf{D}_e)[\mathbf{b} \leq_{T,\omega} \kappa(\mathbf{x}) \to \mathbf{a} \leq_{T,\omega} \kappa(\mathbf{x})], \tag{2.3}$$

and hence, as one can see, $\mathbf{D}_1$ is an automorphism base of $\mathcal{D}_{T,\omega}$.

2.3. THE JUMP OPERATOR

Following the lines of Sariev and Ganchev [6], the ω-Turing jump $\mathcal{A}'$ of $\mathcal{A} \in \mathcal{S}_\omega$ is defined as the sequence $\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \ldots, A_k, \ldots)$.

Note, that $\mathcal{A}' \equiv_{T,\omega} \{P_{k+1}(\mathcal{A})\}_{k<\omega}$, because for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$.

The jump operator is strictly monotone, i.e. $\mathcal{A} \lneq_{T,\omega} \mathcal{A}'$ and $\mathcal{A} \leq_{T,\omega} \mathcal{B} \Rightarrow \mathcal{A}' \leq_{T,\omega} \mathcal{B}'$. This allows to define a jump operation on the ω-Turing degrees by setting

$$\deg_{T,\omega}(\mathcal{A})' = \deg_{T,\omega}(\mathcal{A}').$$

Clearly for all $\mathbf{a}, \mathbf{b} \in \mathbf{D}_{T,\omega}$, $\mathbf{a} <_{T,\omega} \mathbf{a}'$ and $\mathbf{a} \leq_{T,\omega} \mathbf{b} \Rightarrow \mathbf{a}' \leq_{T,\omega} \mathbf{b}'$.

Also the jump operation on ω-Turing degrees agrees with the jump operation on the Turing degrees, i.e. we have

$$\kappa(\mathbf{x}') = \kappa(\mathbf{x})', \text{ for all } \mathbf{x} \in \mathbf{D}_T.$$

We shall denote by $\mathcal{A}^{(n)}$ the n-the iteration of the jump operator on $\mathcal{A}$. Let us note that

$$\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \ldots) \equiv_{T,\omega} \{P_{n+k}(\mathcal{A})\}_{k<\omega}. \tag{2.4}$$

[2]here we present only a *sketch* of the proof: the idea is to use a similar result for the ω-enumeration degrees. First note that there is an embedding $\iota : \mathbf{D}_T \to \mathbf{D}_e$ of the Turing degrees into the enumeration degrees such that $\mathbf{a} \leq_T \mathbf{b} \Leftrightarrow \iota(\mathbf{a}) \leq_e \iota(\mathbf{b})$. Similarly, there is an embedding $\iota_\omega : \mathbf{D}_{T,\omega} \to \mathbf{D}_\omega$ of the ω-Turing degrees into the ω-enumeration degrees such that $\mathbf{a} \leq_{T,\omega} \mathbf{b} \Leftrightarrow \iota_\omega(\mathbf{a}) \leq_\omega \iota_\omega(\mathbf{b})$. And finally, there is an embedding $\kappa_e : \mathbf{D}_e \to \mathbf{D}_\omega$ of the enumeration degrees into the ω-enumeration degrees such that $\mathbf{a} \leq_e \mathbf{b} \Leftrightarrow \kappa_e(\mathbf{a}) \leq_\omega \kappa_e(\mathbf{b})$. More precisely these mappings are described, for example, in [6]. The property we use in the proof is that for each $\mathbf{a} \in \mathbf{D}_T$, $\iota_\omega(\kappa(\mathbf{a})) = \kappa_e(\iota(\mathbf{a}))$. The last part of the proof is the counterpart result of Theorem 1 concerning the ω-enumeration degrees. The main difference in it is that the degree $\mathbf{f}$ is not only in $\kappa_e[\mathbf{D}_e]$, but additionally is in $\kappa_e \circ \iota[\mathbf{D}_T]$. The proof of this result can be found in [10].

It is clear that if $\mathcal{A} \in \mathbf{a}$, then $\mathcal{A}^{(n)} \in \mathbf{a}^{(n)}$, where $\mathbf{a}^{(n)}$ denotes the n-th iteration of the jump operation on the degree $\mathbf{a}$.

In [6] it is proved that the range of the jump operator is exactly the upper cone over the first jump ${\mathbf{0}_{T,\omega}}'$ of the least element. Again in the same paper, it is shown even a stronger jump inversion property, which do not posses neither the Turing degrees, nor the enumeration degrees. Namely, for each natural number n if $\mathbf{b}$ is above $\mathbf{a}^{(n)}$, then there is a least ω-Turing degree $\mathbf{x}$ above $\mathbf{a}$ with $\mathbf{x}^{(n)} = \mathbf{b}$. We shall denote this degree by $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$. An explicit representative of $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$ can be given by setting

$$I^n_{\mathcal{A}}(\mathcal{B}) = (A_0, A_1, \ldots, A_{n-1}, B_0, B_1, \ldots, B_k, \ldots), \tag{2.5}$$

where each $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ are arbitrary.

In the case when $\mathbf{a} = \mathbf{0}_{T,\omega}$ and $n = 1$, for the sake of simplicity, we shall use the notation $\mathbf{I}$ instead of $\mathbf{I}^1_{\mathbf{0}_{T,\omega}}$. Sariev and Ganchev [6] show that the operation $\mathbf{I}$ is monotone,

$${\mathbf{0}_{T,\omega}}' \leq_{T,\omega} \mathbf{x} \leq_{T,\omega} \mathbf{y} \Rightarrow \mathbf{I}(\mathbf{x}) \leq_{T,\omega} \mathbf{I}(\mathbf{y}).$$

3. THE TURING DEGREES GENERATE $\mathbf{D}_{T,\omega}$

Our goal in this section is to prove that the isomorphic copy $\mathbf{D}_1$ of the Turing degrees under the natural embedding κ generates $\mathbf{D}_{T,\omega}$ under the greatest lower bound operation $\wedge$. More specifically, we will prove that for every ω-Turing degree $\mathbf{a}$ there exist degrees $\mathbf{g}$ and $\mathbf{f}$ from $\mathbf{D}_1$ such that $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$. We begin with the simple observation that each ω-Turing degree is bounded by a degree in $\mathbf{D}_1$.

Lemma 2 *Let* $\mathbf{a} \in \mathbf{D}_{T,\omega}$. *Then there is a degree* $\mathbf{g} \in \mathbf{D}_1$ *such that* $\mathbf{a} \leq_{T,\omega} \mathbf{g}$.

Proof. Recall that $\mathbf{a} \lneq_{T,\omega} \mathbf{a}'$. Then by Theorem 1 applied for $\mathbf{C} = \{\mathbf{a}'\}$, there is $\mathbf{g} \in \mathbf{D}_1$, such that $\mathbf{a} \leq_{T,\omega} \mathbf{g}$, but $\mathbf{a}' \nleq_{T,\omega} \mathbf{g}$. So $\mathbf{g}$ is a degree from $\mathbf{D}_1$, which bounds $\mathbf{a}$. □

Lemma 3 *Let* $\mathbf{a}, \mathbf{g} \in \mathbf{D}_{T,\omega}$ *and* $\mathbf{a} \leq_{T,\omega} \mathbf{g}$. *Then there is a degree* $\mathbf{f} \in \mathbf{D}_1$ *such that* $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$.

Proof. Let $\mathbf{a} \leq_{T,\omega} \mathbf{g}$. Consider the set $\mathbf{C} = \{\mathbf{x} \in \mathbf{D}_{T,\omega} \mid \mathbf{x} \leq_{T,\omega} \mathbf{g} \,\&\, \mathbf{x} \nleq_{T,\omega} \mathbf{a}\}$. Clearly $\mathbf{C}$ is countable and, hence, by Theorem 1, there exists a degree $\mathbf{f}$ in $\mathbf{D}_1$ such that $\mathbf{a} \leq_{T,\omega} \mathbf{f}$ and for every $\mathbf{x} \in \mathbf{C}$, $\mathbf{x} \nleq_{T,\omega} \mathbf{f}$.

Finally, let $\mathbf{b} \leq_{T,\omega} \mathbf{g}, \mathbf{f}$. Then $\mathbf{b} \notin \mathbf{C}$ and so $\mathbf{b} \leq_{T,\omega} \mathbf{a}$. Thus $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$. □

Combining the above lemmas, we have the following.

Theorem 4 *Let* $\mathbf{a} \in \mathbf{D}_{T,\omega}$. *Then there are degrees* $\mathbf{g}, \mathbf{f} \in \mathbf{D}_1$ *such that* $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$.

As a corollary we also have that (the isomorphic copy of) the Turing degrees form an automorphism base for the ω-Turing degrees.

Note that for each degree $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$ with $\mathbf{g}, \mathbf{f} \in \mathbf{D}_1$, the jump $\mathbf{a}'$ can be expressed as the greatest lower bound of two degrees $\mathbf{g}_1$ and $\mathbf{f}_1$ from $\mathbf{D}_1$. The next lemma shows that $\mathbf{g}'$ and $\mathbf{f}'$ are such a pair.

Lemma 5 *Let $\mathbf{a}, \mathbf{g}$ and $\mathbf{f}$ are ω-Turing degrees such that $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$. Then $\mathbf{a}' = \mathbf{g}' \wedge \mathbf{f}'$.*

Proof. Let $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$. Then $\mathbf{a} \leq_{T,\omega} \mathbf{g}, \mathbf{f}$ and by the monotonicity of the jump, $\mathbf{a}' \leq_{T,\omega} \mathbf{g}', \mathbf{f}'$.

Now let $\mathbf{b}$ is a lower bound of $\mathbf{g}'$ and $\mathbf{f}'$. Let $\mathbf{b}_1 = \mathbf{b} \vee \mathbf{0}_{T,\omega}{}'$. Then $\mathbf{b} \leq_{T,\omega} \mathbf{b}_1 \leq_{T,\omega} \mathbf{g}', \mathbf{f}'$ and $\mathbf{0}_{T,\omega}{}' \leq_{T,\omega} \mathbf{b}_1$. Let $\mathbf{c} = \mathbf{I}(\mathbf{b}_1)$. Since the jump inversion operation is monotone, we have that $\mathbf{c} = \mathbf{I}(\mathbf{b}_1) \leq_{T,\omega} \mathbf{I}(\mathbf{g}') \leq_{T,\omega} \mathbf{g}$ and $\mathbf{c} = \mathbf{I}(\mathbf{b}_1) \leq_{T,\omega} \mathbf{I}(\mathbf{f}') \leq_{T,\omega} \mathbf{f}$. But $\mathbf{a} = \mathbf{g} \wedge \mathbf{f}$, so $\mathbf{c} \leq_{T,\omega} \mathbf{a}$. Thus $\mathbf{b} \leq_{T,\omega} \mathbf{b}_1 = \mathbf{c}' \leq_{T,\omega} \mathbf{a}'$ by the monotonicity of the jump. $\square$

4. A PROPERTY OF THE LEAST TURING DEGREE

The aim of this section is to provide a characterizing property of the least Turing degree $\mathbf{0}_T$, which shall help us later to find a definition of $\mathbf{D}_1$ in the terms of $\mathbf{0}_{T,\omega}{}'$. We start by showing that $\mathbf{0}_T$ is the only degree $\mathbf{x}$ in $\mathcal{D}_T$, such that for each Turing degree $\mathbf{b}$, if $\mathbf{x} \vee \mathbf{b} \geq_T \mathbf{0}_T'$ then necessary $\mathbf{b} \geq_T \mathbf{0}_T'$. In order to do so, we first need the following notion of *minimal complementation*.

Definition 6 *We shall say that the (Turing) degree $\mathbf{d} >_T \mathbf{0}_T$ satisfies the minimal complementation property (MCP) if for every degree $\mathbf{0}_T <_T \mathbf{a} <_T \mathbf{d}$ there exists a minimal degree $\mathbf{m} <_T \mathbf{d}$ such that $\mathbf{a} \vee \mathbf{m} = \mathbf{d}$ (and therefore $\mathbf{a} \wedge \mathbf{m} = \mathbf{0}_T$):*

$$MCP(\mathbf{d}) \rightleftharpoons (\forall \mathbf{a} < \mathbf{d})[\mathbf{a} \neq \mathbf{0}_T \to (\exists \mathbf{m})[\mathbf{m} \textit{ is minimal} \ \& \ \mathbf{a} \vee \mathbf{m} = \mathbf{d}]].$$

In [1] Lewis proves that every degree $\mathbf{d} \geq_T \mathbf{0}_T'$ satisfies the minimal complementation property.

From here, one can easily derive that if $\mathbf{x}$ is a nonzero Turing degree, then there is a degree $\mathbf{y}$ such that $\mathbf{x} \vee \mathbf{y} \geq_T \mathbf{0}_T'$, but $\mathbf{y}$ is not above $\mathbf{0}_T'$. Indeed, let $\mathbf{x} \in \mathbf{D}_T$ be a nonzero. Then $\mathbf{x}' \geq_T \mathbf{0}_T'$, and hence MCP($\mathbf{x}'$). Since $\mathbf{0}_T <_T \mathbf{x} <_T \mathbf{x}'$ we have a minimal degree $\mathbf{y} <_T \mathbf{x}'$ such that $\mathbf{x} \vee \mathbf{y} = \mathbf{x}'$. But $\mathbf{y}$ is not above $\mathbf{0}_T'$ because it is minimal. Thus $\mathbf{0}_T' \not\leq_T \mathbf{y}$.

Note also, that the formula: $\varphi(\mathbf{x}) \rightleftharpoons (\forall \mathbf{y})[\mathbf{x} \vee \mathbf{y} \geq_T \mathbf{0}_T' \to \mathbf{y} \geq_T \mathbf{0}_T']$ is satisfied by the Turing degree $\mathbf{0}_T$ of the recursive sets. Thus, we have proven the following proposition.

Lemma 7 *The least element $\mathbf{0}_T$ is the only Turing degree $\mathbf{x}$ such that*

$$(\forall \mathbf{y})[\mathbf{x} \vee \mathbf{y} \geq_T \mathbf{0}_T' \to \mathbf{y} \geq_T \mathbf{0}_T'].$$

As an end of this section we move to the structure of the ω-Turing degrees, where we shall investigate the degrees defined by the formula φ. Namely, we shall describe all the ω-Turing degrees $\mathbf{x}$ such that

$$(\forall \mathbf{y})[\mathbf{x} \vee \mathbf{y} \geq_{T,\omega} \mathbf{0}_{T,\omega}{}' \to \mathbf{y} \geq_{T,\omega} \mathbf{0}_{T,\omega}{}'].$$

First let us consider a sequence $\mathcal{X} = \{X_k\}_{k<\omega}$ such that $\mathcal{D}_{T,\omega} \models \varphi(\deg_{T,\omega}(\mathcal{X}))$. In other words, $\mathcal{X}$ is such that for each sequence $\mathcal{Y} = \{Y_k\}_{k<\omega}$ if $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{X} \oplus \mathcal{Y}$ then $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{Y}$. Noting that for each sequence $\mathcal{A} = \{A_k\}_{k<\omega}$, $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{A}$ is equivalent to $\emptyset' \leq_T A_0$, and then using Lemma 7, we conclude that $X_0 \equiv_T \emptyset$.

Now, let $\mathcal{X} = \{X_k\}_{k<\omega}$ be such that $X_0 \equiv_T \emptyset$ and the sequence $\mathcal{Y} = \{Y_k\}_{k<\omega}$ be such that $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{X} \oplus \mathcal{Y}$. Then we have that $\emptyset' \leq_T X_0 \oplus Y_0 \equiv_T Y_0$, and hence $\emptyset_\omega{}' \leq_{T,\omega} \mathcal{Y}$.

