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ПРИВЕЖДАНЕ НА ДЕФИНИЦИОННИТЕ ОБЛАСТИ НА ОСНОВНИТЕ ЕЛЕМЕНТАРНИ ФУНКЦИИ КЪМ ПРОИЗВОЛНО МАЛКИ ИНТЕРВАЛИ

ДИМИТЪР ШИШКОВ

Димитър Шишков. ПРИВЕДЕНИЕ ОБЛАСТЕЙ ОПРЕДЕЛЕНИЯ БАЗОВЫХ ЭЛЕМЕНТАРНЫХ ФУНКЦИЙ К ПРОИЗВОЛЬНО МАЛЫМ ИНТЕРВАЛАМ

Рассматриваются алгоритмы приведения уже стесненных неограниченных областей определения базовых элементарных функций системы $\{e^x, \ln x, \operatorname{tg} x, \operatorname{arc tg} x, \sin x, \cos x, \arcsin x\}$ к достаточно малым интервалам.

Dimitar Shishkov. REDUCTION OF THE DOMAINS OF BASIC ELEMENTARY FUNCTIONS TO ARBITRARY SMALL INTERVALS

Algorithms for reduction of the initially constraint unlimited domains of basic elementary functions of the system $\{e^x, \ln x, \operatorname{tg} x, \operatorname{arc tg} x, \sin x, \cos x, \arcsin x\}$ to sufficiently small intervals are treated.

Важна част на (не)апаратното математическо осигуряване на всяка универсална компютърна система (КС), както и на електронните калкулатори (ЕК) с повишени възможности, е пакетът елементарни функции (ЕФ): показателни; логаритмични; степенни; тригонометрични, хиперболични и обратните им.

Програмирането на пакета се предшества от следните етапи:

Първи етап. Избор на ЕФ, които ще бъдат вградени в дадена КС или за които ще бъдат направени стандартни програми (СП). При апаратна реализация най-често се избира система от базови ЕФ (БЕФ), чрез която се пресмятат останалите. Една такава система е $[e^x,\ \ln x,\ \sin x$ и $\operatorname{tg} x]$ [1]. При неапаратна реализация практически всички програмирани ЕФ са БЕФ, т. е. никоя не се пресмята чрез друга.

Втори етап. Избор на алгоритми за самостоятелно пресмятане на БЕФ с дадена, обикновено голяма и еднаква (където е възможно) за всички относителна точност $\varepsilon_{\text{отн}}$ в достатъчно малки интервали.

Трети етап. Избор на алгоритми за първоначално стесняване на неограничените дефиниционни области (ДО) на БЕФ (от ЕФ само $\operatorname{arc}\begin{bmatrix}\sin x\\ \cos x\end{bmatrix}$ и $\operatorname{arth} x$ имат ограничени ДО). $\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ означава x или y, или z.

Четвърти етап. Избор на алгоритми за допълнително привеждане на тези първоначално получени интервали към достатъчно малки, в които БЕФ ще се пресмятат съгласно резултатите от втория етап.

Тук се разглежда *само четвъртият етап* за ЕФ $f_j(x)$, $j = \overline{1, 7}$, от системата $[e^x,\ \ln x,\ \operatorname{tg} x,\ \operatorname{arctg} x,\ \sin x,\ \cos x$ и $\arcsin x]$, които ще наричаме основни (ОЕФ). При неапаратна реализация обикновено те са БЕФ, докато при апаратна — само част от тях. Резултатите от третия етап са взети от [1], където е разгледано пресмятането на ЕФ чрез верижни дроби. Алгоритмите от четвъртия етап в [1] са разработени на базата на резултатите оттук.

Предполага се, че стойностите на аргументите на ОЕФ и стойностите на последните съответно се задават и получават с полулогаритмичен запис и m-разрядна нормализирана мантиса, m — цяло, $m \geqq 2$, в p-ична бройна система, p — цяло, $p \geqq 2$. Също така времето за изпълнение на аритметичните инструкции (АИ) от тип умножение ($\times$, :, $\sqrt{\ }$) $t_\times$, $t_:$ и $t_{\sqrt{\ }}$ е значително по-голямо от това на тип събиране ($+$, $-$) t_+ и t_- например, както е при последователна аритметика.

Важността на четвъртия етап нарасна много с появата на КС с променливо m (например от серията МИР). Ще покажем, че при алгоритми от втория етап с еднаква степен на сложност, приблизително равносилна на пресмятане на ОЕФ с помощта на първите $n+1$ члена от степенното им развитие (n — фиксирано) и нарастваща с m точност $\varepsilon_{\text{отн}} < C'p^{-m}$ (която намалява като число с растенето на m), интервалът, в който трябва да се пресмятат ОЕФ, намалява. Избраните ОЕФ имат маклореново развитие (МЛР) $f_j(x) = \sum\limits_{i=0}^{\infty} f_j^{(i)}(0)\dfrac{x^i}{i!}$ с изключение на $\ln x$, която не е дефинирана за $x = 0$, но има удобно от изчислителен аспект (ИА) тейлърово развитие около точката 1. Поради това само сега, за удобство вместо $\ln x$ ще смятаме за ОЕФ $\ln(1+x)$, която има МЛР. Нека означим с $S_{j,n}(x) = \sum\limits_{i=0}^{n} f_j^{(i)}(0)\dfrac{x^i}{i!}$ парциалните суми от първите $n+1$ члена от МЛР на ОЕФ. За достатъчно голямо n $S_{j,n}(x) = f_j(x)$ с точност $\varepsilon_{\text{отн}}$. Лесно може да се покаже, че при $|x| < 1$ за всички $f_j(x)$ $\varepsilon_{\text{отн}} = \dfrac{|f_j(x) - s_{j,n}(x)|}{|f_j(x)|} < C''\dfrac{|x|^{l_1(n)}}{|f_j(x)|}$, където $l_1(n)$ е линейна функция на n, $C'' < 1$ (при $\arcsin x$ — за $n \geqq 4$) и $|f_j(x)| > \frac{1}{2}|x|$ с изключение на $e^x (e^x > e^{-1})$, т. е. $\varepsilon_{\text{отн}} < C'''|x|^{l_2(n)}$, $C''' < 1$ и $l_2(n) = \begin{bmatrix} n \\ n+1 \\ 2n+1 \\ 2n+2 \end{bmatrix}$. Тогава при фиксирани $\varepsilon_{\text{отн}}$ и n, и растящо m от $\varepsilon_{\text{отн}} < C'''|x|^{l_2(n)} < C'p^{-m}$ се получава, че $|x|$ трябва да намалява.

Също така придобиват важност алгоритми за стесняване на интервали като например централна хомотетия от вида $\frac{x}{n}$, n — цяло, които имат по-малко бързодействие (заедно с пресмятането на ОЕФ) от други, но с това неоценимо при променливо m свойство да не изискват съхраняването на константи, чийто брой и дължина на мантисата зависят от m. При променливо, но ограничено m трябва да се пазят константите за $m_{\max}$, които са максимално на брой и с максимална дължина на мантисата m_{max}, а това води до голяма загуба на оперативна или пасивна памет в сравнение с „обикновените" стойности $m \ll m_{\max}$. При голямо $m_{\max}$ константите за разумно избрано $m_1 < m_{\max}$ с дължина на мантисата m_1 трябва да се пазят в паметта, докато тези — повече на брой, с разрядност $m_{\max}$ — във външната памет. При неограничено m проблемът е алгоритмично нерешим, ако без да се съхраняват съответните константи, те не се получават по някакъв прост начин, например рекурентно, по време на всяко пресмятане на ОЕФ. Тези разсъждения имат смисъл, понеже са известни редица задачи — например от физика на плазмата, които се решават само чрез обработка на числа с дължина на мантисата десетки десетични разряди (с такива задачи са били тествани аритметиката и пресмятането на ЕФ с произволно m на КС от серията МИР).

Ще въведем някои понятия, които ще използваме по-нататък.

„Къси" числа ще наричаме тези с $\approx \frac{m}{5}$ или по-малко p-цифри, което изисква по-малко памет за съхраняването им. Освен това, като множители, с тях се извършва „късо" микропрограмно умножение в сравнение с обикновеното, средното време на което зависи от средновероятната сума $\frac{p}{2} + (m-1)\frac{p-1}{2} = 0{,}5(m(p-1)+1)$ на цифрите на нормализираната мантиса на множителя. Късите числа и късите микропрограмни аритметични операции имат важно значение при десетичната аритметика, особено когато тя се изпълнява чрез интерпретация.

Ще дефинираме индуктивно s-ти базов интервал (БИ) $B_{j,s} = [I'_{j,s}; I''_{j,s}]$ на ОЕФ $f_j(x)$ с ДО D_j^∞, ако е неограничена, и D_j, ако е ограничена. $B_{j,0} = D_j^\infty$, $B_{j,1} = D_j$. Тогава $B_{j,s+1} \subset B_{j,s}$ съответно за $s > 0$ и $s > 1$, като включването е строго. Ограничения интервал $B_{j,1}$, в който се трансформира D_j^∞ (по-точно $D_j^\infty \setminus B_{j,1}$) чрез специална за всяка ОЕФ (с D_j^∞) първа субституция, ще наричаме основен БИ (ОБИ) на функцията. Привеждането на D_j^∞ на ОЕФ в $B_{j,1}$ е разгледано в [1].

В някои случаи (вж. Т-метод) D_j ще се привежда в $B_{j,s}$ чрез s последователни смени на променливата на ЕФ, като в $B_{j,s}$ се привежда само $B_{j,s-1} \setminus B_{i,s}$, $s \geqq 1$. Стойността на s, съответна на последния (крайния) БИ(КБИ), в който $f_j(x)$ ще се пресмята по някакъв алгоритъм с дадена точност, ще бележим с f_j или само f при конкретна ЕФ. При фиксирано $B_{j,f}$ f зависи от закона, по който се определят $B_{j,s}$. Ще изследваме само случаите, когато $B_{j,1} \neq B_{j,f}$, т. е. $f \geqq 2$. С $\mathrm{tr}_j^{\max}$ ще бележим максималния брой последователни трансформации, чрез които D_j^{∞} се привежда в $B_{j,f}$. Броят $\mathrm{tr}_j^{\max}$ е f или понякога $f-1$ за $f \geqq 4$. Очевидно $\mathrm{tr}_j^{\max}$ при D_j (например при $\arcsin x$) е най-много $f-1$.

В други случаи (вж. С-метод) $B_{j,s} \setminus B_{j,s+1}$, $s = \overline{s_1, s_2}$, $1 \leqq s_1 < s_2 \leqq f-1$, $f \geqq 3$, се привеждат само с една, различна за всяко s трансформация, направо в B_{j,s_2+1}. Ако $s_1 = 1$, $s_2 = f-1$ и $f > 2$, то $\mathrm{tr}_j^{\max} = 2 < f$ при D_j^{∞}.

БИ $B_{j,s}$, $2 \leqq s \leqq f-1$, $f \geqq 3$, ще наричаме междинни (МБИ). С x_s ще бележим $x_s \in B_{j,s} \setminus B_{j,s+n}$, $s = \overline{0, f-1}$, $x_f \in B_{j,f}$, а $x_{s\setminus f} \in B_{j,s} \setminus B_{j,f}$.

Когато $f_j(x)$ е фиксирана, навсякъде индекса j ще заместваме с идентификатора на ЕФ, например $B_{\exp,1}$ е ОБИ на e^x, $B_{\ln,f}$ е КБИ на $\ln x$, а $B_{\mathrm{tg},s}$, $2 \leqq s \leqq f-1$, е МБИ на $\mathrm{tg}\, x$.

Нека разгледаме системата $[B_{j,s}]_{s=1}^{f}$, $f \geqq 2$, от вложените БИ $B_{j,s}$ (при $f=2$ не съществуват МБИ — $B_{j,1}$ е ОБИ, а $B_{j,2} = B_{j,f}$ е КБИ) на $f_j(x)$, $B_{j,s+1} \subset B_{j,s}$, $s = \overline{1, f-1}$. За краткост ще бележим $I_s = I_{j,s} = I''_{j,s}$, $I_s > I_{s+1}$, и ако $B_{j,s}$ са симетрични относно началото, то $I'_{j,s} = -I''_{j,s} = -I_s$. Само при $\ln x$ $I_s = I_{j,s} = I'_{\ln,s}$, $I''_{\ln,s} = I''_{\ln,f}$, $I_s < I_{s+1}$. $I_{j,1}$ (с изключение на $I_{\ln,1}$) не зависи от p и m, а само от ОЕФ чрез специалната първа субституция. $I_{j,f}$ зависи от m чрез $\varepsilon_{\text{отн}}$. МБИ се определят изобщо по еднообразен начин в зависимост от вида на избран$\left[\begin{smallmatrix}\text{ата}\\\text{ите}\end{smallmatrix}\right]$ субституци$\left[\begin{smallmatrix}\text{я}\\\text{и}\end{smallmatrix}\right]$ и $I_{j,f}$.

$[B_{j,s}]_{s=1}^{f}$ поражда върху положителната полуос системата от непресичащи се интервали (сегменти) $I^j = [I_s^j]_{s=1}^{f-1}$, $I_s^j = (I_{s+1}, I_s]$, като само $I_s^{\ln} = [I_s, I_{s+1})$. Тогава за всяко $x_{1\setminus f}$ може да се намери точно един индекс s, $1 \leqq s \leqq f-1$, такъв че $x_1 = x_s$, $|x_1| = |x_s| \in I_s^j$.

За трансформирането на ОБИ в КБИ на ОЕФ ще бъдат използвани различни трансформации:

— линейна:

— нехомогенна с ъглов коефициент 1 (транслация) при e^x, $\mathrm{tg}\, x$;

— хомогенна с нулев отрез (централна хомотетия, накратко хомотетия) и ъглов коефициент $k = \dfrac{1}{n} < 1$, n — цяло, при e^x, $\sin x$, $\cos x$ и $\mathrm{tg}\, x$, и k — дробно — при $\ln x$;

— дробно-линейна от вида $f_a(x) = \dfrac{ax-1}{x+a}$, $\left[\begin{smallmatrix}a\\x\end{smallmatrix}\right] > 0$, при $\mathrm{arc\,tg}\, x$;

— от вида $f(x) = 0{,}5(\sqrt{1+x} - \sqrt{1-x})$, $|x| \leqq 1$, при $\arcsin x$.

С изключение на хомотетия с $k = \dfrac{1}{n}$ това е възможно поради наличието на удобни от ИА псевдосъбирателни формули за съответните ЕФ:

$$e^x = e^{(x-C)+C} = e^C e^{x-C}; \quad \operatorname{tg} x = \frac{\operatorname{tg}(x-C)+\operatorname{tg} C}{1-\operatorname{tg}(x-C)\operatorname{tg} C};$$

$$\ln x = \ln Cx - \ln C; \quad \operatorname{arc\,tg} x = \operatorname{arc\,tg}\frac{1}{C} + \operatorname{arc\,tg}\frac{Cx-1}{x+C};$$

$$\arcsin x + \arcsin x = \arcsin 2x\sqrt{1-x^2} \quad \text{или, по-точно,}$$

$$\arcsin x = 2\arcsin 0{,}5(\sqrt{1+x} - \sqrt{1-x}).$$

При хомотетия с $k = \dfrac{1}{n}$ съответните ОЕФ се изразяват рационално чрез $f_j\left(\dfrac{x}{n}\right)$: $\left[\begin{smallmatrix}\sin x\\ \cos x\end{smallmatrix}\right]$ — чрез полиноми на $\left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right]\dfrac{x}{n}$, $\operatorname{tg} x$ — чрез дробно-линейна функция на $\operatorname{tg}\dfrac{x}{n}$, а $e^x = \left(e^{\frac{x}{n}}\right)^n$.

Ще използваме три метода за определяне на БИ $B_{j,s}$, $s = \overline{2, f-1}$, $f \geqq 3$.

При първия **С-метод** $B_{j,s} \setminus B_{j,s+1}$, $s = \overline{1, f-1}$, се трансформират чрез една, различна за всяко s, но еднотипна трансформация в $B_{j,f}$. Ще казваме, че С-методът е приложен за $B_{j,1}$ и $B_{j,f}$, като $B_{j,1} \setminus B_{j,f}$ е Сегментиран чрез системата I^j, $\mathrm{tr}_j^{\max} = 2$.

Ако се използва транслация (e^x, $\operatorname{tg} x$), дължината $d_{j,s}$ на I_s^j, $s = \overline{1, f-1}$, е изобщо $2I_{j,f} = d_{j,f}$. Ако $\dfrac{|I_{j,1} - I_{j,f}|}{d_{j,f}}$ не е цяло, то или $d_{j,1} < d_{j,f}$, или $d_{j,f-1} < d_{j,f}$, като дали I_1^j или I_{f-1}^j е с $d < d_{j,f}$, е без значение от ИА.

Ако се използва хомотетия и $f = 2$, то $B_{j,1} \setminus B_{j,f}$ се привежда в $B_{j,f}$ чрез субституцията $x_f = \dfrac{x_1}{n}$, $n \geqq \dfrac{I_{j,1}}{I_{j,f}} = q_{j,1}$, n — цяло (e^x, $\sin x$, $\cos x$, $\operatorname{tg} x$) и $x_f = q_2 x_1$, $q_{\ln,2} = \dfrac{I'_{\ln,f}}{I'_{\ln,1}} = q_2$ при $\ln x$ ($f = 2$, ако $I'_{\ln,2} = \sqrt{I'_{\ln,1} I''_{\ln,2}}$).

При хомотетия и $f > 2$ $I_{j,s}$ се определят при $s = \overline{f-1, 1}$, за да може при дробно q_2 по-малък сегмент (I_1^j) да се трансформира с най-лошата от ИА субституция $x_f = \dfrac{x_1}{n_{\max}}$. Една възможна (но не оптимална) система е тази с $I_{j,s} = (f-s+1)I_{j,f}$, $s = \overline{f-1, 2}$, $f =]q_1[+1$. $]x[$ е таван за x — най-малкото цяло число (НМЦЧ), по-голямо или равно на x. $]x[= [x]$ за x — цяло, и $]x[= [x]+1$ за x — дробно. Тогава $B_{j,s} \setminus B_{j,s+1}$, $s = \overline{1, f-1}$, се трансформират в $B_{j,f}$ чрез $x_f = \dfrac{x_s}{f-s+1}$.

В някои случаи С-методът има следния недостатък. При растене на m и/или увеличаване $\varepsilon_{\text{отн}}$ $d_{j,f}$ намаляват, f_j нарастват, а с това и броят на МБИ и на сегментите I_s^j. При транслация (или $\ln x$) това води до нарастване на максималния брой на проверките (по една проверка) за установяване индекса на $x \in B_{j,1}$, т. е. принадлежността на x към някой от сегментите I_s^j, както и до увеличаване на броя на константите за съхранение (по три константи на сегмент) — $I_{j,s}$, $d_{j,s} = I_{j,s} - I_{j,f}$ и $f_j(d_{j,s}), s = \overline{1, f-1}$.

При втория **Т–метод** $B_{j,s} \setminus B_{j,s+1}$ се трансформират в $B_{j,s+1}$, $s = \overline{0, f-1}$, т. е. $B_{j,0}$ (или $B_{j,1}$) се свива Телескопически в $B_{j,f}$ чрез $\mathrm{tr}_j^{\max} = f$ трансформации. Тук е съществено определянето на $I_{j,s}$ именно при $s = \overline{f-1, 1}$, понеже при $f \geqq 4$ в някои случаи (зависи от $I_{j,s}$ и $I_{j,f}$) $B_{j,1} \setminus B_{j,2}$ може да се трансформира направо в $B_{j,s}$, $s \geqq 3$ (а не в $B_{j,2}$), при което $\mathrm{tr}_j^{\max} = f - 1$.

Ако се използва транслация и $f > 2$, $I_{j,s} = 3^{f-s} I_{j,f}$, $s = \overline{f-1, 2}$, f се определя като НМЦЧ, за което $I_{j,s} \leqq 3^{f-1} I_{j,f}$, $f = \left]\dfrac{\ln q_1}{\ln 3}\right[+ 1$. $B_{j,1} \setminus B_{j,2}$ се трансформира вB_{j,s_1}, s_1 се определя като НМЦЧ, удовлетворяващо $I_{j,1} - I_{j,2} \leqq 2.3^{f-s} I_{j,f}$, $s_1 = \left]\ln\left(\dfrac{I_{j,1} - I_{j,2}}{2I_{j,f}}\right) / \ln 3\right[+ 1$. Ако при $f \geqq 4$ $s_1 \geqq 3$, то $B_{j,1} \setminus B_{j,2}$ се трансформира поне в $B_{j,3}$ и $\mathrm{tr}_j^{\max} = f - 1$. Ако $s_1 = 2$, то $\mathrm{tr}_j^{\max} = f$.

Когато при голямо f не е изгодно да се приложи направо С-методът, може да се използва трети **К-метод**, Комбинация на първите два. При него С-методът се прилага не за $B_{j,1}$ и $B_{j,f}$, а за $B_{j,1}$ и някой МБИ (сега стъпката на сегментиране не е $d_{j,f}$, а дължината на МБИ), след това отново по отношение на този МБИ и друг, вложен в него, и т. н. Тогава при D_j^∞ $\mathrm{tr}_j^{\max}$ е равен на броя на тези МБИ, увеличен с две.

Тук се разглежда привеждането на $B_{j,1}$ в $B_{j,f}$ (при $\mathrm{arc\,tg}\, x$ — на D_j^∞ в $B_{j,f}$). Привеждането на D_j^∞ в $B_{j,1}$ е разгледано в [1].

1. $y = e^{x_1}$, $x_1 \in B_{\exp,1} = \left[-\dfrac{\ln p}{2}, \dfrac{\ln p}{2}\right]$.

1.1. Ако приложим С-метода с транслация, $f = \left]\dfrac{I_1 - I_f}{2I_f}\right[+ 1$ и за системата сегменти $I^{\exp}$ при $f \geqq 3$ се получава $d_s = d = I_s - I_{s+1} = 2I_f$, $I_s = I_1 - (s-1)d$, $s = \overline{2, f-1}$ и $I_1 - I_2 \leqq d$. Ако $f = 2$, то $I^{\exp} = I_1^{\exp}$ (тогава $I_1 \leqq 3I_f = 3I_2$). Нека определим константите $c_s > 1$ от $\ln c_s = I_s - I_f = I_1 - (2s-1)I_f > 0$, $s = \overline{1, f-1}$. За всяко x_1 $\exists s$ — единствено, $s = f - \left]\dfrac{x_1 - I_f}{2I_f}\right[$, $1 \leqq s \leqq f$, така че $x_1 = x_s$, т. е. $x_1 = x_s = \mathrm{sgn}\, x_1(I_s - 2tI_f)$, $0 \leqq t < 1$. Ако за $x_1 \notin B_{\exp,f}$ извършим субституцията $x_f = x_1 - \mathrm{sgn}\, x_1 \ln c_s$, $x_f = \mathrm{sgn}\, x_1(I_s - 2tI_f) - \mathrm{sgn}\, x_1(I_s - I_f) = \mathrm{sgn}\, x_1(I_f - 2tI_f)$, т. е. наистина $x_f \in B_{\exp,f}$, $\mathrm{tr}^{\max} = 2$, $e^{x_1} = e^{x_f + \mathrm{sgn}\, x_1 \ln c_s}$ и $e^{x_1} = c_s^{\mathrm{sgn}\, x_1} e^{x_f}$ (общо със субституцията 1+, 1× късо).

Желателно е c_s и c_s^{-1} да са къси за даденото p. Затова практически се избира късо $c_s^* \gtrapprox c_s$, а $I_s^{\exp}$ съответно се намалява, като I_{s+1} се заменя с

$I^*_{s+1} \gtrapprox I_{s+1}$. Когато e^{x_f} се пресмята чрез дроб, достатъчно е само едното от c_s и c_s^{-1} да е късо, за да има винаги късо умножение. Това следва от

$$e^{x_f} = \frac{A}{B}, \quad e^{x_1} = c_s^{\operatorname{sgn} x_1} e^{x_f} = \left[\begin{array}{ll} (c_sA)/B = A(c_s^{-1}B), & x_1 > I_f > 0, \\ A/(c_sB) = (c_s^{-1}A)/B, & x_1 < -I_f < 0. \end{array}\right.$$

Пример. За $p = m = 10$ $f = 2$, $I_f = I_2 = 0{,}46$, $\ln c_1 = I_1 - I_2 = \frac{\ln 10}{2} - 0{,}46 \approx 0{,}691$. Избираме $c_1^* = 2 > e^{0{,}691}$, $\ln c_1^* = \ln 2 \approx 0{,}693$, $c_1^{*-1} = 0{,}5$.

$f-1$ зависи от $\left(\frac{\ln p}{2} - I_f\right)/2I_f$. Понеже I_f не зависи от p, $f-1$ расте с p, а при фиксирано p — с m, понеже тогава I_f и стъпката на сегментирането d намаляват.

1.2. Прилагането на Т-метода с транслация не се различава от описаното във въведението, съчетано с резултатите от т. 1.1. Сега $\ln c_s = I_s - I_{s+1} = d_{s+1} = 2.3^{f-s-1}I_f$.

1.3. Ако f е много голямо, може да се използва К-методът с транслация, като С-методът се приложи за $B_{\exp,1}$ и един МБИ, например $B_{\exp,f-1} = [-3I_f, 3I_f]$. Сега сегментите на $B_{\exp,1} \setminus B_{\exp,f-1}$ ще имат дължина $6I_f$, а $B_{\exp,f-1} \setminus B_{\exp,f}$ се трансформира в $B_{\exp,f}$ чрез една транслация. $\mathrm{tr}^{\max} = 3$. При много малко I_f К-методът се прилага за няколко МБИ.

1.4. Прилагането на С-метода с хомотетия при $f = 2$ е съгласно въведението. Извършва се субституцията $x_f = x_2 = kx_1$, $k = \frac{1}{n}$, $n \geqq$ НМЦЧ, за което $\left|\frac{x_1}{n}\right| \leqq I_f$, $n \geqq n_{\min} =]q_1[= \left]\frac{\ln p}{2I_f}\right[$, $e^{x_1} = (e^{x_2})^n$.

Този алгоритъм изисква съхраняването само на $\frac{1}{n}$. Недостатък на метода е намаляването на бързодействието в сравнение с основния метод с транслация — изисква се $\times\frac{1}{n}$, което не винаги е късо, и повдигане на степен n. Съществува оптимален алгоритъм за пресмятане на x^n като псевдоадитивна функция (ПАФ) [2]. Една функция е ПАФ спрямо целочисления си първи аргумент, ако $f(n_1+n_2, x) = g(f(n_1, x), f(n_2, x))$. Оптималният алгоритъм за пресмятане на $f(n, x)$ чрез $f(1, x)$ се базира на тази зависимост и двоичното представяне на n. Функциите $x^n : x^{n_1+n_2} = x^{n_1}x^{n_2}$, $e^{nx} : e^{nx} = (e^x)^n$ и $\operatorname{tg} nx$ са ПАФ спрямо n. При ПАФ най-изгоден е случаят $n = 2^r$, $2^{r-1} < n_{\min} \leqq 2^r$, при който x^n се пресмята за $r_\times$ и $1 \times \frac{1}{n}$, което може да е късо. Случаят $n = n_{\min}$, $2^{r-1} < n < 2^r$, е по-неизгоден, понеже се изискват не по-малко от $r_\times$ и $1 \times \frac{1}{n}$.

Пресмятането на e^x чрез $\left(e^{\frac{x}{n}}\right)^n$ е най-благоприятно при $\left(\frac{\ln p}{2}\right)_{\min}$ за $p = 2$ и $n = 2$. Тогава $\frac{\ln p}{4} = 0{,}25\ln 2 \approx 0{,}173$ ($n = 2$, ако $0{,}173 \leqq I_f$). За пресмятането на e^x се изисква $1\times$ и намаляване с 1 на порядъка на x (: 2).

Сравнение. При $p = m = 10$ С-методът с транслация изисква 2+ (1— — за проверката $|x_1| \in B_{\exp,f}$, и 1+ с $\ln c_s^{\pm 1}$) и $1\times$ късо с $c_s^{\pm 1}$, докато при С-метода с хомотетия и $f = 2\frac{\ln p}{2} \approx 1{,}15$, $I_f = 0{,}46$ и $n \geqq 3$. За $n = 3$ са необходими $3\times$, а за $n = 4$ — $2\times$ и $1\times$ късо.

1.5. При прилагане на С-метода с хомотетия при $f = 2$ $I_s = 2^{f-s} I_f$, $s = \overline{f-1, 2}$, f се определя като НМЦЧ, за което $I_1 \leqq 2^{f-1} I_f$, $f = \left]\frac{\ln g_1}{\ln 2}\right[+ 1$. За всяко $x_{1\backslash f} = x_s$ се прилага субституцията $x_f = \frac{x_s}{2^{f-s}}$, $1 \leqq s \leqq f-1$, но първоначално в цикъл трябва да се намери съответното s — извършват се проверките $x_1 \leqq I_s^* \approx I_s$, I_s^* са къси, $2 \leqq s \leqq f$, като се почне от $s = f$. Броят на проверките с изваждане, увеличен с 1, дава $f - s$: (: 2^{f-s} е особено изгодно при $p = 2$). Тогава са необходими $(f - s)\times$ за получаването на $e^{x_{1\backslash f}}$ чрез e^{x_f}. За да не се пазят I_s^* за проверките, може да се използва следният рекурентен алгоритъм: $X_r \leqq I_f$, $X_{r+1} = 0{,}5X_r$ ($\times 0{,}5$ са къси, особено при $p = 2$), $r \geqq 0$, $X_0 = x_{1\backslash f}$. При първото r_1, което удовлетворява неравенството, процесът се прекратява, $X_{r_1} = x_f = \frac{x_{1\backslash f}}{2^{r_1}}$ и $e^{x_1} = (e^{x_f})^{2^{r_1}}$ за $(r_1)\times$.

1.6. При променливо m транслацията е практически невъзможна, понеже $\ln c_s$, а оттам и c_s зависят от I_f, което намалява с растенето на m. Следователно за групи от последователни стойности на m, за които I_f е едно и също, трябва предварително да бъдат изчислени и запазени по три масива от числата $\ln c_s$, c_s и I_s с дължина съответното максимално m за групата, което е недопустимо.

Хомотетията е особено удобна, но с по-малко бързодействие поради $r\times$ при повдигането на степен $n = 2^r$.

2. $\boldsymbol{y = \ln x_1}$, $x_1 \in B_{\ln,1} = \left[\frac{1}{p}, I''_{\ln,f}\right]$.

От МЛР на $\ln(1+x)$ се вижда, че $\ln(1 \pm x) \approx \pm x \approx \ln(1 \mp x)$, т. е. за малки $|x|$ $\ln(1+x)$ е почти нечетна, следователно за стойности на x, близки и симетрични около 1, сложността на алгоритъма за пресмятане на $\ln x$ е почти еднаква. Поради това $B_{\ln,f}$ могат да се избират приблизително симетрично около 1 с отказ от традиционното $I''_f = 1$. При това, понеже $\ln x$ е по-стръмна за $x < 1$, ако $\ln(1-x)$ и $\ln(1+x)$ се пресмятат чрез някакъв алгоритъм, точността, с която се получава $\ln(1+x)$, не е по-малка от тази на $\ln(1-x)$.

Да припомним, че $I_s = I'_s$, $I_s < I_{s+1}$, $I''_s = I''_f$, $s = \overline{1,\ f-1}$, $q_2 = \dfrac{I_f}{I_1}$. При $\ln x$ транслация не може да се приложи, понеже не е псевдосъбстрактна функция (ПСФ). ПСФ $f(x)$ е тази, за която $f(x_1 - x_2) = g(f(x_1),\ f(x_2))$.

2.1. Ако приложим С-метода с хомотетия, системата $I^{\ln}$ се определя заедно с константите l_s от: $I_s l_s = I_f$, $s = \overline{1,\ f-1}$, $f \geqq 2$, и $I_{s+1} l_s = I''_f$, $s = \overline{1,\ f-2}$, $f \geqq 3$.

Ако означим $q_3 = \dfrac{I''_f}{I_f}$, тогава $l_s = \dfrac{I''_f}{I_{s+1}}$ и $I_{s+1} = q_3 I_s$, откъдето индуктивно се намира $I_{s+1} = q_3^s I_1$, $s = \overline{1,\ f-2}$, $f \geqq 3$, и $l_s = q_2/q_3^{s-1}$, $s = \overline{1,\ f-1}$, $f \geqq 2$, или (за рекурентно пресмятане) $l_{s+1} = \dfrac{l_s}{q_3}$, $l_1 = q_2$ и $I_{s+1} = q_3 I_s$, $s = \overline{1,\ f-2}$, $f \geqq 3$. f е НМЦЧ, за което $I_f \leqq q_3^{f-1} I_1$, $f =]q_4[+1$, $q_4 = \dfrac{\ln q_2}{\ln q_3}$.

За всяко x_1 $\exists s$ — единствено, $s = \left]\dfrac{\ln(x_1/I_1)}{\ln q_3}\right[+ 1$, $1 \leqq s \leqq f$, така че $x_1 = x_s$, т. е. $x_1 = x_s = t I_s$, $1 \leqq t < \dfrac{I_{s+1}}{I_s} = q_3$. Ако за x_s, $s < f$, извършим субституцията $x_f = l_s x_s$, $x_f = l_s(t I_s) = t(l_s I_s) = t I_f$, т. е. наистина $x_f \in B_{\ln,f}$. $\mathrm{tr}^{\max} = 2$. $\ln x_f = \ln l_s x_1$ и $\ln x_1 = \ln x_f - \ln l_s$ (общо $1+$, $1\times$ късо).

На пръв поглед дори е вредно стойности x_1, далечни от 1, да се трансформират в близки до него, поради загуба на точност от изваждане ($\ln x = \ln(1 + (x-1))$ се пресмята от МЛР на $\ln(1+x)$ чрез $x - 1$). Но това е илюзия.

Пример. За $x_1 = \dfrac{1}{2} + \varepsilon$, $\varepsilon = p^{-m}$, $\ln x_1 \approx \ln\dfrac{1}{2} + \varepsilon' = -\ln 2 + \varepsilon'$, $\varepsilon' \approx \varepsilon$, а
$$\ln x_1 = \ln\frac{2x_1}{2} = \ln 2\left(\frac{1}{2} + \varepsilon\right) - \ln 2 = -\ln 2 + \ln(1 + 2\varepsilon) \approx -\ln 2 + 2\varepsilon,\ 2\varepsilon \approx \varepsilon',$$
т. е. дали $\ln x$ ще се пресмята за $x = 0{,}5 + p^{-m}$ директно или чрез $\ln 2x - \ln 2$ е все едно от гледна точка на грешката от пресмятане.

Броят на сегментите $f - 1$ зависи от $q_4 = \dfrac{\ln p \ln I_f}{\ln(I''_f/I_f)}$. Понеже I_f и I''_f не зависят от p, $f - 1$ расте с p, а при фиксирано p — с m, тъй като тогава $\begin{bmatrix} I''_f \\ I_f \end{bmatrix} \to 1$ $\begin{bmatrix}\text{намалявайки} \\ \text{растейки}\end{bmatrix}$, а $\dfrac{I''_f}{I_f}$ намалява. При $p = m = 10$ $I_f = 0{,}8$, $I''_f = 1$, $f - 1 = 10$, а при $p = 16$, $m = 10$, $I_1 = 1/16$, $I_f = 0{,}8$ и $I''_f = 1$, т. е. $f - 1 = 12$. Това означава, че трябва да се съхраняват три масива от по $f - 1$ константи: $\ln l_s$ (дълги), l_s и I_s, $s = \overline{2,\ f}$ (числата на последните два масива могат да бъдат избрани къси). Чрез I_s с най-много $f-1$ проверки с изваждане се намира индексът s на x_1. При $I''_f > 1$ $f - 1$ е по-малко, отколкото при $I''_f = 1$.

И така С-методът с транслация е с по-добро бързодействие (намаляващо при $x_1 \to I_f$ с проверки $s = \overline{2,\ f}$ и $x_1 \to I_1$ — при $s = \overline{f,\ 2}$), но при големи p и m изисква съхраняването на голям брой константи.

2.2. Ако приложим Т-метода с хомотетия, системата $I^{\ln}$ се определя заедно с l_s от: $I_s l_s = I_{s+1}$, $s = \overline{f-1, 2}$, $f \geqq 3$, и $I_{s+1} l_s = I''_f$, $s = \overline{f-1, 1}$, $f \geqq 2$. Оттук $l_s = \dfrac{I''_f}{I_{s+1}}$, $s = \overline{1, f-1}$, $f \geqq 2$, и $I_s = \dfrac{I^2_{s+1}}{I''_f}$, $s = \overline{2, f-1}$, $f \geqq 3$, окончателно $l_s = q_3 \uparrow 2^{f-s-1}$, $s = \overline{1, f-1}$, $f \geqq 2$, и $I_s = I''_f/(q_3 \uparrow 2^{f-s})$, $s = \overline{2, f-1}$, $f \geqq 3$, или (за рекурентно пресмятане) $l_{s-1} = l_s^2$, $l_{f-1} = q_3$, $s = \overline{f-1, 2}$, $f \geqq 3$, и $I_{s-1} = \dfrac{I_s}{l_{s-1}}$, $s = \overline{f, 3}$, $f \geqq 3$. f е НМЦЧ, за което $I''_f/(q_3 \uparrow 2^{f-1}) \leqq I_1$, $f = \left]\dfrac{\ln(q_4+1)}{\ln 2}\right[+ 1$. Сега I_1 се трансформира в $I_1(q_3 \uparrow 2^{f-2}) \geqq I_2 = I''_f/(q_3 \uparrow 2^{f-2})$. Ако при $f \geqq 4$ $I_1(q_3 \uparrow 2^{f-2}) \geqq I_3 = I''_f/(q_3 \uparrow 2^{f-3})$, то $\mathrm{tr}^{\max} = f-1$, в противен случай $\mathrm{tr}^{\max} = f$. „$\uparrow$“ е програмисткото означение за степен.

Пример. За $p = m = 10$, $I_1 = 0{,}1$, $I_f = 0{,}8$, $I''_f = 1{,}25$, f се получава 4. $I_1 = 0{,}1$ $(l_1 = 0{,}59)$ $I_2 = 0{,}21$ $(l_2 = 2{,}43)$ $I_3 = 0{,}51$ $(l_3 = 1{,}56)$ $I_4 = 0{,}8$. Ако изберем къси числа: $I_1 = 0{,}1$ $(l_1 = 6)$ $I_2 = 0{,}2$ $(l_2 = 2{,}5)$ $I_3 = 0{,}5$ $(l_3 = 1{,}6)$ $I_4 = 0{,}8$.

Разсъжденията за $f-1$ от т. 2.1 могат да се повторят дословно и тук. За сравнение на f_{T} (при Т-метода) и f_{C} (при С-метода) трябва да се сравнят $\dfrac{\ln(1+x)}{\ln 2}$ и x за $\overline{x} = q_4$. $q_4 = \dfrac{\ln q_2}{\ln q_3} \geqq 0$, понеже $\ln q_2 = 0$ при $I_f = I_1$, и $\overline{x} \xrightarrow[m\to+\infty]{} +\infty$, понеже $q_3 \xrightarrow[m\to+\infty]{} 1$ при $\left[\dfrac{I''_f}{I_f}\right] \xrightarrow[m\to+\infty]{} 1$. Да изследваме $g_0(x) = x - \dfrac{\ln(1+x)}{\ln 2}$. $g'_0(x) > 0$ за $x > x_0 = \dfrac{1-\ln 2}{\ln 2} \lessapprox 0{,}5 < 1$, тогава $g_0(x) > g(1) = 0$. И така при $\overline{x} > 1$, т. е. $q_2 > q_3$ и $I_f > \sqrt{I_1 I''_f}$, $\dfrac{\ln(1+\overline{x})}{\ln 2} < \overline{x}$. Това не е достатъчно, за да твърдим, че $f_{\mathrm{T}} < f_{\mathrm{C}}$, понеже $]x[$ е стъпаловидна функция и може $]x_1[=]x_2[$ за близки $x_1 < x_2$ и $]x_1[<]x_2[$ при по-силно различаващи се.

Съвсем сигурно $f_{\mathrm{T}} + (k-2) < f_{\mathrm{C}}$, $k \geqq 2$, ако $\dfrac{\ln(1+\overline{x})}{\ln 2} \leqq \overline{x} - (k-1)$. Да разгледаме $g_{k-1}(x) = x - k + 1 - \dfrac{\ln(1+x)}{\ln 2}$, $g'_{k-1}(x) = g'_0(x)$ и $g'_{k-1}(x) \geqq 0$ при $x \geqq 1$. Нека $x_k = 2^k - 1 \geqq 3$, $k \geqq 2$. Тогава $g_{k-1}(x) \geqq g_{k-1}(x_k) = 2^k - 2k \geqq 0$. И така при $x \geqq x_k \geqq 3$, $k \geqq 2$, $f_{\mathrm{T}} + (k-2) < f_{\mathrm{C}}$. В частност $f_{\mathrm{T}} < f_{\mathrm{C}}$ при $x \geqq x_2 = 3$, т. е. $q_2 \geqq q_3^3$ и $I_f \geqq \sqrt[4]{I_1 I''_f}$. Това е достатъчно, но не и необходимо условие, понеже $f_{\mathrm{T}} < f_{\mathrm{C}}$ дори и за някои x, $1 < x < 3$.

И така при Т-метода са необходими по-малко константи за съхранение и проверки за намиране на индекса s на x_1, но изобщо повече трансформации за трансформирането на x_1 в x_f.

2.3. При Т-метода с хомотетия и променливо m

$$\begin{bmatrix} I''_f \\ I_f \end{bmatrix} \xrightarrow[m\to+\infty]{} 1 \begin{bmatrix} \text{намалявайки} \\ \text{растейки} \end{bmatrix},$$

а $f \xrightarrow[m\to+\infty]{} +\infty$. Тук I_s не трябва да се намират от I_{f-1} към I_1, понеже I_f се мени с m и не може да бъде начало за рекурентно намиране на сегментите при произволно m. Тъй като I_1 зависи от p, но не и от m, I_1 се избира за такова начало. Тогава може да се приложи алгоритъм, изискващ съхраняването на пет масива от по $f_{\max} - 1$ константи ($f_{\max}$ съответства на $m_{\max}$) с дължина на мантисата $m_{\max}$: горните граници на интервалите за m, в които I_f е едно и също; съответните I_f; I_s, l_s и $\ln l_s$, $s = \overline{1, f_{\max}}$, като последните три масива зависят само от $m_{\max}$ и то чрез дължината си. Само $\ln l_s$ са дълги. Вместо $I''_s = I''_f$, което се мени с m, нека $I_s + I''_s = 2$, $I''_s = 2 - I_s$, т. е. $B_{\ln,s}$ са симетрични относно 1.

Нека определим I_s и l_s от $I_s l_s = I_{s+1}$ и $I_{s+1} l_s = I''_{s+1}$, $s = \overline{1, f-1}$ (полученото по този начин I_f е изобщо по-голямо от зададеното, но последното участва в проверките). Тогава $l_s = \dfrac{I_{s+1}}{I_s}$, $\dfrac{I^2_{s+1}}{I_s} = 2 - I_{s+1}$, $I^2_{s+1} + I_s I_{s+1} - 2I_s = 0$ и $I_{s+1} = \dfrac{1}{2}(\sqrt{I_s^2 + 8I_s} - I_s)$ — избираме знак плюс пред корена, иначе $I_{s+1} < 0$.

Ще докажем индуктивно, че $I_s < 1$. $I_1 = \dfrac{1}{p} < 1$. От допускането, че $I_s < 1$ при $s > 1$, следва $I_{s+1} < 1$ като равносилно на $\sqrt{I_s^2 + 8I_s} < I_s + 2$, а то — на $I_s < 1$. От това следва, че $I_s < I_{s+1}$, защото от допускане на противното $I_s \geqq I_{s+1}$ се получава чрез преобразувания невярното $I_s \geqq 1$. Тогава $l_s = \dfrac{I_{s+1}}{I_s} > 1$, $s \geqq 1$. Понеже редицата с общ член I_s е монотонно растяща и ограничена, тя е сходяща заедно с подредицата си с общ член I_{s+1} и след граничен преход в израза за I_{s+1} (чрез I_s) се получава: $\lim\limits_{s\to+\infty} I_s = 1$.

$$l_s = \frac{\sqrt{I_s^2 + 8I_s} - I_s}{2I_s} = \frac{1}{2}\sqrt{\frac{8}{I_s} + 1} - \frac{1}{2} \xrightarrow[s\to+\infty]{} 1,\ I_s = \frac{2}{l_s^2 + l_s},\ l_s = \frac{l_s^2 + l_s}{l_{s+1}^2 + l_{s+1}}$$ и

окончателно $l_{s+1} = \sqrt{l_s + 1{,}25} - 0{,}5$, като $l_1 = \dfrac{1}{2}(\sqrt{8p+1} - 1)$.

Ще докажем, че с l_s $B_s \setminus B_{s+1} = [I_s, I_{s+1}) \cup (I''_{s+1}, I''_s]$ се трансформира в B_{s+1}, $s = \overline{1, f-1}$. $[I_s, I_{s+1}) \xrightarrow{\times l_s} [I_{s+1}, I''_{s+1})$ поради основните равенства за l_s, I_s, I_{s+1} и I''_{s+1}. $(I''_{s+1}, I''_s] \xrightarrow{:l_s} (I_{s+1}, \dfrac{I''_s}{l_s}]$. Лесно се вижда, че $I''_{s+1} > \dfrac{I''_s}{l_s}$. Наистина след изразяване на I''_s и I''_{s+1} чрез l_{s-1} и l_s и заместване на l_{s-1} с $l_s^2 + l_s - 1$, което се намира от израза за l_s (чрез l_{s-1}), се получава вярното неравенство $(l_s - 1)^2(l_s + 1) > 0$, понеже $l_s > 1$. Нещо повече, оттук следва, че заедно с $[l_s, I_s, I''_s] \to 1$, $I''_{s+1} - \frac{I''_s}{l_s} \to 0$. И така $(I_{s+1}, \frac{I''_s}{l_s}] \subset (I_{s+1}, I''_{s+1}]$ и $B_s \setminus B_{s+1} \xrightarrow{l_s} [I_{s+1}, I''_{s+1}) = B_{s+1} \setminus I''_{s+1} \subset B_{s+1}$.

Ако е недопустимо съхраняването на петте масива, I_s и l_s могат да се намират рекурентно $\left(I_{s+1} = I_s l_s,\ l_{s+1} = \frac{I_{s+1}}{I_s}\right)$, но това намалява бързодействието не само поради пресмятането им, но и по следната причина. Ако се използват пет масива и x_s се трансформира не в x_{s+1}, а в x_{s_1}, $s_1 > s+1$, с проверка може да се намери l_{s_1} и процесът да продължи. Ако I_s и l_s не се съхраняват, за всяко x_1 те трябва да се пресмятат последователно до индекса s, за който $x_s \to x_f$ само с една трансформация. Това значи, че за $x \gtreqqless I_s$, $s = \overline{1, f-1}$, трябва да се пресмятат всичките I_s и l_s, $s = \overline{1, f-1}$. Ако при това се изисква рекурентно да бъдат пресмятани къси I_s^* и l_s^* (като се пазят само „дългите“ $\ln l_s^*$), това трябва да става по един и същ начин от резултатите, получавани рекурентно.

При променливо m първоначално се намира съответното му I_f и се използват някои от началните $f-1$ константи I_s, l_s и $\ln l_s$ с не повече от m цифри на мантисата, които се получават например чрез отрязване до m-ия разряд на константите, зададени с $m_{\max}$ цифри (I_s и l_s се пазят без порядък). Понеже при $\frac{1}{p} \leqq x < 1$ p-порядъкът на x е нула, а при $1 \leqq x \leqq 2 - \frac{1}{p}$ — единица, лесно се определя в коя половина се намира x, оттам чрез сравнение съответно с I_s или $I''_s = 2 - I_s$ (I''_s се пресмятат динамично) се определят съответните на дадения етап s и l_s и $x = x_s$ се умножава или дели с l_s в зависимост от това, дали $x_s < 1$ или $x_s > 1$. Този „макроалгоритъм“ за стесняване на $\left[\frac{1}{p}, 2 - \frac{1}{p}\right]$ трябва да се детайлизира за всяка конкретна КС с променливо m.

3. $\boldsymbol{y = \operatorname{tg} x_1}$, $x_1 \in B_{\mathrm{tg},1} = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

3.1. С-методът с транслация може да се приложи съвсем аналогично на това в т. 1.1 — трябва да се замени $\ln c_s$ с t_s, а c_s — с $\operatorname{tg} t_s$. Ако извършим субституцията $x_f = x_1 - (\operatorname{sgn} x_1) t_s$, $x_f \in B_{\mathrm{tg},f}$, $\mathrm{tr}^{\max} = 2$ и

$$\operatorname{tg} x_1 = \frac{\operatorname{tg} x_f + (\operatorname{sgn} x_1) \operatorname{tg} t_s}{1 - (\operatorname{sgn} x_1) \operatorname{tg} x_f \operatorname{tg} t_s} \quad (\text{общо} \quad 1:,\ 1\times \ \text{и}\ 3+).$$

t_s могат да се заменят с $t_s^* \approx t_s$, така подбрани, че $\operatorname{tg} t_s^*$, с които се умножава $\operatorname{tg} x_f$, да са къси. Тогава $t_s^* = \operatorname{arc\,tg}(\operatorname{tg} t_s^*)$.

Пример. За $p = m = 10$ $f = 2$, $I_f = I_2 = \frac{\pi}{12}$, $t_1 = I_1 - I_2 = \frac{\pi}{6}$, $t_1^* \approx 0{,}524$ и $\operatorname{tg} t_1^* = 0{,}564$.

Разсъжденията за $f-1$ са същите като в т. 1.1, но $I_1 = \frac{\ln p}{2}$ трябва да се замени с $I_1 = \frac{\pi}{4}$.

3.2. Прилагането на Т-метода с транслация не се различава от описаното във въведението, съвместно с резултатите от т. 3.1. Сега $t_s = I_s - I_{s+1} = d_{s+1} = 2.3^{f-s-1} I_f$.

3.3. При голямо f К-методът с транслация може да се приложи аналогично на т. 1.3, като индексът exp се замени с tg.

3.4. С-методът с хомотетия и $f = 2$ се прилага аналогично на т. 1.4, като $\frac{\ln p}{2}$ се замени с $\frac{\pi}{4}$, „повдигане на степен n" и x^n — с $\operatorname{tg} nx_2$, а разсъждението за $p = n = 2$ отпада. Остава да се дадат алгоритми за пресмятане на $\operatorname{tg} x_1 = \operatorname{tg} nx_2$ чрез $\operatorname{tg} \frac{x_1}{n} = \operatorname{tg} x_2$.

Нека са дадени хармоничните полиноми [3]:

$$H_n^{(0)}(x, y) = \sum_{i=0}^{[\frac{n}{2}]} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i},$$

$$H_n^{(1)}(x, y) = \sum_{i=0}^{[\frac{n-1}{2}]} (-1)^i \binom{n}{2i+1} x^{n-2i-1} y^{2i+1}.$$

Тогава

$$\sin nx = H_n^{(1)}(\cos x, \sin x) = \sum_{i=0}^{[\frac{n-1}{2}]} (-1)^i \binom{n}{2i+1} \cos^{n-2i-1} x \sin^{2i+1} x,$$

$$\cos nx = H_n^{(0)}(\cos x, \sin x) = \sum_{i=0}^{[\frac{n}{2}]} (-1)^n \binom{n}{2i} \cos^{n-2i} x \sin^{2i} x$$

и

$$\operatorname{tg} nx = \frac{\sin nx}{\cos nx} = \frac{H_n^{(1)}(1, \operatorname{tg} x)}{H_n^{(0)}(1, \operatorname{tg} x)} = \frac{\sum_{i=0}^{[\frac{n-1}{2}]} (-1)^i \binom{n}{2i+1} \operatorname{tg}^{2i+1} x}{\sum_{i=0}^{[\frac{n}{2}]} (-1)^i \binom{n}{2i} \operatorname{tg}^{2i} x}.$$

Окончателно получаваме

$$\operatorname{tg} x_1 = \operatorname{tg} nx_2 = \frac{\operatorname{tg} x_2 R_{[\frac{n-1}{2}]}(\operatorname{tg}^2 x_2)}{S_{[\frac{n}{2}]}(\operatorname{tg}^2 x_2)},$$

където

$$R_{[\frac{n-1}{2}]}(x) = \sum_{i=0}^{[\frac{n-1}{2}]} (-1)^i \binom{n}{2i+1} x^i, \qquad S_{[\frac{n}{2}]}(x) = \sum_{i=0}^{[\frac{n}{2}]} (-1)^i \binom{n}{2i} x^i.$$

Ако вместо $n = n_0 =]q_1[$, $q_1 = \dfrac{I_{\text{tg},1}}{I_{\text{tg},f}}$ изберем $n_r = 2^r \geqq n_0 > 2^{r-1}$, $r = \left]\dfrac{\log_p n}{\log_p 2}\right[$, $\operatorname{tg} n_r x = \operatorname{tg} 2^r x$ може да се пресметне рекурентно от $\operatorname{tg} x$ за r итерации чрез $\operatorname{tg} 2^{i+1}x = \dfrac{2\operatorname{tg} 2^i x}{1-\operatorname{tg}^2 2^i x}$, $i = \overline{0, r-1}$ ($r_\times$, $r_:$, r_+, $r_\times$ са къси).

Пресмятането на $\operatorname{tg} n_0 x_2$ чрез R и S (за всяко n точно един от двата полинома има старши коефициент 1, а другият — $\neq 1$) при $r \geqq 2$ ($n_0 \geqq 3$) изисква $1:$, $2\times$, $(N-1)_\times$ къси, N_+ и съхраняването на $N+2$ къси константи, $N = \left[\dfrac{n_0-1}{2}\right] + \left[\dfrac{n_0}{2}\right] = n_0 - 1$. Ако означим с A, M, M_k и D съответно броя на събиранията, умноженията, късите умножения и деления, и като използваме, че $2^{r-1} \leqq N = n_0 - 1 \leqq 2^r - 1$, получаваме

$$D(n_0) = 1,\ M(n_0) = 2,\ 2^{r-1} - 1 \leqq M_k(n_0) \leqq 2^r - 2,\ 2^{r-1} \leqq A(n_0) \leqq 2^r - 1,$$

$$D(n_r) = r,\ M(n_r) = r,\ M_k(n_r) = r \,(\text{при}\, p > 2),\ A(n_r) = r.$$

Не е взето предвид умножението $\dfrac{1}{n}x_1$, което може да е късо, а при $n = 2^r$ и $p = 2$ се извършва с изваждане на r от порядъка на x_1.

Ако $t_: \approx t_\times$ не отличаваме късите от обикновените умножения и присъединим $M_k(n_r)$ към $A(n_r)$, получаваме

$$M(n_r) + D(n_r) = 2r < 2^{r-1} + 2 \leqq M(n_0) + M_k(n_0) + D(n_0) \leqq 2^r + 1 \text{ при } r \geqq 4,$$

$$A(n_r) + M_k(n_r) = 2r < 2^{r-1} \leqq A(n_0) \leqq 2^r - 1 \quad \text{при} \quad r > 4.$$

Сравнение. Нека $r = 2$, $n_r = 4$. Ако $B_{\text{tg},f} = \left[-\dfrac{\pi}{16}, \dfrac{\pi}{16}\right]$, $x_2 = \dfrac{x_1}{4} = 0{,}25x_1$ и $\operatorname{tg} x_1$ се пресмята с итерации, необходими са $2:$, $2\times$, $2\times$ къси $\times 2$ и $\times 0{,}25$) и $2-$. Ако $\operatorname{tg} x_1 = \dfrac{4\operatorname{tg} x_2(1-\operatorname{tg}^2 x_2)}{(1-\operatorname{tg}^2 x_2)^2 - 4\operatorname{tg}^2 x_2}$, необходими са $1:$, $3\times$, $3\times$ къси ($\times 4$, $\times 0{,}25$) и $2-$.

При $p = m = 10$ $B_{\text{tg},f} = \left[-\dfrac{\pi}{12}, \dfrac{\pi}{12}\right]$, $n_0 = 3$, $x_2 = \dfrac{1}{3}x_1$ и $\operatorname{tg} x_1 = \dfrac{\operatorname{tg} x_2(3-\operatorname{tg}^2 x_2)}{1-3\operatorname{tg}^2 x_2}$ ($1:$, $3\times$, $1\times$ късо ($\times 3$), $2-$).

Очевидно при $n_0 = 3$ и 4 бързодействието и при двата метода е практически еднакво.

3.5. C-методът с хомотетия и $f > 2$ се прилага съвсем аналогично на т. 1.5, като се използват резултатите от т. 3.4 и $\operatorname{tg} x_1$ се пресмята рекурентно чрез $\operatorname{tg} \dfrac{x_1}{2^r}$. От т. 1.5 отпадат разсъжденията, свързани с $e^x = \left(e^{\frac{x}{n}}\right)^n$, което тук не е приложимо. За $x_1 \in B_{\text{tg},1} \setminus B_{\text{tg},2}$ при $I_1 - I_2 \leqq 2I_f$ може вместо хомотетия, която в случая ($n = 2^{r_{\max}}$) ще бъде най-неблагоприятна, да се използва транслация.

3.6. При променливо m важат разсъжденията от т. 4.6, адаптирани към $\operatorname{tg} x$.

При Т-метода с транслация I_s се определят от $I_s = \dfrac{I_1}{3^{s-1}}$, $s \geqq 2$, $B_{\mathrm{tg},1}$ се стеснява 3^s пъти за s итерации, но трябва да се съхраняват два масива с t_s и $\operatorname{tg} t_s$ с дължина на мантисата $m_{\max}$ и брой, съответен на $m_{\max}$.

При С-метода с хомотетия $k = \dfrac{1}{n}$, $n = 2^r$, $I_s = \dfrac{I_1}{2^{s-1}}$, $s \geqq 2$, $\operatorname{tg} x_1$ се пресмята рекурентно. Стесняването на $B_{\mathrm{tg},1}$ е по-бавно — 2^s пъти за s итерации с практически еднакво бързодействие при всяка итерация, но не трябва да се съхраняват никакви константи.

4. $\boldsymbol{y = \operatorname{arc\,tg} x}$, $x \in D^{\infty}_{\operatorname{arc\,tg}} = (-\infty, +\infty)$.

4.1. Ще казваме, че една функция (трансформация) $f(x)$ стеснява по модул (СМ) интервала $[0, \left[{}^{+\infty}_{a_1}\right]]$ до интервала $[0, a_2]$, $0 < a_2 < a_1$, ако $[a_2, \left[{}^{+\infty}_{a_1}\right]] \xrightarrow{f} [a_2^*, a_1^*] \subseteq [-a_2, a_2]$, т. е. ако $x \in [a_2, \left[{}^{+\infty}_{a_1}\right]]$, $|f(x)| = |x^*| \leqq a_2$. За всяка такава трансформация, преди да се свърже с пресмятането на $\operatorname{arc\,tg} x$, трябва да се отговори на няколко въпроса: за дадени $f(x)$ и $\left[{}^{+\infty}_{a_1}\right]$ има ли такива a_2; ако има, има ли едно най-малко между тях; за дадени $f(x)$ и a_2 има ли $\left[{}^{+\infty}_{a_1}\right]$, за които $f(x)$ да СМ $[0, \left[{}^{+\infty}_{a_1}\right]]$ до $[0, a_2]$; ако има, има ли едно най-голямо между тях и др. Понеже $\operatorname{arc\,tg} x$ е нечетна, достатъчно е да $\exists f(x)$, която да СМ $D^{\infty}_{\operatorname{arc\,tg}|x|} = D^{+\infty}_{\mathrm{at}} = [0, +\infty)$ до $[0; I_{\mathrm{at},f}]$ (отсега долните индекси $\operatorname{arc\,tg}$ ще заместваме с at). $f(x)$ не може да бъде линейна, понеже $D^{+\infty}_{\mathrm{at}}$ е неограничена. Ще разгледаме най-близката до линейната, достатъчно проста и удобна при пресмятането на $\operatorname{arc\,tg} x$ дробно–линейна функция, свързана със субституцията $x_{s+1} = \dfrac{ax_{s-1}}{x_s + a} = a - \dfrac{a^2+1}{x_s+a}$, $a > 0$ (1 :, 2+; a^2+1 се пресмята предварително), с цел да бъде използвана при $x_s > 0$ събирателната формула

$$\operatorname{arc\,tg} x_s = \operatorname{arc\,tg} \frac{1}{a} + \operatorname{arc\,tg} x_{s+1}$$

(заедно със субституцията са необходими 1 :, 3+; $\operatorname{arc\,tg} \dfrac{1}{a}$ се пресмята предварително), като a се избира така, че $|x_{s+1}| < |x_s|$.

Граничен случай се получава при $a > 0$, $a \to 0$:

$$x_{s+1} = -\frac{1}{x_s}, \quad \operatorname{arc\,tg} x_s = \frac{\pi}{2} + \operatorname{arc\,tg} x_{s+1}, \quad x_s > 0 \quad (1\text{: и } 1+),$$

по-общо:

$$f_0(x) = -\frac{1}{x},\ (\pm 1, \pm\infty) \xrightarrow{f_0} (\mp 1, 0),\ \operatorname{arc\,tg} x_s = (\operatorname{sgn} x_s)\frac{\pi}{2} + \operatorname{arc\,tg} x_{s+1},\ x_s \neq 0.$$

Ще изследваме по-подробно дробно-линейната трансформация $f(a, x) = f_a(x) = f_a = \dfrac{ax-1}{x+1} = f_x(a)$, $a \neq 0$. Понеже $\dfrac{\partial f_a}{\partial \left[{}^a_x\right]} = \dfrac{\left[{}^x_a\right]^2 + 1}{(x+a)^2} > 0$, f_a е строго монотонно растяща функция (с. м. р. ф.) и на двата си аргумента. Нека отбележим, че няма да отделяме определянето на $B_{\mathrm{at},1}$ от другите БИ, както при другите ОЕФ. Ще ни интересува как f_a трансформира полубезкрайни интервали с лява положителна граница и крайни — с положителни граници. Поради наличието на събирателната формула f_a е удобна от ИА, но не е ясно дали при $\forall a$ СМ такива интервали, в частност дали $\exists x$, $x \neq 0$, които f_a намалява по модул (НМ), т. е. $|f_a(x)| < |x|$ при $a \neq 0$.

Ще изследваме кога $|f_a(x)| = \left|\dfrac{ax-1}{x+a}\right| < |x|$. Възможни са два случая:

1) $ax = -|a||x|$.

В този случай в събирателната формула може да фигурира като ново събираемо $\pm\pi$ в зависимост от a и x. Те са с обратни знакове, $|x+a| = ||x|-|a||$ и неравенството е еквивалентно на $\dfrac{|a||x|+1}{||x|-|a||} < |x|$.

От двата еквивалентни по отношение на неравенството подслучая $a < x$, $x > 0$, и $a > 0$, $x < 0$, ще разгледаме първия и ще докажем следното

Твърдение 1. При $a < 0$ и $x > 0$ $\overline{\exists} f_a$, която да НМ $x \in [0, 1]$.

Доказателство. Сега неравенството е равносилно на $\dfrac{1-ax}{|x+a|} < x$.

а) $x > -a > 0$. Тогава $1-ax < x^2+ax$, $x^2+2ax-1 > 0$, $x_{1,2} = -a\pm\sqrt{a^2+1}$, но $x > 0$, следователно неравенството е изпълнено при $x > \sqrt{a^2+1}+|a| > 1$.

б) $x < -a$. $1-ax < -x^2-ax$ не е вярно.

И така при $a < 0$ и $x > 0$ евентуално само за $x > 1$ $\exists f_a$, която ги НМ, т. е. $\overline{\exists} f_a$, която СМ $[0, 1]$ в частност до $[0; B_{\mathrm{at},f}]$, ако $B_{\mathrm{at},f} < 1$.

2) $ax = |a||x|$.

a и x са с еднакви знакове, $|x+a| = |x|+|a| > 0$ и неравенството е еквивалентно на $||a||x|-1| < |x|(|x|+|a|)$. Аналогично избираме $a > 0$ и $x > 0$. Тогава $f_a(x) = \dfrac{ax-1}{x+a} < x$, но не винаги $|f_a(x)| < x$, т. е. f_a намалява $x > 0$, но не винаги по модул. Ще докажем

Твърдение 2. За $\forall x > 0$ $\exists f_a$, $a > 0$, която го НМ.

Доказателство. В този случай $|f_a(x)| < x$ е еквивалентно на $|ax-1| < x^2+ax$.

а) $x \geqq \dfrac{1}{a}$. $ax-1 < x^2+ax$ е вярно.

б) $x < \dfrac{1}{a}$. $x^2+2ax-1 > 0$, $x_{1,2} = -a\pm\sqrt{a^2+1}$ и понеже $x > 0$, неравенството е еквивалентно на $\sqrt{a^2+1}-a < x < \dfrac{1}{a}$, което заедно с подслучай а) дава, че за $x > \sqrt{a^2+1}-a > 0$ f_a НМ x. Но $\sqrt{a^2+1}-a \xrightarrow[a\to+\infty]{} 0$, т. е. за $\forall x$, $x > 0$, x — произволно малко, $\exists a$, $a > 0$, a — достатъчно голямо, $|f_a(x)| < x$.

И така при дадено $a > 0$, f_a НМ само $x > \sqrt{a^2+1} - a$. Числото $\sqrt{a^2+1} - a$ ще играе важна роля по-нататък. Ще го наричаме положителна неподвижна по модул точка (ПНМТ) на f_a, понеже може да се получи от условието $\dfrac{ax-1}{x+a} = -x$, $x > 0$, т. е. f_a го трансформира в противоположното му число. Очевидно $\forall f_a$, $a > 0$, има точно една ПНМТ.

Окончателно ще изследваме $f_a(x)$ само за $\left[{a \atop x}\right] > 0$. В т. 4.2 и 4.3 ще се търсят a и $\left[{+\infty \atop a_1}\right]$, за които $\exists a_2$, f_a СМ $[0, \left[{-\infty \atop a_1}\right]]$ до $[0, a_2]$.

4.2. При СМ на $[0, +\infty)$ до $[0, a_2]$ е естествено да се изследва за кои a $a_2 = a$, понеже $f_a(x) \xrightarrow[x\to+\infty]{} a$. Изобщо $[a, +\infty) \xrightarrow{f_a} [a^*, a)$, $a^* = f_a(a) = \dfrac{a^2-1}{2a} < a$ (с x^* ще означаваме образа на x чрез f_a, която се подразбира от текста). Нека $M_0^\infty(f_a) = \max(|a^*|, a) = \max(|f_a(a)|, a)$. Понеже $\dfrac{\partial f_a}{\partial x} > 0$, $|f_a(x)| \leqq M_0^\infty(f_a)$ за $x \in [a, +\infty)$. Ако $a \geqq 1$, $a > a^* \geqq 0$ и $M_0^\infty(f_a) = a$. По-интересен е случаят $a < 1$, тогава $a^* < 0$. Ще казваме, че f_a повече НМ $[a, +\infty)$ отколкото f_b — $[b, +\infty)$, ако $M_0^\infty(f_a) < M_0^\infty(f_b)$, при това не се изисква нито f_a да СМ $[0, +\infty)$ до $[0, a]$, нито f_b — до $[0, b]$. Ако и f_a, и f_b го СМ съответно до $[0, a]$ и $[0, b]$, тогава $M_0^\infty(f_a) = a$, $M_0^\infty(f_b) = b$ и от $M_0^\infty(f_a) < M_0^\infty(f_b)$ следва, че $a < b$ и f_a СМ $[0, +\infty)$ до $[0, a]$ повече, отколкото f_b — до $[0, b]$ ($[0, a] \subset [0, b]$). Ще покажем, че $\exists\tilde{f}$ — единствена, с най-малък $M_0^\infty(\tilde{f})$, която освен това СМ $[0, +\infty)$. Ето защо ще я наречем най-стесняваща по модул трансформация (НСМТ) за $[0, +\infty)$. Само за нея $M_0^\infty(f_a)$ достига точната си долна граница (т. д. г.) — $M_0^\infty(f_a)$ е ограничено отдолу — $M_0^\infty(f_a) \geqq a > 0$, а интервалът, до който $\tilde{f}$ СМ $[0, +\infty)$, е най-малкият възможен. Елементите на $\tilde{f}$ ще бележим с $\sim$.

Теорема 3. $\tilde{f} = f_{\tilde{a}_\infty}$ с ПНМТ $\tilde{a}_\infty$ е единствената НСМТ за $[0, +\infty)$.

Доказателство. $\tilde{a}_\infty$ се получава от условието $f_{\tilde{a}_\infty}(\tilde{a}_\infty) = -\tilde{a}_\infty$, $\dfrac{\tilde{a}_\infty^2-1}{2\tilde{a}_\infty} = -\tilde{a}_\infty$, $\tilde{a}_\infty = \pm\dfrac{1}{\sqrt{3}}$, но от $\tilde{a}_\infty > 0$ следва $\tilde{a}_\infty = \dfrac{1}{\sqrt{3}} \approx 0,57735\ldots$

От $[\tilde{a}_\infty, +\infty) \xrightarrow{f_{\tilde{a}_\infty}} [-\tilde{a}_\infty, \tilde{a}_\infty]$ следва, че $f_{\tilde{a}_\infty}$ СМ $[0, +\infty)$ до $[0, \tilde{a}_\infty]$.

Ако $a > \tilde{a}_\infty$, $M_0^\infty(f_a) = \max(|a^*|, a) \geqq a > \tilde{a}_\infty = \max(|-\tilde{a}_\infty|, \tilde{a}_\infty) = M_0^\infty(f_{\tilde{a}_\infty})$.

Ако $0 < a < \tilde{a}_\infty$, $a^* = \dfrac{a^2-1}{2a} < -\tilde{a}_\infty$, понеже $a^2 + 2a\tilde{a}_\infty < 3\tilde{a}_\infty^2 = 1$. Тогава $|a^*| > \tilde{a}_\infty$ и $M_0^\infty(f_a) = \max(|a^*|, a) \geqq |a^*| > \tilde{a}_\infty = M_0^\infty(f_{\tilde{a}_\infty})$. С това теоремата е доказана.

Следствие 3.1. $\forall f_a$, $a > \tilde{a}_\infty$, f_a СМ $[0, +\infty)$ до $[0, a]$; $\overline{\exists} f_a$, $0 < a < \tilde{a}_\infty$, f_a СМ $[0, +\infty)$ до $[0, a]$; за $\forall f_a$, $0 < a < \tilde{a}_\infty$, и за $\forall a_2$, $a_2 \geqq \sqrt{a^2+1} - a$, f_a СМ $[0, +\infty)$ до $[0, a_2]$.

Доказателство. Ако $a > \tilde{a}_\infty$, от $\dfrac{\partial f_a}{\partial \left[\frac{x}{a}\right]} > 0$ следва $-a < -\tilde{a}_\infty = f_{\tilde{a}_\infty}(\tilde{a}_\infty)$ $< f_{\tilde{a}_\infty}(a) < f_a(a) < a$, т. е. $|a^*| < a$ и $[a, +\infty) \xrightarrow{f_a} [a^*, a) \subset [-a, a]$. f_a СМ $[0, +\infty)$ до $[0, a]$, но СМ е по-малко, понеже от $a > \tilde{a}_\infty$ следва $[0, \tilde{a}_\infty] \subset [0, a]$.

Ако $0 < a < \tilde{a}_\infty$ и допуснем, че $\exists f_a$, $[a, +\infty) \xrightarrow{f_a} [f_a(a), a) \subseteq [-a, a]$, то $f_a(a) < f_{\tilde{a}_\infty}(a) < f_{\tilde{a}_\infty}(\tilde{a}_\infty) = -\tilde{a}_\infty$, $|f_a(a)| > \tilde{a}_\infty > a$ и $[-a, a] \subset [f_a(a), a]$, което противоречи на допускането.

Ако се откажем f_a да СМ $[0, \infty)$ до $[0, a]$, $[0, +\infty)$ може да се СМ чрез f_a, $0 < a < \tilde{a}_\infty$, до $[0, a_2]$, $a_2 \geqq \sqrt{a^2+1} - a > \tilde{a}_\infty$ ($a_2 \geqq$ ПНМТ на f_a), понеже е еквивалентно на $a^2 + 2a\tilde{a}_\infty < 3\tilde{a}_\infty^2 = 1$. Наистина от $a < \tilde{a}_\infty$ $< a_2 \leqq x < +\infty$ следва, че $-a_2 = f_a(a_2) \leqq f_a(x) < f_a(+\infty) = a < \tilde{a}_\infty < a_2$, т. е. $|f_a(x)| \leqq a_2$ и f_a СМ $[0, +\infty)$ до $[0, a_2]$.

Следствие 3.2. При $I_{at,f} < \tilde{a}_\infty$ С-методът е неприложим за $\operatorname{arc\,tg} x$ чрез f_a. Следва от второто твърдение на следствие 3.1.

4.3. Ще докажем, че при $I_{\mathrm{at},f} < \tilde{a}_\infty$ Т-методът е приложим за каскадно СМ на $[0, +\infty)$ до $[0; I_{\mathrm{at},f}]$. За тази цел ще изследваме за кои a и a_1 f_a СМ $[0, a_1]$ до $[0, f_a(a_1)]$. Аналогично на т. 4.2 въвеждаме $M_0^{a_1}(f_a)$ $= \max(|f_a(f_a(a_1))|, |f_a(a_1)|)$. Понеже $\dfrac{\partial f_a}{\partial x} > 0$, $|f_a(x)| \leqq M_0^{a_1}(f_a)$ за x $\in [f_a(a_1), a_1]$. Ще казваме, че f_a НМ $[f_a(a_1), a_1]$ повече, отколкото f_b — $[f_b(a_1), a_1]$, ако $M_0^{a_1}(f_a) < M_0^{a_1}(f_b)$, като не се изисква f_a да СМ $[0, a_1]$ до $[0, f_a(a_1)]$, нито f_b — до $[0, f_b(a_1)]$. Ако f_a и f_b го СМ съответно до $[0, f_a(a_1)]$ и $[0, f_b(a_1)]$, то $f_a(a_1) > 0$, $f_b(a_1) > 0$, $M_0^{a_1}(f_a) = f_a(a_1)$, $M_0^{a_1}(f_b) = f_b(a_1)$ и от $M_0^{a_1}(f_a) < M_0^{a_1}(f_b)$ следва $f_a(a_1) < f_b(a_1)$ (тогава и $a < b$ от $\dfrac{\partial f_a}{\partial a} > 0$) и f_a СМ $[0, a_1]$ до $[0, f_a(a_1)]$ повече, отколкото f_b — до $[0, f_b(a_1)]([0, f_a(a_1)] \subset [0, f_b(a_1)])$. Ще докажем, че $\exists \tilde{f}$ — единствена, с най-малък $M_0^{a_1}(\tilde{f})$, която освен това СМ $[0, a_1]$, и ще я наречем НСМТ за $[0, a_1]$. Само за нея $M_0^{a_1}(f_a)$ достига т. д. г., а интервалът, до който се СМ $[0, a_1]$, е най-малкият възможен.

Теорема 4. $f_{\tilde{a}^{(1)}}$, която трансформира a_1 в своята ПНМТ $\tilde{a}_2^{(1)}$, е единствената НСМТ за $[0, a_1]$.

Единственост на НСМТ. Ако $0 < a < \tilde{a}^{(1)}$, от $\dfrac{\partial f_a}{\partial \left[\frac{x}{a}\right]} > 0$ следва $f_a(a_1)$ $< f_{\tilde{a}^{(1)}}(a_1) = \tilde{a}_2^{(1)}$ и $f_a(f_a(a_1) < f_a(\tilde{a}_2^{(1)}) < f_{\tilde{a}^{(1)}}(\tilde{a}_2^{(1)}) = -\tilde{a}_2^{(1)}$. Тогава $M_0^{a_1}(f_a) = \max(|f_a(f_a(a_1))|, |f_a(a_1)|) \geqq |f_a(f_a(a_1))| > \tilde{a}_2^{(1)} = \max(|-\tilde{a}_2^{(1)}|, \tilde{a}_2^{(1)})$ $= M_0^{a_1}(f_{\tilde{a}^{(1)}})$.

Ако $a > \tilde{a}^{(1)}$, $M_0^{a_1}(f_a) \geqq |f_a(a_1)| > |f_{\tilde{a}^{(1)}}(a_1)| = \tilde{a}_2^{(1)} = M_0^{a_1}(f_{\tilde{a}^{(1)}})$.

Съществуване на НСМТ. Ще докажем следната

Лема 5. Ако $a_1 > 0$, $\tilde{a}^{(1)} = \operatorname{ctg}\left(\dfrac{2}{3}\operatorname{arc\,tg} a_1\right)$, $\tilde{a}_2^{(1)} = \operatorname{tg}\left(\dfrac{1}{3}\operatorname{arc\,tg} a_1\right)$, $(2-\sqrt{3})a_1 \leqq \tilde{a}_2^{(1)} < \dfrac{1}{3}a_1$ за $a_1 \leqq 1$ и $2-\sqrt{3} < \tilde{a}_2^{(1)} < \tilde{a}_\infty$ за $a_1 > 1$.

За удобство в т. 4.3 ще използваме $\left[\begin{smallmatrix}\tilde{a}\\ \tilde{a}_2\end{smallmatrix}\right]$ вместо $\left[\begin{smallmatrix}\tilde{a}^{(1)}\\ \tilde{a}_2^{(1)}\end{smallmatrix}\right]$.

Доказателство. Понеже $0 < \tilde{a}_2 = \sqrt{\tilde{a}^2+1} - \tilde{a} < 1$, от $f_{\tilde{a}}(\tilde{a}_2) = -\tilde{a}_2$ получаваме $\tilde{a} = \dfrac{1-\tilde{a}_2^2}{2\tilde{a}_2} > 0$. Ако заместим израза за $\tilde{a}$ в условието $f_{\tilde{a}}(a_1) = \tilde{a}_2$, получаваме

$$g_{a_1}(\tilde{a}_2) = \tilde{a}_2^3 - 3a_1\tilde{a}_2^2 - 3\tilde{a}_2 + a_1 = 0.$$

$g_{a_1}(\tilde{a}_2) = 0$ има винаги три реални корена, два от които не отговарят на условията $0 < \tilde{a}_2 < a_1$:

$g(-\infty) < 0$, $\quad g(0) = a_1 > 0$ — $g_{a_1}(\tilde{a}_2)$ има винаги нула $\tilde{a}_2 < 0$;

$g(a_1) = -2a_1(1+a_1^2) < 0$, $g(+\infty) > 0$ — $g_{a_1}(\tilde{a}_2)$ има винаги нула $\tilde{a}_2 > a_1$.

а) $a_1 \leqq 1$.

При $a_1 < 1$ $g((2-\sqrt{3})a_1) = (5-3\sqrt{3})a_1(a_1^2-1) > 0$, $g\left(\dfrac{a_1}{3}\right) = -\dfrac{8a_1^3}{27} < 0$.

И така $\exists\tilde{a}_2$, $(2-\sqrt{3})a_1 < \tilde{a}_2 < \dfrac{a_1}{3}$. Оценката отдолу се достига при $a_1 = 1$, $g_1(2-\sqrt{3}) = 0$. Оценката отгоре не се достига, понеже $\lim\limits_{\substack{a_1\to 0\\ a_1>0}} \dfrac{\tilde{a}_2}{a_1} = \dfrac{1}{3}$.

б) $a_1 > 1$.

$g(2-\sqrt{3}) = 4(3\sqrt{3}-5)(a_1-1) > 0$, $\left[\begin{smallmatrix} g(a_1/3)\\ g(1/\sqrt{3}) = -\frac{8\sqrt{3}}{9}\end{smallmatrix}\right] < 0$. Следователно $2-\sqrt{3} < \tilde{a}_2 < \min\left(\dfrac{1}{\sqrt{3}}, \dfrac{a_1}{3}\right) \leqq \tilde{a}_\infty$. От $g_{a_1}(\tilde{a}_2) = 0$ a_1 се определя като функция на $\tilde{a}_2$, $a_1 = \dfrac{\tilde{a}_2(3-\tilde{a}_2^2)}{1-3\tilde{a}_2^2} = \tilde{a}_1^{(2)}$. Ако съпоставим този израз с $\operatorname{tg} x$ $= \dfrac{\operatorname{tg}\frac{x}{3}(3-\operatorname{tg}^2\frac{x}{3})}{1-3\operatorname{tg}^2\frac{x}{3}}$ и означим $x = \operatorname{arc\,tg} a_1$, получаваме $\tilde{a}_2 = \operatorname{tg}\left(\dfrac{1}{3}\operatorname{arc\,tg} a_1\right)$, а от $\operatorname{tg} 2x = \dfrac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x}$ получаваме $\tilde{a} = 1\Big/\dfrac{2\tilde{a}_2}{1-\tilde{a}_2^2} = 1\Big/\operatorname{tg}\left(\dfrac{2}{3}\operatorname{arc\,tg} a_1\right)$ $= \operatorname{ctg}\left(\dfrac{2}{3}\operatorname{arc\,tg} a_1\right)$.

Експлицитното изразяване на параметрите $\tilde{a}$ и $\tilde{a}_2$ на НСМТ чрез a_1 доказва съществуването ѝ и с това окончателно теорема 4.

Два важни частни случая. Ако $a_1 = \tilde{a}_\infty = \dfrac{1}{\sqrt{3}} = \operatorname{tg}\dfrac{\pi}{6} \approx 0{,}577\ldots$, $\tilde{a}_2 = \operatorname{tg}\left(\dfrac{1}{3}\operatorname{arc\,tg}\dfrac{1}{\sqrt{3}}\right) = \operatorname{tg}\dfrac{\pi}{18} \approx 0{,}173\ldots$, $\tilde{a} = \operatorname{ctg}\dfrac{\pi}{9} \approx 2{,}8$. Това е НСМТ за $[0, \tilde{a}_\infty]$ (след като $[0, +\infty)$ е СМ до него с НСМТ $f_{\tilde{a}_\infty}$).

Ако $a_1 = 1$, вече получихме $\tilde{a}_2 = 2 - \sqrt{3}$. Отново, съгласно лемата, $\tilde{a}_2 = \operatorname{tg}\left(\frac{1}{3}\operatorname{arc\,tg} 1\right) = \operatorname{tg}\frac{\pi}{12} \approx 0{,}268\ldots = 2 - \sqrt{3}$, $\tilde{a} = \frac{1-(2-\sqrt{3})^2}{2(2-\sqrt{3})} = \sqrt{3}$ $= \operatorname{ctg}\frac{\pi}{6} \approx 1{,}732$. Чрез $\tilde{a}$ за трети път получаваме $\tilde{a}_2 = \frac{\sqrt{3}.1 - 1}{1+\sqrt{3}} = 2 - \sqrt{3}$.

Следствие 4.1. $\bar{\exists} f_a$, която СМ произволно много $[0, \left[{}^{+\infty}_{a_1}\right]]$.

Доказателство. От теорема 3 и следствие 3.1 следва, че $[0, +\infty)$ може да се СМ най-много до $[0, \tilde{a}_\infty)$ чрез $f_{\tilde{a}_\infty}$. От теорема 4 следва, че $[0, a_1]$ може да се СМ най-много до $[0, \tilde{a}_2]$, $g_{a_1}(\tilde{a}_2) = 0$. С това следствието е доказано.

Поради $\tilde{a}_2 < \min\left(\frac{1}{\sqrt{3}}, \frac{a_1}{3}\right) \leqq \frac{a_1}{3}$, е вярно

Следствие 4.2. Чрез прилагане на Т-метода интервалът $[0, +\infty)$ може да се СМ произволно много.

Доказателство. Ако приложим серия от $k \geqq 1$ последователни НСМТ към $[0, +\infty)$, той ще се СМ до $[0; a_{2,(k-1)}]$, $a_{2,(k-1)} < \left(\frac{1}{3}\right)^{k-1} \tilde{a}_\infty \xrightarrow[k\to\infty]{} 0$, т. е. $a_{2,(k-1)}$ може да стане произволно малко при достатъчно голямо k.

С това окончателно се решава въпросът за годността на трансформациите $f_a(x)$ по отношение СМ на $[0, +\infty)$ до $[0; I_{\text{at},f}]$.

В т. 4.4 ÷ 4.6 ще бъдат разгледани въпроси, свързани с избора на най-подходяща от ИА серия от последователни трансформации от вида f_a, която СМ $[0, +\infty)$ при дадени p и $m(m_{\max})$.

4.4. Нека изразим всеки два от трите параметъра $\tilde{a}$, $\tilde{a}_1$ и $\tilde{a}_2$ на една НСМТ $f_{\tilde{a}}$ чрез третия и ги изследваме като функция от него.

В т. 4.3 намерихме как $\tilde{a}$ и $\tilde{a}_1$ се изразяват чрез $\tilde{a}_2$ и как $\tilde{a}$ и $\tilde{a}_2$ — чрез $\tilde{a}_1$. За дадено $\tilde{a}$ $\tilde{a}_2$ се намира като ПНМТ на $f_{\tilde{a}}$. Остава да намерим $\tilde{a}_1$, $f_{\tilde{a}}(\tilde{a}_1) = \tilde{a}_2$. Ако $\tilde{a} = \tilde{a}_\infty = \frac{1}{\sqrt{3}}$, то $\tilde{a}_2 = \frac{1}{\sqrt{3}}$ и $\tilde{a}_1 = +\infty$. Ако $\tilde{a} \neq \frac{1}{\sqrt{3}}$, ще използваме $f_{\tilde{a}}^{-1} = \frac{\tilde{a}x+1}{-x+\tilde{a}}$, $x \neq \tilde{a}$, която е обратна на $f_{\tilde{a}}$ и се намира от тъждеството $f_{\tilde{a}}(f_{\tilde{a}}^{-1}(x)) = x$, $\frac{\tilde{a}f_{\tilde{a}}^{-1} - 1}{f_{\tilde{a}}^{-1} + \tilde{a}} = x$. $\tilde{a}_1 = f_{\tilde{a}}^{-1}(\tilde{a}_2) = \frac{\tilde{a}\tilde{a}_2 + 1}{-\tilde{a}_2 + \tilde{a}}$ $= \frac{1 - \tilde{a}_2^2 + \tilde{a}\sqrt{\tilde{a}^2+1}}{2\tilde{a} - \sqrt{\tilde{a}^2+1}} = \frac{\sqrt{\tilde{a}^2+1} + 2\tilde{a}}{\tilde{a}\sqrt{\tilde{a}^2+1} + \tilde{a}^2 - 1}$. $\tilde{a}_1 > 0$ при $\tilde{a} > \tilde{a}_2$, което е равносилно на $\tilde{a} > \frac{1}{\sqrt{3}}$, т. е. само за $\tilde{a} \geqq \frac{1}{\sqrt{3}}$ $\exists\left[{}^{+\infty}_{\tilde{a}_1}\right]$ така, че $\tilde{a}$, $\tilde{a}_1$ и $\tilde{a}_2$ са елементи на НСМТ $f_{\tilde{a}}$.

Пример. Нека $\tilde{a} = 1$, $f_1 = \frac{x-1}{x+1}$, $f_1^{-1} = \frac{x+1}{-x+1}$, $\tilde{a}_2 = \sqrt{2} - 1$, $\tilde{a}_1 = \sqrt{2} + 1$, $[\sqrt{2} - 1, \sqrt{2} + 1] \xrightarrow{f_1} [-(\sqrt{2} - 1), \sqrt{2} - 1]$.

Да рекапитулираме и изследваме изменението на всеки от параметрите на НСМТ като функция поотделно на останалите два.

4.4.1. Дадено е $a = \tilde{a} \geqq \tilde{a}_\infty$. Тогава $0 < \tilde{a}_2 = \sqrt{\tilde{a}^2+1} - \tilde{a} < 1$ и

$$\left[\begin{matrix} \infty \\ a_1 = \dfrac{\sqrt{\tilde{a}^2+1}+2\tilde{a}}{\tilde{a}\sqrt{\tilde{a}^2+1}+\tilde{a}^2-1} > 0 \end{matrix}\right], \quad \tilde{a}\begin{bmatrix} = \\ > \end{bmatrix}\tilde{a}_\infty.$$

$\tilde{a}_2$ е с. м. н. ф. на $\tilde{a}$ $\left(\dfrac{d\tilde{a}_2}{d\tilde{a}} = -\dfrac{\tilde{a}_2}{\sqrt{\tilde{a}^2+1}} < 0\right)$, $\tilde{a}_{2\max} = \tilde{a}_2\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{1}{\sqrt{3}}$, $\tilde{a}_2 \xrightarrow[\tilde{a}\to+\infty]{} 0$.

$\tilde{a}_1$ е с. м. н. ф. на $\tilde{a}$ $\left(\dfrac{d\tilde{a}_1}{d\tilde{a}} = -\dfrac{3\tilde{a}_2\sqrt{\tilde{a}^2+1}}{(\tilde{a}-\tilde{a}_2)^2} < 0\right)$, $\tilde{a}_1 \xrightarrow[\tilde{a}\to\tilde{a}_\infty]{} +\infty$, $\tilde{a}_1 \xrightarrow[\tilde{a}\to+\infty]{} 0$.

4.4.2. Дадено е $a_2 = \tilde{a}_2$, $0 < \tilde{a}_2 \leqq \dfrac{1}{\sqrt{3}} < 1$. Ако положим $\tilde{a}_2 = \operatorname{tg} x$, $x = \operatorname{arc\,tg} a_2$, $\tilde{a}^{(2)} = \dfrac{1-\tilde{a}_2^2}{2\tilde{a}_2} = \operatorname{ctg} 2x = \operatorname{ctg}(2\operatorname{arc\,tg}\tilde{a}_2) > 0$ и

$$\left[\begin{matrix} \infty \\ \tilde{a}_1^{(2)} = \dfrac{\tilde{a}_2(3-\tilde{a}_2^2)}{1-3\tilde{a}_2^2} = \operatorname{tg} 3x = \operatorname{tg}(3\operatorname{arc\,tg}\tilde{a}_2) > 0 \end{matrix}\right], \quad \tilde{a}_2\begin{bmatrix} = \\ < \end{bmatrix}\frac{1}{\sqrt{3}} = \tilde{a}_\infty.$$

$\tilde{a}^{(2)}$ е с. м. н. ф. на $\tilde{a}_2$ $\left(\dfrac{d\tilde{a}^{(2)}}{d\tilde{a}_2} = -\dfrac{1}{2}\dfrac{\tilde{a}_2^2+1}{\tilde{a}_2^2} < 0\right)$, $\tilde{a}^{(2)}_{\min} = \tilde{a}^{(2)}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{1}{\sqrt{3}}$, $\tilde{a}^{(2)} \to +\infty$, когато $\tilde{a}_2 \to 0$.

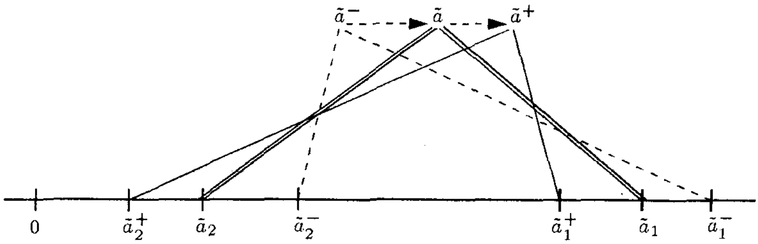
$\tilde{a}_1^{(2)}$ е с. м. р. ф. на $\tilde{a}_2$ $\left(\dfrac{d\tilde{a}_1^{(2)}}{d\tilde{a}_2} = \dfrac{3(\tilde{a}_2^2+1)^2}{(1-3\tilde{a}_2^2)^2} > 0\right)$, $\tilde{a}_1^{(2)} \to +\infty$, когато $\tilde{a}_2 \to \dfrac{1}{\sqrt{3}}$; $\tilde{a}_1^{(2)} \to 0$, когато $\tilde{a}_2 \to 0$.

4.4.3. Дадено е $a_1 = \tilde{a}_1 > 0$. Тогава $\tilde{a}^{(1)} = \operatorname{ctg}\left(\dfrac{2}{3}\operatorname{arc\,tg}\tilde{a}_1\right)$, $\tilde{a}_2^{(1)} = \operatorname{tg}\left(\dfrac{1}{3}\operatorname{arc\,tg}\tilde{a}_1\right)$.

$\tilde{a}^{(1)}$ е с. м. н. ф. на $\tilde{a}_1$ $\left(\dfrac{d\tilde{a}^{(1)}}{d\tilde{a}_1} = -\dfrac{2}{3}\dfrac{(\tilde{a}^{(1)})^2+1}{\tilde{a}_1^2+1} < 0\right)$, $\tilde{a}^{(1)} \to \dfrac{1}{\sqrt{3}}$, когато $\tilde{a}_1 \to +\infty$; $\tilde{a}^{(1)} \to +\infty$, когато $\tilde{a}_1 \to 0$.

$\tilde{a}_2^{(1)}$ с. м. р. ф. на $\tilde{a}_1$ $\left(\dfrac{d\tilde{a}_2^{(1)}}{d\tilde{a}_1} = \dfrac{(\tilde{a}_2^{(1)})^2+1}{3(\tilde{a}_1^2+1)} > 0\right)$, $\tilde{a}_2^{(1)} \to 0$, когато $\tilde{a}_1 \to 0$; $\tilde{a}_2^{(1)} \to \dfrac{1}{\sqrt{3}}$, когато $\tilde{a}_1 \to +\infty$.

Графично това може да се представи за $\underset{\begin{bmatrix}\tilde{a}_2^-\\ \tilde{a}_1^-\end{bmatrix}}{\tilde{a}^-} < \underset{\begin{bmatrix}\tilde{a}_2\\ \tilde{a}_1\end{bmatrix}}{\tilde{a}} < \underset{\begin{bmatrix}\tilde{a}_2^+\\ \tilde{a}_1^+\end{bmatrix}}{\tilde{a}^+}$ така:



4.5. Ще разгледаме връзките между a, $\left[{+\infty \atop a_1}\right]$ и a_2, когато f_a СМ $[0,\ \left[{+\infty \atop a_1}\right]]$ до $[0,\ a_2]$, но не е НСМТ.

4.5.1. Дадено е $a > 0$. За кои $\left[{+\infty \atop a_1}\right]$ и a_2 f_a СМ $[0,\ \left[{+\infty \atop a_1}\right]]$ до $[0,\ a_2]$?

Ако $a \geqq \tilde a_\infty = \dfrac{1}{\sqrt{3}}$, $\exists \left[{+\infty \atop \tilde a_1}\right]$ и $\exists \tilde a_2$, които заедно с a са елементи на НСМТ f_a за $[0,\ \left[{+\infty \atop a_1}\right]]$. Тогава f_a ще СМ всеки интервал $[0,\ \left[{+\infty \atop a_1}\right]]$ до $[0,\ a_2]$, ако $\tilde a_2 \leqq a_2 < \left[{+\infty \atop a_1}\right] \leqq \tilde a_1$. Това следва от $-\tilde a_2 = f_a(\tilde a_2) \leqq f_a(a_2) \leqq f_a(x) \leqq f_a\left(\left[{+\infty \atop a_1}\right]\right) \leqq \left[{\tilde a_2 \atop f_a(\tilde a_1) \leqq \tilde a_2}\right]$, т. е. $|f_a(x)| \leqq \tilde a_2 < a_2$ за $x \in [a_2,\ a_1]$. Понеже f_a е НСМТ за $[0,\ \left[{+\infty \atop a_1}\right]]$, тя не може да СМ $[0, a_1]$, $\tilde a_1 < a_1$, до $[0, a_2]$, $a_2 < \tilde a_2$.

Ако $0 < a < \tilde a_\infty$, $\tilde a_2 > \dfrac{1}{\sqrt{3}}$. Тогава f_a СМ $[0,\ a_1]$ до $[0,\ a_2]$, $\tilde a_2 \leqq a_2 < a_1$, понеже от $a < \dfrac{1}{\sqrt{3}} < \tilde a_2 \leqq a_2 \leqq x \leqq a_1 < +\infty$ следва $-a_2 < -\tilde a_2 = f_a(\tilde a_2) \leqq f_a(x) < f_a(+\infty) = a < \dfrac{1}{\sqrt{3}} < \tilde a_2 \leqq a_2$, т. е. $|f_a(x)| \leqq a_2$ за $x \in [a_2,\ a_1]$.

4.5.2. Дадено е $a_2 > 0$. За кои a и a_1 f_a СМ $[0,\ a_1]$ до $[0,\ a_2]$?

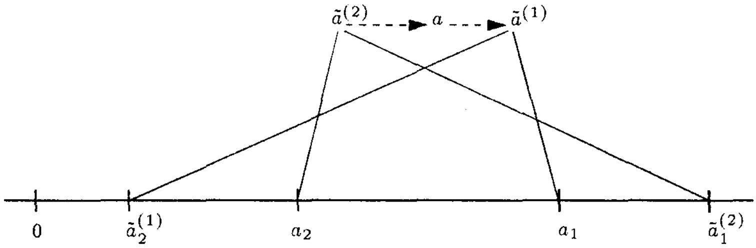
Ако $a_2 \geqq \dfrac{1}{\sqrt{3}}$, за всяко $a_1 > a_2$ поне $f_{\tilde a_\infty}$ е СМТ, понеже $\tilde a_\infty \leqq a_2 < a_1 < +\infty$, а $f_{\tilde a_\infty}$ съгласно т. 4.5.1 освен $[0,\ +\infty)$ СМ и $[0,\ a_1]$ до $[0,\ a_2]$. Но има и други a, за които f_a СМ $[0,\ a_1]$ до $[0,\ a_2]$. Ще покажем, че те удовлетворяват неравенствата $\tilde a_\infty \leqq a \leqq \tilde a^{(1)} = \operatorname{ctg}\left(\dfrac{2}{3}\operatorname{arc\,tg} a_1\right)$. Наистина от $\tilde a_2^{(1)} < \tilde a_\infty \leqq a_2 \leqq x \leqq a_1 < +\infty$ и $\dfrac{\partial f_a}{\partial a} > 0$ следва, че $|f_a(x)| \leqq a_2$ за $x \in [a_2,\ a_1]$:

$$-a_2 \leqq -\tilde a_\infty = f_{\tilde a_\infty}(\tilde a_\infty) \leqq f_{\tilde a_\infty}(x) \leqq f_a(x) \leqq f_{\tilde a^{(1)}}(x) \leqq f_{\tilde a^{(1)}}(a_1) = \tilde a_2^{(1)} < a_2.$$

Нека отбележим, че при $a_2 \geqq \dfrac{1}{\sqrt{3}}$, $\dfrac{1-a_2^2}{2a_2} \leqq \dfrac{1}{\sqrt{3}}$ и условието $\tilde a_\infty < a$ е еквивалентно на $\max\left(\tilde a_\infty,\ \dfrac{1-a_2^2}{2a_2}\right) \leqq a$.

Ако $0 < a_2 < \dfrac{1}{\sqrt{3}}$, $\exists\tilde{a}^{(2)}$ и $\exists\tilde{a}_1^{(2)}$ такива, че $\tilde{a}^{(2)}$, $\tilde{a}_1^{(2)}$ и a_2 са елементи на НСМТ. Тогава за $[0,\, a_1]$, $a_2 < a_1 \leqq \tilde{a}_1^{(2)}$, $f_{\tilde{a}^{(2)}}$ е СМ до $[0,\, a_2]$. Наистина, след като $f_{\tilde{a}^{(2)}}$ е НСМТ за $[0,\, \tilde{a}_1^{(2)}]$ до $[0,\, a_2]$, от $a_2 \leqq x \leqq a_1 \leqq \tilde{a}_1^{(2)}$ следва $-a_2 = f_{\tilde{a}^{(2)}}(a_2) \leqq f_{\tilde{a}^{(2)}}(x) \leqq f_{\tilde{a}^{(2)}}(\tilde{a}_1^{(2)}) = a_2$, т. е. $|f_{\tilde{a}^{(2)}}(x)| \leqq a_2$ за $x \in [a_2,\, a_1]$.

Но ако a_1 и a_2 не са елементи на НСМТ, т. е. $a_2 = \tilde{a}_2^{(1)}$ и $a_1 = \tilde{a}_1^{(2)}$, има и други $a \neq \tilde{a}^{(1)} = \tilde{a}^{(2)}$, за които f_a СМ $[0,\, a_1]$ до $[0,\, a_2]$. Наистина за a, $\dfrac{1-a_2^2}{2a_2} = \tilde{a}^{(2)} \leqq a \leqq \tilde{a}^{(1)} = \operatorname{ctg}\left(\dfrac{2}{3}\operatorname{arc\,tg} a_1\right)$, $\tilde{a}_2^{(1)} \leqq a_2 \leqq x \leqq a_1 \leqq \tilde{a}_1^{(2)}$ и $\dfrac{\partial f_a}{\partial a} > 0$ получаваме $-a_2 = f_{\tilde{a}^{(2)}}(a_2) \leqq f_{\tilde{a}^{(2)}}(x) \leqq f_a(x) \leqq f_{\tilde{a}^{(1)}}(x) = f_{\tilde{a}^{(1)}}(a_1)$ $= \tilde{a}_2^{(1)} \leqq a_2$, т. е. $|f_a(x)| \leqq a_2$ за $x \in [a_2,\, a_1]$, или графически:



Нека отбележим, че при $0 < a_2 < \dfrac{1}{\sqrt{3}}$, $\tilde{a}^{(2)} = \dfrac{1-a_2^2}{2a_2} > \tilde{a}_\infty$, т. е. условието $\tilde{a}^{(2)} \leqq a$ е еквивалентно на $\max\left(\tilde{a}_\infty,\, \dfrac{1-a_2^2}{2a_2}\right) \leqq a$.

Като обединим двата случая $a_2 \geqq \dfrac{1}{\sqrt{3}}$ и $0 < a_2 < \dfrac{1}{\sqrt{3}}$, получаваме, че f_a СМ $[0,\, a_1]$ до $[0,\, a_2]$, ако $0 < a_2 < a_1 < \tilde{a}_1^{(2)}$ и

$$\max\left(\tilde{a}_\infty,\, \frac{1-a_2^2}{2a_2}\right) \leqq a \leqq \operatorname{ctg}\left(\frac{2}{3}\operatorname{arc\,tg} a_1\right).$$

4.5.3. Дадено е $a_1 > 0$. За кои a и a_2 f_a СМ $[0,\, a_1]$ до $[0,\, a_2]$?

Нека $\tilde{a}_2^{(1)} \leqq a_2 < a_1$. Съгласно т. 4.5.2 f_a СМ $[0,\, a_1]$ до $[0,\, a_2]$, ако $\max(\tilde{a}_\infty,\, \tilde{a}^{(2)}) \leqq a \leqq \tilde{a}^{(1)}$.

4.5.4. Дадени са a_1 и a_2. За кои a f_a СМ $[0,\, a_1]$ до $[0,\, a_2]$?

Ако $\tilde{a}_2^{(1)} \leqq a_2 < a_1 \leqq \left[{+\infty \atop \tilde{a}_1^{(2)}}\right]$ и $0 < a_2\left[{\geqq \atop <}\right]\dfrac{1}{\sqrt{3}}$, от т. 4.5.2 и 4.5.3 следва, че f_a СМ $[0,\, a_1]$ до $[0,\, a_2]$, ако a удовлетворява условието от т. 4.5.3.

4.5.5. Дадени са a_1 и a_2, $0 < a_2 < a_1$. Тогава $\exists a^{(1,2)}$, $f_{a^{(1,2)}}(a_1) = a_2$, $\dfrac{a^{(1,2)}a_1^{-1}}{a_1 + a^{(1,2)}} = a_2$, т. е. $a^{(1,2)} = \dfrac{1+a_1a_2}{a_1-a_2}$. Кога $f_{a^{(1,2)}}$ СМ $[0,\, a_1]$ до $[0,\, a_2]$?

Съгласно т. 4.5.1 при $a^{(1,2)} \geqq \tilde{a}_\infty$ $f_{a^{(1,2)}}$ СМ $[0,\ a_1]$ до $[0,\ a_2]$, ако $\tilde{a}_2^{(1)} \leqq a_2 < a_1 \leqq \tilde{a}_1^{(2)}$, и при $0 < a^{(1,2)} < \tilde{a}_\infty$, ако $\tilde{a}_2^{(1,2)} \leqq a_2 < a_1$.

4.6. От гледна точка на икономия на ресурси при програмирането е важно да се изследва кога се получава последователно СМ на един интервал чрез многократно прилагане на една и съща f_a (програмата на няколко различни трансформации, заедно с константите им, е по-дълга от тази на една, използвана в цикъл, най-малко поради елиминиране на масива от параметрите $a_{,i}$ на $f_{a,i}$). Ако $I_{\text{at},f} \geqq \tilde{a}_\infty$, това се извършва еднократно поне с $f_{\tilde{a}_\infty}$.

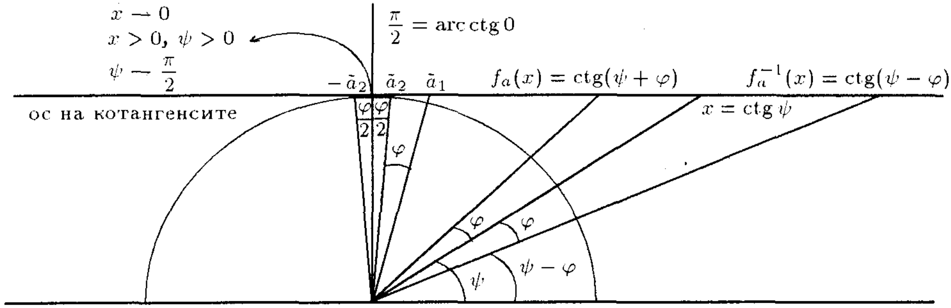
4.6.1. Теорема 6. Чрез многократно прилагане на f_a $[0,\ \left[{+\infty \atop a_1}\right]]$ може да се СМ до $[0;\ I_{\text{at},f}]$, $I_{\text{at},f} < \tilde{a}_\infty$, за такова a, че $\tilde{a}_2 \leqq I_{\text{at},f}$.

Доказателство. f_a не може да СМ $[0,\ \tilde{a}_2]$ до $[0,\ a_2]$, $a_2 < \tilde{a}_2$, понеже от $\dfrac{\partial f_a}{\partial x} > 0$ следва, че $f_a(a_2) < f_a(\tilde{a}_2) = -\tilde{a}_2$, $|f_a(a_2)| > \tilde{a}_2 > a_2$ и следователно няма СМ.

Нека $\tilde{a}_2 \leqq I_{\text{at},f}$, $\sqrt{a^2+1} - a \leqq I_{\text{at},f} < \tilde{a}_\infty$, $a \geqq \dfrac{1 - I_{\text{at},f}^2}{2I_{\text{at},f}} > \tilde{a}_\infty > I_{\text{at},f}$.

След първото прилагане на f_a $[0,\ +\infty)$ се СМ до $[0, a]$, $a > I_{\text{at},f} > \tilde{a}_2$. Ако $\tilde{a}_2 \leqq f_a(a)$, при второто прилагане $[0,\ a]$ се СМ до $[0,\ f_a(a)]$. Наистина от $\tilde{a}_2 \leqq f_a(a) \leqq x \leqq a$ следва, че $-f_a(a) \leqq -\tilde{a}_2 \leqq f_a(\tilde{a}_2) \leqq f_a(x) \leqq f_a(a)$, т. е. $|f_a(x)| \leqq f_a(a)$ и $[0,\ a]$ се СМ до $[0,\ f_a(a)]$. Аналогично, ако $\tilde{a}_2 \leqq f_a(f_a(a))$, при третото прилагане на f_a $[0,\ f_a(a)]$ се СМ до $[0,\ f_a(f_a(a))]$ и т. н. Нека означим $a_{2,k} = a_{1,(k+1)} = f_a(a_{1,k}) = \dfrac{aa_{1,k} - 1}{a_{1,k} + a}$, $k \geqq 1$, $a_{1,1} = +\infty$, $a_{2,1} = a_{1,2} = a$. Ако $a_{2,k} \leqq I_{\text{at},f} < a_{2,(k-1)}$, $k > 1$, след k-кратно прилагане на f_a $[0,\ +\infty)$ се СМ до $[0;\ I_{\text{at},f}]$. Но съществуването на такова k не е очевидно, понеже f_a не е НСМТ за интервалите $[0;\ a_{1,k}]$ с изключение най-много на един, не важат оценките $\dfrac{1}{4}a_{1,k} < a_{2,k} < \dfrac{1}{3}a_{1,k}$ и евентуално би могло $a_{2,k} \to \overline{a}_2 > I_{\text{at},f}$, когато $k \to +\infty$. За да покажем, че това не е възможно, ще използваме тригонометричната интерпретация на f_a.

4.6.2. Нека $a = \operatorname{ctg}\varphi > 0$, $0 < \varphi = \operatorname{arcctg} a < \dfrac{\pi}{2}$, $x = \operatorname{ctg}\psi$, $0 < \psi = \operatorname{arcctg} x < \dfrac{\pi}{2}$. Тогава $f_a(x) = \dfrac{\operatorname{ctg}\varphi \operatorname{ctg}\psi - 1}{\operatorname{ctg}\psi + \operatorname{ctg}\varphi} = \operatorname{ctg}(\psi + \varphi)$, аналогично $f_a^{-1}(x) = \operatorname{ctg}(\psi - \varphi)$. Оттук индуктивно се получава $f_{a,n}(+\infty) = \underbrace{f_a(\ldots(f_a}_{n \text{ пъти}}(+\infty))\ldots) = f_{a,(n-1)}(a) = \operatorname{ctg} n\varphi$ и $f_{a,n}^{-1}(x) = \underbrace{f_a^{-1}(\ldots(f_a^{-1}}_{n \text{ пъти}}(x))\ldots) = \operatorname{ctg}(\psi - n\varphi)$. Също така $\tilde{a}_2 = \sqrt{a^2+1} - a = \operatorname{ctg}\left(\dfrac{\pi}{2} - \dfrac{\varphi}{2}\right) = \operatorname{tg}\dfrac{\varphi}{2}$ и $\tilde{a}_1 = f_a^{-1}(\tilde{a}_2) = \operatorname{ctg}\left(\left(\dfrac{\pi}{2} - \dfrac{\varphi}{2}\right) - \varphi\right) = \operatorname{ctg}\left(\dfrac{\pi}{2} - \dfrac{3\varphi}{2}\right) = \operatorname{tg}\dfrac{3\varphi}{2}$, или графически:



От тригонометричната интерпретация на $f_a(x)$ могат да се изведат всичките ѝ свойства, както и резултатите, получени дотук.

Сега ще покажем, че търсеното k съществува. То се определя от уравнението $f_{a,k}(+\infty) = \operatorname{ctg} k\varphi = I_{\mathrm{at},f} > 0$, $0 < k\varphi < \dfrac{\pi}{2}$, $k = \dfrac{\operatorname{arc\,ctg} I_{\mathrm{at},f}}{\operatorname{arc\,ctg} a} > 1$, понеже $I_{\mathrm{at},f} < a$. И така след k-кратно прилагане на f_a $+\infty$ се трансформира в $I_{\mathrm{at},f}$, $[0, +\infty)$ се СМ до $[0; I_{\mathrm{at},f}]$, с което теорема 6 е доказана.

4.6.3. Практически интерес представляват f_a, които се „изчерпват" след двукратно прилагане при СМ на $[0, +\infty)$. За да има СМ на $[0, +\infty)$ до $[0, a]$ при първото прилагане на f_a, необходимо е $a \geqq \tilde{a}_\infty$. За да има максимално СМ на $[0, a]$ до $[0, \tilde{a}_2]$ при второто прилагане на f_a, трябва $a^* \leqq \tilde{a}_2$, т. е. $f_a(a) = \dfrac{a^2-1}{2a} \leqq \sqrt{a^2+1} - a$, $0 \leqq 3a^2 - 1 \leqq 2a\sqrt{a^2+1}$, $5a^4 - 10a^2 + 1 \leqq 0$. Това е изпълнено за $\sqrt{1 - \dfrac{2}{\sqrt{5}}} \leqq a \leqq \sqrt{1 + \dfrac{2}{\sqrt{5}}}$, но $\sqrt{1 - \dfrac{2}{\sqrt{5}}} < \dfrac{1}{\sqrt{3}}$, остава окончателно $\dfrac{1}{\sqrt{3}} \leqq a \leqq \sqrt{1 + \dfrac{2}{\sqrt{5}}} \lessapprox 1{,}376\ldots$ Понеже $\tilde{a}_2$ е с. м. н. ф. на a, при двукратно прилагане на f_a ще има максимално СМ на $[0, +\infty)$ за $\tilde{a}_{2\min} = \tilde{a}_2\Big|_{a=\sqrt{1+\frac{2}{\sqrt{5}}}} = \sqrt{2 + \dfrac{2}{\sqrt{5}}} - \sqrt{1 + \dfrac{2}{\sqrt{5}}}$ $\approx 0{,}326$. И така f_a, $a = \sqrt{1 + \dfrac{2}{\sqrt{5}}}$, СМ $[0, +\infty)$ до $[0; 0{,}326]$, което е максимално за двукратно прилагане на една и съща f_a.

4.6.4. Многократното използване на една и съща f_a скъсява програмата за сметка на бързодействие. Например за $I_{\mathrm{at},f} = 0{,}1$, $a > 4{,}95$, нека $a = 5$. Тогава f_5 трябва да се приложи до 8 пъти, за да се СМ $[0, +\infty)$ до $[0; 0{,}1]$. Ако се приложат различни трансформации, например НСМТ, достатъчни са само три: $[0, +\infty) \xrightarrow{f_{\tilde{a}_\infty}} [0, \tilde{a}_\infty] \xrightarrow{f_{\operatorname{tg}\frac{\pi}{9}}} \left[0, \operatorname{tg}\dfrac{\pi}{18}\right]$, $\operatorname{tg}\dfrac{\pi}{9} \approx \dfrac{1}{28}$,

$\operatorname{tg}\frac{\pi}{18} \approx 0{,}173$, и накрая трета и последна f_a, $a > 4{,}95$, която СМ $[0;\ 0{,}173]$ до $[0;\ 0{,}1]$ например f_5. С намаляването на $I_{\mathrm{at},f}$ от гледна точка на бързодействие става все по–изгодно използването на серия от НСМТ.

4.7. Ще покажем как практически се прилага Т-методът при $\operatorname{arc\,tg} x$ за СМ на $[0, +\infty)$ до $[0;\ I_{\mathrm{at},f}]$.

Ако $I_{\mathrm{at},f} \geqq \tilde{a}_\infty$, достатъчна е само една трансформация, например $f_{\tilde{a}_\infty}$, за СМ на $[0, +\infty)$ до $[0, \tilde{a}_\infty] \subseteq [0;\ I_{\mathrm{at},f}]$. В частност, ако $I_{\mathrm{at},f} > \tilde{a}_\infty$, $f_{I_{\mathrm{at},f}}$ СМ $[0, +\infty)$ до $[0;\ I_{\mathrm{at},f}]$ или, ако се избере a, $\tilde{a}_\infty < a < I_{\mathrm{at},f}$, f_a СМ $[0, +\infty)$ до $[0, a] \subset [0;\ I_{\mathrm{at},f}]$.

Нека $I_{\mathrm{at},f} < \tilde{a}_\infty$. Да разгледаме системата I^{at}, определена от $I_f = \tilde{a}_{2,f}$, $I_s = \tilde{a}^{(2)}_{1,s+1} = \tilde{a}_{2,s} = f^{-1}_{\tilde{a}^{(2)}_{(s+1)}}(I_{s+1}) = \operatorname{tg}(3 \operatorname{arc\,tg} \tilde{a}_{2,(s+1)} = \operatorname{tg}(3^{f-s} \operatorname{arc\,tg} I_f)$, т. е. $[0, I_s]$ се СМ до $[0, I_{s+1}]$ чрез НСМТ, $s = \overline{f-1,\ 1}$. f се определя като НМЦЧ, за което $\operatorname{tg}\frac{\pi}{6} = \frac{1}{\sqrt{3}} \leqq I_1 = \operatorname{tg}(3^{f-1} \operatorname{arc\,tg} I_f)$, $f = \left]\frac{\ln(\pi/6 \operatorname{arc\,tg} I_f)}{\ln 3}\right[+ 1$.

Остава да се разгледа първоначалното СМ на $[0, +\infty)$ до $[0, a] \subseteq [0, I_1]$ чрез f_a. $f_{\tilde{a}_\infty}$ СМ $[0, +\infty)$ до $[0, \tilde{a}_\infty] \subset [0, I_1]$. f_{I_1} го СМ до $[0, I_1]$. И в двата случая $\mathrm{tr}^{\max} = f$. $(I_1, +\infty)$ се трансформира чрез f_a в (I_1^*, a). В някои случаи $-a \leqq I_1^* = f_a(I_1)$ например за $a \geqq \sqrt{I_1^2+1} - I_1 = \tilde{a}_2$ на f_{I_1}. Ако $\sqrt{I_1^2+1} - I_1 \leqq a \leqq I_k$, $2 \leqq k \leqq f-1$, тогава за точките от $[I_1, +\infty)$ ще са необходими не повече трансформации, отколкото за тези от $[0, I_{k-1}]$, и тогава $\mathrm{tr}^{\max} = f-1$. Това е възможно, ако $\sqrt{I_1^2+1} - I_1 \leqq I_k$, $\frac{1-I_k^2}{2I_k} \leqq I_1$. Понеже $I_k = \tilde{a}_{2,k}$, $I_{k-1} = \tilde{a}_{1,k}$, $\frac{1-I_k^2}{2I_k} = \frac{1-\tilde{a}^2_{2,k}}{2\tilde{a}_{2,k}} = \tilde{a}^{(2)}_{,k} = \tilde{a}^{(1)}_{,k} = \operatorname{ctg}\left(\frac{2}{3} \operatorname{arc\,tg} \tilde{a}_{1,k}\right)$.
Следователно неравенството е еквивалентно на

$$\operatorname{ctg}\left(\frac{2}{3} \operatorname{arc\,tg} I_{k-1}\right) \leqq I_1, \text{ т. е. } \operatorname{ctg}(2.3^{f-k} \operatorname{arc\,tg} I_f) \leqq \operatorname{tg}(3^{f-1} \operatorname{arc\,tg} I_f),$$

$$\frac{\pi}{2} - 2.3^{f-k} \operatorname{arc\,tg} I_f \leqq 3^{f-1} \operatorname{arc\,tg} I_f, \quad \frac{\pi}{2} - 3^{f-1} \operatorname{arc\,tg} I_f \leqq 2.3^{f-k} \operatorname{arc\,tg} I_f,$$

откъдето $k = f - \left]\frac{\ln\left(\frac{\pi}{4 \operatorname{arc\,tg} I_f} - \frac{3^{f-1}}{2}\right)}{\ln 3}\right[$.

Ако $I_1 = \tilde{a}_\infty$, т. е. $I_{\mathrm{at},f} = f_{\tilde{a},f}(f_{\tilde{a},(f-1)}(\ldots(f_{\tilde{a}_\infty}(+\infty))\ldots))$, всичките трансформации са различни и НСМТ. Ако бъдат избрани други трансформации, общият им брой ще се увеличи поне с единица.

На практика $I_1 > a_\infty$, т. е.

$$f_{\tilde{a},f}(\ldots(f_{\tilde{a}_\infty}(+\infty))\ldots) < I_{\mathrm{at},f} < f_{\tilde{a},(f-1)}(\ldots(f_{\tilde{a}_\infty}(+\infty))\ldots).$$

Ако $\tilde{a}_{,s}$ или $\operatorname{arc\,tg}\dfrac{1}{\tilde{a}_{,s}}$ трябва да са къси, НСМТ трябва да се заменят с близки до тях, без да се нарушат неравенствата за $I_{\mathrm{at},f}$. В този случай трябва да се използват резултатите от т. 4.4 ÷ 4.6. Колкото $I_{\mathrm{at},f}$ е по-близо до $f_{\tilde{a},(f-1)}(\ldots(f_{\tilde{a}_\infty}(+\infty))\ldots)$, толкова възможността за изменение на трансформациите без увеличаване на броя им е по-голяма.

Ако серията трансформации е само от НСМТ, интервалите $[\tilde{a}_{2,s};\,\tilde{a}_{1,s}]$, съответни на $f_{\tilde{a}_{,s}}$, съвпадат с тези от системата I^{at} и по същия начин са долепени (конкатенирани) един за друг. Ако поне една от трансформациите не е НСМТ, ще има поне едно припокриване на интервали, съответни на $f_{\tilde{a}_{,s}}$, което е без значение, ако никъде няма разстояние между интервалите на две последователни трансформации.

СМ на $[0,+\infty)$ до $[0;\,I_{\mathrm{at},f}]$, $I_{\mathrm{at},f}$ — съответно на $p=m=10$, е разгледано подробно в [1] на базата на получените дотук резултати.

4.8. При променливо m (f расте с m, а $I_{\mathrm{at},f}$ намалява) трябва да се съхраняват четири масива от константите $\tilde{a}_{,s}$, $\tilde{a}^2_{,s}+1$ (може да се пресмятат с $1\times$ и $1+$), $\tilde{a}_{2,s}$ и $\operatorname{arc\,tg}\tilde{a}^{-1}_{,s}$ с дължина на мантисата $m_{\max}$, всеки с дължина $f_{m_{\max}}$, като се използват НСМТ или близки до тях. $|x|\leqq 1$ се намалява от 3^s до 4^s пъти след s трансформации.

5. $y=\begin{bmatrix}\sin\\ \cos\end{bmatrix}x_1$, $x_1\in B_{\left[\substack{\sin\\ \cos}\right],1}=\left[-\dfrac{\pi}{2},\,\dfrac{\pi}{2}\right]$.

Известно е [3], че $\sin(2n+1)x=(-1)^nT_{2n+1}(\sin x)$, $\cos nx=T_n(\cos x)$ и $\sin 2nx$ не може да се изрази рационално чрез $\sin x$.

$T_n(x)=\sum\limits_{i=0}^{[\frac{n}{2}]}(-1)^i2^{n-2i-1}\dfrac{n}{n-i}\dbinom{n-i}{i}x^{n-2i}$ са полиномите на Чебишев от I-и род (от ИА е по-добре $\dfrac{n}{n-i}\dbinom{n-i}{i}=\dfrac{n}{i}\dbinom{n-1-i}{i-1}$, но трябва да се приеме, че $\dfrac{n}{0}\dbinom{n-1}{-1}=1$ за $i=0$). Тогава $\cos(2n+1)x=T_{2n+1}(\cos x)$,

$$T_{2n+1}(x)=\sum_{i=0}^{n}(-1)^i2^{2(n-i)}\frac{2n+1}{2n+1-i}\binom{2n+1-i}{i}x^{2(n-i)+1}=xP^*_n(x^2),$$

където $P^*_n(x)=\dfrac{T_{2n+1}(\sqrt{x})}{x}=\sum\limits_{i=0}^{n}(-1)^i2^{2(n-i)}\dfrac{2n+1}{2n+1-i}\dbinom{2n+1-i}{i}x^{n-i}$.

$$\begin{bmatrix}\sin\\ \cos\end{bmatrix}(2n+1)x=\begin{bmatrix}(-1)^n\\ 1\end{bmatrix}T_{2n+1}\left(\begin{bmatrix}\sin x\\ \cos x\end{bmatrix}\right)=\begin{bmatrix}(-1)^n\sin x\\ \cos x\end{bmatrix}P^*_n\left(\begin{bmatrix}\sin x\\ \cos x\end{bmatrix}^2\right),$$

т. е. СМ на $\left[0,\,\dfrac{\pi}{2}\right]$ е практически еднакво за $\sin x$ и $\cos x$.

$$\cos 2nx=T_{2n}(\cos x)=P^{**}_n(\cos^2 x),$$

където $T_{2n}(x)=\sum\limits_{i=0}^{n}(-1)^i2^{2(n-i)-1}\dfrac{2n}{2n-i}\dbinom{2n-i}{i}x^{2(n-i)}=P^{**}_n(x^2)$, а $P_n^{**}(x) = T_{2n}(\sqrt{x}) = \sum_{i=0}^{n}(-1)^i 2^{2(n-i)-1}\frac{2n}{2n-i}\binom{2n-i}{i}x^{n-i}$.

И така при $\sin x$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ трябва да се намалява нечетен брой пъти, докато при $\cos x$ няма такова ограничение.

От $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ лесно се получава

$$T_n(x) = 2(2x^2-1)T_{n-2}(x) - T_{n-4}(x).$$

Тогава $\quad T_{2n+1}(\cos x) = 2(2\cos^2 x - 1)T_{2n-1}(\cos x) - T_{2n-3}(\cos x),$

т. е. $\quad \cos(2n+1)x = 2(2\cos^2 x - 1)\cos(2n-1)x - \cos(2n-3)x.$

Можем да получим следната, както и съответната ѝ за $\sin(2n+1)x$, рекурентна зависимост и от представянето на $\left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right](2n+1)x + \left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right](2n-3)x$ като произведение:

$$x_n = Ax_{n-1} - x_{n-2},\ n \geqq 1,\ x_{-1} = \begin{bmatrix}\sin(-1)x\\ \cos(-1)x\end{bmatrix} = \begin{bmatrix}-\sin x\\ \cos x\end{bmatrix},\ x_0 = \begin{bmatrix}\sin(1)x\\ \cos(1)x\end{bmatrix} = \begin{bmatrix}\sin x\\ \cos x\end{bmatrix},$$

$$A(x) = 2(2x^2-1), \quad A = \begin{bmatrix}-A(\sin x)\\ A(\cos x)\end{bmatrix}, \quad x_n = \begin{bmatrix}\sin\\ \cos\end{bmatrix}(2n+1)x,$$

или $x_n = \left[\begin{smallmatrix}\sin x\\ \cos x\end{smallmatrix}\right]$, като x в x_{-1}, x_0 и A е заменено с $\dfrac{x}{2n+1}$.

Аналогично $T_{2n}(\cos x) = 2(2\cos^2 x - 1)T_{2n-2}(\cos x) - T_{2n-4}(\cos x)$. $x_n = \cos 2nx$ при начални условия $x_{-1} = \cos(-2)x = \dfrac{A(\cos x)}{2}$, $x_0 = \cos 0x = 1$, или $x_n = \cos x$, като x в x_{-1}, x_0 и A е заменено с $\dfrac{x}{2n}$.

За пресмятането на $A\left(\left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right](2n+1)x\right)$ се изисқват еднократно $1:$, $1\times$ и $3+$, а за всяка итерация — по $1\times$ и $1+$. За $x_n = \left[\begin{smallmatrix}\sin x\\ \cos x\end{smallmatrix}\right]$ са необходими n итерации, които заедно с A изискват $1:$, $(n+1)_\times$ и $(n+3)_+$, докато за пресмятането им чрез $P_n^*(x)$ се изискват $1:$, $2\times$, $(n+1)_+$ и $n_\times$ къси, заедно със съхраняването на $n+1$ къси коефициента на $P_n^*(x)$. И така рекурентното пресмятане на $\left[\begin{smallmatrix}\sin x\\ \cos x\end{smallmatrix}\right]$ чрез $\left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right]\dfrac{x}{2n+1}$ е по-бавно от това чрез $P_n^*(x)$ по Хорнер, но липсата на константи е голямо преимущество при променливо m, при което x се дели на $2n+1$, $n = n(m)$. В този случай трябва да се пазят коефициентите на $n+1$ полинома $P_s^*(x)$, $s = \overline{0, n(m_{\max})}$, или за всяко x, зададено с m цифри, да се изчисляват рекурентно коефициентите на $P_n^*(x)$, където n съответства на това m. Дори при достатъчно голямо фиксирано m и съответно също голямо n има смисъл $\left[\begin{smallmatrix}\sin x\\ \cos x\end{smallmatrix}\right]$ да се пресмятат рекурентно, особено при $p = 2$, когато $t_\times \gtrapprox t_+$ и понятията късо число и късо умножение са без значение.

Пример. При $p = m = 10$ $I_{\left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right], f} \lessapprox \frac{\pi}{6}$, $f = 2$, $B_{\left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right], f} = B_{\left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right], 2} = \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$, $x_2 = \frac{1}{3}x_1$, $\left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right]x_1 = \left[\begin{smallmatrix}-\sin\\ \cos\end{smallmatrix}\right]x_2[4\left[\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right]^2 x_2 - 3]$, ($3\times$, $1+$ и 1×4 късо). При $\sin x$

не може да се избере $B_{\sin,f} = \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$, понеже $\sin x$ не се изразява рационално чрез $\sin\frac{x}{4}$. При $\cos x$ x може да се намали 4 пъти, като $\cos x = 8\cos^2 0{,}25x(\cos^2 0{,}25x - 1) + 1$ (2×, 2+ и 2× къси). При това $B_{\cos,f} = \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$ за $p = 10$ и $m = 11$.

6. $\boldsymbol{y = \arcsin x_1}$, $x_1 \in D_{\arcsin} = B_{\arcsin,1} = [-1, 1]$.

Понеже $\arcsin x$ е нечетна, $B_{\arcsin,s} = B_{as,s}$ са симетрични относно нулата и е достатъчно да се покаже как $[0,1]$ се СМ до $[0; I_{as,f}]$.

С-методът е неприложим при $\arcsin x$.

Ще приложим Т-метода за СМ на $[0,1]$. Нека разгледаме формулата $\arcsin x + \arcsin x = 2\arcsin x = \arcsin(2x\sqrt{1-x^2}), |x| \leqq \frac{1}{\sqrt{2}}$.

Нека извършим субституцията $X = 2x\sqrt{1-x^2}$. От $|x| \leqq \frac{1}{2}$ следва $|X| \leqq 1$. Намираме $x^2 = \frac{1 \pm \sqrt{1-X^2}}{2}$, но $x^2 \leqq \frac{1}{2}$ по условие, а $\frac{1+\sqrt{1-X^2}}{2} \geqq \frac{1}{2}$, следователно $x^2 = \frac{1-\sqrt{1-X^2}}{2} = \frac{X^2}{2(1+\sqrt{1-X^2})} \leqq \frac{1}{2}$.
Ако сега разменим страните на равенството и заменим X с x, получаваме друга събирателна формула, която ще използваме по-нататък:

$$\arcsin x = (\text{sgn}x)2\arcsin\frac{|x|}{\sqrt{2(1+\sqrt{1-x^2})}},$$

$$|x| \leqq 1, \quad \frac{1}{2}|x| \leqq \frac{|x|}{\sqrt{2(1+\sqrt{1-x^2})}} \leqq \frac{1}{\sqrt{2}}|x|$$

(равенството отляво се достига за $x = 0$, а отдясно — за $|x| = 1$).

Редицата $\{x_n\}_{n=1}^{\infty}$, $x_n \geqq 0$, зададена рекурентно с $0 \leqq x_1 \leqq 1$ и $x_{n+1} = \frac{x_n}{\sqrt{2(1+\sqrt{1-x_n^2})}}$, $n \geqq 1$, се мажорира от нулевата редица с общ член $\left(\frac{1}{\sqrt{2}}\right)^{n-1} \to 0$ при $n \to +\infty$, понеже $x_1 \leqq 1$, $x_2 = \frac{x_1}{\sqrt{2(1+\sqrt{1-x_1^2})}} \leqq \frac{1}{\sqrt{2}}x_1 \leqq \frac{1}{\sqrt{2}}$, а от допускането, че $x_n \leqq \left(\frac{1}{\sqrt{2}}\right)^{n-1}$, аналогично следва, че и $x_{n+1} \leqq \left(\frac{1}{\sqrt{2}}\right)^{n}$. Следователно и $x_n \to 0$. Ако $|x_1| \leqq 1$,

$$\arcsin x_1 = (\text{sgn}x_1)2\arcsin x_2 = \cdots = (\text{sgn}x_1)2^n \arcsin x_{n+1}.$$

Окончателно $\arcsin x_1 = (\operatorname{sgn} x_1) 2^n \arcsin x_{n+1}$ за $|x_1| \leqq 1$, $x_n \geqq 0$, $n \geqq 1$, $x_{n+1}^2 = \dfrac{x_n^2}{2(1+\sqrt{1-x_n^2})}$ ($1_{\sqrt{}}$, $1_{:}$, 1_{+} и $1_{\times}$ късо).

От ИА е по-добре вместо x_{n+1} да се получава от x_n и да се проверява дали $x_{n+1} \leqq I_{\mathrm{as},f}$, x_{n+1}^2, да се получава от x_n^2 и да се проверява дали $x_{n+1}^2 \leqq I_{\mathrm{as},f}^2$. Това пести по $1_{\sqrt{}}$ $\left(x_{n+1} = \sqrt{\dfrac{x_n^2}{\sqrt{\cdots}}}\right)$ и $1\times(x_n^2$ в $\sqrt{1-x_n^2})$ на всяка итерация, като само накрая $x_{n+1} = \sqrt{x_{n+1}^2}$. Или, понеже $x_{n+1} = 0,5(\sqrt{1+x_n} - \sqrt{1-x_n})$, ($2_{\sqrt{}}$, 3_{+} и $1\times$ късо), извършват се $1_{\sqrt{}}$ и 2_{+} срещу $1:$ на всяка итерация.

Сходимостта на $\{x_n\}_{n=1}^{\infty}$ намалява с $|x_1| \to 1$, като е най-бавна за $I_1 = |x_1| = 1$. Тогава $I_2 = x_2 = \dfrac{1}{\sqrt{2}} \approx 0,707$, $I_3 = x_3 \approx 0,366$, $I_4 = x_4 \approx 0,186$, $I_5 = x_5 \approx 0,009$, $I_6 = x_6 \approx 0,047$, $I_7 = x_7 \approx 0,023$, $I_8 = x_8 \approx 0,012$, $I_9 = x_9 \approx 0,006$ и т. н. При $p = m = 10$ $I_{\mathrm{as},f} = 0,1$, $f = 5$ и $\mathrm{tr}^{\max} = 4$. В най-лошия случай коефициентът на намаляване k_1 на $x_1 = 1$ е $\dfrac{1}{\sqrt{2}}$, а при x_s, $s \geqq 2$ — приблизително $\dfrac{1}{2}$, т. е. при s итерации k_s на x_1 е $k_s \lessapprox \dfrac{1}{\sqrt{2}}\left(\dfrac{1}{2}\right)^{s-1} = \dfrac{\sqrt{2}}{2^s}$. Това води до по-бавна сходимост, отколкото при $\operatorname{arctg} x$, където $k_s > \dfrac{1}{3^s}$.

За намиране на точния брой итерации при $|x| \leqq 1$ може да се използва обратната рекурентна формула $x_n^2 = 4x_{n+1}^2(1 - x_{n+1}^2)$. Ако $x_{n+1} = x_f = I_{\mathrm{as},f}$, при $p = m = 10$ получаваме $x_{f-1} \approx 0{,}2$, т. е. за $x_{f-1} \in (0{,}1;\ 0{,}2]$ е необходима само една итерация. Чрез двукратно прилагане на формулата получаваме $x_{f-2} \approx 0{,}392$, т. е. за $x_{f-2} \in [0{,}2;\ 0{,}392]$ са необходими две итерации и т. н.

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ПРИЛОЖЕНИЕ НА МЕТОДИТЕ НА МАТЕМАТИЧЕСКАТА МОРФОЛОГИЯ В ТЕОРИЯТА НА РАЗМИТИТЕ МНОЖЕСТВА

АНТОНИЙ ПОПОВ

Антоний Попов. ПРИЛОЖЕНИЕ МЕТОДОВ МАТЕМАТИЧЕСКОЙ МОРФОЛОГИИ В ТЕОРИИ НЕЧЕТКИХ МНОЖЕСТВ

В настоящей работе отмечены основные свойства морфологических операций над полных решетках — дилатации, эрозии, открытия и замыкания. Доказывается, что определения Вермана и Пелега нечетких морфологических операторов (см. [10]) вписываются в более общую концепцию нечеткой морфологии, которая введена в работе [9] и базируется на понятии индикатора. В последней части, с помощью дилатации Вермана и Пелега, рассматривается понятие нечеткой дифференцируемой функции. Это определение аналогично определению из работы [5]. Здесь исправлена ошибка, допущенная в [5], которая возникает из-за того, что нельзя определить Хаусдорфовое расстояние между пустым и непустым множеством.

Antony Popov. AN APPLICATION OF THE MATHEMATICAL MORPHOLOGY METHODS IN FUZZY SETS THEORY

In this paper we briefly recall the definition of a complete lattice and some basic properties of morphological operations — dilations, erosions, openings and closings, used in the sequel. For completeness, we recall the properties of the dilations and erosions in the family of convex compact sets in $\Re^n$. In this paper special emphasis is set on the fuzzy dilation and fuzzy erosion. It is shown also how Werman and Peleg's operations could be defined in the more general indicator framework. In the last section of the paper it is shown how the Werman and Peleg's notion of fuzzy dilation can be used to define the concept of Frechet-type derivative of a fuzzy function. The presented approach is analogous to those in [5], but a contradiction in Puri and Ralescu's definition is overcome.

1. ВЪВЕДЕНИЕ

Методът на математическата морфология е създаден от Сера и Матерон [4, 7], работещи по проблемите на минералогията и петрографията, по-точно по определяне на свойствата на порести материали в зависимост от тяхната геометрична структура. Основната идея на техния подход е сравняването на геометричната структура на изображението с малки образци, налагайки ги на различни места по изображението.

Първоначално математическата морфология е разработена за анализ на двоични изображения, които могат да бъдат интерпретирани математически като множества. Съответните морфологични оператори се изграждат на базата на теоретико-множествените операции обединение, сечение, допълнение и на геометричното преобразование транслация. Но по-нататък се появява нуждата от по-мощна теория, работеща върху пространства от затворените множества на дадено топологично пространство, от изпъкналите подмножества на линейно пространство, а също така и върху функционални пространства, като целта е да се анализират полутонови изображения. Пръв Сера [8] забелязва, че пространството, върху което действат морфологичните оператори, трябва да има свойствата на пълна решетка.

Настоящата работа използва дефиницията на морфологичните оператори дилатация и ерозия във вида, даден в [2]. За пълнота са представени някои широко известни примери на такива операции. Основното ударение е поставено върху описанието на морфологичните операции върху пространства от размити множества. Както е известно от работите на Верман и Пелег [10], полутоновите изображения могат да се интерпретират като размити множества. Дефинирането на морфологични операции върху размити множества дава друг поглед върху анализа на полутонови изображения с помощта на методите на математическата морфология и предоставя възможност за директно пренасяне на методи за анализ от двоични изображения към полутонови изображения. В светлината на морфологичните операции елегантно се изследват изпъкнали размити множества и се въвежда понятието диференцируема размита функция. Въведена е производна от тип на Фреше, която е аналог на производна на множественозначна функция [5].

2. ОСНОВНИ ПОНЯТИЯ НА МАТЕМАТИЧЕСКАТА МОРФОЛОГИЯ[1]

Едно от основните свойства на математическата морфология е това, че дефиниционната област на морфологичните оператори е пълна решетка, която ще означаваме с $\mathcal{L}$. Това означава, че в $\mathcal{L}$ е зададена частична

[1] Детайлно описание на основните свойства на морфологичните оператори читателят може да намери в работите на Матерон [4], Сера [7, 8], Хайманс и Ронс [2].

наредба „$\leq$" и всяко подмножество $\mathcal{H}$ на $\mathcal{L}$ притежава точна горна и точна долна граница, означавани съответно с $\bigvee \mathcal{H}$ и $\bigwedge \mathcal{H}$. Идентитета върху $\mathcal{L}$ ще означаваме с e, а минималния и максималния елемент на $\mathcal{L}$ — съответно с O и I. Операторите, действащи върху $\mathcal{L}$, също притежават наредба $\varphi \leq \psi$, означаваща $\varphi(X) \leq \psi(X)$ за всяко $X \in \mathcal{L}$. Тук ще разгледаме само дистрибутивни решетки, т.е. такива, за които

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z),$$
$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z).$$

Дефиниция. Операторът $\psi : \mathcal{L} \mapsto \mathcal{L}$ се нарича:

- *дилатация*, ако $\psi(\bigvee_{i\in I} X_i) = \bigvee_{i\in I} \psi(X_i)$ и $\psi(O) = O$;
- *ерозия*, ако $\psi(\bigwedge_{i\in I} X_i) = \bigwedge_{i\in I} \psi(X_i)$ и $\psi(I) = I$.

Дилатациите и ерозиите са монотонно растящи оператори. Казва се, че двойката (ε, δ) е *спрегната*, ако за всеки два елемента $X, Y \in \mathcal{L}$ е изпълнено

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y).$$

Следващите твърдения са доказани в [2].

Теорема 2.1. *Ако (ε, δ) е спрегната двойка оператори над решетката $\mathcal{L}$, то ε е ерозия, а δ е дилатация.*

Теорема 2.2. *Ако δ е дадена дилатация над решетката $\mathcal{L}$, то тя притежава единствена спрегната ерозия ε, и обратно, ако ε е дадена ерозия над решетката $\mathcal{L}$, то тя притежава единствена спрегната дилатация δ.*

Теорема 2.3. *Ако (ε, δ) е спрегната двойка, то $e \leq \varepsilon\delta$ и $\delta\varepsilon \leq e$.*

Лесно могат да се получат следните следствия (вж. [2]):

2.4. Ако (ε, δ) и (ε', δ') са спрегнати двойки, то $(\varepsilon'\varepsilon, \delta\delta')$ е също спрегната двойка.

2.5. Ако $(\varepsilon_i, \delta_i)$ са спрегнати двойки, то $(\bigwedge_i \varepsilon, \bigvee_i \delta)$ е също спрегната двойка.

2.6. Ако (ε, δ) е спрегната двойка, то $\varepsilon\delta\varepsilon = \varepsilon$ и $\delta\varepsilon\delta = \delta$.

Последното от тези твърдения следва от веригите неравенства

$$\delta = \delta e \leq \delta(\varepsilon\delta) = (\delta\varepsilon)\delta \leq e\delta = \delta, \tag{2.1}$$

$$\varepsilon = e\varepsilon \leq (\varepsilon\delta)\varepsilon = \varepsilon(\delta\varepsilon) \leq \varepsilon e = \varepsilon \tag{2.2}$$

и от монотонността на дилатацията и ерозията.

Операторът $\varphi = \varepsilon\delta$ се нарича *затворена обвивка* и има свойствата $X \leq \varphi(X)$ и $\varphi^2 = \varphi$. Операторът $\psi = \delta\varepsilon$ се нарича *отворено ядро* и притежава свойствата $\psi(X) \leq X$ и $\psi^2 = \psi$. Тези свойства на отворените ядра и затворените обвивки подсказват възможността за използването им при филтриране на сигнали [1]. По-нататък под морфологични оператори над дадена решетка ще разбираме ерозиите, дилатациите, отворените ядра и затворените обвивки.

3. ПРИМЕРИ НА ПЪЛНИ РЕШЕТКИ И МОРФОЛОГИЧНИ ОПЕРАТОРИ, ДЕФИНИРАНИ В ТЯХ

Разглеждаме група от автоморфизми T на $\mathcal{L}$. Един оператор $\psi : \mathcal{L} \mapsto \mathcal{L}$ се нарича Т-инвариантен, или просто Т-оператор, ако комутира с всеки елемент на T, т.е. $\psi\tau = \tau\psi$ за всяко $\tau \in T$. Лесно се доказва, че ако единият оператор от спрегнатата двойка (ε, δ) е Т-инвариантен, то и другият е такъв. Ако $\tau \in T$, то двойката (τ^{-1}, τ) е Т-инвариантна спрегната двойка. Доказателства на тези прости твърдения могат да бъдат намерени в [2, 8].

Едно подмножество $l \subset \mathcal{L}$ се нарича супремум-генериращо, ако всеки елемент на $\mathcal{L}$ може да се представи като точна горна граница на съвкупност от елементи на l. По-нататък ще разглеждаме само решетки $\mathcal{L}$, притежаващи абелева група автоморфизми T и супремум-генериращо подмножество l със следните свойства:

(i) T запазва l.

(ii) За всеки два елемента $x \in l$ и $y \in l$ съществува $\tau \in T$, така че $\tau(x) = y$.

В свойство (ii) τ е единствено. Да допуснем, че съществуват два автоморфизма τ_1 и τ_2, такива че $\tau_1(x) = \tau_2(x) = y$. Тогава $\tau_1^{-1}\tau_2(x) = x$. Нека z е произволен елемент от l. Следователно съществува $\tau_3 \in T$, тъй че $\tau_3(x) = z$. Тогава

$$\tau_1^{-1}\tau_2(z) = (\tau_1^{-1}\tau_2)\tau_3(x) = \tau_3(\tau_1^{-1}\tau_2)(x) = \tau_3(x) = z,$$

т.е. $\tau_1^{-1}\tau_2$ фиксира всеки елемент от l. Но тъй като l е супремум-генериращо множество и автоморфизмите комутират с операцията точна горна граница, следва, че $\tau_1^{-1}\tau_2 = e$, с което единствеността е доказана.

Като следваме [2], нека фиксираме елемент $o \in l$, който ще наричаме *начало*. Нека $h \in l$ е произволен елемент. С τ_h означаваме единствения автоморфизъм от T, довеждащ o в h. Нека A е произволен елемент на $\mathcal{L}$. Означаваме $l(A) = \{a \in l : a \leq A\}$. Тогава можем да дефинираме следните операции:

$$\delta_A = \bigvee_{a \in l(A)} \tau_a, \tag{3.1}$$

$$\varepsilon_A = \bigwedge_{a \in l(A)} \tau_a^{-1}. \tag{3.2}$$

От твърдение (2.5) и от факта, че (τ_a^{-1}, τ_a) е Т-инвариантна спрегната двойка, следва, че $(\varepsilon_A, \delta_A)$ е спрегната Т-инвариантна двойка оператори. В [2] е доказано, че при така направените предположения за решетката $\mathcal{L}$ всяка Т-инвариантна двойка има вида (3.1)–(3.2). По-нататък A ще наричаме *структурен елемент*. Освен това при произволни A и B от $\mathcal{L}$ имаме $\delta_A(B) = \delta_B(A)$.

Пример 1. Нека $\mathcal{L}$ е съвкупността от всички подмножества на дадено линейно пространство M над полето от реалните числа с релация на наредба $A \subset B$. Точна горна и точна долна граница се задават посредством операциите обединение и сечение, а именно:

$$\bigvee\{H_i : i \in I\} = \bigcup_{i \in I} H_i,$$

$$\bigwedge\{H_i : i \in I\} = \bigcap_{i \in I} H_i.$$

Супремум-генериращо подмножество може да се зададе чрез

$$l = \{\{x\} : x \in M\}.$$

Като група от автоморфизми T може да се разгледа групата на транслациите [2]. Ще бележим $X_a = \tau_a(X) = \{x + a : x \in X\}$. За начало вземаме нулевия елемент на пространството M. Тогава всяка T-инвариантна дилатация представлява сума на Минковски с някакво фиксирано множество A (вж. [2]), а именно:

$$\delta(X) = X \oplus A = \bigcup_{a \in A} X_a = \{x + a : x \in X,\ a \in A\}.$$

Съответната спрегната ерозия е разлика на Минковски със същото множество, т.е.

$$\varepsilon(X) = X \ominus A = \bigcap_{a \in A} X_{-a}.$$

В случая $\mathcal{L}$ представлява Булева решетка, т.е. всеки елемент A притежава допълнителен елемент $A^c = \{x : x \in M, x \notin A\}$. От законите на Де Морган за обединението и сечението следва следната връзка между операциите на Минковски:

$$(X \ominus A)^c = X^c \oplus (-A),$$

където $-A$ означава централно симетричния образ на A. Подробен обзор на свойствата на събирането и изваждането на Минковски е даден в [6]. Разгледаните в този пример оператори се наричат *бинарни морфологични оператори* [7].

Пример 2. Нека $\mathcal{L}$ е съвкупността от всички функции с дефиниционна област линейното пространство над полето от реалните числа M и стойности в $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$, като

$$(\bigvee_{i \in I} F_i)(x) = \sup_{i \in I} F_i(x),$$

$$(\bigwedge_{i \in I} F_i)(x) = \inf_{i \in I} F_i(x)$$

за всяко $x \in M$. За всяко $x \in M$ и всяко $t \in \mathbf{R}$ дефинираме функциите

$$f_{x,t}(y) = \begin{cases} t, & \text{при } y = x, \\ -\infty, & \text{при } y \neq x. \end{cases}$$

По този начин дефинираме супремум-генериращо семейство $l = \{f_{x,t} : x \in M,\ t \in \mathbf{R}\}$. За всяко $h \in M$ и всяко $v \in \Re$ задаваме автоморфизма $\tau_{h,v}$ чрез

$$(3.3) \qquad (\tau_{h,v}(F))(x) = F(x-h) + v$$

и $T = \{\tau_{h,v} : h \in M,\ v \in \mathbf{R}\}$. T действа транзитивно върху $\mathcal{L}$, защото

$$\tau_{h,v}(f_{x,t}) = f_{x+h,t+v}.$$

Въвеждаме аналози на операциите на Минковски в $\mathcal{L}$:

$$(3.4) \qquad (F \oplus G)(x) = \sup_{x \in M} (F(x-h) + G(h)),$$

$$(3.5) \qquad (F \ominus G)(x) = \inf_{x \in M} (F(x+h) - G(h)),$$

при допълнителни предположения, че $F(x-h) + G(h) = -\infty$, когато $F(x-h) = -\infty$ или $G(h) = -\infty$, и $F(x+h) - G(h) = \infty$, когато $F(x+h) = \infty$ или $G(h) = -\infty$. Поради свойства 3.1 и 3.2 всяка Т-дилатация има вида (3.4), а всяка Т-ерозия — вида (3.5). Разгледаните в този пример оператори се наричат *полутонови морфологични оператори* [7].

Пример 3. Нека $\mathcal{L}$ е съвкупността от всички изпъкнали компактни множества от $\mathbf{R}^n$ с релация на наредба $A \subset B$. Точна горна и точна долна граница се задават така:

$$\bigvee \{H_i : i \in I\} = \mathrm{cl}(\mathrm{co}(\bigcup_{i \in I} H_i)),$$

където с $\mathrm{co}(A)$ означаваме изпъкналата обвивка на A, а $\mathrm{cl}(A)$ е най-малкото затворено множество, съдържащо A,

$$\bigwedge \{H_i : i \in I\} = \bigcap_{i \in I} H_i.$$

Супремум-генериращо подмножество може да се зададе чрез

$$l = \{\{x\} : x \in \mathbf{R}^n\}.$$

Като група от автоморфизми T ще разгледаме групата на транслациите както в пример 1. Събиране и изваждане на Минковски се дефинират също както в пример 1, а именно:

$$(3.6) \qquad X \oplus A = \{x + a : x \in X, a \in A\},$$

$$(3.7) \qquad X \ominus A = \bigcap_{a \in A} X_{-a}.$$

Очевидно е, че сечението на изпъкнали множества е изпъкнало. Ще покажем, че сумата на Минковски на две изпъкнали множества е изпъкнало. Нека p и q са две точки от $A \oplus B$, където A и B са изпъкнали. Тогава $p = x + y$, $q = u + v$, където x и u принадлежат на A, а y и v — на B. Тогава $\alpha p + (1-\alpha)q = (\alpha x + (1-\alpha)u) + (\alpha y + (1-\alpha)v)$. От изпъкналостта на A и

B следва, че $\alpha x + (1-\alpha)u \in A$ и $\alpha y + (1-\alpha)v \in B$, откъдето получаваме, че $\alpha p + (1-\alpha)q \in A \oplus B$, т.е. $A \oplus B$ е изпъкнало. Лесно се проверява, че сумата на две компактни множества е компактно.

Следователно и в този случай всяка Т-инвариантна дилатация представлява сума на Минковски с подходящ структурен елемент, както и всяка Т-ерозия е разлика на Минковски с подходящ структурен елемент:

$$\delta(X) = X \oplus A = \mathrm{cl}(\mathrm{co}(\bigcup_{a \in A} X_a)), \tag{3.8}$$

$$\varepsilon(X) = X \ominus A = \{y : A_y \subset X\}. \tag{3.9}$$

Нека P и Q са две непразни компактни множества в $\mathbf{R}^n$. Въвеждаме функцията

$$d_h(P,Q) = \inf\{\varepsilon : Q \subset P \oplus B_\varepsilon, P \subset Q \oplus B_\varepsilon\}, \tag{3.10}$$

където $B_\varepsilon = \{x : \|x\| \leq \varepsilon\}$ е кръгът с център в началото $(0, 0, \ldots, 0)$ и радиус ε. Тази функция притежава следните свойства:

$$d_h(P,P) = 0, \quad d_h(P,Q) > 0 \text{ за } P \neq Q, \tag{3.11}$$

$$d_h(P,Q) = d_h(Q,P), \tag{3.12}$$

$$d_h(P,Q) + d_h(Q,R) \geq d_h(P,R), \tag{3.13}$$

което показва, че тя задава метрика. По-точно това е Хаусдорфовото разстояние [7] и по този начин множеството от компактните множества от $\mathbf{R}^n$ се превръща в пълно метрично пространство [6].

Нека P и Q са две изпъкнали компактни множества в $\mathbf{R}^n$. *От свойствата* на операцията затворена обвивка знаем, че

$$(P \oplus Q) \ominus Q \supset P. \tag{3.14}$$

В [4], като се използват свойствата на опорните хиперравнини на изпъкналите множества, е доказано и обратното включване:

$$(P \oplus Q) \ominus Q \subset P. \tag{3.15}$$

От двете включвания получаваме равенството

$$(P \oplus Q) \ominus Q = P. \tag{3.16}$$

От равенството $P \oplus Q = R \oplus Q$ следва $P = R$, защото

$$P = (P \oplus Q) \ominus Q = (R \oplus Q) \ominus Q = R.$$

4. РАЗМИТИ МНОЖЕСТВА И МОРФОЛОГИЧНИ ОПЕРАТОРИ ВЪРХУ ТЯХ

Размитите множества са въведени за пръв път от Лотфи Заде и намират широко приложение в системите за управление с елементи на изкуствен интелект, в социологията, психологията, лингвистиката и др. Дефиниция на понятието *размито множество* както и основните свойства могат да бъдат намерени в [11]. Да разгледаме множество $\mathcal{U}$, което наричаме *универсално.* Едно размито подмножество A на универсалното множество $\mathcal{U}$ се отъждествява с произволна функция $\mu_A : \mathcal{U} \mapsto [0,1]$, която се нарича *характеристична функция на* A, а $\mu_A(x)$ се нарича *степен на принадлежност на точката* x *към* A. Обичайните подмножества на $\mathcal{U}$ могат да се разглеждат като частен случай на размити множества, при които характеристичната функция взема стойности само в крайните точки на интервала $[0,1]$. Теоретико-множествените операции и релации се обобщават естествено за размити множества по следния начин:

- $A \subset B$, ако $\mu_A(x) \leq \mu_B(x) \forall x \in \mathcal{U}$;
- $A \cap B = C$, ако $\mu_C(x) = \min\{\mu_A(x), \mu_B(x)\} \forall x \in \mathcal{U}$;
- $A \cup B = C$, ако $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\} \forall x \in \mathcal{U}$;
- допълнение на A е множеството A^c с характеристична функция

$$\mu_{A^c}(x) = 1 - \mu_A(x) \quad \forall x \in \mathcal{U}.$$

Празно размито множество $\varnothing$ се нарича множеството с характеристична функция константата 0, а универсалното множество $\mathcal{U}$ се отъждествява с размитото множество с характеристична функция константата 1. Точна горна и точна долна граница на система размити множества $\{H_i : i \in I\}$ се дефинират по следния начин:

$$(4.1) \qquad U = \bigvee_{i \in I} H_i, \quad \mu_U(x) = \sup\{\mu_{H_i}(x) : i \in I\},$$

$$(4.2) \qquad L = \bigwedge_{i \in I} H_i, \quad \mu_L(x) = \inf\{\mu_{H_i}(x) : i \in I\}.$$

Точна горна граница на система от две множества е тяхното обединение, а точна долна граница — тяхното сечение. По този начин съвкупността $\mathcal{L}$ от размити подмножества на $\mathcal{U}$ се превръща в Булева решетка. Може да се дефинира и група $T = \{\tau_a : A \in \mathbf{R}^n\}$ от автоморфизми-транслации $\tau_a(X) = X_a$, $\mu_{X_a}(y) = \mu_X(y-a)$. Представлява проблем обаче намирането на супремум-генериращо семейство размити множества. За дефинирането на двойки спрегнати дилатации и ерозии помага понятието *индикатор* [9]. Индикатор се нарича функцията $I : \mathcal{L} \times \mathcal{L} \mapsto [0,1]$ със следните 8 свойства:

1. $I(A,B) = 0 \iff \{x : \mu_A(x) = 1\} \cap \{x : \mu_B(x) = 0\} \neq 0$.
2. От $B \subset C$ следва $I(A,B) \leq I(A,C)$.
3. От $B \subset C$ следва $I(C,A) \leq I(B,A)$.
4. $I(A,B) = I(\tau_x(A), \tau_x(B))$.
5. $I(A,B) = I(B^c, A^c)$.
6. $I(\bigvee_{i \in J} B_i, A) = \inf_{i \in J} I(B_i, A)$.
7. $I(A, \bigwedge_{i \in J} B_i) = \inf_{i \in J} I(A, B_i)$.
8. $I(A, \bigvee_{i \in J} B_i) \geq \sup_{i \in J} I(A, B_i)$.

Разглеждаме следните функции:

$$(4.3) \qquad I_1(A,B) = \inf_{x \in \mathcal{U}} \min[1, \lambda(\mu_A(x)) + \lambda(1 - \mu_B(x))],$$

където $\lambda : [0,1] \mapsto [0,1]$ е вдлъбната, строго монотонно намаляваща функция, за която $\lambda(0) = 1$ и $\lambda(1) = 0$, и

$$I_2(A,B) = \inf_{x \in \mathcal{U}} \max[\mu_B(x), 1 - \mu_A(x)]. \tag{4.4}$$

Теорема 4.1. *Условията 1–8 се удовлетворяват от функциите I_1 и I_2.*

Доказателство. За верността на твърдението за I_1 вж. работата на Синха и Дохерти [9]. Тук ще докажем твърдението и за I_2. Свойство 1, монотонните свойства 2 и 3, както и свойства 4 и 5 следват непосредствено от дефиницията. Ще докажем свойство 7. Свойства 6 и 8 следват аналогично. За краткост на записа ще изпускаме индекса 2.

$$\begin{aligned} I(A, \bigwedge_i B_i) &= \inf_x \max(\inf_i \mu_{B_i}(x), 1 - \mu_A(x)) \\ &= \inf_x \{\inf_i [\max(\mu_{B_i}(x), 1 - \mu_A(x))]\} \\ &= \inf_i \{\inf_x [\max(\mu_{B_i}(x), 1 - \mu_A(x))]\}. \end{aligned}$$

Предпоследното от равенствата следва от равенството

$$\max(\inf_i a_i, c) = \inf_i \max(a_i, c),$$

изпълнено за произволни реални числа a_i и c, което представлява обобщение на закона на Де Морган. Комутативността на двата инфимума, от която следва последното равенство, е очевидна. С това свойство 7 е доказано. ∎

Индикаторът I_1 в [9] се нарича *индикатор на включването*, защото от $A \subset B$ следва $I_1(A,B) = 1$. Ако $\lambda(x) = 1 - x$, в сила е и обратното твърдение. Така $I_1(A,B)$ може да се разглежда като *лингвистична променлива*, вж. [11], описваща понятието $A \subset B$.

С помощта на понятието *индикатор* ще дефинираме оператор ε_A за всяко размито множество A с равенството

$$\mu_{\varepsilon_A(B)}(x) = I(\tau_x(A), B)$$

и оператор δ_A с равенството

$$\delta_A(B) = (\varepsilon_{-A}(B^c))^c.$$

Оттук нататък, ако е дадено размито множество A, за краткост ще означаваме $A(x)$ вместо $\mu_A(x)$.

Теорема 4.2. *Дефинираните чрез I_1 оператори при дадена функция λ и размито множество A образуват спрегната двойка $(\varepsilon_A, \delta_A)$.*

Доказателството на това твърдение е скицирано в [9].

Теорема 4.3. *От дефинираните чрез I_2 оператори при дадено размито множество A δ_A винаги е дилатация, а ε_A винаги е ерозия.*

Доказателство. Свойствата

$$\delta_A(\bigwedge_i B_i) = \bigwedge_i \delta(B_i), \tag{4.5}$$

$$(4.6) \qquad \varepsilon_A(\bigvee_i B_i) = \bigvee_i \varepsilon(B_i)$$

следват директно от свойство 7 на индикаторите. Тъй като

$$(\delta_A(B))(x) = \sup_b \min(A(b), B(x-b)),$$

то $(\delta_A(\varnothing))(x) = \sup_b \min(A(b), 0) = 0$, т. е. $\delta_A(\varnothing) = \varnothing$. Така получаваме, че δ_A е дилатация.

$$(\varepsilon_A(B))(x) = \inf_b \max(B(b), 1 - A(b-x)),$$

следователно $(\varepsilon_A(\mathcal{U}))(x) = \inf_b \max(1, 1 - A(b-x)) = 1$, т. е. $\varepsilon_A(\mathcal{U}) = \mathcal{U}$. Могат да бъдат дадени примери, когато за някое A двойката $(\varepsilon_A, \delta_A)$ не е спрегната — например когато $\mu(A) = 0{,}5$.

5. РАЗМИТИ ИЗПЪКНАЛИ МНОЖЕСТВА И ДИФЕРЕНЦИРУЕМОСТ НА РАЗМИТИ ФУНКЦИИ

Тук разглеждаме операторите, дефинирани с индикатора I_2 в съвкупността от размити множества над $\mathbf{R}^n$.

$$(5.1) \qquad (\delta_A(B))(x) = \sup_{b\in\mathbf{R}^n} \min(A(b), B(x-b)),$$

$$(5.2) \qquad (\varepsilon_A(B))(x) = \inf_{b\in\mathbf{R}^n} \max(B(b), 1 - A(b-x)).$$

Нека A е размито множество, а $\alpha \in (0,1]$. С ${}_\alpha A$ бележим *α-отреза* на A:

$$(5.3) \qquad {}_\alpha A = \{x \in \mathbf{R}^n : A(x) \geq \alpha\}.$$

Тогава дилатацията се представя като

$$(5.4) \qquad (\delta_A(B))(x) = \sup\{\alpha : x \in {}_\alpha A \oplus {}_\alpha B\}.$$

Размитото множество A се нарича *изпъкнало*, ако

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2))$$

за всеки $x_1, x_2 \in \mathbf{R}^n$ и $\lambda \in [0,1]$.

Лема 5.1. *Едно размито множество е изпъкнало тогава и само тогава, когато неговите α-отрези са изпъкнали при $\alpha \in (0,1]$.*

Доказателство. Нека A е изпъкнало. Нека $\alpha \in (0,1]$ е такова, че ${}_\alpha A \neq \varnothing$, и $x_1, x_2 \in {}_\alpha A$. Тогава $A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha$.

Обратно, нека ${}_\alpha A$ е изпъкнало за всяко $\alpha \in (0,1]$. Нека $x_1, x_2 \in \mathbf{R}^n$, $\alpha = \min(A(x_1), A(x_2))$ и $0 < \lambda < 1$. Тогава $\lambda x_1 + (1-\lambda)x_2 \in {}_\alpha A$, т.е.

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = \min(A(x_1, A(x_2)),$$

което показва, че A е изпъкнало.

Разглеждаме пространство $\mathcal{C}(\mathbf{R}^n)$ от размити множества A със следните свойства:

1. Функцията μ_A е полунепрекъсната отгоре.
2. A е изпъкнало.
3. ${}_\alpha A$ е компактно в $\mathbf{R}^n$ за всяко $\alpha \in (0,1]$.
4. $\bigcup\{{}_\alpha A : \alpha > 0\}$ е ограничено.

В $\mathcal{C}(\mathbf{R}^n)$ въвеждаме *събиране* и *умножение с неотрицателен скалар* по следния начин:

$$A + B = \delta_A(B), \tag{5.5}$$

$$(\lambda A)(x) = \begin{cases} A\left(\frac{1}{\lambda}x\right) & \text{при } \lambda \neq 0, \\ 0 & \text{при } \lambda = 0,\ x \neq 0, \\ 1 & \text{при } \lambda = 0,\ x = 0. \end{cases} \tag{5.6}$$

Непосредствено се проверява, че

$$0A = \Omega, \quad \Omega(x) = \begin{cases} 1 & \text{при } x = 0, \\ 0 & \text{при } x \neq 0, \end{cases}$$

и за всяко $B \in \mathcal{C}(\mathbf{R}^n)$ имаме $B + \Omega = B$.

В $\mathcal{C}(\mathbf{R}^n)$ можем да въведем метрика [5]

$$d(A,B) = \sup_{\alpha>0} d_h({}_\alpha A, {}_\alpha B), \tag{5.7}$$

където d_h е дефинираното чрез (3.10) хаусдорфово разстояние. Като използваме пълнотата на пространството от непразните компактни множества, снабдено с хаусдорфовата метрика [4], лесно се проверява, че $\mathcal{C}(\mathbf{R}^n)$ е също пълно метрично пространство с метрика d.

Теорема 5.2. *Ако A и B принадлежат на $\mathcal{C}(\mathbf{R}^n)$, то ${}_\alpha(A + B) = {}_\alpha A \oplus {}_\alpha B$.*

Доказателство. $(A+B)(x) = \sup\{\alpha : x \in {}_\alpha A \oplus {}_\alpha B\}$. Нека $x \in {}_\alpha A \oplus {}_\alpha B$. Тогава съществува $\beta \geq \alpha$ такова, че $(A+B)(x) = \beta$, т. е. $x \in {}_\beta(A+B) \subset {}_\alpha(A+B)$, или

$${}_\alpha A \oplus {}_\alpha B \subset {}_\alpha(A+B). \tag{5.8}$$

Обратно, нека $x \in {}_\alpha(A+B)$, като $(A+B)(x) = \alpha$.

$$(A+B)(x) = \sup\{\beta : x \in {}_\beta A \oplus {}_\beta B\}. \tag{5.9}$$

Разглеждаме строго монотонно растяща редица $\{\alpha_i\}_{i=1,2,\dots}$, клоняща към α. α_i не е горна граница в (5.9), следователно за всеки номер i съществуват $x_i \in \mathbf{R}^n$ и $y_i \in \mathbf{R}^n$ такива, че

$$x = x_i + y_i, \quad A(x_i) \geq \alpha_i, \quad B(y_i) \geq \alpha_i.$$

Редицата $\{x_i\}$ е ограничена, защото $x_i \in {}_{\alpha_1}A$, което е компактно по условие. Тогава можем да изберем сходяща подредица $\{x_{i_k}\}$ и нека означим с x_0 нейната граница. Следователно съществува и границата

$$y_0 = \lim_{k\to\infty} y_{i_k} = x - \lim_{k\to\infty} x_{i_k}.$$

Тогава $x = x_0 + y_0$, а от полунепрекъснатостта отгоре на μ_A и μ_B следва $A(x_0) \geq \alpha$ и $B(x_0) \geq \alpha$, т. е. $x \in {}_\alpha A \oplus {}_\alpha B$. Оттук получаваме

$$ {}_\alpha A \oplus {}_\alpha B \supset {}_\alpha(A + B). \tag{5.10}$$

От двете включвания (5.8) и (5.10) следва верността на теоремата. ∎

Следствие 5.3. *Нека A, B и C са размити множества от $\mathcal{C}(\mathbf{R}^n)$. Тогава от $A + C = B + C$ следва $A = B$.*

Следствие 5.4. *Нека A, B и C са размити множества от $\mathcal{C}(\mathbf{R}^n)$. Тогава $d(A + C, B + C) = d(A, B)$.*

Разглеждаме множеството $\mathcal{C}(\mathbf{R}^n) \times \mathcal{C}(\mathbf{R}^n)$. Като следваме познатата конструкция (вж. [5]), ще изградим линейно нормирано пространство. В $\mathcal{C}(\mathbf{R}^n) \times \mathcal{C}(\mathbf{R}^n)$ въвеждаме релация на еквивалентност

$$(A, B) \sim (A', B') \iff A + B' = B + A'.$$

Означаваме с $\mathcal{B}$ факторизираното пространство по отношение на тази релация. Ще покажем, че $\mathcal{B}$ е пълно линейно нормирано пространство. Да означим с $\langle A, B\rangle$ съвкупността от елементи на $\mathcal{C}(\mathbf{R}^n) \times \mathcal{C}(\mathbf{R}^n)$, еквивалентни на (A, B). Сума на два елемента на $\mathcal{B}$ дефинираме чрез равенството

$$\langle A, B\rangle + \langle C, D\rangle = \langle A + C, B + D\rangle.$$

Противоположен елемент на $\langle A, B\rangle$ наричаме $\langle B, A\rangle$. Произведението със скалар $c \in \mathbf{R}$ се задава така:

$$c\langle A, B\rangle = \begin{cases} \langle cA, cB\rangle & \text{при } c \geq 0, \\ \langle (-c)B, (-c)A\rangle & \text{при } c < 0. \end{cases}$$

Нулев елемент на $\mathcal{B}$ е $\langle A, A\rangle$, защото

$$\langle A, A\rangle + \langle B, C\rangle = \langle A + B, A + C\rangle = \langle B, C\rangle,$$

тъй като $(A + B, A + C) \sim (B, C)$.