Thus, the degrees in $\mathcal{D}_{T,\omega}$, which satisfy the formula φ, are exactly these that contain a sequences whose zeroth element is the empty set. Further we shall denote the set of all these degrees by $\widetilde{\mathbf{D}_1}$,

$$\widetilde{\mathbf{D}_1} = \{\mathbf{x} \in \mathbf{D}_{T,\omega} \mid (\exists \{A_k\}_{k<\omega} \in \mathbf{x})[\ A_0 = \emptyset\ ]\}.$$

5. DEFINABILITY IN THE ω-TURING DEGREES

In [6] Sariev and Ganchev show the first-order definability of the natural copy $\mathbf{D}_1$ of the Turing degrees in $\mathcal{D}_{T,\omega}$ in the terms of the structure order and the jump operation. In this section we shall improve this result by showing that only in the language of structure order and using $\mathbf{0}_{T,\omega}{}'$ as a parameter, we can define $\mathbf{D}_1$ in $\mathcal{D}_{T,\omega}$. As a consequence, we derive that the definability of $\mathbf{0}_{T,\omega}{}'$ implies this one of the whole jump operator.

Theorem 8 *The following are equivalent:*

1. *the jump operator is first-order definable in $\mathcal{D}_{T,\omega}$;*
2. *the jump $\mathbf{0}_{T,\omega}{}'$ of the least element is first-order definable in $\mathcal{D}_{T,\omega}$;*
3. *the isomorphic copy $\mathbf{D}_1$ of the Turing degrees is first-order definable in $\mathcal{D}_{T,\omega}$*

Proof. (1) $\Rightarrow$ (2): obvious;

(2) $\Rightarrow$ (3): Note that the first-order definability of $\mathbf{0}_{T,\omega}{}'$ implies the first-order definability of the set $\widetilde{\mathbf{D}_1}$, defined in the previous section. But using the set $\widetilde{\mathbf{D}_1}$ a simple definition of $\mathbf{D}_1$ can be derived. Indeed, for each $\mathbf{a} \in \mathbf{D}_{T,\omega}$, denote by $\mu(\mathbf{a})$ the least (ω-Turing) degree $\mathbf{x}$, for which exists degree $\mathbf{y} \in \widetilde{\mathbf{D}_1}$ such that $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$. It is not difficult to see that the operation μ is correctly defined. Moreover, for each $\mathbf{a}$, if $\{A_k\}_{k<\omega} \in \mathbf{a}$ then $\mu(\mathbf{a})$ contains the sequence $(A_0, \emptyset, \ldots, \emptyset, \ldots)$. In order to prove this, first note that

$$(A_0, \emptyset, \ldots, \emptyset, \ldots) \oplus (\emptyset, A_1, A_2, \ldots, A_n, \ldots) \equiv_{T,\omega} \{A_k\}_{k<\omega}.$$

Thus the degree of $(A_0, \emptyset, \ldots, \emptyset, \ldots)$ is such that there is a degree in $\widetilde{\mathbf{D}_1}$ which cups it to $\mathbf{a}$. Suppose now that $\mathbf{x}$ and $\mathbf{y} \in \widetilde{\mathbf{D}_1}$ are such that $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$. Let us fix sequences $\{X_k\}_{k<\omega} \in \mathbf{x}$ and $\{Y_k\}_{k<\omega} \in \mathbf{y}$ with $Y_0 = \emptyset$. Then $\{X_k\}_{k<\omega} \oplus \{Y_k\}_{k<\omega} \equiv_{T,\omega} \{A_k\}_{k<\omega}$. In particular, $X_0 \oplus Y_0 \equiv_T X_0 \equiv_T A_0$. Therefore,

$$(A_0, \emptyset, \ldots, \emptyset, \ldots) \leq_{T,\omega} (X_0, \emptyset, \ldots, \emptyset, \ldots) \leq_{T,\omega} \{X_k\}_{k<\omega}.$$

Hence, the range of μ is exactly the copy $\mathbf{D}_1$ of the Turing degrees under the embedding κ:

$$\mathbf{D}_1 = \{\mu(\mathbf{a}) \mid \mathbf{a} \in \mathbf{D}_{T,\omega}\}.$$

Thus the Turing degrees are first-order definable in the structure $\mathcal{D}_{T,\omega}$ of the ω-Turing degrees.

(3) $\Rightarrow$ (1): By Theorem 4 and Lemma 5, for each ω-Turing degree $\mathbf{a}$ there are ω-Turing degrees $\mathbf{g}, \mathbf{f} \in \mathbf{D}_1$, such that

$$\mathbf{a} = \mathbf{g} \wedge \mathbf{f} \quad \text{and} \quad \mathbf{a}' = \mathbf{g}' \wedge \mathbf{f}', \tag{5.1}$$

and if there is another pair of degrees, whose greatest lower bound exists and is equal to $\mathbf{a}$, then the greatest lower bound of their jumps also exists and is equal exactly to $\mathbf{a}'$.

As we stated in the preliminaries, $\mathbf{D}_1$ is closed under the jump and the ω-Turing jump agrees with the Turing jump. Also, by Shore and Slaman [7], the jump operator is definable in the structure $\mathcal{D}_T$ of the Turing degrees. Hence the restriction of the ω-Turing jump operator over $\mathbf{D}_1$ is definable in the structure $(\mathbf{D}_1, \leq_{T,\omega}, \vee)$. Thus, by (5.1), we conclude that the definability of $\mathbf{D}_1$ implies this of the jump. $\square$

The definability of $\mathbf{0}_{T,\omega}{}'$, alas, still remains an open question.

Question 9 *Is the jump $\mathbf{0}_{T,\omega}{}'$ of the least element first-order definable in $\mathcal{D}_{T,\omega}$?*

One of the main consequences of the definability of the jump operator will be that each automorphism of $\mathcal{D}_{T,\omega}$ is jump preserving[3], i.e. $\mathrm{Aut}(\mathcal{D}'_{T,\omega})$=$\mathrm{Aut}(\mathcal{D}_{T,\omega})$. This combined with the previously mentioned result by Sariev and Ganchev [6] stating the isomorphicity of the groups of the automorphism of the Turing degrees and of the jump preserving automorphism of the ω-Turing degrees, implies that the groups $\mathrm{Aut}(\mathcal{D}_T)$ and $\mathrm{Aut}(\mathcal{D}_{T,\omega})$ are isomorphic.

[3]a mapping $\pi : \mathbf{D}_{T,\omega} \to \mathbf{D}_{T,\omega}$ is said to be *jump preserving*, if for each degree $\mathbf{a} \in \mathbf{D}_{T,\omega}$, $\pi(\mathbf{a}') = \pi(\mathbf{a})'$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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STRONG CONVERSE INEQUALITIES FOR THE WEIGHTED MULTIVARIATE BERNSTEIN-DURRMEYER OPERATOR ON THE SIMPLEX VIA MULTIPLIERS

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It is demonstrated that multiplier methods naturally yield better constants in strong converse inequalities for the Bernstein-Durrmeyer operator. The absolute constants obtained in some of the inequalities are independent of the weight and the dimension. The estimates are stated in terms of the K-functional that is naturally associated to the operator.

Keywords: Bernstein-Durrmeyer operator, strong converse inequality, K-functional, orthogonal expansion, multipliers

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1. A CHARACTERIZATION OF THE RATE OF APPROXIMATION OF THE BERNSTEIN-DURRMEYER OPERATOR

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we set $|x| := \sum_{i=1}^d |x_i|$. Let S be the standard simplex in $\mathbb{R}^d$ given by

$$S := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \ge 0,\ i = 1, \dots, d,\ |x| \le 1\}.$$

The Jacobi weights on S are defined by

$$w_\alpha(x) := x_1^{\alpha_1} \cdots x_d^{\alpha_d} (1 - |x|)^{\alpha_{d+1}}, \quad \alpha_i > -1,\ i = 1, \dots, d+1. \tag{1.1}$$

We have set $\alpha := (\alpha_1, \dots, \alpha_{d+1})$. For $p \in [1, \infty)$ and a Jacobi weight w_α we consider the space $L_{p,w_\alpha}(S)$ of Lebesgue measurable functions f defined on S such that

$$\|f\|_{p,w_\alpha} := \left(\int_S |f(x)|^p w_\alpha(x)\, dx \right)^{1/p} < \infty.$$

Let, as usual, $L_\infty(S)$ denote the space of the essentially bounded Lebesgue measurable functions on S, equipped with the sup-norm on S. For brevity we set $L_{\infty,w_\alpha}(S) := L_\infty(S)$ and $\|f\|_{\infty,w_\alpha} := \operatorname{ess\,sup}_{x\in S}|f(x)|$.

We proceed to the definition of the multivariate Bernstein-Durrmeyer operators with Jacobi weights given by Ditzian [13]. For $n \in \mathbb{N}_0$ and $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$ with $|\mathbf{k}| \le n$ we define the polynomials

$$p_{n,\mathbf{k}}(x) := \frac{n!}{k_1! \cdots k_d!(n - |\mathbf{k}|)!} \prod_{i=1}^{d} x_i^{k_i} (1 - |x|)^{n-|\mathbf{k}|}.$$

The Jacobi-weighted Bernstein-Durrmeyer operators on $L_{p,w_\alpha}(S)$ are defined by

$$M_{n,\alpha} f(x) := \sum_{|\mathbf{k}|\le n} p_{n,\mathbf{k}}(x) \left(\int_S p_{n,\mathbf{k}}(y) w_\alpha(y)\, dy \right)^{-1} \int_S f(y)\, p_{n,\mathbf{k}}(y) w_\alpha(y)\, dy.$$

These operators in the univariate case and with no weight, i.e. $w_\alpha = 1$, were introduced independently by Durrmeyer [16] and Lupaş [19]; their multivariate generalization was given by Derriennic [11]; and their univariate weighted form was considered by Berens and Xu [2, 3]. These operators were extensively studied by many authors and it is very difficult to summarize all the results. That is why we shall restrict our attention only to those which are directly and most closely related to the subject of the present paper. In the next section we shall recall several of their basic properties. They were proved by Ditzian [13] in the general case, and earlier by Derriennic [11] and Berens and Xu [2, 3] respectively in the multivariate unweighted case and the univariate weighted case.

Ditzian [13] introduced the K-functional

$$K_\alpha(f, t)_p := \inf_{g\in C^2(S)} \{\|f - g\|_{p,w_\alpha} + t\, \|P_\alpha(D)g\|_{p,w_\alpha}\}$$

in order to characterize the rate of approximation of the Bernstein-Durrmeyer operator in $L_{p,w_\alpha}(S)$. Here $P_\alpha(D)$ is the differential operator that is naturally associated to the multivariate Bernstein-Durrmeyer operators with the weight w_α. It is defined by

$$P_\alpha(D) := \sum_{\xi\in E_S} w_\alpha(x)^{-1} \frac{\partial}{\partial\xi}\, \tilde{d}(\xi, x) w_\alpha(x)\, \frac{\partial}{\partial\xi},$$

where E_S is the set of the directions parallel to the edges of S and $\tilde{d}(\xi, x)$ is the distance introduced by Ditzian [12]

$$\tilde{d}(\xi, x) := \sup_{\substack{\lambda\ge0\\ x+\lambda\xi\in S}} d(x, x + \lambda\xi) \sup_{\substack{\lambda\ge0\\ x-\lambda\xi\in S}} d(x, x - \lambda\xi),$$

as $d(x,y)$ is the Euclidean distance.

Ditzian [13] proved that there exist positive constants c_1 and c_2 such that for all $f \in L_p(w_\alpha)(S)$ and all $n \in \mathbb{N}$ there holds

$$c_1 K_\alpha(f, n^{-1})_p \le \|M_{n,\alpha} f - f\|_{p,w_\alpha} \le c_2 K_\alpha(f, n^{-1})_p. \tag{1.2}$$

The direct estimate, i.e. the right-hand side inequality, was established with $c_2 = 2$ independently by Chen and Ditzian [6] (see also [7, p. 38]) and by Berens, Schmid and Xu [1, Theorem 2] in the unweighted case, and by Berens and Xu [2, Theorem 3] in the univariate weighted case. A closer look at the proof of [13, Theorem 3.3] shows that we can take c_2 independent of the dimension d and the weight w_α. Actually, a slight modification of this argument shows that the direct estimate holds with $c_2 = 2$ in the general case. More precisely, we have

$$\|M_{n,\alpha} f - f\|_{p,w_\alpha} \le 2\, K_\alpha(f, n^{-1})_p. \tag{1.3}$$

For the sake of completeness we give its proof in Section 3.

As for the converse estimate, that is, the left inequality in (1.2), Chen, Ditzian and Ivanov [7, Theorems 6.1 and 6.3] established it in the unweighted case for all d if $1 < p < \infty$ and for $d \le 3$ if $p = 1, \infty$ (a little bit weaker result was verified in the larger dimensions). Then Knoop and Zhou [18, Theorem 3.1] proved it for all d and $1 \le p \le \infty$ in the unweighted case. Both proofs give constants c_1 that decrease to 0 when d increases. Heilmann and M. Wagner [17, Theorem 1] improved c_1 for $d \le 3$. Ditzian's proof of the general weighted case also yields a constant c_1 that decreases to 0 when d or $\max_i |\alpha_i|$ increase. All these treatments are based on the quite general and efficient method developed by Ditzian and Ivanov [14]. It enables us to derive converse inequalities like the one on the left-hand side of (1.2) by means of Voronovskaya and Bernstein-type inequalities. These inequalities are important in themselves but their consecutive application leads to decreasing c_1.

The main purpose of this paper is to demonstrate that by means of the multiplier theory we can derive strong converse inequalities with better absolute constants than the methods previously used. Moreover, the arguments are very short. The first result we state contains a strong converse inequality of a form that is a combination of types B and C (according to the terminology introduced in [14]). Quite similar results were previously established by Berens and Xu [2, Theorem 3] (see also [2, Theorem 2]).

Set $\rho := d + \sum_{i=1}^{d+1} \alpha_i$.

Theorem 1.1. *Let $d \in \mathbb{N}$, $1 \le p \le \infty$ and w_α be given by (1.1) with $\alpha_i > -1$, $i = 1, \ldots, d+1$. Then for all $f \in L_p(w_\alpha)(S)$ and all $n \in \mathbb{N}$ there hold*

$$K_\alpha(f, n^{-1})_p \le \left(4 + \frac{2\rho}{n}\right) \left(\|M_{n,\alpha} f - f\|_{p,w_\alpha} + \|M_{2n,\alpha} f - f\|_{p,w_\alpha}\right) + \frac{4}{n} \sum_{k=n+1}^{2n} \|M_{k,\alpha} f - f\|_{p,w_\alpha}.$$

Remark 1.2. Let us explicitly note that the constant on the right-hand side above is asymptotically independent of any parameters unlike the strong converse inequalities obtained in [7], [13], [18]. More precisely, if $n \geq |\rho|$, then

$$K_\alpha(f, n^{-1})_p \leq 6\big(\|M_{n,\alpha} f - f\|_{p,w_\alpha} + \|M_{2n,\alpha} f - f\|_{p,w_\alpha}\big) \\ + \frac{4}{n} \sum_{k=n+1}^{2n} \|M_{k,\alpha} f - f\|_{p,w_\alpha}.$$

However, the inequalities established in [7, 13, 18] are of a stronger type than the one above.

Let us mention that the K-functional $K_\alpha(f,t)_p$ was characterized by a simpler one in [8] for $1 < p < \infty$ (see also the references cited there).