Зависимостите

$$\alpha\langle A, B\rangle + \alpha\langle C, D\rangle = \alpha\langle A + C, B + D\rangle, \tag{5.11}$$

$$\alpha\langle A, B\rangle + \beta\langle A, B\rangle = (\alpha + \beta)\langle A, B\rangle \tag{5.12}$$

се проверяват непосредствено. Оттук следва, че $\mathcal{B}$ е линейно пространство. Разлика на два елемента на $\mathcal{B}$ се определя стандартно по следния начин:

$$\langle A, B\rangle - \langle C, D\rangle = \langle A, B\rangle + (-1)\langle C, D\rangle = \langle A + D, B + C\rangle.$$

Норма в $\mathcal{B}$ ще дефинираме чрез метриката в $\mathcal{C}(\mathbf{R}^n)$:

$$\|\langle A, B\rangle\| = d(A, B), \tag{5.13}$$

където d е разстоянието, дефинирано чрез (5.7), вж. [5]. Ще се уверим, че тази функция е действително норма:

1. $0 = \|\langle A, B\rangle\| \iff d(A, B) = 0 \iff A = B \iff \langle A, B\rangle = 0$.
2. $\|c\langle A, B\rangle\| = d(|c|A, |c|B) = |c|d(A, B) = |c|\,\|\langle A, B\rangle\|$.
3. $\|\langle A, B\rangle + \langle C, D\rangle\| = \|\langle A + C, B + D\rangle\| = d(A + C, B + D)$

$\leq d(A+C, B+C) + d(C+B, D+B) = \|\langle A, B\rangle\| + \|\langle C, D\rangle\|.$

Пълнотата на $\mathcal{B}$ следва от пълнотата на метричното пространство $\mathcal{C}(\mathbf{R}^n)$ с метрика d. $\mathcal{C}(\mathbf{R}^n)$ ще вложим изометрично в подпространството $\mathcal{Y}$ на $\mathcal{B}$, породено от елементите $\{\langle A, \Omega\rangle : A \in \mathcal{C}(\mathbf{R}^n)\}$, т. е. въвеждаме изображението $Y : \mathcal{C}(\mathbf{R}^n) \mapsto \mathcal{Y}$ чрез равенството

$$Y(A) = \langle A, \Omega\rangle.$$

Изометричността следва от равенството

$$d(A, B) = \|\langle A, B\rangle\| = \|\langle A, \Omega\rangle + \langle \Omega, B\rangle\| = \|Y(A) - Y(B)\|.$$

Нека X е подмножество на $\mathbf{R}^m$, съдържащо точката a заедно с една нейна околност $B_\varepsilon(a)$. Размита функция f наричаме изображение $f : X \mapsto \mathcal{C}(\mathbf{R}^n)$. Функцията f ще наричаме диференцируема в точката a, ако съществува линейно изображение $K : \mathbf{R}^m \mapsto \mathcal{Y}$ такова, че

$$\lim_{x \to a} \frac{\|Y(f(x)) - Y(f(a)) - K(x-a)\|}{|x-a|} = 0, \tag{5.14}$$

където $|\cdot|$ е произволна норма в $\mathbf{R}^m$. Ще наречем K производна на f в точката a и ще бележим с $f'(a)$. Така дефинираната производна съвпада с тази, дефинирана в [5]. Избегнати са някои противоречия, например там се задава влагането $Y(A) = \langle A, \varnothing\rangle$, което не е подходящо, тъй като хаусдорфовото разстояние не е дефинирано за празни множества. Така дефинираната диференцируемост може да послужи при построяване на размити управления на динамични системи и изследване на тяхната устойчивост посредством построяването на аналог на функцията на Ляпунов [3].

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Summary

Mathematical morphology has been created firstly as an approach in image processing based on geometrical concepts as transformation groups and metric spaces. It is a method which strength lies in the quantitative description of geometrical structure and shape, and has proved to be useful in many image processing tasks [1]. It has been created originally by Matheron and Serra [4, 7], studying the properties of porous media with respect of their geometrical structure. First, morphological techniques are applied to binary images which can be interpreted mathematically as sets. Corresponding morphological operations are based on set-theoretical concepts and operations like union, intersection, complement, and on geometrical transforms like translations or rotations. Later, the method is extended to spaces of numerical functions for the study of grey-tone images. The initial framework is later replaced by a more general one, namely the framework of complete lattices. (For a number of related results see [2, 8].) So, mathematical morphology is a theory, which appeals to mathematicians since it allows a rigorous mathematical description and derives its tools from several disciplines like algebra, topology and geometry. In this paper we briefly recall the definition of a complete lattice and some basic properties of morphological operations — dilations, erosions, openings and closings, used in the sequel. We use the fact that in the case of binary morphology every translation-invariant dilation is the Minkowski addition and

every translation-invariant erosion is the Minkowski subtraction with an appropriate set, called structuring element [1]. The dilations and erosions in the family of convex compact sets in $\mathbf{R}^n$ are considered also.

It is shown by Werman and Peleg in [10] that grey-tone images could be successfully interpreted as fuzzy sets. Werman and Peleg have made an attempt to apply morphological techniques on the complete lattice of fuzzy sets in $\mathbf{R}^n$. They have defined a fuzzy dilation by equation (5.1) and erosion by equation (5.2). In our paper special emphasis is set on the fuzzy dilation and fuzzy erosion. We denote a fuzzy subset A of an universal set $\mathcal{U}$ via its membership function $\mu_A(x)$. For any element $x \in \mathcal{U}$ $\mu_A(x)$ *denotes the degree to which the element* x *belongs to* A, $\mu_A(x) \in [0,1]$. A general construction of fuzzy morphological operators, based on the so-called indicators, is presented by Sinha and Dougherty in [9]. The indicator $I(A, B)$, where A and B are fuzzy subsets of $\mathbf{R}^n$, is a function defined by a collection of eight axioms (nine axioms in [9]), described in the fourth section of our work. Our paper shows how Werman and Peleg's operations can be defined in the more general indicator framework. In the last section of the paper it is shown how the Werman and Peleg's notion of fuzzy dilation can be used to define the concept of Frechet-type derivative of fuzzy function. The presented approach is analogous to those in [5], but a contradiction in Puri and Ralescu's definition is overcome. This contradiction is based on the fact that the Hausdorff distance between an empty set and a non-empty one is not defined [6]. The proposed notion of differentiable fuzzy function is useful for the theory and the practice of fuzzy control systems [3].

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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AN EXTERNAL APPROACH TO ABSTRACT DATA TYPES I: COMPUTABILITY ON ABSTRACT DATA TYPE*

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Алексанра Соскова. ВНЕШНЫЙ ПОДХОД К АБСТРАКТНЫМ ТИПАМ ДАННЫХ I: ВЫЧИСЛИМОСТЬ В АБСТРАКТНОМ ТИПЕ ДАННЫХ

Представлена характеризация эффективных абстрактных типов данных с точки зрения теории рекурсии. Основным инструментом является понятие вычислимости многосортных абстрактных структур. Это понятие имеет некоторые максимальные свойства при естественных условиях.

Рассмотрены связи между специальными свойствами одного определенного класса вычислимых функций в абстрактной структуре и существованием некоторых специальных нумераций.

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A characterization of the effective abstract data types from the recursion theoretical point of view is presented. The main tool is a notion of computability on many-sorted abstract structures. This notion has certain maximal properties under natural conditions.

The relationships between certain special properties of the class of the computable functions in an abstract structure and the existence of some special enumerations of it are considered.

1. INTRODUCTION

An abstract data type (**ADT**) is usually considered as a class of many-sorted first order structures closed with respect to isomorphism [1, 2, 4, 6]. Using only this property, we are going to discuss the following problems:

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- Define the class of computable functions on an **ADT**.
- Characterize those **ADT** which are effective.

It is natural to insist that the notion of effective computability on an **ADT** agrees with the classical notion of computability on the natural numbers. In other words, over all structures on the natural numbers in our class the computable functions should be among the relatively partial recursive functions. We call this property *effectiveness*.

Considering the **ADT** as a class of structures, the second condition is that our notion of computability should be *invariant* with respect to isomorphisms, i.e. the class of computable functions of a given structure is preserved under isomorphisms.

Our third assumption concerns the use of the sorts during the computation. We consider two kinds of sorts — "*effectively enumerable*" and "*general*" ones. During the computation of a function θ we allow a search trough the data of the effectively enumerable sorts while for the general sorts a search is not allowed. This idea is described by the so-called *substructure property* of the computability, defined in the next section.

In the first part of the paper we present a notion of computability having the above properties. From the normal form of the computable functions on an **ADT**, given in Section 4, it will be clear that the so defined functions are effective in the intuitive sense. Moreover, each computability having the above three properties is weaker than our notion.

Having an appropriate notion of computability on an **ADT**, in the second part of the paper we shall define the so-called *effective data types* with respect to this computability. It will be proven that a data type is effective with respect to all programming languages iff it admits an effective enumeration.

2. PRELIMINARIES

Let a many-sorted signature $\Sigma = (\mathbb{S}, \mathbb{E}, \mathbb{F}, \mathbb{P}, \rho)$ with equality be fixed. Here $\mathbb{S} = \{1, \ldots, m\}$ is the set of sorts; $\mathbb{E} \subseteq \mathbb{S}$ is the set of the effectively enumerable sorts; $\mathbb{F} = \{f_1, \ldots, f_n\}$ is the set of functional symbols; $\mathbb{P} = \{T_1, \ldots, T_k\}$ is the set of predicate symbols; and ρ is a mapping which assigns to each f_i of $\mathbb{F}$ a type $\rho(f_i)$ over $\mathbb{S}$ of the form $(s_1, \ldots, s_{a_i}, s)$, where $s_1, \ldots, s_{a_i}$ are the sorts of the arguments and s is the sort of the result, and it assigns to each T_j of $\mathbb{P}$ a type $\rho(T_j)$ over $\mathbb{S}$ of the form $(s_1, \ldots, s_{b_j})$ for some $s_1, \ldots, s_{b_j}$ of $\mathbb{S}$. The equality for each sort is supposed. The only difference from the usual definition is that we include the set $\mathbb{E}$ of the effectively enumerable sorts as a part of Σ.

Let $\mathfrak{A} = (A_1, A_2, \ldots, A_m;\ \theta_1, \theta_2, \ldots, \theta_n;\ \Sigma_1, \Sigma_2, \ldots, \Sigma_k)$ be a many-sorted structure of signature Σ, where for all $s \in \mathbb{S}$ the initial set A_s of sort s is denumerable and non empty; $\theta_1, \theta_2, \ldots, \theta_n$ are the initial functions, $\theta_i: A_{s_1} \times \ldots \times A_{s_{a_i}} \rightharpoonup A_s$; $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ are the initial predicates, $\Sigma_j : A_{s_1} \times \ldots \times A_{s_{b_j}} \rightharpoonup \{0, 1\}$ (0 for true, 1 for false). If $\theta: A_{s_1} \times \ldots \times A_{s_a} \rightharpoonup A_s$ for some $s_1, \ldots, s_a, s \in \mathbb{S}$, then we shall call the function θ of type $(s_1, \ldots, s_a, s)$ *correctly defined*. By $\mathcal{F}_{\mathfrak{A}}$ we shall denote the set of all correctly defined partial functions on $\mathfrak{A}$, i.e. of a fixed type on $\mathfrak{A}$.

Let $\mathcal{A}$ be a class of many-sorted structures of signature Σ.

Computability on $\mathcal{A}$ we shall call every mapping C on $\mathcal{A}$ such that if $\mathfrak{A} \in \mathcal{A}$, then $C(\mathfrak{A}) \subseteq \mathfrak{F}_{\mathfrak{A}}$, i.e. $C(\mathfrak{A})$ is a set of correctly defined functions on $\mathfrak{A}$.

Denote by N the set of all natural numbers. If the structure $\mathfrak{A}$ is on N, i.e. $A_1 = A_2 = \cdots = A_m = N$, then a function θ of $\mathfrak{F}_{\mathfrak{A}}$ is called *partial recursive in* $\mathfrak{A}$ iff there exists an enumeration operator Γ such that if the graph of θ is G_θ, then $G_\theta = \Gamma(\theta_1, \theta_2, \ldots, \theta_n, \ \Sigma_1, \Sigma_2, \ldots, \Sigma_k)$ [9].

2.1. Definition. A computability C is called *effective* if whenever $\mathfrak{A} \in \mathcal{A}$ and $\mathfrak{A}$ is a structure on N, then all elements of $C(\mathfrak{A})$ are partial recursive in $\mathfrak{A}$.

Let $\mathfrak{A} = (A_1, \ldots, A_m; \ \theta_1, \theta_2, \ldots, \theta_n; \ \Sigma_1, \Sigma_2, \ldots, \Sigma_k)$ and $\mathfrak{B} = (B_1, \ldots, B_m; \ \varphi_1, \varphi_2, \ldots, \varphi_n; \ \sigma_1, \sigma_2, \ldots, \sigma_k)$ be many-sorted structures of signature Σ. Consider an one-to-one mapping α_s from B_s onto A_s for all s of $\mathbb{S}$.

The m-tuple $\langle \alpha_1, \ldots, \alpha_m \rangle$ is called Σ-*isomorphism* from $\mathfrak{B}$ to $\mathfrak{A}$ iff the following conditions hold:

(i) $\alpha_s(\varphi_i(x_1, \ldots, x_{a_i})) \simeq \theta_i(\alpha_{s_1}(x_1), \ldots, \alpha_{s_{a_i}}(x_{a_i}))$ for all $x_1 \in B_{s_1}, \ldots, x_{a_i} \in B_{s_{a_i}}$;

(ii) $\sigma_j(x_1, \ldots, x_{b_j}) \simeq \Sigma_j(\alpha_{s_1}(x_1), \ldots, \alpha_{s_{b_j}}(x_{b_j}))$ for all $x_1 \in B_{s_1}, \ldots, x_{b_j} \in B_{s_{b_j}}$.

2.2. Definition. A computability C is called *invariant* if whenever $\mathfrak{A}$ and $\mathfrak{B}$ belong to $\mathcal{A}$, $\langle \alpha_1, \ldots, \alpha_m \rangle$ is a Σ-isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ and $\theta \in C(\mathfrak{A})$, θ is of type $(s_1, \ldots, s_a, s)$, then there exists a function $\varphi \in C(\mathfrak{B})$ of the same type as θ such that for all $x_1 \in B_{s_1}, \ldots, x_a \in B_{s_a}$

$$(*) \qquad \alpha_s(\varphi(x_1, \ldots, x_a)) \simeq \theta(\alpha_{s_1}(x_1), \ldots, \alpha_{s_a}(x_a)).$$

The structure $\mathfrak{B}$ is called *an extension* of $\mathfrak{A}$ iff the following conditions hold:

(i) $A_s \subseteq B_s$ for all $s \in \mathbb{S}$, but $A_s = B_s$ for all $s \in \mathbb{E}$;

(ii) $\theta_i(t_1, \ldots, t_{a_i}) \simeq \varphi_i(t_1, \ldots, t_{a_i})$ for all $t_1 \in A_{s_1}, \ldots, t_{a_i} \in A_{s_{a_i}}$;

(iii) $\Sigma_j(t_1, \ldots, t_{b_j}) \simeq \sigma_j(t_1, \ldots, t_{b_j})$ for all $t_1 \in A_{s_1}, \ldots, t_{b_j} \in A_{s_{b_j}}$.

By $\mathfrak{A} \subseteq \mathfrak{B}$ we denote the fact that the many-sorted structure $\mathfrak{B}$ is an extension of $\mathfrak{A}$.

Let $|\mathfrak{A}| = A_1 \cup \ldots \cup A_m$.

2.3. Definition. A computability C has *a substructure property* if whenever $\mathfrak{A}$ and $\mathfrak{B}$ are elements of $\mathcal{A}$, $\mathfrak{A} \subseteq \mathfrak{B}$ and $\theta \in C(\mathfrak{A})$, then there exists a function $\varphi \in C(\mathfrak{B})$ of the same type as θ such that for all $t_1, \ldots, t_a$ of $|\mathfrak{A}|$

$$(**) \qquad \theta(t_1, \ldots, t_a) \simeq \varphi(t_1, \ldots, t_a).$$

To explain the last property assume that $\theta \in C(\mathfrak{A})$. Now the above condition follows from the assumption that in the course of the computation of θ if an additional information is needed, then it consists only of elements belonging to some of the effectively enumerable sorts.

Let C_1 and C_2 be two computabilities on $\mathcal{A}$. C_1 is said to be *weaker than* C_2 *on* $\mathcal{A}$ ($C_1 \subseteq_{\mathcal{A}} C_2$) iff $C_1(\mathfrak{A}) \subseteq C_2(\mathfrak{A})$ for all $\mathfrak{A}$ of $\mathcal{A}$.

In the next section we shall present a concept of computability satisfying these properties and such that if the class $\mathcal{A}$ is rich enough, then each computability, which has the above properties, is weaker than ours.

3. A MAXIMAL CONCEPT OF COMPUTABILITY ON MANY-SORTED STRUCTURES

Let $\mathfrak{A} = (\overline{A};\ \overline{\theta};\ \overline{\Sigma})$ be a many-sorted structure of signature Σ.

Combining the assumptions from the previous section, we come to the following technical notion. Suppose that $\mathfrak{A}$ is denumerable and A_s is infinite for each $s \in \mathbb{E}$.

For each sort s consider an one-to-one mapping α_s from a subset of N onto A_s. Let $\mathfrak{B} = (\overline{N};\ \overline{\varphi};\ \overline{\sigma})$ be a partial many-sorted structure of signature Σ on the natural numbers.

3.1. Definition. The tuple $\langle \alpha_1, \ldots, \alpha_m;\ \mathfrak{B}\rangle$ is called *an enumeration of* $\mathfrak{A}$ iff the following conditions hold:

(i) if $x_1 \in \mathrm{dom}(\alpha_{s_1}), \ldots, x_{a_i} \in \mathrm{dom}(\alpha_{s_{a_i}})$ and $\varphi_i(x_1, \ldots, x_{a_i})$ is defined, then $\varphi_i(x_1, \ldots, x_{a_i}) \in \mathrm{dom}(\alpha_s)$;

(ii) if $x_1 \in \mathrm{dom}(\alpha_{s_1}), \ldots, x_{a_i} \in \mathrm{dom}(\alpha_{s_{a_i}})$, then $\alpha_s(\varphi_i(x_1, \ldots, x_{a_i})) \simeq \theta_i(\alpha_{s_1}(x_1), \ldots, \alpha_{s_{a_i}}(x_{a_i}))$;

(iii) if $x_1 \in \mathrm{dom}(\alpha_{s_1}), \ldots, x_{b_j} \in \mathrm{dom}(\alpha_{s_{b_j}})$, then $\sigma_j(x_1, \ldots, x_{b_j}) \simeq \Sigma_j(\alpha_{s_1}(x_1), \ldots, \alpha_{s_{b_j}}(x_{b_j}))$;

(iv) for all effectively enumerable sorts $s \in \mathbb{E}$: $\mathrm{dom}(\alpha_s) = N$.

In fact, $\langle \alpha_1, \alpha_2, \ldots, \alpha_m\rangle$ is a Σ-isomorphism from the structure $(\mathrm{dom}(\alpha_1), \ldots, \mathrm{dom}(\alpha_m);\ \overline{\varphi};\ \overline{\sigma})$ to $\mathfrak{A}$.

Let $\langle \alpha_1, \alpha_2, \ldots, \alpha_m,\ \mathfrak{B}\rangle$ be an enumeration of $\mathfrak{A}$.

3.2. Definition. A function θ is *admissible* in $\langle \alpha_1, \alpha_2, \ldots, \alpha_m,\ \mathfrak{B}\rangle$ if there exists a function φ over N, partial recursive in $\mathfrak{B}$, such that if $x_1 \in \mathrm{dom}(\alpha_{s_1}), \ldots, x_a \in \mathrm{dom}(\alpha_{s_a})$, then:

(i) if $\varphi(x_1, \ldots, x_a)$ is defined, then $\varphi(x_1, \ldots, x_a) \in \mathrm{dom}(\alpha_s)$;

(ii) $\alpha_s(\varphi(x_1, \ldots, x_a)) \simeq \theta(\alpha_{s_1}(x_1), \ldots, \alpha_{s_a}(x_a))$.

3.3. Definition. θ is *computable* in $\mathfrak{A}$ iff θ is admissible in every enumeration of $\mathfrak{A}$.

The class of all computable functions in $\mathfrak{A}$ we shall denote by $\mathcal{C}^*(\mathfrak{A})$.

From the above definitions and the Normal Form Theorem in Section 4 the next proposition follows directly.

3.4. Proposition. *The computability $\mathcal{C}^*$ on $\mathcal{A}$ is effective, invariant and has the substructure property.* ∎

Moreover, we have the following theorem.

The class $\mathcal{A}$ is closed under isomorphisms if whenever $\mathfrak{A} \in \mathcal{A}$ and $\langle \alpha_1, \ldots, \alpha_m \rangle$ is a Σ-isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$, then $\mathfrak{B} \in \mathcal{A}$.

The class $\mathcal{A}$ is closed with respect to extensions if whenever $\mathfrak{A} \in \mathcal{A}$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{B} \in \mathcal{A}$.

3.5. Theorem. *Let $\mathcal{A}$ be a class closed under isomorphisms and with respect to extensions. Every computability C over $\mathcal{A}$ which is effective, invariant and has the substructure property is weaker than $\mathfrak{C}^*$ on $\mathcal{A}$ ($C \subseteq_{\mathcal{A}} \mathfrak{C}^*$).*

Proof. Let C be a computability on $\mathcal{A}$ with the desired properties, $\mathfrak{A} \in \mathcal{A}$ and $\theta \in C(\mathfrak{A})$. Consider an enumeration $\langle \alpha_1, \alpha_2, \ldots, \alpha_m, \mathfrak{B} \rangle$ on $\mathfrak{A}$, where $\mathfrak{B} = (\overline{N}; \overline{\varphi}; \overline{\sigma})$. Let $\mathfrak{B}' = (\mathrm{dom}(\alpha_1), \ldots, \mathrm{dom}(\alpha_m); \overline{\varphi}; \overline{\sigma})$. So $\langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ is a Σ-isomorphism from $\mathfrak{B}'$ to $\mathfrak{A}$. Hence, by the invariant property of C, there exists a function $\varphi' \in C(\mathfrak{B}')$ such that $(*)$ is true. And $\mathfrak{B}' \subseteq \mathfrak{B}$. By the substructure property there exists $\varphi \in C(\mathfrak{B})$ such that $(**)$ holds. But $\mathfrak{B}$ is a structure on the natural numbers and by the effectiveness of C φ is partial recursive in $\mathfrak{B}$. So θ is admissible in $\langle \alpha_1, \alpha_2, \ldots, \alpha_m, \mathfrak{B} \rangle$, and hence $\theta \in \mathfrak{C}^*(\mathfrak{A})$. ∎

4. NORMAL FORM THEOREM

The presented approach to the notion of computability is called "external". It was used first by Lacombe in [5]. The equivalence between Lacombe's notion of "$\forall$-admissibility" and search computability on total structures with equality was considered by Moschovakis in [7, 8]. This approach over arbitrary structures (single-sorted) was extended by Soskov in [11] and further external characterizations of other well-known concepts of abstract computability as prime computability [8], computability by means of effective definitional schemes [3, 10] and definability by logic programs were presented in [12, 14, 15]. The idea to consider the behavior of a computability on a class of structures and the concepts of maximal computabilities among these ones satisfying some natural conditions, were introduced in [13]. Our concept of maximal computability on many-sorted structures combines two maximal concepts — of search computability of Moschovakis (over the effectively enumerable sorts) and Friedman's computability (over the general ones).

While the external approach leads usually to maximal concepts of computability, it is necessary to show that the computable functions are "effective" in the intuitive sense. So we need a normal form of the computable functions on $\mathfrak{A}$. From this form it will be clear that these functions are computable by means of some reasonable algorithms.

Suppose that an infinite list of variables of sort s, for each sort s of $\mathbb{S}$, is fixed. *Terms of given sort* in Σ are defined as usual.

4.1. Definition. Let Π be a finite conjunction of atomic formulae and negated atomic formulae, τ — a term of sort s, and $Y_1, \ldots, Y_b$ — variables with sorts of $\mathbb{E}$. The expression of the form $\exists Y_1 \ldots \exists Y_b (\Pi \supset \tau)$ is called an *s-conditional expression.*

Let $Q = \exists Y_1 \ldots \exists Y_b (\Pi \supset \tau)$ be an s-conditional expression with free variables among $X_1, \ldots, X_a$ and $t_1, \ldots, t_a \in |\mathfrak{A}|$. The value $Q_{\mathfrak{A}}(X_1/t_1, \ldots, X_a/t_a)$ of Q is the set

$$\{\tau_{\mathfrak{A}}(Y_1/p_1,\dots,Y_b/p_b,X_1/s_1,\dots,X_a/s_a) : \\ \Pi_{\mathfrak{A}}(Y_1/p_1,\dots,Y_b/p_b,X_1/t_1,\dots,X_a/t_a)\cong 0, \\ \text{for some } p_1,\dots,p_b \text{ with sorts as } Y_1,\dots,Y_b\}.$$

4.2. Proposition. *If $\langle\overline{\alpha},\ \mathfrak{B}\rangle$ is an enumeration of $\mathfrak{A}$, $\tau(X_1,\dots,X_a)$ is a term of sort s, $\Pi(X_1,\dots,X_a)$ is an atomic formula and $x_1\in\mathrm{dom}(\alpha_{s_1}),\dots,x_a\in\mathrm{dom}(\alpha_{s_a})$, then*:

(i) $\alpha_s(\tau_{\mathfrak{B}}(X_1/x_1,\dots,X_a/x_a))\simeq\tau_{\mathfrak{A}}(X_1/\alpha_{s_1}(x_1),\dots,X_a/\alpha_{s_a}(x_a))$;

(ii) $\Pi_{\mathfrak{B}}(X_1/x_1,\dots,X_a/x_a)\simeq\Pi_{\mathfrak{A}}(X_1/\alpha_{s_1}(x_1),\dots,X_a/\alpha_{s_a}(x_a))$. ∎

4.3. Definition. A function θ is said to be *definable on* $\mathfrak{A}$ iff for some recursively enumerable set $\{Q^v\}_{v\in V}$ of s-conditional expressions with free variables among $Z_1,\dots,Z_r,X_1,\dots,X_a$ and for some fixed elements $q_1,\dots,q_r$ of $|\mathfrak{A}|$ the following equivalence is true:

$$\theta(t_1,\dots,t_a)\simeq t\iff\exists v(v\in V\ \&\ t\in Q^v_{\mathfrak{A}}(Z_1/q_1,\dots,Z_r/q_r,X_1/t_1,\dots,X_a/t_a)).$$

4.4. Theorem (Normal Form Theorem). *The function θ is computable in $\mathfrak{A}$ iff θ is definable on $\mathfrak{A}$.*

Proof. The fact that every definable function on $\mathfrak{A}$ is admissible in all enumerations of $\mathfrak{A}$ follows from the last Proposition 4.2.

To prove the other direction, we will actually prove the contrapositive. Thus we suppose that θ is admissible in $\mathfrak{A}$, but it is not definable on $\mathfrak{A}$. We use this fact to construct an enumeration $\langle\overline{\alpha},\ \mathfrak{B}\rangle$ of $\mathfrak{A}$ such that θ is not admissible in $\langle\overline{\alpha},\ \mathfrak{B}\rangle$. The basic idea of the construction is to ensure that the pullback of θ (by $\overline{\alpha}$) is not partial recursive in $\mathfrak{B}$ by diagonalizing over all possible partial recursive in $\mathfrak{B}$ functions.

For ease of exposition we will suppose that all functions and predicates of the signature Σ are unary and $\theta : A_{s_d}\rightharpoonup A_{s_r}$ is of type (s_d,s_r). The general case is a trivial rewriting of the argument below. If $\langle\overline{\alpha},\ \mathfrak{B}\rangle$ is an enumeration of $\mathfrak{A}$, where $\mathfrak{B}=(\overline{N};\ \varphi_1,\varphi_2,\dots,\varphi_n;\ \sigma_1,\sigma_2,\dots,\sigma_k)$, denote by

$$\langle\mathfrak{B}\rangle=\{\langle i,x,\varepsilon\rangle : (1\le i\le n\ \ \&\ \ \varphi_i(x)\simeq\varepsilon)\vee(n+1\le i\le n+k\ \&\ \sigma_{i-n}(x)\simeq\varepsilon)\}.$$

It is clear that a function φ is partial recursive in $\{\varphi_1,\varphi_2,\dots,\varphi_n;\sigma_1,\sigma_2,\dots,\sigma_k\}$ iff φ is partial recursive in $\langle\mathfrak{B}\rangle$.

The enumeration $\langle\overline{\alpha},\ \mathfrak{B}\rangle$ we shall construct by stages.

On each stage l we find a finite approximation (called *finite part*) Δ_l of $\langle\overline{\alpha},\ \mathfrak{B}\rangle$, so that on even stages we assure that $\mathrm{range}(\alpha_s)=A_s$ for every sort $s\in\mathbb{S}$ and $\mathrm{dom}(\alpha_s)=N$ for $s\in\mathbb{E}$ (effectively enumerable sorts).

On odd stages $l=2n+1$, if Γ_n is the n-th enumeration operator, then for every enumeration $\langle\overline{\alpha},\ \mathfrak{B}\rangle\supseteq\Delta_{l+1}$ there exists $x\in\mathrm{dom}(\alpha_{s_d})$ such that one of the following conditions is not fulfilled:

(A) $\forall y(\langle x, y\rangle \in \Gamma_n(\langle\mathfrak{B}\rangle) \Longrightarrow y \in \mathrm{dom}(\alpha_{s_r}))$;

(B) $\forall t(t \in \theta(\alpha_{s_d}(x)) \Longleftrightarrow \exists y(\alpha_{s_r}(y) \simeq t \;\&\langle x, y\rangle \in \Gamma_n(\langle\mathfrak{B}\rangle)))$.

First let us fix some notations.

4.5. Definition. *A finite part* Δ_l is called each tuple $\Delta_l = \langle\overline{\delta}^l,\ \overline{H}^l,\ \overline{\varphi}^l,\ \overline{\sigma}^l\rangle$, where:

(1) δ_s^l is one-to-one partial mapping $N \rightharpoonup A_s$ and $\mathrm{dom}(\delta_s^l)$ is finite for each $s \in \mathbb{S}$;

(2) H_s^l is a finite subset of N, so that $\mathrm{dom}(\delta_s^l) \cap H_s^l = \emptyset$ for each $s \in \mathbb{S}$, but $H_s^l = \emptyset$ for $s \in \mathbb{E}$;

(3) $\varphi_i^l : H_{s_i}^l \rightharpoonup H_s^l \cup \mathrm{dom}(\delta_s^l)$, where $\rho(f_i) = (s_i, s)$ for $1 \le i \le n$;

(4) $\sigma_j^l : H_{s_j}^l \rightharpoonup \{0, 1\}$, where $\rho(T_j) = (s_j)$ for $1 \le j \le k$.

Given Δ_l and Δ_q — finite parts, denote by $\Delta_l \subseteq \Delta_q$ the fact that $\delta_s^l \le \delta_s^q$, $H_s^l \subseteq H_s^q$ for $s \in \mathbb{S}$ and $\varphi_i^l \le \varphi_i^q$, $i = 1 \ldots n$, $\sigma_j^l \le \sigma_j^q$, $j = 1 \ldots k$. Here by $\varphi \le \psi$ we mean that if $\varphi(x)$ is defined, then $\psi(x)$ is also defined and $\varphi(x) \simeq \psi(x)$.

4.6. Definition. If Δ_l is a finite part and $\langle\overline{\alpha},\ \mathfrak{B}\rangle$ is an enumeration, then $\Delta_l \subseteq \langle\overline{\alpha},\ \mathfrak{B}\rangle$ iff:

(1) $\delta_s^l \le \alpha_s$ for $s \in \mathbb{S}$;

(2) $\mathrm{dom}(\alpha_s) \cap H_s^l = \emptyset$ for each $s \in \mathbb{S}$;

(3) $\varphi_i^l \le \varphi_i$ for $1 \le i \le n$;

(4) $\sigma_j^l \le \sigma_j$ for $1 \le j \le k$.

Construction:

Stage $l = 0$. $H_s^0 = \emptyset$ and $\delta_s^0, \varphi_i^0, \sigma_j^0$ are totally undefined.

Stage $l = 2n$, $n > 0$. For each sort s consider the first $x_s \notin \mathrm{dom}(\delta_s^l) \cup H_s^l$ and $t_s \notin \mathrm{range}(\delta_s^l)$. If there is no such t_s, do nothing. Otherwise, let $\delta_s^{l+1}(x_s) = t_s$ and $\Delta^{l+1} = \langle\overline{\delta}^{l+1},\ \overline{H}^l,\ \overline{\varphi}^l,\ \overline{\sigma}^l\rangle$.

Stage $l = 2n + 1$. Let $\Delta_l = \langle\overline{\delta}^l,\ \overline{H}^l,\ \overline{\varphi}^l,\ \overline{\sigma}^l\rangle$, where $\mathrm{dom}(\delta_s^l) = \{w_1^s, \ldots, w_{k_s}^s\}$ and $\mathrm{range}(\delta_s^l) = \{t_1^s, \ldots, t_{k_s}^s\}$.

Let Γ_n be the n-th enumeration operator. So, if $R \subseteq N$, then

$$z \in \Gamma_n(R) \Longleftrightarrow \exists v(\langle v, z\rangle \in W_n \;\&\; E_v \subseteq R),$$

where W_n is the recursively enumerable set with code n and E_v is the finite set with code v.

Let $x \in N$ and $x \notin H_{s_d}^l$. Then

$$\forall y(\langle x, y\rangle \in \Gamma_n(R) \Longleftrightarrow \exists v(\langle v, x, y\rangle \in W_n \;\&\; E_v \subseteq R)).$$

Denote by $U_{n,x} = \{\langle v, y\rangle \ :\ \langle v, x, y\rangle \in W_n\}$.

The main tool is the construction of a definable function $\xi : A_{s_d} \rightharpoonup A_{s_r}$, based on Γ_n, using the following translation. By $\overline{t}$ we denote the list of all elements of $\mathrm{range}(\delta_1^l) \cup \ldots \cup \mathrm{range}(\delta_m^l)$. For each sort s consider an one-to-one mapping var_s from

N onto the set of all variables of sort s. Let $\mathrm{var}_{s_d}(x) = X$ and $\overline{W} = \{W_1, \dots, W_a\}$ be the corresponding variables to the elements of the set $\mathrm{dom}(\delta_1^l) \cup \dots \cup \mathrm{dom}(\delta_m^l)$. Let c_{s_r} be a new constant of the sort s_r.

4.7. Lemma. *Let $p \in A_{s_d}, t \in A_{s_r}$. There exists an effective way, given $v, y \in N$, to define an s_r-conditional expression $Q^{\langle v,y\rangle}(X, \overline{W})$ such that:*

(1) *if $t \in Q_{\mathfrak{A}}^{\langle v,y\rangle}(X/p, \overline{W}/\overline{t})$, then there exists a finite part $\Lambda \supseteq \Delta_l$ such that for every enumeration $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Lambda$: $E_v \subseteq \langle \mathfrak{B}\rangle$ & $\alpha_{s_d}(x) = p$ & $\alpha_{s_r}(y) = t$;*

(2) *if $t \notin Q_{\mathfrak{A}}^{\langle v,y\rangle}(X/p, \overline{W}/\overline{t})$, then one of the following is true:*
   (a) *there exists $\Lambda \supseteq \Delta_l$ such that for every $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Lambda$*
   $(E_v \subseteq \langle \mathfrak{B}\rangle$ & $\alpha_{s_d}(x) = p \Longrightarrow y \notin \mathrm{dom}(\alpha_{s_r}))$, or
   (b) *for every $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Delta_l$ $(E_v \subseteq \langle \mathfrak{B}\rangle$ & $\alpha_{s_d}(x) = p \Longrightarrow \alpha_{s_r}(y) \not\simeq t)$.*

Proof. Let $v, y \in N$ be fixed. Consider the finite set E_v with the code v.

4.8. Definition. The finite set E_v is called *correct* if the following conditions hold:

(1) each element u of E_v is of the form $u = \langle i, z, \varepsilon\rangle$ and $1 \le i \le n$ or $(n+1 \le i \le k+n$ and $\varepsilon \in \{0,1\})$;
(2) if $\langle i, z, \varepsilon_1\rangle \in E_v$ and $\langle i, z, \varepsilon_2\rangle \in E_v$, then $\varepsilon_1 = \varepsilon_2$;
(3) if $\langle i, z, \varepsilon\rangle \in E_v$, $1 \le i \le n$, $z \in \mathrm{dom}(\delta_{s_i}^l)$, where $\rho(f_i) = (s_i, s)$, and $\varphi_i^l(z)$ is defined, then $\varphi_i^l(z) \simeq \varepsilon$;
(4) if $\langle j, z, \varepsilon\rangle \in E_v$, $n+1 \le j \le n+k$, $z \in \mathrm{dom}(\delta_{s_j}^l)$, where $\rho(T_{j-n}) = (s_j)$, and $\sigma_{j-n}^l(z)$ is defined, then $\sigma_{j-n}^l(z) \simeq \varepsilon$.

If E_v is not correct, then put $Q^{\langle v,y\rangle} = (X \ne X \supset c_{s_r})$.
Let E_v is correct. Denote by

$$\begin{aligned} M_s = &\{z : \exists i, \varepsilon(\langle i, z, \varepsilon\rangle \in E_v \,\&\, (i \le n \,\&\, \rho(f_i) = (s, s_1)) \vee (n < i \,\&\, \rho(T_{i-n}) = (s)\} \\ &\cup \{z : \exists i, z_1(\langle i, z_1, z\rangle \in E_v \,\&\, i \le n \,\&\, \rho(f_i) = (s_i, s)\}. \end{aligned}$$

$$D_s = \begin{cases} N & \text{if } s \in \mathbb{E}, \\ \mathrm{dom}(\delta_s^l) \cup \{x\} & \text{if } s = s_d \,\&\, s_d \notin \mathbb{E}, \\ \mathrm{dom}(\delta_s^l) & \text{otherwise.} \end{cases}$$

Let $K_s^0 = M_s \cap D_s$. Suppose that for each s of $\mathbb{S}$ the set K_s^q is defined. Then

$$\begin{aligned} K_s^{q+1} = \{z : \exists i, z_1(\langle i, z_1, z\rangle \in E_v \,\&\, z \notin K_s^0 \cup \dots \cup K_s^q \,\&\, i \le n \\ \&\, \rho(f_i) = (s_i, s) \,\&\, z_1 \in K_{s_i}^q)\}. \end{aligned}$$

Since the set E_v is finite, there exists an r so that $K_s^{r+1} = \emptyset$.

Let $K_s = K_s^0 \cup \dots \cup K_s^r$ and if $z \in K_s^q$, define $|z| = q$. For each element z of K_s we define a term τ^z by induction on $|z|$.

Let $|z| = 0$, then $\tau^z = \mathrm{var}_s(z)$. Let $|z| = q+1$, $\langle i, z_1, z\rangle \in E_v$ for some z_1, so that $|z_1| = p < q+1$. Then $\tau^z = f_i(\tau^{z_1})$.

Some of the elements of E_v we shall call *appropriate*. Namely, $u = \langle i, z, z_1\rangle$ from E_v is appropriate if:

(1) $i \leq n$ and $\rho(f_i) = (s, s_1)$, $z \in K_s$, $z_1 \in K_{s_1}$;

(2) $n < i$ and $\rho(T_{i-n}) = (s)$, $z \in K_s$.

So we are ready to define the desired conditional expression $Q^{\langle v,y\rangle}(X, \overline{W})$. If for some $s \in \mathbb{S}$ $H^l_s \cap K_s \neq \emptyset$, then we define $Q^{\langle v,y\rangle} = (X \neq X \supset c_{s_r})$ as in the case where E_v is not correct. We keep in mind that the initial functions of an enumeration on $\mathfrak{A}$ should be closed under the domain of the enumeration (Definition 3.1 (i)).

Let for all $s \in \mathbb{S}$ $H^l_s \cap K_s = \emptyset$. Then we check if $y \in K_{s_r}$. If not, we put again $Q^{\langle v,y\rangle} = (X \neq X \supset c_{s_r})$.

Otherwise, for each appropriate element $u = \langle i, z, \varepsilon\rangle \in E_v$ we find the corresponding formula Π^u under the following rule:

(1) if $i \leq n$, then Π^u is $f_i(\tau^z) = \tau^\varepsilon$;

(2) if $n < i$, then Π^u is $T_{i-n}(\tau^z)$ for $\varepsilon = 0$ and Π^u is $\neg T_{i-n}(\tau^z)$ for $\varepsilon = 1$.

Let $u_1, \ldots, u_q$ be all appropriate elements of E_v. Denote by Π the conjunction Π^{u_1} & $\cdots$ & Π^{u_q} & V, where

$$V = \begin{cases} \underset{\substack{z_i \neq z_j, s \in \mathbb{S} \\ z_i, z_j \in K_s}}{\&} \mathrm{var}_s(z_i) \neq \mathrm{var}_s(z_j) \;\&\; \underset{z \in K_{s_d} \setminus \{x\}}{\&} X \neq \mathrm{var}_{s_d}(z) & \text{if } x \notin \mathrm{dom}(\delta^l_{s_d}); \\ \underset{\substack{z_i \neq z_j, s \in \mathbb{S} \\ z_i, z_j \in K_s}}{\&} \mathrm{var}_s(z_i) \neq \mathrm{var}_s(z_j) \;\&\; X = \mathrm{var}_{s_d}(w) & \text{if } x = w \in \mathrm{dom}(\delta^l_{s_d}). \end{cases}$$

Let $y_1, \ldots, y_b$ be all elements of K_s for those $s \in \mathbb{E}$ (effectively enumerable sorts) not belonging to $\mathrm{dom}(\delta^l_1) \cup \ldots \cup \mathrm{dom}(\delta^l_m) \cup \{x\}$ and $\mathrm{var}_{s_1}(y_1) = Y_1, \ldots, \mathrm{var}_{s_b}(y_b) = Y_b$. From the construction it follows that the variables of Π are among $X, W_1, \ldots, W_a, Y_1, \ldots, Y_b$.

Define

$$Q^{\langle v,y\rangle}(X, \overline{W}) = \exists Y_1 \ldots \exists Y_b(\Pi \supset \tau^y).$$

Let consider some properties of the constructed in this way conditional expression $Q^{\langle v,y\rangle}$. From the above construction and Proposition 4.2 follows:

4.9. Proposition. *Let* $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Delta_l$, $\alpha_{s_d}(x) = p$, $E_v \subseteq \langle \mathfrak{B}\rangle$ *and* $\alpha_{s_i}(y_i) = q_i$, $i = 1, \ldots, b$. *Then*:

(1) E_v *is correct*;

(2) $K_s \subseteq \mathrm{dom}(\alpha_s)$ *and* $H^l_s \cap K_s = \emptyset$ *for each* $s \in \mathbb{S}$;

(3) $\forall z \in K_s(\alpha_s(z) \simeq \tau^z_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}, Y_1/q_1, \ldots, Y_b/q_b))$ *for* $s \in \mathbb{S}$;

(4) $\Pi_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}, Y_1/q_1, \ldots, Y_b/q_b) \simeq 0$. ∎

We are ready to prove that $Q^{\langle v,y\rangle}$ satisfies the conditions of Lemma 4.7.

Case 1. Let $t \in Q^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t})$. So E_v is correct, $H^l_s \cap K_s = \emptyset$ for each sort s and $y \in K_{s_r}$. Thus there exist different elements $q_1, \ldots, q_b$ of $|\mathfrak{A}|$ with sorts $s_1, \ldots, s_b$ from $\mathbb{E}$, so that

$$\Pi_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}, Y_1/q_1, \ldots, Y_b/q_b) \simeq 0,\; t \simeq \tau^y_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}, Y_1/q_1, \ldots, Y_b/q_b).$$

Define a finite part $\Lambda \supseteq \Delta^l$, where $\Lambda = \langle \overline{\lambda}, \overline{H}, \overline{\varphi}, \overline{\sigma} \rangle$, as follows: Let $\lambda_{s_d}(x) = p$, $\lambda_{s_1}(y_1) = q_1, \ldots, \lambda_{s_b}(y_b) = q_b$ and $\lambda_s(z) \simeq \delta_s^l(z)$ for all other z, $s \in \mathbb{S}$. Let $H_s = H_s^l \cup (M_s \setminus K_s)$ for each sort s.

Define $\varphi_i : \varphi_i^l \leq \varphi_i$ for $i = 1, \ldots, n$ and $\sigma_j : \sigma_j^l \leq \sigma_j$ for $j = 1, \ldots, k$, so that the extensions are defined under the rule that for each element $u \in E_v$, which is not appropriate:

if $u = \langle i, z_1, z_2 \rangle$, $i \leq n$, then $\varphi_i(z_1) \simeq z_2$, and,

if $u = \langle j, z, \varepsilon \rangle$, $n < j$, then $\sigma_{j-n}(z) \simeq \varepsilon$.

Let $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Lambda$. From the definition of Λ and Proposition 4.2 we have $E_v \subseteq \langle \mathfrak{B} \rangle$, $\alpha_{s_d}(x) = p$, and since $y \in K_{s_r}$, then from Proposition 4.9 it follows that $\alpha_{s_r}(y) = t$. So the condition (1) from Lemma 4.7 is satisfied.

Case 2. Let $t \notin Q_{\mathfrak{A}}^{\langle v,y \rangle}(X/p, \overline{W}/\overline{t})$.

We have the following possibilities:

(a) E_v is not correct. Then, since for every $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Delta_l : E_v \not\subseteq \langle \mathfrak{B} \rangle$, the condition (2)(b) from Lemma 4.7 is trivially satisfied.

(b) Let E_v is correct:

(b.1) If $H_s^l \cap K_s \neq \emptyset$ for some s, then there is $z \in K_s$ and $z \in H_s$. Therefore for some $i \in \{1 \ldots n\}, z_1 \in K_{s_1}$, $\langle i, z_1, z \rangle \in E_v$. If $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Delta_l$, then it is not possible that $E_v \subseteq \langle \mathfrak{B} \rangle$, since $H_s \cap \mathrm{dom}(\alpha_s) = \emptyset$. Then (2)(b) is fulfilled because of the same argument as in (a).

(b.2) Let $H_s^l \cap K_s = \emptyset$ for all $s \in \mathbb{S}$. Denote by $C^{\langle v,y \rangle} = \exists Y_1 \ldots \exists Y_b(\Pi)$.

(b.2.1) $C_{\mathfrak{A}}^{\langle v,y \rangle}(X/p, \overline{W}/\overline{t}) \not\simeq 0$. Then suppose that $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Delta_l$, $E_v \subseteq \langle \mathfrak{B} \rangle$ and $\alpha_{s_d}(x) = p$. Let

$$\{y_1, \ldots, y_b\} = (K_{s_1} \cup \ldots \cup K_{s_b}) \setminus (\mathrm{dom}(\delta_1^l) \cup \ldots \cup \mathrm{dom}(\delta_m^l) \cup \{x\}),$$

$s_1, \ldots, s_b \in \mathbb{E}$. From the definition of enumeration of $\mathfrak{A}$ we have that $\alpha_{s_i}(y_i) = q_i$ for $i = 1, \ldots, b$. Then from Proposition 4.9: $C_{\mathfrak{A}}^{\langle v,y \rangle}(X/p, \overline{W}/\overline{t}) \simeq 0$, which is a contradiction. So (2)(b) holds trivially.

(b.2.2) Let $C_{\mathfrak{A}}^{\langle v,y \rangle}(X/p, \overline{W}/\overline{t}) \simeq 0$:

(b.2.2) (i) $y \notin K_{s_r}$. Then we construct a finite part $\Lambda \supseteq \Delta_l$ in the same way as in the Case 1, but with only one difference — we put $y \in H_{s_r}$. If $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Lambda$, then we have again that $E_v \subseteq \langle \mathfrak{B} \rangle, \alpha_{s_d}(x) = p$, but since $H_{s_r} \cap \mathrm{dom}(\alpha_{s_r}) = \emptyset$, then $y \notin \mathrm{dom}(\alpha_{s_r})$. Thus the condition (2)(a) from Lemma 4.7 is fulfilled.

(b.2.2) (ii) Let $y \in K_{s_r}$. Suppose that $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Delta_l$ and $E_v \subseteq \langle \mathfrak{B} \rangle, \alpha_{s_d}(x) = p$. Then by Proposition 4.9 it follows that

$$y \in \mathrm{dom}(\alpha_{s_r}) \quad \text{and} \quad \alpha_{s_r}(y) \in Q_{\mathfrak{A}}^{\langle v,y \rangle}(X/p, \overline{W}/\overline{t}).$$

So $\alpha_{s_r}(y) \not\simeq t$, i.e. the condition (2)(b) of Lemma 4.7 is true. ■

We return to the proof of Theorem 4.4. For each $u \in U_{n,x}$ denote by $Q^u(X, \overline{W})$ the conditional expression found effectively in Lemma 4.7.

Let fix an $x \notin \mathrm{dom}(\delta_{s_d}^l) \cup H_{s_d}^l$ and $\mathrm{dom}(\delta_{s_d}^l) = \{w_1, \ldots, w_r\}$. Consider the function ξ with the following definition:

$$\begin{aligned} t \in \xi(p) \quad \iff \quad & \exists u \in U_{n,x}(t \in Q^u_{\mathfrak{A}}(X/p, \overline{W}/\bar{t})) \\ & \vee \exists u_1 \in U_{n,w_1}(t \in Q^{u_1}_{\mathfrak{A}}(X/p, \overline{W}/\bar{t})) \\ & \dots \\ & \vee \exists u_r \in U_{n,w_r}(t \in Q^{u_r}_{\mathfrak{A}}(X/p, \overline{W}/\bar{t})). \end{aligned}$$

Then ξ is definable and therefore $\xi \neq \theta$. Using this fact we ensure that one of the conditions (A) or (B) is not satisfied. There are two possibilities:

(a) There exist $p \in A_{s_d}$ and $t \in A_{s_r}$ such that $\xi(p) \simeq t$, but $\theta(p) \not\simeq t$.

Let $p \notin \text{range}(\delta^l_{s_d})$. From $\xi(p) \simeq t$ it follows that for some $u \in U_{n,x}$ we have $t \in Q^u_{\mathfrak{A}}(X/p\overline{W}/\bar{t})$. Let $u = \langle v, y\rangle$. From Lemma 4.7 we know that there exists a finite part $\Lambda \supseteq \Delta_l$ such that for every enumeration $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Lambda$: $E_v \subseteq \langle \mathfrak{B}\rangle$ & $\alpha_{s_d}(x) = p$ & $\alpha_{s_r}(y) = t$. Define $\Delta_{l+1} = \Lambda$. Then if $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Delta_{l+1}$, there are $x \in \text{dom}(\alpha_{s_d})$ and $y \in \text{dom}(\alpha_{s_r})$ such that $\alpha_{s_r}(y) = t$, $\langle x, y\rangle \in \Gamma_n(\langle \mathfrak{B}\rangle)$, but $\theta(\alpha_{s_d}(x)) \not\simeq t$. So the condition (B) is not valid.

If $p = \delta^l_{s_d}(w)$, then for some $u \in U_{n,w}$ we have $t \in Q^u_{\mathfrak{A}}(X/p, \overline{W}/\bar{t})$. Then using Lemma 4.7 we prove that (B) is not valid analogously.

(b) Let $\xi(p) \not\simeq t$, but $\theta(p) \simeq t$, $p \in A_{s_d}$, $t \in A_{s_r}$. We shall consider the case when $p \notin \text{range}(\delta^l_{s_d})$.

Let have the following situation:

There exist v and y such that $\langle v, y\rangle \in U_{n,x}$, $Q^{\langle v,y\rangle}(X, \overline{W}) = \exists Y_1 \dots \exists Y_b(\Pi \supset \tau^y)$, $C^{\langle v,y\rangle} = \exists Y_1 \dots \exists Y_b(\Pi)$, $C^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\bar{t}) \simeq 0$ and $y \notin K_{s_r}$. In this case we choose $\Delta_{l+1} = \Lambda$ from the proof of Lemma 4.7, Case 2, (b.2.2) (i). We know that if $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Delta_{l+1}$, then $x \in \text{dom}(\alpha_{s_d})$, $E_v \subseteq \langle \mathfrak{B}\rangle$ and therefore $\langle x, y\rangle \in \Gamma_n(\langle \mathfrak{B}\rangle)$, but $y \notin \text{dom}(\alpha_{s_r})$, i.e. the condition (A) is not valid.

Otherwise,

$$\forall u \in U_{n,x}(C^u_{\mathfrak{A}}(X/p, \overline{W}/\bar{t}) \simeq 0 \Longrightarrow y \in K_{s_r}).$$

Put $\delta^{l+1}_{s_d}(x) = p$ and $\delta^{l+1}_s(z) \simeq \delta^l_s(z)$ for all other z and s and $\Delta_{l+1} = \langle \overline{\delta}^{l+1}, \overline{H}^l, \overline{\varphi}^l, \overline{\sigma}^l\rangle$. We shall prove that in this case the condition (B) is not valid. Let $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Delta_{l+1}$. So $\alpha_{s_d}(x) = p$. Suppose that there exists $y \in \text{dom}(\alpha_{s_r})$ such that $\alpha_{s_r}(y) = t$ and $\langle x, y\rangle \in \Gamma_n(\langle \mathfrak{B}\rangle)$. Then for some v: $\langle v, x, y\rangle \in W_n$, i.e. $\langle v, y\rangle \in U_{n,x}$ and $E_v \subseteq \langle \mathfrak{B}\rangle$. From Proposition 4.9 we have that E_v is correct, $H^l_s \cap K_s = \emptyset$ for each $s \in \mathbb{S}$ and $C^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\bar{t}) \simeq 0$. Hence $y \in K_{s_r}$. Since $t \notin Q^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\bar{t})$, from the proof of Lemma 4.7 it follows that the condition (2)(b) is valid, i.e. $\alpha_{s_r}(y) \neq t$, which is a contradiction.

The case when $p \in \text{range}(\delta^l_{s_d})$ is considered similarly.

End of construction.

Consider an enumeration $\langle \alpha_1, \alpha_2, \dots, \alpha_m, \mathfrak{B}\rangle$, where $\mathfrak{B} = (\overline{N}; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k)$, defined as follows:

$$\alpha_s = \bigcup_{l=0}^{\infty} \delta^l_s \quad \text{for each } s \in \mathbb{S},$$

$$\varphi_i^* = \bigcup_{l=0}^{\infty} \varphi_i^l \text{ for } 1 \le i \le n, \quad \sigma_j^* = \bigcup_{l=0}^{\infty} \sigma_j^l \text{ for } 1 \le j \le k.$$

From the construction it is clear that range$(\alpha_s) = A_s$ for all sorts, dom$(\alpha_s) = N$ for every $s \in \mathbb{E}$ and dom$(\alpha_s) \cap H_s^l = \emptyset$ for $s \in \mathbb{S}$ and $l \in N$.

Let

$$\varphi_i(x) \simeq \begin{cases} \theta_i(\alpha_{s_i}(x)) & \text{if } x \in \text{dom}(\alpha_{s_i}), \\ \varphi_i^*(x) & \text{otherwise,} \end{cases}$$

and

$$\sigma_j(x) \simeq \begin{cases} \Sigma_j(\alpha_{s_j}(x)) & \text{if } x \in \text{dom}(\alpha_{s_j}), \\ \sigma_j^*(x) & \text{otherwise.} \end{cases}$$

Suppose that the function θ is admissible in $\langle \overline{\alpha}, \mathfrak{B} \rangle$. Then for some n, if Γ_n is the enumeration operator with number n, for each $x \in$ dom(α_{s_d}) we have:

(A) $\forall y(\langle x, y \rangle \in \Gamma_n(\langle \mathfrak{B} \rangle) \Longrightarrow y \in \text{dom}(\alpha_{s_r}))$;

(B) $\forall t(t \in \theta(\alpha_{s_d}(x)) \Longleftrightarrow \exists y(\alpha_{s_r}(y) \simeq t \;\&\; \langle x, y \rangle \in \Gamma_n(\langle \mathfrak{B} \rangle)))$.

Let $l = 2n + 1$. Then there exists $x \in$ dom$(\delta_{s_d}^{l+1})$ such that for every enumeration $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Delta_{l+1}$ the condition (A) or (B) is not valid. Hence, θ is not admissible in $\langle \overline{\alpha}, \mathfrak{B} \rangle$. ∎

4.10. Corollary. *If the structure $\mathfrak{A}$ is single-sorted, then*:

(i) *if the only sort is effectively enumerable, then $\mathcal{C}^*(\mathfrak{A})$ is the class of search computable functions of Moschovakis on $\mathfrak{A}$;*

(ii) *otherwise $\mathcal{C}^*(\mathfrak{A})$ is the class of computable functions of Friedman.* ∎

In [17, 18] several notions of computability on **ADT** were considered. A generalized variant of Church-Turing thesis for deterministic and non-deterministic computation is announced. It is easy to see that their notion for deterministic computability — *star computability*, coincides with our in case that all sorts are not effectively enumerable, i.e. $\mathbb{E} = \emptyset$. If $\mathbb{E} = \mathbb{S}$, then our computability coincides with the *projective star-computability.*

Remark. All results could be generalized for **ADT** without equality, considering another algebraic transformations — special homomorphisms instead of isomorphisms [16].

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CHARACTERIZATION OF THE EFFECTIVE COMPUTABILITY IN $\mathfrak{f}$-ENUMERATIONS

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Румен Димитров. ХАРАКТЕРИЗАЦИЯ ЭФФЕКТИВНОЙ ВЫЧИСЛИМОСТИ ЧЕРЕЗ $\mathfrak{f}$-НУМЕРАЦИИ

В начале статьи данны дефиниции понятий $\mathfrak{f}$-базис, $\mathfrak{f}$-нумерация и $\mathfrak{f}$-допустимость. Основным результатом является эквивалентность $\mathfrak{f}$-допустимых функций и просто вычислимых функций в структуре $\mathfrak{N}$. В конце доказываются три варианта основной теоремы. Первые две дают обобщения части „$\Leftarrow$" и „$\Rightarrow$" главной теоремы. Третий характеризирует потенциальные $\mathfrak{f}$-допустимые функции.

Rumen Dimitrov. CHARACTERIZATION OF THE EFFECTIVE COMPUTABILITY IN $\mathfrak{f}$-ENUMERATIONS

The main definitions of $\mathfrak{f}$-basis, $\mathfrak{f}$-enumeration and $\mathfrak{f}$-admissibility are given. As a main result the equivalence between $\mathfrak{f}$-admissibility and prime computability in $\mathfrak{N}$ is proved. Finally, three variants of the main theorem are proved. The first two ones are generalizations of the directions "$\Leftarrow$" and "$\Rightarrow$" of the main theorem. In the third variant potentially $\mathfrak{f}$-admissible functions are concerned.

1. INTRODUCTION

The notion of prime and search computability on abstract structures was introduced by Moschovakis [5] in 1969. An equivalent but more natural definition of prime computability was given by Skordev [8]. An important question is to characterize the prime computable functions on structures with domains the set of all natural numbers N. A well-known result is that all functions which can be computed using the functions S (successor), P (predecessor) and the predicate Z (zero recognition) are the μ-recursive functions. Here we study computability in the structure $\mathfrak{N} = (N; P; Z)$.

Our approach is external and is based on the characterizations of abstract computability by means of enumerations. This approach was initiated by Lacombe [4] and studied in [2, 3, 6, 9, 10]. In this paper we study a special class of enumerations of the structure $\mathfrak{N}$ — the $\mathfrak{f}$-enumerations. We prove the equivalence of $\mathfrak{f}$-admissibility and prime computability on $\mathfrak{N}$. As the set of $\mathfrak{f}$-enumerations is a proper subset of all enumerations, where P and Z are effective, in this case the result in one direction is stronger than that proved by Soskov in [10] or in [9].

2. NOTATIONS

Let $\mathfrak{N} = (N; P; Z)$ be the structure with a domain N, a single operation $P(x) = x \dot{-} 1$, and a single predicate $Z(x)$ which gives true for $x = 0$ and false otherwise.

Let p_i be the i-th prime number. We write $(x)_i$ for the primitive recursive function

$$\gamma(x,i) = \begin{cases} \max\{t : p_i^t / x\} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

and $\gamma_0(x)$ for $\gamma(x, 0)$.

We shall fix a coding $\langle\ \rangle$ of the finite sequences of natural numbers such that

$$\langle x_1, x_2, \ldots, x_n\rangle = \mu s\,[s > 0\,\&\,(s)_0 = n\,\&\,(s)_1 = x_1\,\&\,\ldots\,\&\,(s)_n = x_n]\,,$$

i.e. $\langle x_1, x_2, \ldots x_n\rangle = 2^n.3^{x_1}\ldots p_n^{x_n}$.

We shall write $\downarrow f(x_1, \ldots, x_n)$ if $f(x_1, \ldots, x_n)$ is defined, and $\uparrow f(x_1, \ldots, x_n)$ if it is not.

3. BASIC DEFINITIONS

Definition 1. A set $A \subseteq N$ is called $\mathfrak{f}$-basis if there exists a total function $\Psi : N \to N$ such that $A = \{\langle\Psi(0), \ldots, \Psi(i-1)\rangle : i \in N\}$.

Definition 2. The ordered pair $\langle A, \alpha\rangle$ is called $\mathfrak{f}$-enumeration if A is an $\mathfrak{f}$-basis and $\alpha = \gamma_0\lceil A$ (i.e. $\alpha(\langle x_1, x_2, \ldots, x_n\rangle) = n$).

Note. If $\langle A, \alpha\rangle$ is an $\mathfrak{f}$-enumeration, then α is an 1,1 mapping from A to N.

Definition 3. Let $\alpha : A \to B$ be a surjective mapping, where $A \subseteq N$. A function $f : B^n \multimap B$ is called effective in $\langle A, \alpha\rangle$ if there exists a partial recursive function $\varphi : N^n \multimap N$ such that

$$(\forall a_1 \in A)\ldots(\forall a_n \in A)\,(f(\alpha(a_1), \ldots, \alpha(a_n)) \simeq \alpha\,(\varphi(a_1, \ldots, a_n)))\,.$$

Remark. It is clear that given a code $\langle\Psi(0), \ldots, \Psi(i-1)\rangle$ of an i-tuple, we can effectively recognize whether $i = 0$ (i.e. whether the sequence is empty), and if $i \neq 0$, then we can find the code of the sequence $\Psi(0), \ldots, \Psi(i-2)$. It means that in every $\mathfrak{f}$-enumeration P and Z are effective. Notice that in a fixed $\mathfrak{f}$-enumeration $\langle A, \alpha\rangle$ the function S (successor) is effective iff the function Ψ is recursive.

Definition 4. A partial function $f : N^n \twoheadrightarrow N$ is called $\mathfrak{f}$-admissible if it is effective in all $\mathfrak{f}$-enumerations.

Remark. The definition is correct, because for every $\mathfrak{f}$-enumeration $\langle A, \alpha\rangle$ the mapping α is surjective.

4. THE MAIN RESULT

Soskov has proved in [11] that a function f is prime computable (see [5]) in the structure $\mathfrak{N}$ iff it is partial recursive and

$$\forall x_1 \ldots \forall x_n \forall y\, (f(x_1, \ldots, x_n) = y \longrightarrow y \leqq \max(x_1, \ldots, x_n)),$$

i.e.

$$\forall x_1 \ldots \forall x_n\, (\downarrow f(x_1, \ldots, x_n) \longrightarrow f(x_1, \ldots, x_n) \leqq \max(x_1, \ldots, x_n)).$$

Here, for the proof of Skordev's conjecture (the main theorem), we are not going to use the prime computability.

Theorem 1. *A function $f : N^n \twoheadrightarrow N$ is $\mathfrak{f}$-admissible iff it is partial recursive and there exists a natural number c such that the following condition is true:*

$$\forall x_1 \ldots \forall x_n\, (\downarrow f(x_1, \ldots, x_n) \longrightarrow f(x_1, \ldots, x_n) \leqq \max(x_1, \ldots, x_n, c)).$$

Proof. **A.** Let f be a partial recursive function and c be a natural number such that

$$\forall x_1 \ldots \forall x_n\, (\downarrow f(x_1, \ldots, x_n) \longrightarrow f(x_1, \ldots, x_n) \leqq \max(x_1, \ldots, x_n, c)).$$

Given an $\mathfrak{f}$-enumeration $\langle A, \alpha\rangle$, we shall construct a partial recursive function φ, so that

$$f(\alpha(a_1), \ldots, \alpha(a_n)) \simeq \alpha(\varphi(a_1, \ldots, a_n))$$

for all $a_1, \ldots, a_n$ of A. The construction is standard and we shall not go into details. Since α is an 1,1 mapping from A to N, there exists $a \in A$ such that $\alpha(a) = c$. Fix a and let $\mathrm{Max} : N^n \to N$ be the primitive recursive function such that for all a_1, $\ldots$, a_n of A

$$\mathrm{Max}(a_1, \ldots, a_n) = \begin{cases} a_i & \text{if } \gamma_0(a_i) = \max(\gamma_0(a), \gamma_0(a_1), \ldots, \gamma_0(a_n)), \\ a & \text{if } \gamma_0(a) = \max(\gamma_0(a), \gamma_0(a_1), \ldots, \gamma_0(a_n)). \end{cases}$$

Let $\mathrm{PRED} : N \to N$ be the primitive recursive function such that $\mathrm{PRED}(x)$ gives the code $\langle a_0, \ldots, a_{n-1}\rangle$ of the sequence $a_0, \ldots, a_{n-1}$ if x is the code of the sequence $a_0, \ldots, a_{n-1}, a_n$, and $\mathrm{PRED}(x) = x$ if x is the code of the empty sequence.

Define $\mathrm{S1} : N^2 \to N$ by the following equations:

$$\mathrm{S1}(x, 0) = x, \qquad \mathrm{S1}(x, t+1) = \mathrm{PRED}(\mathrm{S1}(x, t)).$$

It is clear that S1 is primitive recursive and for $x \in A$

$$\gamma_0(\mathrm{S1}(x, t)) = \gamma_0(x) \dot{-} t.$$

Let S2 : $N^n \multimap N$ be defined in the following way:

$$S2(a_1,\ldots,a_n) \simeq \mu t\,[(\gamma_0(S1(\mathrm{Max}(a_1,\ldots,a_n),t)) \dot{-} f(\gamma_0(a_1),\ldots,\gamma_0(a_n))) = 0]\,.$$

Finally, if $\varphi : N^n \multimap N$ is defined by the equation

$$\varphi(a_1,\ldots,a_n) \simeq S1(\mathrm{Max}_n(a_1,\ldots,a_n), S2(a_1,\ldots,a_n)),$$

then it is easy to prove that φ is the function we are looking for.

B. In this direction, we shall prove that if f is $\mathfrak{f}$-admissible, then f is partial recursive and

$$(*) \qquad \exists c\forall a_1\ldots\forall a_n\,(\downarrow f(a_1,\ldots,a_n) \longrightarrow f(a_1,\ldots,a_n) \leq \max(a_1,\ldots,a_n,c))\,.$$

From the result in [9] we can obtain that if f is effective in all enumerations of the structure $\mathfrak{N}$, then f is definable in $\mathfrak{N}$. Here we require that f be effective only in $\mathfrak{f}$-enumerations of $\mathfrak{N}$ and we prove something equivalent to definability.

First we shall prove that f is partial recursive. Let $\langle A,\alpha\rangle$ be an $\mathfrak{f}$-enumeration, where $A = \{\langle\Psi(0),\ldots,\Psi(i-1)\rangle \;:\; i\in N\}$ and Ψ is recursive. It is clear from the definitions that α and α^{-1} are partial recursive. We know that there exists a partial recursive function φ such that

$$f(\alpha(a_1),\ldots,\alpha(a_n)) \simeq \alpha(\varphi(a_1,\ldots,a_n)) \quad \text{for all } a_1,\ldots,a_n \text{ of } A.$$

Since α is bijective, we obtain

$$f(x_1,\ldots,x_n) \simeq \alpha\left(\varphi(\alpha^{-1}(x_1),\ldots,\alpha^{-1}(x_n))\right)\,.$$

Hence f is a partial recursive function.

Let us now suppose that $(*)$ is not true, i.e.

$$(\overline{*}) \qquad \forall c\exists a_1\ldots\exists a_n\,(\downarrow f(a_1,\ldots,a_n)\,\&\,(f(a_1,\ldots,a_n) > \max(a_1,\ldots,a_n,c)))\,.$$

First we shall prove the following

Lemma. *The set* $\{(x_1,\ldots,x_n) : f(x_1,\ldots,x_n) > \max(x_1,\ldots,x_n)\}$ *is infinite.*

Proof. Suppose $\left(x_1^{(1)},\ldots,x_n^{(1)}\right)$, $\ldots$, $\left(x_1^{(k)},\ldots,x_n^{(k)}\right)$ are all elements of this set. Let $c_1 = \max\left(f\left(x_1^{(1)},\ldots,x_n^{(1)}\right),\ \ldots,\ f\left(x_1^{(k)},\ldots,x_n^{(k)}\right)\right)$, and $c_1 = 0$ if the set is empty. By $(\overline{*})$ we can find numbers y_1, $\ldots$, y_n such that $(f(y_1,\ldots,y_n) > \max(y_1,\ldots,y_n,\ c_1))$. Then, obviously, $f(y_1,\ldots,y_n) > \max(y_1,\ldots,y_n)$, but $f(y_1,\ldots,y_n) \neq \left(x_1^{(i)},\ldots,x_n^{(i)}\right)$ for all $i = 1,\ldots,k$. We have supposed that the set is finite and obtained a contradiction.

Let φ_0, φ_1, $\ldots$ be a list of all partial recursive functions of n variables.

Definition 5. Let $\langle A,\alpha\rangle$ be an $\mathfrak{f}$-enumeration. An n-tuple $(a_1,\ldots,a_n)$, where a_1, $\ldots$, a_n belong to A, is called witness for the condition

$$(i) \qquad \neg(f(\alpha(a_1),\ldots,\alpha(a_n)) \simeq \alpha(\varphi_i(a_1,\ldots,a_n)))$$

if $\downarrow f(\alpha(a_1),\ldots,\alpha(a_n))$ and one of the following is true:

1) $\uparrow \varphi_i(a_1,\ldots,a_n)$;

2) $\downarrow \varphi_i(a_1,\ldots,a_n)$, but $\varphi_i(a_1,\ldots,a_n) \notin A$, i.e. $\uparrow \alpha(\varphi_i(a_1,\ldots,a_n))$;

3) $\downarrow \alpha(\varphi_i(a_1, \ldots, a_n))$, but $f(\alpha(a_1), \ldots, \alpha(a_n)) \neq \alpha(\varphi_i(a_1, \ldots, a_n))$.

An $\mathfrak{f}$-basis A that consists of the numbers $a_0 < a_1 < a_2 < \cdots$ will be defined in steps. In each step l (for $l = -1, 0, 1, 2, \ldots$) we shall define a finite approximation $A_l = \{a_0, a_1, \ldots, a_{k_l}\}$. In other words, at the step l the values $\Psi(i)$ for $i = 0, \ldots, k_l - 1$ will be defined. In the step $(l+1)$ we shall build a set $A_{l+1} \supset A_l$ such that A_{l+1} will contain a witness for the condition $(l+1)$. Together with the set A we shall also define a set A^- such that $A \cap A^- = \varnothing$. In each step $(l+1)$ the condition $A^-_{l+1} \supseteq A^-_l$ will be met.

We shall prove that the set A is an $\mathfrak{f}$-basis. Next, if $\langle A, \alpha \rangle$ is an $\mathfrak{f}$-enumeration, then we can find a witness for each of the conditions (i), where $i \in N$.

Step -1. Let $k_{-1} = 0$, $a_0 = 1$, $A_{-1} = \{1\}$, $A^-_{-1} = \varnothing$.

Suppose that in step (l) we have built the finite set A^-_l and the finite set A_l which consists of the elements $a_0, a_1, \ldots, a_{k_l}$. Suppose $\Psi(i)$ has been defined for $i < k_l$.

Step $l+1$. We shall define the sets A_{l+1} and A^-_{l+1}, so that A_{l+1} contains a witness for the condition $(l+1)$.

Let $(k^1_{l+1}, k^2_{l+1}, \ldots, k^n_{l+1})$ be an n-tuple such that

$$f(k^1_{l+1}, \ldots, k^n_{l+1}) > \max(k^1_{l+1}, \ldots, k^n_{l+1}) > k_l.$$

The choice of such n-tuple is possible because the set

$$\{(x_1, \ldots, x_n) \ : \ (\forall i \leqq n)(x_i \leqq k_l)\}$$

is finite while the set

$$\{(x_1, \ldots, x_n) \ : \ f(x_1, \ldots, x_n) > \max(x_1, \ldots, x_n)\}$$

is infinite by the previous Lemma. Let $k^1_{l+1} = \max(k^1_{l+1}, \ldots, k^n_{l+1})$. Note that $k_{l+1} > k_l$. Let $h = (k_{l+1} - k_l)$. We shall define $a_{k_l+1}, \ldots, a_{k_{l+1}}$ and $\Psi(k_l), \ldots, \Psi(k_{l+1} - 1)$ successively, so that the following is true for $i = 1, \ldots, h$:

1) $a_{k_l+i} = 2a_{k_l+i-1} p_{k_l+i}^{\Psi(k_l+i-1)}$;

2) $a_{k_l+i} \notin A^-_l$.

The first will ensure that $a_i = \langle \Psi(0), \ldots, \Psi(i-1) \rangle$ for all $i \in N$ and thus A will be an $\mathfrak{f}$-basis. The second will ensure that the requirements $(1), \ldots, (l)$ are not injured for the sake of $(l+1)$.

Since A^-_l is finite, we can define $\Psi(k_l + i - 1)$ and a_{k_l+i} successively for $i = 1, \ldots, h$ in the following way:

$$\Psi(k_l + i - 1) = \mu t \left[2a_{k_l+i-1} p^t_{k_l+i} \notin A^-_l\right] \text{ and } a_{k_l+i} = 2a_{k_l+i-1} p_{k_l+i}^{\Psi(k_l+i-1)}.$$

Note that 1) and 2) are true now.

Let $A_{l+1} = A_l \cup \{a_{k_l+1}, \ldots, a_{k_l+1}\}$ and

$$A^-_{l+1} = \begin{cases} A^-_l \cup \{\varphi_{l+1}(a_{k^1_{l+1}}, \ldots, a_{k^n_{l+1}})\} & \text{if } \downarrow \varphi_{l+1}(a_{k^1_{l+1}}, \ldots, a_{k^n_{l+1}}) \notin A_{l+1}, \\ A^-_l & \text{otherwise.} \end{cases}$$

From these definitions we can see that A_{l+1} and A^-_{l+1} are finite, $A_l \subset A_{l+1}$ and $A^-_l \subseteq A^-_{l+1}$.

Let $A = \bigcup_{i=-1}^{\infty} A_i$ and $A^- = \bigcup_{i=-1}^{\infty} A_i^-$. Now we have to prove that A is an f-basis and $\langle A, \alpha\rangle$ ($\alpha = \gamma_0 \restriction A$) is the f-enumeration that we are looking for. It is clear that $a_0, a_1, \ldots$ are all the elements of A. In the following lemma we prove that A is an f-basis.

Lemma 1. *$a_i = \langle \Psi(0), \ldots, \Psi(i-1)\rangle$ for $i \in N$.*

Proof. Using the definitions, we can prove Lemma 1 easily by induction.

Next we shall see that $A^- \cap A = \varnothing$. For this purpose we shall prove

Lemma 2. *$A_k^- \cap A_k = \varnothing$ for all $k \geqq -1$.*

Proof. An induction is applied.

For $k = -1$ we know that $A_{-1} = \{a_0\}$ and $A_{-1}^- = \varnothing$ and obviously the statement is true.

Let suppose that for some natural l we have $A_l^- \cap A_l = \varnothing$. By construction $A_{l+1} = A_l \cup \{a_{k_l+1}, \ldots, a_{k_{l+1}}\}$ and $a_{k_l+1}, \ldots, a_{k_{l+1}} \notin A_l^-$. Using the induction hypothesis we derive that $A_l^- \cap A_{l+1} = \varnothing$. If $A_l^- = A_{l+1}^-$, then there is nothing to prove. Else

$$A_{l+1}^- = A_l^- \cup \left\{\varphi_{l+1}(a_{k_{l+1}^1}, \ldots, a_{k_{l+1}^n})\right\} \quad \text{and} \quad \varphi_{l+1}(a_{k_{l+1}^1}, \ldots, a_{k_{l+1}^n}) \notin A_{l+1}.$$

In this case it is obvious that $A_{l+1}^- \cap A_{l+1} = \varnothing$.

Now we are ready to prove

Lemma 3. *$A \cap A^- = \varnothing$.*

Proof. Suppose that there exists a number a such that $a \in A \cap A^-$. We can find i and j such that $a \in A_i$ and $a \in A_j^-$. If $k = \max(i, j)$, then $A_k^- \cap A_k \neq \varnothing$, which contradicts Lemma 2.

We shall see next that for each $i \in N$ $(a_{k_i^1}, \ldots, a_{k_i^n})$ is a witness for the condition (i). Let i be a fixed natural number. The n-tuple $(k_i^1, k_i^2, \ldots, k_i^n)$ has been chosen in such a way that $f(k_i^1, \ldots, k_i^n) > \max(k_i^1, \ldots, k_i^n) = k_i$. Note that $f(\alpha(a_{k_i^1}), \ldots, \alpha(a_{k_i^n}))$ is defined. We shall consider the following cases for $\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n})$:

1. $\downarrow \varphi_i(a_{k_i^1}, \ldots, a_{k_i^n}) \in A_i$.

Since $A_i = \{a_0, \ldots, a_{k_i}\}$, then

$$(\forall a \in A_i)(\alpha(a) = \gamma_0(a) \leqq k_i) \quad \text{and} \quad \alpha(\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n})) \leqq k_i.$$

We know that $f(k_i^1, \ldots, k_i^n) > k_i$ and hence

$$\alpha(\varphi_i(a_{k_i^1}, \ldots, a_{k_i^n})) \neq f(\alpha(a_{k_i^1}), \ldots, \alpha(a_{k_i^n})),$$

i.e. $(a_{k_i^1}, \ldots, a_{k_i^n})$ is a witness for the condition (i) by 3) of Definition 5.

2. $\uparrow \varphi_i(a_{k_i^1}, \dots, a_{k_i^n})$.

Since $\uparrow \alpha(\varphi_i(a_{k_i^1}, \dots, a_{k_i^n}))$, we obtain that $(a_{k_i^1}, \dots, a_{k_i^n})$ is a witness for the condition (i) by 1) of Definition 5.

3. $\downarrow \varphi_i(a_{k_i^1}, \dots, a_{k_i^n})$, but $\varphi_i(a_{k_i^1}, \dots, a_{k_i^n}) \notin A_i$.

In this case $\varphi_i(a_{k_i^1}, \dots, a_{k_i^n}) \in A_i^- \subseteq A^-$. By Lemma 3 $A \cap A^- = \varnothing$ and hence $\varphi_i(a_{k_i^1}, \dots, a_{k_i^n}) \notin A$. We obtain $\uparrow \alpha(\varphi_i(a_{k_i^1}, \dots, a_{k_i^n}))$ and then $(a_{k_i^1}, \dots, a_{k_i^n})$ is a witness for the condition (i) by 2) of Definition 5.

We have derived that $(a_{k_i^1}, \dots, a_{k_i^n})$ is a witness for the condition (i) and the theorem is proved.

Remark 1. In the construction of the set A we have not used the partial recursiveness of f.

Remark 2. The construction of the set A^- could be avoided because the requirement 2) (i.e. $a_{k_{l+i}^n} \notin A_l^-$) in step $(l+1)$ of the construction of the set A may be changed by the condition:

$$\text{if for some } l_1 \leqq l \quad y = \varphi_{l_1}(a_{k_{l_1}^1}, \dots, a_{k_{l_1}^n}), \quad \text{then} \quad a_{k_{l+1}^n} \neq y.$$

5. THREE VARIANTS OF THE THEOREM

First we shall prove a stronger result than that proved in the direction "$\Leftarrow$" of the main result.

Definition 6. Let $\alpha : A \to B$ be a surjective mapping, where $A \subseteq N$. A predicate $P(x_1, x_2, \dots, x_n)$ on B is called effective in the enumeration $\langle A, \alpha \rangle$ if there exists a partial recursive function $\varphi : N^n \multimap \{0,1\}$ such that

$$(\forall a_1 \in A) \dots (\forall a_n \in A)(\downarrow \varphi(a_1, \dots, a_n) \,\&\, (P(\alpha(a_1), \dots, \alpha(a_n)) \Leftrightarrow \varphi(a_1, \dots, a_n) = 1)).$$

Theorem 2. *If f is a function such that*

$$\exists c \forall x_1 \dots \forall x_n (\downarrow f(x_1, \dots, x_n) \to f(x_1, \dots, x_n) \leqq \max(x_1, \dots, x_n, c)),$$

then f is effective in all enumerations of the structure $\mathfrak{N}$.

Proof. First we shall note that enumerations of the structure $\mathfrak{N}$ are those for which the functions and predicated of the structure are effective. Obviously, the $\mathfrak{f}$-enumerations are enumerations of $\mathfrak{N}$.

Let $\langle A, \alpha \rangle$ be an enumeration of $\mathfrak{N}$. Let $Z : N \multimap \{0,1\}$ and PRED $: N \multimap N$ be partial recursive functions such that for all $a \in A$:

1) $\alpha(\text{PRED}(a)) \simeq \alpha(a) \dot{-} 1$;

2) $Z(a) = \begin{cases} 1 & \text{if } \alpha(a) = 0, \\ 0 & \text{if } \alpha(a) \neq 0. \end{cases}$

Obviously, $\downarrow \text{PRED}(a) \in A$ and $\downarrow Z(a)$ for all $a \in A$. Thus $\downarrow Z(\text{PRED}^t(a))$ for all $a \in A$ and $t \in N$.

Now we shall see that there exists a partial recursive function γ such that $\gamma\lceil A = \alpha$. Let us define $\gamma : N \longrightarrow N$ in the following way:

$$\gamma(x) \simeq \mu t \left[Z(\mathrm{PRED}^t(x)) = 1\right].$$

We shall prove that $\gamma(a) = \alpha(a)$ for all $a \in A$, i.e.

$$\alpha(a) = \mu t \left[Z(\mathrm{PRED}^t(a)) = 1\right]. \tag{1}$$

Since $\downarrow \alpha(a) \in N$ for $a \in A$, then (1) could be proved by induction on $\alpha(a)$. The proof of this fact is left to the reader.

We are looking for a partial recursive function $\varphi : N^n \longrightarrow N$ such that

$$(\forall a_1 \in A) \ldots (\forall a_n \in A)(f(\alpha(a_1), \ldots, \alpha(a_n)) \simeq \alpha(\varphi(a_1, \ldots, a_n))).$$

The construction of φ is the same as the construction of the function φ in the proof of the main result.

In the proof of the main result we observed that if

$$\forall c \exists x_1 \ldots \exists x_n (\downarrow f(x_1, \ldots, x_n) \,\&\, f(x_1, \ldots, x_n) > \max(x_1, \ldots, x_n, c)),$$

then there exists an $\mathfrak{f}$-enumeration $\langle A, \alpha \rangle$ such that f is not effective in $\langle A, \alpha \rangle$. We built the set A in steps. In each step we built a finite approximation of A. We shall analyze that construction and obtain a stronger result than the one proved in the direction "$\Rightarrow$" of Theorem 1. We have noted that the construction of A could be modified in such a way that the use of A^- be avoided. We shall use that f is partial recursive and modify the construction of A in the following way.

On the step $(l+1)$ we define effectively a code of an n-tuple $(k_{l+1}^1, k_{l+1}^2, \ldots, k_{l+1}^n)$ such that

$$f(k_{l+1}^1, \ldots, k_{l+1}^n) > \max(k_{l+1}^1, \ldots, k_{l+1}^n) > k_l.$$

Later we define the recursive function $S : N \to N$ such that $S(l+1)$ gives the code of the n-tuple $(k_{l+1}^1, k_{l+1}^2, \ldots, k_{l+1}^n)$, which was defined on step $(l+1)$.

Let F be a primitive recursive function such that

$$(x_1, x_2, \ldots, x_n, y) \in G_f \iff \exists z (F(x_1, x_2, \ldots, x_n, y, z) = 0)$$

and let $g(x_1, x_2, \ldots, x_n, t) = F(x_1, x_2, \ldots, x_n, L(t), R(t))$.

We know that f has the normal form

$$f(x_1, x_2, \ldots, x_n) \simeq L(\mu t[g(x_1, x_2, \ldots, x_n, t) = 0]).$$

Let J be a standard coding of ordered pairs in N. We denote

$$J^2 = J, \quad J^{n+1}(x_1, x_2, \ldots, x_n, x_{n+1}) = J(J^n(x_1, x_2, \ldots, x_n), x_{n+1}) \text{ for } n > 2.$$

Let Pr_m^n $(m \leq n)$ be a primitive recursive function such that

$$\mathrm{Pr}_m^n(J^n(x_1, x_2, \ldots, x_n)) = x_m$$

(for $n = 2$ we write L for Pr_1^2 and R for Pr_2^2).

Define the functions $S : N \to N$ and $M : N \to N$ as it follows:

1) $S(-1) = 0$ and $M(-1) = 0$;

2) $S(l+1) = L(\mu s[g(\Pr_1^{n+1}(s), \Pr_2^{n+1}(s), \ldots, \Pr_n^{n+1}(s), \Pr_{n+1}^{n+1}(s)) = 0$
$\& L(\Pr_{n+1}^{n+1}(s)) > \max(\Pr_1^{n+1}(s), \Pr_2^{n+1}(s), \ldots, \Pr_n^{n+1}(s))$
$\& \max(\Pr_1^{n+1}(s), \Pr_2^{n+1}(s), \ldots, \Pr_n^{n+1}(s)) > M(l)])$ and
$M(l+1) = \max(\Pr_1^n(S(l+1)), \Pr_2^n(S(l+1)), \ldots, \Pr_n^n(S(l+1)))$.

Remark. We shall define S and M for $n = -1$ in order to unify the definitions of S and M for $n = 0$ and $n > 0$, but we shall think that they are defined only for $n \geqq 0$.

By induction we shall prove that S and M are totally defined.

1. For $l = -1$ we have $S(-1) = M(-1) = 0$.
2. Let $\downarrow S(l)$ and $\downarrow M(l)$ for some natural l.
3. We know that the set $\{(x_1, \ldots, x_n) : f(x_1, \ldots, x_n) > \max(x_1, \ldots, x_n)\}$ is infinite and there exists $(x_1, \ldots, x_n)$ such that

$$\downarrow f(x_1, \ldots, x_n) > \max(x_1, \ldots, x_n) > M(l).$$

Using the normal form of f, we derive that there exists s such that

$$\begin{aligned} &g(\Pr_1^{n+1}(s), \Pr_2^{n+1}(s), \ldots, \Pr_n^{n+1}(s), \Pr_{n+1}^{n+1}(s)) = 0 \\ &\quad \& L(\Pr_{n+1}^{n+1}(s)) > \max(\Pr_1^{n+1}(s), \Pr_2^{n+1}(s), \ldots, \Pr_n^{n+1}(s)) \\ &\quad \& \max(\Pr_1^{n+1}(s), \Pr_2^{n+1}(s), \ldots, \Pr_n^{n+1}(s)) > M(l). \end{aligned}$$

From here we can easily see that $\downarrow S(l+1)$ and then $\downarrow M(l+1)$. Thus, using the definition of the functions S and M, we derive that they are recursive. We can see that $S(l+1)$ is the code of the n-tuple $(k_{l+1}^1, k_{l+1}^2, \ldots, k_{l+1}^n)$, which was defined on step $(l+1)$, and that $M(l+1) = \max(k_{l+1}^1, k_{l+1}^2, \ldots, k_{l+1}^n) = k_{l+1}$.

Let $\mathfrak{F}$ be a universal for the partial recursive functions and recursive predicate such that: $\varphi_l(x_1, \ldots, x_n) = y \Leftrightarrow \exists z \mathfrak{F}(l, x_1, \ldots, x_n, y, z)$. We shall define a predicate $\mathfrak{B}(x)$ such that $\mathfrak{B}(x) \Leftrightarrow x \in A$. For this purpose first we shall define the predicate $\mathfrak{C}_0$ in the following way:

$$\begin{aligned} \mathfrak{C}_0(s, x, i) \Leftrightarrow \forall l_1 \leqq \mu l[(M(l) < ((x)_0 \dot{-} i)) \& (((x)_0 \dot{-} i) \leqq M(l+1))], \\ (\forall y \forall z(\mathfrak{F}(l_1, \mathrm{PRED}^{(x)_0 \dot{-} \Pr_1^n(S(l_1))}(x), \ldots, \mathrm{PRED}^{(x)_0 \dot{-} \Pr_n^n(S(l_1))}(x), y, z) \\ \longrightarrow (2\,\mathrm{PRED}^{i+1}(x) p^s_{(x)_0 \dot{-} i} \neq y))). \end{aligned}$$

Next define the predicate $\mathfrak{C}_1$ as it follows:

$$\mathfrak{C}_1(s, x, i) \Leftrightarrow \mathfrak{C}_0(s, x, i) \& \forall s_1 < s \neg \mathfrak{C}_0(s_1, x, i).$$

Note that if $x = \langle a_1, \ldots, a_j \rangle$ $(j > i)$, then $\mathfrak{C}_1(s, x, i)$ is true if and only if s is defined just the way $\Psi(j - i - 1)$ is defined in the construction of the set A in the main theorem.

By the expression $(M(l) < ((x)_0 \dot{-} i)) \& (((x)_0 \dot{-} i) \leqq M(l+1))$ we find a number l such that $(l+1)$ is the number of the step, where $\mathrm{PRED}^i(x)$ is defined. Then for all $l_1 \leqq l$ we calculate the code of the first n-tuple $(k_{l_1}^1, k_{l_1}^2, \ldots, k_{l_1}^n)$ (i.e. $S(l_1)$) such that

$$f(k_{l_1}^1, k_{l_1}^2, \ldots, k_{l_1}^n) > \max(k_{l_1}^1, k_{l_1}^2, \ldots, k_{l_1}^n).$$

Further we find the least s such that for all $l_1 \leqq l$ if $\varphi_{l_1}(x_{k^1_{l_1}}, \dots, x_{k^n_{l_1}}) = y$, then $2\,\mathrm{PRED}^{i+1}(x)p^s_{(x)_0 \dot{-} i} \neq y$.

Since M and S are recursive functions and $\mathfrak{F}$ is a recursive predicate, there exists a recursive predicate $\mathfrak{P}_1$ such that

$$\mathfrak{C}_1(s,x,i) \Leftrightarrow \forall q_1 \mathfrak{P}_1(s,x,i,q_1) \,\&\, (\forall s_1 < s)\neg\forall q_2 \mathfrak{P}_1(s_1,x,i,q_2),$$

i.e.

$$(**) \qquad \mathfrak{C}_1(s,x,i) \Leftrightarrow \forall q_1 \mathfrak{P}_1(s,x,i,q_1) \,\&\, \exists q_3 \mathfrak{P}_2(s,x,i,q_3),$$

where $\mathfrak{P}_2$ is again a recursive predicate.

Let us define $\mathfrak{B}$ in the following way:

$$\mathfrak{B}(x) \Leftrightarrow (\mathrm{PRED}^{(x)_0}(x) = 1 \,\&\, (\forall i < (x)_0)(\mathfrak{C}_1((x)_{((x)_0 \dot{-} i)}, x, i)).$$

Let notice that $\mathfrak{B}(x)$ is true if and only if $x = 2^n.3^{\Psi(0)} \dots p_n^{\Psi(n-1)}$, where $\Psi(0)$, $\Psi(1)$, ..., $\Psi(n-1)$ are exactly like those ones, defined in the construction of the set A in the proof of the main result. In other words, $\mathfrak{B}(x) \Leftrightarrow x \in A$.

Using $(**)$, we can find recursive predicates $\mathfrak{P}_3$ and $\mathfrak{P}_4$ such that

$$x \in A \Leftrightarrow \mathfrak{B}(x) \Leftrightarrow \forall q_4 \mathfrak{P}_3(q_4, x) \,\&\, \exists q_5 \mathfrak{P}_4(q_5, x).$$

The set $\{x : \exists q_5 p_4(q_5, x)\}$ is recursively enumerable. The set $\{x : \forall q_4 p_3(q_4, x)\}$ is co-recursively enumerable. Thus the set A could be represented as a difference of two recursively enumerable sets. We proved the following result:

Theorem 3. *If the function $f : N \multimap N$ is effective in every $\mathfrak{f}$-enumeration $\langle A, \alpha\rangle$ such that A can be represented as an intersection of a recursively enumerable and a co-recursively enumerable set, then f is partial recursive and there exists $c \in N$ such that*

$$\forall x_1 \dots \forall x_n (\downarrow f(x_1, \dots, x_n) \longrightarrow f(x_1, \dots, x_n) \leqq \max(x_1, \dots, x_n, c)).$$

For the proof of the main theorem we have defined the term witness for the condition (i) $\neg(f(\alpha(a_1), \dots, \alpha(a_n)) \simeq \alpha(\varphi_i(a_1, \dots, a_n)))$, where the left-hand side of this conditional equality was defined for every witness. Now we shall use this fact to prove another variant of the main theorem.

Definition 7. If $\alpha : A \to B$, where $A \subseteq N$ is a surjective mapping, then $f : B^n \multimap B$ is said to be potentially effective in the enumeration $\langle A, \alpha\rangle$ if there exists a partial recursive function $\varphi : N^n \multimap N$ such that

$$(\forall a_1 \in A) \dots (\forall a_n \in A)(\downarrow f(\alpha(a_1), \dots, \alpha(a_n)) \longrightarrow \downarrow \alpha(\varphi(a_1, \dots, a_n)) \\ \& \, f(\alpha(a_1), \dots, \alpha(a_n)) = \alpha(\varphi(a_1, \dots, a_n))).$$

Definition 8. $f : N^n \multimap N$ is called potentially $\mathfrak{f}$-admissible if f is potentially effective in all $\mathfrak{f}$-enumerations.

Theorem 4. *A function $f : N \multimap N$ is potentially $\mathfrak{f}$-admissible if and only if f is potentially partial recursive and there exists $c \in N$ such that*

$$\forall x_1 \dots \forall x_n (\downarrow f(x_1, \dots, x_n) \longrightarrow f(x_1, \dots, x_n) \leqq \max(x_1, \dots, x_n, c)).$$

Proof. **A.** Let $\langle A, \alpha\rangle$ be a functional enumeration, f be potentially partial recursive, and let exist $c \in N$ such that

$$\forall x_1 \ldots \forall x_n(\downarrow f(x_1, \ldots, x_n) \longrightarrow f(x_1, \ldots, x_n) \leqq \max(x_1, \ldots, x_n, c)).$$

We shall find a partial recursive function φ such that

$$(\forall a_1 \in A) \ldots (\forall a_n \in A)(\downarrow f(\alpha(a_1), \ldots, \alpha(a_n)) \longrightarrow \downarrow \alpha(\varphi(a_1, \ldots, a_n)) \\ \& f(\alpha(a_1), \ldots, \alpha(a_n)) = \alpha(\varphi(a_1, \ldots, a_n))).$$

The construction of the function φ is the same as the construction of φ in the proof of the main result, but here instead of the function f we shall use its partial recursive continuation.

B. In the direction "$\Rightarrow$" for the proof that f is potentially partial recursive we can see that $\alpha(\varphi(\alpha^{-1}(x_1), \ldots, \alpha^{-1}(x_n)))$ is a partial recursive continuation of f. For the proof that $\exists c \in N$ such that

$$\forall x_1 \ldots \forall x_n(\downarrow f(x_1, \ldots, x_n) \longrightarrow f(x_1, \ldots, x_n) \leqq \max(x_1, \ldots, x_n, c))$$

we can construct a set A just the way we built it in the main theorem. We can see that $\downarrow f(\alpha(x_{k_i^1}), \ldots, \alpha(x_{k_i^n}))$, but either

$$\uparrow \alpha(\varphi_i(x_{k_i^1}, \ldots, x_{k_i^n}))$$

or

$$(\downarrow \alpha(\varphi_i(x_{k_i^1}, \ldots, x_{k_i^n})) \,\&\, \alpha(\varphi_i(x_{k_i^1}, \ldots, x_{k_i^n})) \neq f(\alpha(x_{k_i^1}), \ldots, \alpha(x_{k_i^n})))$$

for all natural i. Thus there exists φ such that

$$(\forall a_1 \in A) \ldots (\forall a_n \in A)(\downarrow f(\alpha(a_1), \ldots, \alpha(a_n)) \longrightarrow \downarrow \alpha(\varphi(a_1, \ldots, a_n)) \\ \& f(\alpha(a_1), \ldots, \alpha(a_n)) = \alpha(\varphi(a_1, \ldots, a_n))).$$

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РЕШЕНИЕ НА ПРОБЛЕМ НА ФЬОДОРОВ-ГРЮНБАУМ

НИКОЛА МАРТИНОВ

Никола Мартинов. РЕШЕНИЕ ОДНОЙ ПРОБЛЕМЫ ФЬОДОРОВА-ГРЮНБАУМА

Определяем в полности множество $\mathcal{L}$ пар (n, f) целых чисел, для которых существует размещение из n прямых и f клеток, или в другой интерпретации — для которых существует зоноэдр с n зонами и $2f$ вершинами. Пара (n, f) принадлежит $\mathcal{L}$ тогда и только тогда, когда существуют неотрицательные целые числа k и t, для которых $t \leqq \binom{k}{2}$ и такие, что $n \geqq k+t+2$ и $f = (n-k)(k+1) + \binom{k}{2} - t$. Для каждых k и t точки (n, f), удовлетворяющие этим условиям, образуют множество $R(k,t)$, совпадающее с целочисленными точками одного луча, и эти множества $R(k,t)$ непересекающиеся. Далее, каждая точка $(n, f) \in \mathcal{L}$ может быть получена из легко описываемого размещения прямых. Получены и некоторые характерные свойства размещений, соответствующие одной и той же паре $(n, f) \in \mathcal{L}$.

Nicola Martinov. A SOLUTION TO A PROBLEM OF FEDOROV-GRÜNBAUM

We determine copletely the set $\mathcal{L}$ of pairs (n, f) of integers for which there exist arrangements with n lines and f cells or, in other interpretation, for which there exists a zonohedra with n zones and $2f$ vertices. The pair (n, f) is in $\mathcal{L}$ if and only if there are non-negative integers k and t satisfying $t \leqq \binom{k}{2}$ such that $n \geqq k+t+2$ and $f = (n-k)(k+1) + \binom{k}{2} - t$. For each k and t the points (n, f) satisfying these conditions are all the lattice points on a halfline $R(k,t)$, and these halflines are disjoint. Moreover, each point $(n, f) \in \mathcal{L}$ can be obtained from an easily described arrangement. We have found out also some characteristic properties of the arrangements corresponding to one and the same pair $(n, f) \in \mathcal{L}$.

1. ВЪВЕДЕНИЕ И ТЕРМИНОЛОГИЯ

Ще използваме терминологията на Грюнбаум [4] с малка модификация. Мрежа от прави A наричаме всяка крайна фамилия от $n(A) \geqq 2$ различни прави в реалната проективна равнина P. С мрежата A асоциираме 2-мерния клетъчен комплекс, получен от разбиването на P от правите на A; върховете, ръбовете и клетките (многоъгълниците) на комплекса са върхове, ръбове и клетки на мрежата. С $f(A)$ означаваме броя на клетките на мрежата A, а двойката $(n(A), f(A))$ наричаме допустима двойка, съответстваща на мрежата A. Множеството на допустимите двойки, съответстващи на всевъзможните мрежи, означаваме с $\mathcal{L}$ и наричаме f-диаграма. Проблемът за определяне кога една целочислена двойка (n, f) принадежи на $\mathcal{L}$ се дискутира от Грюнбаум [4], а много преди това (в друга интерпретация) и от руския кристалограф Фьодоров [10]. Основна цел на тази статия е пълното определяне на f-диаграмата $\mathcal{L}$.

Ако v е връх, то $t(v)$ е степента му, т. е. броят на правите, инцидентни с v. С $t(A)$ означаваме максималната степен на върховете на мрежата A. Ако g е права на мрежата, $r(g)$ е броят на върховете, инцидентни с g. С $r(A)$ означаваме максималния брой на върховете, инцидентни с права на мрежата A. Ако $t(A) = 2$, A се нарича проста мрежа, а ако $t(A) = n(A) = n$, A се нарича n-сноп.

Непосредствено се съобразява, че за всяка мрежа A с n прави е изпълнено

$$n \leqq f(A) \leqq \binom{n}{2} + 1,$$

като равенство отляво се достига, когато A е сноп, а равенство отдясно — когато A е проста мрежа.

Целите числа от интервала $J = \left[n, \binom{n}{2} + 1\right]$, които не са измежду стойностите на $f(A)$ за мрежа A с n прави, наричаме недопустими. Лесно се съобразява, че отворените подинтервали на J:

$$J_1 = (n, 2n-2) \text{ за } n \geqq 4 \quad \text{и} \quad J_2 = (2n-2, 3n-6) \text{ за } n \geqq 6,$$

съдържат само недопустими числа. Според хипотеза 2.4 на Грюнбаум [4], доказана по-рано ([1], [10], а за $n \geqq 40$ в [7]), подинтервалът

$$J_3 = (3n-5, 4n-12) \text{ за } n \geqq 9$$

също съдържа само недопустими числа. В [5] са намерени всички подинтервали на J, които съдържат само недопустими числа, а следователно и всички допустими двойки (n, f) при фиксирано n. Тук чрез модифициране и доразвиване на някои детайли на изложението в [5] ще получим както пълно описание на съвкупността $\mathcal{L}$, така и различни характеризации на някои от класовете на получената класификация.

Доказателствата, които тук се дават (с изключение на доказателството на теорема 1), са чисто комбинаторни, което означава, че резултатите са в сила и за мрежите от псевдоправи, а също и за ориентираните матроиди с ранг 3.

Резултатите за мрежи от прави се отнасят пряко и за зоноедрите. Зоноедър се нарича всеки изпъкнал многостен, чиито стени са централно-симетрични многоъгълници (и, както е известно, той е централно-симетричен). Зоноедрите са въведени и частично изучавани от руския кристалограф Фьодоров [10] при изследване на геометричния строеж на кристалите. Зона на зоноедъра се нарича всеки цикъл от последователни съседни стени, чиито общи ръбове са взаимно успоредни. Както е известно [2], мрежите от прави са комбинаторно еквивалентни със зоноедрите: на правите, върховете и клетките на мрежа съответстват зоните, двойките срещуположни стени и двойките срещуположни върхове на зоноедър. Следователно проблемът за намиране на всички възможности за броя на клетките на мрежа с n прави (при произволно n), който проблем тук е решен окончателно, може да се счита, че води началото си от изследванията на Фьодоров от 1885 г. Аналогичният проблем за определяне на възможностите за броя на върховете на мрежа с n прави, т. е. за определяне на възможностите за броя на стените на зоноедрите с n зони, е решаван в [3], [6], [8] и др., но още не е получил окончателно решение.

В [8] се посочва, че резултатите за възможностите за броя на върховете на мрежите (а вероятно и другите резултати за мрежи от прави) са свързани и със статистическата физика.

2. ПРЕДВАРИТЕЛНИ РЕЗУЛТАТИ

От нагледни съображения лесно се стига до следната

Лема 1. *Ако A е мрежа с n прави, за броя на клетките ѝ е изпълнено*

$$f(A) = \binom{n}{2} + 1 - \sum_{k>2} \binom{k-1}{2} t_k(A),$$

където с $t_k(A)$ е означен броят на върховете със степен k.

Доказателство. Ще докажем лемата при по-общото предположение, когато A се състои от псевдоправи, т. е. от непрекъснати линии, всеки две от които се пресичат еднократно. Тогава трансформираме A в проста мрежа A' (пак от псевдоправи) по следния начин. Ако v е връх със степен $k \geqq 3$, а l — псевдоправа през него, заменяме достатъчно малка дъга от l, която съдържа v, с нова, която не минава през v и пресича останалите $k-1$ псевдоправи през v в различни точки. Така получаваме псевдоправа l', която разделя $k-1$ от клетките на по две части. Следователно мрежата, която се получава от A чрез заменяне на l с l', ще има $k-2$ клетки повече от клетките на A. Като постъпим по същия начин със следваща псевдоправа през v, броят на клетките ще се увеличи допълнително с $k-3$

и т. н. Следователно, когато степента на v стане 2, първоначалният брой на клетките ще се е увеличил със

$$k-2+k-3+\cdots+1=\binom{k-1}{2}.$$

Така постъпваме последователно с всички върхове със степен $\geqq 3$ и в резултат получаваме мрежата A'. Като сумираме намерените увеличения на броя на клетките за всеки връх със степен $\geqq 3$ и вземем предвид, че мрежата A' има $\binom{n}{2}+1$ клетки, получаваме твърдението на лемата.

Лесно се съобразява [4, с. 14], че е в сила

Лема 2. *Ако мрежата A не е сноп, то*

$$f(A) \geqq 2n(A)-2.$$

Ще докажем

Лема 3. *Нека мрежата A има $n \geqq 8$ прави и $f(A) \leqq 3n-5$. Тогава*

(1) $$t(A) \geqq n-2.$$

Доказателство. Ако A е сноп, то $t(A)=n>n-2$. По-нататък ще предполагаме, че A не е сноп. Ще приложим индукция по n.

1. Нека $n=8$. Тогава чрез заместване и прилагане на лема 1 получаваме

$$3n-5=19,$$

$$f(A)=\binom{n}{2}+1-\sum_{k>2}\binom{k-1}{2}t_k(A)=29-\sum_{k>2}\binom{k-1}{2}t_k(A).$$

Следователно

(2) $$\sigma=\sum_{k>2}\binom{k-1}{2}t_k(A) \geqq 10.$$

От $n(A)=8$ следва непосредствено (или от [4, с. 22]), че

$$t_3(A) \leqq 7, \quad t_4(A) \leqq 2, \quad t_5(A) \leqq 1.$$

Нека $t(A)=3$. Тогава $\sigma \leqq 7$ и идваме до противоречие с (2).

Нека $t(A)=4$. Тогава, както непосредствено се проверява, ако $t_4(A)=1$, то $t_3(A) \leqq 6$, и ако $t_4(A)=2$, то $t_3(A) \leqq 3$. Следователно и в двата случая получаваме $\sigma \leqq 9$, което противоречи на (2).

Нека $t(A)=5$. Тогава или $t_4(A)=1$ и $t_3(A)=0$, или $t_4(A)=0$ и $t_3(A) \leqq 3$. Отново при двете възможности получаваме $\sigma \leqq 9$ и пак имаме противоречие с (2).

Следователно за $t(A)$ остава единствено възможността $t(A) \geqq 6$, което искахме да докажем, за да е изпълнено (1) в този случай.

2. Нека $n>8$. Приемаме, че лемата е в сила за всяка мрежа A^* с $n-1$ прави, за която е изпълнено $f(A^*) \leqq 3n(A^*)-5$.

Тъй като A не е сноп, има права g на A, която съдържа поне 3 от върховете на A. Тогава за броя на клетките на мрежата $A^* = A\backslash g$ получаваме

$$(3) \qquad f(A^*) = f(A) - r(g) \leqq f(A) - 3 \leqq 3(n-1) - 5.$$

От (3) съгласно индукционното предположение получаваме

$$t(A^*) \geqq n(A^*) - 2 = n - 3.$$

Ако правата g минава през върха на A^*, който има максимална степен (която е поне $n-3$), то очевидно изпълнено е (1). Нека g не минава през този връх. Тогава върху g ще лежат поне $n-3$ от върховете на A, т. е. $r(g) \geqq n-3$, и ще получим следното доуточняване на (3):

$$(3') \qquad f(A^*) = f(A) - r(g) \leqq f(A) - n + 3 \leqq 2(n-1).$$

Оттук съгласно лема 2 получаваме, че A^* е сноп. Следователно $t(A) \geqq t(A^*) = n(A^*) = n-1$. С това доказателството на лемата е завършено.

Както вече споменахме, валидността на хипотеза 2.4 на Грюнбаум [4] е установена. Следователно в сила е

Лема 4. *Ако A е мрежа с $f(A) < 4n(A) - 12$, то $f(A) \leqq 3n(A) - 5$.*

Ще докажем

Лема 5. *Ако мрежата A е различна от сноп и $n(A) \geqq 10$, то $r(A) \geqq 5$.*

Доказателство. 1. Нека $f(A) < 4n(A) - 12$. От лема 4 получаваме $f(A) \leqq 3n(A) - 5$. Оттук и от лема 3 следва, че $t(A) \geqq n(A) - 2 \geqq 8$. Следователно $r(A) \geqq t(A) \geqq 8$.

2. Нека $f(A) \geqq 4n(A) - 12$. За броя на ръбовете на A ще бъде изпълнено $f_1(A) \geqq \frac{3}{2}f(A)$. Следователно

$$r(A) \geqq \left]\frac{f_1(A)}{n(A)}\right[\geqq \left]6 - \frac{18}{10}\right[= 5.$$

Тук с $]x[$ се означава най-малкото цяло число, което не е по-малко от x.

3. МРЕЖИ, СЪСТАВЕНИ ОТ СНОП И ПРОСТА МРЕЖА

Нека p, k и s са неотрицателни цели числа и такива, че $s \leqq \min\{p-2, \binom{k}{2}\}$; приемаме, че $\binom{0}{2} = \binom{1}{2} = 0$. Означаваме с $\mathbf{B}^s_{p,k}$ съвкупността на мрежите, всяка от които е обединение на p-сноп с проста мрежа с k прави, като простата мрежа не съдържа прави от снопа, но s от върховете ѝ са инцидентни с прави от снопа. От въведеното ограничение за s непосредствено се съобразява, че такива мрежи има. Ще покажем, че всички двойки (n, f) от $\mathcal{L}$ съответстват на мрежите, които принадлежат на фамилиите $\mathbf{B}^s_{p,k}$. В тази точка ще опишем съвкупността $\mathcal{F}$ на допустимите двойки (n, f), съответстващи на мрежите от фамилиите $\mathbf{B}^s_{p,k}$, а в т. 4 ще покажем, че $\mathcal{L} = \mathcal{F}$.

Лема 6. *Ако* $A \in \mathbf{B}^s_{p,k}/k \geqq 0, 0 \leqq s \leqq \binom{k}{2}, p \geqq s+2$, *то*

$$f(A) = p(k+1) + \binom{k}{2} - s.$$

Доказателство. От $A \in \mathbf{B}^s_{p,k}$ следва, че $n(A) = p+k$ и че A има (освен върховете си със степен 2) един връх със степен p и s върха със степен 3. Оттук съгласно лема 1 получаваме

$$f(A) = \binom{p+k}{2} + 1 - \binom{p-1}{2} - s = p(k+1) + \binom{k}{2} - s.$$

Съгласно лема 6 всички мрежи от фамилията $\mathbf{B}^s_{p,k}$ определят една и съща допустима двойка $(n, f) : n = p+k, \ f = p(k+1) + \binom{k}{2} - s$. Ще казваме, че тази двойка съответства на фамилията $\mathbf{B}^s_{p,k}$. Непосредствено получаваме

Лема 7. *Нека* n *и* $k \leqq n-2$ *са неотрицателни цели числа, а* S_k *е интервалът* $[a_k, b_k]$, *където*

$$b_k = (n-k)(k+1) + \binom{k}{2}, \quad a_k = b_k - \min\{n-k-2, \binom{k}{2}\}.$$

Тогава за всяко $f \in S_k$ *двойката* $(n, f) \in \mathcal{F}$; *тази двойка съответства на фамилията* $\mathbf{B}^s_{n-k,k}$, *където* $s = b_k - f$.

Лема 8. *Нека* k *и* s *са неотрицателни цели числа и такива, че* $s \leqq \binom{k}{2}$. *Тогава съвкупността*

$$R(k,s) = \{(n(A), f(A)) \mid A \in \mathbf{B}^s_{p,k}, \ p \geqq s+2\}$$

съвпада със съвкупността на целочислените точки (в декартови координати) на лъча

$$H(k,s) = \{(x,y) \mid y = (k+1)x - (k+1)k + \binom{k}{2} - s, \quad x \geqq k+s+2\}. \tag{4}$$

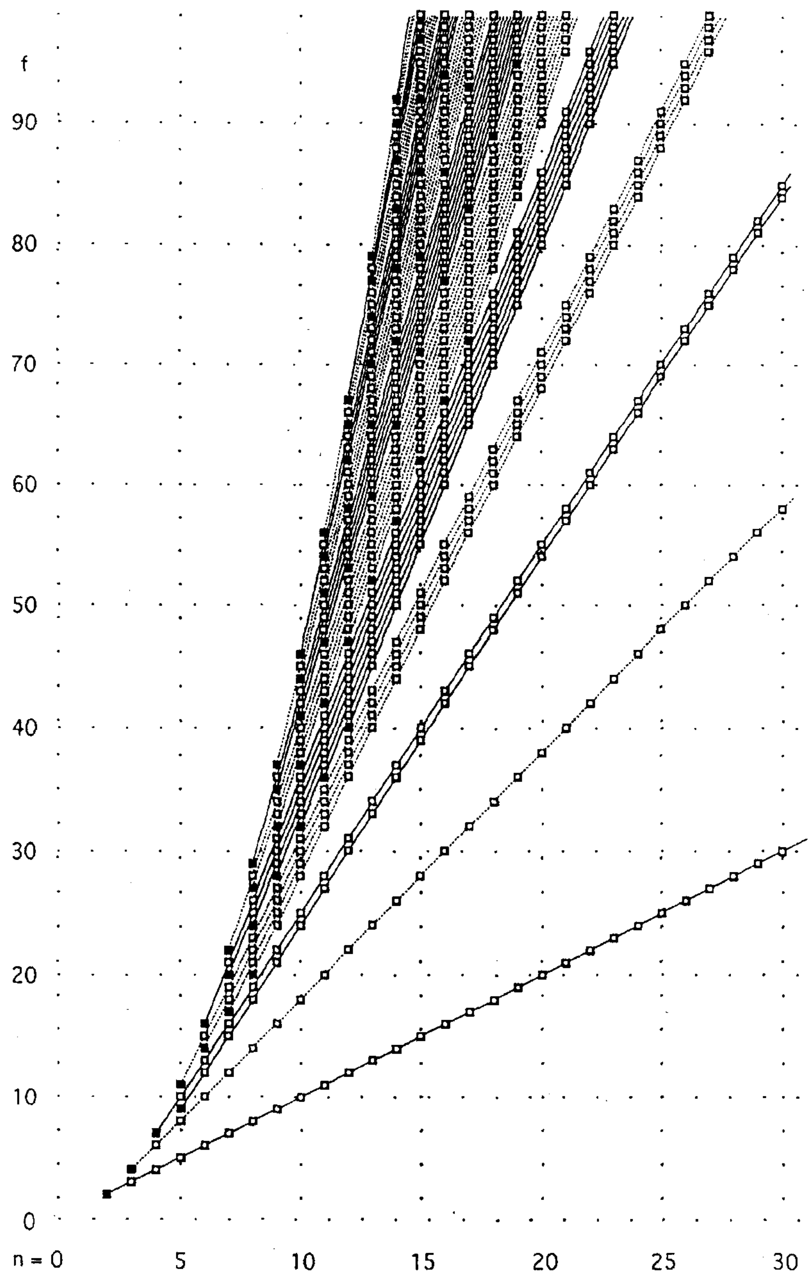
Доказателство. Уравнението на лъча $H(k,s)$ можем да представим във вида

$$y = (k+1)(x-k) + \binom{k}{2} - s, \quad s \leqq x-k-2.$$

Тогава, като положим $x - k = p$ и вземем предвид, че $s \leqq \binom{k}{2}$, получаваме

$$y = p(k+1) + \binom{k}{2} - s, \quad s \leqq \min\{p-2, \binom{k}{2}\}.$$

Оттук се вижда, че целочислените точки на лъча $H(k,s)$ се получават за целочислените стойности на параметъра p, а съгласно лема 6 тези точки са точно допустимите двойки, съответстващи на фамилиите $\mathbf{B}^s_{p,k}$, т. е. тези точки са точно елементите на съвкупността $R(k,s)$.



Представено е изображение на f-диаграма. Точките (n, f) в f-диаграмата са означени с малки квадрати. Линиите, които ги свързват, са лъчите $H(k, s)$, определящи множествата $R(k, s)$. Началните точки на тези лъчи са означени с плътни квадрати. За всяка от последователните стойности на k съответстващите $\binom{k}{2}$ успоредни лъча са означени по един и същ начин — редуващо се с плътни и пунктирани линии.

Следствие 1. *Лъчите $H(k,s)$ и $H(g,t)$ са успоредни точно тогава, когато $g = k$, т. е. когато тези лъчи пресичат правата $x = n$ в един и същ интервал S_k, определен в лема 7.*

Ще покажем, че при $g \neq k$ лъчите $H(k,s)$ и $H(g,t)$ също нямат обща целочислена точка, т. е. че е в сила

Теорема 1. *Всяка допустима двойка $(n,f) \in \mathcal{F}$ съответства само на една фамилия $\mathbf{B}^s_{p,k}/k \geqq 0,\ 0 \leqq s \leqq \binom{k}{2},\ p \geqq s+2$.*

Доказателство. Допускаме, че твърдението не е вярно. Нека допустимата двойка (n,f) е точка както от лъча $H(k,s)$, така и от лъча $H(g,t)$. Тъй като тези лъчи не са успоредни, то $g \neq k$. Приемаме, че $g > k$. От $(n,f) \in H(k,s)$ съгласно (4) получаваме

$$f = (k+1)n - \frac{k^2+3k}{2} - s, \quad n \geqq k+s+2. \tag{5}$$

Аналогично от $(n,f) \in H(g,t)$ получаваме

$$f = (g+1)n - \frac{g^2+3g}{2} - t, \quad n \geqq g+t+2. \tag{6}$$

1. Нека $g = k+1$. Тогава от равенствата при (5) и (6) чрез почленно изваждане получаваме

$$n = k+2+t-s \leqq k+t+2, \tag{7}$$

а от оценките за n при (5) и (6) имаме

$$n \geqq g+t+2 = k+t+3,$$

което противоречи на (7).

2. Нека $g \geqq k+2$. Сега от равенствата при (5) и (6) чрез почленно изваждане получаваме

$$2(t-s) = (g-k)(2n-g-k-3).$$

Оттук, като използваме оценките за n от (5) и (6), намираме следното противоречие:

$$2(t-s) \geqq (g-k)(s+t+1) \geqq 2s+2t+2.$$

С това доказателството на теоремата е завършено.

4. МНОЖЕСТВОТО $\mathcal{L}$ НА ВСИЧКИ ДОПУСТИМИ ДВОЙКИ (n,f)

Теорема 2 (основна теорема). *Нека $k \geqq 3$ е естествено число. За всяка мрежа A с $n(A) \geqq \binom{k+1}{2} + 3$, която е различна от сноп, са в сила следните три твърдения:*

A_k. *Ако* $f(A) \leqq kn(A) - \binom{k+1}{2} + 1$, *то* $t(A) \geqq n(A) - k + 1$;

B_k. $r(A) \geqq k + \frac{1}{2}(k-3) + \frac{15}{k^2+k+6}$;

C_k. *Ако* $f(A) < (k+1)(n(A)-k)$, *то* $f(A) \leqq kn(A) - \binom{k+1}{2} + 1$.

Доказателство. Ще приложим индукция по k. При $k = 3$ теоремата е изпълнена: A_3 и C_3 съвпадат съответно с лемите 3 и 4; B_3 следва от това, че $r(A) \geqq 4$ винаги когато A има поне 7 прави. Предполагаме $k > 3$ и изпълнението на A_{k-1}, B_{k-1} и C_{k-1} за всяка мрежа A^*, *която не е сноп и има* поне $\binom{k}{2} + 3$ прави. Тези две изисквания са изпълнени, ако мрежата A^* е получена от A чрез отстраняване на права и $f(A) < 2n(A) - 2$; по-нататък само в такива случаи ще прилагаме индукционното предположение.

Доказателство на A_k. Ако $f(A) < k(n(A) - k + 1)$, от C_{k-1} получаваме $f(A) \leqq (k-1)n(A) - \binom{k}{2} + 1$. Оттук с използване на A_{k-1} намираме, че $t(A) \geqq n(A) - k + 2$. Следователно остава да докажем твърдението A_k, когато

$$f(A) = k(n(A) - k + 1) + m, \quad 0 \leqq m \leqq \binom{k-1}{2}.$$

Доказателството ще извършим с индукция по m. Нека g е права на A с $r(g) = r(A)$. От B_{k-1} (когато $k > 4$) и от лема 5 (когато $k = 4$) следва, че $r(A) \geqq k + 1$. Следователно за мрежата $A^* = A \backslash g$ ще е изпълнено

$$(8) \quad f(A^*) = f(A) - r(g) \leqq k(n(A) - k + 1) + m - k - 1 = k(n(A) - k) + m - 1.$$

1. Нека $m = 0$. Тогава от (8) следва, че $f(A^*) < k(n(A^*) - k + 1)$, и съгласно C_{k-1} получаваме $f(A^*) \leqq (k-1)n(A^*) - \binom{k}{2} + 1$. Оттук и от A_{k-1} заключаваме, че $t(A) \geqq t(A^*) \geqq n(A^*) - k + 2 = n(A) - k + 1$.

2. Нека сега $m > 0$ и предполагаме, че A_k е изпълнено при по-малки стойности на m. От това индукционно предположение и от (8) следва, че $t(A^*) \geqq n(A^*) - k + 1$, откъдето $t(A) \geqq n(A) - k$. Следователно $r(A) \geqq t(A) \geqq n(A) - k$. Оттук получаваме следната по-точна от (8) оценка за $f(A^*)$:

$$f(A^*) = f(A) - r(A) \leqq k(n(A) - k + 1) + m - n(A) + k = kn(A^*) - k^2 + 2k + m - n(A).$$

Но $n(A) - m \geqq \binom{k+1}{2} + 3 - \binom{k-1}{2} = 2k + 2$. Следователно $f(A^*) < k(n(A^*) - k + 1)$. Оттук съгласно C_{k-1} получаваме $f(A^*) \leqq (k-1)n(A^*) - \binom{k}{2} + 1$ и като приложим A_{k-1}, заключаваме, че $t(A) \geqq t(A^*) \geqq n(A^*) - k + 2 = n(A) - k + 1$.

Доказателство на B_k. 1. Нека $f(A) \leqq kn(A) - \binom{k+1}{2} + 1$. Тогава от доказаното вече твърдение A_k получаваме $t(A) \geqq n(A) - k + 1 \geqq \binom{k+1}{2} + 3 - k + 1 = \binom{k}{2} + 4$. Следователно в този случай B_k е изпълнено.

2. Нека $f(A) > kn(A) - \binom{k+1}{2} + 1$. *Тъй като за броя* $f_1(A)$ *на ръбовете* имаме $f_1(A) \geqq \frac{3}{2} f(A)$ и $r(A) \geqq \frac{f_1(A)}{n(A)}$, то в този случай получаваме

$$r(A) \geqq \frac{3f(A)}{2n(A)} \geqq \frac{3k}{2} - \frac{3(k^2+k-4)}{2(k^2+k+6)} = \frac{3}{2}(k-1) + \frac{15}{k^2+k+6}.$$

Доказателство на $\mathrm{C_k}$. Допускаме, че твърдението $\mathrm{C_k}$ не е вярно. Нека A е мрежа с минимален брой прави, за която е изпълнено

$$(9) \quad n(A) = n \geqq \binom{k+1}{2} + 3 \quad \text{и} \quad kn - \binom{k+1}{2} + 1 < f(A) < (k+1)(n-k).$$

Нека g е права на A с $r(g) = r(A)$. Съгласно вече доказаното $\mathrm{B_k}$ (и лема 5 при $k = 4$) ще е изпълнено $r(A) \geqq k+2$ и за мрежата $A^* = A \backslash g$ получаваме

$$f(A^*) = f(A) - r(g) < (k+1)(n(A)-k) - (k+2) < (k+1)(n(A^*)-k).$$

Нека $n(A) > \binom{k+1}{2} + 3$. Тогава $n(A^*) \geqq \binom{k+1}{2} + 3$ и тъй като (9) не е изпълнено за мрежи с по-малко от $n(A)$ прави, то

$$(10) \qquad f(A^*) \leqq kn(A^*) - \binom{k+1}{2} + 1.$$

Неравенството (10) е изпълнено и когато $n(A) = \binom{k+1}{2} + 3$, защото тогава интервалът $S_k = [kn(A) - \binom{k+1}{2} + 1, (k+1)(n(A)-k)]$ съдържа само едно цяло число и то е $f(A)$. Следователно числото $f(A)$ е с единица по-голямо от левия край на интервала S_k. Оттук и от $f(A^*) = f(A) - r(g) \leqq f(A) - k - 2$ следва, че (10) е изпълнено.

От (10), като приложим доказаното вече твърдение $\mathrm{A_k}$, получаваме $t(A^*) \geqq n(A^*) - k + 1$, откъдето следват $t(A) \geqq n(A) - k$ и $r(A) \geqq t(A) \geqq n(A) - k$. Оттук намираме по-точна от (10) оценка за $f(A^*)$: $f(A^*) < (k+1)(n(A)-k) - (n(A)-k) = k(n(A^*) - k + 1)$. Тогава, като приложим $\mathrm{C_{k-1}}$, получаваме $f(A^*) \leqq (k-1)n(A^*) - \binom{k}{2} + 1$ и $f(A) \leqq f(A^*) + n(A) - 1 \leqq kn(A) - \binom{k+1}{2} + 1$. Полученото противоречие с (9) доказва верността на твърдението $\mathrm{C_k}$.

Ще отбележим, че теоремата е в сила и при $k = 1, 2, 3$, тъй като интервалите J_1, J_2, J_3 съдържат само недопустими числа.

Теорема 3. *Нека n и $k \leqq n-2$ са естествени числа, а J_k е интервалът (b_{k-1}, a_k), където $b_k = (n-k)(k+1) + \binom{k}{2}$, $a_k = b_k - \min\{n-k-2, \binom{k}{2}\}$. Тогава за всяко $f \in J_k$ не съществува мрежа с n прави и f клетки.*

Доказателство. Ще разгледаме отделно два допълващи се случая.

1. Нека $n \geqq \binom{k+1}{2} + 3$. В този случай $n - k - 2 > \binom{k}{2}$ и $a_k = (n-k)(k+1)$. Съгласно твърдението $\mathrm{C_k}$ на теорема 2, ако за произволна мрежа A с n прави е изпълнено $f(A) < a_k$, то $f(A) \leqq kn - \binom{k+1}{2} + 1 = b_{k-1}$. Следователно $f(A)$ е извън интервала J_k, т. е. теоремата е изпълнена за този случай.

2. Нека $n < \binom{k+1}{2} + 3$. Сега $n - k - 2 \leqq \binom{k}{2}$ и $a_k = (n-k)k + \binom{k}{2} + 2$. В този случай за дължината на интервала J_k получаваме

$$a_k - b_{k-1} = (n-k)k + \binom{k}{2} + 2 - kn + \binom{k+1}{2} - 1 = 1.$$

Това означава, че интервалът $J_k = (b_{k-1}, a_k)$ не съдържа цели числа. Следователно и в този случай няма число $f \in J_k$, такова че двойката (n, f) да е допустима, т. е. да има мрежа с n прави и f клетки. С това доказателството на теоремата е завършено.

От теорема 3 и лема 7 непосредствено получаваме

Следствие 2. *Двойката (n, f) принадлежи на $\mathcal{L}$ точно тогава, когато има цяло число k и $0 \leqq k \leqq n-2$, за което е изпълнено*

$$(n-k)(k+1) + \binom{k}{2} - \min\left\{n-k-2, \binom{k}{2}\right\} \leqq f \leqq (n-k)(k+1) + \binom{k}{2}.$$

От следствие 2, като вземем предвид, че $\mathcal{F} \subseteq \mathcal{L}$, и лема 7, получаваме

Следствие 3. *Съвкупностите $\mathcal{L}$ и $\mathcal{F}$ съвпадат.*

5. НЯКОИ СВОЙСТВА НА МРЕЖИТЕ, СЪОТВЕТСТВАЩИ НА ЕДНА И СЪЩА ДОПУСТИМА ДВОЙКА

Съгласно теоремите 3 и 1 на всяка двойка $(n, f) \in \mathcal{L}$ съответства единствена фамилия $\mathbf{B}^s_{p,k}/k \geqq 0, 0 \leqq s \leqq \binom{k}{2}, p \geqq s+2$, като мрежите на фамилията имат по n прави и f клетки. Ще сравним спрямо някои характеристики тези мрежи с останалите, които също имат по n прави и f клетки.

Теорема 4. *Нека (n, f) е допустима двойка, $\Omega(n, f)$ е съвкупността на мрежите, имащи по n прави и f клетки, а B е мрежа от фамилията $\mathbf{B}^s_{p,k} \subset \Omega(n, f)$. Тогава за всяка мрежа $A \in \Omega(n, f)$ е изпълнено $t(A) \leqq t(B)$.*

Доказателство. От $B \in \mathbf{B}^s_{p,k}$ и от лема 1 следва

$$n = n(B) = p + k, \quad f = f(B) = \binom{n}{2} + 1 - \binom{n-k-1}{2} - s,$$

$$s \leqq \min\left\{n-k-2, \binom{k}{2}\right\}.$$

Освен това $t(B) = p = n - k$. Допускаме, че $t(A) > t(B)$, т. е. че $t(A) = n - k + m$, $m > 0$. Тогава от лема 1 получаваме

$$f(A) \leqq \binom{n}{2} + 1 - \binom{n-k+m-1}{2} \leqq \binom{n}{2} + 1 - \binom{n-k-1}{2} - m(n-k-1).$$

Следователно $f(A) = f(B) + s - m(n-k-1) < f(B)$. Полученото противоречие изключва възможността $t(A) > t(B)$. Следователно $t(A) \leqq t(B)$. С това доказателството на теоремата е завършено.

Нека n и k са естествени числа, удовлетворяващи условието

$$n \geqq \binom{k+2}{2} + 3. \tag{11}$$

Тогава за всяко цяло f от интервала $S_k = [a_k, b_k]$, определен в лема 7, двойката $(n, f) \in \mathcal{L}$ и за мрежите от $\Omega(n, f)$, имащи по n прави и f клетки, ще е в сила теорема 2. За всяка мрежа $A \in \Omega(n, f)$ е в сила първото твърдение на теорема 2, формулирано за $k+1$ вместо за k, а именно

A_{k+1}. Ако $f(A) \leqq b_k$, то $t(A) \geqq n-k$.

От друга страна, съгласно лема 7 фамилията $\mathbf{B}^s_{n-k,k}$, където $s = b_k - f$, се съдържа в $\Omega(n, f)$ и съгласно теорема 4 е изпълнено $t(A) \leqq n-k$. Следователно

$$t(A) = n - k. \tag{12}$$

И така (12) ще е изпълнено за всяка мрежа $A \in \Omega(n, f)$, където $f \in S_k$, а k удовлетворява (11), като максималната стойност f_0 на f е стойността на b_k за максималното k, удовлетворяващо (11). Но съгласно теорема 3 и лема 7 допустимите двойки (n, f) се получават само за стойности на f от интервала S_k. Следователно, щом $(n, f) \in \mathcal{L}$ и $f \leqq f_0$, то за мрежите от $\Omega(n, f)$ ще бъде изпълнено (12), т. е. те ще имат една и съща максимална степен на върховете. Като вземем предвид, че $b_k = (n-k)(k+1) + \binom{k}{2}$ расте заедно с k и че за максималната стойност k_0 на k, удовлетворяваща (11), е изпълнено

$$\frac{1}{2}(\sqrt{8n-23} - 5) < k_0 \leqq \frac{1}{2}(\sqrt{8n-23} - 3),$$

получаваме

$$f_0 = b_{k_0} = (n-k_0)(k_0+1) + \binom{k_0}{2} \geqq k_0 n + 4 > \frac{n}{2}(\sqrt{8n-23} - 5) + 4 > n(\sqrt{2n} - 4) + 4.$$

Така установяваме валидността на

Теорема 5. *Нека $(n, f) \in \mathcal{L}$, а $\Omega(n, f)$ е класът мрежи, имащи по n прави и f клетки. Ако $f \leqq n(\sqrt{2n} - 4) + 4$, то всички мрежи от $\Omega(n, f)$ имат една и съща максимална степен на върховете.*

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ON THE CHURCH-ROSSER PROPERTY AND REDUCIBILITY OF NATURAL DERIVATIONS

CHAVDAR ILIEV

Чавдар Илиев. СВОЙСТВО ЧЕРЧ-РОССЕРА И РЕДУКЦИИ ЕСТЕСТВЕННЫХ ВЫВОДОВ

В статье рассмотрено свойство Черч-Россера для некоторых видов редукции. Предложен пример, что редукция дефинирована в [1], не обладает данным свойством, несмотря на то, что оно использовано для доказательства единственности нормальной формы и для отождествление выводов; показан вывод, который редуцируется до различных нормальных. Это свойство действительно для этой редукции в том случае, если отменим требование о приложении комутативных редукции только к максимальным сегментом. Так как свойство действительно для β- и η-редукции, можно предположить, что при добавлении комутативных редукции оно бы сохранилось — здесь показано, что это ведет к утрате свойства.

Chavdar Iliev. ON THE CHURCH-ROSSER PROPERTY AND REDUCIBILITY OF NATURAL DERIVATIONS

The paper contains a treatment of the Church-Rosser property with regard to several kinds of reductions. We give an example that the reducibility relation defined in [1] does not possess the property, although it is used to verify the uniqueness of the normal form and for stating the identity between proofs; we show a derivation that reduces to different normal ones. The property for this relation appears if we deny the restriction over commutative reductions to be applied only for maximal segments. The Church-Rosser property is a well-known fact for reducibility of derivations, and it might be expected that enlarging reducibility with commutative reductions will save the property. Here we illustrate that in that case the property is lost.

0. INTRODUCTION

The reducibility relation of natural derivation can be stated by analogy with λ-calculus: β- and η-reductions are defined for derivations. The question about the Church-Rosser property for these reductions seems to be clear, since they are analogous to reductions in λ-calculus and the Church-Rosser property is well-known fact for λ-terms [3]. According to the Curry-Howard isomorphism we have the same fact for implicative derivations. It is no difficulty to verify the property for derivations in the full language.

Another way of stating the reducibility of derivations is using the inversion principle, as it is made in [1]. Then the question about the Church-Rosser property needs some particularization. According to that principle the corresponding introduction and elimination inference rules are inverses of each other and nothing new is obtained by an elimination immediately following an introduction, so that such a sequence of inferences occuring in a derivation can be dispensed with — in other words, a proof of the conclusion of elimination is already "contained" in the proof of the premisses when the major premiss is inferred by introduction.

The notions *maximum formula* and *maximum segment* are based on this principle and make it explicit for the different cases that can arise; the inversion principle implies that they are unnecessary detours in a derivation which can be removed. A derivation is defined as normal (cf. [1]), when it contains no maximum formula and no maximum segment. A β-reduction removes the maximum formula. To remove maximum segments, commutative reductions are stated. They decrease the length of maximal segments. This allows to define a proper measure and by induction on it to prove that every derivation reduces to a normal form — the well-known Weak normal form theorem. By the Church-Rosser property it follows that the normal form of every derivation is unique — thence derivations with identical normal form can be equivalenced. In studies treating reducibility of derivations ([1, 2]) a detailed proof of the Church-Rosser property is missing, although it is used to verify the uniqueness of the normal form. Here we shall examplify that the normal form of derivations in the full language of intuitionistic logic is not unique when commutative reductions are carried out only for a maximum segment (this requirement for commutative reductions is essential for the proof of the Normal form theorem).

It is not necessary to require the maximum segment to carry out commutative reductions — then we call them free (commutative) reductions. Such a reduction does not always lead to any simplification of the derivation, but sometimes only changes the places of inference rules. Nevertheless, the refusal of the mentioned above requirement is essential for the Church-Rosser property, i.e. the reducibility relation defined by the β-reductions and the free reductions has the Church-Rosser property.

With the help of the Church-Rosser property we can state a natural equivalence relation between the proofs (derivations). As we mentioned above, the Church-Rosser property is valid for $\beta\eta$-reducibility. If we enrich this relation with the commutative reductions, using the Church-Rosser property, we could define a "better" equivalence relation between the proofs. We shall examplify that the adding of the free reductions to the β- and η-reductions leads to the lost of the Church-Rosser property.

1. DEFINITIONS

We shall represent derivations as terms and let denote them by e, f, g, h etc. The formulas that derivations are constructed by are in a language containing $\&$, $\vee$, $\supset$, $\perp$, $\forall$ and $\exists$. Derivations are constructed inductively starting from the atomic ones of the kind $[A]$, where $[A]$ is the trivial derivation of the formula A from a sequence of formulas (assumptions, hypotheses) Γ, with Γ containing A. For each logical constant σ (except $\perp$) we have two inference rules — introduction and elimination which we denote by σ^+ and σ^-, respectively. In certain steps some of the assumptions and parameters may be discharged (or closed). To avoid collisions in substitution operations, we shall put labels on the discharged formulas. We shall use natural numbers for labels, and we shall write them as an upper index of the labelled formula A, i.e. A^k. It is suitable to divide the sequence of assumptions into two ones — Δ with not labelled and Γ with labelled formulas, and to allow discharging formulas only from Γ. We write $d : \Delta\Gamma \longrightarrow A$ for "d is a derivation with conclusion formula A and all uneliminated (or open) assumptions belonging to $\Delta\Gamma$". The relation $d : \Delta\Gamma \longrightarrow A$ is defined inductively as it follows:

1.1. Definition

1) If $A \in \Delta$, then $[A] : \Delta\Gamma \longrightarrow A$; if $B^k \in \Gamma$, then $[B^k] : \Delta\Gamma \longrightarrow B$;
2) If $d_0 : \Delta\Gamma \longrightarrow A$, $d_1 : \Delta\Gamma \longrightarrow B$, then $\&^+ d_0 d_1 : \Delta\Gamma \longrightarrow A \,\&\, B$;
3) If $d : \Delta\Gamma \longrightarrow A_0 \,\&\, A_1$, then $\&_0^- d : \Delta\Gamma \longrightarrow A_0$, $\&_1^- d : \Delta\Gamma \longrightarrow A_1$;
4) If $d : \Delta\Gamma \longrightarrow A$, then $\vee_0^+ d : \Delta\Gamma \longrightarrow A \vee B$, $\vee_1^+ d : \Delta\Gamma \longrightarrow B \vee A$;
5) If $d : \Delta\Gamma \longrightarrow A_0 \vee A_1$, $d_0 : \Delta A_0^{k_0}\Gamma \longrightarrow D$, $d_1 : \Delta A_1^{k_1}\Gamma \longrightarrow D$, then $\vee^- d(A_0^{k_0})d_0(A_1^{k_1})d_1 : \Delta\Gamma \longrightarrow D$, and the occurrences of $[A_0^{k_0}]$ and $[A_1^{k_1}]$ *in* d_0 and d_1, respectively, are closed;
6) If $d : \Delta B^k\Gamma \longrightarrow C$, then $\supset^+ (B^k)d : \Delta\Gamma \longrightarrow B \supset C$, and the occurrences of $[B^k]$ *in* d are closed;
7) If $d : \Delta\Gamma \longrightarrow B \supset C$, $e : \Delta\Gamma \longrightarrow B$, then $\supset^- de : \Delta\Gamma \longrightarrow C$;
8) If $d : \Delta\Gamma \longrightarrow A(u)$, $u \notin \mathrm{Par}(\Delta\Gamma$, then $\forall^+(u)d : \Delta\Gamma \longrightarrow \forall z A_u(z)$, and the occurrences of u *in* d are closed;
9) If $d : \Delta\Gamma \longrightarrow \forall z B_u(z)$, t is a term, then $\forall^- td : \Delta\Gamma \longrightarrow B_u(t)$;
10) If $d : \Delta\Gamma \longrightarrow B_u(t)$, t is a term, then $\exists^+ td : \Delta\Gamma \longrightarrow \exists z B_u(t)$;
11) If $d : \Delta\Gamma \longrightarrow \exists z B(z)$, $e : \Delta B^k(u)\Gamma \longrightarrow C$, $u \notin \mathrm{Par}(\Delta\Gamma C)$, then $\exists^- d(B^k, u)e : \Delta\Gamma \longrightarrow C$, and the occurrences of $[B^k]$ and u *in* e are closed;
12) If $d : \Delta\Gamma \longrightarrow \perp$, then $\perp^- d : \Delta\Gamma \longrightarrow B$.

By $[F]$ we shall denote derivations of the kind $[A]$ or $[B^k]$. The occurrence of $[F]$ in a derivation is said to be *open* (or F is open) if it is not closed in the subderivation in which $[F]$ occurs. *Open parameters* are defined in the same way.

Note. According to the definition a labelled formula can be open, but if a formula is closed, it is surely labelled. Also, a formula or a parameter may have open and closed occurrences in a certain derivation, depending on the subderivations it is met in.

Derivations that only differ with respect to closed parameters or labels of closed formulas should be counted as identical.

A *conclusion of inference rule* is the conclusion of the derivation obtained by immediate applying of the rule. *Major premiss* of an elimination rule σ^- is the conclusion of the derivation written immediately after σ^-. *Length* of a derivation is defined as the number of the occurrences of strings of the kind σ^- or σ^+ in the derivation.

1.2. Substitution operations

By $[d \mid F : e]$ we denote the result of replacing the open occurrences of $[F]$ in d *with the derivation* e. *The result of replacing the open occurrences of a parameter* u in d with a term t is denoted by $[d \mid u : t]$ or shortly $d_u(t)$. We say that a collision appears at the substitution operation if a closed formula or a parameter of the derivation we substitute in occurs as open in the derivation or in the term we substitute with; or when substituting in a derivation or a subderivation of the kind $\exists^- d(B^k, u)e$, $B^k_u(t)$ for some term t occurs as open in the derivation we substitute with (by $B^k_u(t)$ we mean $(B_u(t))^k$). Collisions may be avoided by renaming closed parameters and proper change of labels of closed formulas. Since we have equivalenced derivations that differ with respect to closed parameters and labels of closed formulas, we shall assume that collisions do not appear at substitutions.

By $\Theta_u(t)$ we denote the replacing of the parameter u with the term t in all formulas of Θ (Θ is a sequence of formulas).

Lemma. a) *Let* $h : \Delta\Gamma \longrightarrow D$. *Then* $[h \mid v : t] : \Delta_v(t)\Gamma_v(t) \longrightarrow D_v(t)$.
b) *Let* $h : \Delta F\Gamma \longrightarrow C$ *and* $g : \Delta\Gamma \longrightarrow F$. *Then* $[h \mid F : g] : \Delta\Gamma \longrightarrow C$.

Proof. An induction on the length of h is applied.

2. REDUCIBILITY OF DERIVATIONS

2.1. β-reductions and commutative reductions

We define reductions as formal expressions of the form $g \rightarrowtail g'$ as follows:

($\beta1$) $\&^-_i \&^+ d_0 d_1 \rightarrowtail d_i,\ \ i < 2$;
($\beta2$) $\vee^- \vee^+_i\ d(A^{k_0}_0)d_0(A^{k_1}_1)d_1 \rightarrowtail [d_i \mid A^{k_i}_i : d],\ \ i < 2$;
($\beta3$) $\supset^- \supset^+ (B^k)de \rightarrowtail [d \mid B^k : e]$;
($\beta4$) $\forall^- t\forall^+(u)d \rightarrowtail [d \mid u : t]$;
($\beta5$) $\exists^- \exists^+ td(B^k, u)e \rightarrowtail [[e \mid u : t] \mid B^k_u(t) : d]$;
($c1$) $\sigma^- \exists^- d(B^k, u)eY \rightarrowtail \exists^- d(B^k, u)\sigma^- eY$;
($c2$) $\sigma^- \vee^-\ d(A^{k_0}_0)d_0(A^{k_1}_1)d_1 Y \rightarrowtail \vee^- d(A^{k_0}_0)\sigma^- d_0 Y(A^{k_1}_1)\sigma^- d_1 Y$.

The expressions ($\beta1$)–($\beta5$) we call β-reductions and ($c1$), ($c2$) — free commutative reductions. Y is an expression depending on σ^-. For ($c1$) we assume that u and $B^k_u(t)$ (for any term t) do not occur as open in Y; and for ($c2$) we assume that $A_0^{k_0}$ and $A_1^{k_1}$ are not open in Y (these requirements are not restrictions over $(c1)$ and $(c2)$, since they concern closed parameters and labels of closed formulas). In the expression of the kind $g \rightarrowtail g'$ the right hand side g' is called a reduction of the left hand side g. We write $d \longmapsto_1 d'$ for "d' is obtained from d by replacing a subderivation of d by a reduction of it". The *reducibility relation* is defined as the reflexive and transitive closure of $\longmapsto_1$, and it is denoted by $\longmapsto$. A derivation is said to be *irreducible* if it reduces only to itself.

2.2. Maximum formula and maximum segment

Maximum formula is a formula which is a conclusion of introduction rule and magor premiss of the corresponding elimination rule. A derivation of the form $\sigma^- dY$ is a *maximum segment* if there exists a sequence of derivations d_i $(i \leq n)$ such that d_0 is obtained by immediate applying of σ^+ (the corresponding of σ^-), for every $i > 0$ d_i is obtained from d_{i-1} either by $\vee^-$ or by $\exists^-$, for every $i > 0$ the conclusions of d_i and d_{i-1} are identical, and $d_n = d$. The number n is called *length of the maximum segment.* A derivation is defined as *normal* if it contains no maximum formula and no maximum segment. By β-reductions the maximum formulas are removed (although new ones may appear) and the commutative reductions decrease the length of maximal segments in the cases they occur. It is not necessary to require a maximum segment to carry out a free commutative reduction. When we require the left hand sides of $(c1)$ and $(c2)$ to be maximal segments, the reducibility relation that arises is denoted by $\underset{P}{\longmapsto}$ (as defined by Prawitz in [1]). Obviously, a derivation is normal if and only if it is irreducible (in the sense of $\underset{P}{\longmapsto}$). According to the Weak normal form theorem every derivation reduces to a normal one. If $\underset{P}{\longmapsto}$ had the Church-Rosser property, every derivation should reduce to unique normal form. Here we give an example of derivation which reduces to different normal forms, i.e. $\underset{P}{\longmapsto}$ does not possess the Church-Rosser property and the conjecture ([1]) that two derivations may be equivalenced only if the normal derivations to which they are reduced are identical is not valid.

2.2.1. Example. Let A, B and C be formulas, $u \notin \mathrm{Par}(AC)$, $D = B \,\&\, (A \,\&\, C)$, and $F = \exists z D_u(z)$. We construct the following derivations:

$d_0 = \&^+[A^k]\&_1^-\&_1^-[D^n] : FA^kD^k \longrightarrow A \,\&\, C,$

$d_1 = \&_1^-[D^n] : FB^mD^n \longrightarrow A \,\&\, C,$

$d = \vee_1^+\&_0^-[D^n] : FD^n \longrightarrow A \vee B.$

Then $e = \vee^- d(A^k)d_0(B^m)d_1 : FD^n \longrightarrow A \,\&\, C$. Let

$$h = \&_1^-\exists^-[F](D^n, u)e : F \longrightarrow C.$$

We shall show that h reduces to two different ones.

$A \,\&\, C$ is a conclusion of $\&^+[A^k]\&_1^-\&_1^-[D^n]$, which is obtained by immediate applying of $\&^+$, and next the rules $\vee^-$, $\exists^-$ and $\&_1^-$ are applied consequently, i.e. we have a maximum segment, so we can carry out the following reductions:

$$h = \&_1^-\exists^-[F](D^n, u)e \underset{P}{\longmapsto} \exists^-[F](D^n, u)\&_1^- e \qquad \text{(by } (c1))$$

$$= \exists^-[F](D^n, u)\&_1^- \vee^- d(A^k)d_0(B^m)d_1 \qquad \text{(by } (c2) \text{ in } \&_1^- \vee^- d(A^k)d_0(B^m)d_1)$$

$$\underset{P}{\longmapsto} \exists^-[F](D^n, u) \vee^- d(A^k)\&_1^- d_0(B^m)\&_1^- d_1.$$

$$\vee^- d(A^k)\&_1^- d_0(B^m)\&_1^- d_1 = \vee^- \vee_1^+ \&_0^- [D^n](A^k)\&_1^- d_0(B^m)\&_1^- \&_1^- [D^n] \quad (\text{by } (\beta 2))$$
$$\underset{P}{\vdash\!\!\to} [\&_1^- \&_1^- [D^n] \mid B^m : \&_0^- [D^n]] = \&_1^- \&_1^- [D^n],$$

i.e.

$$\vee^- d(A^k)\&_1^- d_0(B^m)\&_1^- d_1 \underset{P}{\vdash\!\!\to} \&_1^- \&_1^- [D^n].$$

Then

$$\exists^- [F](D^n, u) \vee^- d(A^k)\&_1^- d_0(B^m)\&_1^- d_1 \underset{P}{\vdash\!\!\to} \exists^- [F](D^n, u)\&_1^- \&_1^- [D^n].$$

We have

$$h = \&_1^- \exists^- [F](D^u, u)e \underset{P}{\vdash\!\!\to} \exists^- [F](D^u, u)\&_1^- \&_1^- [D^n] = h_0,$$

and h_0 is normal.

On the other hand,

$$e = \vee^- \vee_1^+ \&_0^- [D^n](A^k)d_0(B^m)\&_1^- [D^n] \qquad (\text{by } (\beta 2))$$
$$\underset{P}{\vdash\!\!\to} [\&_1^- [D^n] \mid B^m : \&_0^- [D^n]] = \&_1^- [D^n].$$

Then

$$h = \&_1^- \exists^- [F](D^n, u)e \underset{P}{\vdash\!\!\to} \&_1^- \exists^- [F](D^n, u)\&_1^- [D^n] = h_1.$$

In h_1 we can not carry out the reduction

$$\&_1^- \exists^- [F](D^n, u)\&_1^- [D^n] \underset{P}{\vdash\!\!\to} \exists^- [F](D^n, u)\&_1^- \&_1^- [D^n],$$

since $\&_1^- \exists^- [F](D^n, u)\&_1^- [D^n]$ is not a maximum segment, hence h_1 is normal. We have

$$h \underset{P}{\vdash\!\!\to} \exists^- [F](D^n, u)\&_1^- \&_1^- [D^n] = h_0 \quad \text{and} \quad h \underset{P}{\vdash\!\!\to} \&_1^- \exists^- [F](D^n, u)\&_1^- [D^n] = h_1,$$

where h_0 and h_1 are normal and not identical.

There may be objection in connection with the derivation $\vee^- d(A^k)d_0(B^m)d_1$, since no assumption is closed in d_1 by $\vee^-$, but if we take $\&_1^- \&^+ [B^m]d_1$ instead of d_1, by a similar way we can reduce h to h_0 and h_1.

As it is seen from the example, the reason for the different normal forms of one and the same derivation is the restriction over commutative reductions to be applied only for maximal segments.

We can redefine the notion "maximum segment" by changing "d_0 is obtained by immediate applying of σ^+ (the corresponding of σ^-)" in the above definition with "d_0 is not of the kind $\exists^- g(B^k, u)e$ or $\vee^- f(C^i)f_0(D^j)f_1$". Then we have: a derivation is normal (contains no maximum formula and no maximum segment) if and only if it is irreducible (in the sense of $\vdash\!\!\to$).

3. CHURCH-ROSSER PROPERTY

In this section we shall give a sketch of the proof of the Church-Rosser property for reducibility relation constructed by β-reductions and the free commutative reductions. Also, we shall examplify that the extention of this reducibility relation with η-reductions leads to the lost of the property.

To verify the property, we shall use the Tait's idea for λ-terms.

3.1. Fast reduction

The relation "fast reduction" is defined inductively using the already defined reductions. We shall denote it by "$\Vdash\!\to$".

Definition.

(R1) $d \Vdash\!\to d$;

(R2) if $d_i \Vdash\!\to e_i$ for every $i < 2$, then $\&^+ d_0 d_1 \Vdash\!\to \&^+ e_0 e_1$;

(R3) if $d \Vdash\!\to d'$, then $\&_i^- d \Vdash\!\to \&_i^- d'$ for every $i < 2$;

(R4) if $d \Vdash\!\to e$, then $\vee_i^+ d \Vdash\!\to \vee_i^+ e$ for every $i < 2$;

(R5) if $d \Vdash\!\to d'$, $d_i \Vdash\!\to e_i$ for every $i < 2$, then

$$\vee^- d(B^k)d_0(C^m)d_1 \Vdash\!\to \vee^- d'(B^k)e_0(C^m)e_1;$$

(R6) if $d \Vdash\!\to d'$, then $\supset^+ (B^k)d \Vdash\!\to \supset^+ (B^k)d'$;

(R7) if $d \Vdash\!\to d'$, $e \Vdash\!\to e'$, then $\supset^- de \Vdash\!\to \supset^- d'e'$;

(R8) if $d \Vdash\!\to d'$, then $\forall^+(u)d \Vdash\!\to \forall^+(u)d'$;

(R9) if $d \Vdash\!\to d'$, then $\forall^- td \Vdash\!\to \forall^- td'$;

(R10) if $d \Vdash\!\to d'$, then $\exists^+ td \Vdash\!\to \exists^+ d'$;

(R11) if $d \Vdash\!\to d'$, $e \Vdash\!\to e'$, then $\exists^- d(B^k, u)e \Vdash\!\to \exists^- d'(B^k, u)e'$;

(R12) if $d \Vdash\!\to d'$, then $\bot^- d \Vdash\!\to \bot^- d'$;

(R13) if $d_i \Vdash\!\to e_i$, then $\&_i^- \&^+ d_0 d_1 \Vdash\!\to e_i$ for every $i < 2$;

(R14) if $d \Vdash\!\to d'$, $d_i \Vdash\!\to e_i$, then

$$\vee^- \vee_i^+ d(A_0^{k_0})d_0(A_1^{k_1})d_1 \Vdash\!\to [e_i \mid A_i^{k_i} : d'] \text{ for every } i < 2;$$

(R15) if $d \Vdash\!\to d'$, $e \Vdash\!\to e'$, then $\supset^- \supset^+ (B^k)de \Vdash\!\to [d' \mid B^k : e']$;

(R16) if $d \Vdash\!\to d'$, then $\forall^- t \forall^+(u)d \Vdash\!\to [d' \mid u : t]$;

(R17) if $d \Vdash\!\to d'$, $e \Vdash\!\to e'$, then $\exists^- \exists^+ td(B^k, u)e \Vdash\!\to [[e' \mid u : t] \mid B_u^k(t) : d']$;

(C1) if $\sigma^- eY \Vdash\!\to e'$, $d \Vdash\!\to d'$, then $\sigma^- \exists^- d(B^k, u)eY \Vdash\!\to \exists^- d'(B^k, u)e'$;

(C2) if $\sigma^- d_i Y \Vdash\!\to d_i'$ for every $i < 2$, $d \Vdash\!\to d'$, then

$$\sigma^- \vee^- d(A_0^{k_0})d_0(A_1^{k_1})d_1 Y \Vdash\!\to \vee^- d'(A_0^{k_0})d_0'(A_1^{k_1})d_1'.$$

As for (c1) and (c2) we have similar requirements for u, B^k, $A_0^{k_0}$ and $A_1^{k_1}$ in (C1) and (C2).

We shall call (R1)–(R2) simple (fast) reductions.

It is almost obvious that the transitive closure of $\Vdash\!\to$ coincides with $\vdash\!\to$. To prove that fact, it is enough to verify:

1) if $d \vdash\!\to_1 g$, then $d \Vdash\!\to g$, and

2) if $d \Vdash\!\to g$, then $d \vdash\!\to g$.

The first condition verifies by induction on d, and the second — by induction on the definition of $\Vdash\!\to$.

Definition. The relation $\vdash\!\to$ has the Church-Rosser property if for every d, d_0, d_1 that $d \vdash\!\to d_0$ and $d \vdash\!\to d_1$ there exists d^* such that $d_0 \vdash\!\to d^*$ and $d_1 \vdash\!\to d^*$.

Using the fact that if a relation has the Church-Rosser property, then its reflexive and transitive closure also has the property (cf. [3]); to verify the Church-Rosser property for $\longmapsto$, it is enough to verify it for $\mathrel{\|\!\!\to}$. We can not prove the property directly for $\longmapsto$, since $\longmapsto_1$ does not possess it (it is easy to give an example).

First we state two commutational lemmata which say that $\mathrel{\|\!\!\to}$ commutates with the substitution operations from Section 1.

3.1.1. Lemma. *If $h \mathrel{\|\!\!\to} h'$, then $[h \mid w : t] \mathrel{\|\!\!\to} [h' \mid w : t]$.*

3.1.2. Lemma. *If $h : \Delta F \Gamma \longrightarrow C$, $h_1 : \Delta\Gamma \longrightarrow F$ and $h \mathrel{\|\!\!\to} h'$, $h_1 \mathrel{\|\!\!\to} h_1'$, then $[h \mid F : h_1] \mathrel{\|\!\!\to} [h' \mid F : h_1']$.*

The proof of the lemmata is carried out by induction on the length of h.

3.2. Theorem. *The relation $\mathrel{\|\!\!\to}$ has the Church-Rosser property:*
if $h \mathrel{\|\!\!\to} h^0$ and $h \mathrel{\|\!\!\to} h^1$, then there exists h^* such that $h^0 \mathrel{\|\!\!\to} h^*$ and $h^1 \mathrel{\|\!\!\to} h^*$.

Proof. An induction on the length of h is applied. If $h = [F]$, then the only possibility for h^0 and h^1 is $h^0 = [F]$ and $h^1 = [F]$. Then $h^* = [F]$. We shall treat in details only one of the cases concerning commutative reductions — $h = \sigma^- \exists^- d(B^k, u)eY$. The case $h = \sigma^- \vee^- d(A_0^{k_0})d_0(A_1^{k_1})d_1 Y$ is similar. A treatment of the cases concerning β-reductions may be found in [2].

Let h be of the kind $\sigma^- \exists^- d(B^k, u)eY$. The following subcases arise:

1. $\sigma^- \exists^- d(B^k, u)eY \mathrel{\|\!\!\to} \exists^- d^0(B^k, u)e^0 - h^0$ (by (C1)) and
$\sigma^- \exists^- d(B^k, u)eY \mathrel{\|\!\!\to} \sigma^- \exists^- d^1(B^k, u)e^1 Y^1 = h^1$ (by simple reduction),
where by hypothesis we have $\sigma^- eY \mathrel{\|\!\!\to} e^0$, $eY \mathrel{\|\!\!\to} e^1 Y^1$, $d \mathrel{\|\!\!\to} d^j$ for $j = 0, 1$.

We have to show that there exists h^* such that $h^0 \mathrel{\|\!\!\to} h^*$ and $h^1 \mathrel{\|\!\!\to} h^*$. The induction hypothesis is valid for $\sigma^- eY$. We have $\sigma^- eY \mathrel{\|\!\!\to} \sigma^- e^1 Y^1$ and $\sigma^- eY \mathrel{\|\!\!\to} e^0$, hence there exists e^* such that $\sigma^- e^1 Y^1 \mathrel{\|\!\!\to} e^*$ and $e^0 \mathrel{\|\!\!\to} e^*$.

By the induction hypothesis for d we have $d^j \mathrel{\|\!\!\to} d^*$ for $j = 0, 1$. Let $h^* = \exists^- d^*(B^k, u)e^*$. Then
$h^0 = \exists^- d^0(B^k, u)e^0 \mathrel{\|\!\!\to} \exists^- d^*(B^k, u)e^*$ (by (R17)),
$h^1 = \sigma^- \exists^- d^1(B^k, u)e^1 Y^1 \mathrel{\|\!\!\to} \exists^- d^*(B^k, u)e^*$ (by (C1)).

2. $\sigma^- \exists^- d(B^k, u)eY \mathrel{\|\!\!\to} \exists^- d^0(B^k, u)e^0 = h^0$ (by (C1)),
$\sigma^- \exists^- d(B^k, u)eY \mathrel{\|\!\!\to} \exists^- d^1(B^k, u)e^1 = h^1$ (by (C1)),
where $\sigma^- eY \mathrel{\|\!\!\to} e^l$ for $l = 0, 1$, and $d \mathrel{\|\!\!\to} d^j$ for $j = 0, 1$. By $\sigma^- eY \mathrel{\|\!\!\to} e^j$ for $j = 0, 1$, and by the induction hypothesis for $\sigma^- eY$ we have that there exists e^* such that $e^l \mathrel{\|\!\!\to} e^*$ for $l = 0, 1$. By the induction hypothesis for $d : d^j \mathrel{\|\!\!\to} d^*$ for $j = 0, 1$. Let $h^* = \exists^- d^*(B^k, u)e^*$. Then by (R17) we have

$$h^0 = \exists^- d^0(B^k, u)e^0 \mathrel{\|\!\!\to} \exists^- d^*(B^k, u)e^*$$

and

$$h^1 = \exists^- d^1(B^k, u)e^1 \mathrel{\|\!\!\to} \exists^- d^*(B^k, u)e^*.$$

3. $h = \sigma^- \exists^- \exists^+ td(B^k, u)eY$.
$h = \sigma^- \exists^- \exists^+ td(B^k, u)eY \mathrel{\|\!\!\to} \exists^- \exists^+ td^0(B^k, u)e'' = h^0$ (by (C1)),
$\exists^- \exists^+ td(B^k, u)e \mathrel{\|\!\!\to} [[e' \mid u : t]B_u^k(t) : d^1]$ (by (R17)), and
$h = \sigma^- \exists^- \exists^+ td(B^k, u)eY \mathrel{\|\!\!\to} \sigma^-[[e' \mid u : t] \mid B_u^k(t) : d^1]Y^1 = h^1$ (by simple reduction), where we have $d \mathrel{\|\!\!\to} d^j$ for $j = 0, 1$, $e \mathrel{\|\!\!\to} e'$, $\sigma^- eY \mathrel{\|\!\!\to} e''$ and $Y \mathrel{\|\!\!\to} Y^1$.

We assume that $B_u^k(t)$ is not an open assumption in Y and u is not open in Y, hence in Y^1. (New open parameters and formulas do not appear by reductions.)

By the induction hypothesis for d: $d^j \Vdash\!\!\to d^*$ for $j = 0, 1$. By $e \Vdash\!\!\to e'$ and $Y \Vdash\!\!\to Y^1$ it follows $\sigma^- eY \Vdash\!\!\to \sigma^- e'Y^1$ by simple reduction.

The induction hypothesis is valid for $\sigma^- eY$ and $\sigma^- eY \Vdash\!\!\to \sigma^- e'Y^1$, $\sigma^- eY \Vdash\!\!\to e''$, hence there exists e^* such that $\sigma^- e'Y^1 \Vdash\!\!\to e^*$ and $e'' \Vdash\!\!\to e^*$.

$B_u^k(t)$ is not open in Y^1; u is not open in Y^1, hence

$$\sigma^-[[e' \mid u : t] \mid B_u^k(t) : d^1]Y^1 = [[\sigma^- e'Y^1 \mid u : t] \mid B_u^k(t) : d^1].$$

By Lemma 3.1.1 and $\sigma^- e'Y^1 \Vdash\!\!\to e^*$ we have $[\sigma^- e'Y^1 \mid u : t] \Vdash\!\!\to [e^* \mid u : t]$, and by Lemma 3.1.2 it follows

$$[[\sigma^- e'Y^1 \mid u : t] \mid B_u^k(t) : d^1] \Vdash\!\!\to [[e^* \mid u : t] \mid B_u^k(t) : d^*],$$

i.e.

$$h^1 = \sigma^-[[e' \mid u : t] \mid B_u^k(t) : d^1]Y \Vdash\!\!\to [[e^* \mid u : t] \mid B_u^k(t) : d^*].$$

Using $e'' \Vdash\!\!\to e^*$ and $d^j \Vdash\!\!\to d^*$, $j = 0, 1$, we have

$$\exists^- \exists^+ t d^0(B^k, u)e'' \Vdash\!\!\to [[e^* \mid u : t] \mid B_u^k(t) : d^*] \quad \text{(by (R7))}.$$

Let $h^* = [[e^* \mid u : t] \mid B_u^k(t) : d^*]$. Then

$$h^0 = \exists^- \exists^+ t d^0(B^k, u)e'' \Vdash\!\!\to [[e^* \mid u : t] \mid B_u^k(t) : d^*]$$

and

$$h^1 = \sigma^-[[e' \mid u : t] \mid B_u^k(t) : d^1]Y \Vdash\!\!\to [[e^* \mid u : t] \mid B_u^k(t) : d^*].$$

4. $h = \sigma^- \exists^- \exists^- d(B^k, u)e(C^m, v)gY$.

$$h = \sigma^- \exists^- \exists^- d(B^k, u)e(C^m, v)gY \Vdash\!\!\to \sigma^- \exists^- d^0(B^k, u)e'Y^1 = h^0$$

(by simple reduction, where $\exists^- e(C^m, v)g \Vdash\!\!\to e'$ and then

$$\exists^- \exists^- d(B^k, u)e(C^m, v)g \Vdash\!\!\to \exists^- d^0(B^k, u)e' \quad \text{by (C1))}$$

and

$$h = \sigma^- \exists^- \exists^- d(B^k, u)e(C^m, v)gY \Vdash\!\!\to \exists^- \exists^- d^1(B^k, u)e^1(C^m, v)g' = h^1 \quad \text{(by (C1))},$$

where $d \Vdash\!\!\to d^j$, $j = 0, 1$, $\exists^- e(C^m, v)g \Vdash\!\!\to e'$, $\sigma^- gY \Vdash\!\!\to g'$, $e \Vdash\!\!\to e^1$ and $Y \Vdash\!\!\to Y^1$.