It seems quite plausible that the strong converse inequality in (1.2) also holds with c_1, which is independent of p, d and w_α. We were not able to show that. However, a short multiplier argument yields a strong converse inequality of that type in a special case. It is based on a result due to H. Pollard. Let $d = 1$ and $w_\alpha = 1$. Let $S_n f$ be the n-th partial sum of the Fourier-Legendre series of f. Pollard [20] proved that if $4/3 < p < 4$, then the operators $S_n : L_p[0,1] \to L_p[0,1]$ are uniformly bounded on n, that is, there exists a constant $\varsigma \geq 1$ such that

$$\|S_n f\|_p \leq \varsigma \|f\|_p, \quad f \in L_p[0,1],\ n \in \mathbb{N}.$$

Here $\| \circ \|_p$ denotes the standard L_p-norm on the interval $[0,1]$. We will omit the subscript α in the notation of the K-functional and the Bernstein-Durrmeyer operator when $w_\alpha = 1$.

We will establish the following result.

Proposition 1.3. *Let $4/3 < p < 4$. Then for all $f \in L_p[0,1]$ and all $n \in \mathbb{N}$ there holds*

$$K(f, n^{-1})_p \leq (1 + 2\varsigma)\, \|M_n f - f\|_p.$$

The contents of the paper are organized as follows. In the next section we collect the basic properties of Bernstein-Durrmeyer operator that we will use. Section 3 contains the proofs of the theorems and the proposition stated above. In the last section we discuss how the same multiplier method can be applied in the general case of weights w_α with $\alpha_i \geq -1/2$ for all i. This proof is not shorter than the ones previously used; but it has the advantage of using elementary calculus and being invariant in its technical part on the dimension—it depends only on that how large ρ is.

2. BASIC PROPERTIES OF THE BERNSTEIN-DURRMEYER OPERATOR

Here we shall recall the properties of the Jacobi-weighted Bernstein-Durrmeyer operator that we need (see [13]).

First of all, it is a contraction on the space $L_{p,w_\alpha}(S)$, that is,

$$\|M_{n,\alpha}f\|_{p,w_\alpha} \le \|f\|_{p,w_\alpha}. \tag{2.1}$$

$M_{n,\alpha}$ is a self-adjoint linear operator w.r.t. the inner product

$$\langle f,g\rangle_{w_\alpha} := \int_S f(x)g(x)w_\alpha(x)\,dx.$$

Its eigenvalues are

$$\mu_{n,\ell} := \frac{n!}{(n-\ell)!}\,\frac{\Gamma(n+\rho+1)}{\Gamma(n+\ell+\rho+1)}, \quad \ell = 0,\dots,n, \tag{2.2}$$

where Γ denotes the gamma function and, to recall, we have set $\rho := d + \sum_{i=1}^{d+1}\alpha_i$. For each ℓ, to $\mu_{n,\ell}$ corresponds the same eigenspace for all n. We denote it by V_ℓ. For $\ell \ge 1$ the space V_ℓ consists of those algebraic polynomials of $x_1,\dots,x_d$ and total degree ℓ that are orthogonal w.r.t. the above inner product to the polynomials of degree $\ell-1$. The eigenspace V_0, corresponding to $\mu_{n,0}=1$, consists of all constants. Now, if we denote the projections on V_ℓ by $\mathcal{P}_\ell$, then $M_{n,\alpha}$ can be represented in the form

$$M_{n,\alpha} = \sum_{\ell=0}^{n} \mu_{n,\ell}\mathcal{P}_\ell. \tag{2.3}$$

The operator $P_\alpha(D)$ is also self-adjoint and its eigenspaces coincide with those of $M_{n,\alpha}$. More precisely, there holds

$$P_\alpha(D)P = -\ell(\ell+\rho)P, \quad P\in V_\ell,\ \ell\in\mathbb{N}_0. \tag{2.4}$$

Finally, let us recall that $M_{n,\alpha}$ and $P_\alpha(D)$ commute on $C^2(S)$:

$$M_{n,\alpha}P_\alpha(D)f = P_\alpha(D)M_{n,\alpha}f, \quad f\in C^2(S). \tag{2.5}$$

3. PROOFS OF THE MAIN RESULTS

First, we will prove the direct estimate stated in (1.3) for the sake of completeness of the exposition.

Proof of (1.3). Z. Ditzian's proof of the direct estimate in (1.2), is based on the elegant formula (see [13, (3.3)])

$$M_{n,\alpha}f - f = \sum_{\ell=n+1}^{\infty} \frac{1}{\ell(\ell+\rho)}\,P_\alpha(D)M_{\ell,\alpha}f, \tag{3.1}$$

valid for all $f \in L_{p,w_\alpha}(S)$. Using that $M_{n,\alpha}$ is a contraction (see (2.1)), we get

$$\|M_{n,\alpha}f - f\|_{p,w_\alpha} \le 2\,\|f - g\|_{p,w_\alpha} + \|M_{n,\alpha}g - g\|_{p,w_\alpha} \tag{3.2}$$

for any $g \in C^2(S)$. Next, we apply (2.1), (2.5) and (3.1) to estimate the second term on the right. Thus we get

$$\|M_{n,\alpha}g - g\|_{p,w_\alpha} \le \sum_{\ell=n+1}^{\infty} \frac{1}{\ell(\ell+\rho)}\,\|P_\alpha(D)g\|_{p,w_\alpha}. \tag{3.3}$$

It is quite straightforward, to see that

$$\sum_{\ell=n+1}^{\infty} \frac{1}{\ell(\ell+\rho)} \le \frac{1}{n}.$$

Now, substituting (3.3) in (3.2) and taking an infimum on $g \in C^2(S)$, we arrive at

$$\|M_{n,\alpha}f - f\|_{p,w_\alpha} \le 2\,K_\alpha(f, n^{-1})_p.$$

Thus the first inequality in (1.3) is verified; the second one is trivial. □

Proof of Theorem 1.1. The proof is a modification of a very short argument due to Berens and Xu (see [2, Theorem 3]). Set

$$g_n := \frac{1}{t_n} \sum_{k=n+1}^{2n} \frac{M_{k,\alpha}f}{k(k+\rho)}, \quad t_n := \sum_{k=n+1}^{2n} \frac{1}{k(k+\rho)}.$$

Clearly, $g_n \in C^2(S)$ for all $n \in \mathbb{N}$ and then

$$K_\alpha(f, n^{-1})_p \le \|f - g_n\|_{p,w_\alpha} + \frac{1}{n}\,\|P_\alpha(D)g_n\|_{p,w_\alpha}. \tag{3.4}$$

We estimate the first term on the right above by means of

$$\begin{aligned} \|f - g_n\|_{p,w_\alpha} &= \left\| f - \frac{1}{t_n} \sum_{k=n+1}^{2n} \frac{M_{k,\alpha}f}{k(k+\rho)} \right\|_{p,w_\alpha} \\ &\le \frac{1}{t_n} \sum_{k=n+1}^{2n} \frac{\|M_{k,\alpha}f - f\|_{p,w_\alpha}}{k(k+\rho)} \\ &\le \frac{4}{n} \sum_{k=n+1}^{2n} \|M_{k,\alpha}f - f\|_{p,w_\alpha}. \end{aligned} \tag{3.5}$$

In order to estimate the second term on the right in (3.4), we apply (2.3) and (2.4) to get the representation

$$P_\alpha(D)g_n = -\frac{1}{t_n} \sum_{k=n+1}^{2n} \sum_{\ell=0}^{k} \frac{\ell(\ell+\rho)}{k(k+\rho)}\,\mu_{k,\ell}\mathcal{P}_\ell.$$

Next, we take into account the remarkable property of the multipliers $\mu_{n,\ell}$

$$\mu_{k,\ell} - \mu_{k-1,\ell} = \frac{\ell(\ell+\rho)}{k(k+\rho)}\,\mu_{k,\ell}$$

to arrive at the formula

$$\begin{aligned} P_\alpha(D) g_n &= \frac{1}{t_n} \sum_{k=n+1}^{2n} (M_{k-1,\alpha} f - M_{k,\alpha} f) \\ &= \frac{1}{t_n}\,(M_{n,\alpha} f - M_{2n,\alpha} f). \end{aligned}$$

Consequently,

$$\frac{1}{n}\,\|P_\alpha(D) g_n\|_{p,w_\alpha} \le \left(4 + \frac{2\rho}{n}\right) \big(\|M_{n,\alpha} f - f\|_{p,w_\alpha} + \|M_{2n,\alpha} f - f\|_{p,w_\alpha}\big). \quad (3.6)$$

Combining (3.4)-(3.6), we complete the proof of the theorem. $\square$

Let us proceed to the proof of the converse inequality in Proposition 1.3. The method we use is quite straightforward. It is based entirely on standard techniques in the multiplier theory and orthogonal series expansions. We will present it in the general case of the multivariate Bernstein-Durrmeyer operator on the simplex. The method is based on constructing a family of uniformly bounded operators $\mathcal{Q}_n$ such that

$$\frac{1}{n}\,P_\alpha(D) M_{n,\alpha}^m f = \mathcal{Q}_n(M_{n,\alpha} f - f)$$

with some fixed $m \in \mathbb{N}$. Then the strong one-term converse inequality in (1.2) easily follows from

$$\begin{aligned} K_\alpha(f, n^{-1})_p &\le \|M_{n,\alpha}^m f - f\|_{p,w_\alpha} + \frac{1}{n}\,\|P_\alpha(D) M_{n,\alpha}^m f\|_{p,w_\alpha} \\ &= \|(M_{n,\alpha}^{m-1} + M_{n,\alpha}^{m-2} + \cdots + I)(M_{n,\alpha} f - f)\|_{p,w_\alpha} + \|\mathcal{Q}_n(M_{n,\alpha} f - f)\|_{p,w_\alpha} \\ &\le (m+q)\,\|M_{n,\alpha} f - f\|_{p,w_\alpha}, \end{aligned}$$

where I denotes the identity and $q > 0$ is such that $\|\mathcal{Q}_n F\|_{p,w_\alpha} \le q\|F\|_{p,w_\alpha}$ for all $F \in L_{p,w_\alpha}(S)$ and $n \in \mathbb{N}$.

That approach to converse inequalities has been applied before (see e.g. [14, (2.13)], and also cf. [p. 32][2]). The proof of the direct inequality, we recalled above, was realized in a similar way (see (3.1)). There is a general comparison principle that underlies this technique. It was formulated independently, in two different settings, by Shapiro [21] (see also [22, Section 9.4]) and Trigub [24, § 4] and [25, § 4] (see also [27, Chapter 7] and [26, p. 4]. The author tried to present systematically that method of verifying direct and converse estimates in terms of K-functionals in [15] (see also the references cited there).

The earlier proofs of the converse inequality of the type given in (1.2) for the Bernstein-Durrmeyer operator also employed orthogonal expansions, but in a lesser degree and within the framework in [14]. Berens and Xu [2] also extensively used multiplier techniques (see also [2, Theorem 2]).

Proof of Proposition 1.3. Let us begin with several observations valid in the general multivariate weighted case. They will be useful for our discussion in the next section.

We first note that (2.3) and (2.4) yield

$$P_\alpha(D)M_{n,\alpha}f = -\sum_{\ell=1}^{n} \ell(\ell+\rho)\mu_{n,\ell}\mathcal{P}_\ell f.$$

We introduce the linear operator on $L_{p,w_\alpha}(S)$

$$Q_n f := \sum_{\ell=1}^{n} \nu_{n,\ell}\mathcal{P}_\ell f,$$

where

$$\nu_{n,\ell} := \frac{\ell(\ell+\rho)\,\mu_{n,\ell}}{n(1-\mu_{n,\ell})}. \tag{3.7}$$

Note that $\mu_{n,\ell} < 1$ for $\ell = 1, 2, \ldots, n$. With that operator we have

$$\frac{1}{n}\,P_\alpha(D)M_{n,\alpha}f = Q_n(M_{n,\alpha}f - f).$$

Thus to establish a one-term strong converse inequality, it is enough to show that

$$\|Q_n f\|_{p,w_\alpha} \le c\,\|f\|_{p,w_\alpha}$$

for all $f \in L_{p,w_\alpha}(S)$ and $n \in \mathbb{N}$.

After this general remark, we proceed to the proof of the proposition. Now, $S_n f$ coincide with the nth partial sum of the orthogonal expansion of f on $\mathcal{P}_\ell$, that is,

$$S_n f := \sum_{\ell=0}^{n} P_\ell f.$$

We use the representation

$$Q_n f = \sum_{\ell=1}^{n-1}(\nu_{n,\ell} - \nu_{n,\ell+1})S_\ell f + \nu_{n,n}S_n f - \nu_{n,1}S_0 f.$$

In Lemma 3.4 below we will show that $\nu_{n,\ell} - \nu_{n,\ell+1} > 0$ for all ℓ. Then, taking also into account that the ν's are positive and $\nu_{n,1} = 1$, we deduce the estimate

$$\begin{aligned}\|Q_n f\|_{p,w_\alpha} &\le \varsigma\left(\sum_{\ell=1}^{n-1}(\nu_{n,\ell} - \nu_{n,\ell+1}) + \nu_{n,n} + \nu_{n,1}\right)\|f\|_{p,w_\alpha}\\ &\le 2\varsigma\nu_{n,1}\|f\|_{p,w_\alpha} = 2\varsigma\,\|f\|_{p,w_\alpha};\end{aligned}$$

hence the assertion of the proposition follows. ☐

Lemma 3.4. *Let $\rho > -1$. For $\nu_{n,\ell}$ defined in* (3.7) *there holds*

$$\nu_{n,\ell} > \nu_{n,\ell+1}, \quad \ell = 1, \dots, n-1, \ n = 2, 3, \dots . \tag{3.8}$$

Proof. Relation (3.8) is equivalent to

$$\frac{1-\mu_{n,\ell}}{\ell(\ell+\rho)\,\mu_{n,\ell}} < \frac{1-\mu_{n,\ell+1}}{(\ell+1)(\ell+\rho+1)\,\mu_{n,\ell+1}},$$

which can be written in the form

$$\frac{1}{\ell(\ell+\rho)\,\mu_{n,\ell}} - \frac{1}{\ell(\ell+\rho)} < \frac{1}{(\ell+1)(\ell+\rho+1)\mu_{n,\ell+1}} - \frac{1}{(\ell+1)(\ell+\rho+1)}.$$

We group the terms with μ's on the left-hand side and those without on the right-hand side, and substitute the value of the μ's given in (2.2). After straightforward calculations, using that $\rho > -1$ and

$$\Gamma(n+\ell+\rho+2) = (n+\ell+\rho+1)\Gamma(n+\ell+\rho+1), \tag{3.9}$$

which follows from $\Gamma(z+1) = z\,\Gamma(z)$, $z > 0$, we deduce that (3.8) is equivalent to

$$(n-\ell-1)!\,\Gamma(n+\ell+\rho+1)[n-\ell(\ell+\rho+1)] < n!\,\Gamma(n+\rho+1)$$

for $\ell = 1, \dots, n-1$, $n = 2, 3, \dots$. To verify this inequality, we shall show that the quantity on the left-hand side is decreasing on ℓ and it is valid for $\ell = 1$. The latter is a matter of a direct check—it reduces to $(\rho+1)(\rho+2) > 0$. To verify the former, we set

$$\xi_{n,\ell} := (n-\ell-1)!\,\Gamma(n+\ell+\rho+1)[n-\ell(\ell+\rho+1)].$$

To see that

$$\xi_{n,\ell} > \xi_{n,\ell+1}, \quad \ell = 1, 2, \dots, n-2, \ n = 3, 4, \dots, \tag{3.10}$$

we again apply (3.9) to deduce that (3.10) is equivalent to

$$(n-\ell-1)[n-\ell(\ell+\rho+1)] > (n+\ell+\rho+1)[n-(\ell+1)(\ell+\rho+2)].$$

Now, direct computations yield

$$\begin{aligned}(n-\ell-1)[n-\ell(\ell+\rho+1)] - (n+\ell+\rho+1)[n-(\ell+1)(\ell+\rho+2)]& \\ = (\ell+1)(\ell+\rho+1)(2\ell+\rho+2) > 0&,\end{aligned}$$

which verifes (3.10) and completes the proof of the lemma. ☐

4. AN EXTENSION

In this section we will demonstrate that the multiplier method can be used to verify the one-term converse inequality in (1.2) in a more general situation than the one considered in Proposition 1.3.