By the induction hypothesis for d we have $d^j \Vdash\!\!\to d^*$, $j = 0, 1$. It is necessary to exist e^* such that $\sigma^- e'Y^1 \Vdash\!\!\to e^*$ and $\exists^- e^1(C^m, v)g' \Vdash\!\!\to e^*$. Using $\exists^- e(C^m, v)g \Vdash\!\!\to e'$ and $Y \Vdash\!\!\to Y^1$, we have $\sigma^- \exists^- e(C^m, v)gY \Vdash\!\!\to \sigma^- e'Y^1$ by simple reduction. Also $\sigma^- gY \Vdash\!\!\to g'$ and $e \Vdash\!\!\to e^1$, hence

$$\sigma^- \exists^- e(C^m, v)gY \Vdash\!\!\to \exists^- e^1(C^m, v)g' \quad \text{(by (C1))}.$$

The induction hypothesis is valid for $\sigma^- \exists^- e(C^m, v)gY$, hence there exists e^* such that $\sigma^- e'Y^1 \Vdash\!\!\to e^*$ and $\exists^- e^1(C^m, v)g' \Vdash\!\!\to e^*$. Let $h^* = \exists^- d^*(B^k, u)e^*$. Using $\sigma^- e'Y^1 \Vdash\!\!\to e^*$, $d^j \Vdash\!\!\to d^*$ for $j = 0, 1$ and (C1), we have

$$h^0 = \sigma^- \exists^- d^0(B^k, u)e'Y^1 \Vdash\!\!\to \exists^- d^*(B^k, u)e^*.$$

By $\exists^- e^1(C^m, v)g' \Vdash\!\!\to e^*$, $d^j \Vdash\!\!\to d^*$ and (C1) we get

$$h^1 = \exists^- \exists^- d^1(B^k, u)e^1(C^m, v)g' \Vdash\!\!\to \exists^- d^*(B^k, u)e^*.$$

5. $h = \sigma^- \exists^- \vee^- f(A_0^{k_0}) f_0(A_1^{k_1}) f_1(B^k, u) e Y$.
$h = \sigma^- \exists^- \vee^- f(A_0^{k_0}) f_0(A_1^{k_1}) f_1(B^k, u) e Y \Vdash\!\rightarrow \exists^- \vee^- f^0(A_0^{k_0}) f_0^0(A_1^{k_1}) f_1^0(B^k, u) e' = h^0$ (by (C1)), $\exists^- f_i(B^k, u) e \Vdash\!\rightarrow f_i'$ for $i = 0, 1$, and then

$$\exists^- \vee^- f(A_0^{k_0}) f_0(A_1^{k_1}) f_1(B^k, u) e \Vdash\!\rightarrow \vee^- f^1(A_0^{k_0}) f_0'(A_1^{k_1}) f_1' \quad \text{(by (C2))},$$

$$h = \sigma^- \exists^- \vee^- f(A_0^{k_0}) f_0(A_1^{k_1}) f_1(B^k, u) e Y \Vdash\!\rightarrow \sigma^- \vee^- f^1(A_0^{k_0}) f_0'(A_1^{k_1}) f_1' Y^1 = h^1$$

(by simple reduction).

We have $\sigma^- e Y \Vdash\!\rightarrow e'$, $f_i \Vdash\!\rightarrow f_i^0$, $\exists^- f_i(B^k, u) e \Vdash\!\rightarrow f_i'$, $i = 0, 1$, $f \Vdash\!\rightarrow f^j$ for $j = 0, 1$, and $Y \Vdash\!\rightarrow Y^1$. We need f_0^* and f_1^* such that

$$\sigma^- f_0' Y^1 \Vdash\!\rightarrow f_0^*, \quad \exists^- f_0^0(B^k, u) e' \Vdash\!\rightarrow f_0^*, \quad \sigma^- f_1' Y^1 \Vdash\!\rightarrow f_1^*, \quad \exists^- f_1^0(B^k, u) e' \Vdash\!\rightarrow f_1^*.$$

By $\exists^- f_0(B^k, u) e \Vdash\!\rightarrow f_0'$ and $Y \Vdash\!\rightarrow Y^1$ we have $\sigma^- \exists^- f_0(B^k, u) e Y \Vdash\!\rightarrow \sigma^- f_0' Y^1$ with a simple reduction.

By $\sigma^- e Y \Vdash\!\rightarrow e'$, $f_0 \Vdash\!\rightarrow f_0^0$ and (C1) we get $\sigma^- \exists^- f_0(B^k, u) e Y \Vdash\!\rightarrow \exists^- f_0^0(B^k, u) e'$. The induction hypothesis is valid for $\sigma^- \exists^- f_0(B^k, u) e Y$, hence there exists f_0^* such that $\sigma^- f_0' Y^1 \Vdash\!\rightarrow f_0^*$ and $\exists^- f_0^0(B^k, u) e' \Vdash\!\rightarrow f_0^*$. Similar, we have f_1^* such that $\sigma^- f_1' Y^1 \Vdash\!\rightarrow f_1^*$ and $\exists^- f_1^0(B^k, u) e' \Vdash\!\rightarrow f_1^*$.

By the induction hypothesis for f we have $f^0 \Vdash\!\rightarrow f^*$ and $f^1 \Vdash\!\rightarrow f^*$. Let $h^* = \vee^- f^*(A_0^{k_0}) f_0^*(A_1^{k_1}) f_1^*$. By $f^0 \Vdash\!\rightarrow f^*$, $\exists^- f_0^0(B^k, u) e' \Vdash\!\rightarrow f_0^*$, $\exists^- f_1^0(B^k, u) e' \Vdash\!\rightarrow f_1^*$ and (C2):

$$h^0 = \exists^- \vee^- f^0(A_0^{k_0}) f_0^0(A_1^{k_1}) f_1^0(B^k, u) e' \Vdash\!\rightarrow \vee^- f^*(A_0^{k_0}) f_0^*(A_1^{k_1}) f_1^*.$$

By $\sigma^- f_0' Y^1 \Vdash\!\rightarrow f_0^*$, $\sigma^- f_1' Y^1 \Vdash\!\rightarrow f_1^*$ and (C2):

$$h^1 = \sigma^- \vee^- f^1(A_0^{k_0}) f_0'(A_1^{k_1}) f_1' Y^1 \Vdash\!\rightarrow \vee^- f^*(A_0^{k_0}) f_0^*(A_1^{k_1}) f_1^*.$$

The proof is completed.

We define the following relation: $d_0 \equiv d_1$ if there exists d^* such that $d_0 \vdash\!\rightarrow d^*$ and $d_1 \vdash\!\rightarrow d^*$. Using the Church-Rosser property it is easy to verify that $\equiv$ is an equivalence relation (cf. [3]).

3.3. η-reductions

We can enrich $\vdash\!\rightarrow$ with the following reductions, called η-reductions:

(η1) $\&^+ \&_0^- d \&_1^- d \rightarrowtail d$;
(η2) $\vee^- d(A_0^{k_0}) \vee_0^+ [A_0^{k_0}](A_1^{k_1}) \vee_1^+ [A_1^{k_1}] \rightarrowtail d$;
(η3) $\supset^+ (B^k) \supset^- d[B^k] \rightarrowtail d$ if B^k is not open in d;
(η4) $\forall^+(u) \forall^- u d \rightarrowtail d$;
(η5) $\exists^- d(B^k, u) \exists^+ u[B^k] \rightarrowtail d$.

It is easy to show that when $\vdash\!\rightarrow$ is enlarged with the η-reductions, it does not possess the Church-Rosser property. Let $D = A \,\&\, (B \,\&\, C)$, $u \notin \mathrm{Par}(BC)$ and $F = \exists z D_u(z)$. Obviously, $g = \exists^- [F](D^k, u) \&_1^- [D^k]$, for which we have $g : \exists z (A \,\&\, (B \,\&\, C))_u(z) \longrightarrow B \,\&\, C$, is irreducible.

Let $h = \&^+ \&_0^- \exists^- [F](D^k, u) \&_1^- [D^k] \&_1^- \exists^- [F](D^k, u) \&_1^- [D^k]$. We have

$$\&^+ \&_0^- \exists^- [F](D^k, u) \&_1^- [D^k] \&_1^- \exists^- [F](D^k, u) \&_1^- [D^k]$$
$$\vdash\!\rightarrow \exists^- [F](D^k, u) \&_1^- [D^k] = g, \qquad \text{(by } (\eta 1))$$

and

$$\&^{+}\&_{0}^{-}\exists^{-}[F](D^{k},u)\&_{1}^{-}[D^{k}]\&_{1}^{-}\exists^{-}[F](D^{k},u)\&_{1}^{-}[D^{k}]$$
$$\longmapsto \&^{+}\exists^{-}[F](D^{k},u)\&_{0}^{-}\&_{1}^{-}[D^{k}]\exists^{-}[F](D^{k},u)\&_{1}^{-}\&_{1}^{-}[D^{k}] \quad \text{(by } (c1)\text{)}.$$

Obviously, $\&^{+}\exists^{-}[F](D^{k},u)\&_{0}^{-}\&_{1}^{-}[D^{k}]\exists^{-}[F](D^{k},u)\&_{1}^{-}\&_{1}^{-}[D^{k}]$ is also irreducible and different from $\exists^{-}[F](D^{k},u)\&_{1}^{-}[D^{k}]$, i.e. h reduces to two different irreducible ones. This verifies that the Church-Rosser property is lost when $\longmapsto$ is enriched with η-reductions. This example also illustrates that the reducibility relation which arises from η-reductions and commutative reductions does not possess the Church-Rosser property.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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ОБОБЩЕННЫЕ РЕШЕНИЯ ЛИНЕЙНЫХ УПРАВЛЯЕМЫХ СИСТЕМ

ВЛАДИМИР ЧАКАЛОВ

Владимир Чакалов. ОБОБЩЕННЫЕ РЕШЕНИЯ ЛИНЕЙНЫХ УПРАВЛЯЕМЫХ СИСТЕМ

В работе рассматривается линейная управляемая система на конечном сегменте. Основной результат содержится в теореме 2, где даются условия для существования обобщенных решений, которые „переносят“ объект от одной точки до другой. При том интенсивность I (измеряющая затраченную энергию для перенесения объекта, значения которой суть нормы функционалов подходящего сопряженного пространства) достигает наименьшее значение. Так как интегралы в решении являются неразложимыми элементами единичного шара этого сопряженного пространства, мы можем в некоторых случаях найти их и таким образом уточнить решение системы. Этот подход иллюстрируется на двух примерах.

Vladimir Chakalov. GENERALIZED SOLUTIONS OF LINEAR CONTROL SYSTEMS

In the present paper we consider a linear control system on a finite interval. The main result is Theorem 2, which gives sufficient conditions for the existence of generalized solutions, that steer given object from one point to another. The intensity I (it measures the energy spent to steer the object, and its values are norms of functionals belonging to a convenient conjugate space) achieves its smallest value. Since the integrals involved in the solution are extreme points of the unit ball of the corresponding conjugate space, in some cases we can determine these integrals and thus obtain an effective solution of the given system. We illustrate the method with two examples.

Существуют различные методы для решения задач теории линейного оптимального управления. Один из них является разновидностью классической теории моментов. Он происходит от одной экстремальной задачи, рассматриваемой А. Марковым в конце XIX столетия (см. [1], [2] и [3, стр. 120–130]). В своем первоначальном виде эту задачу можно сформулировать следующим образом.

Пусть F — множество всех действительных функции f, интегрируемых на сегменте $[a, b]$ и удовлетворяющих условиям

а) $0 \leqq f(x) \leqq L \qquad (x \in [a,b])$,

б) $\int\limits_a^b f(x)\,dx = \alpha_0, \quad \int\limits_a^b xf(x)\,dx = \alpha_1, \ \ldots, \quad \int\limits_a^b x^{n-1}f(x)\,dx = \alpha_{n-1}$,

где $L > 0$, α_0, α_1, ..., α_{n-1} — заданные числа. Найти те функции $f \in F$, для которых интеграл $\int\limits_a^b x^n f(x)\,dx$ достигает наибольшего и наименьшего значения.

А. Марков нашел простое решение этой задачи. Наряду с ней однако он рассмотрел и некоторые далеко идущие обобщения, используя виртуозно аппарат непрерывных дробей. Несмотря на важность полученных результатов, трудность используемого аппарата не способствовала распространению идей А. Маркова среди математиков. Едва в самом начале тридцатых годов настоящего столетия Н. Ахиезер и М. Крейн обратились к этой тематике. Используя разработанную в то время теорию моментов, они продолжили и углубили исследования А. Маркова. Одна из рассмотренных ими задач это так называемая L-проблема. Ее самый простой вариант формулируется следующим образом [4–7].

Каким условиям должны удовлетворять числа $L > 0$, α_0, α_1, ..., α_{n-1} для того, чтобы условия а) и б) были совместимыми, т. е. чтобы множество F не было пусто.

Эта и сходные с ней задачи получили наименование „L-проблема А. Маркова". В последующее время М. Крейн обобщил L-проблему и связанные с ней экстремальные задачи для любых чебышевских систем (см. [8]). Подробное изложение его результатов содержится в его известной работе [9], а так же в [10, гл. VII и IX]. Общая формулировка L-проблемы А. Маркова понастоящем следующая:

Пусть X нормированное (действительное или комплексное) линейное пространство, X^* — сопряженное пространство, пусть $L > 0$, α_1, α_2, ..., α_n — заданные числа (действительные или комплексные вместе с X), а x_1, ..., x_n — элементы X. Найти условия, при которых существуют линейные функционалы $\lambda \in X^*$, удовлетворяющие условиям

$$\lambda(x_k) = \alpha_k \quad (k = 1, \ldots, n), \qquad ||\lambda|| \leqq L.$$

Н. Красовский первый заметил в своих работах [11] и [12], что L-проблема А. Маркова является удобным инструментом для решения одной из задач линейного оптимального управления, а именно задача о нахождении управления, минимизирующего т. наз. интенсивность. Полное изложение этого подхода, сочетанное с большим числом приложений, находим в монографии [13, гл. 5 и 6].

В настоящей работе рассматривается та же самая задача. Для ее решения используется одно уточнение L-проблемы А. Маркова (см. [14]). Это позволяет нам найти простые обобщенные решения управляемой системы как в действительном, так и в комплексном случае.

Постановка задачи. Пусть задана система дифференциальных уравнений

$$\dot{x} = A(t)x + B(t)u \tag{1}$$

с начальным условием

$$x(0) = x_0 \neq 0, \tag{2}$$

где $A(t) = \left(a_{ij}(t)\right)_{i=1}^{n}{}_{,}{}_{j=1}^{n}$, $B(t) = \left(b_{kl}(t)\right)_{k=1}^{n}{}_{,}{}_{l=1}^{m}$ — матрицы, элементы которых суть непрерывные функции на сегменте $[0, T]$, $u = (u_1, \ldots, u_m)$ — m-мерная векторная функция (управление), компонеты которой — суммируемые функции на сегменте $[0, T]$, $x_0 = (x_{01}, \ldots, x_{0n})$ — n-мерный вектор, а $x = (x_1, \ldots, x_n)$ — решение системы (траектория). Рассматриваемые функции и векторы могут быть как действительными, так и комплексными. Мы объединим рассмотрение этих двух случаев, но на соответствующих местах будем отмечать различия между полученными результатами.

Если u — управление, а $X(t)$ — решение матричной системы $\dot{X}(t) = A(t)X(t)$, при чем $X(0)$ — единичная матрица, то соответствующее решение x системы (1), удовлетворяющее начальному условию (2), задается формулой

$$x(t) = X(t)\left[x_0 + \int_0^t X^{-1}(\tau)B(\tau)u(\tau)\,d\tau\right]. \tag{3}$$

Положим для краткости

$$-X^{-1}(t)B(t) = C(t) = \left(c_{il}(t)\right)_{i=1}^{n}{}_{,}{}_{l=1}^{m}. \tag{4}$$

Очевидно, что функции c_{il} непрерывны на сегменте $[0, T]$. Будем обозначать через c_i вектор-функции

$$c_i = (c_{i1}, \ldots, c_{im}) \quad (i = 1, \ldots, n). \tag{5}$$

Имея ввиду (4), можно дать формуле (3) вид

$$x(t) = X(t)\left[x_0 - \int_0^t C(\tau)u(\tau)d\tau\right].$$

Управление u порождает векторную меру на сегменте $[0, T]$. Поясним, что если $Y = (Y_1, \ldots, Y_m)$ — m-мерная (непрерывная) функция, $\mu = (\mu_1, \ldots, \mu_m)$ — m-мерная векторная мера и $C(t)$ — $(n \times m)$-матрица, элементы которой суть непрерывные функции на $[0, T]$, то

$$\int_0^T Y(t)\,d\mu(t) = \int_0^T Y_1(t)\,d\mu_1(t) + \cdots + \int_0^T Y_m(t)\,d\mu_m(t)$$

и

$$\int_0^T C(t)\,d\mu(t) = \left(\int_0^T C_1(t)\,d\mu(t), \ldots, \int_0^T C_n(t)\,d\mu(t)\right).$$

Совершенно естественно считать, что произвольная векторная мера μ является обобщенным управлением (здесь $\mu = (\mu_1, \ldots, \mu_m)$, где μ_l ($l = 1, \ldots, m$) — действительные или комплексные меры вместе с системой (1)). Исходя от этого обобщения понятия управления, мы могли бы заключить, что соответствующее обобщенное решение системы (1) имеет вид

$$x(t) = X(t)\left[x_0 - \int_0^t C(\tau)\,d\mu(\tau)\right]. \tag{6}$$

Такое заключение было бы поспешным, так как не исключено, чтобы мера μ сосредоточивала ненулевую (векторную) „массу" в точку $t = 0$. Тогда очевидно $x(0) \neq x_0$, т. е. траектория не удовлетворяла бы начальному условию (2). Полагая

$$\varepsilon(t) = \begin{cases} 0 & \text{для } t = 0, \\ 1 & \text{для } 0 < t \leqq T, \end{cases}$$

легко получаем, что n-мерная функция $x(t)$, определенная равенством

$$x(t) = X(t)\left[x_0 - \varepsilon(t)\int_0^t C(\tau)\,d\mu(\tau)\right], \tag{7}$$

совпадает с функцией x, определенной равенством (6) всюду на интервале $(0, T]$, и кроме того $x(0) = x_0$. Это дает нам основание называть функцию x в (7) обобщеным решением системы (1), удовлетворяющим начальному условию (2).

Мы ставим перед собой задачу найти обобщенное управление μ таким образом, чтобы траектория x в (7) „перенесла" данный объект от точки x_0 до другой точки, например до начала, т. е. чтобы имели место равенства $x(0) = x_0$ и $x(T) = 0$. Кроме того управление μ должно быть таким, чтобы заданная неотрицательная функция управления I, измеряющая затраченную на перенесения объекта энергию, имела возможно наименьшее значение. Обыкновенно предполагают, что функция I (её называют интенсивностью) имеет свойства нормы, т. е.

1. $I(\mu) \geqq 0$ и от $I(\mu) = 0$ следует, что $\mu = 0$;
2. $I(p\mu) = |p| I(\mu)$;
3. $I(\mu_1 + \mu_2) \leqq I(\mu_1) + I(\mu_2)$.

Решение задачи. Мы решим нашу задачу как следствие одной общей теоремы. Сформулируем без доказательства эту теорему, уточняющую L-проблему А. Маркова.

Теорема 1 [14, теорема 4]. Пусть E — нормированное действительное или комплексное пространство и пусть E_s — s-мерное пространство E. Тогда для любого функционала $l \in E_s^*$, $l \neq 0$, существуют такие неразложимые функционалы λ_k ($k = 1, \ldots, r$) единичного шара K^* пространства E^* и положительные числа $\mu_1, \ldots, \mu_r$ (здесь $1 \leqq r \leqq s$, если E

— действительное пространство и $1 \leqq r \leqq 2s-1$, если E — комплексное пространство), что выполняются следующие условия.

1. Для любого $x \in E_s$ имеет место представление

$$(8) \qquad l(x) = \mu_1\lambda_1(x) + \dots + \mu_r\lambda_r(x) \quad \left(\mu_k > 0,\ k = 1, \dots, r,\ \sum_{k=1}^{r} \mu_k = ||l||\right).$$

2. Если $x_1, \dots, x_s$ — базис E_s, то в действительном случае матрица $\left(\lambda_k(x_l)\right)_{k=1}^{r}{}_{,}{}_{l=1}^{s}$ имеет ранг r $(1 \leqq r \leqq s)$. Если E — комплексное пространство, то матрица

$$\begin{pmatrix} \lambda_1^R(x_1) & \lambda_2^R(x_1) & \dots & \lambda_r^R(x_1) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \lambda_1^R(x_s) & \lambda_2^R(x_s) & \dots & \lambda_r^R(x_s) \\ \lambda_1^R(ix_1) & \lambda_2^R(ix_1) & \dots & \lambda_r^R(ix_1) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \lambda_1^R(ix_s) & \lambda_2^R(ix_s) & \dots & \lambda_r^R(ix_s) \end{pmatrix}$$

имеет ранг r $(1 \leqq r \leqq 2s-1)$. Здесь λ^R — действительная часть функционала λ, а i — мнимая единица.

3. Если $x' \in E_s$, $x' \neq 0$ — экстремальный элемент для функционала l, т. е. $l(x') = ||x'||.||l||$,*) то $\lambda_k(x') = ||x'||$ $(k = 1, \dots, r)$.

Здесь введем некоторые обозначения и предположения, которыми будем пользоватся в следующем изложении.

Обозначим через $\mathbb{R}^m$ действительное нормированное пространство всех действительных m-мерных векторов с нормой $||.||_{\mathbb{R}^m}$. Аналогично, через $\mathbb{C}^m$ обозначим комплексное нормированное пространство всех m-мерных комплексных векторов с нормой $||.||_{\mathbb{C}^m}$. Через $C([0,T])$ будем обозначать как действительное пространство всех действительных непрерывных m-мерных функций $x : [0,T] \to \mathbb{R}^m$, так и комплексное пространство всех комплексных непрерывных m-мерных функций $x : [0,T] \to \mathbb{C}^m$, нормированное при помощи равномерной нормы

$$(9) \qquad ||x||_C = \begin{cases} \sup\limits_{t \in T} ||x(t)||_{\mathbb{R}^m} & \text{в действительном случае,} \\ \sup\limits_{t \in T} ||x(t)||_{\mathbb{C}^m} & \text{в комплексном случае.} \end{cases}$$

Обозначим через E то же самое пространство в случае, когда оно нормированно через норму, отличную от равномерной нормы. Такое объединение действительного и комплексного случая мотивируется тем, что впредь мы будем рассматривать эти два случая вместе.

Чтобы решить нашу задачу, мы должны показать существование такого обобщенного управления $\mu = (\mu_1, \dots, \mu_m)$, для которого обобщенное

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*) *Существование экстремального элемента для l следует из-за конечномерности пространства E_s.*

решение (7) принимало значения $x(0) = x_0$ и $x(T) = 0$. Первое равенство обеспечивается выбором функции $\varepsilon(t)$. Чтобы обеспечить второе равенство, мы должны подобрать μ таким образом, чтобы имело место равенство

$$(x_{01}, x_{02}, \ldots, x_{0n}) = x_0 = \int_0^T C(t)\, d\mu(t) = \left(\int_0^T C_1(t)\, d\mu(t), \ldots, \int_0^T C_n(t)\, d\mu(t) \right),$$

или

$$(10) \qquad \int_0^T C_i(\tau)\, d\mu(\tau) = x_{0i} \quad (i = 1, \ldots, n).$$

Здесь непрерывные m-мерные (действительные или комплексные) функции $c_i(\tau)$ $(i = 1, \ldots, n)$, как видно от (5), являются строками матрицы $C(\tau)$. Нетрудно сообразить, что выбранное таким образом обобщенное управление μ задает адитивный гомогенный функционал λ на пространстве непрерывных m-мерных функций $x : [0,T] \to \mathbb{R}^m$ $(x : [0,T] \to \mathbb{C}^m)$ вида

$$(11) \qquad \lambda(x) = \int_0^T x(t)\, d\mu(t),$$

удовлетворяющий равенствам (10). Мы хотим определить обобщенное управление еще так, чтобы оно переводило заданного объекта с точки $x(0) = x_0$ в точку $x(T) = 0$ с наименьшим затратом энергии, т. е. определить функционал λ (мера μ), заданный через (11) так, чтобы интесивность имела наименьшее значение. Для этой цели предположим (как это общепринято), что в пространстве m-мерных непрерывных функций, определенных на $[0,T]$ (действительных или комплексных) можно ввести норму $||\cdot||$ таким образом, что нормы функционалов λ сопряженного пространства были значениями интенсивности, т. е.

$$I(\lambda) = ||\lambda||$$

для всех λ сопряженного пространства. Как и выше, обозначим через E это нормированное пространство. Предположим еще, что оно удовлетворяет следующему условию.

Будем предполагать, что норма пространства $C([0, T])$ мажорирует норму пространства E. Это означает, что существует такая константа K, что неравенство

$$(12) \qquad ||x|| \leqq K||x||_C$$

выполняется для всех $x \in E$. Неравенство (12) обеспечивает существование меры μ для каждого функционала $\lambda \in E^*$ так, чтобы имело место интегральное представление (11). Действительно, от (12) следует, что любой функционал $\lambda \in E^*$ принадлежит и пространству $C^*([0, T])$, следовательно, согласно теореме Ф. Риса имеет интегральное представление

(11). При этих условиях и обозначениях нашу задачу можно сформулировать следующим образом.

Найти обобщенное управление μ таким образом, что определенный через (11) функционал $\lambda \in E^*$ удовлетворял равенствам (10) и имел наименьшую норму.

Чтобы решить эту задачу, рассмотрим подпространство E_s пространства E (E_s нормированно через норму в E), состоящееся из всех линейных комбинаций (с вещественными или комплексными коэффициентами в зависимости от того, является ли E действительным или комплексным пространством) функций $C_1, \ldots, C_n$, заданных через (5). Пусть s ($s \leqq n$) — максимальное число линейно независимых из них. Не ограничая общности, можно предположить, что первые s из этих функций $c_1, \ldots, c_s$ образуют базис E_s.

Пусть $l \in E_s^*$ — функционал, для которого имеем

$$(13)\qquad (l(c_1), \ldots, l(c_n)) = (x_{01}, \ldots, x_{0n}) = x_0.$$

Если $s = n$, такой функционал наверное существует, а если $s < n$, для существования функционала необходимо и достаточно, чтобы выполнялись равенства

$$(14)\qquad x_{0,s+k} = \alpha_{1k}x_{01} + \cdots + \alpha_{sk}x_{ks} \quad (k = 1, \ldots, n-s),$$

где константы $\alpha_{1k}, \ldots, \alpha_{sk}$ определяются линейными зависимостями

$$(15)\qquad c_{s+k}(t) \equiv \alpha_{1k}c_{1k}(t) + \cdots + \alpha_{sk}c_{sk}(t) \quad (k = 1, \ldots, n-s; \quad t \in [0,1]).$$

Функционал l, определенный через (13), отличен от нуля, так как x_0 — ненулевой вектор. Тогда для него будут выполняться утверждения теоремы 1. В частности l удовлетворяет представлению (8). Продолжим l посредством этого представления до функционала λ_0, определенного на E, полагая для любого $x \in E$

$$(16)\qquad \lambda_0(x) = \mu_1\lambda_1(x) + \cdots + \mu_r\lambda_r(x) \quad \left(\mu_k > 0,\ k = 1, \ldots, r,\ \sum_{k=1}^{r} \mu_k = ||l||\right),$$

где $\lambda_1, \ldots, \lambda_r$ — неразложимые векторы единичного шара K^* пространства E^*. Очевидно λ_0 удовлетворяет равенству

$$(17)\qquad (\lambda_0(c_1), \ldots, \lambda_0(c_n)) = x_0.$$

Так как λ_0 продолжает l, то $||\lambda_0|| \geqq ||l||$. С другой стороны, имея ввиду (16), получим для $||\lambda_0||$

$$||\lambda_0|| \leqq \mu_1||L_1|| + \cdots + \mu_r||\lambda_r|| = \sum_{k=1}^{r} \mu_k = ||l||.$$

Отсюда видно, что $||\lambda_0|| = ||l||$. От определения λ_0 ясно, что среди всех функционалов $\lambda \in E^*$, удовлетворяющих (17), λ_0 имеет наименьшую норму. От представления (16) и равенства (17) получим равенства

$$(18)\quad x_0 = (\lambda_0(c_1), \ldots, \lambda_0(c_n)) = \mu_1(\lambda_1(c_1), \ldots, \lambda_1(c_n)) + \cdots + \mu_r(\lambda_r(c_1), \ldots, \lambda_r(c_n)).$$

Но согласно (11)

$$(19)\quad (\lambda_k(c_1),\dots,\lambda_k(c_n)) = \left(\int\limits_0^T c_1(t)d\nu_k(t),\dots,\int\limits_0^T c_n(t)d\nu_k(t)\right) = \int\limits_0^T C(t)d\nu_k(t),$$

откуда получаем для (18)

$$(20)\qquad x_0 = \mu_1\int\limits_0^T C(t)\,d\nu_1(t) + \dots + \mu_r\int\limits_0^T C(t)\,d\nu_r(t).$$

Здесь $\nu_k = (\nu_{k1},\ \dots,\ \nu_{km})$, $k = 1,\dots,r$ — действительные или комплексные m-мерные меры (вместе с E). Равенства (18) и (19) задают следующее обобщенное решение системы (1):

$$x(t) = X(t)\left[x_0 - \varepsilon(t)\left(\mu_1\int\limits_0^t C(\tau)\,d\nu_1(\tau) + \dots + \mu_r\int\limits_0^t C(\tau)\,d\nu_r(t)\right)\right],$$

а от (20) сразу вытекает, что это решение удовлетворяет равенству $x(T) = 0$.

С изложенными выше рассуждениями мы установили в качестве следствия теоремы 1 следующую теорему.

Теорема 2. При введенных выше обозначениях и предположениях, если $s = n$, то существуют функционалы $\lambda \in E^*$, удовлетворяющие равенству

$$(21)\qquad (\lambda(c_1),\ \dots,\ \lambda(c_n)) = x_0.$$

Если $s < n$, чтобы существовали такие функционалы, необходимо и достаточно выполнение (14), при чем константы $\alpha_{1k},\ \dots,\ \alpha_{sk}$ $(k = 1,\ 2,\ \dots,\ n-s)$ определяются зависимостями (15). Среди функционалов λ, удовлетворяющих (21), имеется хотя бы один — λ_0, для которого $||\lambda_0|| = I(\lambda_0) = \min$. Он удовлетворяет еще следующим условиям.

1. Для любого $x \in E$ имеет место представление

$$(22)\qquad \lambda_0(x) = \mu_1\int\limits_0^T x(t)\,d\nu_1(t) + \dots + \mu_r\int\limits_0^T x(t)\,d\nu_r(t).$$

Здесь $\int\limits_0^T x(t)\,d\nu_k(t)$ $(k = 1,\dots,r)$ — неразложимые элементы единичного шара пространства E^*, $\mu_k > 0$, $\sum\limits_{k=1}^{r}\mu_k = ||\lambda_0||$, $\nu_1,\ \dots,\ \nu_r$ — действительные m-мерные меры, а $1 \leqq r \leqq s$, если E — действительное пространство и $\nu_1,\ \dots,\ \nu_r$ — комплексные m-мерные меры, а $1 \leqq r \leqq 2s-1$, если E — комплексное пространство. Система уравнений (1) имеет обобщенное решение вида

$$x(t) = X(t)\left[x_0 - \varepsilon(t)\left(\mu_1\int\limits_0^t C(\tau)\,d\nu_1(\tau) + \dots + \mu_r\int\limits_0^t C(\tau)\,d\nu_r(\tau)\right)\right],$$

для которого $x(0) = x_0$ и $x(T) = 0$.

2. Если система (1) — действительная вместе с E, то матрица $\left(\int_0^T c_l(t) d\nu_k(t)\right)_{l=1,\, k=1}^{s,\quad r}$ имеет ранг r ($1 \leqq r \leqq s$), а если (1) — комплексная система вместе с E, то матрица

$$\begin{pmatrix} \operatorname{Re}\int_0^T c_1(t)\,d\nu_1(t) & \operatorname{Re}\int_0^T c_1(t)\,d\nu_2(t) & \dots & \operatorname{Re}\int_0^T c_1(t)\,d\nu_r(t) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \operatorname{Re}\int_0^T c_s(t)\,d\nu_1(t) & \operatorname{Re}\int_0^T c_s(t)\,d\nu_2(t) & \dots & \operatorname{Re}\int_0^T c_s(t)\,d\nu_r(t) \\ \operatorname{Re}\int_0^T ic_1(t)\,d\nu_1(t) & \operatorname{Re}\int_0^T ic_1(t)\,d\nu_2(t) & \dots & \operatorname{Re}\int_0^T ic_1(t)\,d\nu_r(t) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \operatorname{Re}\int_0^T ic_s(t)\,d\nu_1(t) & \operatorname{Re}\int_0^T ic_s(t)\,d\nu_2(t) & \dots & \operatorname{Re}\int_0^T ic_s(t)\,d\nu_r(t) \end{pmatrix}$$

имеет ранг r ($1 \leqq r \leqq 2s-1$).

3. Если $c' = a'_1 c_1 + \dots + a'_n c_n = b'_1 c_1 + \dots + b'_s c_s$ — экстремальный элемент для λ_0, т. е. $\lambda_0(c') = ||\lambda_0||\, ||c'||$ (такой элемент существует), то

$$\int_0^T c'(t)\, d\nu_k(t) = ||c'|| \quad (k = 1, \dots, r).$$

Замечание. От точки 3 теоремы 2 сразу следует, что функционал удовлетворяет принцип максимума. Действительно, если c' — экстремальная функция для λ_0 (или т. наз. минимальная функция), то для всякого функционала $\lambda \in E^*$, для которого $||\lambda|| = ||\lambda_0||$, имеем

$$|\lambda(c')| \leqq ||\lambda||\, ||c'|| = ||\lambda_0||\, ||c'|| = \lambda_0(c'),$$

т. е.

$$\lambda_0(c') = \sup_{||\lambda|| = ||\lambda_0||} |\lambda(c')|.$$

Последнее соотношение выполняется как в действительном, так и в комплексном случае.

Утверждения теоремы 2 можно уточнить, если нам известны неразложимые элементы единичного шара пространства E^*. В следующим изложении рассмотрим два примера, в которых дается описание этих неразложимых элементов.

Пример 1. Рассмотрим случай, когда норма пространства E совпадает с $||\cdot||_C$, т. е. когда $E = C([0,T])$. В этом случае неразложимые элементы единичного шара $C^*([0,T])$ имеют вид

$$\lambda(x) = \Lambda(x(t')),$$

где Λ — неразложимый функционал единичного шара пространства $\mathbb{R}^{m^*}$ (или $\mathbb{C}^{m^*}$), а t' — фиксированное число сегмента $[0,T]$ (см. [15, стр. 166–170] и [14, теорему 1]). Но тогда представление (22) функционала λ_0 имеет вид

$$\lambda_0(x) = \mu_1\Lambda_1(x(t_1)) + \cdots + \mu_r\Lambda_r(x(t_r)). \tag{24}$$

Имея ввиду факт, что функционалы Λ_k $(k = 1, \ldots, r)$ и значения $x(t)$ функций пространства $C([0, T])$ суть m-мерные векторы, положив в (24) последовательно $x = c_1$, $x = c_2$, ..., $x = c_n$ и применяя равенство (17), заключаем, что

$$x_0 = \mu_1 C(t_1)\Lambda_1 + \cdots + \mu_r C(t_r)\Lambda_r. \tag{25}$$

От (25) получаем обобщенное решение системы (1)

$$x(t) = X(t)\left[x_0 - \varepsilon(t)\left(\sum_{k:t_k \leqq t} \mu_k C(t_k)\Lambda_k\right)\right],$$

где мера, сосредоточивающая массы μ_k в точках t_k $(k = 1, \ldots, r)$, задает импульсное решение системы (1). Очевидно при том, что $x(0) = x_0$ и $x(T) = 0$.

Чтобы получить в этом частном случае соответствующую теорему, достаточно сделать некоторые очевидные изменения теоремы 2, но мы не будем заниматься этим. Отметим только, что рассмотренный пример обобщает и уточняет один результат Н. Красовского (см. [13, теорему 23.1, стр. 188]), а так же один пример в [10, стр. 513–515].

Пример 2. Рассмотрим теперь случай, когда норма в E задается равенством

$$||x|| = \int_0^T \left(\sum_{l=1}^m |x_l(t)|\right) dt.$$

Так как норма пространства E удовлетворяет неравенству (12), то обобщенные управления μ заданы на всем пространстве E^*, а значения интенсивности являются нормами функционалов $\lambda \in E^*$. По существу известно, что пространство E изоморфно относительно линейных действий, а так же изометрично пространству, которое построим сейчас. Для этой цели перенесем $m - 1$ раз сегмент $[0, T]$ по числовой оси так, что полученние m сегментов не имели общих точек. Обозначим через $E(T)$ объединение этих сегментов. Пусть например

$$E(T) = \bigcup_{l=1}^m [\beta_l,\ \beta_l + T],$$

где $\beta_1 = 0$ и $\beta_{l+1} - \beta_l > T$, $l = 1, \ldots, l - 1$ (см. [10, гл. VIII, § 9, п. 6, стр. 436–437]). Определим на $E(T)$ функции v, положив

$$v(t) = x_l(t - \beta_l) \quad \text{для} \quad t \in [\beta_l,\ \beta_l + T] \quad (l = 1, \ldots, m).$$

Таким образом устанавливается взаимно однозначное соответствие между функциями $x \in E$ и функциями v. Это соответствие является изоморфизмом относительно линейных действий. Добавим, что каждая непрерывная числовая функция на $E(T)$ есть образ некоторой функции $x \in E$. Введя в пространство функций v норму

$$||v|| = \int_{E(T)} |v(t)|\, dt \tag{26}$$

и обозначив через $L_1(E(T))$ соответствующее нормированное пространство, получим равенства

$$||v|| = \int_{E(T)} |v(t)|dt = \int_0^T |x_1(t)|dt + \int_{\beta_2}^{\beta_2+T} |x_2(t-\beta_2)|dt + \cdots + \int_{\beta_m}^{\beta_m+T} |x_m(t-\beta_m)|dt$$

$$= \int_0^T \left(\sum_{l=1}^{m} |x_l(t)|\right) dt = ||x||.$$

И так, установленное соответствие между функциями пространства $L_1(E(T))$ и E является так же изометрией. Это обстоятельство дает нам возможность утверждать, что сопряженные пространства $L_1^*(E(T))$ и E^* совпадают. Очевидно, что их единичные шары, как и неразложимые элементы этих шаров тоже совпадают. Но неразложимые элементы единичного шара $L_1(E(T))$ известны (см. [15, лемму 1.13, стр. 83–84]). Они имеют представление вида

$$\lambda(x) = \lambda(v) = \int_{E(T)} v(t)\alpha(t)\, dt,$$

где α — суммируемая функция на $E(T)$ (действительная или комплексная вместе с E и системой (1)), соответствующая функционалу λ и удовлетворяющая условию

$$|\alpha(t)| = 1 \quad \text{почти всюду на} \quad E(T),$$

а $||\lambda|| = \sup \mathrm{ess}|\alpha(t)| = 1$. Тогда для любой функции $v \in L_1(E(T))$ имеем

$$\int_{E(T)} v(t)\alpha(t)dt = \int_0^T x_1(t)\alpha(t)dt + \int_{\beta_2}^{\beta_2+T} x_2(t-\beta_2)\alpha(t)dt + \cdots + \int_{\beta_m}^{\beta_m+T} x_m(t-\beta_m)\alpha(t)dt$$

$$= \int_0^T \left(\sum_{l=1}^{m} x_l(t)\alpha_l(t)\right) dt,$$

где $\alpha_l(t) = \alpha(t+\beta_l)$ для $t \in [0, T]$, $l = 1, \ldots, m$. Обозначим опять через α m-мерную векторную функцию $\alpha = (\alpha_1, \ldots, \alpha_m)$ и положим

$$\sum_{l=1}^{m} x_l(t)\alpha_l(t) = x(t)\alpha(t).$$

Тогда для функционала λ ($||\lambda|| = 1$) получим

$$\lambda(x) = \int_0^T x(t)\alpha(t)\,dt.$$

Здесь $\alpha_1, \ldots, \alpha_m$ — суммируемые функции на сегменте $[0, T]$ и кроме того

$$|\alpha_l(t)| = 1 \quad \text{почти всюду на} \quad [0, T] \quad (l = 1, \ldots, m).$$

Имея ввиду представление неразложимых функционалов единичного шара пространства $L_1^*(E(T)) = E^*$, легко получаем для представления (22) функционала λ_0 выражение

$$\lambda_0(x) = \mu_1 \int_0^T x(t)\alpha^1(t)\,dt + \cdots + \mu_r \int_0^T x(t)\alpha^r(t)\,dt,$$

где $\alpha^k = (\alpha_1^k, \ldots, \alpha_m^k)$ $(k = 1, \ldots, m)$ — векторные функции на сегменте $[0, T]$, компоненты которых являются суммируемыми функциями на этом сегменте и удовлетворяют условию

$$|\alpha_l^k(t) = 1 \quad \text{почти всюду на} \quad [0, T] \quad (k = 1, \ldots, r; \quad l = 1, \ldots, m).$$

Кроме того α_l^k — действительные или комплексные функции вместе с E. Отсюда получим для системы (1) обобщенное решение

$$x(t) = X(t)\left[x_0 - \mu_1 \int_0^t C(\tau)\alpha^1(\tau)\,d\tau - \cdots - \mu_r \int_0^t C(\tau)\alpha^r(\tau)\,d\tau\right].$$

Так как мера Лебега сосредоточивает массу 0 в точку $t = 0$ (как и во всех остальных точек сегмента $[O, T]$), то умножение интегралов на $\varepsilon(t)$ является излишным. Очевидно, что имеют место равенства $x(0) = x_0$, $x(T) = 0$. И здесь мы ограничимся со сделанными выше заметками и не будем форморулировать соответное уточнение теоремы 2.

Мы могли бы увеличить число примеров, но как нам кажется, приведенные примеры достаточно хорошо иллюстрируют пользу от привлечения понятия неразложимости в этих вопросах.

Мой коллега проф. Н. Хаджииванов просмотрел всю рукопись и сделал ряд полезных замечаний. Считаю своим приятным долгом выразить ему свою сердечную благодарность.

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ON THE CONNECTION BETWEEN THE ABSTRACT RECURSION THEORY AND THE METHOD OF SUCCESSIVE APPROXIMATIONS*

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Йордан Зашев. О СВЯЗИ АБСТРАКТНОЙ ТЕОРИИ РЕКУРСИИ С МЕТОДОМ ПОСЛЕДОВАТЕЛЬНОГО ПРИБЛИЖЕНИЯ

Вводится понятие ортогонализуемого полукольца, охватывающее как частный случай некоторые хорошо известные понятия классической чистой математики и пригодное, с другой стороны, для целей алгебраической теории рекурсии. Для таких полуколец доказывается так называемая теорема кодирования, являющейся фундаментальной для последней теории. Рассматриваются некоторые следствия главным образом для кольца, ограниченных линейных операторов в бесконечномерном пространстве Гильберта.

Jordan Zashev. ON THE CONNECTION BETWEEN THE ABSTRACT RECURSION THEORY AND THE METHOD OF SUCCESSIVE APPROXIMATIONS

We introduce the concept of orthogonalizable semiring which contains as a special case some familiar objects of classical pure mathematics and, on the other hand, is fit for the purposes of the algebraic recursion theory. A fundamental result of the last theory, called code evaluation theorem, is proved for such semirings. Some corollaries are considered, especially for the ring of bounded linear operators over an infinite dimensional Hilbert space.

There is a great deal of similarity between the principal problem of the algebraic recursion theory (that is the problem of fixed-point completion in the sense of [2]), on the one hand, and some problems about the existence of solutions of various kinds of systems of equations in the classical mathematics. Of course, in

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the algebraic recursion theory we have to do with least solutions of systems of inequalities, but this difference seems not to be so principal as another one, namely that in the algebraic recursion theory the emphasis is put on the expressibility of least solutions of those systems of inequalities by means of some basic operations, among which the most important one is the so-called iteration, while the existence problem is comparatively easy. On the other hand, the iteration is the least solution of one simple inequality of one unknown, which is clearly analogous to linear equations of one unknown. In the algebraic recursion theory an arbitrary "nonlinear" system of inequalities is reduced to one such inequality by a typical for the recursion theory process involving coding, so that the first system is interpreted in some sense "internally" as an object of the domain of solutions of the system. In this reducing a similarity may be observed with the iteration method in classical mathematics, but peculiarities also occur, which makes the comparison of the two methods not quite obvious. Such a peculiarity is the mentioned "internal" coding of the iteration process by which it is presented through one inequality.

In this paper we are going to inquire a little bit deeper into this analogy. Our special purpose is to consider corollaries of general results of algebraic recursion theory applied to classical mathematical objects. To this end we introduce the concept of orthogonalizable semiring, which is a kind of topological operative spaces in the sense of [1]. This concept is more suitable for our purposes, being closer to or comprising as a special case familiar objects of classical pure mathematics, for instance, the operator rings over infinite dimensional Hilbert spaces. On the other hand, the main result of the present paper — the code evaluation theorem in orthogonalizable semirings — does not follow from the corresponding result for operative spaces, which is ultimately due to the simpler algebraic structure of the former ones.

1. OTHOGONALIZABLE SEMIRINGS

By a semiring we shall mean an additive (commutative) monoid R in which an associative multiplication with an unit I, satisfying the usual distributive laws, is given, such that $\varphi 0 = 0 = 0\varphi$ for all $\varphi \in R$, where 0 is the zero of the addition in R. For all semirings, considered in the sequel, we shall suppose also that a topology and a partial order "$\leq$" are given in them, such that the addition is continuous as a function of two arguments and the following three conditions are fulfilled:

(i) if $\varphi \in R$ belongs to every neighborhood of $\psi \in R$, then $\psi \leq \varphi$;

(ii) every sequence $\{\varphi_\alpha\}_{\alpha \in D}$ of elements of the given semiring R (where D is a directed poset of indices), which has a limit, has also the greatest limit, i.e. a limit φ of this sequence such that for every other limit ψ of the same sequence we have $\psi \leq \varphi$;

(iii) if $\{\varphi_\alpha\}_{\alpha \in D}$ and $\{\psi_\alpha\}_{\alpha \in D}$ are two sequences of elements of R such that $\varphi_\alpha \leq \psi_\alpha$ for all $\alpha \in D$ and $\{\psi_\alpha\}_{\alpha \in D}$ has a limit, then $\{\varphi_\alpha\}_{\alpha \in D}$ also has a limit, and for the greatest limits ψ and φ of the last two sequences, respectively, we have $\varphi \leq \psi$.

Note that the condition (i) implies that for every sequence $\{\varphi_\alpha\}_{\alpha \in D}$ of elements of R, which is a stationary one, i.e. for some $\alpha_0 \in D$ and $\varphi \in R$ we have $\varphi_\alpha = \varphi$

for all $\alpha \geq \alpha_0$, the element φ is the greatest limit of $\{\varphi_\alpha\}_{\alpha \in D}$.

Now let R be a semiring of this kind and let J be an arbitrary non-empty set of indices. By a *confinal* in J we shall mean a set D of finite subsets of J such that every finite subset of J is included in some element of D; every confinal is directed with respect to the partial order "$\subseteq$" in it. A series of the form $\sum_{j \in J} \varphi_j$, where $\varphi_j \in R$ for all $j \in J$, will be called *convergent* with respect to a confinal D in J iff the sequence $\{\sum_{j \in \alpha} \varphi_j\}_{\alpha \in D}$ of D-partial sums of that series has a limit in R as a directed set indexed sequence. The greatest limit of the last sequence will be called a *sum* of the series in question, corresponding to D. The series in question will be called *conventionally convergent* iff there is a confinal D in J such that it is convergent with respect to D. In the sequel the expressions of the form $\sum_{j \in J} \varphi_j$ will be used to denote also a sum of the series denoted in the same way, and the confinal D, to which this sum corresponds, will be clear from the context or it will be arbitrary otherwise.

If all but a finite number of the members of a series $\sum_{j \in J} \varphi_j$ in R are zeros, then the only sum of the last series is the algebraic sum of the non-zero members of it. This follows from condition (i) since in this case the sequence of D-partial sums of that series is stationary for any confinal D.

A mapping $f : R \to R$ will be called *additive* iff for every series $\sum_{j \in J} \varphi_j$ of elements of R and every confinal D in J such that the last series converges with respect to D in R, the series $\sum_{j \in J} f(\varphi_j)$ also converges with respect to D, and for the corresponding sums we have the equality

$$\sum_{j \in J} f(\varphi_j) = f\Big(\sum_{j \in J} \varphi_j\Big).$$

An element $\varphi \in R$ will be called *left* (respectively, *right*) additive iff the mapping $f : R \to R$, defined by $f(\xi) = \varphi\xi$ (respectively, $f(\xi) = \xi\varphi$), is additive.

A semiring R, satisfying all the suppositions above, will be called an *orthogonalizable* one (or, shortly, an *orthoring*) iff all elements of R are left additive and there is an *orthogonal quadruple* in R, i.e. a quadruple (T_+, T_-, F_+, F_-) of elements of R such that T_+ and F_+ are right additive as elements of R and the following equalities hold:

$$T_-T_+ = I = F_-F_+ \quad \text{and} \qquad T_-F_+ = 0 = F_-T_+.$$

An orthogonal quadruple (T_+, T_-, F_+, F_-) will be called complete iff

$$T_+T_- + F_+F_- = I.$$

More generally, a family $(\varphi^+(t), \varphi^-(t))_{t \in K}$ of pairs of elements of R will be called an *orthogonal system* iff the following three conditions hold:

a) for all $t \in K$ the element $\varphi^+(t)$ is right additive;

b) for all $t \in K$ we have $\varphi^-(t)\varphi^+(t) = I$; and

c) for all $t, s \in K$, for which $t \neq s$, we have $\varphi^-(t)\varphi^+(s) = 0$.

Infinite orthogonal systems exist into every non-trivial orthoring. Indeed, such is, for instance, the system $(\mathbf{n}^+, \mathbf{n}^-)_{n\in\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers and for every $n \in \mathbb{N}$

$$\mathbf{n}^+ = F_+^n T_+ \quad \text{and} \quad \mathbf{n}^- = T_- F_-^n.$$

Example 1. Let M be an infinite set, and let $T_+ : M \to M$ and $F_+ : M \to M$ be two injective functions such that $T_+(M) \cap F_+(M) = \varnothing$. Denote by R the set of all partial multivalued functions from M to M; the elements of R may be identified with (arbitrary) binary relations in M. The set R is a semiring with respect to the union of two relations as the addition operation and the composition of multivalued functions as multiplication; the zero 0 and the unit I are the nowhere defined function and the identity mapping in M, respectively. This semiring is an orthogonalizable one with respect to the inclusion relation "$\subseteq$" as partial order, the Scott topology with respect to this partial order (which may be described as follows: those sets $Y \subseteq R$ are open, for every element φ of which there is a finite $\iota \subseteq \varphi$ such that all $\psi \in R$, containing ι, belong to Y), and the orthogonal quadruple (T_+, T_-, F_+, F_-), where T_- and F_- are the partial mappings defined for $x \in R$ by

$$T_-(x) = \begin{cases} y & \text{if } x = T_+(y), \\ \text{undefined} & \text{if there is no such } y, \end{cases}$$

and

$$F_-(x) = \begin{cases} y & \text{if } x = F_+(y), \\ \text{undefined} & \text{if there is no such } y. \end{cases}$$

Example 2. Let V be the set of all infinite sequences $(x_0, x_1, \ldots)$ of real numbers x_i with the product topology. V is a real vector space and let L be the ring of linear operators $\varphi : V \to V$ with the topology induced by the product one in V^V. Denote by R the ring which differs from L only in the order of writing the multiplication: $\varphi\psi$ in R means $\psi\varphi$ in L. Take the identity "$=$" as partial order "$\leq$" in R. Then R is an orthogonalizable ring (i.e. an orthoring which is a ring with respect to the algebraic operations in it); an orthogonal quadruple in R is, for instance, the following one:

$$T_+((x_0, x_1, \ldots)) = (x_0, x_2, x_4, \ldots);$$
$$F_+((x_0, x_1, \ldots)) = (x_1, x_3, x_5, \ldots);$$
$$T_-((x_0, x_1, \ldots)) = (x_0, 0, x_1, 0, \ldots);$$
$$F_-((x_0, x_1, \ldots)) = (0, x_0, 0, x_1, \ldots).$$

Example 3. Let H be an infinite dimensional (real or complex) Hilbert space and let R be the ring of bounded linear operators in H. Take the trivial partial order, i.e. the identity "$=$", for "$\leq$" in R. Then R is an orthogonalizable ring with respect to any of the known operator topologies. For us, however, the weak operator topology will be the most important one and, unless otherwise indicated, we shall always have in view this topology in the context of this example. Orthogonal quadruples in R may be found, for instance, as it follows: Let T_+ and F_+ be two isometrical (preserving the scalar product) operators in R, the images $\mathrm{Im}T_+$ and $\mathrm{Im}F_+$ of which are orthogonal to each other subspaces of H, and let $T_- = T_+^*$ and $F_- = F_+^*$ be the corresponding adjoint operators. Then (T_+, T_-, F_+, F_-) is an orthogonal quadruple, which is a complete one iff the images of T_+ and

F_+ are orthogonal completions to each other. It will be convenient in the sequel to call the orthogonal quadruples, arising in this way, *isometrical* ones and the positive components of such quadruples (i.e. T_+ and F_+) — *semiunitary* operators. Accordingly, by an *isometrical* orthogonal system we mean an orthogonal system $(K^+(j), K^-(j))_{j\in J}$ such that the operators $K^+(j)$ are isometrical and $K^-(j)$ is the adjoint operator of $K^+(j)$ for all $j \in J$. Here is a simple lemma about such systems.

Lemma 1. *Let $(K^+(j), K^-(j))_{j\in J}$ be an isometrical orthogonal system in R and let D be a confinal in J. Then for any sequence of operators $\varphi_j \in R$, $j \in J$, the following two conditions are equivalent:*

(a) *the sum of $\sum_{j\in J} K^+(j)\varphi_j$, corresponding to D, exists with respect to the strong operator topology in R; and*

(b) *for all vectors $x \in H$ the series $\sum_{j\in J} ||\varphi_j x||^2$ converges.*

Indeed, if (a) holds and S is the sum of the series in question, corresponding to D, then using the isometricality of the operators $K^+(j)$ we have

$$\sum_{j\in J} ||\varphi_j x||^2 = \sum_{j\in J} ||K^+(j)\varphi_j x||^2 = ||Sx||^2 < \infty.$$

Conversely, let we have (b). Then for all finite $a \subseteq J$

$$||\sum_{j\in a} K^+(j)\varphi_j x||^2 = \sum_{j\in a} ||K^+(j)\varphi_j x||^2 = \sum_{j\in a} ||\varphi_j x||^2.$$

Since the series

$$\sum_{j\in J} ||\varphi_j x||^2$$

converges, using the completeness of the space H we conclude that the series

$$Sx = \sum_{j\in J} K^+(j)\varphi_j x$$

converges in it as well; whence by the Banach–Steinhaus theorem we get (a).

Remark. We may avoid quoting the last theorem if we replace the condition (b) by the following one:

(b′) *there is a positive real number C such that for all vectors $x \in H$ we have* $\sum_{j\in J} ||\varphi_j x||^2 \le C||x||^2$,

which may serve our purposes below as well.

2. THE CODE EVALUATION THEOREM

Let R be an orthoring. Suppose a semigroup G with an unit e and a homomorphism $\chi : G \to R'$ into the multiplicative semigroup R' of right additive elements of R are given. Let $G(\overline{X}) = G(X_0, \ldots, X_{n-1})$ be the semigroup of monomials of the variables $X_0, \ldots, X_{n-1}$ with coefficients in G, i.e. the set of all formal expressions of the form

(1) $$q = g_0 Y_0 g_1 \cdots g_{m-1} Y_{m-1} g_m,$$

where $g_0, \ldots, g_m \in G$ and $Y_0, \ldots, Y_{m-1} \in \{X_0, \ldots, X_{n-1}\}$, with the obvious multiplication operation. The homomorphism χ extends uniquely to a homomorphism $\widetilde{\chi} : G(\overline{X}) \to R^{R^n}$ into the semigroup of all functions $f : R^n \to R$ (with respect to the usual multiplication operation $(fg)(\overline{\xi}) = f(\overline{\xi})g(\overline{\xi})$, $\overline{\xi} \in R^n$) such that for all $i < n$, $\widetilde{\chi}(X_i)$ is the i-th projection $R^n \to R$, i.e. $\widetilde{\chi}(X_i)(\xi_0, \ldots \xi_{n-1}) = \xi_i$ for all $(\xi_0, \ldots, \xi_{n-1}) \in R^n$. By a *coding* for $G(\overline{X})$ with respect to χ we shall mean an orthogonal system

(2) $$(k^+(q), k^-(q))_{q \in G(\overline{X})}$$

in the orthoring R for which the following two conditions hold:

a) for all $q \in G(\overline{X})$ both of the series

(3) $$\sum_{g \in G} k^+(qg)k^-(g)$$

and

(4) $$\sum_{s \in G(\overline{X}) \setminus G} k^+(qs)k^-(s)$$

are conventionally convergent; and

b) there is an element $\tau \in R$ such that

$$\tau k^+(g) = \chi(g)$$

for all $g \in G$, and for all $q \in G(\overline{X}) \setminus G$

$$\tau k^+(q) = 0.$$

The last condition b) can be replaced by the conventional convergence of the series

$$\sum_{g \in G} \chi(g)k^-(g).$$

Indeed, if this series is conventionally convergent, then we may take any sum of it for τ and the equalities in b) will follow from the definition of orthogonal system. Conversely, if τ satisfies b), then the last series coincides with

$$\sum_{g \in G} \tau k^+(g)k^-(g),$$

which is conventionally convergent by a) since τ is left additive. The element τ will be called a starting element of the coding in question.

To every coding (2) we naturally assign two idempotent elements $\varkappa'$ and $\varkappa''$ of R defined as the sums of the series

$$\sum_{g \in G} k^+(g)k^-(g) \quad \text{and} \quad \sum_{s \in G(\overline{X}) \setminus G} k^+(s)k^-(s),$$

respectively, the existence of which is supposed in a). The element $\varkappa = \varkappa' + \varkappa''$ is also idempotent and the codings (2), for which $\varkappa = I$, will be called complete. It follows from the above that for any starting element τ of a coding (2) the equality

$$\tau\varkappa = \tau\varkappa' = \sum_{g\in G} \chi(g)k^-(g) \tag{5}$$

holds (where the last sum corresponds to the confinal supposed in condition a) for the case $q = e$); and for complete codings the starting element is unique.

In the sequel we shall usually have the homomorphism χ fixed and we shall write $\widetilde{q}$ for $\widetilde{\chi}(q)$.

Let $G[\![X_0, \ldots, X_{n-1}]\!] = G[\![\overline{X}]\!]$ be the set of all formal series of the form

$$P(\overline{X}) = \sum_{p\in M} \lambda_p p(\overline{X}),$$

where $M \subseteq G(\overline{X})$, and for all $p \in M$ the element $\lambda_p \in R$ belongs to the center of the orthoring R, i.e. commutes with all elements of R. Here in the set M a confinal is supposed to be given, so that to the last series we assign a *value*

$$\widetilde{P}(\overline{\xi}) = \sum_{p\in M} \lambda_p \widetilde{p}(\overline{\xi})$$

for every n-tuple $\overline{\xi} = (\xi_0, \ldots, \xi_{n-1}) \in R^n$ for which the last sum, corresponding to that confinal, exists in R. Consider a formal system of inequalities of the form

$$P_i(\overline{X}) \leq X_i, \quad i < n, \tag{6}$$

where

$$P_i(\overline{X}) = \sum_{p\in M_i} \lambda_{ip} p(\overline{X})$$

is a formal series in $G[\![\overline{X}]\!]$ for all $i \leq n$. By a solution of the system (6) we mean an n-tuple $\overline{\xi} \in R^n$ such that the values $\widetilde{P}_i(\overline{\xi})$ exist and satisfy the inequalities $\widetilde{P}_i(\overline{\xi}) \leq \xi_i$ in R for all $i < n$.

An element $\rho \in R$ will be called a *governing* element of the system (6) with respect to the coding (2) iff the equalities

$$\rho k^+(g) = 0 \tag{7}$$

and

$$\rho k^+(tX_i g) = \sum_{p\in M_i} \lambda_{ip} k^+(tp)\chi(g) \tag{8}$$

are satisfied for all $g \in G$, $t \in G(\overline{X})$ and $i < n$. Here the right hand side of the last equality has to be understood as the sum corresponding to the confinal in M_i, which is supposed in the definition of formal series, and thus the existence of the last sum is supposed in the definition of governing element. A necessary and sufficient condition for the existence of governing element for (6) with respect to (2) is the existence of the sum in (8) and the conventional convergence of the series

$$\sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip} k^+(tp)\chi(g)\Big) k^-(tX_i g). \tag{9}$$

Indeed, if the sum in (8) exists and (9) is conventionally convergent, then any sum ρ of it satisfies (7) and (8). Conversely, if ρ is a governing element, then (9) is conventionally convergent, because it coincides with the series obtained from (4) by putting $q = e$ and multiplying memberwise from left by ρ. We see as well that the governing element ρ satisfies the equalities

$$\rho \varkappa = \rho \varkappa'' = \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip} k^+(tp)\chi(g) \Big) k^-(tX_i g), \tag{10}$$

where the external sum corresponds to the confinal supposed in the condition a) for $q = e$, and for complete codings ρ is unique.

Let ρ be a governing element for the system (6) with respect to the coding (2) with a starting element τ. Then the inequality

$$\tau\varkappa + \xi\rho\varkappa \le \xi, \tag{11}$$

which is a linear one with respect to ξ, will be called *iterational* inequality for (6). For any two elements $\varphi, \psi \in R$, by *iteration* of φ starting from ψ we shall mean an element $\vartheta \in R$ such that

$$\psi + \vartheta\varphi \le \vartheta$$

and for all $\alpha, \xi \in R$ we have

$$\alpha\psi + \xi\varphi \le \xi \implies \alpha\vartheta \le \xi.$$

Theorem 1. *Let ρ be a governing element for the system (6) with respect to a coding (2) for $G(\overline{X})$ with a starting element τ. If ω is an iteration of $\rho\varkappa$ starting from $\tau\varkappa$, then the n-tuple*

$$\omega k^+(\overline{X}) = (\omega k^+(X_0), \ldots, \omega k^+(X_{n-1}))$$

is the least solution of (6) in the set

$$E = \{\overline{\xi} \in R^n \mid \sum_{q\in G(\overline{X})} \widetilde{q}(\overline{\xi})k^-(q) \text{ is conventionally convergent}\},$$

and for all $q \in G(\overline{X})$ it satisfies the equality

$$\widetilde{q}(\omega k^+(\overline{X})) = \omega k^+(q).$$

Conversely, if there is a solution of (6) in E, then the iterational inequality (11) has a solution with respect to ξ in R.

Proof. Since by the suppositions of the theorem ω is the least solution of (11), it should satisfy the equality $\tau\varkappa + \omega\rho\varkappa = \omega$, whence by a multiplication from right we get

$$\chi(g) = \tau k^+(g) = \tau\varkappa k^+(g) = \omega k^+(g) \tag{12}$$

for all $g \in G$, and

$$\sum_{p\in M_i} \lambda_{ip}\omega k^+(qp)\chi(g) = \omega\rho k^+(qX_i g) = \omega k^+(qX_i g) \tag{13}$$

for all $g \in G$, $q \in G(\overline{X})$ and $i < n$. Hence we get also that for all such g and q

$$\omega k^+(q)\chi(g) = \omega k^+(qg). \tag{14}$$

Indeed, if $q \in G$, then by (12) we have

$$\omega k^+(q)\chi(g) = \chi(q)\chi(g) = \chi(qg) = \omega k^+(qg),$$

and if $q = q_1 X_i g_1$, then using (13) we obtain

$$\omega k^+(q)\chi(g) = \sum_{p\in M_i} \lambda_{ip}\omega k^+(q_1p)\chi(g_1)\chi(g)$$

$$= \sum_{p\in M_i} \lambda_{ip}\omega k^+(q_1p)\chi(g_1g) = \omega k^+(q_1Xg_1g) = \omega k^+(qg).$$

We shall show that for all $q, s \in G(\overline{X})$

(15) $$\omega k^+(q)\omega k^+(s) \leq \omega k^+(qs).$$

By condition a) in the definition of coding it follows that there are elements $\zeta \in R$ such that for all $s \in G(\overline{X})$ the equality

$$\zeta k^+(s) = \omega k^+(qs)$$

holds; such is, for instance, the element

$$\zeta = \omega \sum_{g\in G} k^+(qg)k^-(g) + \omega \sum_{s\in G(\overline{X})\setminus G} k^+(qs)k^-(s).$$

Taking an arbitrary such element ζ and using (5), (10), (13) and (14), we have

$$\omega k^+(q)\tau\varkappa + \zeta\rho\varkappa = \omega k^+(q)\tau\varkappa + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip}\zeta k^+(tp)\chi(g)\Big) k^-(tX_ig)$$

$$= \omega k^+(q) \sum_{g\in G} \chi(g)k^-(g) + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip}\omega k^+(qtp)\chi(g)\Big) k^-(tX_ig)$$

$$= \sum_{g\in G} \omega k^+(qg)k^-(g) + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \omega k^+(qtX_ig)k^-(tX_ig) = \zeta',$$

where the last two sums correspond to the confinals D' and D'', respectively, with respect to which the series (3) and (4) are supposed to converge in condition a) for the case $q = e$. Since, however, the element ζ' still satisfies $\zeta' k^+(s) = \omega k^+(qs)$ for all monomials s, the above calculations hold for ζ' as well and prove the equality

$$\omega k^+(q)\tau\varkappa + \zeta'\rho\varkappa = \zeta',$$

whence by the definition of iteration it follows that

$$\omega k^+(q)\omega \leq \zeta',$$

and multiplying from right by $k^+(s)$, we obtain (15). Using (12), (14), (15) and a simple induction on the degree of the monomial $q \in G(\overline{X})$, we get the inequality

(16) $$\widetilde{q}(\omega k^+(\overline{X})) \leq \omega k^+(q),$$

where $\omega k^+(\overline{X}) \in R^n$ is $(\omega k^+(X_0), \ldots, \omega k^+(X_{n-1}))$. Thence by the help of (13) and conditions (ii) and (iii) in Section 1 it follows that $\omega k^+(\overline{X})$ is a solution of (6):

$$\widetilde{P}_i(\omega k^+(\overline{X})) = \sum_{p\in M_i} \lambda_{ip}\widetilde{p}(\omega k^+(\overline{X})) \leq \sum_{p\in M_i} \lambda_{ip}\omega k^+(p) \leq \omega k^+(X_i).$$

Moreover, by (16) the series

$$\sum_{q\in G(\overline{X})} \widetilde{q}(\omega k^+(\overline{X}))k^-(q)$$

is memberwise less or equal to

$$\sum_{q\in G(\overline{X})} \omega k^+(q)k^-(q);$$

and the last series converges with respect to the confinal $\{a\cup b \mid a \in D' \,\&\, b \in D''\}$, as it follows in an obvious way from the convergence of the sums in the definitions of $\varkappa'$ and $\varkappa''$ with respect to D' and D'', respectively, using the condition of continuity of the addition $+$. Therefore by the same conditions (ii) and (iii) it follows that $\omega k^+(\overline{X}) \in E$. To show that this solution is the least one in E, suppose that the n-tuple $\overline{\xi} = (\xi_0, \ldots, \xi_{n-1}) \in E$ is a solution of (6) in E, and let v be the element

$$\sum_{g\in G} \chi(g)k^-(g) + \sum_{q\in G(\overline{X})\setminus G} \widetilde{q}(\overline{\xi})k^-(q),$$

where the last two sums correspond to the confinals D' and D'', respectively. The existence of the second one of those sums follows from that for the sum in the definition of the idempotent $\varkappa''$ by a memberwise multiplication from left by an arbitrary sum ψ of the series

$$\sum_{q\in G(\overline{X})} \widetilde{q}(\overline{\xi})k^-(q);$$

and the existence of the last sum is the condition $\overline{\xi} \in E$. Then, using the supposition that the elements $\chi(g)$ are right additive, we have

$$\begin{aligned}
\tau\varkappa + v\rho\varkappa &= \tau\varkappa + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip} v k^+(tp)\chi(g)\Big) k^-(tX_i g) \\
&= \tau\varkappa + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \Big(\sum_{p\in M_i} \lambda_{ip}\widetilde{t}(\overline{\xi})\widetilde{p}(\overline{\xi})\chi(g)\Big) k^-(tX_i g) \\
&= \sum_{g\in G} \chi(g)k^-(g) + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \widetilde{t}(\overline{\xi})\widetilde{P}_i(\overline{\xi})\chi(g)k^-(tX_i g) \\
&\le \sum_{g\in G} \chi(g)k^-(g) + \sum_{i<n,\, t\in G(\overline{X}),\, g\in G} \widetilde{t}(\overline{\xi})\xi_i\chi(g)k^-(tX_i g) = v.
\end{aligned}$$

Hence by the definition of iteration $\omega \le v$ and

$$\omega k^+(X_i) \le v k^+(X_i) = \widetilde{X}_i(\overline{\xi}) = \xi_i,$$

which means that $\omega k^+(\overline{X})$ is the least solution of (6) in E. In the same time we have shown that if (6) has a solution in E, then there is $v \in R$ such that $\tau\varkappa + v\rho\varkappa \le v$. Finally, taking $\omega k^+(\overline{X})$ for $\overline{\xi}$ and multiplying the inequality $\omega \le v$ from right by $k^+(q)$, we obtain the reverse inequality of (16).

An important special case is that of orthorings in which the order "$\le$" is just the equality "$=$". In this case the system (6) and the inequality (11) become a formal system of equations

(17) $$P_i(\overline{X}) = X_i, \quad i < n,$$

and a linear equation (which we call iterational)

$$\tau\varkappa + \xi\rho\varkappa = \xi, \tag{18}$$

respectively. In this case from Theorem 1 (or rather by a trivial modification of its proof) we obtain the following

Corollary 1. *Let in the orthoring R $\varphi \leq \psi$ be equivalent to $\varphi = \psi$ for all $\varphi, \psi \in R$ and the conditions of Theorem 1 hold. If the equation*

$$\xi\rho\varkappa = \xi$$

has no non-zero solutions with respect to ξ, then the iterational equation (18) has a solution $\omega \in R$ with respect to ξ iff the system (17) has a solution $\overline{\xi}(\xi_0, \dots, \xi_{n-1})$ in the set E in Theorem 1, and in the last case this solution of (17) is unique and is given by $\xi_i = \omega k^+(X_i)$, $i < n$, and for all $q \in G(\overline{X})$ it satisfies

$$\widetilde{q}(\omega k^+(\overline{X})) = \omega k^+(q).$$

3. EXAMPLES AND APPLICATIONS

Example 1 (continued). In this example iterations exist always, i.e. for every two elements $\varphi, \psi \in R$ there is an iteration of φ starting from ψ; and the conditions of Theorem 1 hold for every *finite* system of the form (6) (i.e. such that the sets M_i are finite) with respect to arbitrary G and χ. Usually, a multiplicative subsemigroup of R, produced by a subset $B \subseteq R$, is taken for G, and the identical embedding — for χ. In this case the members of the least solutions of finite systems of the form (6) are called *recursive in B* elements of R. The example is actually well-known in the recursion theory and treats the first order recursion theory of multivalued functions in possibly the most general domain; Theorem 1 implies easily all basic results of the last theory for such functions.

Example 2 (continued). Define for any natural number n

$$\mathbf{n}^+ = F_+^n T_+ \quad \text{and} \quad \mathbf{n}^- = T_- F_-^n.$$

It is not hard to see that all series of the form

$$\sum_{m=0}^{\infty} \varphi_m \mathbf{m}^-,$$

where $\varphi_m \in R$ for all natural m, are convergent in this example and have an unique sum. Then every orthogonal system of the form (2) is, obviously, a coding, provided all its members $(k^+(q), k^-(q))$ have the form $(\mathbf{m}^+, \mathbf{m}^-)$ for a suitable natural m. In this way, for every finite semigroup G one can easily construct codings, using suitable numerations, and let us call such codings numerical. Therefore for numerical codings the set E in Theorem 1 will be the whole orthoring R; and by Corollary 1 with the trivial semigroup, consisting only of the unit e for G, we obtain the following proposition:

Suppose the left hand sides of the system (17) are polinomials with real coefficients, and let ρ be the sum of the series (9) with respect to a numerical coding (2). If the equation

$$\xi\rho = \xi$$

has an unique solution $\xi = 0$ *in* R, *then the system* (17) *has a solution in* R^n *iff the iterational equation* $\tau\varkappa + \xi\rho = \xi$ *has a solution* ω *with respect to* ξ, *and in the last case this solution of* (17) *is unique and is given by* $\xi_i = \omega k^+(X_i)$, $i < n$.

Example 3 (continued). Let G be the trivial semigroup $\{e\}$ and let take an isometrical orthogonal quadruple (T_+, T_-, F_+, F_-). The semigroup $G(X)$ of monomials of one variable X consists of powers X^n, especially X^0 is the unit e, and we have the following coding for $G(X)$:

$$k^+(X^n) = \mathbf{n}^+ = F_+^n T_+, \quad k^-(X^n) = \mathbf{n}^- = T_- F_-^n.$$

The element $\tau = T_-$ is a starting element for this coding. The series

$$\varkappa = \sum_{m=0}^{\infty} k^+(X^m)k^-(X^m) = \sum_{m=0}^{\infty} \mathbf{m}^+\mathbf{m}^- \tag{19}$$

converges in the strong operator topology, because the operators $\mathbf{m}^+$, $\mathbf{m}^-$ are orthogonal projections upon pairwise orthogonal subspaces of H. The same holds for the series (4), because it may be obtained from (19) by a multiplication from left by a suitable element of the form F_+^k and from right by F_- (all elements are right additive in this example). In this case $G[\![X]\!]$ is the set of all formal power series

$$P(X) = \sum_{n=0}^{\infty} \lambda_n X^n \tag{20}$$

with scalar coefficients λ_n. As a corollary, we have the following

Proposition 1. *Let the sum* $f(F_+)$ *of the series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{21}$$

(*where* a_n *are complex numbers*) *exist in the sense of weak operator topology and let the scalar unit* 1 *not belong to the spectrum of the operator* $f(F_+)F_-\varkappa$. *Then there is an unique operator* $\xi \in R$ *such that* $\|\xi\| < 1$, $f(\xi)$ *exists in the same sense and* $f(\xi) = \xi$.

Indeed, if the value $\widetilde{P}(F_+)$ of (20) exists, then it may be seen immediately that the operator $\rho = \widetilde{P}(F_+)F_-\varkappa$ is a governing element with respect to the above coding for the system of one equation

$$P(X) = X. \tag{22}$$

And since 1 does not belong to the spectrum of ρ, the last element has an iteration starting from any element of R. Applying Corollary 1, we conclude that the operator $\omega\mathbf{1}^+$, where ω is the unique solution of the iterational equation

$$\tau + \xi\rho\varkappa = \xi$$

with respect to ξ, is the unique solution of (22) in the set

$$E = \{\xi \in R \mid \sum_{n=0}^{\infty} \xi^n \mathbf{n}^- \text{ converges in the weak operator topology}\}.$$

By Theorem 1 it follows also that $(\omega\mathbf{1}^+)^n = \omega\mathbf{n}^+$ for all natural n. Using the last equality, we see that the operator $\omega\mathbf{1}^+$ belongs to the set

$$E' = \{\xi \in R \mid \sum_{n=0}^{\infty} \mathbf{n}^+(\xi^*)^n \text{ converges in the strong operator topology}\},$$

because

$$\sum_{n=0}^{\infty} \mathbf{n}^+((\omega\mathbf{1}^+)^*)^n = \sum_{n=0}^{\infty} \mathbf{n}^+(\omega\mathbf{n}^+)^* = \sum_{n=0}^{\infty} \mathbf{n}^+\mathbf{n}^-\omega^*.$$

Since $E' \subseteq E$, the operator $\omega\mathbf{1}^+$ is the unique solution of (22) in E'. On the other hand, for every invertible operator $U \in R$ we have $U^{-1}E'U \subseteq E'$, as it follows easily from Lemma 1. Hence we conclude that $U^{-1}\omega\mathbf{1}^+U \in E'$ for any invertible operator U, but since $U^{-1}\omega\mathbf{1}^+U$ is also bound to be a solution of (22), by the uniqueness of the last solution in E' it follows that $U^{-1}\omega\mathbf{1}^+U = \omega\mathbf{1}^+$, i.e. $\omega\mathbf{1}^+$ commutes with all invertible operators and therefore $\omega\mathbf{1}^+ = cI$ for a scalar c. Since $cI \in E'$, i.e. $\sum_{n=0}^{\infty} c^n\mathbf{n}^+$ converges in the strong operator topology, by Lemma 1 the series $\sum_{n=0}^{\infty} |c|^{2n}$ converges and therefore $|c| < 1$, whence $||\omega\mathbf{1}^+|| < 1$.

Applying Corollary 1 to other codings, we obtain other propositions of that kind. Consider, for instance, the following one.

Proposition 2. *Let l be a natural number and for* (21) *we have the inequality*

$$(l+1)^2 \sum_{n=0}^{\infty} l^{-n}|a_n|^2 < 1.$$

Then the function f has an unique fixed point ξ_0 in the ball

$$\{\xi \in R \mid ||\xi|| < (l+1)^{-1/2}\},$$

which is of the form $\xi_0 = cI$ for a suitable scalar c.

Indeed, consider the semigroup G consisting of $l+1$ elements $e_0, \ldots, e_l$ with the following multiplication law: $e_ie_j = e_{\max(i,j)}$, and let $\chi : G \to \{I\}$ be the trivial homomorphism. Suppose

$$(T_+, T_-, F_+(e_0), F_-(e_0), \ldots, F_+(e_l), F_-(e_l))$$

is an isometrical orthogonal system in R, consisting of $l+2$ pairs of operators. To each monomial

$$q = g_0Xg_1 \ldots g_{n-1}Xg_n \tag{23}$$

in $G(X)$ we assign the operator $W^-(q) = F_-(g_0)\ldots F_-(g_n)$ and define $W^+(q) = (W^-(q))^*$, $k^+(q) = W^+(q)T_+$ and $k^-(q) = T_-W^-(q)$. Using Lemma 1, it is easy to show that the orthogonal system $(k^+(q), k^-(q))_{q\in G(X)}$ is a coding for $G(X)$ with respect to χ. Indeed, condition b) in the definition of coding is fulfilled with $\tau = T_- \sum_{i\leq l} F_-(e_i)$. By Lemma 1 the series

$$\sum_{s\in G(X)\setminus G} k^+(s)k^-(qs),$$

which is adjoint to (4), converges in the strong operator topology iff for all $x \in H$ converges the series

$$\sum_{s \in G(X)\setminus G} \|k^-(qs)x\|^2,$$

which, since the operators $k^+(q)$ are isometrical, is the same as

$$\sum_{s \in G(X)\setminus G} \|k^+(qs)k^-(qs)x\|^2,$$

and the sum of the last series is obviously not greater than $l\|x\|^2$. Thus we see that (4) converges in the weak operator topology, i.e. condition a) in the definition of coding holds. A monomial of the form (23) will be called *regular* iff $g_0 = e_l$ and $e_l \notin \{g_1, \ldots, g_n\}$. Denote by M the set of all regular monomials in $G(X)$ and let for every regular monomial $p(X)$ of degree n define $\lambda_p = l^{-n}a_n$. Since the number of regular monomials of degree n is l^n, the value $\widetilde{P}(\xi)$ of the formal series

$$P(X) = \sum_{p \in M} \lambda_p p(X)$$

with respect to χ is just the sum $f(\xi)$ of (21) for the operator ξ in the sense of the weak operator topology. But the system

$$(W^+(p), W^-(p))_{p \in M}$$

is an orthogonal one, whence by Lemma 1 and the inequality, assumed in Proposition 2, we conclude that the series

$$\psi = \sum_{p \in M} \lambda_p W^+(p)$$

converges in the strong operator topology; and for the sum ψ of this series we have

$$\|\psi\|^2 = \sum_{p \in M} |\lambda_p|^2,$$

because the vectors $W^+(p)x$, $p \in M$, are pairwise orthogonal for all $x \in H$. Now define

$$\rho = \psi(T_+T_- + \varkappa) \sum_{g \in G} \sum_{h \in G} F_-(h)F_-(g),$$

where $\varkappa$ is the projection

$$\varkappa = \sum_{q \in G(X)} k^+(q)k^-(q). \tag{24}$$

We shall show that ρ is a governing element for the equation $P(X) = X$ with respect to the coding in question. The equality (7) is obvious for this ρ, and to see (8), we have to consider two cases for the monomial $q \in G(X)$:

Case 1) $q = h \in G$. Then

$$\rho k^+(qXg) = \rho F_+(g)F_+(h)T_+ = \psi(T_+T_- + \varkappa)T_+ = \psi T_+$$
$$= \sum_{p \in M} \lambda_p W^+(p)T_+ = \sum_{p \in M} \lambda_p W^+(hp)T_+ = \sum_{p \in M} \lambda_p k^+(qp),$$

because $p = hp$ for any regular monomial p.

Case 2) $q = q_0 X h$ for some $q_0 \in G(X)$ and $h \in G$. Then, as in the first case,

$$\rho k^+(qXg) = \rho F_+(g)F_+(h)W^+(q_0)T_+ = \psi(T_+T_- + \varkappa)W^+(q_0)T_+$$
$$= \sum_{p\in M} \lambda_p W^+(p)W^+(q_0)T_+ = \sum_{p\in M} \lambda_p W^+(hp)W^+(q_0)T_+ = \sum_{p\in M} \lambda_p k^+(qp).$$

So ρ is indeed a governing element. On the other hand, since the operator $T_+T_- + \varkappa$ is an orthogonal projection, we have

$$||\rho||^2 \le ||\psi||^2 ||(T_+T_- + \varkappa)||^2 \Big\|\sum_{g\in G}\sum_{h\in G} F_-(h)F_-(g)\Big\|^2 = (l+1)^2||\psi||^2$$
$$= (l+1)^2 \sum_{p\in M} |\lambda_p|^2 = (l+1)^2 \sum_{n=0}^{\infty} \sum_{p\in M,\,\deg p = n} |\lambda_p|^2 = (l+1)^2 \sum_{n=0}^{\infty} l^{-n}|a_n|^2,$$

and by the assumptions of the Proposition 2 $||\rho|| < 1$. Therefore the iterational equation has an unique solution ω and by Corollary 1 the operator $\omega k^+(X)$ is the unique solution of $P(X) = X$ in the set E of all operators $\xi \in R$ for which $\sum_{q\in G(X)} \widetilde{q}(\xi)k^-(q)$ is conventionally convergent in R, i.e. in the weak operator topology. We see as above that this set E contains the set E' of those operators ξ for which the adjoint of the last series converges in the strong operator topology. Using again Theorem 1 and the strong convergence of the series in (24), we conclude that $\omega k^+(X) \in E'$. By Lemma 1 the set E' coincides with the set of all operators ξ for which the series

$$\sum_{q\in G(X)} ||(\widetilde{q}(\xi))^* x||^2$$

converges for all vectors x. Hence it follows that $U^{-1}E'U \subseteq E'$ for every invertible operator U and therefore $\omega k^+(X)$ commutes with every such operator, which shows that $\omega k^+(X) = cI$ for a suitable scalar c. By the convergence of the last series with cI for ξ it follows that $\sum_{i=0}^{\infty}(l+1)^{i+1}|c|^{2i}$ also converges, whence $|c| < (l+1)^{-1/2}$. The last characterization of the set E' shows also that it contains the ball $\{\xi \in R \mid ||\xi|| < (l+1)^{-1/2}\}$, whence we get all the conclusions of Proposition 2.

An n-dimensional variant of the last proposition holds as well. Its proof differs from the above one only in trivial details and we shall give here its formulation only. For that purpose we use the following notations: for every n-tuple $w = (i_0, \dots, i_{n-1})$ of natural numbers and every n-dimensional vector $z = (z_0, \dots, z_{n-1})$ ($z_0, \dots, z_{n-1}$ being complex numbers) define $z^w = z_0^{i_0} z_1^{i_1} \dots z_{n-1}^{i_{n-1}}$, $|w| = i_0 + \dots + i_{n-1}$ and $(w)! = i_0! \dots i_{n-1}!$.

Proposition 2′. *Let for a system of n series of the form*

$$f_i(z) = \sum_{w\in\mathbb{N}^n} a_{iw} z^w, \quad i < n,$$

we have the inequality

$$(l+1)^2 \sum_{i<n} \sum_{w \in \mathbb{N}^n} \left(l^{|w|}|w|!\right)^{-1} (w)!|a_{iw}|^2 < 1.$$

Then the system of equations

$$f_i(\xi_0, \ldots, \xi_{n-1}) = \xi_i, \quad i < n,$$

has an unique solution in operators $(\xi_0, \ldots, \xi_{n-1}) \in R^n$ *for which*

$$\sum_{i<n} ||\xi_i||^2 < (l+1)^{-1}$$

and this solution is of the form $(c_0 I, \ldots, c_{n-1} I)$, *where* $c_0, \ldots, c_{n-1}$ *are scalars.*

Some other propositions of this kind can be obtained by using suitable infinite semigroups for G.

Finally, let us note that the method used in the present paper is applicable in various situations. For instance, it holds quite well, promising interesting applications, for orthorings with an additional binary operation having the properties of tensor product. The last orthorings are for the combinatory spaces [2] approximately the same what the orthorings in the present paper are for the operative spaces [1]. It is a more complicated question whether the code evaluation theorem holds under not very restrictive suppositions for orthorings with involution. Here the technique of iterative extensions in the algebraic recursion theory may provide the necessary tool. Perhaps the further investigations will throw more light over this situation.

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ВЪРХУ ДВЕ ИНТЕРВАЛНО-АРИТМЕТИЧНИ СТРУКТУРИ И ТЕХНИТЕ СВОЙСТВА[1]

СВЕТОСЛАВ МАРКОВ

Светослав Марков. О ДВУХ ИНТЕРВАЛЬНО-АРИФМЕТИЧЕСКИХ СТРУКТУР И ИХ СВОЙСТВ

Рассматриваются две интервально-арифметические структуры, которые являются расширениями интервальной арифметики для нормальных интервалов, использующей только обычных (внешных) операциях в двух направлений: а) введением специальных (нестандартных) операций; б) расширением множества нормальных интервалов до совокупности направленных интервалов введением несобственных интервалов. Рассмотрены новые свойства интервальных структур. Означения унифицированны используя новую „плюс-минус“-технику для индексации переменных, которая позволяет легко сравнивать и систематизировать обе структуры. Введена новая нормальная форма представлениях направленных интервалов, и благодаря ей можно свести свойств направленных интервалов к свойствам нормальных интервалов используя внешных и внутренных операциях. Доказаны утверждения, позволяющие вычислять области значений функции при помощи интервально-арифметических выражений.

Svetoslav Markov. ON TWO INTERVAL-ARITHMETIC STRUCTURES AND THEIR PROPERTIES

Two interval-arithmetic structures are considered, which are extensions of the interval arithmetic for normal intervals using only familiar (outer) operations in two directions: i) via additional introduction of special inner (non-standard) operations; ii) via extension of the set of normal intervals up to the set of directed (generalized) intervals by improper intervals. New properties of the interval structures are considered. The notations are unified by introducing new (±)-type indeces for the interval variables for an easy comparison and systematization of both structures. The

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so-called normal form for the representation of directed intervals is introduced, which allows to reformulate propositions for directed intervals into corresponding propositions for normal intervals, using inner and outer operations. Some propositions supporting the computation of functional ranges via interval arithmetic expressions are formulated.

1. УВОД

Настоящата работа е посветена на систематизирано представяне и изследване на две разширения (обобщения) на интервалната аритметика: а) разширението със специални вътрешни (нестандартни) интервални операции [16, 18] и б) разширението с несобствени интервали [10, 11]. Тези две интервални системи са представяни досега независимо, като са използувани разнообразни подходи и означения. Основният принос в настоящата работа е методологически: показано е, че двете разширения могат да се разглеждат в тясно взаимодействие от единна гледна точка и с единни означения и че разширението с несобствени интервали води до необходимост от използуване на вътрешни интервални операции. Практическата стойност на този начин на изложение е както в по-голямата обозримост и простота при използуването на двете системи, така и във възможността за едновременното им програмно реализиране [27] и съответно използуване посредством числени алгоритми.

Интервалната аритметика е в основата на голям брой числени алгоритми с верификация на резултатите, включително алгоритми, които работят с интервални входни данни. Най-важното приложение на интервалната аритметика (и нейните разширения) е в пресмятането на обхвати (области от стойности) на функции [22, 23, 25, 32] и във възможността за автоматизиране на това пресмятане [2, 3, 28]. Друго приложение (на по-абстрактно ниво) интервално-аритметичните структури намират в анализа на интервално-значни функции [17, 19, 33, 35–39].

Нека $C(T)$ е множеството от непрекъснатите реално-значни функции, дефинирани в интервала $T = [t^{(-)}, t^{(+)}]$. За $f \in C(T)$ множеството $f(T) = \{f(t) \mid t \in T\}$ наричаме обхват (множество от стойности) на f върху T. За $f \in C(T)$ множеството $f(T)$ е интервал. Важна задача е пресмятането на обхват на рационална функция. Една помощна задача е следната: ако са дадени обхватите $f(T)$, $g(T)$ на две функции $f, g \in C(T)$, да се представят обхватите на функциите $f+g$, $f-g$, fg, f/g върху T чрез дадените обхвати. Интервалната аритметика дава полезно помощно средство за практическото решаване на тази задача особено в съчетание с техниката на автоматично диференциране, при което изследването за монотонност на функциите се автоматизира.

В интервалната аритметика често се използуват множества от два елемента (двойки) и двоични променливи, които могат да имат различен смисъл. Например (затворен) интервал е наредена двойка, знак на интервал е двоична променлива и т. н. Подобни двойки и двоични променливи възникват при разширените интервални системи, например външна и вътрешна интервална операция, положителна и отрицателна посока на насочен интервал и др. под. В настоящата работа се въвеждат и системно използуват единни означения за използуваните в интервалната аритметика двоични обекти, за които се показва, че са тясно взаимосвързани, въпреки че привидно са твърде различни по своята природа. Тъй като една често срещана двоична променлива е променливата „знак на интервал“, която е обобщение на знака на реално число, и тъй като за стойностите на тази променлива са възприети означенията плюс (+) и минус (−), възприемаме тези означения и за стойностите на всички използувани от нас двоични променливи. Благодарение на така въведените единни означения става възможно да се покаже, че разширената (с вътрешни операции) интервална аритметика е необходим помощен апарат при работа с насочени интервали. Този апарат дава възможност за интерпретиране на резултати, отнасящи се за насочени интервали, в термините на нормални интервали. Показано е също така как твърдения за насочени интервали се преформулират в съответни твърдения за нормални интервали посредством вътрешни и външни интервални операции.

Използуваните в работата символи „+“, „−“ могат да имат различен смисъл в зависимост от свързаните с тях математически обекти, което личи от контекста и от мястото на тези символи (напр. горен или долен индекс на числова променлива, на символ за операция или релация). Ето някои от използуваните двойки обекти и двоични променливи:

- знак на числова или интервална променлива (плюс/минус);
- край на нормален интервал (десен/ляв);
- компонента на насочен интервал (втора/първа);
- посока на насочен интервал (положителна/отрицателна);
- интервална операция (външна/вътрешна);
- наличност на оператор за спрягане (неспрегнат/спрегнат интервал);
- посока на релация включване (се съдържа в/съдържа);
- числова аритметична операция (събиране/изваждане) и др. под.

В т. 2 се въвежда накратко обикновената интервална аритметика за нормални (собствени) интервали. Това въвеждане става с помощта на споменатата „плюс-минус“-техника на означения, което дава основа за сравняване и унифициране на въведените разширения. В т. 3 се разширява множеството от интервални операции чрез въвеждането на специални вътрешни (нестандартни) операции. В т. 4 се въвеждат несобствени интервали и множеството от нормални (собствени) интервали се разширява до множеството на насочените интервали (собствени и несобствени). Т. 5 е посветена на представянето на насочените интервали в т. нар. нормален запис (състоящ се от двойката обикновен интервал и знак). В т. 6 е демонстрирана техника за преформулиране на твърдения за насочени интервали в твърдения за обикновени интервали; за илюстрация са получени твърдения от разширената интервална аритметика, въведена в т. 3. В т. 7 се формулират твърдения, които позволяват интерпретирането и използуването на разгледаните две интервални системи за пресмятане на области от стойности на функции.

В настоящата работа е поставено ударение върху интервално-аритметичните операции, когато краищата (компонентите) на интервалите са реални числа. Не се разглеждат комплексният случай, компютърно-аритметични операции и свързаните с тях закръглявания, необходими за програмна реализация. Теоретичната основа на тези разглеждания е въвеждането на релации на частична наредба и допълването на частично-наредените интервални пространства с безкрайни елементи (срв. [12–14, 21]). За известна пълнота са дадени само дефинициите на основните релации на частична наредба и някои техни свойства. Този материал няма съществено значение за основните връзки между представените интервално-аритметични системи и читателят може да го пропусне (вж. формули (1), (16), (17), (25)–(27) и свойства **S6**, **M6**, $\tilde{\mathbf{M}}$**6**, **K5***ii*), *iii*), **K6**). Но трябва да отбележим, че освен за компютърните закръглявания интервалните релации имат значение за намирането на включвания (вътрешни и външни) на области от стойности на функции. Това е илюстрирано в края на работата (вж. твърдение 9).

2. ОБИКНОВЕНА ИНТЕРВАЛНА АРИТМЕТИКА В „ПЛЮС-МИНУС"-ОЗНАЧЕНИЯ

Ще припомним добре известната интервална аритметика за компактни интервали върху реалната права R, като при това ще въведем нов тип означения, които ще наричаме $(\pm)$-означения.

За дадени $a, b \in R, a \leq b$, нормален (собствен) интервал $[a, b]$ се дефинира като компактно подмножество на реалната права R посредством $[a, b] = \{x \mid a \leq x \leq b\}$. Числата $a, b \in R$ наричаме ляв, съответно десен край на $[a, b]$. Множеството $\{[a, b] \mid a, b \in R, a \leq b\}$ от всички интервали означаваме с $I(R)$. Елементите на $I(R)$ ще означаваме с главни букви. Левият край на $A \in I(R)$ означаваме с $A^{(-)}$, а десния край — с $A^{(+)}$, така че $A = [A^{(-)}, A^{(+)}]$. За краищата $A^{(-)}$, $A^{(+)}$ на интервала A ще използваме и означенията $a^{(-)}$, съответно $a^{(+)}$, тъй като означенията на краищата с малки букви са навлезли трайно в литературата по интервален анализ (но за съжаление се отклоняват от обичайния начин на въвеждане на означения с функционален характер). Така $a^{(s)}$ (или $A^{(s)}$), с $s \in \Lambda = \{+, -\}$ означава ляв или десен край на $A \in I(R)$ в зависимост от стойността на s. По-нататък двоичната променлива s често ще се изразява като произведение на две (или повече) двоични променливи, което дефинираме посредством $++ = -- = +$, $+- = -+ = -$. Да отбележим, че това произведение е асоциативно в Λ, т. е. за $p, q, r \in \Lambda$ имаме $(pq)r = p(qr) = pqr$.

Интервал $A = [a^{(-)}, a^{(+)}]$ с $a^{(-)} = a^{(+)}$ се нарича изроден. За изродения интервал $[a, a]$ ще пишем и просто a, напр. $[0,0] = 0$, $[1,1] = 1$ и т. н.

Нека $A = [a^{(-)}, a^{(+)}]$, $B = [b^{(-)}, b^{(+)}] \in I(R)$. Релацията включване в $I(R)$ има познатия теоретико-множествен смисъл. В термините на краищата включването се изразява посредством (по-нататък символите „$\wedge$“, „$\vee$“ означават „и“, съответно „или“)

$$(1) \qquad A \subseteq B \iff (b^{(-)} \le a^{(-)}) \wedge (a^{(+)} \le b^{(+)}), \qquad A, B \in I(R) .$$

Да означим множеството от интервали, съдържащи нула, със $Z = \{A \in I(R) \mid 0 \in A\} = \{A \mid a^{(-)} \le 0 \le a^{(+)}\}$, а множеството от интервали, за които нула е вътрешна точка, със $Z^* = \{A \in I(R) \mid a^{(-)} < 0 < a^{(+)}\}$. Множеството от интервали, които не съдържат нула, е $I(R) \setminus Z = \{A \in I(R) \mid 0 \notin A\}$; множеството от интервалите, които не съдържат нула като вътрешна точка, означаваме $I(R)^* = I(R) \setminus Z^*$. Множеството $I(R)^*$ се състои от подмножествата $I(R) \setminus Z$, $\{0\}$, $\{[-a, 0], [0, a] \mid a > 0\}$. Функционалът $\sigma : I(R)^* \to \Lambda$ се дефинира за $A \in I(R)^* \setminus \{0\}$ със $\sigma(A) = \{+,$ ако $a^{(-)} \ge 0;$ $-,$ ако $a^{(+)} \le 0\}$, а за аргумент нула — със $\sigma([0,0]) = \sigma(0) = +$; ще го наричаме знак на $A \in I(R)^*$. Знак на интервал не се дефинира за интервал от Z^*, т. е. за интервал, съдържащ едновременно и отрицателни, и положителни числа. В частност знак на изроден интервал е познатият знак на число, $\sigma([a, a]) = \mathrm{sgn}(a)$, с уговорката, че знакът на числото нула се приема за положителен. С това функционалът σ е добре дефиниран върху $I(R)^*$ и в частност върху R.

Забележка. В някои работи функционалът σ се дефинира за всички ненулеви интервали посредством $\sigma(A) = \sigma(a^{(-)} + a^{(-)})$; в тази работа няма да използуваме това разширение.

Операциите $+, -, \times, /$ се дефинират за нормални интервали посредством

$$(2) \qquad A * B = \{a * b \mid a \in A, b \in B\},\ * \in \{+, -, \times, /\},\ A, B \in I(R),$$

като в случая на „$* = /$“ се предполага, че $B \in I(R) \setminus Z$ (вж. [1, 24, 41], вж. също [37–39]). Формула (2) не е удобна за софтуерна реализация. Използувайки ($\pm$)-означения, можем да запишем интервално-аритметичните операции посредством краищата на операндите $A = [a^{(-)}, a^{(+)}]$, $B = [b^{(-)}, b^{(+)}]$. За трите операции събиране, умножение и инверсия (реципрочен интервал) имаме съответно:

$$\begin{aligned}
(3)\ A + B &= [a^{(-)} + b^{(-)}, a^{(+)} + b^{(+)}],\ A, B \in I(R),\\
(4)\ A \times B &= \begin{cases} [a^{(-\sigma(B))} b^{(-\sigma(A))}, a^{(\sigma(B))} b^{(\sigma(A))}], & A, B \in I(R)^*,\\ [a^{(\delta)} b^{(-\delta)}, a^{(\delta)} b^{(\delta)}], \quad \delta = \sigma(A), & A \in I(R)^*,\ B \in Z^*,\\ [a^{(-\delta)} b^{(\delta)}, a^{(\delta)} b^{(\delta)}], \quad \delta = \sigma(B), & A \in Z^*,\ B \in I(R)^*, \end{cases}\\
(5)\ A \times B &= [\min\{a^{(-)} b^{(+)}, a^{(+)} b^{(-)}\},\ \max\{a^{(-)} b^{(-)}, a^{(+)} b^{(+)}\}],\quad A, B \in Z^*,\\
(6)\ \ 1/B &= [1/b^{(+)}, 1/b^{(-)}],\ B \in I(R) \setminus Z.
\end{aligned}$$

Дефиницията на умножение е дадена с две отделно номерирани формули за по-лесно цитиране. В подробен запис формула (4) има вида

$$A \times B = [a^{(-)}, a^{(+)}] \times [b^{(-)}, b^{(+)}] = \begin{cases} [a^{(-)}b^{(-)}, a^{(+)}b^{(+)}], & A \geq 0, \quad B \geq 0, \\ [a^{(+)}b^{(-)}, a^{(-)}b^{(+)}], & A \geq 0, \quad B \leq 0, \\ [a^{(-)}b^{(+)}, a^{(+)}b^{(-)}], & A \leq 0, \quad B \geq 0, \\ [a^{(+)}b^{(+)}, a^{(-)}b^{(-)}], & A \leq 0, \quad B \leq 0, \\ [a^{(+)}b^{(-)}, a^{(+)}b^{(+)}], & A \geq 0, \quad B \in Z^*, \\ [a^{(-)}b^{(+)}, a^{(-)}b^{(-)}], & A \leq 0, \quad B \in Z^*, \\ [a^{(-)}b^{(+)}, a^{(+)}b^{(+)}], & A \in Z^*, \quad B \geq 0, \\ [a^{(+)}b^{(-)}, a^{(-)}b^{(-)}], & A \in Z^*, \quad B \leq 0. \end{cases}$$

В частния случай, когато A е изроден от вида $A = [a, a] = a \in R$, формула (4) дава $A \times B = a \times B = [ab^{(-\sigma(a))}, ab^{(\sigma(a))}] = \{[ab^{(-)}, ab^{(+)}]$, ако $a \geq 0$; $[ab^{(+)}, ab^{(-)}]$, ако $a < 0\}$. при умножение с число понякога ще изпускаме знака $\times$, пишейки aB вместо $a \times B$. За $a = -1$ имаме $(-1)B = -B = [-b^{(+)}, -b^{(-)}]$. Операторът $-B$ се нарича отрицание на B и се означава още с $\mathrm{neg}(B)$ (от *negation*). Дефинираните с (2) операции изваждане $A - B$ и деление A/B могат да се представят като съставни операции чрез $A - B = A + (-B)$, $A/B = A \times (1/B)$. Изразени с краищата на операндите, те имат вида

$$(7) \quad A - B = [a^{(-)} - b^{(+)}, a^{(+)} - b^{(-)}], \; A, B \in I(R),$$

$$(8) \quad A/B = \begin{cases} [a^{(-\sigma(B))}/b^{(\sigma(A))}, a^{(\sigma(B))}/b^{(-\sigma(A))}], & A \in I(R)^*, \; B \in I(R) \setminus Z, \\ [a^{(-\delta)}/b^{(-\delta)}, a^{(\delta)}/b^{(-\delta)}], \quad \delta = \sigma(B), & A \in Z^*, \; B \in I(R) \setminus Z. \end{cases}$$

Подробно записано, делението има вида

$$A/B = [a^{(-)}, a^{(+)}]/[b^{(-)}, b^{(+)}] = \begin{cases} [a^{(-)}/b^{(+)}, a^{(+)}/b^{(-)}], & A \geq 0, \quad B > 0, \\ [a^{(+)}/b^{(+)}, a^{(-)}/b^{(-)}], & A \geq 0, \quad B < 0, \\ [a^{(-)}/b^{(-)}, a^{(+)}/b^{(+)}], & A \leq 0, \quad B > 0, \\ [a^{(+)}/b^{(-)}, a^{(-)}/b^{(+)}], & A \leq 0, \quad B < 0, \\ [a^{(-)}/b^{(-)}, a^{(+)}/b^{(-)}], & A \in Z^*, \quad B > 0, \\ [a^{(+)}/b^{(+)}, a^{(-)}/b^{(+)}], & A \in Z^*, \quad B < 0. \end{cases}$$

Тъй като изваждането е съставна операция, можем да не го включваме в списъка на основните операции на така получената алгебрична система $\mathcal{S} = (I(R), +, \times, /, \subseteq)$. Операциите $+, -, \times, /$, дефинирани с (2), респ. с (3), (7), (4)–(5), (8), ще наричаме обикновени или външни интервални операции или $\mathcal{S}$-операции, а системата $\mathcal{S}$ ще наричаме обикновена или външна интервална аритметика (използуването на наименованието „външен" е мотивирано по-нататък). Свойствата на $\mathcal{S} = (I(R), +, \times, /, \subseteq)$ са добре изучени [1, 24, 29–34, 41]. Ще припомним някои от тях. Ако не е казано нещо друго, A, B, C, ... означават елементи на $I(R)$.

S1. $A + B = B + A, \quad A \times B = B \times A.$

S2. $(A+B)+C = A+(B+C), \quad (A \times B)\times C = A \times (B \times C).$

S3. $X = [0,0] = 0$ и $Y = [1,1] = 1$ са единствени неутрални елементи относно събиране, респ. умножение, т. е.

$$A = X + A \text{ за всяко } A \in I(R) \iff X = [0,0];$$
$$A = Y \times A \text{ за всяко } A \in I(R) \iff Y = [1,1].$$

S4. Никой елемент $A \in I(R)$ с $A^{(-)} \neq A^{(+)}$ не притежава обратен по отношение на операцията $+$ и никой елемент $A \in I(R)\setminus Z$ с $A^{(-)} \neq A^{(+)}$ няма обратен по отношение на $\times$. Елементите $-A$ и $1/A$ (които могат да бъдат заподозрени за обратни, но не са) удовлетворяват включванията

$$0 \in A + (-A) = A - A, \text{ респ. } 1 \in A \times (1/A) = A/A.$$

Тези включвания преминават в равенства само за дегенерирани интервали. Уравнението $A + X = B$ не винаги има единствено решение; то има единствено решение при $b^{(-)} - a^{(-)} \leq a^{(+)} - b^{(+)}$ и в този случай то е $X = [b^{(-)} - a^{(-)}, a^{(+)} - b^{(+)}]$. При $b^{(-)} - a^{(-)} > a^{(+)} - b^{(+)}$ уравнението няма решение. По аналогичен начин стои въпросът с решението на уравнението $A \times X = B$; то съществува и е единствено при определено условие за краищата на интервалите A и B.

В сила са следните закони за съкращаване:
а) за $A, B, C \in I(R)$, $A + C = B + C \Longrightarrow A = B$;
б) за $A, B \in I(R)$, $C \in I(R)^* \setminus \{0\}$, $A \times C = B \times C \Longrightarrow A = B$.

Закон за съкращаване при умножение има и в случая $A, B, C \in Z^*$.

S5. Имаме $(A+B)\times C \subseteq (A\times C)+(B\times C)$ (субдистрибутивен закон [30]). Равенство (дистрибутивност) е налице *в някои частни случаи*, измежду които ще отбележим следните:

$$\begin{aligned} c(A+B) &= cA + cB, \ c \in R, \\ (A+B)\times C &= (A\times C)+(B\times C), \end{aligned}$$

ако $A, B, C, A+B \in I(R)\setminus Z$, $\sigma(A) = \sigma(B)$.

S6. Имаме $X \subseteq X_1 \Longrightarrow X * C \subseteq X_1 * C$ за $* \in \{+,-,\times,/\}$. Като следствие се получава

$$X \subseteq X_1\,,\ Y \subseteq Y_1 \Longrightarrow X * Y \subseteq X_1 * Y_1.$$

Съединение (или свързано обединение) $[A \bigvee B]$ на два интервала A, B се дефинира посредством

$$[A \bigvee B] = [\min\{A^{(-)}, B^{(-)}\}, \max\{A^{(+)}, B^{(+)}\}].$$

В частния случай, когато интервалите A, B са изродени, т. е. $A = a$, $B = b$, $a, b \in R$, съединението $[A \bigvee B] = [a \bigvee b]$ е удобно за представяне на интервал, на който краищата a, b са известни, но евентуално не е известно кой от краищата е ляв и кой десен (такава ситуация възниква, когато краищата са стойности на функции, които не са известни предварително или още не са пресметнати).

Алгебричните системи $(I(R), +)$, $(I(R)^*, \times)$, $(I(R) \setminus Z, +, \times, /)$ имат редица несъвършенства, които ги правят непривлекателни за приложенията. Съгласно **S1**—**S4** $(I(R), +)$ и $(I(R)^*, \times)$ са комутативни полугрупи, но не са групи, тъй като не съществуват обратни елементи по отношение на операциите $+$, съответно $\times$ (вж. напр. [15, гл. 2]). Решенията на уравненията $A + X = B$, съответно $A \times X = B$, когато съществуват, не могат да бъдат изразени чрез интервалните операции (2). В $(I(R) \setminus Z, +, \times, /)$ операцията $1/B$ не може да бъде определена чрез $+$ и $\times$ (напр. не може да се твърди, че е обратна на умножението). В $(I(R)^*, +, \times)$ няма дистрибутивен закон в общия случай и въобще между операциите събиране и умножение в множеството от обикновените интервали има твърде слаба поносимост. Външните интервални операции са неприложими за получаване на вътрешни включвания (вж. т. 8, твърдение 9).

Известни са два пътя за преодоляване на споменатите недостатъци на обикновената интервална аритметика: а) чрез разширяване на множеството от операции в $I(R)$, и б) чрез разширяване на самото множество $I(R)$ и додефиниране на аритметичните операции в новото множество. Разширяването на множеството от операциите посредством специални вътрешни (нестандартни) интервални операции $+^-$, $\times^-$ води до интервално-аритметичната структура $\mathcal{M} = (I(R), +, +^-, \times, \times^-, \subseteq)$ [16–21]. При това разширяване носителят на пространството остава непроменен — това е множеството от нормалните интервали. Алтернативно чрез разширяване на множеството $I(R)$ от нормалните интервали до множеството D от насочените (обобщените) интервали и чрез подходящо разширяване на дефинициите на интервално-аритметичните операции от $I(R)$ до D получаваме интервалното пространство $\mathcal{K} = (D, +, \times)$, в което операциите $+, \times$ имат групови свойства [7–11, 21, 26]. Това разширяване е съществено по-силно от първото, защото при него се засяга носителят на пространството, а именно от нормални интервали се преминава към насочени. По-нататък ще разгледаме последователно тези две разширени интервални системи. Въвеждането на двете системи може да се направи независимо една от друга. Действително, читателят може да прочете следващите т. 3 и 4 в произволен ред. В т. 5, която използва резултати от предишните две точки, са дадени връзки между двете интервални системи $\mathcal{M}$ и $\mathcal{K}$, получени чрез едно ново т. нар. нормално представяне на насочените интервали.

3. РАЗШИРЕНА ИНТЕРВАЛНА АРИТМЕТИКА

В интервалната аритметика $\mathcal{S}$ въвеждаме специални вътрешни (или нестандартни) операции. Така полученото разширение ще наричаме разширена интервална аритметика.

Да дефинираме функционалите $\omega : I(R) \to R^+ = [0, \infty)$, $\chi : I(R)^* \setminus \{0\} \to [0, 1]$ посредством

$$\begin{aligned} \omega(A) &= a^{(+)} - a^{(-)}, \\ \chi(A) &= a^{(-\sigma(A))}/a^{(\sigma(A))} = \begin{cases} a^{(-)}/a^{(+)}, & \sigma(A) = +, \\ a^{(+)}/a^{(-)}, & \sigma(A) = -, \end{cases} \end{aligned}$$

както и $\phi : I(R) \bigotimes I(R) \to \Lambda$, $\psi : (I(R)^* \setminus \{0\}) \bigotimes (I(R)^* \setminus \{0\}) \to \Lambda$ посредством

$$\begin{aligned} \phi(A, B) &= \sigma(\omega(A) - \omega(B)) = \begin{cases} +, & \omega(A) \ge \omega(B), \\ -, & \omega(A) \ge \omega(B); \end{cases} \\ \psi(A, B) &= \sigma(\chi(A) - \chi(B)) = \begin{cases} +, & \chi(A) \ge \chi(B), \\ -, & \chi(A) \ge \chi(B). \end{cases} \end{aligned}$$

С помощта на така въведените функционали дефинираме в $I(R)$ операциите $+^-, \times^-$ посредством

$$(9) \quad A +^- B = [a^{(-\phi)} + b^{(\phi)}, a^{(\phi)} + b^{(-\phi)}], \quad \phi = \phi(A, B), \quad A, B \in I(R),$$

$$(10) \quad A \times^- B = \begin{cases} [a^{(\sigma(B)\psi)} b^{(-\sigma(A)\psi)}, a^{(-\sigma(B)\psi)} b^{(\sigma(A)\psi)}], & \psi = \psi(A, B), \\ & A, B \in I(R)^*, \\ [a^{(-\delta)} b^{(-\delta)}, a^{(-\delta)} b^{(\delta)}], & \delta = \sigma(A), \quad A \in I(R)^*, \ B \in Z^*, \\ [a^{(-\delta)} b^{(-\delta)}, a^{(\delta)} b^{(-\delta)}], & \delta = \sigma(B), \quad A \in Z^*, \ B \in I(R)^* \\ [0, 0], \ A, B \in Z^*. \end{cases}$$

Забележки. В предишни работи [16–22] за случая $A, B \in Z^*$ резултатът от операцията $A \times^- B$ е дефиниран като $[\max\{a^{(-)}b^{(+)}, a^{(+)}b^{(-)}\}, \min\{a^{(-)}b^{(-)}, a^{(+)}b^{(+)}\}]$. При това полагане обаче твърдение 9 не е вярно за $A, B \in Z^*$. Във формула (10) в случая $A, B \in I(R)^*$ (първия ред) може да възникне неяснота, когато някой от интервалите A, B е нулев (поради това, че функционалите χ, съответно ψ, не са дефинирани за нулеви интервали). В тези случаи резултатът е нула, тъй като краищата са нули. Понякога е удобно функционалът χ да се додефинира в $Z^* \setminus \{0\}$, като в дефиниционната му формула $\sigma(A)$ се замени с $\sigma(a^{(-)} + a^{(+)})$. По този начин функционалът χ се дефинира за всеки интервал от $I(R) \setminus \{0\}$, т. е. χ остава дефиниран само за нулевия интервал $[0, 0] = 0$. В тази работа няма да се налага да използуваме такова разширение на дефиниционната област на χ. Читателят ще забележи дуалност между резултатите, отнасящи се до събиране, съответно умножение, при която функционалите ω и χ са съответни един на друг. В тези резултати има малка несиметрия, която може да се премахне, ако функционалът ω се дефинира като $\omega = a^{(-)} - a^{(+)}$, което обаче не е прието в литературата по интервален анализ.

Твърдение 1. *Изразите в десните страни на* (9), (10) *са интервали от* $I(R)$, *т. е. първите им компоненти не надминават съответните втори компоненти и действително са леви, съответно десни краища на интервали.*

Доказателство. Това е очевидно за изразите, в които не участват ϕ и ψ. Остава да покажем, че в изразите за $+^-$, $\times^-$, дефинирани със:

$$(11)\quad A +^- B = [a^{(-\gamma)} + b^{(\gamma)}, a^{(\gamma)} + b^{(-\gamma)}],\quad A, B \in I(R),$$
$$(12)\quad A \times^- B = [a^{(\varepsilon\sigma(B))} b^{(-\varepsilon\sigma(A))}, a^{(-\varepsilon\sigma(B))} b^{(\varepsilon\sigma(A))}],\quad A, B \in I(R)^*,$$

двоичните променливи $\gamma, \varepsilon \in \Lambda$ могат да се изберат по такъв начин, че десните страни да са елементи на $I(R)$, т. е. $a^{(-\gamma)} + b^{(\gamma)} \leq a^{(\gamma)} + b^{(-\gamma)}$, $a^{(\varepsilon\sigma(B))} b^{(-\varepsilon\sigma(A))} \leq a^{(-\varepsilon\sigma(B))} b^{(\varepsilon\sigma(A))}$. От тези условия ще изразим γ, ε посредством ϕ и ψ. Като използуваме, че за $A, B \in I(R) \setminus Z$ неравенствата $\chi(A) \geq \chi(B)$ и $a^{(\sigma(B))} b^{(-\sigma(A))} \leq a^{(-\sigma(B))} b^{(\sigma(A))}$ са еквивалентни (!), получаваме, че γ, ε в (11)—(12) се представят с $\gamma = \phi(A, B)$, $\varepsilon = \psi(A, B)$, както това се изисква в (9), (10). □

Ако редът на краищата не ни е нужен, можем да използуваме представяне с помощта на съединение. Тогава формули (11), (12) добиват вида

$$A +^- B = [(a^{(-)} + b^{(+)}) \vee (a^{(+)} + b^{(-)})],\ A, B \in I(R),$$
$$A \times^- B = [(a^{(\sigma(B))} b^{(-\sigma(A))}) \vee (a^{(-\sigma(B))} b^{(\sigma(A))})],\ A, B \in I(R)^*.$$

Интервално-аритметичната структура $\mathcal{M} = (I(R), +, +^-, \times, \times^-, \subseteq)$ е разширение на интервалната аритметика $\mathcal{S} = (I(R), +, \times, /, \subseteq)$; ще я наричаме разширена интервална аритметика [16–22]. Да отбележим, че $A +^- (-A) = 0$, $A \times^- (1/A) = 1$, което означава, че $-A = [-a^{(+)}, -a^{(-)}]$ и $1/A = [1/a^{(+)}, 1/a^{(-)}]$ са обратни елементи съответно относно операциите $+^-$ и $\times^-$. Ще припомним, че операторът $1/A$ не може да се определи с помощта на основните интервално-аритметични операции „+" и „×" и следователно трябва да се разглежда като основен в $\mathcal{S}$, но същият оператор $1/A$ не е основен в $\mathcal{M}$, тъй като $1/A$ може да се определи като обратния на A по отношение на основната операция $\times^-$. Алгебричната структура $\mathcal{M}$, дефинирана посредством (3)–(5), (9)–(10), позволява въвеждане на операции, определени чрез основните аритметичните операции $+, +^-, \times, \times^-$ и чрез обратните елементи $-A$ и $1/A$ по отношение на операциите $+^-$ и $\times^-$. Такива са съставните операции $A - B = A + (-B)$ и $A/B = A \times (1/B)$, дефинирани посредством (7), съответно (8). Можем да дефинираме и съставните операции $A -^- B = A +^- (-B)$ и $A/^- B = A \times^- (1/B)$, които се изразяват чрез краищата на операндите посредством

$$(13)\quad A -^- B = [a^{(-\phi)} - b^{(-\phi)}, a^{(\phi)} - b^{(\phi)}],\qquad \phi = \phi(A, B),\ A, B \in I(R),$$

$$(14)\quad A/^- B = \begin{cases} [a^{(\sigma(B)\psi)}/b^{(\sigma(A)\psi)}, a^{(-\sigma(B)\psi)}/b^{(-\sigma(A)\psi)}],\ \psi = \psi(A, B), \\ \qquad\qquad A \in i(R)^*,\ B \in I(R) \setminus Z, \\ [a^{(-\delta)}/b^{(\delta)}, a^{(\delta)}/b^{(\delta)}],\quad \delta = \sigma(B),\quad A \in Z^*,\ B \in I(R) \setminus Z. \end{cases}$$

Като използуваме операцията „съединение", можем да запишем изразите, в които участват променливите ϕ и ψ, в следната по-обозрима форма:

$$A -^- B = [(a^{(-)} - b^{(-)}) \vee (a^{(+)} - b^{(+)})],\quad A, B \in I(R),$$
$$A/^- B = [(a^{(\sigma(B))}/b^{(\sigma(A))}) \vee (a^{(-\sigma(B))}/b^{(-\sigma(A))})],\quad A, B \in I(R) \setminus Z.$$

Четирите основни операции (3)–(5), (9), (10) заедно с четирите съставни операции (7), (8), (13), (14) дават осем интервално-аритметични операции в $\mathcal{M}$, които ще наричаме $\mathcal{M}$-операции. Четири от тези $\mathcal{M}$-операции са $\mathcal{S}$-операциите $+, -, \times, /$, които наричаме още външни интервални операции. Останалите четири операции $+^-, \times^-, -^-, /^-$, дефинирани съответно с (9), (10), (13), (14), ще наричаме вътрешни (това наименование се използува напр. в [40], друго използувано наименование е „специални" [2] или „нестандартни" [21]. Ще припомним, че $\mathcal{S}$-операцията „$/$" може да се определи чрез операцията „$\times$" и оператора „обратен елемент" $1/A$ относно „$\times^-$" (аналогично на случая с изваждането „$-$"). Това ни позволява да записваме алгебричната структура $\mathcal{M}$ във вида $(I(R), +, +^-, \times, \times^-, \subseteq)$, при който $\mathcal{S}$-изваждането и $\mathcal{S}$-делението се изключват от списъка на основните операции на $\mathcal{M}$.

В някои случаи е възможна следната нагледна интерпретация на операциите в разширената интервална аритметика. Нека „$*$" е коя да е от четирите числови операции $* \in \{+, -, \times, /\}$ и $A = [a^{(-)}, a^{(+)}]$, $B = [b^{(-)}, b^{(+)}] \in I(R)$. Като извършим числовата операция „$*$" между край на единия интервал и край на другия интервал, получаваме четирите реални числа $a^{(-)} * b^{(-)}, a^{(-)} * b^{(+)}, a^{(+)} * b^{(-)}, a^{(+)} * b^{(+)}$. Да предположим, че тези числа са две по две различни помежду си. Да ги подредим по големина, означавайки ги с c_1, c_2, c_3, c_4, където $c_1 < c_2 < c_3 < c_4$. Тогава външният интервал $[c_1, c_4]$ е резултатът от съответната външна интервална операция върху интервалите A, B, т. е. $A * B = [c_1, c_4]$, а вътрешният интервал $[c_2, c_3]$ е резултатът от съответната вътрешна интервална операция, т. е. $A *^- B = [c_2, c_3]$. Изключение от това правило е случаят на умножение на два интервала, които едновременно съдържат нула във вътрешността си; в този случай вътрешното произведение съгласно (10) е нулев интервал.

На двете $\mathcal{M}$-операции за събиране може да се гледа като на една „двумодусна" операция $+^\theta$ с параметър $\theta \in \Lambda$, който определя типа (модуса) на операцията (външен/вътрешен) по формулата

$$A +^\theta B = [(a^{(-)} + b^{(-\theta)}) \bigvee (a^{(+)} + b^{(\theta)})], \ A, B \in I(R).$$

Аналогично на двете $\mathcal{M}$-операции за умножение може да се гледа като на една операция $\times^\theta$ с параметър $\theta \in \Lambda$, която за $A, B \in I(R) \setminus Z$ може да се запише във вида

$$A \times^\theta B = [(a^{(-\sigma(B))} b^{(-\theta\sigma(A))}) \bigvee (a^{(\sigma(B))} b^{(\theta\sigma(A))})], \ A, B \in I(R)^*.$$

При тези означения можем да записваме системата $\mathcal{M}$ още във вида $\mathcal{M} = (I(R), +^\theta, \times^\theta, \subseteq)$.

Забележка. Възможно е в $\mathcal{M}$ да се вземе друг набор от основни операции, например $+, -^-, \times$ и $/^-$. В този случай операциите $+^-, \times^-$, които приехме за основни, ще се определят чрез избраните за основни операции $+, -^-, \times, /^-$ посредством $A +^- B = A -^- (-B)$, $A \times^- B = A /^- (1/B)$. В [16–20] взетите за основни операции $-^-$ и $/^-$ се означават съответно с „$-$" и „$/$", а производните операции $+^-, -, \times^-$ и $/$ се означават съответно с $\oplus, \ominus, \otimes$ и $\oslash$. Един недостатък на този подход е, че означенията „$-$" и „$/$" съответно за $\mathcal{S}$-изваждането и $\mathcal{S}$-делението са широко възприети в литературата по интервален анализ и има опасност от объркване.

Съгласно дефинираното в началото на т. 2 произведение на двоични символи имаме $+^{++} = +^{--} = +^{+} = +$, $+^{+-} = +^{-+} = +^{-}$, $\times^{++} = \times^{--} = \times^{+} = \times$, $\times^{+-} = \times^{-+} = \times^{-}$. За $A \in I(R)^*$ ще означаваме $|A| = \sigma(A)A = \{A,$ ако $\sigma(A) = +; -A,$ ако $\sigma(A) = -\}$.

Освен релациите **S1**—**S6** в $\mathcal{M}$ в сила са следващите закони **M1**—**M7**. Непосредствената проверка на тези закони е елементарна, макар и в някои от случаите доста трудоемка. В т. 6 даваме просто извеждане на някои от формулираните по-долу твърдения.

M1. За $A, B \in I(R)$ имаме $A +^{-} B = B +^{-} A$, $A \times^{-} B = B \times^{-} A$.

M2 (условно асоциативен закон за „$+^{\theta}$"). За $A, B, C \in I(R)$ имаме

$$\begin{aligned}
(A + B) +^{-} C &= A +^{\phi(B,C)} (B +^{-} C); \\
(A +^{-} B) + C &= \begin{cases} A +^{-\phi(B,C)} (B +^{-} C), & \omega(A) \geq \omega(B), \\ A +^{-} (B + C), & \omega(A) < \omega(B); \end{cases} \\
(A +^{-} B) +^{-} C &= \begin{cases} A +^{-\phi(B,C)} (B +^{-} C), & \omega(A) < \omega(B), \\ A +^{-} (B + C), & \omega(A) \geq \omega(B). \end{cases}
\end{aligned}$$

Условно асоциативен закон за „$\times^{\theta}$". За $A, B, C \in I(R) \setminus Z$ имаме

$$\begin{aligned}
(A \times B) \times^{-} C &= A \times^{-\psi(B,C)} (B \times C); \\
(A \times^{-} B) \times C &= \begin{cases} A \times^{-\psi(B,C)} (B \times^{-} C), & \chi(A) \leq \chi(B), \\ A \times^{-} (B \times C), & \chi(A) > \chi(B); \end{cases} \\
(A \times^{-} B) \times^{-} C &= \begin{cases} A \times^{-\psi(B,C)} (B \times^{-} C), & \chi(A) > \chi(B), \\ A \times^{-} (B \times C), & \chi(A) \leq \chi(B). \end{cases}
\end{aligned}$$

M3. $X = [0,0] = 0$ и $Y = [1,1] = 1$ **са единствени неутрални елементи относно вътрешните $\mathcal{M}$-операции за събиране и умножение, т. е.**
$A = X +^{-} A$ за всяко $A \in I(R)$ $\iff X = [0,0]$,
$A = Y \times^{-} A$ за всяко $A \in I(R)$ $\iff Y = [1,1]$.

M4. Всеки елемент $A \in I(R)$ има единствен обратен относно $+^{-}$ и всеки елемент $A \in I(R) \setminus Z$ има единствен обратен относно $\times^{-}$. Това са елементите $-A$, съответно $1/A$, т. е. $0 = A +^{-} (-A) = A -^{-} A$ и $1 = A \times^{-} (1/A) = A /^{-} A$. За $\lambda \in \Lambda$ уравнението $A +^{\lambda} X = B$ има единствено решение при $\omega(A) \leq \omega(B)$, то е $X = B -^{(-\lambda)} A$. При $\omega(A) > \omega(B)$ единствено решение не е налице — може да има две решения или нито едно. По-точно при $\omega(A) > \omega(B)$ уравнението $A + X = B$ няма решение, а уравнението $A +^{-} X = B$ има две решения: $X = B - A$ и $X = B -^{-} A$.

Аналогично уравнението $A \times^\lambda X = B$ при $A, B \in I(R) \setminus Z$, $\lambda \in \Lambda$, има единствено решение при $\chi(A) \geq \chi(B)$, то е $X = B/^{-\lambda}A$. При $\chi(A) < \chi(B)$ единствено решение не е налице – може да има две решение или нито едно. По-точно при $\chi(A) < \chi(B)$ уравнението $A \times X = B$ няма решение, а уравнението $A \times^- X = B$ има две решения: $X = B/A$ и $X = B/^- A$.

В сила е условен закон за съкращаване, който за вътрешното събиране гласи:

$$A +^- C = A +^- D, \quad \omega(C) \leq \omega(A), \quad \omega(D) \leq \omega(A) \Longrightarrow C = D,$$

където $\leq$ може да се замени с $\geq$. В общия случай при липса на допълнително условие от типа $\omega(C) \leq \omega(A)$, $\omega(D) \leq \omega(A)$ закон за съкращаване не е в сила, тъй като при произволно $B \in I(R)$ интервалите $C = B - A$, $D = B -^- A$ удовлетворяват $A +^- C = A +^- D$ и при това $\omega(C) > \omega(A)$, $\omega(D) < \omega(A)$ при $A \neq 0$.

M5 (**условно-дистрибутивен закон**). За $A, B, C, A + B \in I(R) \setminus Z$ имаме следните равенства:

$$(A + B) \times C = \begin{cases} (A \times C) + (B \times C), \ \sigma(A) = \sigma(B); \\ (A \times C) +^{\psi(C,B)} (B \times^- C), \\ \qquad \sigma(A) = -\sigma(B) = \sigma(A + B), \\ (A \times^- C) +^{\psi(C,A)} (B \times C), \\ \qquad \sigma(A) = -\sigma(B) = -\sigma(A + B); \end{cases}$$

$$(A + B) \times^- C = \begin{cases} (A \times^- C) +^{\psi(A,C)\psi(B,C)} (B \times^- C), \ \sigma(A) = \sigma(B); \\ (A \times^- C) +^{\psi(C,A)} (B \times C), \\ \qquad \sigma(A) = -\sigma(B) = \sigma(A + B), \\ (A \times C) +^{\psi(C,B)} (B \times^- C), \\ \qquad \sigma(A) = -\sigma(B) = -\sigma(A + B); \end{cases}$$

$$(A +^- B) \times C = \begin{cases} (A \times C) +^{-\psi(C,B)} (B \times^- C), \ \sigma(A) = \sigma(B), \\ \qquad \omega(A) \geq \omega(B), \\ (A \times^- C) +^{\psi(C,A)} (B \times C), \ \sigma(A) = \sigma(B), \\ \qquad \omega(A) < \omega(B), \\ (A \times C) +^- (B \times C), \ \sigma(A) = -\sigma(B), \\ \qquad \xi(A, B) \geq 0, \\ (A \times^- C) +^{\psi(C,A)\psi(C,B)} (B \times^- C), \ \sigma(A) = -\sigma(B), \\ \qquad \xi(A, B) < 0; \end{cases}$$

$$(A +^- B) \times^- C \;=\; \begin{cases} (A \times^- C) +^{-\psi(C,A)} (B \times C), & \sigma(A) = \sigma(B), \\ & \omega(A) \geq \omega(B), \\ (A \times C) +^{\psi(C,B)} (B \times^- C), & \sigma(A) = \sigma(B), \\ & \omega(A) < \omega(B), \\ (A \times^- C) +^{-\psi(C,A)\psi(C,B)} (B \times^- C), & \sigma(A) = -\sigma(B), \\ & \xi(A,B) \geq 0, \\ (A \times C) +^- (B \times^- C), & \sigma(A) = -\sigma(B),\ \xi(A,B) < 0, \end{cases}$$

където $\xi(A,B) = \sigma(A \times (A +^- B))\phi(A,B)$.

В частност в $\mathcal{M}$ е в сила $a(B +^- C) = aB +^- aC$, $a \in R$, което комбинирано с $a(B + C) = aB + aC$, $a \in R$, дава

$$a(B +^\theta C) = aB +^\theta aC,\ a \in R,\ \theta \in \Lambda,$$

Като друг частен случай получаваме

$$(a+b)C = (aC) +^{\sigma(a)\sigma(b)} (bC) = \begin{cases} (aC) + (bC), & \sigma(a) = \sigma(b); \\ (aC) +^- (bC), & \sigma(a) = -\sigma(b). \end{cases}$$

M6. Нека $* \in \{+^-, -^-\}$ и $X, X_1, Y, Y_1 \in I(R)$. Предполагаме $X \supseteq X_1$, $Y \subseteq Y_1$ и получаваме:

$$\text{ако } \omega(X) \leq \omega(Y), \text{ то } X * Y \subseteq X_1 * Y_1;$$
$$\text{ако } \omega(X_1) \geq \omega(Y_1), \text{ то } X * Y \supseteq X_1 * Y_1.$$

Нека $* \in \{\times^-, /^-\}$ и $X, X_1, Y, Y_1 \in I(R) \setminus Z$ са такива, че $X \subseteq X_1$, $Y \subseteq Y_1$. При тези предположения имаме:

$$\text{ако } \min\{\chi(X), \chi(X_1)\} \geq \max\{\chi(Y), \chi(Y_1)\}, \text{ то } X * Y \subseteq X_1 * Y_1;$$
$$\text{ако } \max\{\chi(X), \chi(X_1)\} \leq \min\{\chi(Y), \chi(Y_1)\}, \text{ то } X * Y \subseteq X_1 * Y_1.$$

Понякога е удобно да се използва релацията $\preceq$, дефинирана посредством

(15) $$A \preceq B \iff (a^{(-)} \leq b^{(-)}) \wedge (a^{(+)} \leq b^{(+)}),\quad A, B \in I(R).$$

Релацията $\preceq$ удовлетворява следните свойства, аналогични на свойствата **S6**, **M6**, отнасящи се до релацията $\subseteq$.

$\widetilde{\mathbf{M6}}$. Нека $* \in \{+, +^-\}$, $X, X_1, Y, Y_1 \in I(R)$. Тогава $X \preceq X_1$, $Y \preceq Y_1 \Rightarrow X * Y \preceq X_1 * Y_1$.
Нека $* \in \{-, -^-\}$, $X, X_1, Y, Y_1 \in I(R)$. Тогава $X \preceq X_1$, $Y_1 \preceq Y \Rightarrow X * Y \preceq X_1 * Y_1$.
Нека $* \in \{\times, \times^-\}$, $X, X_1, Y, Y_1 \in I(R)^*$. Тогава $|X| \preceq |X_1|$, $|Y| \preceq |Y_1| \Rightarrow |X * Y| \preceq |X_1 * Y_1|$.
Нека $* \in \{/, /^-\}$ и $X, X_1, Y, Y_1 \in I(R) \setminus Z$. Тогава $|X| \preceq |X_1|$, $|Y_1| \preceq |Y| \Rightarrow |X * Y| \preceq |X_1 * Y_1|$.

M7. За $A, B \in I(R)$ имаме $A *^- B \subseteq A * B$, $* \in \{+, -, \times, /\}$. Равенство е налице точно тогава, когато поне единият от интервалите A, B е изроден. Това оправдава изпускането на знаците $\times, \times^-$ при умножение на интервал с число.

Ако $A \preceq B$, ще записваме съединението $C = [A \bigvee B]$ още във вида $C = [A, B]$ и ще казваме, че A, B са ляв, съответно десен (интервален) край на C.

Представяне в CR-форма. Използваното до този момент представяне на интервалите с помощта на техните краища ще наричаме EP-представяне (от англ. *end-point*). Ще използуваме и представяне с центрове и радиуси, което ще наричаме CR-представяне за нормални интервали.

Да означим центъра и радиуса на $A \in I(R)$ съответно с $c(A)$ и $\rho(A)$, т. е. $c(A) = (a^{(-)} + a^{(+)})/2$, $\rho(A) = (a^{(+)} - a^{(-)})/2$. Преходът от CR-представяне $A = (c(A), \rho(A))$ в EP-представяне става посредством $a^{(-)} = c(A) - \rho(A)$, $a^{(+)} = c(A) + \rho(A)$. Всяка двойка (c, ρ) с $\rho \geq 0$ представя по единствен начин един нормален интервал в CR-форма. По-нататък ще дадем формулите за аритметичните операции в CR-форма [4]. Ако не е казано друго, A, B са елементи на $I(R)$. Използвани са и следните означения: $|c(A)| = \sigma(A)c(A)$, $\overline{\rho}(A) = \sigma(A)\rho(A)$. Имаме представянията:

$$\begin{aligned}
A + B &= (c(A) + c(B), \rho(A) + \rho(B)),\\
A - B &= (c(A) - c(B), \rho(A) + \rho(B)),\\
A \times B &= (c(A)c(B) + \overline{\rho}(A)\overline{\rho}(B), |c(A)|\rho(B) + |c(B)|\rho(A)),\ A, B \in I(R)^*,\\
A/B &= (\delta^2(c(A)c(B) + \overline{\rho}(A)\overline{\rho}(B)), (\delta^2(|c(A)|\rho(B) + |c(B)|\rho(A))),\\
&\quad A \in I(R)^*, B \in I(R) \setminus Z,
\end{aligned}$$

където тук и по-нататък $\delta^2 = \delta^2(B) = (c^2(B) - \rho^2(B))^{-1}$.

За вътрешните операции имаме представянията [4, 6]:

$$\begin{aligned}
A +^- B &= (c(A) + c(B), |\rho(A) - \rho(B)|),\\
A -^- B &= (c(A) - c(B), |\rho(A) - \rho(B)|),\\
A \times^- B &= (c(A)c(B) - \overline{\rho}(A)\overline{\rho}(B), ||c(A)|\rho(B) - |c(B)|\rho(A)|),\ A, B \in I(R)^*,\\
A/^- B &= (\delta^2(c(A)c(B) - \overline{\rho}(A)\overline{\rho}(B)), (\delta^2||c(A)|\rho(B) - |c(B)|\rho(A)|),\\
&\quad A \in I(R)^*, B \in I(R) \setminus Z.
\end{aligned}$$

В CR-запис е удобно да разглеждаме двете операции $+^\theta$, $\theta = \pm$, като една операция-функция с параметър двоичната променлива $\theta \in \Lambda$, която задава двата възможни начина на изпълнение на операцията: външен при $\theta = +$ и вътрешен при $\theta = -$. Като гледаме по подобен начин и на останалите операции, можем да резюмираме формулите за външните и вътрешните интервални операции, както следва:

$$\begin{aligned}
A +^\theta B &= (c(A) + c(B), |\rho(A)\ \theta\ \rho(B)|),\\
A -^\theta B &= (c(A) - c(B), |\rho(A)\ \theta\ \rho(B)|),\\
A \times^\theta B &= (c(A)c(B)\ \theta\ \overline{\rho}(A)\overline{\rho}(B), ||c(A)|\rho(B)\ \theta\ |c(B)|\rho(A)|),\ A, B \in I(R)^*,\\
A/^\theta B &= (\delta^2(c(A)c(B)\ \theta\ \overline{\rho}(A)\overline{\rho}(B)), (\delta^2||c(A)|\rho(B)\ \theta\ |c(B)|\rho(A)|),\\
&\quad A \in I(R)^*, B \in I(R) \setminus Z,
\end{aligned}$$

където $\theta = \pm$ в десните страни означава числова аритметична операция (събиране/изваждане).

Да изразим съотношенията $\subseteq$ и $\preceq$ в CR-форма. Имаме

$$A \subseteq B \Longleftrightarrow |c(B) - c(A)| \leq \rho(B) - \rho(A). \tag{16}$$

Наистина

$$\begin{aligned} A \subseteq B &\Longleftrightarrow b^{(-)} \leq a^{(-)} \wedge a^{(+)} \leq b^{(+)} \\ &\Longleftrightarrow \begin{cases} c(B) - \rho(B) \leq c(A) - \rho(A) \\ c(A) + \rho(A) \leq c(B) + \rho(B) \end{cases} \\ &\Longleftrightarrow \begin{cases} c(B) - c(A) \leq \rho(B) - \rho(A) \\ c(A) - c(B) \leq \rho(B) - \rho(A) \end{cases} \\ &\Longleftrightarrow |c(A) - c(B)| \leq \rho(B) - \rho(A). \end{aligned}$$

От свойство (16) следва $\rho(B) - \rho(A) \geq 0$, т. е. $A \subseteq B \Longrightarrow \rho(A) \leq \rho(B)$.

Аналогично за релацията $\preceq$ имаме

$$A \preceq B \Longleftrightarrow |\rho(B) - \rho(A)| \leq c(A) - c(B). \tag{17}$$

Наистина

$$\begin{aligned} A \preceq B &\Longleftrightarrow a^{(-)} \leq b^{(-)} \wedge a^{(+)} \leq b^{(+)} \\ &\Longleftrightarrow \begin{cases} c(A) - \rho(A) \leq c(B) - \rho(B) \\ c(A) + \rho(A) \leq c(B) + \rho(B) \end{cases} \\ &\Longleftrightarrow \begin{cases} c(A) - c(B) \leq \rho(A) - \rho(B) \\ c(A) - c(B) \leq \rho(B) - \rho(A) \end{cases} \\ &\Longleftrightarrow c(B) - c(A) \geq |\rho(A) - \rho(B)|. \end{aligned}$$

От свойство (17) следва $c(B) - c(A) \geq 0$, т. е. $A \preceq B \Longrightarrow c(A) \leq c(B)$. Свойства (16) и (17) се получават едно от друго чрез формална размяна на c и ρ.

Още съотношения в разширената интервална аритметика ще бъдат получени в т. 6 с помощта на алгебричната система на насочените интервали, която ще разгледаме в следващата точка.

4. НАСОЧЕНИ ИНТЕРВАЛИ В ПОКОМПОНЕНТНА ФОРМА

Алгебричните системи $(I(R), +)$ и $(I(R), \times)$ са комутативни полугрупи, като в първата има закон за съкращаване, а във втората може да се отдели подполугрупа със съкращаване. Поради това тези системи могат да бъдат вложени в групи (вж. [15, с. 62]). Така получените влагания са изследвани основно от Е. Каухер [9–11]. По-нататък представяме накратко някои резултати на Е. Каухер, като използваме отново $(\pm)$-означения.

Ще разширим множеството $I(R)$ до множеството $D = \{[a, b] \mid a, b \in R\}$ от наредените двойки реални числа, като по този начин въвеждаме и „интервали", на които „левият край" е по-голям от „десния". За да не става объркване със случая на обикновени интервали, елементите на D ще наричаме насочени (или обобщени) интервали, а „краищата" на насочените интервали ще наричаме компоненти. Първата компонента на $\mathbf{A} \in D$ ще означаваме с a^- или с $\mathbf{A}^-$, а втората — с a^+ или с $\mathbf{A}^+$, така че $\mathbf{A} = [a^-, a^+] = [\mathbf{A}^-, \mathbf{A}^+]$. Липсата на скобки, заграждащи $+$ и $-$ в горните индекси, ще означава, че неравенството $a^- \leq a^+$ вече не е задължително (за разлика от $a^{(-)} \leq a^{(+)}$, което е задължително за краищата на собствените интервали). Използуваната форма на представяне наричаме покомпонентна или *CW*-форма (от англ. *component-wise*). На насочения интервал $\mathbf{A} = [a^-, a^+] \in D$ съпоставяме двоичната променлива „посока" $\tau(\mathbf{A})$, дефинирана с $\tau(\mathbf{A}) = \sigma(a^+ - a^-) = \{+, \text{ ако } a^- < a^+;\ -, \text{ ако } a^- > a^+\}$. Съгласно дадената дефиниция изродените интервали (тези с $a^- = a^+$) имат положителна посока. Множеството от всички елементи на D, които са с положителна посока, т. е. множеството от собствените интервали, е еквивалентно на $I(R)$ (поради което ще го означаваме отново с $I(R)$); множеството от интервалите с отрицателна посока, които ще наричаме несобствени, ще означаваме с $\overline{I(R)}$, така че $D = I(R) \cup \overline{I(R)}$. На всеки насочен интервал $\mathbf{A} = [a^-, a^+] \in D$ съпоставяме собствения интервал $p(\mathbf{A}) = \{[a^-, a^+],$ ако $\tau(\mathbf{A}) = +;\ [a^+, a^-],$ ако $\tau(\mathbf{A}) = -\}$; имаме $p(\mathbf{A}) = [a^{-\tau(\mathbf{A})}, a^{\tau(\mathbf{A})}]$. Вместо $p(\mathbf{A})$ ще пишем понякога просто A, ако това не води до неяснота. Интервала A ще наричаме собствена (нормална) част на насочения интервал $\mathbf{A}$.

Ще разширим дефиниционните области на операциите $+, \times$ от $I(R)$ в D. Да означим

$$\begin{aligned}
\overline{Z} &= \{\mathbf{A} \in \overline{I(R)} \mid a^+ \leq 0 \leq a^-\}, \\
\overline{Z}^* &= \{\mathbf{A} \in \overline{I(R)} \mid a^+ < 0 < a^-\}, \\
\mathcal{T} &= Z \cup \overline{Z} = \{\mathbf{A} \in D \mid 0 \in p(\mathbf{A})\} \\
&= \{\mathbf{A} \in D \mid (a^- < 0 < a^+) \vee (a^+ < 0 < a^-)\}.
\end{aligned}$$

С $\mathcal{T}^*$ означаваме насочените интервали, които съдържат както отрицателни, така и положителни числа, т. е.

$$\mathcal{T}^* = Z^* \cup \overline{Z}^* = \{\mathbf{A} \in \mathcal{T} \mid (a^- < 0 < a^+) \vee (a^+ < 0 < a^-)\}.$$

В $D^* = D \setminus \mathcal{T}^*$ дефинираме знак на насочен интервал $\sigma : D^* \to \{+, -\}$ посредством $\sigma(\mathbf{A}) = \sigma(A) = \{+, \text{ ако } a^- \geq 0, a^+ \geq 0\ ;\ -, \text{ ако } a^- \leq 0,\ a^+ \leq 0,\ a^- + a^+ \neq 0\}$. (Ще отбележим, че от $\mathbf{A} \in D^*$ следва $A = p(\mathbf{A}) \in I(R)^*$, а за интервали от $I(R)^*$ знакът σ е дефиниран.)

Формалното заместване на краищата (леви/десни) с компонентите (първи/втори), както и на $I(R)^*$ с D^* и на Z^* с $\mathcal{T}^*$ в (3)–(4) дава формули за операциите $+$, $\times$ в D:

$$\mathbf{A} + \mathbf{B} = [a^- + b^-, a^+ + b^+], \quad \mathbf{A}, \mathbf{B} \in D, \tag{18}$$

$$\mathbf{A} \times \mathbf{B} = \begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)}, a^{\sigma(B)} b^{\sigma(A)}], & \mathbf{A}, \mathbf{B} \in D^*, \\ [a^{\delta} b^{-\delta}, a^{\delta} b^{\delta}], \quad \delta = \sigma(A), & \mathbf{A} \in D^*,\ \mathbf{B} \in \mathcal{T}^*, \\ [a^{-\delta} b^{\delta}, a^{\delta} b^{\delta}], \quad \delta = \sigma(B), & \mathbf{A} \in \mathcal{T}^*,\ \mathbf{B} \in D^*. \end{cases} \tag{19}$$

Завършваме разширяването на дефиницията на $\mathbf{A} \times \mathbf{B}$ за случая, когато $\mathbf{A}, \mathbf{B} \in \mathcal{T}^*$, като положим (срв. [9, 11]):

$$(20) \qquad \mathbf{A} \times \mathbf{B} = \begin{cases} [\min\{a^- b^+, a^+ b^-\}, \max\{a^- b^-, a^+ b^+\}], & \mathbf{A}, \mathbf{B} \in Z^*, \\ [\max\{a^- b^-, a^+ b^+\}, \min\{a^- b^+, a^+ b^-\}], & \mathbf{A}, \mathbf{B} \in \overline{Z}^*, \\ 0, \quad \mathbf{A} \in Z^*, \ \mathbf{B} \in \overline{Z}^* \quad \text{или} \quad \mathbf{A} \in \overline{Z}^*, \ \mathbf{B} \in Z^* . \end{cases}$$

Формули (18)–(20) дефинират алгебричната система $\mathcal{K} = (D, +, \times)$, която ще наричаме насочено (обобщено) интервално пространство или насочена (обобщена) интервална аритметика.

От (19) за $\mathbf{A} = [a, a] = a$, $\mathbf{B} \in D$ имаме $a \times \mathbf{B} = a\mathbf{B} = [ab^{-\sigma(a)}, ab^{\sigma(a)}]$. Полагайки $a = -1$, получаваме $(-1)\mathbf{B} = [-b^+, -b^-]$. Оператора $(-1)\mathbf{B}$ ще наричаме отрицание на $\mathbf{B}$ и ще го означаваме с $-\mathbf{B}$ или $\mathrm{neg}(\mathbf{B})$. Очевидно $-\mathbf{B}$ е разширение на $\mathcal{S}$-оператора отрицание в D. Съставната операция $\mathbf{A} + (-1)\mathbf{B} = \mathbf{A} + (-\mathbf{B}) = [a^- - b^+, a^+ - b^-]$ за $\mathbf{A}, \mathbf{B} \in D$ е разширение на $\mathcal{S}$-изваждането в D и ще се означава и занапред с $\mathbf{A} - \mathbf{B}$ както в (7).

Системите $(D, +)$ и $(D \setminus \mathcal{T}, \times)$ са групи [10]; в тях съществуват обратни елементи. Да означим обратния адитивен елемент на $\mathbf{A} \in D$ с $-_h\mathbf{A}$, а обратния елемент на $\mathbf{A} \in D \setminus \mathcal{T}$ по отношение на операцията „$\times$“ с $1/_h\mathbf{A}$. За обратните елементи имаме покомпонентните представяния $-_h\mathbf{A} = [-a^-, -a^+]$ за $\mathbf{A} \in D$ и $1/_h\mathbf{A} = [1/a^-, 1/a^+]$ за $\mathbf{A} \in D \setminus \mathcal{T}$.

Обратният адитивен елемент $-_h\mathbf{A}$ не трябва да се смесва с отрицанието на $\mathbf{A}$, т. е. с $-\mathbf{A} = (-1) \times \mathbf{A} = [-a^+, -a^-]$. Използувайки операторите $-\mathbf{A} = [-a^+, -a^-]$ и $-_h\mathbf{A} = [-a^-, -a^+]$, образуваме оператора

$$(21) \qquad \overline{\mathbf{A}} = -_h(-\mathbf{A}) = -(-_h\mathbf{A}) = [a^+, a^-],$$

който наричаме спрягане. Операторът спрягане в D е съставен оператор, производен на операциите $+, \times$ и техните обратни. Ще означаваме оператора спрягане по три възможни начина: $\overline{\mathbf{A}} = \mathbf{A}_- = \mathrm{con}(\mathbf{A})$. Операторите „спрягане“ и „обратен адитивен“ променят посоката на аргумента, а операторът „отрицание“ я запазва.

Равенствата (21) подсказват, че можем да търсим оператор $1/\mathbf{A}$ в $D \setminus \mathcal{T}$, който (аналогично на оператора $-\mathbf{A}$ в (21)) евентуално удовлетворява съотношенията

$$(22) \qquad 1/_h(1/\mathbf{A}) = 1/(1/_h\mathbf{A}) = \overline{\mathbf{A}}.$$

Единственият такъв оператор е „инверсията“ $1/\mathbf{A} = \overline{1/_h\mathbf{A}} = 1/_h\overline{\mathbf{A}} = [1/a^+, 1/a^-]$ за $\mathbf{A} \in D \setminus \mathcal{T}$; противоположно на случая в $\mathcal{S}$ тук инверсията $1/\mathbf{A}$ е съставен оператор. От друга страна, операцията $\mathbf{A} \times (1/\mathbf{B})$ за $\mathbf{A} \in D$, $\mathbf{B} \in D \setminus \mathcal{T}$, която ще означаваме по-нататък с $\mathbf{A}/\mathbf{B}$, е разширение в D на $\mathcal{S}$-операцията $\mathbf{A}/\mathbf{B}$, дефинирана с (8); имаме

$$\mathbf{A}/\mathbf{B} = \mathbf{A} \times (1/\mathbf{B}) = \begin{cases} [a^{-\sigma(B)}/b^{\sigma(A)}, a^{\sigma(B)}/b^{-\sigma(A)}], \ \mathbf{A} \in D^*, \ \mathbf{B} \in D \setminus \mathcal{T}, \\ [a^{-\delta}/b^{-\delta}, a^{\delta}/b^{-\delta}], \ \delta = \sigma(B), \ \mathbf{A} \in \mathcal{T}^*, \ \mathbf{B} \in D \setminus \mathcal{T}. \end{cases}$$

От $\overline{\mathbf{A}} = -(-_h\mathbf{A})$ и $\overline{\mathbf{A}} = 1/(1/_h\mathbf{A})$ (вж. (21) и (22)) получаваме следните изрази за обратните оператори: $-_h\mathbf{A} = -\overline{\mathbf{A}}$, $1/_h\mathbf{A} = 1/\overline{\mathbf{A}}$. Обратните елементи $-_h\mathbf{A}$, $1/_h\mathbf{A}$ пораждат операциите $\mathbf{A}+(-_h\mathbf{B}) = \mathbf{A}+(-\overline{\mathbf{B}}) = \mathbf{A}-\overline{\mathbf{B}}$, $\mathbf{A}\times(1/_h\mathbf{B}) = \mathbf{A}\times(1/\overline{\mathbf{B}}) = \mathbf{A}/\overline{\mathbf{B}}$, които ще означаваме с $\mathbf{A} -_h \mathbf{B}$ и $\mathbf{A}/_h\mathbf{B}$:

$$\mathbf{A} -_h \mathbf{B} = \mathbf{A} + (-_h\mathbf{B}) = \mathbf{A} - \overline{\mathbf{B}} = [a^- - b^-, a^+ - b^+],\ \mathbf{A},\mathbf{B}\in D,$$
$$\mathbf{A} /_h \mathbf{B} = \mathbf{A}\times(1/_h\mathbf{B}) = \mathbf{A}/\overline{\mathbf{B}}$$
$$= \begin{cases} [a^{-\sigma(B)}/b^{-\sigma(A)}, a^{\sigma(B)}/b^{\sigma(A)}], & \mathbf{A}\in D^*,\ \mathbf{B}\in D\setminus\mathcal{T}, \\ [a^{-\delta}/b^{\delta}, a^{\delta}/b^{\delta}],\ \delta=\sigma(B), & \mathbf{A}\in\mathcal{T}^*,\ \mathbf{B}\in D\setminus\mathcal{T}. \end{cases}$$

От последното равенство получаваме $\mathbf{A}/\mathbf{B} = \mathbf{A}/_h\overline{\mathbf{B}} = \mathbf{A}/_h(-_h((-1)\times\mathbf{B}))$, което показва, че делението „$/$“ може да се определи чрез операциите „$\times$“, $-_h$ и $/_h$ (обратно на ситуацията в $\mathcal{S}$, където „$/$“ е независима операция). Следователно символът „$/$“ може да не се включва в списъка от основните операции на така получената алгебрична система, която ще означаваме с $\mathcal{K} = (D, +, \times)$. Виждаме, че системата $\mathcal{K}$ съдържа съставните операции изваждане $\mathbf{A}-\mathbf{B}$, деление $\mathbf{A}/\mathbf{B}$, спрягане $\overline{\mathbf{A}}$, операциите $\mathbf{A}-\overline{\mathbf{B}}$, $\mathbf{A}/\overline{\mathbf{B}}$ и техните спрегнати $\overline{\mathbf{A}}-\mathbf{B}$, $\overline{\mathbf{A}}/\mathbf{B}$. Други съставни операции са $\mathbf{A}+\overline{\mathbf{B}}$, $\mathbf{A}\times\overline{\mathbf{B}}$, $\overline{\mathbf{A}}+\mathbf{B}$, $\overline{\mathbf{A}}\times\mathbf{B}$; покомпонентните представяния на последните са съответно (вземайки предвид, че $\sigma(\mathbf{A}) = \sigma(\overline{\mathbf{A}}) = \sigma(A)$):

$$\mathbf{A}+\overline{\mathbf{B}} = [a^- + b^+, a^+ + b^-],\quad \mathbf{A},\mathbf{B}\in D;$$
$$\overline{\mathbf{A}}+\mathbf{B} = [a^+ + b^-, a^- + b^+],\quad \mathbf{A},\mathbf{B}\in D;$$
$$\mathbf{A}\times\overline{\mathbf{B}} = \begin{cases} [a^{-\sigma(B)}b^{\sigma(A)}, a^{\sigma(B)}b^{-\sigma(A)}], & \mathbf{A},\mathbf{B}\in D^*, \\ [a^{\delta}b^{\delta}, a^{\delta}b^{-\delta}],\ \delta=\sigma(A), & \mathbf{A}\in D^*,\ \mathbf{B}\in\mathcal{T}^*, \\ [a^{-\delta}b^{-\delta}, a^{\delta}b^{-\delta}],\ \delta=\sigma(B), & \mathbf{A}\in\mathcal{T}^*,\ \mathbf{B}\in D^*; \end{cases}$$
$$\overline{\mathbf{A}}\times\mathbf{B} = \begin{cases} [a^{\sigma(B)}b^{-\sigma(A)}, a^{-\sigma(B)}b^{\sigma(A)}], & \mathbf{A},\mathbf{B}\in D^*, \\ [a^{-\delta}b^{-\delta}, a^{-\delta}b^{\delta}],\ \delta=\sigma(A), & \mathbf{A}\in D^*,\ \mathbf{B}\in\mathcal{T}^*, \\ [a^{\delta}b^{\delta}, a^{-\delta}b^{\delta}],\ \delta=\sigma(B), & \mathbf{A}\in\mathcal{T}^*,\ \mathbf{B}\in D^*. \end{cases}$$

Операциите $\tilde{+}$ и $\tilde{\times}$, дефинирани с $\mathbf{A}\tilde{+}\mathbf{B} = \mathbf{A}+\overline{\mathbf{B}}$, $\mathbf{A}\tilde{\times}\mathbf{B} = \mathbf{A}\times\overline{\mathbf{B}}$, удовлетворяват $\mathbf{B}\tilde{+}\mathbf{A} = \mathbf{B}+\overline{\mathbf{A}} = \overline{\mathbf{A}\tilde{+}\mathbf{B}}$, $\mathbf{B}\tilde{\times}\mathbf{A} = \mathbf{B}\times\overline{\mathbf{A}} = \overline{\mathbf{A}\tilde{\times}\mathbf{B}}$, което показва, че „$\tilde{+}$“ и „$\tilde{\times}$“ не са комутативни.

По-общо, като използуваме означенията $\mathbf{A}_- = \overline{\mathbf{A}}$, $\mathbf{A}_+ = \mathbf{A}$, можем да запишем, че за $t, s\in\{+,-\}$

$$\mathbf{A}_t + \mathbf{B}_s = [a^{-t} + b^{-s}, a^t + b^s],\quad \text{за } \mathbf{A},\mathbf{B}\in D, \tag{23}$$
$$\mathbf{A}_t\times\mathbf{B}_s = \begin{cases} [a^{-t\sigma(B)}b^{-s\sigma(A)}, a^{t\sigma(B)}b^{s\sigma(A)}], & \mathbf{A},\mathbf{B}\in D^*, \\ [a^{t\delta}b^{-s\delta}, a^{t\delta}b^{s\delta}],\ \delta=\sigma(A), & \mathbf{A}\in D^*,\ \mathbf{B}\in\mathcal{T}^*, \\ [a^{-t\delta}b^{s\delta}, a^{t\delta}b^{s\delta}],\ \delta=\sigma(B), & \mathbf{A}\in\mathcal{T}^*,\ \mathbf{B}\in D^*. \end{cases} \tag{24}$$

Формули (23), (24) позволяват пресмятането на изразите $\mathbf{A}+\mathbf{B}$, $\overline{\mathbf{A}}+\mathbf{B}$, $\mathbf{A}+\overline{\mathbf{B}}$, $\overline{\mathbf{A}}+\overline{\mathbf{B}}$, $\mathbf{A}\times\mathbf{B}$, $\overline{\mathbf{A}}\times\mathbf{B}$, $\mathbf{A}\times\overline{\mathbf{B}}$, $\overline{\mathbf{A}}\times\overline{\mathbf{B}}$.

Нововъведеното означение $\mathbf{A}_{\pm}$, показващо наличие или липса на спрягане върху $\mathbf{A}$, е много удобно за работа и ще се използува често по-нататък. С негова помощ например формула (20) може да се запише за $\mathbf{A}, \mathbf{B} \in \mathcal{T}^*$ така:

$$\mathbf{A} \times \mathbf{B} \;=\; \begin{cases} [\min\{a^- b^+, a^+ b^-\},\ \max\{a^- b^-, a^+ b^+\}]_{\tau(\mathbf{A})}, & \tau(\mathbf{A}) = \tau(\mathbf{B}), \\ 0, \quad \tau(\mathbf{A}) = -\tau(\mathbf{B}). \end{cases}$$

Операциите $+, \times$ запазват свойствата комутативност и асоциативност при тяхното разширение (18)–(20) от $I(R)$ в D [10, 11, 26]. Освен това в D разширените операции имат обратни и съответните алгебрични системи са групи. Особено привлекателно е това, че в $\mathcal{K}$ има прости дистрибутивно-подобни връзки, които могат да се формулират в следното

Твърдение 2 (условно-дистрибутивен закон). *За* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A} + \mathbf{B} \in D^*$ *е в сила*

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \times \mathbf{C}_{\sigma(\mathbf{A})\sigma(\mathbf{A}+\mathbf{B})}) + (\mathbf{B} \times \mathbf{C}_{\sigma(\mathbf{B})\sigma(\mathbf{A}+\mathbf{B})}).$$

Доказателство. Като използуваме дефинициите за умножение и събиране, получаваме

$$\begin{aligned} (\mathbf{A} + \mathbf{B}) \times \mathbf{C} &= [(\mathbf{A} + \mathbf{B})^{-\sigma(\mathbf{C})} \mathbf{C}^{-\sigma(\mathbf{A}+\mathbf{B})},\ (\mathbf{A} + \mathbf{B})^{\sigma(\mathbf{C})} \mathbf{C}^{\sigma(\mathbf{A}+\mathbf{B})}] \\ &= [(\mathbf{A}^{-\sigma(\mathbf{C})} + \mathbf{B}^{-\sigma(\mathbf{C})}) \mathbf{C}^{-\sigma(\mathbf{A}+\mathbf{B})},\ (\mathbf{A}^{\sigma(\mathbf{C})} + \mathbf{B}^{\sigma(\mathbf{C})}) \mathbf{C}^{\sigma(\mathbf{A}+\mathbf{B})}]. \end{aligned}$$

Поради $\sigma(\mathbf{A})\sigma(\mathbf{A}) = +$ и формула (24) намираме за дясната страна

$$\begin{aligned} &[(\mathbf{A})^{-\sigma(\mathbf{C})} \mathbf{C}^{-\sigma(\mathbf{A}+\mathbf{B})},\ (\mathbf{A})^{\sigma(\mathbf{C})} \mathbf{C}^{\sigma(\mathbf{A}+\mathbf{B})}] \\ &\qquad + [(\mathbf{B})^{-\sigma(\mathbf{C})} \mathbf{C}^{-\sigma(\mathbf{A}+\mathbf{B})},\ (\mathbf{B})^{\sigma(\mathbf{C})} \mathbf{C}^{\sigma(\mathbf{A}+\mathbf{B})}] \\ &= [(\mathbf{A}^{-\sigma(\mathbf{C})} + \mathbf{B}^{-\sigma(\mathbf{C})}) \mathbf{C}^{-\sigma(\mathbf{A}+\mathbf{B})},\ (\mathbf{A}^{\sigma(\mathbf{C})} + \mathbf{B}^{\sigma(\mathbf{C})}) \mathbf{C}^{\sigma(\mathbf{A}+\mathbf{B})}]. \end{aligned}$$

Навсякъде тук аргументите на σ са насочени интервали, но вместо тях могат да се пишат съответните им собствени части, например $\sigma(\mathbf{A} + \mathbf{B}) = \sigma(A + B) = \sigma(A +^- B)$ и т. н. Още едно просто извеждане на условно-дистрибутивния закон ще бъде направено в края на следващата точка. В по-подробен запис условно-дистрибутивният закон има вида

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \begin{cases} (\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \mathbf{C}), & \sigma(\mathbf{A}) = \sigma(\mathbf{B})\ \ (= \sigma(\mathbf{A} + \mathbf{B})), \\ (\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \overline{\mathbf{C}}), & \sigma(\mathbf{A}) = -\sigma(\mathbf{B}) = \sigma(\mathbf{A} + \mathbf{B}), \\ (\mathbf{A} \times \overline{\mathbf{C}}) + (\mathbf{B} \times \mathbf{C}), & \sigma(\mathbf{A}) = -\sigma(\mathbf{B}) = -\sigma(\mathbf{A} + \mathbf{B}). \end{cases}$$

Условно-дистрибутивният закон може да се запише и в следните форми:

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C}_{\sigma(A+B)} = (\mathbf{A} \times \mathbf{C}_{\sigma(A)}) + (\mathbf{B} \times \mathbf{C}_{\sigma(B)}),\ \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A} + \mathbf{B} \in D^*,$$

$$\mathbf{C}_{\sigma(A+B)} \times (\mathbf{A} + \mathbf{B}) = (\mathbf{C}_{\sigma(A)} \times \mathbf{A}) + (\mathbf{C}_{\sigma(B)} \times \mathbf{B}),\ \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A} + \mathbf{B} \in D^*,$$

които показват, че интервалът $\mathbf{C}$ при множителите $\mathbf{A} + \mathbf{B}, \mathbf{A}, \mathbf{B}$ се спряга, ако съответният множител е отрицателен.

Релации за включване. Удобно е да се използуват две релации за включване.

Релацията h-включване [11] е удобна за работа, но има необичаен смисъл; тя се дефинира така:

$$\mathbf{A} \subseteq_h \mathbf{B} \iff (b^- \le a^-) \wedge (a^+ \le b^+), \quad \mathbf{A}, \mathbf{B} \in D. \tag{25}$$

Релацията p-включване се дефинира за два насочени интервала $\mathbf{A} = [a^-, a^+]$, $\mathbf{B} = [b^-, b^+]$ по следния естествен начин:

$$\mathbf{A} \subseteq_p \mathbf{B} \iff p(\mathbf{A}) \subseteq p(\mathbf{B}), \quad \mathbf{A}, \mathbf{B} \in D, \tag{26}$$

като в случая $\mathbf{A} = \mathbf{B}_-$, т. е. $b^- = a^+$, $b^+ = a^-$, приемаме за определеност, че интервалът с положителна посока съдържа този с отрицателна посока. Формула (26) изразява, че компонентите на $\mathbf{A}$ се намират между компонентите на $\mathbf{B}$ (или че краищата на A са между краищата на B). Друг еквивалентен запис на (26) е $\mathbf{A} \subseteq_p \mathbf{B} \iff b^{-\tau(\mathbf{B})} \le a^{-\tau(\mathbf{A})} \le a^{\tau(\mathbf{A})} \le b^{\tau(\mathbf{B})}$, $\mathbf{A}, \mathbf{B} \in D$.

Връзките между релациите $\subseteq_h$ и $\subseteq_p$ се дават за интервали $\mathbf{A}, \mathbf{B}$ с еднаква посока посредством

$$\mathbf{A} \subseteq_h \mathbf{B} \iff \mathbf{A} \subseteq_p^\theta \mathbf{B}, \ \mathbf{A} \subseteq_p \mathbf{B} \iff \mathbf{A} \subseteq_h^\theta \mathbf{B}, \ \tau(\mathbf{A}) = \tau(\mathbf{B}) = \theta, \tag{27}$$

където $\subseteq_p^+ = \subseteq_p$, $\subseteq_p^- = \supseteq_p$, $\subseteq_h^+ = \subseteq_h$ и $\subseteq_h^- = \supseteq_h$.

По-нататък обобщаваме основните свойства на интервалната структура $\mathcal{K} = (D, +, \times, \subseteq_p)$. Ако не е казано нещо друго, с $\mathbf{A}, \mathbf{B}, \mathbf{C}, ..., \mathbf{X}, \mathbf{Y}$ ще означаваме елементи на D. С изключение на свойства **K5** и **K6** изброените по-нататък свойства могат да се намерят в [11].

K1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \quad \mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$.

K2. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}), \quad (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

K3. $\mathbf{X} = [0, 0] = 0$ и $\mathbf{Y} = [1, 1] = 1$ са единствените нулев и съответно единичен елемент по отношение на събирането и умножението, т. е.

$$\mathbf{A} = \mathbf{X} + \mathbf{A} \text{ за всяко } \mathbf{A} \in D \iff \mathbf{X} = [0, 0];$$
$$\mathbf{A} = \mathbf{Y} \times \mathbf{A} \text{ за всяко } \mathbf{A} \in D \iff \mathbf{Y} = [1, 1].$$

K4. Всеки елемент $\mathbf{A} \in D$ има единствен обратен елемент по отношение на $+$ и всеки елемент $\mathbf{A} \in D \setminus \mathcal{T}$ притежава единствен обратен елемент по отношение на $\times$. Това са елементите $-\overline{\mathbf{A}}$, съответно $1/\overline{\mathbf{A}}$, т. е. имаме

$$0 = \mathbf{A} + (-\overline{\mathbf{A}}) = \mathbf{A} - \overline{\mathbf{A}} \text{ и } 1 = \mathbf{A} \times (1/\overline{\mathbf{A}}) = \mathbf{A}/\overline{\mathbf{A}}.$$

K5. *i*) За $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A}+\mathbf{B} \in D^*$, $(\mathbf{A}+\mathbf{B}) \times \mathbf{C}_{\sigma(A+B)} = (\mathbf{A} \times \mathbf{C}_{\sigma(A)}) + (\mathbf{B} \times \mathbf{C}_{\sigma(B)})$;
ii) За $\mathbf{A}, \mathbf{B}, \mathbf{C} \in I(R)$ имаме $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) \subseteq_p \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$;
iii) За $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \overline{I(R)}$ имаме $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) \supseteq_p \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.

K6. Нека $* \in \{+, -, \times, /\}$. Тогава $\mathbf{X} \subseteq_p \mathbf{X}_1, \implies \mathbf{X} * \mathbf{C} \subseteq_p \mathbf{X}_1 * \mathbf{C}$. Като обобщение получаваме $\mathbf{X} \subseteq_p \mathbf{X}_1, \mathbf{Y} \subseteq_p \mathbf{Y}_1 \implies \mathbf{X} * \mathbf{Y} \subseteq_p \mathbf{X}_1 * \mathbf{Y}_1$.