To this end, we represent Q_n as a linear combination of the Cesàro means of the partial sums of the orthogonal expansion of f on $\mathcal{P}_\ell$ (see [4, Theorem 3.2]). We set

$$\widetilde{S}_n f := \frac{1}{n+1}\sum_{k=0}^{n} S_k f,$$

where

$$S_k f := \sum_{\ell=0}^{k} P_\ell f.$$

Then we have

$$\begin{aligned} Q_n f = \sum_{\ell=1}^{n-2}(\ell+1)(\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell})\widetilde{S}_\ell f \\ + n(\nu_{n,n-1} - 2\nu_n)\widetilde{S}_{n-1} f + (n+1)\nu_{n,n}\widetilde{S}_n f + (\nu_{n,2} - 2\nu_{n,1})\widetilde{S}_0 f. \end{aligned} \tag{4.1}$$

As usually, if the range of summation is empty, we set the sum to be equal to zero.

Dai and Xu [9, Theorem 2.8 with $\delta = 1$] (or see [10, Theorem 13.4.4], as we also apply the Riesz-Thorin interpolation theorem) showed that if $1 \le p \le \infty$, $\alpha_i \ge -1/2$, $i = 0, \dots, d+1$, and $\rho - \min_i \alpha_i < 3/2$, then the Cesàro means are uniformly bounded on n, i.e. there exists a constant κ such that

$$\|\widetilde{S}_n f\|_{p,w_\alpha} \le \kappa\, \|f\|_{p,w_\alpha}, \quad f \in L_p(w_\alpha)(S),\ n \in \mathbb{N}. \tag{4.2}$$

Lemma 3.4 yields $\nu_{n,2} \le \nu_{n,1} = 1$. Then we have by (4.1) and (4.2)

$$\|Q_n f\|_{p,w_\alpha} \le \kappa \left(\sum_{\ell=1}^{n-2}(\ell+1)|\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell}| \right. \\ \left. + (4n+1)\nu_{n,n-1} + 3 \right) \|f\|_{p,w_\alpha}.$$

We will prove that

$$\sum_{\ell=1}^{n-2}(\ell+1)|\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell}| \le c$$

and

$$n\,\nu_{n,n-1} \le c.$$

Above and henceforward, c denotes a positive constant, not necessarily the same at each occurrence, whose value is independent of n.

Thus we will have shown that if $1 \le p \le \infty$, $\alpha_i \ge -1/2$, $i = 1, \dots, d+1$, and

$$d + \sum_{i=1}^{d+1} \alpha_i - \min_{1 \le i \le d+1} \alpha_i < \frac{3}{2},$$

then for all $f \in L_p(w_\alpha)(S)$ and all $n \in \mathbb{N}$ there holds

$$K_\alpha(f, n^{-1})_p \le c \, \|M_{n,\alpha} f - f\|_{p,w_\alpha}.$$

In order to treat the general case, we can still apply the same method but use Cesàro means of higher order (see [5, Theorem 7.1] or [23, Theorem 3.3]). Their uniform boundedness was established by Dai and Xu [9] (or see [10, Theorems 13.2.7 and 13.4.6]).

We proceed to establishing the auxiliary results.

We set for $\tau \in (0, n]$

$$\mu_n(\tau) := \frac{\Gamma(n+1)\Gamma(n+\rho+1)}{\Gamma(n-\tau+1)\Gamma(n+\tau+\rho+1)}, \quad \nu_n(\tau) := \frac{\tau(\tau+\rho)\,\mu_n(\tau)}{n(1-\mu_n(\tau))}.$$

We will make use of the following formula of the derivative of the gamma function

$$\Gamma'(z) = \Gamma(z)\psi(z),$$

where $\psi(z)$ is the digamma function, defined as the logarithmic derivative of the gamma function

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}.$$

We have

$$\mu_n'(\tau) = -\mu_n(\tau) C_n(\tau), \tag{4.3}$$

where

$$C_n(\tau) := \psi(n+\tau+\rho+1) - \psi(n-\tau+1).$$

We will use the following estimates.

Lemma 4.5. *Let $\rho \ge 0$. Then:*

$$C_n(\tau) \le \frac{2\tau+\rho}{n-\tau}, \quad \tau \in (0,n); \tag{4.4}$$

$$C_n(\tau) \ge \frac{2\tau+\rho}{2(n-\tau+1)}, \quad \tau \in (0,(n-\rho)/3),\ n > \rho; \tag{4.5}$$

$$C_n'(\tau) \le \frac{2n+\rho}{(n+\tau+\rho)(n-\tau)}, \quad \tau \in (0,n); \tag{4.6}$$

$$C_n'(\tau) \ge \frac{2n+\rho+2}{(n+\tau+\rho+1)(n-\tau+1)}, \quad \tau \in (0,n); \tag{4.7}$$

$$C_n''(\tau) \ge \frac{2(2\tau+\rho-1)(2n+\rho+1)}{(n+\tau+\rho)^2(n-\tau+1)^2}, \quad \tau \in (0,n). \tag{4.8}$$

Proof. As is known,

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(k+x)}, \quad x > 0, \tag{4.9}$$

where γ is Euler's constant. Therefore

$$C_n(\tau) = (2\tau + \rho) \sum_{k=1}^{\infty} \frac{1}{(n-\tau+k)(n+\tau+\rho+k)}. \tag{4.10}$$

Interpreting the sum above as a Darboux sum, we arrive at the estimates

$$\log\left(1 + \frac{2\tau+\rho}{n-\tau+1}\right) \le C_n(\tau) \le \log\left(1 + \frac{2\tau+\rho}{n-\tau}\right). \tag{4.11}$$

To complete the proof of the first two estimates, it remains to take into account the inequalities

$$\begin{aligned} &\log(1+x) \le x, \quad x \in \mathbb{R}, \\ &\log(1+x) \ge x - \frac{x^2}{2} \ge \frac{x}{2}, \quad x \in [0,1]. \end{aligned}$$

In order to estimate the derivatives of C_n, we use that for $m \ge 1$ we have

$$\psi^{(m)}(x) = (-1)^{m+1} m! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{m+1}}, \quad x > 0.$$

Therefore

$$\frac{1}{x} \le \psi'(x) \le \frac{1}{x-1}; \tag{4.12}$$

$$-\frac{2}{(x-1)^2} \le \psi''(x) \le -\frac{2}{x^2}; \tag{4.13}$$

$$\tag{4.14}$$

for $x > 1$. These inequalities directly yield (4.6)-(4.8). $\square$

Lemma 4.6. *Let $\rho \ge 0$, $b > 0$ and $0 < \delta \le 1$. Let also $n \in \mathbb{N}$ be such that $n \ge 3$ and $1 \le \sqrt{bn} \le n-1$. Then*

$$n^2 \nu_{n,\ell} \le c, \quad \delta n \le \ell \le n, \tag{4.15}$$

$$\tau|\nu_n'(\tau)| \le c, \quad \tau \in [1, n-1], \tag{4.16}$$

and

$$\tau^2 |\nu_n''(\tau)| \le c, \quad \tau \in [1, \sqrt{bn}], \tag{4.17}$$

where the constant c is independent of n.

Proof. First, we estimate from below the difference $1 - \mu_{n,\ell}$.

By means of the property $\Gamma(z+1) = z\Gamma(z)$, $z > 0$, we represent $\mu_{n,\ell}$ in the form

$$\mu_{n,\ell} = \frac{n(n-1)\cdots(n-\ell+1)}{(n+\rho+1)(n+\rho+2)\cdots(n+\rho+\ell)}.$$

Consequently,

$$1 - \mu_{n,\ell} \geq \frac{(n+\rho+1)(n+\rho+2)\cdots(n+\rho+\ell) - n^\ell}{(n+\rho+1)(n+\rho+2)\cdots(n+\rho+\ell)}.$$

We expand the numerator, take into account that $\rho \geq 0$, and use the well-known formulas for sums of powers of consecutive positive integers, to arrive at the estimate

$$(n+\rho+1)(n+\rho+2)\cdots(n+\rho+\ell) - n^\ell \geq c(\ell^2 n^{\ell-1} + \ell^6 n^{\ell-3}).$$

Hence we get the inequalities

$$1 - \mu_{n,\ell} \geq \frac{c\,\ell^2 n^{\ell-1}}{(n+\rho+1)(n+\rho+2)\cdots(n+\rho+\ell)} \tag{4.18}$$

and

$$1 - \mu_{n,\ell} \geq \frac{c\,\ell^6 n^{\ell-3}}{(n+\rho+1)(n+\rho+2)\cdots(n+\rho+\ell)} \tag{4.19}$$

for $3 \leq \ell \leq n$.

Inequality (4.15) for $\ell \geq 3$ follows directly from (4.19) and $\ell \geq \delta n$:

$$n^2 \nu_{n,\ell} \leq c\, \frac{n^{\ell+2}}{\ell^4 n^{\ell-2}} \leq c.$$

For $\ell = 1, 2$ (4.15) is trivial.

We proceed to the second assertion of the lemma. Making use of (4.3), we arrive at

$$\tau \nu_n'(\tau) = \frac{\tau(2\tau+\rho)\,\mu_n(\tau)}{n(1-\mu_n(\tau))} - \frac{\tau^2(\tau+\rho)\,\mu_n(\tau)\,C_n(\tau)}{n(1-\mu_n(\tau))^2}. \tag{4.20}$$

The function $\mu_n(\tau)$ is monotone decreasing on τ for each fixed n. For the rest of the proof let $\ell \in \{1, \ldots, n-2\}$ be such that $\ell \leq \tau \leq \ell + 1$. Then

$$\mu_n(\tau) \leq \mu_{n,\ell}, \tag{4.21}$$

$$1 - \mu_n(\tau) \geq 1 - \mu_{n,\ell}. \tag{4.22}$$

These two inequalities, the property $\Gamma(z+1) = z\Gamma(z)$, $z > 0$, and (4.18) imply the following estimate of the first term on the right in (4.20)

$$\begin{aligned} 0 \leq \frac{\tau(2\tau+\rho)\,\mu_n(\tau)}{n(1-\mu_n(\tau))} &\leq \frac{(\ell+1)(2\ell+\rho+2)\,\mu_{n,\ell}}{n(1-\mu_{n,\ell})} \\ &\leq c\,\frac{(\ell+1)(2\ell+\rho+2)}{\ell^2}\,\frac{n!}{n^\ell(n-\ell)!} \\ &\leq c, \quad \tau \in [1, n-1]. \end{aligned} \tag{4.23}$$

To estimate the second term we argue in a similar way, as we also use (4.4). We have

$$\begin{aligned}\frac{\tau^2(\tau+\rho)\,\mu_n(\tau)\,nC_n(\tau)}{(n(1-\mu_n(\tau)))^2} &\le c\,\frac{(\ell+1)^2(\ell+\rho+1)(2\ell+\rho+2)}{\ell^4}\\ &\quad\times\frac{n!\,(n+\rho+1)\cdots(n+\rho+\ell)}{n^{2\ell-1}(n-\ell)!\,(n-\ell-1)}\\ &\le c\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{\ell-2}{n}\right)\left(1+\frac{\rho+1}{n}\right)\cdots\left(1+\frac{\rho+\ell}{n}\right)\\ &\le c\prod_{i=1}^{\ell-2}\left(1-\frac{i}{n}\right)\left(1+\frac{i+\rho}{n}\right).\end{aligned}$$

As usually, we set an empty product to be equal to 1.

Next, we take into account that

$$\left(1-\frac{i}{n}\right)\left(1+\frac{i+\rho}{n}\right)=1-\frac{i^2}{n^2}+\frac{\rho}{n}\left(1-\frac{i}{n}\right)\le 1+\frac{\rho}{n} \tag{4.24}$$

and the inequality $(1+\rho/n)^n\le e^\rho$ to deduce

$$0\le\frac{\tau^2(\tau+\rho)\,\mu_n(\tau)\,C_n(\tau)}{n(1-\mu_n(\tau))^2}\le c,\quad \tau\in[1,n-1]. \tag{4.25}$$

Relations (4.20), (4.23) and (4.25) imply the second inequality in the lemma.

In order two prove the last assertion of the lemma, we use the representation

$$\begin{aligned}\nu''(\tau)&=\frac{2\mu_n(\tau)}{n(1-\mu_n(\tau))}-\frac{2(2\tau+\rho)\,\mu_n(\tau)\,C_n(\tau)}{n(1-\mu_n(\tau))^2}\\ &\quad-\frac{\tau(\tau+\rho)\,\mu_n(\tau)\,C_n'(\tau)}{n(1-\mu_n(\tau))^2}+\frac{\tau(\tau+\rho)(1+\mu_n(\tau))\,\mu_n(\tau)\,C_n(\tau)^2}{n(1-\mu_n(\tau))^3}.\end{aligned} \tag{4.26}$$

Just similarly to (4.23) and (4.25), we establish

$$0\le\frac{\tau^2\mu_n(\tau)}{n(1-\mu_n(\tau))}\le c, \tag{4.27}$$

$$0\le\frac{\tau^2(2\tau+\rho)\mu_n(\tau)nC_n(\tau)}{(n(1-\mu_n(\tau)))^2}\le c \tag{4.28}$$

for $\tau\in[1,n-1]$.

Again, similarly to the proof of (4.25), but this time using (4.6), we get

$$\begin{aligned}\frac{\tau^3(\tau+\rho)\,\mu_n(\tau)\,C_n'(\tau)}{n(1-\mu_n(\tau))^2}&\le c\,\frac{(\ell+1)^3(\ell+\rho+1)}{\ell^4}\\ &\quad\times\frac{(2n+\rho)\,n!\,(n+\rho+1)\cdots(n+\rho+\ell-1)}{n^{2\ell-1}(n-\ell)!\,(n-\ell-1)}\\ &\le c\prod_{i=1}^{\ell-2}\left(1-\frac{i}{n}\right)\left(1+\frac{i+\rho}{n}\right)\le c.\end{aligned}$$

Consequently,

$$0 \le \frac{\tau^3(\tau+\rho)\,\mu_n(\tau)\,C_n'(\tau)}{n(1-\mu_n(\tau))^2} \le c, \quad \tau \in [1, n-1]. \tag{4.29}$$

In order to estimate the last term in the representation of ν_n'' we use (4.4) and $\mu_{n,\ell} \le 1$ to deduce

$$\begin{aligned}
\frac{\tau^3(\tau+\rho)(1+\mu_n(\tau))\,\mu_n(\tau)\,C_n(\tau)^2}{n(1-\mu_n(\tau))^3} &\le c\,\frac{(\ell+1)^3(\ell+\rho+1)(2\ell+\rho+2)^2}{\ell^6} \\
&\quad \times \frac{n!\,(n+\rho+1)^2\cdots(n+\rho+\ell)^2}{n^{3\ell-2}(n-\ell)!\,(n-\ell-1)^2} \\
&\le c\prod_{i=1}^{\ell-3}\left(1-\frac{i}{n}\right)\left(1+\frac{i+\rho}{n}\right)^2.
\end{aligned}$$

It remains to observe that, by virtue of (4.24) and the inequality $(1+\rho/n)^n \le e^\rho$, we have

$$\prod_{i=1}^{\ell-3}\left(1-\frac{i}{n}\right)\left(1+\frac{i+\rho}{n}\right)^2 \le c\left[\left(1+\frac{\ell+\rho}{n}\right)^n\right]^{\ell/n} \le c\,e^{\ell^2/n} \le c.$$

$\square$

Lemma 4.7. *Let $\rho \ge 0$. There holds*

$$\ell(\nu_{n,\ell}-\nu_{n,\ell+1}) \le c, \quad \ell = 1,\dots,n-1,$$

where the constant c is independent of n.