K7. Операцията спрягане удовлетворява следните свойства:
$i)$ $\overline{\mathbf{A}+\mathbf{B}} = \overline{\mathbf{A}} + \overline{\mathbf{B}}$, $\overline{\mathbf{A} \times \mathbf{B}} = \overline{\mathbf{A}} \times \overline{\mathbf{B}}$;
$ii)$ $\mathbf{A} + \overline{\mathbf{A}} = a^- + a^+ \in R$,
$\mathbf{A} \times \overline{\mathbf{A}} = \{a^-a^+, \text{ ако } \mathbf{A} \in D \setminus \mathcal{T};\ 0, \text{ ако } \mathbf{A} \in \mathcal{T}\} \in R$.

K8. За $\mathbf{A}, \mathbf{B} \in D \setminus \mathcal{T}$, $1/(\mathbf{A} \times \mathbf{B}) = (1/\mathbf{A}) \times (1/\mathbf{B})$, $1/(\mathbf{A}/\mathbf{B}) = \mathbf{B}/\mathbf{A}$.

За прегледност резюмираме по-важните аритметични закони в D, съответно в D^*:

$i)$ За $\mathbf{A}, \mathbf{B}, \mathbf{C} \in D$, $\quad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \quad (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$;

$ii)$ За $\mathbf{A}, \mathbf{B}, \mathbf{C} \in D$, $\quad \mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}, \quad (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$;

$iii)$ За $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A}+\mathbf{B} \in D^*$, $\quad (\mathbf{A}+\mathbf{B}) \times \mathbf{C}_{\sigma(A+B)} = (\mathbf{A} \times \mathbf{C}_{\sigma(A)}) + (\mathbf{B} \times \mathbf{C}_{\sigma(B)})$.

Хиперболично умножение. Операцията „хиперболично умножение" (h-умножение) на насочени интервали се дефинира посредством [10]

$$\mathbf{A} \times_h \mathbf{B} = [a^-b^-, a^+b^+],\ \mathbf{A}, \mathbf{B} \in D. \tag{28}$$

Нека $\mathbf{A} \in D \setminus \mathcal{L}$, където $\mathcal{L} = \{[a^-, a^+] \mid a^-a^+ = 0\}$. Лесно се проверява, че обратният елемент на $\mathbf{A} = [a^-, a^+] \in D \setminus \mathcal{L}$ по отношение на хиперболичното умножение е $[1/a^-, 1/a^+]$, който означихме с $1/_h\mathbf{A}$. Забелязваме, че за $\mathbf{A} \in D \setminus \mathcal{T}$ *обратният* елемент на $\mathbf{A}$ относно „$\times_h$" съвпада с обратния елемент на $\mathbf{A}$ относно „$\times$", който съгласно **K4** е $1/\overline{\mathbf{A}} = [1/a^-, 1/a^+]$.

Следващото твърдение се отнася до свойства на хиперболичното умножение и мотивира наименованието *хиперболично полутяло на системата* $(D, +, \times_h)$.

Твърдение 3 ([10]). *В сила са следните свойства:*

а) $(D, \times_h)$ *е комутативна полугрупа и* $(D \setminus \mathcal{L}, \times_h)$ *е група;*

б) *операциите* $+$ *и* $\times_h$ *са дистрибутивни, т. е. в* D^* *имаме* $(\mathbf{A}+\mathbf{B}) \times_h \mathbf{C} = (\mathbf{A} \times_h \mathbf{C}) + (\mathbf{B} \times_h \mathbf{C})$;

в) $(D, +, \times_h)$ *е тяло с нулев делител в* $\mathcal{L}$.

Алгебричната система $(D, +, \times_h)$ е проста и лесна за използуване. Но в тази система липсват важните интервални оператори отрицание $-\mathbf{A}$ и спрягане, т. е. те не могат да бъдат определени чрез операциите $+, \times_h$ и техните обратни или съставни. Въвеждането на кой да е от операторите отрицание или спрягане допълва $(D, +, \times_h)$ до система, еквивалентна на обобщеното интервално пространство $\mathcal{K} = (D, +, \times)$. Наистина следващите съотношения показват, че всеки един от тези оператори се определя чрез другия и чрез хиперболичното умножение: $-\mathbf{A} = (-1) \times_h \overline{\mathbf{A}}$; $\overline{\mathbf{A}} = (-1) \times_h (-\mathbf{A})$. Записани в термините на neg и con тези съотношения имат вида $\text{neg}(\mathbf{A}) = (-1) \times_h \text{con}(\mathbf{A})$, $\text{con}(\mathbf{A}) = (-1) \times_h \text{neg}(\mathbf{A})$. Това показва, че е в сила следното

Твърдение 4. *Системите* $(D, +, \times_h, \mathrm{con})$, $(D, +, \times_h, \mathrm{neg})$ *и* $(D, +, \times)$ *са еквивалентни.*

Можем да намерим връзки между интервалното и хиперболичното умножение, съответно деление. За $\mathbf{A}$, $\mathbf{B} \in D^*$ имаме

$$\text{(29)} \qquad \mathbf{A} \times \mathbf{B} = \mathbf{A}_{\sigma(B)} \times_h \mathbf{B}_{\sigma(A)}, \ \mathbf{A} \times_h \mathbf{B} = \mathbf{A}_{\sigma(B)} \times \mathbf{B}_{\sigma(A)};$$

$$\text{(30)} \qquad \mathbf{A}/\mathbf{B} = \mathbf{A}_{\sigma(B)}/_h\mathbf{B}_{-\sigma(A)}, \ \mathbf{A}/_h\mathbf{B} = \mathbf{A}_{\sigma(B)}/\mathbf{B}_{-\sigma(A)}.$$

Към горните връзки можем да причислим и съотношенията

$$\text{(31)} \qquad \mathbf{A} - \mathbf{B} = \mathbf{A} -_h \mathbf{B}_-, \quad \mathbf{A} -_h \mathbf{B} = \mathbf{A} - \mathbf{B}_-, \quad \mathbf{A}, \mathbf{B} \in D,$$

които допълват възможността за преход между трите системи $(D, +, \times_h, \mathrm{con})$, $(D, +, \times_h, \mathrm{neg})$ и $(D, +, \times)$.

Като приложение на хиперболичното полутяло ще изведем условно-дистрибутивния закон от твърдение 2. Съгласно дистрибутивния закон в $(D^*, +, \times_h)$ (твърдение 3б) имаме $(\mathbf{A} + \mathbf{B}) \times_h \mathbf{C} = (\mathbf{A} \times_h \mathbf{C}) + (\mathbf{B} \times_h \mathbf{C})$. Като заместим h-произведенията с интервални умножения с помощта на (29) (което имаме право да извършим в D^*) и преозначим $\mathbf{A}_{\sigma(C)}, \mathbf{B}_{\sigma(C)}$ отново с $\mathbf{A}$, съответно $\mathbf{B}$, получаваме $(\mathbf{A}+\mathbf{B}) \times \mathbf{C}_{\sigma(A+B)} = (\mathbf{A} \times \mathbf{C}_{\sigma(A)}) + (\mathbf{B} \times \mathbf{C}_{\sigma(B)})$. Това демонстрира ползата от $(\pm)$-означенията, които позволяват наличието, респ. липсата, на операцията спрягане да се постави в зависимост от стойността на двоичната променлива „знак на интервал“. Табличното представяне, използувано от Е. Каухер, не е така удобно за работа.

5. НАСОЧЕНИ ИНТЕРВАЛИ В НОРМАЛНА ФОРМА

Дотук използувахме покомпонентната форма (CW-форма) на представяне на насочените интервали. Ще използуваме и едно друго представяне на насочените интервали $\mathbf{A} = [a^-, a^+] \in D$, което ще наричаме представяне в нормална форма. Елементите на декартовото произведение $I(R) \bigotimes \Lambda$, $\Lambda = \{+, -\}$, са двойки $\{A, \alpha\}$, състоящи се от нормален (собствен) интервал $A \in I(R)$ и знак $\alpha \in \Lambda$, които ще интерпретираме съответно като собствена част на насочен интервал $\mathbf{A}$ и неговата посока $\tau(\mathbf{A})$. Това ще записваме $\mathbf{A} = \{A, \alpha\} = [A; \alpha] = [a^{(-)}, a^{(+)}; \alpha]$ с $A = [a^{(-)}, a^{(+)}] \in I(R)$ и $\alpha = \tau(\mathbf{A}) \in \Lambda$. Само двойката $\{[0,0], -\} = [0, 0; -]$ не съответствува на никой насочен интервал, поради което ще я изключим от множеството $I(R) \bigotimes \Lambda$. По такъв начин между множествата D и $(I(R) \bigotimes \Lambda) \setminus \{[0,0], -\}$ има едно-еднозначно съответствие, поради което последното множество ще означаваме отново с D.

Формулите за преход от покомпонентна форма $\mathbf{A} = [a^-, a^+]$ към нормална форма $\mathbf{A} = [a^{(-)}, a^{(+)}; \alpha]$ имат вида

$$\text{(32)} \quad \alpha = \sigma(a^+ - a^-), \quad a^{(-)} = a^{-\alpha}, \quad a^{(+)} = a^{\alpha}, \quad a^- = a^{(-\alpha)}, \quad a^+ = a^{(\alpha)}.$$

Да намерим израз за сума $\mathbf{C} = \mathbf{A} + \mathbf{B}$ на два насочени интервала $\mathbf{A}$, $\mathbf{B}$, представени в нормална форма. За посоката $\gamma = \tau(\mathbf{C})$ на сумата $\mathbf{C} = [c^-, c^+] = [a^- + b^-, a^+ + b^+]$ получаваме

$$\begin{aligned} \gamma = \tau(\mathbf{A}+\mathbf{B}) &= \sigma(c^+ - c^-) = \sigma(a^+ + b^+ - a^- - b^-) \\ &= \sigma((a^{(\alpha)} - a^{(-\alpha)}) + (b^{(\beta)} - b^{(-\beta)})) \\ &= \sigma\left(\alpha\omega(A) + \beta\omega(B)\right), \end{aligned} \tag{33}$$

където $\omega(A) = a^{(+)} - a^{(-)}$. В израза $\sigma\left(\alpha\omega(A) + \beta\omega(B)\right)$ символите $\alpha, \beta \in \Lambda$ пред реалните положителни числа $\omega(A)$, съответно $\omega(B)$, се интерпретират като знаци на тези числа, т. е. $\alpha, \beta = \pm 1$.

Подробно записан, последният израз има вида

$$\begin{aligned} \gamma = \tau(\mathbf{A}+\mathbf{B}) &= \begin{cases} \alpha, & \alpha = \beta, \\ \alpha, & \alpha = -\beta,\ \omega(A) > \omega(B), \\ \beta, & \alpha = -\beta,\ \omega(A) < \omega(B), \\ +, & \alpha = -\beta,\ \omega(A) = \omega(B), \end{cases} \\ &= \begin{cases} \alpha, & \omega(A) > \omega(B), \\ \beta, & \omega(A) < \omega(B), \\ \alpha, & \omega(A) = \omega(B),\ \alpha = \beta, \\ +, & \omega(A) = \omega(B),\ \alpha = -\beta. \end{cases} \end{aligned}$$

Сега пресмятаме краищата: $c^{(-)} = c^{-\gamma} = a^{-\gamma} + b^{-\gamma} = a^{(-\alpha\gamma)} + b^{(-\beta\gamma)}$, $c^{(+)} = c^{\gamma} = a^{\gamma} + b^{\gamma} = a^{(\alpha\gamma)} + b^{(\beta\gamma)}$. Следователно

$$\mathbf{A} + \mathbf{B} = [a^{(-)}, a^{(+)}; \alpha] + [b^{(-)}, b^{(+)}; \beta] = [a^{(-\alpha\gamma)} + b^{(-\beta\gamma)}, a^{(\alpha\gamma)} + b^{(\beta\gamma)}; \gamma], \tag{34}$$

където $\gamma = \tau(\mathbf{A}+\mathbf{B})$ се дава от (33).

От израза (34) следва, че нормалната част $p(\mathbf{A}+\mathbf{B}) = [a^{(-\alpha\gamma)} + b^{(-\beta\gamma)}, a^{(\alpha\gamma)} + b^{(\beta\gamma)}]$ на сумата $\mathbf{A}+\mathbf{B}$ в случая $\alpha = \beta$ $(= \gamma)$ е равна на $[a^{(-)} + b^{(-)}, a^{(+)} + b^{(+)}] = A + B = p(\mathbf{A}) + p(\mathbf{B})$. При $\alpha \neq \beta$ нормалната част на $\mathbf{A}+\mathbf{B}$ е равна на

$$\begin{aligned} C &= \begin{cases} [a^{(-)} + b^{(+)}, a^{(+)} + b^{(-)}], & a^{(-)} + b^{(+)} \le a^{(+)} + b^{(-)}, \\ [a^{(+)} + b^{(-)}, a^{(-)} + b^{(+)}], & a^{(-)} + b^{(+)} > a^{(+)} + b^{(-)}, \end{cases} \\ &= \begin{cases} [a^{(-)} + b^{(+)}, a^{(+)} + b^{(-)}], & \omega(A) \ge \omega(B), \\ [a^{(+)} + b^{(-)}, a^{(-)} + b^{(+)}], & \omega(A) < \omega(B), \end{cases} \end{aligned} \tag{35}$$

което е собствен интервал с краища $a^{(-)} + b^{(+)}$ и $a^{(+)} + b^{(-)}$. Интервалът (35) е вътрешната (нестандартна) сума $A +^- B$. Следователно можем да запишем

$$p(\mathbf{A}+\mathbf{B}) = \begin{cases} A + B, & \tau(\mathbf{A}) = \tau(\mathbf{B}), \\ A +^- B, & \tau(\mathbf{A}) \neq \tau(\mathbf{B}). \end{cases}$$

Виждаме, че за да се представи собствената част на сума от два насочени интервала посредством собствените части на тези интервали, са необходими двата типа суми за нормални интервали: външната сума $A + B$ и вътрешната сума $A +^- B$.

За униформеност при представянето на $C = p(\mathbf{A}+\mathbf{B})$ да използваме означението $+^{+} = +$; тогава можем да обобщим двата случая $\alpha = \beta$, $\alpha \neq \beta$, като запишем $C = A +^{\alpha\beta} B$. И така C е или външна, или вътрешна сума на A и B. Имаме .

$$
\begin{aligned}
[A;\alpha]+[B;\beta] &= [A +^{\alpha\beta} B; \tau([A;\alpha]+[B;\beta])],\ A,B \in I(R), \alpha,\beta \in \Lambda; \\
\mathbf{A}+\mathbf{B} &= [A +^{\tau(\mathbf{A})\tau(\mathbf{B})} B; \tau(\mathbf{A}+\mathbf{B})],\ \mathbf{A},\mathbf{B} \in D,
\end{aligned} \tag{36}
$$

където $\tau(\mathbf{A}+\mathbf{B}) = \gamma$ се дава с (33).

За посоката на сумата на насочените интервали $\mathbf{A}$, $\mathbf{B}$ в случай, че сумата не е изроден интервал, може да се вземе функционалът $\tau_1 : D \bigotimes D \to \Lambda$, дефиниран посредством

$$
\tau_1(\mathbf{A},\mathbf{B}) = \tau_1([A;\alpha],[B;\beta]) = \begin{cases} \alpha, & \omega(A) \geq \omega(B), \\ \beta, & \omega(A) < \omega(B). \end{cases} \tag{37}
$$

При $\omega(C) \neq 0$ имаме $\tau_1(\mathbf{A},\mathbf{B}) = \tau(\mathbf{A}+\mathbf{B})$. Стойността на $\tau_1(\mathbf{A},\mathbf{B})$ обаче може да бъде отрицателна при $\omega(C) = 0$, а съгласно уславянето, което направихме, в този случай знакът е положителен. Поради това формула (37) не съвпада напълно с (33). В някои случаи обаче стойността на $\tau_1(\mathbf{A},\mathbf{B})$ при $\omega(C) = 0$ е без значение, поради което вместо $\tau(\mathbf{A}+\mathbf{B})$ можем да използваме $\tau_1(\mathbf{A},\mathbf{B})$ (такъв е случаят с твърдения 6 и 7 от следващата точка).

За да представим разликата $\mathbf{C} = \mathbf{A}-\mathbf{B} = \mathbf{A}+(-\mathbf{B}) = [a^{-}-b^{+}, a^{+}-b^{-}]$ в нормална форма, най-напред пресмятаме $\tau(\mathbf{C}) = \tau(\mathbf{A}-\mathbf{B}) = \tau(\mathbf{A}+(-\mathbf{B})) = \sigma(\alpha(\omega(A) + \beta(\omega(-B)) = \sigma(\alpha(\omega(A) + \beta(\omega(B)) = \tau(\mathbf{A}+\mathbf{B})$ (използваме, че $\omega(-(B)) = \omega(B)$ и $\tau(\mathbf{B}) = \tau(-\mathbf{B}) = \beta$). След това с помощта на (32) пресмятаме $c^{(-)} = c^{-\gamma} = a^{-\gamma}-b^{\gamma} = a^{(-\alpha\gamma)}-b^{(\beta\gamma)}$, $c^{(+)} = c^{\gamma} = a^{\gamma}-b^{-\gamma} = a^{(\alpha\gamma)}-b^{(-\beta\gamma)}$, така че

$$
\begin{aligned}
\mathbf{A}-\mathbf{B} &= [a^{(-)}, a^{(+)}; \alpha] - [b^{(-)}, b^{(+)}; \beta] \\
&= [a^{(-\alpha\gamma)} - b^{(\beta\gamma)}, a^{(\alpha\gamma)} - b^{(-\beta\gamma)}; \tau(\mathbf{A}-\mathbf{B})],
\end{aligned} \tag{38}
$$

където $\tau(\mathbf{A}-\mathbf{B}) = \tau(\mathbf{A}+\mathbf{B}) = \gamma$ се дава с (33).

От (38) следва, че при $\alpha = \beta\ (= \gamma)$ собствената част $p(\mathbf{A}-\mathbf{B})$ на разликата $\mathbf{A}-\mathbf{B}$ е равна на $[a^{(-)} - b^{(+)}, a^{(+)} - b^{(-)}] = A - B = p(\mathbf{A}) - p(\mathbf{B})$. При $\alpha \neq \beta$ собствената част $p(\mathbf{A}-\mathbf{B})$ е равна на интервала

$$
\begin{cases} [a^{(-)} - b^{(-)}, a^{(+)} - b^{(+)}], & \omega(A) \geq \omega(B), \\ [a^{(+)} - b^{(+)}, a^{(-)} - b^{(-)}], & \omega(A) < \omega(B), \end{cases}
$$

т. е. на собствения интервал с краища $a^{(-)} - b^{(-)}$ и $a^{(+)} - b^{(+)}$. Както знаем от т. 3, последният е точно вътрешната (нестандартната) разлика $A -^{-} B$. Следователно можем да запишем

$$
p(\mathbf{A}-\mathbf{B}) = \begin{cases} A - B, & \tau(\mathbf{A}) = \tau(\mathbf{B}), \\ A -^{-} B, & \tau(\mathbf{A}) \neq \tau(\mathbf{B}). \end{cases}
$$

Това показва, че за представянето на собствената част на разликата на два насочени интервала са необходими двата типа разлики: външната разлика $A - B$ и вътрешната разлика $A -^- B$.

За да получим общ израз за разлика на два насочени интервала, използуваме означението $-^+ = -$; тогава можем да запишем

$$\begin{aligned} [A;\alpha] - [B;\beta] &= [A -^{\alpha\beta} B; \tau([A;\alpha] - [B;\beta])],\ A, B \in I(R), \alpha, \beta \in \Lambda; \\ \text{(39)} \qquad \mathbf{A} - \mathbf{B} &= [A -^{\tau(\mathbf{A})\tau(\mathbf{B})} B; \tau(\mathbf{A} - \mathbf{B})],\ \mathbf{A}, \mathbf{B} \in D, \end{aligned}$$

където $\tau(\mathbf{A} - \mathbf{B}) = \tau(\mathbf{A} + \mathbf{B}) = \sigma(\alpha\omega(A) + \beta\omega(B)) = \gamma$ се дава с (33).

Аналогично получаваме

$$\text{(40)} \quad [A;\alpha] \times [B;\beta] = [A \times^{\alpha\beta} B; \tau([A;\alpha] \times [B;\beta])] \text{ за } A, B \in I(R)^*, \alpha, \beta \in \Lambda,$$

$$\text{(41)} \qquad \mathbf{A} \times \mathbf{B} = [A \times^{\tau(\mathbf{A})\tau(\mathbf{B})} B; \tau(\mathbf{A} \times \mathbf{B})],\ \mathbf{A}, \mathbf{B} \in D^*,$$

където $\times^+ = \times$ е външното произведение, а $\times^-$ — вътрешното произведение, дефинирано в $I(R)^*$ с

$$\begin{aligned} A \times^- B &= \begin{cases} [a^{(-\sigma(B))} b^{(\sigma(A))}, a^{(\sigma(B))} b^{(-\sigma(A))}], & \chi(A) \geq \chi(B), \\ [a^{(\sigma(B))} b^{(-\sigma(A))}, a^{(-\sigma(B))} b^{(\sigma(A))}], & \chi(A) > \chi(B), \end{cases} \\ \chi(A) &= a^{(-\sigma(A))} / a^{(\sigma(A))}. \end{aligned}$$

За участвуващата в (41) посока на произведението имаме $\tau(\mathbf{A} \times \mathbf{B}) = +$ в случай, че някой от интервалите $\mathbf{A}, \mathbf{B}$ е нула, а за интервали A, B от $D^* \setminus \{0\}$ посоката се дава със:

$$\begin{aligned} \gamma = \tau(\mathbf{A} \times \mathbf{B}) &= \tau([A;\alpha] \times [B;\beta]) \\ &= \begin{cases} \alpha, & \alpha = \beta, \\ \alpha, & \alpha = -\beta,\ \chi(A) < \chi(B), \\ \beta, & \alpha = -\beta,\ \chi(A) > \chi(B), \\ +, & \alpha = -\beta,\ \chi(A) = \chi(B), \end{cases} \\ &= \begin{cases} \alpha, & \chi(A) < \chi(B), \\ \beta, & \chi(A) > \chi(B), \\ \alpha, & \chi(A) = \chi(B),\ \alpha = \beta, \\ +, & \chi(A) = \chi(B),\ \alpha = -\beta. \end{cases} \end{aligned}$$

Последната формула може да се представи в компактна форма така: $\gamma = \sigma(\alpha\chi(B) + \beta\chi(A))$. (Същото представяне е в сила и когато и двата интервала съдържат нула във вътрешността си, стига в този случай да се използува разширената дефиниция на χ.)

Получихме, че собствената част на произведение на два насочени интервала от D^* е или външното, или вътрешното произведение на собствените им части в зависимост от посоките на насочените интервали.

Ако произведението на насочените интервали $\mathbf{A}$, $\mathbf{B} \in D \setminus \mathcal{T}$ не е изроден интервал, то неговата посока може да се определи посредством функционала $\tau_2 : D \setminus \mathcal{T} \bigotimes D \setminus \mathcal{T} \to \Lambda$, дефиниран със

$$\text{(42)} \qquad \tau_2(\mathbf{A}, \mathbf{B}) = \tau_2([A;\alpha], [B;\beta]) = \begin{cases} \alpha, & \chi(A) \leq \chi(B), \\ \beta, & \chi(A) > \chi(B). \end{cases}$$

В някои случаи вместо $\tau(\mathbf{A}\times\mathbf{B})$ можем да използуваме $\tau_2(\mathbf{A},\mathbf{B})$ (вж. твърдения 6 и 7 от следващата точка).

За да представим собствената част на частно на два насочени интервала $\mathbf{A}/\mathbf{B} = \mathbf{A} \times (1/\mathbf{B})$, са необходими двата типа интервално деление за собствени интервали — външното деление „$/$", дефинирано с (2), респ. (8), и вътрешното деление „$/^-$", дефинирано за $A \in I(R)^*, B \in I(R) \setminus Z$ посредством

$$A/^- B = \begin{cases} [a^{(-\sigma(B))}/b^{(\sigma(A))}, a^{(\sigma(B))}/b^{(-\sigma(A))}], & \chi(A) \geq \chi(B), \\ [a^{(\sigma(B))}/b^{(-\sigma(A))}, a^{(-\sigma(B))}/b^{(\sigma(A))}], & \chi(A) > \chi(B). \end{cases}$$

Използувайки външно и вътрешно деление, можем да запишем

$$(43)\qquad \begin{aligned} \mathbf{A}/\mathbf{B} &= [A/^{\tau(\mathbf{A})\tau(\mathbf{B})}B; \tau(\mathbf{A}/\mathbf{B})],\ \mathbf{A} \in D^*, \mathbf{B} \in D \setminus T, \\ \tau(\mathbf{A}/\mathbf{B}) &= \tau([A;\alpha]/[B;\beta]) = \tau(\mathbf{A} \times \mathbf{B}). \end{aligned}$$

Всички пресмятания с насочени интервали могат да бъдат извършвани в нормална форма. Да дадем някои примери. Съгласно (41) произведението с изроден интервал a се изразява посредством $a \times [B;\beta] = a \times [b^{(-)}, b^{(+)}; \beta] = [ab^{(-\sigma(a))}, ab^{(\sigma(a))}; \beta]$. Ако $a = -1$, имаме $(-1) \times [b^{(-)}, b^{(+)}; \beta] = -[b^{(-)}, b^{(+)}; \beta] = [-b^{(+)}, -b^{(-)}; \beta]$. Вижда се, че операторът отрицание $\mathrm{neg}[B;\beta] = -[B;\beta] = [-B;\beta] = [-b^{(+)}, -b^{(-)}; \beta]$ запазва посоката и сменя знака на аргумента.

Обратният адитивен на $\mathbf{A} = -\overline{\mathbf{A}} = [A;\alpha] = [a^{(-)}, a^{(+)}; \alpha]$ е насоченият интервал $\mathrm{inva}(\mathbf{A}) = [-A; -\alpha] = [-a^{(+)}, -a^{(-)}; -\alpha]$. Наистина от (34) имаме $[a^{(-)}, a^{(+)}; \alpha] + [-a^{(+)}, -a^{(-)}; -\alpha] = [0, 0; \pm] = 0$. Обратният адитивен променя както посоката, така и знака на насочения интервал.

Обратният адитивен на отрицателния е спрегнатият интервал

$$\mathrm{con}([a^{(-)}, a^{(+)}; \alpha]) = [a^{(-)}, a^{(+)}; \alpha]_- = \mathrm{inva}(-[A;\alpha]) = [a^{(-)}, a^{(+)}; -\alpha].$$

Операторът спрягане обръща посоката, но запазва знака. Имаме

$$[a^{(-)}, a^{(+)}; \alpha]_- = [a^{(-)}, a^{(+)}; -\alpha],$$

съответно $[A;\alpha]_- = [A; -\alpha]$. По-общо за $\lambda \in \Lambda$ имаме $\mathbf{A}_\lambda = [A;\alpha]_\lambda = [A; \lambda\alpha]$. Двоичната променлива λ е индикатор за наличност или липса на оператор за спрягане. Това означение е важно за софтуерни реализации; вместо да се разменят краищата на интервала, сменя се само стойността на двоичната променлива „посока".

Аналогично, обратният мултипликативен на $[A;\alpha] = [a^{(-)}, a^{(+)}; \alpha]$ е насоченият интервал

$$\mathrm{invm}(\mathbf{A}) = 1/\overline{\mathbf{A}} = [1/A; \alpha]_- = [1/A; -\alpha] = [1/a^{(+)}, 1/a^{(-)}; -\alpha].$$

Имаме $[a^{(-)}, a^{(+)}; \alpha] \times [1/a^{(+)}, 1/a^{(-)}; -\alpha] = [1, 1; \pm] = 1$.

Видяхме, че за насочените интервали от D^* има прости връзки между хиперболичното и интервалното умножение (вж. (29)). В нормален запис връзките между $\times_h$ и $\times$ се дават със

$$[A;\alpha] \times [B;\beta] = [A;\alpha\sigma(B)] \times_h [B;\beta\sigma(A)], \quad (44)$$

$$[A;\alpha] \times_h [B;\beta] = [A;\alpha\sigma(B)] \times [B;\beta\sigma(A)], \quad (45)$$

където A, B са произволни интервали от $I(R) \setminus Z$.

Всяко съотношение за насочени интервали, записано като интервално-аритметичен израз, поражда съотношение за собствените части на участвуващите в него интервали, т. е. израз за нормални интервали. Ще демонстрираме смисъла на това твърдение, като изведем някои основни закони за нормални интервали, изхождайки от съответни закони за насочени интервали.

6. ОСНОВНИ ЗАКОНИ ЗА НОРМАЛНИ ИНТЕРВАЛИ, ИЗВЕДЕНИ ЧРЕЗ НАСОЧЕНИ ИНТЕРВАЛИ

Свойствата на нормалните интервали (вж. свойства **S1**–**S5**, **M1**–**M5**) могат да се получат просто като частен случай на съответни свойства на насочените интервали. Следващите твърдения демонстрират техника за получаване на такива свойства.

Твърдение 5 (**комутативен закон в** $\mathcal{M}$).
а) *За* $A, B \in I(R)$ *и* $\lambda \in \Lambda$ *имаме* $A +^{\lambda} B = B +^{\lambda} A$;
б) *За* $A, B \in I(R)^*$ *и* $\lambda \in \Lambda$ *имаме* $A \times^{\lambda} B = B \times^{\lambda} A$.

Доказателство. Съгласно **K1** имаме $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, $\mathbf{A}, \mathbf{B} \in D$. Като използуваме (33), (36) и разгледаме поотделно случаите, когато $\mathbf{A}, \mathbf{B}$ имат еднакви и различни посоки, получаваме съответно за собствените им части равенствата $A + B = B + A$, $A +^{-} B = B +^{-} A$.

Твърдение 6 (**условно-асоциативен закон в** $\mathcal{M}$).
а) *За* $A, B, C \in I(R)$ *и* $\alpha, \beta, \gamma \in \Lambda$ *имаме*

$$(A +^{\alpha\beta} B) +^{\gamma\tau_1(\mathbf{A},\mathbf{B})} C = A +^{\alpha\tau_1(\mathbf{B},\mathbf{C})} (B +^{\beta\gamma} C); \quad (46)$$

б) *За* $A, B, C, D \in I(R)$ *и* $\alpha, \beta, \gamma, \delta \in \Lambda$ *имаме*

$$\begin{aligned}(A +^{\alpha\beta} B) &+^{\tau_1(\mathbf{A},\mathbf{B})\tau_1(\mathbf{C},\mathbf{D})} (C +^{\gamma\delta} D) \\ &= (A +^{\alpha\gamma} C) +^{\tau_1(\mathbf{A},\mathbf{C})\tau_1(\mathbf{B},\mathbf{D})} (B +^{\beta\gamma} D);\end{aligned} \quad (47)$$

в) *За* $A, B, C \in I(R)^*$ *и* $\alpha, \beta, \gamma \in \Lambda$ *имаме*

$$(A \times^{\alpha\beta} B) \times^{\gamma\tau_2(\mathbf{A},\mathbf{B})} C = A \times^{\alpha\tau_2(\mathbf{B},\mathbf{C})} (B \times^{\beta\gamma} C); \quad (48)$$

г) *За* $A, B, C, D \in I(R)^*$ *и* $\alpha, \beta, \gamma, \delta \in \Lambda$ *имаме*

$$\begin{aligned}(A \times^{\alpha\beta} B) &\times^{\tau_2(\mathbf{A},\mathbf{B})\tau_2(\mathbf{C},\mathbf{D})} (C \times^{\gamma\delta} D) \\ &= (A \times^{\alpha\gamma} C) \times^{\tau_2(\mathbf{A},\mathbf{C})\tau_2(\mathbf{B},\mathbf{D})} (B \times^{\beta\gamma} D),\end{aligned} \quad (49)$$

където τ_1, τ_2 *са дефинираните с* (33), (42) *знакови функции за сума, съответно произведение на насочените интервали* $\mathbf{A} = [A;\alpha]$, $\mathbf{B} = [B;\beta]$, $\mathbf{C} = [C;\gamma]$, $\mathbf{D} = [D;\delta]$.

Доказателство. Ще докажем a). Като заместим $\mathbf{A} = [A;\alpha]$, $\mathbf{B} = [B;\beta]$, $\mathbf{C} = [C;\gamma]$ в $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$ и използуваме (36), получаваме $[A+^{\alpha\beta} B; \tau(\mathbf{A}+\mathbf{B})]+[C;\gamma] = [A;\alpha]+[B+^{\beta\gamma} C; \tau(\mathbf{B}+\mathbf{C})]$. Първо сумираме, после сравняваме собствените части на изразите от двете страни на равенството и получаваме асоциативния закон (46). Асоциативният закон (47) се извежда по аналогичен начин; той играе важна роля в интервалния анализ [19]. Съотношенията (48) и (49) се получават аналогично с помощта на (41), (42). □

Забележка. Навсякъде във формули (46)–(49) можем да пишем вместо τ_1, τ_2 посоката τ на съответната сума или произведение.

Тъждеството (46) обобщава условно-асоциативните връзки, отнасящи се до всевъзможните комбинации между операциите $+, +^-$ (вж. правило **M2** от т. 3). Използувайки (46), можем да сменим реда на извършване на операциите в произволен израз, в който участвуват две събирания (външни или вътрешни) на нормални интервали.

Твърдение 7 (условно-дистрибутивен закон в $\mathcal{M}$). *За произволни* $A, B, C \in I(R)^*$, *такива че* $A + B \in I(R)^*$, *и за произволни* $\alpha, \beta, \gamma \in \Lambda$ *е изпълнено*

$$\begin{aligned}(A +^{\alpha\beta} B) \times^{\gamma\tau_1(\mathbf{A},\mathbf{B})} C \\ = (A \times^{\alpha\gamma\sigma(A)\sigma(A+B)} C) +^{\tau_2(\mathbf{A},\mathbf{C})\tau_2(\mathbf{B},\mathbf{C})} (B \times^{\beta\gamma\sigma(B)\sigma(A+B)} C).\end{aligned}$$

Доказателство. Като положим $\mathbf{A} = [A;\alpha]$, $\mathbf{B} = [B;\beta], \mathbf{C} = [C;\gamma]$ в твърдение 2 и използуваме (36), (41), получаваме

$$[A +^{\alpha\beta} B; \tau(\mathbf{A} + \mathbf{B})] \times [C;\gamma]$$

$$= [A \times^{\alpha\gamma\sigma(A)\sigma(A+B)} C; \tau(\mathbf{A} \times \mathbf{C})] + [B \times^{\beta\gamma\sigma(B)\sigma(A+B)} C; \tau(\mathbf{B} \times \mathbf{C})].$$

Като използуваме отново нормалните форми за операцията „$\times$“ в лявата страна и за операцията „$+$“ в дясната и сравним собствените части в двете страни на горното уравнение, получаваме твърдението. □

Твърдение 7 резюмира дистрибутивния закон на разширената интервална аритметика (вж. правило **M5** от т. 3).

Методът на изследване на множеството $I(R)$ от нормалните интервали, произтичащ от структурата $\mathcal{K}$ на насочените интервали, води до фундаментални релации в $I(R)$, представени в сбита форма. Известните досега подобни релации имат твърде необозрима форма и не са така удобни за автоматична символна обработка (вж. напр. [4, 6, 29–31]).

Алгебричните свойства на системата $\mathcal{M} = (I(R), +, +^-, \times, \times^-, \subseteq)$ са изучени в [4–7, 16–18]; тези свойства съдържат и разширяват свойствата на стандартните интервално-аритметични операции. Нестандартните операции се използуват в интервалния анализ [17, 19, 20]. Смисълът на нестандартните операции става прозрачен, когато те се използуват за пресмятане на обхвати на монотонни функции, което ще разгледаме в следващата точка.

Представяне на насочени интервали в CR-форма. Ще използуваме два начина за представяне на насочени интервали с помощта на техните центрове и радиуси: единия начин ще наричаме (обикновено) CR-представяне, а другия — CR-представяне в нормална форма.

CR-представяне на насочени интервали. Насочените интервали се представят във вида $(c(\mathbf{A}), r(\mathbf{A}))$, където $c(\mathbf{A})$ и $r(\mathbf{A})$ са реални числа, които могат да вземат и отрицателни стойности ($r(\mathbf{A}) < 0$ означава, че интервалът $\mathbf{A}$ е несобствен). Формулите за преход от CW-форма в CR-форма и обратно се дават със

$$c(\mathbf{A}) = (a^- + a^+)/2,\ r(\mathbf{A}) = (a^+ - a^-)/2; \quad a^- = c(\mathbf{A}) - r(\mathbf{A}),\ a^+ = c(\mathbf{A}) + r(\mathbf{A}).$$

Тук привеждаме операциите в D, записани в CR-форма. Използувани са следните означения: $|c(\mathbf{A})| = \sigma(\mathbf{A})c(\mathbf{A})$, $\overline{r}(\mathbf{A}) = \sigma(\mathbf{A})r(\mathbf{A})$.

$$\begin{aligned}
\mathbf{A} + \mathbf{B} &= (c(\mathbf{A}) + c(\mathbf{B}), r(\mathbf{A}) + r(\mathbf{B})),\\
\mathbf{A} - \mathbf{B} &= (c(\mathbf{A}) - c(\mathbf{B}), r(\mathbf{A}) + r(\mathbf{B})),\\
\mathbf{A} \times \mathbf{B} &= (c(\mathbf{A})c(\mathbf{B}) + \overline{r}(\mathbf{A})\overline{r}(\mathbf{B}), |c(\mathbf{A})|r(\mathbf{B}) + |c(\mathbf{B})|r(\mathbf{A})),\ \mathbf{A}, \mathbf{B} \in D^*,\\
\mathbf{A}/\mathbf{B} &= (\delta^2(c(\mathbf{A})c(\mathbf{B}) + \overline{r}(\mathbf{A})\overline{r}(\mathbf{B})), (\delta^2(|c(\mathbf{A})|r(\mathbf{B}) + |c(\mathbf{B})|r(\mathbf{A}))),\\
&\qquad \mathbf{A} \in D^*, \mathbf{B} \in D \setminus \mathcal{T},
\end{aligned}$$

където $\delta^2 = \delta^2(\mathbf{B}) = (c^2(\mathbf{B}) - r^2(\mathbf{B}))^{-1}$.

За хиперболичното умножение имаме

$$\mathbf{A} \times_h \mathbf{B} = (c(\mathbf{A})c(\mathbf{B}) + r(\mathbf{A})r(\mathbf{B}), |c(\mathbf{A})|r(\mathbf{B}) + |c(\mathbf{B})|r(\mathbf{A})),\ \mathbf{A}, \mathbf{B} \in D^*.$$

За нестандартните операции са в сила представянията

$$\begin{aligned}
\mathbf{A} +^- \mathbf{B} &= (c(\mathbf{A}) + c(\mathbf{B}), |r(\mathbf{A}) - r(\mathbf{B})|),\\
\mathbf{A} -^- \mathbf{B} &= (c(\mathbf{A}) - c(\mathbf{B}), |r(\mathbf{A}) - r(\mathbf{B})|),\\
\mathbf{A} \times^- \mathbf{B} &= (c(\mathbf{A})c(\mathbf{B}) - \overline{r}(\mathbf{A})\overline{r}(\mathbf{B}), ||c(\mathbf{A})|\rho(\mathbf{B}) - |c(\mathbf{B})|r(\mathbf{A})|),\ \mathbf{A}, \mathbf{B} \in I(R)^*,\\
\mathbf{A}/^-\mathbf{B} &= (\delta^2(c(\mathbf{A})c(\mathbf{B}) - \overline{r}(\mathbf{A})\overline{r}(\mathbf{B})), (\delta^2||c(\mathbf{A})|r(\mathbf{B}) - |c(\mathbf{B})|r(\mathbf{A})|),\\
&\qquad \mathbf{A} \in I(R)^*, \mathbf{B} \in I(R) \setminus Z.
\end{aligned}$$

Формулите за стандартните и нестандартните операции могат да се обобщят по следния начин:

$$\begin{aligned}
\mathbf{A} +^\theta \mathbf{B} &= (c(\mathbf{A}) + c(\mathbf{B}), |r(\mathbf{A})\ \theta\ r(\mathbf{B})|),\\
\mathbf{A} -^\theta \mathbf{B} &= (c(\mathbf{A}) - c(\mathbf{B}), |r(\mathbf{A})\ \theta\ r(\mathbf{B})|),\\
\mathbf{A} \times^\theta \mathbf{B} &= (c(\mathbf{A})c(\mathbf{B})\ \theta\ \overline{r}(\mathbf{A})\overline{r}(\mathbf{B}), ||c(\mathbf{A})|r(\mathbf{B})\ \theta\ |c(\mathbf{B})|r(\mathbf{A})|),\ \mathbf{A}, \mathbf{B} \in I(R)^*,\\
\mathbf{A}/^\theta\mathbf{B} &= (\delta^2(c(\mathbf{A})c(\mathbf{B})\ \theta\ \overline{\rho}(\mathbf{A})\overline{r}(\mathbf{B})), (\delta^2||c(\mathbf{A})|r(\mathbf{B})\ \theta\ |c(\mathbf{B})|r(\mathbf{A})|),\\
&\qquad \mathbf{A} \in I(R)^*, \mathbf{B} \in I(R) \setminus Z,
\end{aligned}$$

където $\theta = \pm$.

CR-представяне в нормална форма за насочени интервали. Това представяне е много близко до CR-представянето. Разликата е в това, че за знака на радиуса се предвижда отделна променлива $\alpha = \sigma(r(\mathbf{A})) = \tau(\mathbf{A})$. Тогава радиусът $r(\mathbf{A})$ се записва във вида $r(\mathbf{A}) = \alpha\rho(\mathbf{A})$, с $\rho(\mathbf{A}) \geq 0$, т. е. имаме $\mathbf{A} = (c(\mathbf{A}), \alpha\rho(\mathbf{A})) = (c(\mathbf{A}), \tau(\mathbf{A})\rho(\mathbf{A})$. За аргумента на c и ρ не е от значение дали е насочен интервал или нормалната му част, т. е. $c(\mathbf{A}) = c(A)$, $\rho(\mathbf{A}) = \rho(A)$, поради което ще пишем още $\mathbf{A} = (c(A), \alpha\rho(A))$. Като означим както по-рано $\overline{r}(\mathbf{A}) = \sigma(\mathbf{A})r(\mathbf{A}) = \sigma(A)\alpha\rho(A) = \alpha\sigma(A)\rho(A) = \alpha\overline{\rho}(A)$, изразите за интервалните операции могат да се запишат във вида

$$\begin{aligned}
\mathbf{A}+\mathbf{B} &= (c(A)+c(B), \alpha\rho(A)+\beta\rho(B)),\\
\mathbf{A}-\mathbf{B} &= (c(A)-c(B), \alpha\rho(A)+\beta\rho(B)),\\
\mathbf{A}\times\mathbf{B} &= (c(A)c(B)+\alpha\overline{\rho}(A)\beta\overline{\rho}(B), |c(A)|\beta\rho(B)+|c(B)|\alpha\rho(A)),\ \mathbf{A},\mathbf{B}\in D^*,\\
\mathbf{A}/\mathbf{B} &= (\delta^2(c(A)c(B)+\alpha\overline{\rho}(A)\beta\overline{\rho}(B)), (\delta^2(|c(A)|\beta\rho(B)+|c(B)|\alpha\rho(A))),\\
&\qquad \mathbf{A}\in D^*, \mathbf{B}\in D\setminus \mathcal{T},
\end{aligned}$$

където $\delta^2 = \delta^2(\mathbf{B}) = (c^2(B) - \rho^2(B))^{-1}$.

От тези формули веднага се получава посоката на резултантния интервал, именно тя е равна на знака на втората компонента. Например $\tau(\mathbf{A}+\mathbf{B}) = \sigma(\alpha\rho(A)+\beta\rho(B))$ съвпада с (33), $\tau(\mathbf{A}\times\mathbf{B}) = \sigma(|c(A)|\beta\rho(B) + |c(B)|\alpha\rho(A))$ съвпада с (42), т. е. с израза $\sigma(\alpha\chi(B)+\beta\chi(A))$. Това показва, че роля, подобна на функционала χ, играе функционал от вида $(a^{(-)}+a^{(+)})/(a^{(-)}-a^{(+)})$.

7. ПРЕДСТАВЯНЕ И ПРЕСМЯТАНЕ НА ОБХВАТИ И НАСОЧЕНИ ОБХВАТИ НА МОНОТОННИ ФУНКЦИИ

Нека $CM(T)$ е множеството от всички непрекъснати и монотонни функции, дефинирани в $T = [t^{(-)}, t^{(+)}] \in I(R)$. Множеството $f(T) = \{f(t)|t\in T\}$ от функционални стойности на f ще наричаме обхват на f (върху T). Ако $f \in CM(T)$, то за обхвата на f имаме или $f(T) = [f(t^{(-)}), f(t^{(+)})]$, или $f(T) = [f(t^{(+)}), f(t^{(-)})]$ в зависимост от типа монотонност. На всяка $f \in CM(T)$ съпоставяме двоична променлива $\tau_f = \tau(f;T) \in \Lambda = \{+,-\}$, която определя типа на монотонност на f посредством

$$\tau(f;T) = \sigma(f(x^+ - f(x^-) = \begin{cases} +, & f(t^-) \leq f(t^+);\\ -, & f(t^-) > f(t^+).\end{cases}$$

За $f(t^{(-)}) = f(t^{(+)})$, т. е. $f = \mathrm{const}$, приемаме за определеност $\tau(f,T) = +$. За $f, g \in CM(T)$ равенството $\tau_f = \tau_g$ означава, че двете функции f, g са изотонни (ненамаляващи) или двете са антитонни (нерастящи) в T; $\tau_f = -\tau_g$ означава, че едната от двете функции е изотонна, а другата — антитонна. В следващото твърдение използуваме и означението $\tau_{|f|} = \tau(|f|;T)$ за монотонни функции, които не си менят знака в T.

Твърдение 8 ([20]). *Нека $f, g \in CM(T)$. Тогава за $X \subseteq T$ имаме:*

i) ако $f+g \in CM(T)$, *то* $(f+g)(X) = f(X) +^{\tau_f \tau_g} g(X)$;

ii) ако $f-g \in CM(T)$, *то* $(f-g)(X) = f(X) -^{-\tau_f \tau_g} g(X)$.

Нека в допълнение на предположението $f, g \in CM(T)$ *функциите* f, g *не променят знака си в* T. *Тогава за* $X \subseteq T$:

iii) ако $fg \in CM(T)$, *то* $(fg)(X) = f(X) \times^{\tau_{|f|} \tau_{|g|}} g(X)$;

iv) ако $f/g \in CM(T)$, $g(x) \neq 0$, $x \in T$, *то* $(f/g)(X) = f(X) /^{-\tau_{|f|} \tau_{|g|}} g(X)$.

Пример 1. Да означим $\exp(-X) = \{\exp(-x) \mid x \in X\}$, $\mathrm{arctg} X = \{\mathrm{arctg} x \mid x \in X\}$. С помощта на твърдение 8 получаваме за обхвата на сумата $h(x) = \exp(-x) + \mathrm{arctg} x$ израза $h(X) = \exp(-X) +^- \mathrm{arctg} X$ за всяко $X \in I(R), 0 \notin X$; ще отбележим, че в интервалната аритметика $\mathcal{S}$ не можем да представим $h(X)$ посредством обхватите $\exp(-X)$ и $\mathrm{arctg} X$.

Нека $X \in I(R)$ е фиксиран интервал. Ако f е непрекъсната в X, то $\min_{x \in X} f(x)$, $\max_{x \in X} f(x)$ съществуват; вместо тези означения ще пишем кратко $\min f$, съответно $\max f$. Нека f, g са непрекъснати в X. Имаме:

а) $\min f + \min g \leq \min(f+g)$, $\max(f+g) \leq \max f + \max g$;

б) $\min(f+g) \leq \min f + \max g \leq \max(f+g)$, $\min(f+g) \leq \max f + \min g \leq \max(f+g)$.

Горните неравенства означават, че интервалът $(f+g)(X) = [\min(f+g), \max(f+g)]$:

а) се съдържа в интервала с краища $\min f + \min g$, $\max f + \max g$, т. е. в интервала $f(X) + g(X)$;

б) съдържа интервала с краища $\min f + \max g$, $\max f + \min g$, т. е. интервала $f(X) +^- g(X)$.

В символен запис получаваме $f(X) +^- g(X) \subseteq (f+g)(X) \subseteq f(X) + g(X)$. С аналогични разсъждения за останалите операции (изваждане, умножение и деление) намираме:

Твърдение 9. *Функциите* f, g *са непрекъснати в* $D \subseteq R$. *За* $* \in \{+, -, \times, /\}$ *и за всяко* $X \subseteq D, X \in I(R)$ *имаме* $f(X) *^- g(X) \subseteq (f * g)(X) \subseteq f(X) * g(X)$.

Горното твърдение показва, че външните операции са удобни за получаване на външни включвания, докато вътрешните интервални операции служат за вътрешни включвания. Това оправдава наименованието „вътрешни интервални оперéции", използувано от някои автори (вж. напр. [40]). Пример за използуване на вътрешни включвания има в [5].

Ще формулираме аналог на твърдение 8 за насочени интервали. Тук ще дефинираме насочен обхват чрез допускане на несобствени интервали за аргументи на функцията по следния начин:

Дефиниция. Нека $T \in I(R)$, $f \in CM(T)$. Нека $\mathbf{X} = [x^-, x^+] \in D$, $\mathbf{X} \subseteq T$. Насоченият обхват на f върху $\mathbf{X}$ е насоченият интервал $f(\mathbf{X}) = [f(x^-), f(x^+)]$. Неговата посока $\tau(f(\mathbf{X})) = \sigma(f(x^+ - f(x^-)$ ще наричаме монотонност на f върху $\mathbf{X}$ и ще означаваме още с $\tau(f; \mathbf{X})$.

Твърдение 10. *Нека* $f, g \in CM(T)$. *За* $\mathbf{X} \subseteq T$ *имаме:*

i) ако $f+g \in CM(T)$, *то* $(f+g)(\mathbf{X}) = f(\mathbf{X}) + g(\mathbf{X})$;

ii) ако $f-g \in CM(T)$, *то* $(f-g)(\mathbf{X}) = f(\mathbf{X}) - g(\mathbf{X})_- = f(\mathbf{X}) -_h g(\mathbf{X})$.

Да предположим в допълнение на $f, g \in CM(T)$, че функциите f, g не си променят знака в T. Тогава за $\mathbf{X} \subseteq T$:

iii) ако $fg \in CM(T)$, то

$$(fg)(\mathbf{X}) = f(\mathbf{X})_{\sigma(g(X))} \times g(\mathbf{X})_{\sigma(f(X))} = f(\mathbf{X}) \times_h g(\mathbf{X});$$

iv) ако $f/g \in CM(T)$, $0 \notin g(X)$, то

$$(f/g)(\mathbf{X}) = f(\mathbf{X})_{\sigma(g(X))}/g(\mathbf{X})_{-\sigma(f(X))} = f(\mathbf{X})/_h g(\mathbf{X}).$$

В последните две равенства $\sigma(f(X))$ означава знака на интервала $f(X)$ (в случая е без значение дали ще пишем $\sigma(f(\mathbf{X}))$ или $\sigma(f(X))$). Така например имаме $g(\mathbf{X})_{\sigma(f(X))} = \{g(\mathbf{X})$, ако $f \geq 0$; $\mathrm{con}(g(\mathbf{X}))$, ако $f \leq 0\}$.

Твърдение 10 ни дава посоката на резултантните интервали, следователно и допълнителна информация (в сравнение с твърдение 8) за типа на монотонност (изотонност или антитонност) на резултата $f * g$, $* \in \{+, -, \times, /\}$.

Нека $\mathbf{X}$ е насочен интервал и X е собствената му част, а ξ е посоката му. От твърдение 10 *iii*) виждаме, че ако f не променя знака си в X, то насоченият обхват на функцията $-f = (-1)f$ върху $\mathbf{X}$ удовлетворява $(-f)(\mathbf{X}) = -f(\mathbf{X})_- = -[f(X); \tau(f; \mathbf{X})]_- = [-f(X); -\tau(f; \mathbf{X})]$. В частност насоченият обхват на функцията $f(x) = -x$ върху $\mathbf{X}$ е насоченият интервал $-(\mathbf{X})_- = [-X; -\tau(\mathbf{X})] = [-X; -\xi]$, т. е. обратно-адитивният насочен интервал на насочения обхват $[X; \xi]$ на функцията x върху $\mathbf{X}$. Сега можем да направим обичайната в интервалния анализ илюстрация с пресмятането на обхвата на функцията $x - x$ (както знаем, резултатът в обикновената интервална аритметика $\mathcal{S}$ е $X - X = [x^{(-)} - x^{(+)}, x^{(+)} - x^{(-)}]$, който е интервал с дължина $2\omega(X)$). В насочена интервална аритметика обхватът на функцията $x - x = x + (-x)$ върху насочения интервала $\mathbf{X} = [x^-, x^+]$ дава съгласно твърдение 10 *i*) $[X; \xi] + [-X; -\xi] = [X -^- X; +] = 0$.

Като друго следствие на твърдение 10 за обхвата на функцията $h(x) = 1/g(x)$ върху $\mathbf{X}$ получаваме $h(\mathbf{X}) = 1/g(\mathbf{X})_-$. В частност, ако $h(x) = 1/x$, то за $\mathbf{X} = [x^-, x^+]$, такъв че $0 \notin X$, имаме $h(\mathbf{X}) = 1/(\mathbf{X})_- = 1/[x^-, x^+; -] = [1/X; -\xi] = [1/x^+, 1/x^-; -\xi]$.

Пример 2. Както в пример 1 да означим $\exp(-X) = \{\exp(-x) | x \in X\}$ и $\mathrm{arctg}X = \{\mathrm{arctg}x | x \in X\}$. За съответните насочени обхвати имаме $\exp(-\mathbf{X}) = [\exp(-X); -\xi]$, съответно $\mathrm{arctg}\mathbf{X} = [\mathrm{arctg}X; \xi]$. От твърдение 8 получаваме за насочения обхват на $h(x) = \exp(-x) + \mathrm{arctg}x$ върху $\mathbf{X}$ израза $h(\mathbf{X}) = \exp(-\mathbf{X}) + \mathrm{arctg}\mathbf{X}$ за всяко $\mathbf{X} \in D$, върху което $h(x)$ е монотонна, т. е. за $\mathbf{X} \in D^*$. Сравнен с резултата за $h(X)$ от пример 1, тук $h(\mathbf{X})$ съдържа допълнителна информация за типа на монотонност на h върху $\mathbf{X}$.

Този пример подсказва следното правило за работа с функция от функция. Ако са дадени функции $g \in CM(T)$, $f \in CM(g(T))$, $h = f(g) \in CM(T)$, то очевидно имаме за посоката на h върху $\mathbf{X}$ равенството $\tau(h, \mathbf{X}) = \tau(f, g(\mathbf{X}))$.

Пример 3. Да разгледаме функцията $h(x) = 1 - x + x^2$, $x \in X = [0, 1]$. Означавайки $f(x) = 1 - x$, $g(x) = x^2$, имаме $F(X) = 1 - X$, $G(X) = X \times X =$

X^2. Разделяме интервала $[0,1]$ на два подинтервала $[0,1/2]$, $[1/2,1]$, такива че f, g и h са монотонни във всеки подинтервал. Съгласно твърдение 8 можем да пресметнем точно обхвата на функцията h във всеки подинтервал, използувайки вътрешна сума $H([0,1/2]) = (1-X) +^- X^2 = [3/4,1]$, $H([1/2,1]) = (1-X) +^- X^2 = [3/4,1]$, и получаваме окончателно $H(X) = H([0,1/2]) \cup H([1/2,1]) = [3/4,1]$. Ако не използуваме съображения за монотонност, то можем да получим съотношения на включване; външното събиране води до външно включване, докато вътрешното събиране дава вътрешно включване. Наистина имаме $F(X) + G(X) = [0,1] + [0,1] = [0,2]$ и $F(X) +^- G(X) = [0,1] +^- [0,1] = [1,1]$.

Разгледаните обобщени интервални структури позволяват не само да получаваме (евентуално груби) външни включвания с помощта на $\mathcal{S}$-операциите, но и да намираме вътрешни включвания, както и да представяме точно обхвати на функции (при отчитане на монотонности). Възможността да се използуват равенства вместо включвания може да се разпространи върху рационални функции, тъй като те имат свойството да са монотонни в даден интервал с евентуално изключение на краен брой точки от него. В [2] е показано как вътрешните операции в $\mathcal{M}$ могат да се използуват за получаване на тесни включвания за обхвати на функции.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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TWO CUT-FREE MODAL SEQUENT CALCULI

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Андрей Арсов. ДВЕ МОДАЛЬНЫЕ СЕКВЕНЦИАЛЬНЫЕ ИСЧИСЛЕНИЯ И УСТРАНЕНИЕ СЕЧЕНИЙ ДЛЯ НИХ

В этой статье рассматривается модальный подход к терминологическим языкам. Вводятся две секвенциальные исчисления для $\mathcal{AL}$ и $\mathcal{ALN}$ и доказывается теорема об устранении сечений для них.

Andrei Arsov. TWO CUT-FREE MODAL SEQUENT CALCULI

In this paper the modal approach to concept languages is considered. Two sequent-style calculi for the modal systems $\mathcal{AL}$ and $\mathcal{ALN}$ are introduced and the cut-elimination property is proved.

1. INTRODUCTION

In recent years there has been a growing interest in presenting formalisms and languages that will be able to express various knowledge in a domain of discourse. One such example are the so-called terminological or concept languages. The brief overview that we will present uses the conventions established in [1]. In the concept languages expressions are built from concepts and roles, which are interpreted as subsets and binary relations on a given universe. Further one can define compound expressions from the primitive concepts using a number of constructs. Two such constructs are intersection and complement of concepts (restricted and unrestricted). Roles are used in the so-called restricted quantification. The restricted quantification of a concept C over a role R gives a concept whose elements x are such that if x is R-connected to an element y, this y is in C.

Another construct used in most concept languages is the number restriction. The number restriction over a role R gives a set of objects or a concept, the elements of which have at least or at most a certain number of R-connections.

As far as we know, the almost obvious connection between modal languages and concept languages was considered for the first time in [5]. In [1] a whole hierarchy of such languages is built — the $\mathcal{AL}$-languages — and their relation with modal languages and systems is used to obtain some complexity results on the satisfaction for the $\mathcal{AL}$-languages. In [4] special modal axiomatic systems are developed for these languages and they are also viewed from the perspective of generalized quantifiers. In the next section we present the precise definition and semantics of the languages we shall be interested in, as well as the axiomatic systems taken from the above mentioned paper.

The axiomatic systems, known up to now, for the terminological languages lack one important feature. What we have in mind is that they are not very suitable for practical derivations. This is quite important since we would like to be able to derive in practice some knowledge that is implicitly embedded on the facts that are known up to a certain moment.

In section 3 we present sequent-style axiomatizations of the validity we have in mind, and we prove one important property of the systems — namely, that the cut-rule can be eliminated from them. It is this feature that makes the derivations in such systems somewhat more feasible.

2. THE MODAL SYSTEMS $\mathcal{AL}$ AND $\mathcal{ALN}$

Definition 2.1. Following Van der Hoek & De Rijke, [4], let us define the basic modal language $\mathcal{AL}$. It has the following elements:

- VAR — a denumerable set of propositional variables;
- $\top$ and $\bot$ — propositional constants;
- $\neg$ and $\wedge$ — classical propositional connectives;
- $\langle R\rangle$ and $[R]$ — modal operators for every binary relation R taken from a collection $\mathcal{R}$.

Now we can define the set Φ of formulas in this language in the following way (we denote the formulas by ϕ, ψ, ...):

- if $p \in VAR$, then $p \in \Phi$ and $\neg p \in \Phi$;
- $\top \in \Phi$ and $\bot \in \Phi$;
- if $\phi_1, \phi_2 \in \Phi$, then also $(\phi_1 \wedge \phi_2) \in \Phi$;
- if $R \in \mathcal{R}$, then $\langle R\rangle\top \in \Phi$;

- if $R \in \mathcal{R}$ and $\phi \in \Phi$, then $[R]\phi \in \Phi$.

Models for $\mathcal{AL}$ have the form $\mathcal{M} = (W, \{ R \}_{R \in \mathcal{R}}, V)$, where W is a non-empty set, each R is a binary relation on W, and V is a valuation, that is: a function assigning subsets of W to proposition letters in the language. Next we define the truth value of a formula ϕ in Φ at a given point x in a model $\mathcal{M}$ (this fact is designated by $\mathcal{M}, x \models \phi$) We have that for every x $\mathcal{M}, x \models \top$, and for no x $\mathcal{M}, x \models \bot$. The other elements of Φ are treated as follows:

$$\begin{aligned}
&\mathcal{M}, x \models p \iff x \in V(p),\\
&\mathcal{M}, x \models \neg p \iff \mathcal{M}, x \not\models p,\\
&\mathcal{M}, x \models (\phi_1 \wedge \phi_2) \iff \mathcal{M}, x \models \phi_1 \text{ and } \mathcal{M}, x \models \phi_2,\\
&\mathcal{M}, x \models [R]\varphi \iff \forall y(Rxy \Rightarrow \mathcal{M}, y \models \phi),\\
&\mathcal{M}, x \models \langle R \rangle \top \iff \exists y Rxy.
\end{aligned}$$

Observe that $\mathcal{AL}$ is a very weak language; it lacks full complementation and full disjunction. It also lacks a full dual to the modal operator $[R]$.

Van der Hoek and De Rijke have proposed an axiomatic system for the language $\mathcal{AL}$ and have proved that it completely captures all validities of the form $\Gamma \vdash \phi$, where $\Gamma \cup \phi$ is a finite set. Such sequents are considered valid in the models if the following is true:

$$\mathcal{M}, x \models \Gamma \vdash \phi \text{ iff } (\mathcal{M}, x \models \Gamma \Rightarrow \mathcal{M}, x \models \phi).$$

They call this the system **AL**. The axioms of this system are:

(A1) $\phi \vdash \phi$,
(A2) $p, \neg p \vdash \bot$,
(A3) $\phi \vdash \top$,
(A4) $\bot \vdash \phi$,
(A5) $\phi, \psi \vdash \phi \wedge \psi$,
(A6) $\phi \wedge \psi \vdash \phi$ and $\phi \wedge \psi \vdash \psi$,
(A7) $[R]\bot, \langle R \rangle \top \vdash \bot$.

Further this system has some rules. Since they also appear in our systems, we refer the reader to the rules (Mon), (Cut), (Distr) and (Compl), presented in subsection 3.1 of section 3.

The next language that we shall consider is given *in the following definition:*

Definition 2.2. The language $\mathcal{ALN}$ has all the elements of $\mathcal{AL}$ plus two other modal operators for every n, which we write as $[R]_n$ and $\langle R \rangle_n$. The set of formulas is further expanded by adding for every n and every relation R in $\mathcal{R}$ the formulas $[R]_n \bot$ and $\langle R \rangle_n \top$ with the following semantics:

$$\begin{aligned}
\mathcal{M}, x \models [R]_n \bot \quad &\text{iff} \quad |\{ y : Rxy\}| \leq n,\\
\mathcal{M}, x \models \langle R \rangle_n \top \quad &\text{iff} \quad |\{ y : Rxy\}| > n.
\end{aligned}$$

Observe that in $\mathcal{ALN}$ the standard modal diamond $\langle R \rangle \cdot$ is the special case of $\langle R \rangle_n \cdot$ with $n = 0$.

To capture again all valid sequents, Van der Hoek and De Rijke have devised the following below axiomatic system and called it **ALN**. It has as axioms all the axioms of the system **AL** plus the following ones:

$(A8)$ $\langle R\rangle_{n+1}\top \vdash \langle R\rangle_n\top$ and $[R]_n\bot \vdash [R]_{n+1}\bot$,

$(A9)$ $[R]_n\bot, \langle R\rangle_n\top \vdash \bot$.

The rules of **ALN** are the same as those of **AL** with the exception that the rule (Compl) in section 3.1 is replaced by the respective rule (Compl) in section 3.2.

3. SEQUENT CALCULI AND CUT ELIMINATION

In this section, which is the heart of the paper, we present the promised cut-free systems in the languages $\mathcal{AL}$ and $\mathcal{ALN}$. These new systems are connected to the systems developed by Van der Hoek and De Rijke and further it is shown in what way.

3.1. THE SYSTEM **SAL** AND THE CUT ELIMINATION PROOF FOR IT

As before, we use the capital Greek letters Γ, Δ, ..., to refer to finite sets of formulas, and ϕ, ψ, ..., to refer to single formulas. A *sequent* in the language $\mathcal{AL}$ is an expression $\Gamma \vdash \phi$. The axiomatic system **SAL** is given by the next axioms and rules:

Axioms:

(A1) $\phi \vdash \phi$,

(A2) $p, \neg p \vdash \phi$,

(A3) $[R]\bot, \langle R\rangle\top \vdash \phi$.

Rules:

(Mon) $$\frac{\Gamma \vdash \chi}{\Gamma \cup \{\phi\} \vdash \chi},$$

(Cut) $$\frac{\Gamma_1 \vdash \phi \qquad \Gamma_2, \{\phi\} \vdash \chi}{\Gamma_1, \Gamma_2 \vdash \chi},$$

($\wedge$-intr-L) $$\frac{\Gamma, \phi \vdash \chi}{\Gamma, \phi \wedge \psi \vdash \chi}, \qquad \frac{\Gamma, \psi \vdash \chi}{\Gamma, \phi \wedge \psi \vdash \chi},$$

($\wedge$-intr-R) $$\frac{\Gamma_1 \vdash \phi \qquad \Gamma_2 \vdash \psi}{\Gamma_1, \Gamma_2 \vdash \phi \wedge \psi},$$

($\top$-drop) $$\frac{\Gamma, \top \vdash \phi}{\Gamma \vdash \phi}, \quad \text{provided } \Gamma \text{ is non-empty},$$

($\bot$-use) $$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi},$$

$$\text{(Compl)}\quad \frac{\Gamma, p \vdash \phi \qquad \Gamma, \neg p \vdash \phi}{\Gamma \vdash \phi}, \qquad \frac{\Gamma, [R]\bot \vdash \phi \qquad \Gamma, \langle R\rangle\top \vdash \phi}{\Gamma \vdash \phi},$$

$$\text{(Distr)}\quad \frac{\Gamma \vdash \chi}{[R]\Gamma \vdash [R]\chi},$$

$$\text{(Comb)}\quad \frac{\Gamma \vdash \bot}{[R]\Gamma, \langle R\rangle\top \vdash \bot}.$$

Remark 3.1. Using the completeness of **AL** it is easy to see that the both systems **SAL** and **AL** prove the same sequents.

3.1.1. Cut elimination for SAL. Now we turn to presenting the proof for cut elimination of the system **SAL**. Here is a brief outline of the proof. We define a notion of *weight* of a derivation in the system **SAL**, which will associate to every derivation a natural number. By induction on this weight we show that every derivation can be transformed into a cut-free proof of the same conclusion. During the induction we shall need to make quite a few case distinctions, most of which will be left out, however, either because they are trivial or because they are similar to cases that we do consider.

Convention 3.2. If d is an arbitrary derivation, then by $r(d)$ we mean the conclusion, or the last sequent in d.

If Γ is a set of formulas, then $[R]\Gamma = \{[R]\phi : \phi \in \Gamma\}$.

In an application of the cut rule as in the derivation d below we call the sequent $\Gamma_1 \vdash \phi$ the *left premise* of the cut rule, and the sequent $\Gamma_2, \phi \vdash \chi$ its *right premise*:

$$d: \frac{\dfrac{\vdots}{\Gamma_1 \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2, \phi \vdash \chi}}{\Gamma_1, \Gamma_2 \vdash \chi}.$$

We need the next lemma.

Lemma 3.3. *Assume that the derivation d satisfies the following conditions:*

1. *The last rule applied in d is the cut rule, and this is the only application of cut in d.*
2. *The left premise of the last rule is an axiom.*

Then there is a derivation d' of $r(d)$, which does not use the cut rule.

Proof. There are 3 possibilities for the axiom occurring as the left premise of the cut rule. Suppose first that it is (A1). Then the derivation has the form

$$d: \frac{\phi \vdash \phi \qquad d': \dfrac{\vdots}{\Gamma, \phi \vdash \chi}}{\Gamma, \phi \vdash \chi}.$$

But then d' is already a cut-free derivation of $r(d)$, as required. The cases when the axiom is (A2) or (A3) are similar, so we consider only one of them: (A2). Then the derivation d has the form displayed below, which can be transformed into a derivation d' with the same conclusion:

$$d: \frac{p, \neg p \vdash \phi \qquad \dfrac{\vdots}{\Gamma, \phi \vdash \chi}}{\Gamma, p, \neg p \vdash \chi} \quad \Rightarrow \quad d': \ (\text{Mon})\frac{p, \neg p \vdash \chi}{\Gamma, p, \neg p \vdash \chi}.$$

This completes the proof. ⊣

As announced before, we use a notion of weight to carry through our proof of cut elimination.

Definition 3.4 (weight of a derivation). We define the weight ω of a derivation d in **SAL** by induction:

- If d consists of a single axiom, then $\omega(d) = 1$.
- If the last rule which is applied in d has only one premise, that is if d has the form

  $$d: \frac{d': \dfrac{\vdots}{\Gamma \vdash \chi'}}{\Gamma \vdash \chi},$$

  then $\omega(d) = \omega(d') + 1$.
- If the last rule which is applied in d has two premises, that is if d has the form

  $$d: \frac{d': \dfrac{\vdots}{\Gamma' \vdash \chi'} \qquad d'': \dfrac{\vdots}{\Gamma'' \vdash \chi''}}{\Gamma \vdash \chi},$$

  then $\omega(d) = \omega(d') + \omega(d'') + 1$.

Theorem 3.5. SAL *admits a cut elimination.*

Proof. The proof is by induction on the weight of derivations. As every derivation d has $\omega(d) \geq 1$, the induction starts with $\omega(d) = 1$. In that case the derivation consists of a single cut-free axiom.

Next, suppose that for every derivation d of weight less than n, a cut-free derivation can be found of the same conclusion; we proceed to show that the same is true also for derivations of weight n. Let d be any derivation of weight n. Let (R) be the last rule applied in d. If (R) is not the cut rule, then the derivation without the last rule (R) will be of smaller weight, so by our inductive hypothesis it can be transformed into a cut-free derivation(s) of the same conclusion(s). Subsequently applying (R) yields a cut-free derivation of the conclusion of d.

Now for the main case: the last rule applied in a derivation of weight n is the cut rule. By our inductive hypothesis we can assume that the derivations of the premises of the cut rule are cut-free. We distinguish several cases. If the derivation of the left premise of the cut rule consists of a single axiom, we need only to apply Lemma 3.3 to find a cut-free derivation of the same conclusion.

So assume that the derivation of the left premise does not consist of a single axiom, and consider the different cases for the last rule in this derivation. Let us first consider the case when this rule is such that the formula on the right hand side of the conclusion is the same as the formula on the right hand side of the premises. Such rules are (Mon), ($\wedge$-intr-L), ($\top$-drop), (Compl), and (Comb). Since the required transformations in these cases are similar, we consider only one of them, that of (Mon). In this case we perform the following transformation, denoted by $\Rightarrow$:

$$d:\ \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \phi}}{\Gamma_1 \cup \{\psi\} \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2,\ \{\phi\} \vdash \chi}}{\Gamma_1,\ \Gamma_2,\ \psi \vdash \chi} \quad \Rightarrow \quad (\mathrm{Mon})\dfrac{d':\ \dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2,\ \{\phi\} \vdash \chi}}{\Gamma_1,\ \Gamma_2 \vdash \chi}}{\Gamma_1,\ \Gamma_2,\ \psi \vdash \chi}.$$

We have that $\omega(d') < \omega(d)$, because d' consists of one step less than d, so using the induction hypothesis we can transform d' into a cut-free derivation, but this yields a cut-free derivation of $r(d)$ as well.

The remaining cases are ones in which the last rule applied in the derivation of the left sequent is either ($\bot$-use), ($\wedge$-intr-R), or (Distr). Let us first see what happens when the rule is ($\bot$-use). We can apply the following transformation:

$$d:\ \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \bot}}{\Gamma_1 \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2,\ \phi \vdash \chi}}{\Gamma_1,\ \Gamma_2 \vdash \chi} \quad \Rightarrow \quad (\mathrm{Mon})\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \bot}}{\Gamma_1 \vdash \chi}}{\Gamma_1,\ \Gamma_2 \vdash \chi}.$$

As the derivation of $\Gamma_1 \vdash \bot$ is cut-free, the derivation d can be transformed into a cut-free one.

In case one of ($\wedge$-intr-R) and (Distr) is the last rule applied to obtain the left premise of the cut rule, we have to dig into the derivation of the right premise of the cut rule. As the arguments for ($\wedge$-intr-R) and (Distr) are similar, we present the details for only one of them, viz. (Distr). First, suppose that the right premise of the cut rule is an axiom. If this axiom is (A1), then the derivation has the form

$$d:\ \dfrac{d':\ \dfrac{\dfrac{\vdots}{\Gamma \vdash \chi}}{[R]\Gamma \vdash [R]\chi} \qquad [R]\chi \vdash [R]\chi}{[R]\Gamma \vdash [R]\chi}.$$

Now d' is a cut-free derivation of $r(d)$, so we are done. Clearly, the axiom cannot be (A2), since the left part of the right sequent should contain a formula of the form $[R]\chi$. Hence, the next possibility is (A3), then the derivation has the form of the derivation d below:

$$d: \frac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \bot}}{[R]\Gamma \vdash [R]\bot} \qquad [R]\bot, \langle R\rangle\top \vdash \chi}{[R]\Gamma, \langle R\rangle\top \vdash \chi} \quad \Rightarrow \quad d': \ (\bot\text{-use})\frac{(\text{Comb})\dfrac{\dfrac{\vdots}{\Gamma \vdash \bot}}{[R]\Gamma, \langle R\rangle\top \vdash \bot}}{[R]\bot, \langle R\rangle\top \vdash \chi}.$$

Since the derivation of $\Gamma \vdash \bot$ is cut-free, the above derivation d can be transformed into the cut-free derivation d' of $r(d)$.

Next we proceed to deal with the cases when the right premise of the cut rule is the result of applying one of the rules of **SAL**. We consider all these rules one at a time.

(Mon) The derivation must have the form of the derivation d below.

$$d: \frac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \phi}}{[R]\Gamma_1 \vdash [R]\phi} \qquad \dfrac{\dfrac{\vdots}{\Gamma_2 \vdash \chi}}{\Gamma_2 \cup \{\psi\} \vdash \chi}}{[R]\Gamma_1, (\Gamma_2 \cup \{\psi\})\backslash[R]\phi \vdash \chi}.$$

To be able to apply the cut rule, we must have $[R]\phi \in \Gamma_2 \cup \{\psi\}$. We again distinguish two cases: $[R]\phi \in \Gamma_2$ or $[R]\phi = \psi$. Suppose first that $\Gamma_2 = \Gamma_2' \cup \{[R]\phi\}$. Then we can transform d into the derivation d' below. As the sub-derivation d'' of d' has $\omega(d'') < \omega(d)$, it can be transformed into a cut-free derivation, and hence, so can d:

$$d': \frac{d'': \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \phi}}{[R]\Gamma_1 \vdash [R]\phi} \qquad \dfrac{\vdots}{\Gamma_2', [R]\phi \vdash \chi}}{[R]\Gamma_1, \Gamma_2' \vdash \chi}}{[R]\Gamma_1, \Gamma_2', \psi \vdash \chi}.$$

Next, assume that $\psi = [R]\phi$. Then we have $(\Gamma_2 \cup \{\psi\})\backslash[R]\phi = \Gamma_2$, so we can derive $r(d)$ as in the derivation d''':

$$d''': \ (\text{Mon})\frac{\dfrac{\vdots}{\Gamma_2 \vdash \chi}}{[R]\Gamma_1, \Gamma_2 \vdash \chi}.$$

(∧-intr-R), (⊤-drop), (⊥-use), (Compl) Since the transformations we perform on the derivations in these cases are similar, we consider only one of them. Suppose the rule that is last applied in the derivation of the right premise of the cut rule is (∧-intr-R), i.e. the derivation d has the form

$$d:\ \frac{\dfrac{\dfrac{\vdots}{\Gamma\vdash\chi}}{[R]\Gamma\vdash[R]\chi}\qquad \dfrac{\dfrac{\vdots}{\Gamma_1\vdash\phi}\qquad\dfrac{\vdots}{\Gamma_2\vdash\psi}}{\Gamma_1,\,\Gamma_2\vdash\phi\wedge\psi}}{[R]\Gamma,\,(\Gamma_1\cup\Gamma_2)\backslash[R]\chi\vdash\phi\wedge\psi}.$$

As the last step in d is an application of the cut rule, we must have that $[R]\chi\in\Gamma_1$ or $[R]\chi\in\Gamma_2$. Assume first that $[R]\chi\in\Gamma_1$ and $[R]\chi\notin\Gamma_2$; then Γ_1 has the form $\Gamma_1=\Gamma_1'\cup\{[R]\chi\}$ and $(\Gamma_1\cup\Gamma_2)\backslash[R]\chi=\Gamma_1'\cup\Gamma_2$. Now we can derive $r(d)$ as follows:

$$d':\ \frac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma\vdash\chi}}{[R]\Gamma\vdash[R]\chi}\qquad\dfrac{\vdots}{\Gamma_1',\,[R]\chi\vdash\phi}}{[R]\Gamma,\,\Gamma_1'\vdash\phi}\qquad\dfrac{\vdots}{\Gamma_2\vdash\psi}}{[R]\Gamma,\,\Gamma_1',\Gamma_2\vdash\phi\wedge\psi}.$$

As before we can transform d' into a cut-free derivation, and thus get a cut-free derivation of $r(d)$. For the cases when $[R]\chi\notin\Gamma_1$, $[R]\chi\in\Gamma_2$, and $[R]\chi\in\Gamma_1$ and $[R]\chi\in\Gamma_2$, we can perform similar transformations to arrive at the required result.

(∧-intr-L) We only consider one of the rules of this kind, the first one. The form of our initial derivation d in this case is

$$d:\ \frac{\dfrac{\dfrac{\vdots}{\Gamma_1\vdash\lambda}}{[R]\Gamma_1\vdash[R]\lambda}\qquad\dfrac{\dfrac{\vdots}{\Gamma_2,\,\phi\vdash\chi}}{\Gamma_2,\,\phi\wedge\psi\vdash\chi}}{[R]\Gamma_1,\,\Gamma_2\backslash[R]\lambda,\,\phi\wedge\psi\vdash\chi}.$$

Since $[R]\lambda\neq\phi\wedge\psi$, we must have $\Gamma_2=\Gamma_2'\cup\{[R]\lambda\}$. Using this, we can also derive $r(d)$ in the following way:

$$d':\ \frac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1\vdash\lambda}}{[R]\Gamma_1\vdash[R]\lambda}\qquad\dfrac{\vdots}{\Gamma_2'\cup\{[R]\lambda\},\,\phi\vdash\chi}}{[R]\Gamma_1,\,\Gamma_2',\,\phi\vdash\chi}}{[R]\Gamma_1,\,\Gamma_2',\,\phi\wedge\psi\vdash\chi}.$$

As before, this derivation can be turned into a cut-free one of $r(d)$.

(Distr), (Comb) In these cases the transformations that we perform on the derivations are again similar, so we treat only one of them: (Distr). Then d has the form

$$d: \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \chi_1}}{[R]\Gamma_1 \vdash [R]\chi_1} \qquad \dfrac{\dfrac{\vdots}{\Gamma_2 \vdash \chi_2}}{[R]\Gamma_2 \vdash [R]\chi_2}}{[R]\Gamma_1,\, [R]\Gamma_2 \backslash [R]\chi_1 \vdash [R]\chi_2}.$$

Since $[R]\chi_1 \in [R]\Gamma_2$, we must have $\chi_1 \in \Gamma_2$, so we can transform the derivation to the following one, which can be turned into a cut-free one using the induction hypothesis:

$$d': \quad (\text{Distr})\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \chi_1} \qquad \dfrac{\vdots}{\Gamma_2 \cup \{\chi_1\} \vdash \chi_2}}{\Gamma_1,\, \Gamma_2' \vdash \chi_2}}{[R]\Gamma_1,\, [R]\Gamma_2' \vdash [R]\chi_2}.$$

Summing up, any derivation in the system **SAL** can be transformed into a derivation of the same conclusion in which the cut rule is not used. ⊣

3.2. THE SYSTEM SALN AND ITS CUT ELIMINATION

In this section we present a cut-free sequential system, the logic **ALN** in [4]. Since the system **ALN** is in fact the system **AL** with added a few more axioms, our system **SALN** will be very much like the system **SAL**. We shall only present in detail the axioms and rules that are new or differ from the corresponding ones in **SAL**.

The axioms (A1) and (A2) are the same as in **SAL**, only (A3) is changed to the following form:

(A3) $\qquad [R]_n\bot,\, \langle R\rangle_n\top \vdash \phi, \quad n \geq 1.$

The rules (Mon), ($\wedge$-intr-L), ($\wedge$-intr-R), ($\bot$-drop), ($\top$-use) and (Distr) are again the same in **SALN** as in **SAL**. The second part of the rule (Compl) and the rule (Comb) are changed to the following forms:

$$(\text{Compl}) \qquad \frac{\Gamma, [R]_n\bot \vdash \chi \qquad \Gamma, \langle R\rangle_n\top \vdash \chi}{\Gamma \vdash \chi}.$$

$$(\text{Comb}) \qquad \frac{\Gamma \vdash \bot}{[R]_n\Gamma,\, \langle R\rangle_n\top \vdash \bot}, \quad n \geq 1.$$

Now the next three rules that we present are new ones, specific of the system **SALN**.

$$(\langle R\rangle+) \qquad \frac{\Gamma, \langle R\rangle_n\top \vdash \phi}{\Gamma, \langle R\rangle_{n+1}\top \vdash \phi}, \quad n \geq 1,$$

$$([\mathrm{R}]-) \qquad \frac{\Gamma, [R]_{n+1}\top \vdash \phi}{\Gamma, [R]_n\top \vdash \phi}, \quad n \geq 1,$$

$$([\mathrm{R}]+) \qquad \frac{\Gamma \vdash [R]_n\bot}{\Gamma \vdash [R]_{n+1}\bot}, \quad n \geq 1.$$

Before beginning the proof that the system **SALN** admits a cut elimination, let us briefly outline how we are going to proceed. First, we proof a lemma which in effect claims that a somewhat restricted form of the cut rule can be eliminated and then we proof that also the general form of the (Cut) is dispensable. To prove both of these claims, we use the technique that we used to prove the cut elimination for the system **SAL**, namely induction on the weight of the derivation containing (Cut). Since the most cases will emerge as similar to the respective ones in the cut elimination proof of **SAL**, we shall be more concise in our exposition and we shall treat in detail only some of the different cases.

Lemma 3.6. *The following rule can be eliminated from every* **SALN**-*derivation.*

$$(\mathrm{Cut}_{mn})\quad \frac{\Gamma_1 \vdash [R]_m\bot \qquad \Gamma_2, [R]_n\bot \vdash \phi}{\Gamma_1, \Gamma_2 \vdash \phi}, \quad \textit{where } m \leq n.$$

Proof. We prove this claim following the same pattern we used before. Suppose d is some derivation and (Cut_{mn}) is applied only at the last step of this derivation. Further suppose that the left premise of (Cut_{mn}) is an axiom. The only case of any interest is the case of (A1). Then d has the form

$$d\colon\ \frac{[R]_m\bot \vdash [R]_m\bot \qquad \dfrac{\vdots}{[R]_n\bot,\ \Gamma \vdash \phi}}{[R]_m\bot,\ \Gamma \vdash \phi}.$$

Now, since we have applied the rule (Cut_{mn}), we have $m \leq n$. If $m = n$, then we can use the derivation of the right premise as the Cut_{mn}-free derivation of $r(d)$. If $m < n$, then to the right premise we apply $(n - m)$ times the rule ([R]-) and we shall have the desired derivation.

Further we have to consider the cases when the left premise appeared by an application of some rule. The transformations we do are similar to those presented so far, so to diminish the risk of becoming boring we shall present only those we think most unusual.