Proof. The inequality follows readily from (4.15) for $\ell = n-1$. Let $\ell \le n-2$. Then, by virtue of (4.16), we have

$$\begin{aligned}
\ell(\nu_{n,\ell}-\nu_{n,\ell+1}) &= -\ell\int_\ell^{\ell+1} \nu_n'(\tau)\,d\tau \\
&\le \sup_{1\le\tau\le n-1} |\tau\nu_n'(\tau)| \le c.
\end{aligned}$$

$\square$

Lemma 4.8. *Let $\rho \ge 0$. There holds*

$$\sum_{\ell=1}^{n-2}(\ell+1)|\nu_{n,\ell+2}-2\nu_{n,\ell+1}+\nu_{n,\ell}| \le c,$$

where the constant c is independent of n.

Proof. Clearly, it is sufficient to verify the lemma for large n. Its assertion for $n \le n_0$, where $n_0 \in \mathbb{N}$ is fixed, is trivial.

We split the sum into four parts:

$$\begin{aligned}
&1 \le \ell \le \sqrt{an} - 2,\\
&\sqrt{an} - 2 < \ell \le \sqrt{bn},\\
&\sqrt{bn} < \ell \le \frac{n}{4},\\
&\frac{n}{4} < \ell \le n - 2,
\end{aligned}$$

where $0 < a < b$ will be fixed in appropriate way to be indicated in the course of the proof. We denote these parts with Σ_i, $i = 1, \ldots, 4$, respectively.

As is known

$$\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell} = \int_{\ell}^{\ell+2} M(\tau - \ell)\nu_n''(\tau)\, d\tau, \quad \ell = 1, \ldots, n-2,$$

where

$$M(\tau) := \begin{cases} \tau, & 0 \le \tau \le 1,\\ 2 - \tau, & 1 \le \tau \le 2. \end{cases}$$

By virtue of (4.17), we have

$$\begin{aligned}
\Sigma_2 &:= \sum_{\sqrt{an}-2 < \ell \le \sqrt{bn}} (\ell + 1)|\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell}|\\
&\le c \int_{\sqrt{an}-2}^{\sqrt{bn}+2} \tau|\nu_n''(\tau)|\, d\tau \le c.
\end{aligned}$$

Let m_n be the integer part of $n/4$. We apply (4.15) to get

$$\Sigma_4 := \sum_{n/4 < \ell \le n-2} (\ell + 1)|\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell}| \le c\, n^2 \nu_{n,m_n} \le c.$$

We proceed to estimating Σ_3. Let $\sqrt{bn} \le \tau \le n/4 + 2$. Let n be so large that we have $n/4 + 2 \le (n - \rho)/3$. We will show that if b is fixed large enough, then $\nu_n''(\tau) > 0$ for all large n. Hence $\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell} \ge 0$ if $\sqrt{bn} < \ell \le n/4 - 2$. Let ℓ_n be the smallest integer greater than $\sqrt{bn}$. Then, by virtue also of Lemmas 3.4 and 4.7, we deduce that

$$\begin{aligned}
\Sigma_3 &:= \sum_{\sqrt{bn} < \ell \le n/4} (\ell + 1)|\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell}|\\
&= \ell_n(\nu_{n,\ell_n} - \nu_{n,\ell_n+1}) + \nu_{n,\ell_n} - (m_n + 1)\nu_{n,m_n} + m_n\nu_{n,m_n+1}\\
&\le c.
\end{aligned}$$

Thus, to complete the proof of the estimate of Σ_3 it remains to show that $\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell} \geq 0$ if $\sqrt{bn} \leq \ell \leq n/4 - 2$ for all n large enough an appropriately fixed b. By (4.26) we have

$$\nu_n''(\tau) = \frac{\mu_n(\tau)}{n(1-\mu_n(\tau))^3} N(\tau),$$

where we have set

$$\begin{aligned} N(\tau) := 2(1-\mu_n(\tau))^2 &- 2(2\tau+\rho)C_n(\tau)(1-\mu_n(\tau)) \\ &- \tau(\tau+\rho)C_n'(\tau)(1-\mu_n(\tau)) + \tau(\tau+\rho)C_n^2(\tau)(1+\mu_n(\tau)). \end{aligned}$$

By virtue of Lemma 4.5, we arrive at the estimate

$$\begin{aligned} N(\tau) \geq \mu_n(\tau) &\left(\frac{2(2\tau+\rho)^2}{n-\tau} + \frac{\tau(\tau+\rho)(2n+\rho)}{(n+\tau+\rho)(n-\tau)} + \frac{\tau(\tau+\rho)(2\tau+\rho)^2}{4(n-\tau+1)^2} \right) \\ &- \frac{2(2\tau+\rho)^2}{n-\tau} - \frac{\tau(\tau+\rho)(2n+\rho)}{(n+\tau+\rho)(n-\tau)} + \frac{\tau(\tau+\rho)(2\tau+\rho)^2}{4(n-\tau+1)^2}. \end{aligned} \tag{4.30}$$

In order to show that $N(\tau) > 0$ it is enough to prove that the quantity on the right-hand side of the last relation is positive. Using that $n-\tau+1 < n+\tau+\rho$, we see that this follows from

$$\begin{aligned} &\mu_n(\tau)[8(2\tau+\rho)^2(n+\tau+\rho)^2 + 4\tau(\tau+\rho)(2n+\rho)(n+\tau+\rho) + \tau(\tau+\rho)(2\tau+\rho)^2(n-\tau)] \\ &> 8(2\tau+\rho)^2(n+\tau+\rho)^2 + 4\tau(\tau+\rho)(2n+\rho)(n+\tau+\rho) - \tau(\tau+\rho)(2\tau+\rho)^2(n-\tau). \end{aligned}$$

To complete the proof it remains to observe that if b is fixed large enough, then the quantity on the right-hand side of the inequality above is negative for large n. To see this, we observe that the sum of the terms in the polynomial on the variables τ and n on the right-hand side that determine its sign for large τ and n is

$$40n^2\tau^2 + 72n\tau^3 - 8\rho n\tau^3 - 4n\tau^4 + 4\tau^5.$$

Since

$$40n^2\tau^2 + 72n\tau^3 - 8\rho n\tau^3 - 4n\tau^4 + 4\tau^5 \leq 4\tau^2(10n^2 + 18n\tau - n\tau^2 + \tau^3),$$

to complete the proof it is sufficient to show that

$$10n^2 + 18n\tau - n\tau^2 + \tau^3 < 0$$

if $\sqrt{bn} \leq \tau \leq n/4$ with an appropriately fixed b. But this readily becomes clear from the estimate

$$\begin{aligned} 10n^2 + 18n\tau - n\tau^2 + \tau^3 &\leq 10n^2 + \frac{9}{2}n^2 - n\tau^2 + \frac{1}{4}n\tau^2 \\ &\leq \frac{29}{2}n^2 - \frac{3b}{4}n^2. \end{aligned}$$

To estimate Σ_1 we use similar but more lengthy considerations than those for Σ_3. They are based on the inequalities stated in Lemma 4.5 as we have to use instead of (4.5) its refinement that follows from $\log(1+x) \geq x - x^2/2$. This time we show that there exists $a \in (0,1)$ such that $N'(\tau) < 0$ at least for large n if $1 \leq \tau \leq \sqrt{an}$; hence $N(\tau) \leq N(1) < 0$. Consequently, $\nu_{n,\ell+2} - 2\nu_{n,\ell+1} + \nu_{n,\ell} \leq 0$ if $1 \leq \ell \leq \sqrt{an} - 2$ and n is large. $\square$

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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WEIGHTED APPROXIMATION BY KANTOROVICH TYPE MODIFICATION OF MEYER-KÖNIG AND ZELLER OPERATOR

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We investigate the weighted approximation of functions in L_p-norm by Kantorovich modifications of the classical Meyer-König and Zeller operator, with weights of type $(1-x)^{\alpha}$, $\alpha \in \mathbb{R}$. By defining an appropriate K-functional we prove direct theorems for them.

Keywords: Meyer-König and Zeller operator, K-functional, direct theorem, moduli of smoothness.

2010 Math. Subject Classification: 41A36, 41A25, 41A17.

1. INTRODUCTION

In order to approximate unbounded functions in uniform norm in $[0,1)$, Meyer-König and Zeller (see [15]) introduced a new operator by the formula

$$M_n(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\Big(\frac{k}{n+k}\Big), \tag{1.1}$$

where

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}. \tag{1.2}$$

But this operator cannot be used to approximate functions in L_p-norm because it is not bounded operator in L_p. Some kind of modification is needed. In this paper we investigate the weighted approximation of functions in L_p-norm by Kantorovich modifications of the classical Meyer-König and Zeller (MKZ) operator.

In 1930, Kantorovich [13] suggested a modification of the classical Bernstein operator, replacing the function values by mean values. Analogously, Totik [16] introduced Kantorovich type modification of MKZ operator:

$$\tilde{M}_n^*(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(n+k)(n+k+1)}{n} \int_{\frac{k}{n+k}}^{\frac{k+1}{n+k+1}} f(u)\,du,$$

and proved direct and converse theorems of weak type in terminology of Ditzian and Ivanov [4] for it. Although this definition looks as the most natural one, the operator $\tilde{M}_n^*$ is not a contraction, hence it is not very suitable for approximating functions in L_p-norm for $p < \infty$.

In [14] Müller defined a Kantorovich modification of MKZ operator in a slightly different way, so that the resulting operator is a contraction:

$$\tilde{M}_n(f;x) = \tilde{M}_n f(x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(n+k+1)(n+k+2)}{n+1} \int_{\frac{k}{n+k+1}}^{\frac{k+1}{n+k+2}} f(u)\,du. \quad (1.3)$$

Recently, in [11] by introducing an appropriate K-functional the first author proved a direct theorem for the operators $\tilde{M}_n(f;x)$. Our goal in this paper is to extend this result for the case of weighted approximation of functions in L_p-norm by $\tilde{M}_n(f;x)$ operator.

Let us introduce some notations. For the sake of simplicity and brevity of our presentation we set

$$\gamma_{n,k} = \frac{(n+k+1)(n+k+2)}{n+1}, \qquad \Delta_{n,k} = \Big[\frac{k}{n+k+1}, \frac{k+1}{n+k+2}\Big]. \quad (1.4)$$

Then, the Kantorovich modification of MKZ operator (1.3) takes the form

$$\tilde{M}_n(f;x) = \sum_{k=0}^{\infty} \gamma_{n,k}\, m_{n,k}(x) \int_{\Delta_{n,k}} f(u)\,du.$$

The weights under consideration in our survey are

$$w(x) = (1-x)^{\alpha}, \qquad \alpha \in \mathbb{R}. \quad (1.5)$$

By $\varphi(x) = x(1-x)^2$ we denote the weight which is naturally related to the second derivative of MKZ operator. The usual first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^k g(x) = g^{(k)}(x)$ for every $k \in \mathbb{N}$.

We define a differential operator $\tilde{D}$ by the formula

$$\tilde{D} = \frac{d}{dx}\Big(\varphi(x)\frac{d}{dx}\Big) = D\varphi D.$$

The space $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b]\subset(0,1)$. For $1\le p\le\infty$ and weight function $w(x)$ as in (1.5) we set

$$L_p(w)=\{f: wf\in L_p[0,1)\},$$

$$W_p(w)=\begin{cases}\big\{f: f, Df\in AC_{loc}(0,1),\ w\tilde{D}f\in L_p[0,1),\ \lim\limits_{x\to 0^+,1^-}\varphi(x)Df(x)=0\big\}, & \alpha<0,\\ \big\{f: f, Df\in AC_{loc}(0,1),\ w\tilde{D}f\in L_p[0,1),\ \lim\limits_{x\to 0^+}\varphi(x)Df(x)=0\big\}, & \alpha\ge 0,\end{cases}$$

$$L_p(w)+W_p(w)=\big\{f: f=f_1+f_2,\ f_1\in L_p(w),\ f_2\in W_p(w)\big\}.$$

Also, we define a K-functional $\tilde{K}_w(f,t)_p$ for $t>0$ by

$$\tilde{K}_w(f,t)_p=\inf\big\{\|w(f-g)\|_p+t\|w\tilde{D}g\|_p : f-g\in L_p(w),\ g\in W_p(w)\big\}. \tag{1.6}$$

Our main result is the following theorem.

Theorem 1. *For $1\le p\le\infty$, w defined by* (1.5)*, $\tilde{M}_n$ defined by* (1.3)*, and the K-functional given by* (1.6) *there exists a positive constant C such that for every $n>|\alpha|$, $n\in\mathbb{N}$, and for all functions $f\in L_p(w)+W_p(w)$ there holds*

$$\|w(\tilde{M}_nf-f)\|_p\le C\tilde{K}_w\Big(f,\frac{1}{n}\Big)_p. \tag{1.7}$$

Remark 1. Converse theorem remains an open problem even for the non-weighted case, i.e., for $w(x)=1$ in (1.5).

Problems on characterization of weighted K-functionals by moduli of smoothness were considered by Draganov and Ivanov in [6, 7, 9]. Particularly, they characterized the K-functional

$$K_w(f,t)_p=\inf\big\{\|w(f-g)\|_p+t\|w\varphi D^2g\|_p : g, Dg\in AC_{loc}(0,1),\ f-g, \varphi D^2g\in L_p(w)\big\}. \tag{1.8}$$

In this paper we also show that the same moduli of smoothness can be used for computing the K-functional $\tilde{K}_w(f,t)_p$. So, we prove the next statement.

Theorem 2. *For $1<p<\infty$ and w, $\tilde{K}_w(f,t)_p$, $K_w(f,t)_p$, defined by* (1.5)*,* (1.6) *and* (1.8)*, respectively, there exists a positive constant C such that for all $f\in L_p(w)+W_p(w)$ there holds*

$$\tilde{K}_w(f,t)_p\le C\big(K_w(f,t)_p+tE_0(f)\big), \tag{1.9}$$

where $E_0(f)=\inf_{c\in\mathbb{R}}\|w(f-c)\|_p$ is the best weighted approximation to f by a constant.

Remark 2. For $p=1$ and $p=\infty$ new moduli are needed. Also, a problem on characterization of the K-functional $\tilde{K}_w(f,t)_p$ arises, but it is not the subject of our survey here.

Henceforth, the constant C will always be an absolute positive constant, which means it does not depend on f and n. Also, it may be different on each occurrence. The relation $\theta_1(f,t) \sim \theta_2(f,t)$ means that there exists a constant $c \geq 1$, independent of f and t, such that

$$c^{-1}\theta_1(f,t) \leq \theta_2(f,t) \leq c\,\theta_1(f,t).$$

2. AUXILIARY RESULTS

In this section we present some properties of the operators M_n, $\tilde{M}_n$, basis functions $m_{n,k}$ (see [1, 10, 12]), and prove auxiliary lemmas that we need further.

The operators M_n and $\tilde{M}_n$ are linear positive operators with $\|M_n f\|_\infty \leq \|f\|_\infty$ and $\|\tilde{M}_n\|_1 = 1$. Moreover,

$$\|\tilde{M}_n\|_p \leq 1, \qquad 1 \leq p \leq \infty, \tag{2.1}$$

$$M_n(1;x) = 1, \qquad M_n(t-x;x) = 0, \tag{2.2}$$

$$\tilde{M}_n(1;x) = 1. \tag{2.3}$$

A direct integration yields the identity:

$$\int_0^1 m_{n,k}(x)dx = \frac{1}{\gamma_{n,k}}. \tag{2.4}$$

We shall need the next three properties of the functions $\{m_{n,k}\}_{k=0}^\infty$, defined by (1.2) (for proofs, see e.g., [11]).

Lemma 1. *If $n \in \mathbb{N}$, then*

$$\frac{1}{1-x} = \frac{1}{n+1}\sum_{k=0}^\infty (n+k+1)m_{n,k}(x), \qquad x \in [0,1). \tag{2.5}$$

Lemma 2. *If $n \in \mathbb{N}$, then*

$$\sum_{k=1}^n \frac{(1-x)^k}{k} = \sum_{k=0}^\infty m_{n,k}(x)\sum_{j=1}^n \frac{1}{k+j}, \qquad x \in [0,1). \tag{2.6}$$

Lemma 3. *There exists an absolute constant C such that for every $n \in \mathbb{N}$ the following inequality holds true:*

$$\left|\ln(1-x) + \sum_{k=0}^\infty m_{n,k}(x)\sum_{j=1}^{k+1}\frac{1}{n+j}\right| \leq \frac{C}{n}, \qquad x \in [0,1). \tag{2.7}$$

In [16, Lemma 3] Totik proved that for $1 \le p < \infty$,

$$\|(1-x)Df(x)\|_p \le C\big(\|f\|_p + \|\varphi D^2 f\|_p\big). \tag{2.8}$$

In order to prove our main results we need a few additional lemmas.

Lemma 4. *For every integer ν there exists a constant $C = C(\nu)$, such that*

$$\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\nu} m_{n,k}(x) \le C(1-x)^{\nu}, \qquad x \in [0,1), \tag{2.9}$$

for all $n > |\nu|$, $n \in \mathbb{N}$.

Proof. We have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\nu} m_{n,k}(x)\\
&\quad= \sum_{k=0}^{\infty}\Big(\frac{n+1}{n+k+1}\Big)^{\nu}\binom{n+k}{k}x^k(1-x)^{n+1}\\
&\quad= (1-x)^{\nu}\sum_{k=0}^{\infty}\frac{(n+1)^{\nu}\,(n+k-\nu+1)\cdots(n+k)}{(n-\nu+1)\cdots n\,(n+k+1)^{\nu}}\, m_{n-\nu,k}(x)\\
&\quad\le (1-x)^{\nu}\sum_{k=0}^{\infty} C(\nu)\, m_{n-\nu,k}(x)\\
&\quad= C(\nu)(1-x)^{\nu}.
\end{aligned}$$

□

Lemma 5. *For every $\alpha \in \mathbb{R}$ there exists a constant $C = C(\alpha)$, such that the following inequality is satisfied:*

$$\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\alpha} m_{n,k}(x) \le C(1-x)^{\alpha}, \qquad x \in [0,1), \tag{2.10}$$

for all $n > |\alpha|$, $n \in \mathbb{N}$.

Proof. Let ν be the smallest positive integer such that $\nu \ge |\alpha|$. Then, by Hölder's inequality it follows that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\alpha} m_{n,k}(x)\\
&\quad\le \left(\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\nu\,\mathrm{sign}\,(\alpha)} m_{n,k}(x)\right)^{|\alpha|/\nu}\left(\sum_{k=0}^{\infty} m_{n,k}(x)\right)^{1-|\alpha|/\nu}.
\end{aligned}$$

Applying Lemma 4 we obtain

$$\left(\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\nu\ \mathrm{sign}(\alpha)} m_{n,k}(x)\right)^{|\alpha|/\nu} \le \big(C(1-x)^{\nu\ \mathrm{sign}(\alpha)}\big)^{|\alpha|/\nu} = C(\alpha)(1-x)^{\alpha}.$$

Therefore,

$$\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{\alpha} m_{n,k}(x) \le C(\alpha)(1-x)^{\alpha}$$

and the lemma is proved. $\square$

The next lemma is a weighted variant of (2.1).

Lemma 6. *Let $1 \le p \le \infty$ and $\alpha \in \mathbb{R}$. Then, there exists an absolute constant C such that for all $n > |\alpha|$, $n \in \mathbb{N}$, and $f \in L_p(w)$, we have*

$$\|w\tilde{M}_n f\|_p \le C\|wf\|_p. \tag{2.11}$$

Proof. First we prove (2.11) for $p = 1$ and $p = \infty$. Then, by applying Riesz-Thorin theorem we obtain the estimation for every $1 < p < \infty$.

The case $p = 1$. We have

$$\begin{aligned}
\|w\tilde{M}_n f\|_1 &= \int_0^1 w(x)\left|\sum_{k=0}^{\infty}\gamma_{n,k}\, m_{n,k}(x)\int_{\Delta_{n,k}} f(t)\,dt\right| dx\\
&\le \int_0^1 w(x)\left[\sum_{k=0}^{\infty}\gamma_{n,k}\, m_{n,k}(x)\int_{\Delta_{n,k}} \frac{|(wf)(t)|}{w(t)}\,dt\right] dx\\
&\le C\int_0^1 \left[\sum_{k=0}^{\infty}\gamma_{n,k}\,\frac{w(x)}{w\big(\frac{k}{n+k+1}\big)}\, m_{n,k}(x)\int_{\Delta_{n,k}} |(wf)(t)|\,dt\right] dx\\
&= C\int_0^1 \sum_{k=0}^{\infty}\left(\frac{1-x}{1-\frac{k}{n+k+1}}\right)^{\alpha} a_{n,k} m_{n,k}(x)\,dx,
\end{aligned}$$

where we set

$$a_{n,k} = \gamma_{n,k}\int_{\Delta_{n,k}} |(wf)(t)|\,dt.$$

Let $\nu = \lceil |\alpha| \rceil$ be the smallest positive integer such that $\nu \ge |\alpha|$. Applying Hölder's inequality twice we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{1-x}{1-\frac{k}{n+k+1}}\right)^{\alpha} a_{n,k} m_{n,k}(x)\\
&\quad\le \left[\sum_{k=0}^{\infty}\left(\frac{1-x}{1-\frac{k}{n+k+1}}\right)^{\nu\ \mathrm{sign}(\alpha)} a_{n,k} m_{n,k}(x)\right]^{|\alpha|/\nu}\left[\sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x)\right]^{1-|\alpha|/\nu},
\end{aligned}$$

thus

$$\|w\tilde{M}_n f\|_1 \le C \left\| \sum_{k=0}^{\infty} \left(\frac{1-x}{1-\frac{k}{n+k+1}} \right)^{\nu \,\mathrm{sign}(\alpha)} a_{n,k} m_{n,k}(x) \right\|_1^{|\alpha|/\nu} \times \left\| \sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x) \right\|_1^{1-|\alpha|/\nu}. \tag{2.12}$$

Now, we estimate the first nonconstant multiplier in the right-hand side of inequality (2.12). Let $\ell = \nu \,\mathrm{sign}(\alpha)$. For every integer number ℓ we have

$$\begin{aligned} \left(\frac{1-x}{1-\frac{k}{n+k+1}} \right)^{\ell} m_{n,k}(x) &= \frac{(n+k+1)^{\ell}\,(n+1)\cdots(n+\ell)}{(n+k+1)\cdots(n+k+\ell)\,(n+1)^{\ell}}\, m_{n+\ell,k}(x) \\ &\le C(\ell) m_{n+\ell,k}(x), \end{aligned}$$

hence

$$\sum_{k=0}^{\infty} \left(\frac{1-x}{1-\frac{k}{n+k+1}} \right)^{\ell} a_{n,k}\, m_{n,k}(x) \le C(\ell) \sum_{k=0}^{\infty} a_{n,k}\, m_{n+\ell,k}(x).$$

Then, by (2.4),

$$\begin{aligned} \left\| \sum_{k=0}^{\infty} \left(\frac{1-x}{1-\frac{k}{n+k+1}} \right)^{\ell} a_{n,k} m_{n,k}(x) \right\|_1 &\le C \left\| \sum_{k=0}^{\infty} a_{n,k}\, m_{n+\ell,k}(x) \right\|_1 \\ &\le C \sum_{k=0}^{\infty} a_{n,k} \|m_{n+\ell,k}(x)\|_1 = C \sum_{k=0}^{\infty} \frac{a_{n,k}}{\gamma_{n+\ell,k}} \\ &= C \sum_{k=0}^{\infty} \frac{\gamma_{n,k}}{\gamma_{n+\ell,k}} \int_{\Delta_{n,k}} |(wf)(t)|\, dt \\ &\le C \sum_{k=0}^{\infty} \int_{\Delta_{n,k}} |(wf)(t)|\, dt = C\|wf\|_1. \end{aligned}$$

Since $\sum_{k=0}^{\infty} a_{n,k} m_{n,k}(x) = \tilde{M}_n(wf;x)$ and $\|\tilde{M}_n(wf)\|_1 \le \|wf\|_1$ by (2.1), then for the last multiplier in the right-hand side of (2.12) we obtain the inequality $\|\sum_{k=0}^{\infty} a_{n,k} m_{n,k}\|_1 \le \|wf\|_1$. Therefore,

$$\|w\tilde{M}_n f\|_1 \le C \|wf\|_1^{|\alpha|/\nu} \|wf\|_1^{1-|\alpha|/\nu} = C\|wf\|_1$$

and the proof of the estimate (2.11) for $p = 1$ is complete.

The case $p=\infty$. We obtain

$$\begin{aligned}\Big|w(x)\sum_{k=0}^{\infty}\gamma_{n,k}\,m_{n,k}(x)\int_{\Delta_{n,k}}f(t)\,dt\Big| &\le w(x)\sum_{k=0}^{\infty}\gamma_{n,k}\,m_{n,k}(x)\int_{\Delta_{n,k}}\frac{|(wf)(t)|}{w(t)}\,dt\\ &\le Cw(x)\sum_{k=0}^{\infty}\frac{\gamma_{n,k}\,m_{n,k}(x)}{w\big(\frac{k}{n+k+1}\big)}\int_{\Delta_{n,k}}|(wf)(t)|\,dt\\ &\le Cw(x)\sum_{k=0}^{\infty}m_{n,k}(x)\,\frac{\|wf\|_\infty}{w\big(\frac{k}{n+k+1}\big)}\\ &= Cw(x)\|wf\|_\infty\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{-\alpha}m_{n,k}(x).\end{aligned}$$

Now, by Lemma 5 we have

$$\sum_{k=0}^{\infty}\Big(1-\frac{k}{n+k+1}\Big)^{-\alpha}m_{n,k}(x)\le C(1-x)^{-\alpha}.$$

Hence,

$$\|w\tilde{M}_nf\|_\infty\le Cw(x)\|wf\|_\infty(1-x)^{-\alpha}=C\|wf\|_\infty,$$

which proves (2.11) in the case $p=\infty$.

Finally, the inequality (2.11) follows for all $1\le p\le\infty$ by the Riesz-Thorin interpolation theorem. □

The crucial result in our investigation is the following Jackson type inequality.

Lemma 7. *Let $1\le p\le\infty$ and $\alpha\in\mathbb{R}$. Then there exists an absolute constant C, such that for all $n>|\alpha|$, $n\in\mathbb{N}$, and $f\in W_p(w)$, the following estimate holds true:*

$$\big\|w(\tilde{M}_nf-f)\big\|_p\le\frac{C}{n}\big\|w\tilde{D}f\big\|_p. \tag{2.13}$$

(Let us note that the lemma implies that $\tilde{M}_nf-f\in L_p(w)$ for $f\in W_p(w)$.)

Proof. Let us set

$$\phi(x)=\ln\frac{x}{1-x}+\frac{1}{1-x},\qquad x\in(0,1),$$

with $\phi'(x)=\frac{1}{x(1-x)^2}=\frac{1}{\varphi(x)}>0$, i.e., $\phi(x)$ is an increasing function. Then we have

$$f(t)=f(x)+\varphi(x)[\phi(t)-\phi(x)]Df(x)+\int_x^t[\phi(t)-\phi(u)]\tilde{D}f(u)\,du,\qquad t\in(0,1).$$

Applying the operator $\tilde{M}_n$ to both sides of the latter equality and multiplying by $w(x)$ we obtain

$$w(x)\big(\tilde{M}_n f(x) - f(x)\big) = w(x)\varphi(x)Df(x)[\tilde{M}_n\phi(x) - \phi(x)] \\ + w(x)\tilde{M}_n\bigg(\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)]\tilde{D}f(u)\,du; x\bigg). \qquad (2.14)$$

First we prove the lemma for $p = 1$ and $p = \infty$. Then we apply the Riesz-Thorin theorem to obtain (2.13) for every $1 < p < \infty$.

The case $p = 1$. In order to prove that

$$\big\|w\varphi Df\,[\tilde{M}_n\phi - \phi]\big\|_1 \le \frac{C}{n}\|w\tilde{D}f\|_1 \qquad (2.15)$$

for all weights (1.5), we shall make use of the estimate

$$\big\|\tilde{M}_n\phi - \phi\big\|_1 \le \frac{C}{n} \qquad (2.16)$$

(see [11, Proof of Theorem 1] for a complete proof).

Let $\alpha > 0$ be fixed. Then, for all $n > \alpha$ and $f \in W_1(w)$ we have

$$\varphi(x)Df(x) = \int_0^x (\varphi Df)'(u)\,du = \int_0^x \tilde{D}f(u)\,du, \qquad x \in (0,1).$$

Hence,

$$|w(x)\varphi(x)Df(x)| \le w(x)\int_0^x |\tilde{D}f(u)|\,du \le \int_0^x |(w\tilde{D}f)(u)|\,du \le \int_0^1 |(w\tilde{D}f)(u)|\,du,$$

i.e.,

$$|w(x)\varphi(x)Df(x)| \le \|w\tilde{D}f\|_1, \qquad x \in (0,1).$$

Thus,

$$\big\|w\varphi Df\,[\tilde{M}_n\phi - \phi]\big\|_1 \le \|w\tilde{D}f\|_1\,\big\|\tilde{M}_n\phi - \phi\big\|_1$$

and (2.15) follows from (2.16).

Similarly, let $\alpha < 0$ be fixed. Then, for all $n > -\alpha$ we have $-n < \alpha < 0$ and for $f \in W_1(w)$, we consecutively obtain

$$\varphi(x)Df(x) = \int_x^1 (\varphi Df)'(u)\,du = \int_x^1 \tilde{D}f(u)\,du, \qquad x \in (0,1),$$

$$|w(x)\varphi(x)Df(x)| \le w(x)\int_x^1 |\tilde{D}f(u)|\,du \le \int_x^1 |(w\tilde{D}f)(u)|\,du \le \int_0^1 |(w\tilde{D}f)(u)|\,du,$$

i.e.,

$$|w(x)\varphi(x)Df(x)| \le \|w\tilde{D}f\|_1, \qquad x \in (0,1).$$

Hence, (2.16) yields (2.15).

Therefore, for arbitrary $\alpha \in \mathbb{R} \setminus \{0\}$ and $f \in W_1(w)$ the estimate (2.15) holds true for $n > |\alpha|$. The case $\alpha = 0$ was considered by the first author in [11].

Now, we estimate the L_1-norm of the second summand in the right-hand side of (2.14). More precisely, we will prove

$$\left\| w(x)\tilde{M}_n\left(\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)]\tilde{D}f(u)\,du\,; x\right)\right\|_1 \leq \frac{C}{n}\|w\tilde{D}f\|_1. \tag{2.17}$$

Having in mind (1.4), for $x \in (0,1)$ we have

$$\begin{aligned}
&\left| w(x)\tilde{M}_n\left(\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)]\tilde{D}f(u)\,du\,; x\right)\right| \\
&\quad \leq w(x)\sum_{k=0}^{\infty} \gamma_{n,k}\, m_{n,k}(x) \int_{\Delta n,k} \left(\int_x^t [\phi(t) - \phi(u)]\frac{|(w\tilde{D}f)(u)|}{w(u)}\,du\right) dt \\
&\quad \leq Cw(x)\sum_{k=0}^{\infty} \gamma_{n,k}\, m_{n,k}(x) \\
&\qquad \times \left(\frac{1}{w\left(\frac{k}{n+k+1}\right)} + \frac{1}{w(x)}\right) \int_{\Delta n,k} \left(\int_x^t [\phi(t) - \phi(u)]|(w\tilde{D}f)(u)|du\right) dt \\
&\quad \leq C\sum_{k=0}^{\infty} \left(\frac{w(x)}{w\left(\frac{k}{n+k+1}\right)} + 1\right) b_{n,k}\, m_{n,k}(x),
\end{aligned}$$

where

$$b_{n,k} = \gamma_{n,k} \int_{\Delta n,k} \left(\int_x^t [\phi(t) - \phi(u)]|(w\tilde{D}f)(u)|du\right) dt.$$

Let ν be the smallest positive integer such that $\nu \geq |\alpha|$. Applying twice Hölder's inequality we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{w(x)}{w\left(\frac{k}{n+k+1}\right)}\, b_{n,k}m_{n,k}(x) \leq{}& \left[\sum_{k=0}^{\infty} \left(\frac{w(x)}{w\left(\frac{k}{n+k+1}\right)}\right)^{\nu/|\alpha|} b_{n,k}m_{n,k}(x)\right]^{|\alpha|/\nu} \\
&\times \left[\sum_{k=0}^{\infty} b_{n,k}m_{n,k}(x)\right]^{1-|\alpha|/\nu},
\end{aligned}$$

thus

$$\begin{aligned}
&\left\| w(x)\tilde{M}_n\left(\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)]\tilde{D}f(u)\,du\,;\right)\right\|_1 \\
&\quad \leq C\left\| \sum_{k=0}^{\infty} \left(\frac{w(x)}{w\left(\frac{k}{n+k+1}\right)}\right)^{\nu/|\alpha|} b_{n,k}m_{n,k}\right\|_1^{|\alpha|/\nu} \left\| \sum_{k=0}^{\infty} b_{n,k}m_{n,k}\right\|_1^{1-|\alpha|/\nu}.
\end{aligned} \tag{2.18}$$

For estimation of the last factor in (2.18) we apply the estimate from [11] (see Proof of Theorem 1, Case 1, therein), by simply replacing $\tilde{D}f$ with $w\tilde{D}f$. So, we obtain

$$\left\| \sum_{k=0}^{\infty} b_{n,k} m_{n,k} \right\|_1 \le \frac{C}{n} \|w\tilde{D}f\|_1. \tag{2.19}$$

Next, we focus on the estimating of the other multiplier in (2.18). Clearly,

$$\sum_{k=0}^{\infty} \left(\frac{w(x)}{w\big(\frac{k}{n+k+1}\big)} \right)^{\nu/|\alpha|} b_{n,k} m_{n,k}(x) = \sum_{k=0}^{\infty} \Big(\frac{(1-x)(n+k+1)}{n+1} \Big)^{\nu \ \mathrm{sign}(\alpha)} b_{n,k} m_{n,k}(x).$$

Let us set for simplicity $\ell = \nu \ \mathrm{sign}(\alpha) = \lceil |\alpha| \rceil \ \mathrm{sign}(\alpha)$. We have

$$\begin{aligned} \Big(\frac{(1-x)(n+k+1)}{n+1} \Big)^{\ell} m_{n,k}(x) &= \frac{(n+k+1)^{\ell}\,(n+1)\cdots(n+\ell)}{(n+k+1)\cdots(n+k+\ell)\,(n+1)^{\ell}}\, m_{n+\ell,k}(x) \\ &\le C(\ell)\, m_{n+\ell,k}(x) \\ &\le C(\ell)\, \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\, m_{n+\ell,k}(x). \end{aligned}$$

Observe that the constant $C(\ell)$ depends only on α.

We shall make use of the following operator defined by

$$\tilde{M}_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} \gamma_{n+\ell,k}\, m_{n+\ell,k}(x) \int_{\Delta_{n,k}} f(u)\, du. \tag{2.20}$$

Then,

$$\sum_{k=0}^{\infty} \left(\frac{w(x)}{w\big(\frac{k}{n+k+1}\big)} \right)^{\nu/|\alpha|} b_{n,k} m_{n,k}(x) \le C\tilde{M}_{n,\alpha}\bigg(\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)] |(w\tilde{D}f)(u)|\, du;\, x \bigg). \tag{2.21}$$

In order to estimate the L_1-norm of the right-hand side in (2.21) we follow an approach applied, e.g., in [2, pp. 41–43]. The proof in our case is much more complicated, because the operator $\tilde{M}_{n,\alpha}$ does not preserve the constant functions. More precisely, it has the properties

$$\|\tilde{M}_{n,\alpha}\|_1 = 1, \qquad \tilde{M}_{n,\alpha}(1;x) = \sum_{k=0}^{\infty} \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\, m_{n+\ell,k}(x).$$

Let us write the operator $\tilde{M}_{n,\alpha}$ from (2.20) in the form

$$\tilde{M}_{n,\alpha}(f;x) = \int_0^1 K_n(x,t) f(t)\, dt,$$

where $K_n(\cdot,\cdot)$ is the related kernel. Introducing the functions

$$\phi_1(x) = \ln x, \qquad \phi_2(x) = -\ln(1-x), \qquad \phi_3(x) = \frac{1}{1-x},$$

we have $\phi(x) = \phi_1(x) + \phi_2(x) + \phi_3(x)$ and for $j = 1, 2, 3$,

$$\begin{aligned}
&\tilde{M}_{n,\alpha}\biggl(\int_x^{(\cdot)} [\phi_j(\cdot) - \phi_j(u)]|(w\tilde{D}f)(u)|\,du; x\biggr) \\
&\qquad = \int_0^x K_n(x,t)\int_x^t [\phi_j(t) - \phi_j(u)]|(w\tilde{D}f)(u)|\,du\,dt \\
&\qquad\quad + \int_x^1 K_n(x,t)\int_x^t [\phi_j(t) - \phi_j(u)]|(w\tilde{D}f)(u)|\,du\,dt.
\end{aligned}$$

Then, by Fubini's theorem we obtain:

$$\begin{aligned}
&\biggl\|\tilde{M}_{n,\alpha}\int_x^{(\cdot)} [\phi(\cdot) - \phi(u)]|(w\tilde{D}f)(u)|\,du\biggr\|_1 \\
&\quad = \int_0^1 |(w\tilde{D}f)(u)| \sum_{j=1}^{3} \biggl(\int_u^1 \tilde{M}_{n,\alpha}\bigl([\phi_j(u) - \phi_j(\cdot)]_+\,; x\bigr)\,dx \\
&\qquad\qquad + \int_0^u \tilde{M}_{n,\alpha}\bigl([\phi_j(\cdot) - \phi_j(u)]_+\,; x\bigr)\,dx\biggr)\,du. \qquad (2.22)
\end{aligned}$$

To estimate the right-hand side of (2.22) we need estimations for the expressions in the sum for each of the functions ϕ_j, $j = 1, 2, 3$.

First, for ϕ_1, using

$$\begin{aligned}
\int_0^1 \tilde{M}_{n,\alpha}\bigl([\phi_1(u) - \phi_1(\cdot)]_+\,; x\bigr)\,dx &= \|\tilde{M}_{n,\alpha}\bigl([\phi_1(u) - \phi_1(\cdot)]_+\,; x\bigr)\|_1 \\
&\le \|[\phi_1(u) - \phi_1(x)]_+\|_1 \\
&= \int_0^u (\phi_1(u) - \phi_1(x))\,dx,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_u^1 \tilde{M}_{n,\alpha}\bigl([\phi_1(u) - \phi_1(\cdot)]_+\,; x\bigr)\,dx + \int_0^u \tilde{M}_{n,\alpha}\bigl([\phi_1(\cdot) - \phi_1(u)]_+\,; x\bigr)\,dx \\
&\quad = \int_0^1 \tilde{M}_{n,\alpha}\bigl([\phi_1(u) - \phi_1(\cdot)]_+\,; x\bigr)\,dx - \int_0^u \tilde{M}_{n,\alpha}\bigl([\phi_1(u) - \phi_1(\cdot)]_+\,; x\bigr)\,dx \\
&\qquad + \int_0^u \tilde{M}_{n,\alpha}\bigl([\phi_1(\cdot) - \phi_1(u)]_+\,; x\bigr)\,dx
\end{aligned}$$

$$\le \int_0^u (\phi_1(u)-\phi_1(x))\,dx + \int_0^u \tilde{M}_{n,\alpha}\big([\phi_1(\cdot)-\phi_1(u)]_+ - [\phi_1(u)-\phi_1(\cdot)]_+\,;x\big)\,dx$$
$$= u\phi_1(u) - \int_0^u \phi_1(x)\,dx + \int_0^u \tilde{M}_{n,\alpha}(\phi_1;x)\,dx - \phi_1(u)\int_0^u \tilde{M}_{n,\alpha}(1;x)\,dx$$
$$= \int_0^u \big(\tilde{M}_{n,\alpha}(\phi_1;x)-\phi_1(x)\big)\,dx - \phi_1(u)\int_0^u \big(\tilde{M}_{n,\alpha}(1;x)-1\big)\,dx. \tag{2.23}$$

Analogously, for ϕ_j, $j=2,3$, we obtain

$$\int_u^1 \tilde{M}_{n,\alpha}\big([\phi_j(u)-\phi_j(\cdot)]_+\,;x\big)\,dx + \int_0^u \tilde{M}_{n,\alpha}\big([\phi_j(\cdot)-\phi_j(u)]_+\,;x\big)\,dx$$
$$\le \int_u^1 \big(\tilde{M}_{n,\alpha}(\phi_j;x)-\phi_j(x)\big)\,dx - \phi_j(u)\int_u^1 \big(\tilde{M}_{n,\alpha}(1;x)-1\big)\,dx. \tag{2.24}$$

Since for $x,u\in(0,1)$,

$$|\tilde{M}_{n,\alpha}(1;x)-1| = \Big|\sum_{k=0}^{\infty}\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\,m_{n+\ell,k}(x)-1\Big| \le \frac{C}{n},$$
$$|u\phi_1(u)|\le C, \qquad |(1-u)\phi_2(u)|\le C, \qquad |(1-u)\phi_3(u)|\le C,$$

then

$$\begin{aligned}\Big|\phi_1(u)\int_0^u \big(\tilde{M}_{n,\alpha}(1;x)-1\big)\,dx\Big| &\le \frac{C}{n},\\ \Big|\phi_j(u)\int_u^1 \big(\tilde{M}_{n,\alpha}(1;x)-1\big)\,dx\Big| &\le \frac{C}{n}, \qquad j=2,3.\end{aligned} \tag{2.25}$$

1. Estimation of $\big|\int_0^u \big(\tilde{M}_{n,\alpha}(\phi_1;x)-\phi_1(x)\big)\,dx\big|$. We have

$$\int_{\Delta_{n,k}} \phi_1(t)\,dt = \frac{k+1}{n+k+2}\ln\frac{k+1}{n+k+2} - \frac{k}{n+k+1}\ln\frac{k}{n+k+1} - \frac{1}{\gamma_{n,k}},$$

and for $x\in(0,1)$,

$$\phi_1(x) = -\sum_{k=1}^{n+\ell}\frac{(1-x)^k}{k} - \sum_{k=n+\ell+1}^{\infty}\frac{(1-x)^k}{k}.$$

By Lemma 2,

$$\sum_{k=1}^{n+\ell}\frac{(1-x)^k}{k} = \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\sum_{i=1}^{n+\ell}\frac{1}{k+i},$$

and therefore

$$
\begin{aligned}
&\Big|\int_0^u \big(\tilde{M}_{n,\alpha}(\phi_1;x)-\phi_1(x)\big)\,dx\Big| \\
&\quad = \Big|\int_0^u \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\Big[\gamma_{n+\ell,k}\int_{\Delta_{n,k}}\phi_1(t)\,dt+\sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big]dx+\int_0^u\sum_{k=n+\ell+1}^{\infty}\frac{(1-x)^k}{k}\,dx\Big| \\
&\quad \le \Big|\int_0^u \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\Big[\gamma_{n+\ell,k}\int_{\Delta_{n,k}}\phi_1(t)\,dt+\sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big]dx\Big|+\frac{C}{n}.
\end{aligned}
$$

For $k \ge 1$,

$$
\begin{aligned}
\ln\frac{k+1}{n+k+2} &= -\ln\prod_{i=1}^{n+1}\frac{k+i+1}{k+i} = -\sum_{i=1}^{n+1}\ln\Big(1+\frac{1}{k+i}\Big) \\
&= -\sum_{i=1}^{n+1}\Big[\frac{1}{k+i}-\frac{1}{2(k+i)^2}+\mathcal{O}\Big(\frac{1}{(k+i)^3}\Big)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{n+1}\frac{1}{(k+i)^2} &= \sum_{i=1}^{n+1}\Big[\frac{1}{(k+i)(k+i+1)}+\mathcal{O}\Big(\frac{1}{(k+i)^3}\Big)\Big] \\
&= \frac{n+1}{(k+1)(n+k+2)}+\sum_{i=1}^{n+1}\mathcal{O}\Big(\frac{1}{(k+i)^3}\Big),
\end{aligned}
$$

hence

$$
\ln\frac{k+1}{n+k+2} = -\sum_{i=1}^{n+1}\frac{1}{k+i}+\frac{n+1}{2(k+1)(n+k+2)}+\mathcal{O}\Big(\frac{1}{k^2}\Big).
$$

Since

$$
\frac{k+1}{n+k+2}\,\mathcal{O}\Big(\frac{1}{k^2}\Big)=\mathcal{O}\Big(\frac{1}{k^2}\Big),
$$

then

$$
\frac{k+1}{n+k+2}\,\ln\frac{k+1}{n+k+2} = -\frac{k+1}{n+k+2}\sum_{i=1}^{n+1}\frac{1}{k+i}+\frac{n+1}{2(n+k+2)^2}+\mathcal{O}\Big(\frac{1}{k^2}\Big).
$$

Similarly,

$$
\frac{k}{n+k+1}\,\ln\frac{k}{n+k+1} = -\frac{k}{n+k+1}\sum_{i=0}^{n}\frac{1}{k+i}+\frac{n+1}{2(n+k+1)^2}+\mathcal{O}\Big(\frac{1}{k^2}\Big).
$$

Therefore,

$$\begin{aligned}\int_{\Delta_{n,k}} \phi_1(t)\,dt &= \frac{k}{n+k+1}\sum_{i=0}^{n}\frac{1}{k+i} - \frac{k+1}{n+k+2}\sum_{i=1}^{n+1}\frac{1}{k+i} \\ &\quad - \frac{n+1}{2}\Big[\frac{1}{(n+k+1)^2} - \frac{1}{(n+k+2)^2}\Big] + \mathcal{O}\Big(\frac{1}{k^2}\Big) - \frac{1}{\gamma_{n,k}} \\ &= -\frac{1}{\gamma_{n,k}}\sum_{i=0}^{n}\frac{1}{k+i} + \mathcal{O}\Big(\frac{1}{k^2}\Big).\end{aligned}$$

Now, we have

$$\begin{aligned}\big|\tilde{M}_{n,\alpha}(\phi_1;x) - \phi_1(x)\big| &\le m_{n+\ell,0}(x)\Big|\ln(n+2) + 1 - \sum_{i=1}^{n+\ell}\frac{1}{i}\Big| \\ &\quad + \sum_{k=1}^{\infty} m_{n+\ell,k}(x)\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\sum_{i=1}^{n}\frac{1}{k+i} - \sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big| + \frac{C}{n}.\end{aligned}$$

From

$$\Big|\ln(n+2) + 1 - \sum_{i=1}^{n+\ell}\frac{1}{i}\Big| \le C, \qquad \|m_{n+\ell,0}\|_1 \le \frac{C}{n},$$

it follows

$$\Big\|m_{n+\ell,0}(x)\Big|\ln(n+2) + 1 - \sum_{i=1}^{n+\ell}\frac{1}{i}\Big|\Big\|_1 \le \frac{C}{n}.$$

Moreover,

$$\begin{aligned}&\sum_{k=1}^{\infty} m_{n+\ell,k}(x)\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\sum_{i=1}^{n}\frac{1}{k+i} - \sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big| \\ &\qquad \le \sum_{k=1}^{\infty} m_{n+\ell,k}(x)\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1\Big|\sum_{i=1}^{n}\frac{1}{k+i} + \sum_{k=1}^{\infty} m_{n+\ell,k}(x)\sum_{i=n+1}^{n+\ell}\frac{1}{k+i}.\end{aligned}$$

Now, the inequalities

$$\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1\Big| \le \frac{C}{n}, \qquad \sum_{k=1}^{\infty} m_{n+\ell,k}(x)\sum_{i=n+1}^{n+\ell}\frac{1}{k+i} \le \frac{C}{n}\sum_{k=1}^{\infty} m_{n+\ell,k}(x) \le \frac{C}{n},$$

yield

$$\sum_{k=1}^{\infty} m_{n+\ell,k}(x)\Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\sum_{i=1}^{n}\frac{1}{k+i} - \sum_{i=1}^{n+\ell}\frac{1}{k+i}\Big| \le \frac{C}{n}\sum_{k=1}^{\infty} m_{n+\ell,k}(x)\sum_{i=1}^{n}\frac{1}{k+i} + \frac{C}{n}.$$

By Lemma 2 we obtain

$$\sum_{k=1}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{n} \frac{1}{k+i} \le \sum_{k=1}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{n+\ell} \frac{1}{k+i} \le |\ln x|.$$

Therefore,

$$\Big|\int_0^u \sum_{k=1}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{n} \frac{1}{k+i}\, dx\Big| \le \Big|\int_0^u \ln x\, dx\Big| \le \Big|\int_0^1 \ln x\, dx\Big| \le C,$$

and we conclude that

$$\Big|\int_0^u \big(\tilde{M}_{n,\alpha}(\phi_1;x) - \phi_1(x)\big)\, dx\Big| \le \frac{C}{n}. \tag{2.26}$$

<u>*2. Estimation of* $\big|\int_u^1 \big(\tilde{M}_{n,\alpha}(\phi_2;x) - \phi_2(x)\big)\, dx\big|$</u>. We have

$$\int_{\Delta_{n,k}} \phi_2(t)\, dt = \frac{n+1}{n+k+2} \ln \frac{n+1}{n+k+2} - \frac{n+1}{n+k+1} \ln \frac{n+1}{n+k+1} + \frac{1}{\gamma_{n,k}},$$

$$\begin{aligned}\gamma_{n,k} \int_{\Delta_{n,k}} \phi_2(t)\, dt &= 1 - (n+k+1) \ln\Big(1 + \frac{1}{n+k+1}\Big) - \ln \frac{n+1}{n+k+1} \\ &= \ln \frac{n+k+1}{n+1} + \mathcal{O}\Big(\frac{1}{n+k}\Big),\end{aligned}$$

hence,

$$M_{n,\alpha}(\phi_2;x) = \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\, \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} \Big[\ln \frac{n+k+1}{n+1} + \mathcal{O}\Big(\frac{1}{n+k}\Big)\Big].$$

Applying Lemma 3 we obtain

$$\Big|\phi_2(x) - \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k+1} \frac{1}{n+\ell+i}\Big| \le \frac{C}{n},$$

and then

$$\big|\tilde{M}_{n,\alpha}(\phi_2;x) - \phi_2(x)\big| \le \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big|\frac{\gamma_{n+\ell,k}}{\gamma_{n,k}}\, \ln \frac{n+k+1}{n+1} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i}\Big| + \frac{C}{n}.$$

Taking into account that

$$\ln \frac{n+k+1}{n+1} = \sum_{i=1}^{k} \ln\Big(1 + \frac{1}{n+i}\Big) = \sum_{i=1}^{k} \frac{1}{n+i} + \sum_{i=1}^{k} \mathcal{O}\Big(\frac{1}{(n+i)^2}\Big)$$

and

$$\sum_{i=1}^{k} \frac{1}{(n+i)^2} \leq \frac{C}{n},$$

we estimate

$$\begin{aligned}
\big|\tilde{M}_{n,\alpha}(\phi_2;x) - \phi_2(x)\big| &\leq \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| + \frac{C}{n} \\
&\leq \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \Big| \sum_{i=1}^{k} \frac{1}{n+i} \\
&\quad + \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| + \frac{C}{n}.
\end{aligned}$$

Since

$$\Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \Big| \leq \frac{C}{n}\,,$$

it follows that

$$\begin{aligned}
&\sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \Big| \sum_{i=1}^{k} \frac{1}{n+i} \leq \frac{C}{n} \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k} \frac{1}{n+i} \\
&\quad \leq \frac{C}{n} \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| + \frac{C}{n} \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k+1} \frac{1}{n+\ell+i}.
\end{aligned}$$

Observe that

$$\sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \sum_{i=1}^{k} \frac{1}{n+i} - \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| \leq \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{\ell} \frac{1}{n+i} \leq \frac{C}{n}.$$

We recall that $\ell = \lceil |\alpha| \rceil \operatorname{sign}(\alpha)$ and $C = C(\alpha)$, i.e. C is an absolute constant for a fixed α. Then, by Lemma 3 we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} m_{n+\ell,k}(x) \Big| \frac{\gamma_{n+\ell,k}}{\gamma_{n,k}} - 1 \Big| \sum_{i=1}^{k} \frac{1}{n+i} \\
&\qquad \leq \frac{C}{n^2} + \frac{C}{n} \Big| \ln(1-x) + \sum_{k=0}^{\infty} m_{n+\ell,k}(x) \sum_{i=1}^{k+1} \frac{1}{n+\ell+i} \Big| + \frac{C}{n}\, |\ln(1-x)| \\
&\qquad \leq \frac{C}{n^2} + \frac{C}{n^2} + \frac{C}{n}\, |\ln(1-x)|.
\end{aligned}$$

Therefore,

$$\Big| \int_u^1 \big(\tilde{M}_{n,\alpha}(\phi_2;x) - \phi_2(x)\big)\, dx \Big| \leq \frac{C}{n} \int_0^1 (2 - \ln(1-x))\, dx \leq \frac{C}{n}. \tag{2.27}$$

3. Estimation of $\left|\int_u^1 \big(\tilde{M}_{n,\alpha}(\phi_3;x)-\phi_3(x)\big)\,dx\right|$. The last estimation we need concerns the function $\phi_3(x)=\frac{1}{1-x}$. We have

$$\int_{\Delta_{n,k}} \phi_3(t)\,dt = \ln\Big(1+\frac{1}{n+k+1}\Big) = \frac{1}{n+k+1} + \mathcal{O}\Big(\frac{1}{(n+k)^2}\Big),$$

$$\gamma_{n,k}\int_{\Delta_{n,k}} \phi_3(t)\,dt = \frac{n+k+2}{n+1} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

By Lemma 1,

$$\phi_3(x) = \frac{1}{n+\ell+1}\sum_{k=0}^{\infty}(n+\ell+k+1)m_{n+\ell,k}(x),$$

hence

$$\begin{aligned}\big|\tilde{M}_{n,\alpha}(\phi_3;x)-\phi_3(x)\big| &\le \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\frac{n+k+\ell+1}{n+\ell+1}\Big(\frac{n+k+\ell+2}{n+k+1}-1\Big)+\mathcal{O}\Big(\frac{1}{n}\Big)\\ &= \sum_{k=0}^{\infty} m_{n+\ell,k}(x)\frac{n+k+\ell+1}{n+\ell+1}\cdot\frac{\ell+1}{n+k+1}+\mathcal{O}\Big(\frac{1}{n}\Big)=\mathcal{O}\Big(\frac{1}{n}\Big).\end{aligned}$$

Then

$$\Big|\int_u^1 \big(\tilde{M}_{n,\alpha}(\phi_3;x)-\phi_3(x)\big)\,dx\Big| \le \frac{C}{n}\int_u^1 dx \le \frac{C}{n}. \tag{2.28}$$

Now, from inequalities (2.22)–(2.28) it follows that

$$\Big\|\tilde{M}_{n,\alpha}\int_x^{(\cdot)}[\phi(\cdot)-\phi(u)]|(w\tilde{D}f)(u)|\,du\Big\|_1 \le \frac{C}{n}. \tag{2.29}$$

The estimate (2.17) is a consequence of (2.18), (2.19), (2.21), and (2.29).

Finally, the estimate (2.13) for the case $p=1$ follows from (2.14), (2.15) and (2.17).

The case $p=\infty$.

We proceed similarly to the case $p=1$: applying Holder's inequality for the smallest integer $\ge\alpha$, considering again the operator $\tilde{M}_{n,\alpha}$ and using the following estimation

$$\begin{aligned}&\tilde{M}_{n,\alpha}\Big(\int_x^{(\cdot)}[\phi(\cdot)-\phi(u)]|(w\tilde{D}f)(u)|\,du;x\Big)\\ &\quad\le \|w\tilde{D}f\|_\infty\,\tilde{M}_{n,\alpha}\Big(\int_x^{(\cdot)}[\phi(\cdot)-\phi(u)]\,du;x\Big)\\ &\quad\le x\big|\tilde{M}_{n,\alpha}(\ln t;x)-\ln x\big|\|w\tilde{D}f\|_\infty + (1-x)\Big|\tilde{M}_{n,\alpha}\Big(\frac{1}{1-t};x\Big)-\frac{1}{1-x}\Big|\|w\tilde{D}f\|_\infty\\ &\qquad + x\big|\tilde{M}_{n,\alpha}(\ln(1-t);x)-\ln(1-x)\big|\|w\tilde{D}f\|_\infty.\end{aligned}$$ □

For the proof of Theorem 2 we need a weighted variant of (2.8).

Lemma 8. *Let $1 < p < \infty$. Then, for all functions $f \in L_p(w)$ such that $\varphi D^2 f \in L_p(w)$, there exists a constant C such that the next inequality is true*

$$\|wD\varphi Df\|_p \le C\big(\|wf\|_p + \|w\varphi D^2 f\|_p\big).$$

Proof. The proof is analogous to the proof of [16, Lemma 3], using the obvious

$$|D\varphi(x)| = |(1-x)(1-3x)| < 2(1-x), \qquad 0 \le x < 1,$$

and $w(x) \sim w(1-2^{-k})$ for $x \in (1-2^{-k}, 1-2^{-k-1})$. □

3. PROOFS OF THEOREM 1 AND THEOREM 2

Proof of Theorem 1. We establish the direct inequality by means of a standard argument.

Let $1 \le p \le \infty$. For any $g \in W_p(w)$ such that $f - g \in L_p(w)$ we have, by virtue of (2.11) and Lemma 7,

$$\begin{aligned}\|w(f-\tilde{M}_n f)\|_p &\le \|w(f-g)\|_p + \|w(g-\tilde{M}_n g)\|_p + \|w\tilde{M}_n(f-g)\|_p\\ &\le 2\|w(f-g)\|_p + \frac{C}{n}\,\|w\tilde{D}g\|_p\\ &\le C\Big(\|w(f-g)\|_p + \frac{1}{n}\,\|w\tilde{D}g\|_p\Big).\end{aligned}$$

Taking the infimum on g we obtain the inequality (1.7) in the theorem. □

Proof of Theorem 2. For every $c \in \mathbb{R}$, by virtue of Lemma 8, we have

$$\begin{aligned}\|wD\varphi Dg\|_p &= \|wD\varphi D(g-c)\|_p\\ &\le C\big(\|w\varphi D^2(g-c)\|_p + \|w(g-c)\|_p\big)\\ &= C\big(\|w\varphi D^2 g\|_p + \|w(g-c)\|_p\big).\end{aligned}$$

Using the latter inequality and the obvious

$$\|w\tilde{D}g\|_p \le \|wD\varphi Dg\|_p + \|w\varphi D^2 g\|_p$$

we have for $t > 0$

$$\begin{aligned}&\|w(f-g)\|_p + t\|w\tilde{D}g\|_p\\ &\quad\le \|w(f-g)\|_p + t\|wD\varphi Dg\|_p + t\|w\varphi D^2 g\|_p\\ &\quad= \|w(f-g)\|_p + Ct\big(\|w\varphi D^2 g\|_p + \|w(g-c)\|_p\big) + t\|w\varphi D^2 g\|_p\\ &\quad= C\big(\|w(f-g)\|_p + t\|w\varphi D^2 g\|_p\big) + Ct\|w(g-f+f-c)\|_p\\ &\quad\le C\big(\|w(f-g)\|_p + t\|w\varphi D^2 g\|_p\big) + Ct\|w(g-f)\|_p + Ct\|w(f-c)\|_p\\ &\quad\le C\big(\|w(f-g)\|_p + t\|w\varphi D^2 g\|_p + t\|w(f-c)\|_p\big).\end{aligned}$$

By taking infimum over all functions $g \in W_p(w)$ and all real constants c we obtain the inequality

$$\tilde{K}_w(f,t)_p \le C \inf \left\{ \|w(f-g)\|_p + t\|w\varphi D^2 g\|_p \,:\, f-g \in L_p(w),\, g \in W_p(w) \right\} \\ + CtE_0(f).$$

To complete the proof in the case $\alpha \ge 0$, it remains to take into consideration that in the definition of $K_w(f,t)_p$ we can, equivalently, assume that g is in C^2 in a neighbourhood of 0 if $f \in L_p(w)$ (see [3, p. 110]).

To complete the proof for $\alpha < 0$, we will show that if $g, Dg \in AC_{loc}(0,1)$ and $wg, w\varphi D^2 g \in L_p[0,1)$, then

$$\lim_{x \to 1^-} \varphi(x) Dg(x) = 0.$$

To this end, we first apply [5, Lemma 1] to get $(1-x)^{\alpha+1} Dg(x) \in L_p[1/2,1)$.

Next, we use [8, Lemma 3.1(a)], transformed for a singularity at $x = 1$, with $G = \varphi Dg$ and $\gamma = \alpha - 1 < -1$ to derive

$$\lim_{x \to 1^-} G(x) = \lim_{x \to 1^-} \varphi(x) Dg(x) = 0. \qquad \square$$

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