([R]+) In this case we do the following transformation:

$$d\colon\ \frac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash [R]_m\bot}}{\Gamma_1 \vdash [R]_{m+1}\bot} \qquad \dfrac{\vdots}{\Gamma_2,\ [R]_n\bot \vdash \phi}}{\Gamma_1,\ \Gamma_2 \vdash \phi}$$

$$\Rightarrow \quad d'\colon\ \frac{\dfrac{\vdots}{\Gamma_1 \vdash [R]_m\bot} \qquad \dfrac{\vdots}{\Gamma_2,\ [R]_n\bot \vdash \phi}}{\Gamma_1,\ \Gamma_2 \vdash \phi}.$$

Now, since we have applied (Cut_{mn}) in the first place, it is true that $m+1 \leq n$, so $m \leq n$, *and the application of* (Cut_{mn}) *in* d' is legal. Using also that $\omega(d') < \omega(d)$, we can conclude the result.

(Distr) Our derivation which we want to turn into a Cut_{mn}-free one in this case has the form

$$d: \frac{\dfrac{\overline{\Gamma_1 \vdash \bot}}{[R]\Gamma_1 \vdash [R]\bot} \qquad \overline{\Gamma_2,\ [R]_n\bot \vdash \phi}}{[R]\Gamma_1,\ \Gamma_2 \vdash \phi}.$$

In this case as before we have to dig into the derivation of the right premise. To deal with the axioms (A1) and (A3), we use the rules ([R]+) and (Comb), respectively. Next we should turn to considering the cases when we have applied one of the rules in the derivation of the right premise. As an example we shall consider only one case which is the most instructive. Suppose the last rule applied in the derivation of the right premise is the rule ([R]–). We do the following transformation on the derivation d:

$$d: \frac{\dfrac{\overline{\Gamma_1 \vdash \bot}}{[R]\Gamma_1 \vdash [R]\bot} \qquad \dfrac{\overline{\Gamma_2,\ [R]_{n+1}\bot \vdash \phi}}{\Gamma_2,\ [R]_n\bot \vdash \phi}}{[R]\Gamma_1,\ \Gamma_2 \vdash \phi}$$

$$\Rightarrow \quad d': \frac{\dfrac{\overline{\Gamma_1 \vdash \bot}}{[R]\Gamma_1 \vdash [R]\bot} \qquad \overline{\Gamma_2,\ [R]_{n+1}\bot \vdash \phi}}{[R]\Gamma_1,\ \Gamma_2 \vdash \phi}.$$

We have that the derivation d' is of lesser weight than d, so we can apply the induction hypothesis to get the result.

Now all the cases for the type of the last rule in the derivation of the left premise of the (Cut_{mn}) are considered and the proof is finished. ⊣

Further we turn to the proof that the whole version of the cut rule can be eliminated in the system **SALN**, that is to the proof of the next theorem.

Theorem 3.7. SALN *admits a cut elimination.*

Proof. We shall use again in the proof our well-worked induction on the weight of the derivation with the different cases for the structure of the last steps of the derivations of the premises of the cut rule.

First, we consider the form of teh derivation of the left premise. The case when this derivation consists of a single axiom presents no difficulty. Let us turn to the case when the last step in the derivation of the left premise consists of applying some rule. The rules that are the same as in **SAL** or the formula on the right hand side of the conclusion is the same as that on the right hand side of the premise(s), create no difficulty. Such rules are (Mon), ($\wedge$-intr-L), ($\wedge$-intr-R), ($\top$-drop), ($\perp$-use), (Compl), ($\langle R\rangle+$), ($[R]-$), and (Comb).

If the last rule is ([R]+), then since the cut formula is of the form $[R]_m\perp$, we must only apply Lemma 3.6.

Now we turn to the most difficult to treat rule, namely (Distr). As usual, in this case we consider the cases for the form of the derivation of the right premise of the cut rule. The axioms and the rules that are present in **SAL** are treated as in this system. We need only to show that nothing goes wrong if the last rule in the derivation of the right premise is one of the new or changed rules. For ($\langle R\rangle+$), ([R]+) and (Comb) we do quite obvious transformations, using the fact that we can locate where the cut formula belongs. The only case that remains is that of ($[R]+$). But we can use the fact that the cut formula in this case is of the form $[R]_m\perp$, and applying Lemma 3.6 we conclude the proof. ⊣

So we can now state that the presented sequent system **SALN** indeed admits elimination of the cut rule.

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INFINITESIMAL BENDINGS OF ROTATIONAL SURFACES WITH CHANGING SIGNS CURVATURE*

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Иванка Иванова-Каратопраклиева. БЕСКОНЕЧНО МАЛЫЕ ИЗГИБАНИЯ ПОВЕРХНОСТЕЙ ВРАЩЕНИЯ ЗНАКОПЕРЕМЕННОЙ КРИВИЗНЫ

Исследовано множество либманновых параллелей первого порядка нежесткой поверхности вращения S знакопеременной кривизны K. S — замкнутая (рода 0 либо 1) либо с краем. Доказано, что на S, вне её частей, которые являются круговыми цилиндрами, имеется счетное множество либманновых параллелей, если S имеет бесконечное число нетривиальных фундаментальных полей изгибания. На каждом поясе с $K < 0$ эти параллели расположены везде плотно. На каждом поясе $S_0 = S_{L_0 L_1}$ с $K \geqq 0$, ограниченной асимптотической параллелью L_0, существуют либманновые параллели тогда и только тогда, когда S_0 содержит подпояс $\widehat{S}_0 = S_{L^* L_1}$ (L^* самая правая максимальная параллель на S_0). Все эти параллели образуют счетное множество, принадлежат $\widehat{S}_0$ и сгущаются к L^*. Даны достаточные условия для жесткости S.

Ivanka Ivanova-Karatoprаklieva. INFINITESIMAL BENDINGS OF ROTATIONAL SURFACES WITH CHANGING SIGNS CURVATURE

The set of Liebmann's parallels of first order on a non-rigid rotational surface S with changing signs curvature K is investigated. S is closed (of genus 0 or 1) or with a boundary. It is proved that there is a countable set of Liebmann's parallels on S outside of its parts which are circular cylinders if S has got an infinite number non-trivial fundamental fields of bending. On each belt with $K < 0$ these parallels are everywhere densely. On each belt $S_0 = S_{L_0 L_1}$ with $K \geqq 0$, bordered by an asymptotic parallel L_0, there exist Liebmann's parallels if and only if S_0 contains a subbelt $\widehat{S}_0 = S_{L^* L_1}$ (L^* is the most right maximal parallel of S_0). The Liebmann's parallels

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a subbelt $\widehat{S}_0 = S_{L^*L_1}$ (L^* is the most right maximal parallel of S_0). The Liebmann's parallels on S_0 are a countable set, belong to $\widehat{S}_0$ and are condensed to L^*. Some sufficient conditions for rigidity of S are given.

1. PRELIMINARIES

If S is a rotational surface with changing signs curvature, then the domains with positive Gaussian curvature on S are separated from the domains with negative Gaussian curvature by belts with zero curvature or by parabolic parallels, i.e. parallels on which the Gaussian curvature of S is zero. Those parallels are from first, second or third type [1]. A parallel from first type is described by a point of rectification of the meridian c of S (a point of inflection or not) at which the tangent of c is not perpendicular to the rotational axis. The principal curvatures of S at an arbitrary point of a parabolic parallel from first type are $\nu_{\text{mer}} = 0$, $\nu_{\text{par}} \neq 0$. A parabolic parallel from second type is described by such a point of c which is not a point of rectification but the tangent of c at this point is perpendicular to the rotational axis. We have $\nu_{\text{mer}} \neq 0$, $\nu_{\text{par}} = 0$ at an arbitrary point of such a parallel. A parabolic parallel from third type is described by a point of rectification of c (a point of inflection or not) at which the tangent to c is perpendicular to the rotational axis too. Any point of such a parallel is planar one for S ($\nu_{\text{mer}} = \nu_{\text{par}} = 0$). Any parabolic parallel L_0 from second (respectively third) type is an asymptotic line of S because the plane of L_0 is tangent of first (respectively higher) order to the surface at any point of L_0. That is why we shall call the parabolic parallels from second and third type in short asymptotic parallels.

Let S be an arbitrary rotational surface with not more than a finite number of asymptotic parallels. S can be closed (of genus zero or one) or with a boundary (consisting of one or two parallels). Let S be from the class C^q, $q \geqq 2$, out of its poles (if it has such ones).If the surface has not got any planar domains, so its meridian c can be represented as a union of a finite number of arcs such that each of them can be projected one-to-one on the rotational axis. If the surface has got some planar domains, so such a representation is possible for c without those of its parts which are segments, perpendicular to the rotational axis (exactly, they describe the planar domains of S by the rotation of c around the rotational axis).

Let the meridian c of S be in the co-ordinate plane Ouy and let it has got a finite number of points of inflection. If the point $P_0 \in c$ describes an asymptotic parallel L_0, then either a) P_0 is a point of inflection (see Fig. 1), or b) P_0 is not a point of inflection (see Fig. 2). Let us note that in the case a) there is a two-sided neighbourhood on c which can be projected one-to-one on the rotation axis and in the case b) there is not such a neighbourhood. We denote by c_1 and c_2 the arcs of c bordering on P_0 which can be projected one-to-one on the rotation axis. We shall consider only the case when c_1 and c_2 have not inner points which describe asymptotic parallels because the other case obviously is reduced to that one. We assume that in a neighbourhood of the point $P_0(u_0, r_0)$ the meridian c, i. e. c_1 and

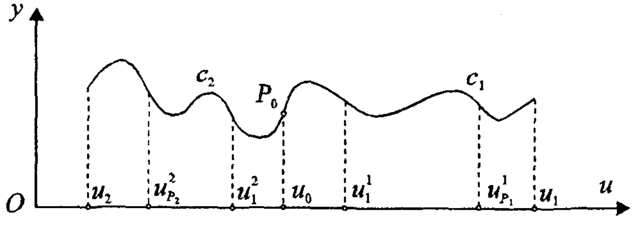

Fig. 1

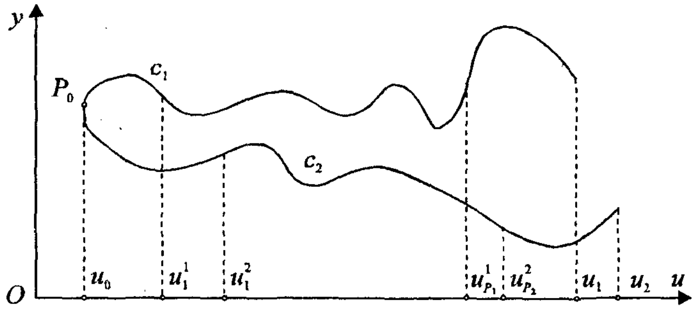

Fig. 2

c_2, has a representation

$$u = (\pm y \mp r_0)^n f_{1,2}(y) + u_0,\ n \geqq 2, \quad f_{1,2}(r_0) \neq 0, \tag{1}$$
$$f_1 \in C^A[r_0,\, r_0+\varepsilon], \quad f_2 \in C^A[r_0-\varepsilon,\, r_0].$$

Then we have

$$c_j\ :\ y = r_j(u),\ j=1,\,2,\ r_1(u_0) = r_2(u_0),\ r_1(u) \in C[u_0,\, u_1] \cap C^q(u_0,\, u_1), \tag{2}$$
$$r_2(u) \in C[u_2,\, u_0] \cap C^q(u_2,\, u_0), \quad \lim_{u \to u_0} r'_{1,2}(u) = +\infty,$$

and in a neighbourhood of u_0

$$r_1(u) = (u-u_0)^{n_1}\, \tilde{\varphi}_1(u) + r_0, \quad r_2 = (u_0-u)^{n_1}\, \tilde{\varphi}_2(u) + r_0, \quad n_1 = \frac{1}{n},$$

$$\tilde{\varphi}_j(u_0) \neq 0,\ j=1,\,2, \quad \tilde{\varphi}_1(u) \in C^q[u_0,\, u_0+\varepsilon], \quad \tilde{\varphi}_2(u) \in C^q[u_0-\varepsilon,\, u_0],\ q \geqq 2,$$

when P_0 is a point of inflection, and

$$c_j\ :\ y = r_j(u),\ j=1,\,2,\ r_1(u_0) = r_2(u_0),\ r_1(u) > r_2(u) \tag{3}$$
$$\text{for } u \in (u_0,\, u_1] \cap (u_0,\, u_2],$$
$$r_{1,2}(u) \in C[u_0,\, u_{1,2}] \cap C^q(u_0,\, u_{1,2}),\ \lim_{u \to u_0} r'_{1,2}(u) = \pm\infty,$$

and in a neighbourhood of u_0

$$r_j(u) = (u-u_0)^{n_1}\, \tilde{\varphi}_j(u) + r_0,\ \tilde{\varphi}_j(u_0) \neq 0,\ n_1 = \frac{1}{n},\ \tilde{\varphi}_j(u) \in C^q[u_0,\, u_0+\varepsilon],$$
$$j = 1,\,2,\ q \geqq 2,$$

when P_0 is not a point of inflection.

If the meridian c has other points which describe asymptotic parallels, then in a neighbourhood of any of them we assume that analogical conditions to those in (1) are satisfied. Finally, let us amplify that if the surface S has got one or two poles $P_0^{1,2}(u_0^{1,2}, 0)$ — smooth or conic, then we assume that in a neighbourhood of any of them analogical conditions to those for c_1 in (1) (see [3, 5]) are satisfied.

We assume that the surface S is non-rigid of first order with a field U of infinitesimal bending (inf.b.) which is continuous on the whole surface and belongs to the class C^1 out of its poles (if S has such ones). It is well-known that such non-rigid of first order rotational surfaces — closed or with a boundary, exist and each of them, which has got asymptotic parallels, is rigid of second order (see for example [1 — 4]).

In this paper we shall investigate the set of Liebmann's parallels of first order on S, i. e. those parallels which remain in their planes by inf.b. of first order. We shall give some sufficient conditions for rigidity of S too.

2. PROPERTIES OF THE FUNDAMENTAL FIELDS $U_k(u, v)$, $k \geqq 2$

We represent the parts $S_j \subset S$ obtained by rotation of the arcs $c_j \subset c$, $j = 1, 2$ (see Fig. 1 and 2) with the vectorial parametric equation

$$x(u, v) = u.e + r(u).a(v) \tag{4}$$

(here for simplicity we have denoted $r_j(u)$, $j = 1, 2$, with $r(u)$), where: u belongs to the indicated in (2) and (3) intervals, $v \in [0, 2\pi]$, e is the unit vector of the rotational axis Ou, and $a(v)$ is a unit vector perpendicular to Ou and twisted at an angle v from Oy). Let $U_k(u, v)$, $k \geqq 2$ be a non-trivial fundamental field of inf. b. of first order of the surface S. Then [1] we have on S_j, $j = 1, 2$,

$$\begin{aligned} U_k(u, v) = {} & e^{ikv}\,[\varphi_k(u).e + \chi_k(u).a + \psi_k(u).a'] \\ & + e^{-ikv}\,[\overline{\varphi}_k(u).e + \overline{\chi}_k(u).a + \overline{\psi}_k(u).a'] , \end{aligned} \tag{5}$$

$$\begin{aligned} & \varphi_k'(u) + r'(u)\,\chi_k'(u) = 0, \\ & \chi_k(u) + i\,k\,\psi_k(u) = 0, \\ & i\,k\,\varphi_k(u) + r'(u)\,[i\,k\,\chi_k(u) - \psi_k(u)] + r(u)\,\psi_k'(u) = 0, \quad k \geqq 2, \end{aligned} \tag{6}$$

from where we obtain for the function $\chi_k(u)$ the differential equation

$$r(u)\,\chi_k''(u) + (k^2 - 1)\,r''(u)\,\chi_k(u) = 0, \quad k \geqq 2. \tag{7}$$

Using the condition (1), we obtain

$$\begin{aligned} & u'(y)\,\varphi_k'(y) + \chi_k'(y) = 0, \\ & \chi_k(y) + i\,k\,\psi_k(y) = 0, \\ & i\,k\,u'(y)\,\varphi_k(y) + i\,k\,\chi_k(y) - \psi_k(y) + y\,\psi_k'(y) = 0, \quad k \geqq 2, \end{aligned} \tag{6'}$$

and

$$(7') \qquad y\,u'(y)\,\chi_k''(y) - y\,u''(y)\,\chi_k'(y) - (k^2-1)\,u''(y)\,\chi_k(y) = 0, \quad k \geqq 2,$$

in a neighbourhood of the point $P_0(u_0,\, r_0)$.

From the equalities (6) and from the assumption that the field U of inf. b. of S belongs to class C^1 out of the poles and the meridian $c \in C^q$, $q \geqq 2$, it follows immediately that the fundamental field $U_k(u,\, v)$, $k \geqq 2$, of S_j, $j = 1,\, 2$, belongs to the class C^q, $q \geqq 2$, out of the asymptotic parallel L_0. It is seen from (6′) that $\chi_k(y)|_{y=r_0} = \chi_k'(y)|_{y=r_0} = 0$ and therefore the fundamental field $U_k(u,\, v)$, $k \geqq 2$, satisfies the equality

$$(8) \qquad \chi_k(u_0) = 0, \quad k \geqq 2,$$

i. e.

$$(8') \qquad U_k(u_0,\, v) = \left[e^{ikv}\,\varphi_k(u_0) + e^{-ikv}\,\overline{\varphi}_k(u_0)\right].e, \quad k \geqq 2,$$

along the asymptotic parallel L_0.

Since the function $\chi_k(u)$, $k \geqq 2$, is a solution of the equation (7), so in the intervals, where $r''(0) \leqq 0$, it is not oscillating, i. e. it has not more than one null, it has neither a positive maximum nor a negative minimum and its graph is convex to the rotational axis Ou. Let us remind that in these intervals the meridian c is convex above and the corresponding belt of the surface S has got Gaussian curvature $K \geqq 0$. The equation (7) has a singularity in the point u_0. Taking y for an independent variable in a neighbourhood of u_0, (7) passes to the equation (7′) which is from Fuchs' type. We have proved in [4] that the problem (7), (8) has got a non-trivial solution $\chi_k(u)$ and in a neighbourhood of u_0 it has the form

$$(9) \qquad \chi_k(u) = (u-u_0)\,\chi_k^0(u), \quad \chi_k^0(u_0) \neq 0,$$

where $\chi_k^0(u) = \tilde{\varphi}_1^n(u)\,\tilde{P}_1\,[\,r_0 + (u-u_0)^{n_1}\,\tilde{\varphi}_1(u)\,]$, $\tilde{P}_1$ is an analytic function of $y = r_0 + (u-u_0)^{n_1}\,\tilde{\varphi}_1(u)$.

Remark 1. If the surface S has got a planar domain $\tilde{S}_0$, so $\tilde{S}_0$ is a disk or an annulus bounded by asymptotic parallels $L_0^{1,2}$: $u = u_0^{1,2}$ of S (even all the parallels on $\tilde{S}_0$ are asymptotic). In this case $U_k|_{\tilde{S}_0} \perp \tilde{S}_0$ (see [6]), i. e. $\chi_k(y)|_{\tilde{S}_0} = \psi_k(y)|_{\tilde{S}_0} = 0$ and consequently the condition (8) is satisfied on $L_0^{1,2}$.

Remark 2. If the rotational surface S has got poles $P_0^{1,2}$ so the function $\chi_k(u)$, which correspondes to the non-trivial fundamental field $U_k(u,\, v)$ of S, also satisfies the equality (8) (see for example [3, 5]).

Lemma 1. *Let $u = \alpha$ and $u = \beta$ be two sequential nulls of $\chi_k(u)$, $k \geqq 2$.*

a) *If the belt $S_{\alpha\beta}$ of S does not contain a subbelt with extremal parallels of S, so the function $\varphi_k(u)$, $k \geqq 2$, has got exactly one null in $(\alpha,\, \beta)$.*

b) *If S has got a subbelt $S_{u_1^* u_2^*}$ with extremal parallels, then either $\varphi_k(u)$, $k \geqq 2$, has got exactly one null in $(\alpha,\, \beta)$ and it is in $(\alpha,\, \beta) \setminus [u_1^*,\, u_2^*]$ or $\varphi_k(u) \equiv 0$, $k \geqq 2$, in $[u_1^*,\, u_2^*]$ but $\varphi_k(u) \neq 0$ in $(\alpha,\, \beta) \setminus [u_1^*,\, u_2^*]$.*

Proof. From (6) we find

$$(10) \qquad \varphi_k(u) = -\frac{r(u)\,\chi_k(u)}{k^2}\,f_k(u), \quad f_k(u) = \frac{\chi_k'(u)}{\chi_k(u)} + \frac{(k^2-1)\,r'(u)}{r(u)}$$

in the interval (α, β), wherefrom we obtain directly

$$f_k'(u) = -\left[\left(\frac{\chi_k'(u)}{\chi_k(u)}\right)^2 + (k^2-1)\left(\frac{r'(u)}{r(u)}\right)^2\right]. \tag{11}$$

From here and from $f_k(\alpha+0) = +\infty$, $f_k(\beta-0) = -\infty$ it follows that in the case a) $\varphi_k(u)$ has got exactly one null in the interval (α, β). The statement in b) follows directly from (6′), (10) and (11).

Lemma 2. *Let the belt $S_{\overline{u}_1 \overline{u}_2} \subset S$ has got negative Gaussian curvature and u', u'' are two arbitrary points from the interval $[\overline{u}_1, \overline{u}_2]$. There exists $k_0 \geqq 2$ such that the function $\varphi_k(u)$, $k \geqq k_0$, has got a null in (u', u'') if U_k, $k \geqq k_0$, is a non-trivial field of bending of S.*

Proof. We write the equation (7) for the interval $[\overline{u}_1, \overline{u}_2]$ in the form

$$\chi_k''(u) + G_k(u)\,\chi_k(u) = 0, \quad k \geqq 2, \tag{12}$$

where

$$G_k(u) = \frac{(k^2-1)\,r''(u)}{r(u)}.$$

We have

$$\min_{\overline{u}_1 \leqq u \leqq \overline{u}_2} G_k(u) \geqq \frac{(k^2-1)\,m}{M}, \tag{13}$$

where $m = \min r''(u)$, $M = \max r(u)$ when $u \in [\overline{u}_1, \overline{u}_2]$. We choose k_0 so that

$$\frac{(k_0^2-1)\,m}{M} > \left(\frac{N\,\Pi}{u''-u'}\right)^2, \tag{14}$$

where $N \geqq 2$. We consider the equation

$$Y''(u) + \mu^2\,Y(u) = 0 \tag{15}$$

with $\mu = \dfrac{N\,\Pi}{u''-u'}$. Since the solution $Y = \sin\mu(u-u')$ of (15) has got $N+1 \geqq 3$ nulls in $[u', u'']$ and $G_k(u) > \mu^2$ holds for $k \geqq k_0$ in $[u', u'']$ because of (13) and (14), then from the Sturm's theorem it follows that every solution $\chi_k(u)$, $k \geqq k_0$, of (12) has $M_k \geqq N \geqq 2$ nulls in $[u', u'']$. Then from Lemma 1 it follows that every function $\varphi_k(u)$, $k \geqq k_0$, has got at least one null in (u', u'').

Lemma 3. *Let the belt $S_0 \subset S$ has a non-negative Gaussian curvature and $\partial S_0 = L_0 \cup L_1$, where L_0 is the asymptotic parallel described by the point $P_0(u_0, r_0)$ and L_1 is the parallel*[1] *described by the neighbour point of inflection $P_1(u_1^1, r_1^1)$ of P_0. At each fixed $u \in (\tilde{u}_0, u_1^1]$, $\tilde{u}_o > u_0$, for which $r(u) > r(\tilde{u}_0)$ the following property is true:*

$$\text{the number sequence } \frac{\chi_k'(u)}{\sqrt{(k^2-1)}\,\chi_k(u)} \text{ is bounded} \tag{16}$$

[1] L_1 is a non-asymptotic parallel since the surface S is non-rigid (see [2]).

if each U_k, $k \geqq 2$, is a non-trivial field of bending of S.

Proof. Let $\chi_{k_1}(u)$ and $\chi_{k_2}(u)$ are solutions of the problem (7), (8) at $k = k_1$ and $k = k_2$, correspondingly, $k_1 < k_2$. We multiply (7) at $k = k_1$ by $(k_2^2 - 1)\,\chi_{k_2}(u)$ and (7) at $k = k_2$ by $(k_1^2 - 1)\,\chi_{k_1}(u)$, subtract the obtained equalities and integrate the result from u_0 to $u \in (u_0,\, u_1^1]$. We obtain

$$(k_2^2 - 1)(k_1^2 - 1)\,\chi_{k_2}(u)\,\chi_{k_1}(u)\left(\frac{\chi'_{k_1}(u)}{(k_1^2 - 1)\,\chi_{k_1}(u)} - \frac{\chi'_{k_2}(u)}{(k_2^2 - 1)\,\chi_{k_2}(u)}\right)$$
$$= (k_2^2 - k_1^2)\int_{u_0}^{u} \chi'_{k_1}(u)\,\chi'_{k_2}(u)\,du.$$

Since $\chi_k(u)\,\chi'_k(u) > 0$ in $(u_0,\, u_1^1]$, we conclude from here that at each fixed $u \in (u_0,\, u_1^1]$ the inequality

$$\frac{\chi'_{k_1}(u)}{(k_1^2 - 1)\,\chi_{k_1}(u)} > \frac{\chi'_{k_2}(u)}{(k_2^2 - 1)\,\chi_{k_2}(u)} \quad \text{for } k_1 < k_2 \tag{17}$$

holds.

Multiplying (7) by $2\chi'_k(u)$ we obtain

$$(r(u)\,\chi_k'^{\,2}(u))' = \left[r'(u)\,\frac{\chi'_k(u)}{(k^2 - 1)\,\chi_k(u)} - 2\,r''(u)\right](k^2 - 1)\,\chi_k(u)\,\chi'_k(u). \tag{18}$$

Let $\tilde{u}_0 > u_0$, $\overline{u} \in (\tilde{u}_0,\, u_1^1]$, and $N \geqq 2$ be an integer. Since $\chi_k(u)$, $k \geqq 2$, has not a null in $(u_0,\, u_1^1]$, so there exists a constant $M > 0$ such that

$$\frac{\chi'_N(u)}{(N^2 - 1)\,\chi_N(u)} < M \quad \text{in } [\tilde{u}_0,\, \overline{u}].$$

From here because of (17) we have

$$\frac{\chi'_k(u)}{(k^2 - 1)\,\chi_k(u)} < M \quad \text{in } [\tilde{u}_0,\, \overline{u}] \text{ for each } k \geqq N. \tag{19}$$

From (18) and (19) we obtain

$$\begin{aligned}(r(u)\,\chi_k'^{\,2}(u))' &< (|r'(u)|\,M - 2\,r''(u))\,(k^2 - 1)\,\chi_k(u)\,\chi'_k(u)\\ &< 2\,M_1(k^2 - 1)\,\chi_k(u)\,\chi'_k(u),\end{aligned} \tag{20}$$

where M_1 is a suitable constant. Integrating (20) from $\tilde{u}_0$ to $u \in (\tilde{u}_0,\, u_1^1]$ we find

$$\frac{\chi_k'^{\,2}(u)}{(k^2 - 1)\,\chi_k^2(u)}\left[1 - \frac{r(\tilde{u}_0)}{r(u)}\,\frac{\chi_k'^{\,2}(\tilde{u}_0)}{\chi_k'^{\,2}(u)}\right] < \frac{M_1}{r(u)}\left[1 - \frac{\chi_k^2(\tilde{u}_0)}{\chi_k^2(u)}\right], \quad k \geqq N. \tag{21}$$

From here and from

$$\frac{\chi_k'^{\,2}(\tilde{u}_0)}{\chi_k'^{\,2}(u)} < 1, \quad \frac{\chi_k^2(\tilde{u}_0)}{\chi_k^2(u)} < 1$$

we obtain

$$\frac{\chi_k'^{\,2}(u)}{(k^2 - 1)\,\chi_k^2(u)}\,[r(u) - r(\tilde{u}_0)] < M_1, \quad u \in (\tilde{u}_0,\, u_1^1],\ k \geqq N, \tag{22}$$

from where the statement in Lemma 3 follows immediately.

Corollary 1. *For each fixed* $u \in (\tilde{u}_0,\, u_1^1]$, $\tilde{u}_0 > u_0$, *such that* $r(u) > r(\tilde{u}_0)$ *it is valid*

$$\lim_{k\to\infty} \frac{\chi_k'(u)}{(k^2-1)\,\chi_k(u)} = 0. \tag{23}$$

Remark 3. If we multiply (7) at $k = k_1$ and at $k = k_2$ by $\chi_{k_2}(u)$ and $\chi_{k_1}(u)$, correspondingly, subtract the obtained equalities and integrate the result from u_0 to $u \in (u_0,\, u_1^1]$, we obtain that the inequality

$$\frac{\chi_{k_1}'(u)}{\chi_{k_1}(u)} < \frac{\chi_{k_2}'(u)}{\chi_{k_2}(u)}, \quad k_1 < k_2, \tag{24}$$

holds at each fixed $u \in (u_0,\, u_1^1]$.

Remark 4. The properties (24), (17), (16) and (23) of the fundamental fields $U_k(u,\, v)$, $k \geqq 2$, are proved by E. Rembs [7] for the case when the belt S_0 is simply connected, i. e. when $\partial S_0 = L_1$, and the point P_0 is a smooth non-parabolic pole of S_0. They are also valid in the case when the pole P_0 is parabolic or conic (see [5] and for a generalization see [2]). Proving these properties here for the doubly-connected belt S_0, we have used the equalities (8). That is why these properties will also be valid in the case when the tangent at P_0 to c is not perpendicular to the rotational axis, i.e. when the parallel L_0 is not asymptotic but the fields of the bending satisfy the conditions (8). From Remark 1 it is clear that we can ensure the conditions (8) sticking the boundary of a disk $\tilde{S}_0$ along L_0 and assuming that the field of inf.b. of the surface $S \cup \tilde{S}_0$ is continuous on it and from class C^1 on S and $\tilde{S}_0$.

Remark 5. If the belt S_0 is obtained for $u \in [u_1^1,\, u_0]$, i. e. if instead of c_1 and c_2 at Fig. 1 and 2 we have theirs orthogonally symmetric curves with respect to the line by P_0, which is parallel to the axis Oy, then obviously Lemma 3 and Corollary 1 — the properties (16) and (23), are valid too, but $\chi_k(u)\,\chi_k'(u) < 0$ and the inequalities (24) and (17) are inverted.

3. MAIN RESULTS

If $U_k(u,\, v)$, $k \geqq 2$, is a fundamental field of inf.b. of the surface S for which the parallel $\widehat{L} : u = \widehat{u}$ is Liebmann's, i.e. $\widehat{L}$ remains in its plane, then we say that $U_k(u,\, v)$ is a field of inf.b. with sliding along $\widehat{L}$. It is clear from (5) and (6) that $U_k(u,\, v)$ is a field of inf.b. with sliding along $\widehat{L}$ exactly when

$$\varphi_k(u)|_{\widehat{L}} = 0, \quad k \geqq 2. \tag{25}$$

The following statements are valid:

Theorem 1. *On the rotational surface S, outside of her belts of extremal parallels (if S has got such belts) there exists a countable set of Liebmann's parallels. Moreover:*

a) *On each belt with negative Gaussian curvature the Liebmann's parallels are everywhere densely;*

b) *There are Liebmann's parallels on every belt S_0 with non-negative Gaussian curvature, which belt is simply connected with a pole and a boundary $\partial S_0 = L_1$, or doubly connected with a boundary $\partial S_0 = L_0 \cup L_1$, where L_0 is an asymptotic parallel, if and only if S_0 contains a subbelt $\widehat{S}_0 = S_{L^* L_1}$ (respectively, $\widehat{S}_0 = S_{L_1 L^*}$) bounded by the most right (respectively, the most left) maximal parallel L^* of S_0 and the parallel L_1. All these Liebmann's parallels are a countable set, belong to $\widehat{S}_0$ and are condensed to L^* if S has got an infinite number non-trivial fundamental fields of bending*[2].

Corollary 2. *The surface S is rigid with respect to inf.b. with sliding along an asymptotic parallel of S.*

Corollary 3. *The surface S is rigid with respect to inf.b. with sliding along a parallel $\widehat{L} \in S_0$ if the belt S_0 has not got a subbelt $\widehat{S}_0$, and along a parallel $\widehat{L} \in S_0 \backslash \widehat{S}_0$ if the belt S_0 has a subbelt $\widehat{S}_0$.*

Proof. These statements follow directly from the lemmas. In fact, the existance of a countable set of Liebmann's parallels on S follows from the condition (25), Lemma 1 and from the facts that the non-trivial fundamental fields $U_k(u, v)$, $k \geqq 2$, of S are a countable set and any function $\chi_k(u)$, $k \geqq 2$, can have only a finite number nulls in a closed interval. The statement a) follows from (25) and Lemma 2. We shall pause in detail on the proof of the statement b).

For concreteness let S_0 be obtained by $u \in (u_0,\, u_1^1]$. If the belt S_0 is simply connected, so the statement b) is well-known (see [3, 5, 7]). Let S_0 be a doubly connected belt. It is seen from (10) that the function $\varphi_k(u)$, $k \geqq 2$, is annuled in $(u_0,\, u_1^1)$ if and only if $f_k(u)$, $k \geqq 2$, is annuled. Because of (11) the function

$$f_k(u) = (k^2 - 1)\left[\frac{\chi_k'(u)}{(k^2 - 1)\,\chi_k(u)} + \frac{r'(u)}{r(u)}\right] \tag{26}$$

is monotonically decreasing, as $f_k(u_0 + 0) = +\infty$ and $f_k(u^*) > 0$ for each $k \geqq 2$, where $L^* \,:\, u = u^*$ is the most right maximal parallel of S_0. Since (26) and Corollary 1 hold and $\dfrac{r'(u)}{r(u)} < 0$ in $(u^*,\, \overline{u}]$, $\overline{u} \leqq u_1^1$, it follows that for each fixed $\overline{u} \in (u^*,\, u_1^1]$ such that $r(\overline{u}) > r(\tilde{u}_0)$ there exists an integer $N_1 \geqq 2$ such that $f_k(\overline{u}) < 0$ for any $k \geqq N_1$. Consequently, for each $k \geqq N_1$ there exists $u_k \in (u^*,\, \overline{u})$ such that $f_k(u_k) = 0$, i. e. $\varphi_k(u_k) = 0$. Moreover, if $k_1 < k_2$, then from (17) and from the fact that $\dfrac{r'(u)}{r(u)}$ is a monotonically decreasing function in $[u^*,\, u_1^1]$ it follows $u_{k_2} < u_{k_1}$. Thus for each $k \geqq N_1$ there exists a Liebmann's parallel L_k in $(u^*,\, \overline{u})$. In addition, for $k_2 > k_1$ the Liebmann's parallel L_{k_2}, corresponding to

[2] Such surfaces exist — for example, if S is simply connected and it has not an asymptotic parallel, or S is doubly connected with not more than one asymptotic parallel, then it has got a countable number non-trivial fundamental fields of bending (see [2], [3]).

the fundamental field $U_{k_2}(u, v)$, is situated more to the left than the Liebmann's parallel L_{k_1}, corresponding to the fundamental field $U_{k_1}(u, v)$. All these Liebmann's parallels condense to the most right maximal parallel $L^* : u = u^*$ of S_0. In order to verify this it is sufficient to take $\overline{u} = u^* + \varepsilon < u_1^1$, where ε is a small enough positive number, and to repeat the considerations which we have just done.

Remark 6. The statement a) for a rotational surface S with a negative curvature is proved in [8]. There are such investigations in [9] and [11] too, but the formulated results contradict to [8] and to our statement a) here.

Remark 7. The statement in Corollary 2 follows from the well-known lemma of Minagawa and Rado (see [2]). It is proved in [10] (see also [3, 9, 11]) and here we formulate it for completeness. That statement is proved in [9] (see Theorem 5 there) by the method "a, b, c" under a lot of restrictions on the surface.

Remark 8. The statement in Corollary 3 is proved as well in [12] and [9]. In [12] the asymptotic parallel is not of second type (as it is said there) — it is of third type, and in [9] (see Corollary 3 there) the statement is proved by the method "a, b, c" under a lot of restrictions on the surface.

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ADMISSIBLE FUNCTIONALS IN ABSTRACT STRUCTURES WITH ARBITRARY POWER*

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Стела Николова. ДОПУСТИМЫЕ ФУНКЦИОНАЛЫ В АБСТРАКТНОЙ СТРУКТУРЕ ПРОИЗВОЛЬНОЙ МОЩНОСТИ

В работе предлагается один способ обобщения понятия допустимой ($\forall$-рекурсивной [2]) функции для абстрактных структур произвольной мощности. Для этого здесь вводится и изучается понятия допустимого функционала. При его помощи определяется понятие допустимости для частично-многозначной функции в произвольно мощной абстрактной структуре. Установляется, что частично-многозначная функция допустима тогда и только тогда, когда она абсолютно поисково вычислима.

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A way for generalizing the notion of admissible (or $\forall$-recursive [2]) function for the case of arbitrary (not only denumerable) abstract structure is considered. For this purpose a notion of admissible functional is introduced and studied in the paper. Using this notion, a concept of admissibility for partial multiple valued function over arbitrary structure is introduced. It is established that a function is admissible if and only if it is absolutely search computable.

In the present paper we suggest a certain way for generalizing the notion of $\forall$-recursiveness for the case of abstract structures with arbitrary power. The notion of $\forall$-recursive predicate, introduced by Lacombe [2], is aimed at describing effectively definable relations on denumerable structures with equality. Later on Moschovakis

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proved that it is equivalent to search computability with constants [3]. In some recent works of Soskov [11–13] a certain modification of the original definition of $\forall$-recursiveness is suggested. It is used there as a basis for an uniform and very natural classification of several well-known concepts of abstract computability — prime and search computability [4], computability by means of effectively definable schemes (EDS) of Friedman [1] etc. What is more, the equality relation is not already supposed to be among the initial predicates.

Since the notion of $\forall$-recursiveness incorporates different numberings of the data domain, in each of the above mentioned works the structures are supposed to be at most denumerable. Exactly, to embrace the general case of arbitrary domains, the notion of admissible functional is designed.

There are several versions of the notion in question, depending on the kind of relative recursiveness over the naturals that we have taken as basic and according to our understanding of the expression "function, computable in a given structure $\mathfrak{A}$". Here we study one, in a sense the easiest to examine, of these versions, the other cases being considered in [5, 6]. We shall regard as basic the broadest notion of relative recursiveness over the set of all natural numbers — the partial recursiveness. Moreover, given $\mathfrak{A}$, we shall assume that some "oracle" for the data domain (along with "oracles" for the initial functions and predicates of $\mathfrak{A}$) is available.

1. PRELIMINARIES

Assume that a partial structure $\mathfrak{A} = (B; \varphi_1, \ldots, \varphi_a; R_1, \ldots, R_b)$ of some fixed signature $\sigma = (f_1, \ldots, f_a; P_1, \ldots, P_b)$ is given. Suppose first that B is at most denumerable. An *enumeration* of $\mathfrak{A}$ is any ordered pair $(\varkappa, \mathfrak{B})$, where $\mathfrak{B} = (N; \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b)$ is a σ-structure over the set N of all natural numbers and $\varkappa$ is a total mapping from N onto B, the following conditions being satisfied:

$\varkappa(\psi_i(x_1, \ldots, x_{l_i})) \simeq \varphi_i(\varkappa(x_1), \ldots, \varkappa(x_{l_i}))$ for all $x_1, \ldots, x_{l_i}$ in N, $1 \leqq i \leqq a$;

$Q_j(x_1, \ldots, x_{m_j}) \simeq R_j(\varkappa(x_1), \ldots, \varkappa(x_{m_j}))$ for all $x_1, \ldots, x_{m_j}$ in N, $1 \leqq j \leqq b$.

Suppose now that $\mathfrak{A}$ is a structure with arbitrary power. It is clear that in this case we cannot speak about enumerations of $\mathfrak{A}$. Nevertheless a notion of admissibility in $\mathfrak{A}$ is still possible. The key to it is the observation that every computational path is at most countable, hence no more than countably many elements of B can be involved in the course of the computation. Our idea is to break $\mathfrak{A}$ into some suitable denumerable parts, to enumerate them and to combine all these parts in some reasonable way.

We begin with some notational conventions. The elements of the basic set B will be denoted by s, p, r, possibly with indexes; as usual $(s_1, \ldots, s_k)$ will be abbreviated to $\overline{s}$. We shall use small greek letters to denote sequences — β, δ will range over the class of all infinite sequences of elements of B (to be denoted by B^N), while α, γ will denote infinite sequences of natural numbers (which sometimes will be viewed as total functions in N). For any $A \subseteq B$ set $\mathcal{T}(A) = \{p \mid p \simeq \tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_n/s_n)$, where $\tau(X_1, \ldots, X_n)$ is a σ-term with variables among $X_1, \ldots, X_n$ and $s_1, \ldots, s_n$ are elements of $A\}$. In particular, the set $\mathcal{T}(\{s_1, \ldots, s_k\} \cup \{\beta(n) \mid n \in N\})$ will be denoted by $\mathcal{T}(\overline{s}, \beta)$. This set is closed under the initial functions of $\mathfrak{A}$, so there exists a substructure of $\mathfrak{A}$ with domain $\mathcal{T}(\overline{s},\beta)$, which we shall denote by $\mathfrak{A}(\overline{s},\beta)$. Notice that $\mathfrak{A}(\overline{s},\beta)$ is at most denumerable and hence it can be enumerated.

Let $\mathfrak{B} = (N;\psi_1,\ldots,\psi_a;Q_1,\ldots,Q_b)$ be some structure over the natural numbers (further $\mathfrak{B}$ will always denote such structures). Whenever Γ is an enumeration operator (cf. [8]) of appropriate arity, we shall write $\Gamma(\mathfrak{B})$ to denote the set $\Gamma(\psi_1,\ldots,\psi_a,Q_1^*,\ldots,Q_b^*)$, where Q_i^* is the characteristic function of the predicate Q_i. Further $\Gamma(\mathfrak{B})$ will be looked upon rather as the partial multiple-valued function with a graph $\Gamma(\mathfrak{B})$. If α is a function in N, we shall set also $\mathfrak{B}_\alpha = (N;\alpha,\psi_1,\ldots,\psi_a;Q_1,\ldots,Q_b)$.

Let $\varkappa$ be a total mapping from N into B, $M \subseteq N$, $\overline{x} \in N^k$ and $\alpha \in N^N$. By $\varkappa(M)$, $\varkappa(\overline{x})$ and $\varkappa(\alpha)$ we shall denote $\{\varkappa(x) \mid x \in M\}$, $(\varkappa(x_1),\ldots,\varkappa(x_k))$ and $\{\varkappa(\alpha(n))\}_n$, respectively.

Suppose now that a computational process over $\mathfrak{A}$ *at some input* $\overline{s}$ *is initiated.* As we have mentioned above, we assume that an oracle $\mathbb{O}$ for the data domain B may also be used during this computation, operating in the following way: whenever some question to $\mathbb{O}$ is asked, it returns (generates) an arbitrary element of B and its answers do not depend on the current configuration. *Now it is clear that if the* answers of $\mathbb{O}$ are $t_1,\ldots,t_m$ (or $t_1,t_2,\ldots$ if the process is infinite and infinitely many questions are put to $\mathbb{O}$), then each $p \in B$, which appears during the execution, is an element of $\mathcal{T}(\overline{s},\beta)$, where $\beta(n) = t_n$ for $n = 1,2,\ldots$

We shall use mappings F of the set $B^k \times B^N$ to describe mathematically the behavior of non-deterministic algorithms over $\mathfrak{A}$ with k input variables. For every $(\overline{s},\beta) \in B^k \times B^N$ $F(\overline{s},\beta)$ will be interpreted as the result obtained when such an algorithm is applied at the input $\overline{s}$, provided for every $n = 1,2,\ldots$ the answer to the n-th question to the oracle has been equal to $\beta(n)$ (see also [10, ch. 2, § 5.1] for additional motivation).

The intuitive remarks just maid justify the introduction of the following definition:

$F : B^k \times B^N \to 2^B$ is said to be *admissible* (*in* $\mathfrak{A}$) iff there exists an enumeration operator Γ such that for every $(\overline{s},\beta) \in B^k \times B^N$ and for every enumeration $(\varkappa,\mathfrak{B})$ of $\mathfrak{A}(\overline{s},\beta)$ the equality

$$\varkappa(\Gamma(\mathfrak{B}_\alpha)(\overline{x})) = F(\overline{s},\beta)$$

holds for each $(\overline{x},\alpha) \in N^k \times N^N$ such that $\varkappa(\overline{x}) = \overline{s}$ and $\varkappa(\alpha) = \beta$.

Now taking into account the interpretation of F, we come to the following notion of admissible function:

The partial multiple-valued (p.m.v.) function $\varphi : B^k \to 2^B$ is said to be *admissible* (*in* $\mathfrak{A}$) iff there exists an admissible functional F such that for every $(s_1,\ldots,s_k,p) \in B^{k+1}$ the following is true:

$$p \in \varphi(s_1,\ldots,s_k) \Leftrightarrow \exists\beta(p \in F(s_1,\ldots,s_k,\beta)).$$

In order to formulate an explicit characterization of the notions introduced so far, some syntactical constructions will be needed.

Let P_0 be a new unary predicate symbol which is intended to represent the predicate $\lambda x.\mathtt{true}$. Throughout the paper a *σ-formula* will be any finite conjunction of atomic formulas in the extended signature $(f_1, \ldots, f_a; P_0, P_1, \ldots, P_b)$ or their negations. An expression of the form $\Phi \Rightarrow \tau$, where Φ is a σ-formula and τ is a σ-term, *will be called a σ-clause. Every recursively enumerable set of σ-clauses we* shall call, following [9], *recursively enumerable* (r.e.) *scheme.*

Let Ψ be some expression with variables $X_1, \ldots, X_k$ and $s_1, \ldots, s_k$ be arbitrary elements of B. We shall write $\Psi_{\mathfrak{A}}(X_1/s_1, \ldots, X_k/s_k)$ ($\Psi_{\mathfrak{A}}(\bar{s})$ for short) to denote the value (if it exists) of Ψ on $\mathfrak{A}$ when X_i is replaced by s_i, $i = 1, \ldots, k$. If Ψ is a term or a formula, the meaning of $\Psi_{\mathfrak{A}}(\bar{s})$ is the usual one. In the case when Ψ is $\Phi \Rightarrow \tau$, $\Psi_{\mathfrak{A}}(\bar{s})$ *is introduced by the equivalence*

$$(\Phi \Rightarrow \tau)_{\mathfrak{A}}(\bar{s}) \simeq p \quad \text{iff} \quad \Phi_{\mathfrak{A}}(\bar{s}) \simeq \mathrm{t} \text{ and } \tau_{\mathfrak{A}}(\bar{s}) \simeq p.$$

Here and further "t" stands for "`true`".

Let $\Delta = \{\Phi^w \Rightarrow \tau^w \mid w \in W\}$ be some set of clauses with variables $X_1, \ldots, X_k, Y_0, Y_1, \ldots$ (i.e. for every $w \in W$ the variables of $\Phi^w \Rightarrow \tau^w$ are among $X_1, \ldots, X_k, Y_0, Y_1, \ldots$). Δ determines a mapping $\Delta_{\mathfrak{A}} : B^k \times B^N \to 2^B$ defined by the following condition:

$$p \in \Delta_{\mathfrak{A}}(\bar{s}, \beta) \Leftrightarrow \exists w_{w \in W}((\Phi^w \Rightarrow \tau^w)_{\mathfrak{A}}(\bar{s}, \beta) \simeq p).$$

Here $(\Phi^w \Rightarrow \tau^w)_{\mathfrak{A}}(\bar{s}, \beta)$ is an abbreviation for $(\Phi^w \Rightarrow \tau^w)_{\mathfrak{A}}(X_1/s_1, \ldots, X_k/s_k, Y_{j_1}/\beta(j_1), \ldots, Y_{j_n}/\beta(j_n))$, where $X_1, \ldots, X_k, Y_{j_1}, \ldots, Y_{j_n}$ is a list containing all variables of $\Phi^w \Rightarrow \tau^w$.

We shall say that the functional $F : B^k \times B^N \to 2^B$ is *definable* iff there exists a r.e. scheme Δ with variables $X_1, \ldots, X_k, Y_0, Y_1, \ldots$ such that $F = \Delta_{\mathfrak{A}}$.

In order to save space, in the following we shall assume that the initial functions and predicates of $\mathfrak{A}$ are unary.

2. CONSTRUCTING A RECURSIVELY ENUMERABLE SCHEME FOR A GIVEN ADMISSIBLE FUNCTIONAL

Denote by σ^+ the signature $\sigma \cup \{\boldsymbol{S}, \boldsymbol{0}, =\}$ with the commonly accepted semantics of the additional symbols $\boldsymbol{S}$, $\boldsymbol{0}$, $=$ over the naturals. Set also $\sigma_0^+ = \sigma^+ \cup \{f_0\}$, f_0 being a new unary function symbol.

From now on we shall suppose that some admissible functional $F : B^k \times B^N \to 2^B$ is fixed. Without any loss of generality we may assume that $k = 1$. In this section we are going to construct a r.e. scheme Δ, for which we shall establish later that $F = \Delta_{\mathfrak{A}}$.

Indeed, since F is admissible, there exists an enumeration operator Γ such that for every (s, β) and every enumeration $(\varkappa, \mathfrak{B})$ of $\mathfrak{A}(s, \beta)$ the equality

$$\varkappa(\Gamma(\mathfrak{B}_\alpha)(x)) = F(s, \beta)$$

is satisfied for each x, α: $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$. In the present paper we shall use an equivalent characterization of enumeration operators from [9, Thm. 7.5]. According to it there exists a r.e. set Δ^0 of σ_0^+-clauses with one variable X, such that for every x, α and $\mathfrak{B}$

$$\Gamma(\mathfrak{B}_\alpha)(x) = \Delta^0_{\mathfrak{B}_\alpha}(x). \tag{2.0}$$

Moreover, each clause of Δ^0 is of the form

$$X = \underline{m} \,\&\, f_{i_1}(\underline{m}_1) = \underline{n}_1 \,\&\, \dots \,\&\, f_{i_e}(\underline{m}_e) = \underline{n}_e \,\&\, \Phi \Rightarrow \underline{m}_0,$$

where $e \geqq 0$ and Φ is a (possibly empty) conjunction of formulas of the kind $P_j(\underline{n})$ or $\neg P_j(\underline{n})$ (here as usual $\underline{k}$ stands for $S^k(0)$). We may also suppose that in one and the same clause of Δ^0 there are not repeating conjuncts as well as conjuncts of the kind $f_i(\underline{k}) = \underline{n}$ and $f_i(\underline{k}) = \underline{n}'$ with $n \neq n'$.

Our main task in this section will be to remove from Δ^0 function and predicate symbols which are not in σ, preserving at the same time (a part of) the information about F that Δ^0 bears. This will be done in a few steps.

At the first step we eliminate f_0. For that purpose choose some sequence $Y_0, Y_1, \dots$ of different variables and replace each conjunct of the kind $f_0(\underline{n}) = \underline{n}'$ by $Y_n = \underline{n}'$. This reduces Δ^0 to a r.e. scheme Δ^1 which variables are among $X, Y_0, Y_1, \dots$ Furthermore, for every x, α, $\mathfrak{B}$

$$\Delta^0_{\mathfrak{B}_\alpha}(x) = \Delta^1_{\mathfrak{B}}(x, \alpha), \tag{2.1}$$

which follows immediately from the appropriate definitions and the equivalences

$$(f_0(\underline{n}) = \underline{n}')_{\mathfrak{B}_\alpha}(x) \simeq \mathrm{t} \Leftrightarrow \alpha(n) = n' \Leftrightarrow (Y_n = \underline{n}')_{\mathfrak{B}}(x, \alpha) \simeq \mathrm{t}.$$

Let us now fix some injective recursive function α such that $N \setminus \mathrm{Range}(\alpha)$ is infinite and decidable. Fix also an arbitrary x which is not in $\mathrm{Range}(\alpha)$. Let set for brevity $y_i = \alpha(i)$, $i = 0, 1, \dots$, and $M = \{x, y_0, y_1, \dots\}$. In Δ^1 we make the following transformations: first remove each clause containing conjuncts $X = \underline{m}$ ($Y_i = \underline{n}$), where $m \neq x$ ($n \neq y_i$); then delete every conjunct of the kind $X = \underline{x}$ or $Y_i = \underline{y}_i$, or replace it by $P_0(X)$ if it is the unique conjunct in the clause. This procedure yields a set $\Delta^2 = \{A^{(w)} \Rightarrow a^{(w)} \mid w \in W\}$ which is also r.e. (since α is recursive), the following condition being satisfied for all σ-structures $\mathfrak{B}$ and for already fixed x and α:

$$\Delta^1_{\mathfrak{B}}(x, \alpha) = \Delta^2_{\mathfrak{B}}(x, \alpha). \tag{2.2}$$

Now let us fix some clause $A^{(w)} \Rightarrow a^{(w)}$ from Δ^2. It has the form $f_{i_1}(\underline{m}_1) = \underline{n}_1 \,\&\, \dots \,\&\, f_{i_e}(\underline{m}_e) = \underline{n}_e \,\&\, \Phi \Rightarrow \underline{m}_0$, where $f_{i_1}, \dots, f_{i_e}$ are already function symbols from σ.

We shall say that the index w of $A^{(w)} \Rightarrow a^{(w)}$ is *suitable* iff the following conditions hold:

(i) $\{n_1, \dots, n_e\} \subseteq N \setminus M$;

(ii) $m_j < n_j$ for every $j = 1, \dots, e$;

(iii) $n_1, \dots, n_e$ are different natural numbers.

Now set $\Delta^3 = \{A^{(w)} \Rightarrow a^{(w)} \mid w \text{ is suitable}\}$. Each of the conditions (i), (ii), (iii) is decidable, hence Δ^3 is r.e., too. Clearly, for every structure $\mathfrak{B}$ over the naturals we have

$$\Delta^2_{\mathfrak{B}}(x, \alpha) \supseteq \Delta^3_{\mathfrak{B}}(x, \alpha). \tag{2.3}$$

The opposite inclusion of (2.3) is not always true. In what follows we shall define a non-empty class of enumerations $(\varkappa, \mathfrak{B})$ such that $\Delta^2_{\mathfrak{B}}(x, \alpha) = \Delta^3_{\mathfrak{B}}(x, \alpha)$ *holds for every* $\mathfrak{B}$ *from this class.*

Indeed, for an arbitrary $(s,\beta) \in B \times B^N$ denote by $\mathcal{K}_{s,\beta}$ the class of those enumerations $(\varkappa, \mathfrak{B} = (N;\psi_1,\ldots,\psi_a;Q_1,\ldots,Q_b))$ of $\mathfrak{A}(s,\beta)$ which satisfy the conditions:

0) $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$;

1) $\mathrm{Range}(\psi_i) \subseteq N \setminus M$, $1 \leqq i \leqq a$;

2) if $\psi_i(m)$ is defined, then $\psi_i(m) > m$, $1 \leqq i \leqq a$;

3) $\mathrm{Range}(\psi_i) \cap \mathrm{Range}(\psi_j) = \varnothing$, $i,j \in \{1,\ldots,a\}$ and $i \neq j$;

4) ψ_i is injective, $1 \leqq i \leqq a$.

Let us first check the following

Lemma 2.1. *For every s, β the class $\mathcal{K}_{s,\beta}$ is not empty.*

Proof. Clearly, one can choose $\varkappa$ satisfying 0) and the requirement: for every $q \in \mathcal{T}(s,\beta)$ the set $N^{(q)} = \{n \mid n \in N \setminus M \,\&\, \varkappa(n) = q\}$ is infinite. Let $n_1^{(q)}$, $n_2^{(q)}$, ... be the list of all elements of $N^{(q)}$ put in ascending order. Now for every $i = 1,\ldots,a$ define ψ_i as follows: if $\varphi_i(\varkappa(m)) \simeq q$, set $\psi_i(m) = n_k^{(q)}$, where $k = 2^i.3^m$; if $\varphi_i(\varkappa(m))$ is undefined, set $\psi_i(m)$ to be undefined, too. Set also $Q_j(m) \simeq R_j(\varkappa(m))$ for every $m \in N$ and $j \in \{1,\ldots,b\}$.

We have by definition

$$!\psi_i(m) \Leftrightarrow !\varphi_i(\varkappa(m)) \quad \text{and} \quad !\psi_i(m) \Rightarrow \varkappa(\psi_i(m)) = \varkappa(n_k^{(q)}) = q = \varphi_i(\varkappa(m)),$$

which together with the choice of Q_1, ..., Q_b shows that the pair

$$(\varkappa, \mathfrak{B} = (N;\psi_1,\ldots,\psi_a;Q_1,\ldots,Q_b))$$

is an enumeration of $\mathfrak{A}(s,\beta)$. Now the conditions 0) and 1) are obviously true and the validity of 2) follows from the observation $n_k^{(q)} \geqq k > (k)_1 = m$. We shall simultaneously prove that 3) and 4) are also true. To do this, we assume that $\psi_i(m) = \psi_j(m') = n$. We have to prove that $i = j$ and $m = m'$. Indeed, using the equalities $\varkappa(\psi_i(m)) = \varkappa(\psi_j(m')) = \varkappa(n) = q$ we get $n_k^{(q)} = \psi_i(m) = \psi_j(m') = n_{k'}^{(q)}$, where $k = 2^i.3^m$ and $k' = 2^j.3^{m'}$. Since there are no repetitions in the sequence $n_1^{(q)}, n_2^{(q)}, \ldots$, the equality $n_k^{(q)} = n_{k'}^{(q)}$ implies $k = k'$, i.e. $i = j$ and $m = m'$.

Now define $\mathcal{K}$ as $\bigcup\{\mathcal{K}_{s,\beta} \mid (s,\beta) \in B \times B^N\}$. We are going to check that

$$\Delta^2_{\mathfrak{B}}(x,\alpha) = \Delta^3_{\mathfrak{B}}(x,\alpha) \tag{2.4}$$

for every $\mathfrak{B}$ such that $(\varkappa,\mathfrak{B}) \in \mathcal{K}$. Indeed, take some $(\varkappa,\mathfrak{B})$ from $\mathcal{K}$ and hence from $\mathcal{K}_{s,\beta}$ for some s,β. Clearly, (2.4) will be proven if we succeed in verifying the following statement: whenever $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x,\alpha) \simeq y$, then w is suitable. Assume $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x,\alpha) \simeq y$; then in particular $A^{(w)}_{\mathfrak{B}}(x,\alpha) \simeq \mathrm{t}$, i.e. $(f_{i_1}(\underline{m}_1) = \underline{n}_1 \,\&\, \ldots \,\&\, f_{i_e}(\underline{m}_e) = \underline{n}_e \,\&\, \Phi)_{\mathfrak{B}}(x,\alpha) \simeq \mathrm{t}$. Hence $(f_{i_j}(\underline{m}_j) = \underline{n}_j)_{\mathfrak{B}}(x,\alpha) \simeq \mathrm{t}$, which means that $\psi_{i_j}(m_j) = n_j$ for every $1 \leqq j \leqq e$. These equalities together with conditions 1) and 2) of the definition of $\mathcal{K}_{s,\beta}$ ensure the validity of (i) and (ii). To see that (iii) is also true, use along with 3) and 4) the supposition made in the beginning of this section, namely that there are no two conjuncts of the kind $f_i(\underline{k}) = \underline{n}$ and $f_i(\underline{k}) = \underline{n}'$ in one and the same clause of Δ^0.

Now we gather up (2.0)–(2.4) to conclude:

Lemma 2.2. *For every σ-structure $\mathfrak{B}$ the following is true:*
a) $\Gamma(\mathfrak{B}_\alpha)(x) \supseteq \Delta^3_{\mathfrak{B}}(x,\alpha)$;
b) *if there exists $\varkappa$ such that $(\varkappa, \mathfrak{B}) \in \mathcal{K}$, then* $\Gamma(\mathfrak{B}_\alpha)(x) = \Delta^3_{\mathfrak{B}}(x,\alpha)$.

Our next task is to remove the symbols $\boldsymbol{S}$, $\mathbf{0}$ and $=$ from Δ^3. To this end let us fix some suitable w. We are going to introduce certain auxiliary notion of depth of $n \in N$ with respect to this index w. Let us remind that $A^{(w)} \Rightarrow a^{(w)}$ is of the form $f_{i_1}(\underline{m}_1) = \underline{n}_1 \,\&\, \ldots \,\&\, f_{i_e}(\underline{m}_e) = \underline{n}_e \,\&\, \Phi \Rightarrow \underline{m}_0$. Now set

$$N^0 = N \setminus \{n_1, \ldots, n_e\};$$
$$N^{i+1} = \{n_j \mid \exists j (1 \leqq j \leqq e \;\&\; m_j \in N^i)\} \text{ for } i = 0, 1, \ldots$$

Lemma 2.3. $N^0 \cup N^1 \cup \ldots = N$ *and* $N^i \cap N^k = \varnothing$ *whenever* $i \neq k$.

Proof. Suppose first that for some $i \neq k$ $N^i \cap N^k \neq \varnothing$. We may assume that i is the least number satisfying this condition. Take some $n \in N^i \cap N^k$. We have $k > i \geqq 0$ and therefore $k \geqq 1$, hence there exists some $j \in \{1, \ldots, e\}$ such that $n = n_j$ and $m_j \in N^{k-1}$. Now $n = n_j \in N^i$ gives us $N^i \cap \{n_1, \ldots, n_e\} \neq \varnothing$, i.e. $i > 0$. It means that there exists some $q \in \{1, \ldots, e\}$ such that $n = n_q$ and $m_q \in N^{i-1}$. The equalities $n = n_j = n_q$ imply $j = q$ (w is a suitable index!). Thus we get $m_j \in N^{i-1} \cap N^{k-1}$, which contradicts the choice of i.

Now let us assume that there exists some $n \in N$ such that $n \notin N^0 \cup N^1 \cup \ldots$ and let n be the least with this property. Apparently, $n = n_j$ for some $j \in \{1, \ldots, e\}$. Again from the fact that w is suitable we obtain $m_j < n_j = n$ and therefore $m_j \in N^i$ for some i. This implies $n_j \in N^{i+1}$ — a contradiction with the choice of n.

Lemma 2.3 makes the following definition correct. Let us call a *depth of n with respect to w* (in symbols: $|n|_w$) the unique natural number i such that $n \in N^i$. Let us notice here the obvious observation that there is an effective way for every $n \in N$ and every suitable index w to find $|n|_w$.

Suppose now that some additional list of different variables Z_0, Z_1, ... is chosen. Fix some suitable w. By induction on $|n|_w$ we define a sequence $\{\tau^n\}_n$ of terms in the following way:

$$\text{If } |n|_w = 0, \text{ set } \tau^n = \begin{cases} X, & \text{if } n = x, \\ Y_i, & \text{if } n = y_i, \\ Z_n & \text{otherwise.} \end{cases}$$

When $|n|_w = i > 0$, then by definition $n = n_j$, the depth of m_j being $i - 1$. Set in this case $\tau^n = f_{i_j}(\tau^m j)$.

Before explaining the basic property of τ^n we introduce a notational convention. Whenever Ψ is a σ-expression with variables X, Y_{j_1}, ..., Y_{j_k}, Z_{l_1}, ..., Z_{l_m}, $\mathfrak{M}$ is an arbitrary σ-structure and $(s, \beta, \delta) \in |\mathfrak{M}| \times |\mathfrak{M}|^N \times |\mathfrak{M}|^N$; we shall write $\Psi_{\mathfrak{M}}(s, \beta, \delta)$ as an abbreviation for

$$\Psi_{\mathfrak{M}}(X/s, Y_{j_1}/\beta(j_1), \ldots, Y_{j_k}/\beta(j_k), Z_{l_1}/\delta(l_1), \ldots, Z_{l_m}/\delta(l_m)).$$

From now on γ will denote the identity function on N. Let $\mathfrak{B} = (N; \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b)$ be a structure over N. A routine induction on the depth of n convince us that

$$(2.5) \qquad \tau_{\mathfrak{B}}^{(n)}(x, \alpha, \gamma) = n \ \text{ provided } \ \psi_{i_1}(m_1) = n_1, \ldots, \psi_{i_e}(m_e) = n_e.$$

Now we are in position to remove the symbols $\boldsymbol{S}$, $\mathbf{0}$ and $=$ from $A^{(w)} \Rightarrow a^{(w)}$. Indeed, let Φ' is obtained from Φ by replacing each conjunct $P_i(\underline{n})$ $(\neg P_i(\underline{n}))$ of Φ by $P_i(\tau^n)$ $(\neg P_i(\tau^n))$ and denote by $B^{(w)} \Rightarrow b^{(w)}$ the clause $P_0(\tau^{n_1})\ \&\ \ldots\ \&\ P_0(\tau^{n_e})\ \&\ \Phi' \Rightarrow \tau^{m_0}$. Notice that $B^{(w)} \Rightarrow b^{(w)}$ is already a σ-clause. Moreover, $(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma)$ is equal to $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x, \alpha)$ under certain condition about $\mathfrak{B} = (N; \psi_1, \ldots, \psi_a; Q_1, \ldots, Q_b)$, namely:

Lemma 2.4. *Suppose that* $\psi_{i_1}(m_1) = n_1, \ldots, \psi_{i_e}(m_e) = n_e$. *Then*

$$(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma) \simeq (A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x, \alpha).$$

Proof. It is obvious if we take into account the observation (2.5)

Let us mention here the following almost obvious fact, which will be used in the next section:

Lemma 2.5. *If* Z_n *is a variable of* $B^{(w)} \Rightarrow b^{(w)}$, *then* $n \notin \{x, y_0, y_1, \ldots\} \cup \{n_1, \ldots, n_e\}$.

Proof. If Z_n is a variable of $B^{(w)} \Rightarrow b^{(w)}$, then Z_n is a variable of $\tau^{(m)}$ for some m. Now use a straightforward induction on the depth of m with respect to w.

Variables of each clause $B^{(w)} \Rightarrow b^{(w)}$ are among $X, Y_0, Y_1, \ldots, Z_0, Z_1, \ldots$ Our final transformation aims to eliminate the variables from the list $Z_0, Z_1, \ldots$ Let us fix some effective enumeration $\rho^0, \rho^1, \ldots$ of all σ-terms with variables among X, $Y_0, Y_1, \ldots$ (i.e. such that the function which assigns to each n the Gödel number of ρ^n is recursive). For every $m > 0$ denote by $C_m^{(w)} \Rightarrow c_m^{(w)}$ the clause which is obtained from $B^{(w)} \Rightarrow b^{(w)}$ by replacing each variable Z_n by $\rho^{(m)_n}$. Now set $\Delta = \{C_m^{(w)} \Rightarrow c_m^{(w)} \mid m > 0 \text{ and } w \text{ is suitable}\}$. Due to the choice of the sequence $\{\rho^n\}_n$, Δ is a r.e. scheme. In the next section we shall establish that $F = \Delta_{\mathfrak{A}}$.

3. EXPLICIT CHARACTERIZATION OF ADMISSIBILITY

Theorem 3.1. *F is admissible in $\mathfrak{A}$ iff F is definable.*

Proof. The converse part is almost obvious. Let $F = \Delta_{\mathfrak{A}}$ for some r.e. scheme $\Delta = \{\Phi^w \Rightarrow \tau^w \mid w \in W\}$. Define $\Gamma(\mathfrak{B}_\alpha)(x)$ as $\{y \mid \exists w_{w \in W}(\Phi^w \Rightarrow \tau^w)_{\mathfrak{B}}(x, \alpha) \simeq y\}$. A straightforward verification convinces us that for every $(s, \beta) \in B \times B^N$ the equality

$$(3.0) \qquad \varkappa(\Gamma(\mathfrak{B}_\alpha)(x)) = F(s, \beta)$$

holds for every enumeration $(\varkappa, \mathfrak{B})$ of $\mathfrak{A}(s,\beta)$ and for every (x,α) such that $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$.

Now suppose that F is admissible. Then there exists an enumeration operator Γ such that the equality (3.0) is true. Let Δ be the r.e. scheme for F, constructed in the previous section. Fix some (s,β) — an arbitrary element of $B \times B^N$. We are going to prove that

$$p \in F(s,\beta) \iff p \in \Delta_{\mathfrak{A}}(s,\beta). \tag{3.1}$$

Indeed, assume first that $p \in F(s,\beta)$. Let choose some enumeration $(\varkappa, \mathfrak{B} = (N;\ \psi_1,\dots,\psi_a; Q_1,\dots,Q_b))$ from $\mathcal{K}_{s,\beta}$ which is not empty according to Lemma 2.1. By the choice of $(\varkappa,\mathfrak{B})$ we have $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$. Therefore by (3.0) $\varkappa(\Gamma(\mathfrak{B}_\alpha)(x)) = F(s,\beta)$ and hence there exists $y \in N$ such that $y \in \Gamma(\mathfrak{B}_\alpha)(x)$ and $\varkappa(y) = p$. By Lemma 2.2 $y \in \Delta^3_{\mathfrak{B}}(x,\alpha)$ or, equivalently, $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x,\alpha) \simeq y$ for some suitable index w. In particular, $\psi_{i_1}(m_1) = n_1, \dots, \psi_{i_e}(m_e) = n_e$, hence Lemma 2.4 can be applied. As a result we obtain $(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x,\alpha,\gamma) \simeq (A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x,\alpha) = y$, that is

$$B^{(w)}_{\mathfrak{B}}(x,\alpha,\gamma) \simeq \mathrm{t} \quad \text{and} \quad b^{(w)}_{\mathfrak{B}}(x,\alpha,\gamma) \simeq y. \tag{3.2}$$

Define the sequence δ as follows: $\delta(n) = \varkappa(n)$ for $n = 0,1,\dots$ We have $\varkappa(\gamma) = \delta$ as well as $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$, and therefore $B^{(w)}_{\mathfrak{A}}(s,\beta,\delta) \simeq B^{(w)}_{\mathfrak{B}}(x,\alpha,\gamma)$ and $b^{(w)}_{\mathfrak{A}}(s,\beta,\delta) \simeq \varkappa(b^{(w)}_{\mathfrak{B}}(x,\alpha,\gamma))$. These equalities combined with (3.2) and the fact that $\varkappa(y) = p$ give us

$$(B^{(w)}, \Rightarrow b^{(w)})_{\mathfrak{A}}(s,\beta,\delta) \simeq p. \tag{3.3}$$

Let $Z_{k_1}, \dots, Z_{k_n}$ be all variables of $B^{(w)} \Rightarrow b^{(w)}$ from the list $Z_0, Z_1, \dots$ For $i = 1,\dots,n$ we set for short $\varkappa(k_i) = r_i$. Each r_i belongs to $\mathrm{Range}(\varkappa) = \mathcal{T}(s,\beta)$, hence $r_i = \rho^{m_i}_{\mathfrak{A}}(s,\beta)$ for some term ρ^{m_i} from the sequence $\{\rho^n\}_n$, fixed in the end of the previous section. Now take some $m \in N$ such that $(m)_{k_i} = m_i$ for $1 \leqq i \leqq n$ and consider the clause $C^{(w)}_m \Rightarrow c^{(w)}_m$. By definition it is obtained from $B^{(w)} \Rightarrow b^{(w)}$ by simultaneous replacement of each Z_{k_i} by $\rho^{(m)_{k_i}}$, i.e. by ρ^{m_i} in our case. Notice that

$$(Z_{k_i})_{\mathfrak{A}}(s,\beta,\delta) = \delta(k_i) = r_i = \rho^{m_i}_{\mathfrak{A}}(s,\beta).$$

So we get

$$(C^{(w)}_m \Rightarrow c^{(w)}_m)_{\mathfrak{A}}(s,\beta) \simeq (B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{A}}(s,\beta,\delta)$$

and by (3.3) $(C^{(w)}_m \Rightarrow c^{(w)}_m)_{\mathfrak{A}}(s,\beta) \simeq p$. The last, according to our choice of Δ, means that $p \in \Delta_{\mathfrak{A}}(s,\beta)$, which completes the verification of the first direction of (3.1).

Assume now that $p \in \Delta_{\mathfrak{A}}(s,\beta)$. The only way to force $p \in F(s,\beta)$ is to show that there is some enumeration $(\varkappa,\mathfrak{B})$ of $\mathfrak{A}(s,\beta)$ such that

$$\varkappa(x) = s, \quad \varkappa(\alpha) = \beta \quad \text{and} \quad p \in \varkappa(\Gamma(\mathfrak{B}_\alpha)(x)), \tag{3.4}$$

where, of course, x and α are again those already fixed in Sec. 2, because Δ convey certain information about Γ at the point (x,α) only.

The assumption $p \in \Delta_{\mathfrak{A}}(s,\beta)$ is equal to $(C_m^{(w)} \Rightarrow c_m^{(w)})_{\mathfrak{A}}(s,\beta) \simeq p$ for some suitable index w, whence in particular each of the expressions $C_{m\mathfrak{A}}^{(w)}(s,\beta)$ and $c_{m\mathfrak{A}}^{(w)}(s,\beta)$ is defined. Now having in mind the construction of $C_m^{(w)} \Rightarrow c_m^{(w)}$, we may conclude that for each n, such that Z_n is a variable of $B^{(w)} \Rightarrow b^{(w)}$, $\rho_{\mathfrak{A}}^{(m)_n}(s,\beta)$ is defined. Let us define a sequence $\delta \in B^N$ in the following way:

$$\delta(n) = \begin{cases} \rho_{\mathfrak{A}}^{(m)_n}(s,\beta), & \text{if } Z_n \text{ is a variable of } B^{(w)} \Rightarrow b^{(w)}, \\ s & \text{otherwise.} \end{cases}$$

It is clear that $(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{A}}(s,\beta,\delta) \simeq (C_m^{(w)} \Rightarrow c_m^{(w)})_{\mathfrak{A}}(s,\beta)$ and hence

$$(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{A}}(s,\beta,\delta) \simeq p. \tag{3.5}$$

We have in particular that $B_{\mathfrak{A}}^{(w)}(s,\beta,\delta)$ is defined. From here, since each $P_0(\tau^{n_i})$, $1 \leqq i \leqq e$, is a conjunct of $B^{(w)}$,

$$\tau_{\mathfrak{A}}^{n_i}(s,\beta,\delta) \text{ is defined for every } i = 1,\dots,e. \tag{3.6}$$

Now we can explain how to construct an enumeration $(\varkappa, \mathfrak{B})$ of $\mathfrak{A}(s,\beta)$ satisfying the requirement (3.4). We begin with the definition of $\varkappa$. Set $\varkappa(x) = s$, $\varkappa(y_i) = \beta(i)$ for $i = 0, 1, \dots$, $\varkappa(n) = \delta(n)$ for every n such that n is a variable of $B^{(w)} \Rightarrow b^{(w)}$ and $\varkappa(n_j) = \tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta)$ for $j = 1,\dots,e$ (notice that by (3.6) each $\tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta)$ is defined). To see that this settings are correct, recall our choice of α (α is injective and $x \notin \text{Range}(\alpha)$), take into account the fact that w is a suitable index (and hence $n_1, \dots, n_e$ are different natural numbers which do not belong to $M = \{x, y_0, y_1, \dots\}$) and consider Lemma 2.5.

Now we have to extend the definition of $\varkappa$ onto the whole N. Since $N \setminus \text{Range}(\alpha)$ is infinite, this can be done in such a way that $\{\varkappa(n) \mid n \in N\} = \mathfrak{T}(s,\beta)$, i.e. so that $\varkappa$ is a mapping onto $\mathfrak{T}(s,\beta)$. In addition, $\varkappa$ has one very important property, namely

$$\varkappa(n_j) = \varphi_{i_j}(\varkappa(m_j)) \text{ for every } j = 1,\dots,e. \tag{3.7}$$

We shall separately consider the two cases for the depth of m_j (with respect to w): it is 0; and it is positive. In the second case by definition $m_j \in \{n_1, \dots, n_e\}$ and hence $\varkappa(m_j) = \tau_{\mathfrak{A}}^{m_j}(s,\beta,\delta)$. We have also $\tau^{n_j} = f_{i_j}(\tau^{m_j})$ and therefore $\tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta) = (f_{i_j}(\tau^{m_j}))_{\mathfrak{A}}(s,\beta,\delta) = \varphi_{i_j}(\tau_{\mathfrak{A}}^{m_j}(s,\beta,\delta)) = \varphi_{i_j}(\varkappa(m_j))$. This, combined with $\varkappa(n_j) = \tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta)$, completes the verification of (3.7) for the case when $|m_j|_w > 0$. If $|m_j|_w = 0$, there are three possibilities: $m_j = x$, $m_j \in \{y_0, y_1, \dots\}$, and $m_j \in N \setminus \{x, y_0, y_1, \dots\} \cup \{n_1, \dots, n_e\}$. If $m_j = x$, then by definition $\tau^{m_j} = X$ and $\tau^{n_j} = f_{i_j}(X)$. Further

$$\varkappa(n_j) = \tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta) = (f_{i_j}(X))_{\mathfrak{A}}(s,\beta,\delta) = \varphi_{i_j}(s) = \varphi_{i_j}(\varkappa(x)) = \varphi_{i_j}(\varkappa(m_j)).$$

In the case $m_j \in \{y_0, y_1, \dots\}$ we proceed analogously. In the last case we have $\tau^{m_j} = Z_{m_j}$ and $\tau^{n_j} = f_{i_j}(Z_{m_j})$, respectively. Let us notice that by construction $P_0(\tau_{n_j})$ is a conjunct of $B^{(w)}$ and Z_{m_j} is a variable of $B^{(w)}$. Therefore by definition $\varkappa(m_j) = \delta(m_j)$. Now, similarly to the previous case, we get

$$\tau_{\mathfrak{A}}^{n_j}(s,\beta,\delta) = (f_{i_j}(Z_{m_j}))_{\mathfrak{A}}(s,\beta,\delta) = \varphi_{i_j}(\delta(m_j)) = \varphi_{i_j}(\varkappa(m_j)).$$

Now define the structure $\mathfrak{B}$ as follows: $\psi_{i_j}(m_j) = n_j$ for every $j = 1, \dots, e$; $\psi_i(m) \simeq \mu n[\varkappa(n) \simeq \varphi_i(\varkappa(m))]$ in the remaining cases, i.e. when $(i, m) \neq (i_j, m_j)$ for each $1 \leqq j \leqq e$; $Q_i(m) \simeq R_i(\varkappa(m))$ for every $m \in N$ and $1 \leqq i \leqq b$.

In order to show that $(\varkappa, \mathfrak{B})$ is an enumeration of $\mathfrak{A}(s, \beta)$ it suffices to check that $\varkappa(\psi_i(m)) \simeq \varphi_i(\varkappa(m))$. Indeed, whenever $(i, m) = (i_j, m_j)$ for some $j \in \{1, \dots, e\}$, this equality is true on the grounds of (3.7). Suppose now that $(i, m) \neq (i_j, m_j)$ for every $j \in \{1, \dots, e\}$ and assume, first, that $\varphi_i(\varkappa(m))$ is defined. We have $\varkappa(m) \in \mathcal{T}(s, \beta)$ and next $\varphi_i(\varkappa(m)) \in \mathcal{T}(s, \beta)$. Since $\varkappa$ is a mapping onto $\mathcal{T}(s, \beta)$, there exists n such that $\varkappa(n) = \varphi_i(\varkappa(m))$ and hence $\psi_i(m)$ is defined. Further

$$\varkappa(\psi_i(m)) = \varkappa(\mu n[\varkappa(n) = \varphi_i(\varkappa(m))]) = \varphi_i(\varkappa(m)).$$

Whenever $\varphi_i(\varkappa(m))$ is undefined, $\psi_i(m)$ is undefined by definition.

A straightforward verification convinces us that $\varkappa(V_{\mathfrak{B}}(x, \alpha, \gamma)) = V_{\mathfrak{A}}(s, \beta, \delta)$ for every variable V of $B^{(w)} \Rightarrow b^{(w)}$. Thus we may conclude that

$$\varkappa((B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma)) \simeq (B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{A}}(s, \beta, \delta)$$

and by (3.5) $\varkappa((B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma)) \simeq p$. It means that there exists $y \in N$ such that

$$(B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma) \simeq y \text{ and } \varkappa(y) = p. \tag{3.8}$$

On the other hand, owing to the special construction of $\mathfrak{B}$, Lemma 2.4 can be applied. So we get $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x, \alpha) \simeq (B^{(w)} \Rightarrow b^{(w)})_{\mathfrak{B}}(x, \alpha, \gamma)$ and therefore by (3.8) $(A^{(w)} \Rightarrow a^{(w)})_{\mathfrak{B}}(x, \alpha) \simeq y$. Since $A^{(w)} \Rightarrow a^{(w)}$ is a clause of Δ^3, $y \in \Delta^3_{\mathfrak{B}}(x, \alpha)$. Finally, using Lemma 2.2 we come to the conclusion that $y \in \Gamma(\mathfrak{B}_\alpha)(x)$. We have also $\varkappa(x) = s$ and $\varkappa(\alpha) = \beta$ and hence by (3.0) $\varkappa(y) \in F(s, \beta)$, that is $p \in F(s, \beta)$ by (3.8). This completes the verification of the first direction of the theorem. As we noticed above, the opposite is straightforward. Thus the proof of the theorem is completed.

As a consequence of the above theorem we get the following characterizations of the admissible functions, which agree with the corresponding result from [11, Thm. 4], obtained for the case of denumerable $\mathfrak{A}$:

Theorem 3.2. *$\varphi : B \to 2^B$ is admissible in $\mathfrak{A}$ iff it is absolutely search computable over $\mathfrak{A}$.*

Proof. We shall use the following normal form theorem for absolutely search computable (ASC) functions [7, Cor. 3]. A k-ary p.m.v. function φ is ASC in $\mathfrak{A}$ iff there exists r.e. scheme $\Delta = \{\exists Y_{j_1} \dots \exists Y_{j_{w_n}}(\Phi^w \Rightarrow \tau^w) \mid w \in W\}$ with free variables $X_1, \dots, X_k$ such that for every $\bar{s} \in B^k$ the following is true: $\varphi(\bar{s}) \ni p$ iff

$$\exists p_{j_1} \dots \exists p_{j_{w_n}}(\Phi^w \Rightarrow \tau^w)_{\mathfrak{A}}(X_1/s_1, \dots, X_k/s_k, Y_{j_1}/p_{j_1}, \dots, Y_{j_{w_n}}/p_{j_{w_n}}) \simeq p.$$

Now the proof of the theorem is straightforward if we take into account Theorem 3.1.

4. STABLE FUNCTIONALS

Admissible functionals were introduced as a mathematical description of algorithms over $\mathfrak{A}$, which use an additional input from the data domain B. A certain way to separate amongst them those F which do not depend on B is to postulate

$$F(\bar{s}, \beta) = F(\bar{s}, \beta') \text{ for every } \bar{s} \in B^k \text{ and } \beta, \beta' \in B^N. \tag{4.0}$$

We shall say that F is stable if it is admissible and satisfy (4.0).

The next characterization of the stable functionals follows directly from Theorem 3.1.

Proposition 4.1. *$F : B^k \times B^N \to 2^B$ is stable iff there is a r.e. scheme Δ with variables $X_1, \dots, X_k$ such that*

$$F(\bar{s}, \beta) = \Delta_{\mathfrak{A}}(\bar{s}) \text{ for every } (\bar{s}, \beta) \in B^k \times B^N.$$

Proof. If F is stable, then F is admissible and according to Theorem 3.1 it is definable by some r.e. scheme Θ with variables $X_1, \dots, X_k, Y_0, Y_1, \dots$ Replace in Θ each variable Y_i by X_1 and denote the scheme obtained in this way by Δ.

Now let us fix some $(s_1, \dots, s_k)$ and define $\beta_0 \in B^N$ as $\beta_0(n) = s_1$ for every n. Clearly, $\Theta(\bar{s}, \beta_0) = \Delta_{\mathfrak{A}}(\bar{s})$. We have $F(\bar{s}, \beta) = F(\bar{s}, \beta_0) = \Theta(\bar{s}, \beta_0) = \Delta_{\mathfrak{A}}(\bar{s})$. If the right hand side of the proposition holds, then F is definable and therefore admissible. Obviously, $F(\bar{s}, \beta) = F(\bar{s}, \beta')$ for every β, β' in B^N, which means that F is stable.

For any stable F we define φ_F by setting

$$\varphi_F(\bar{s}) = F(\bar{s}, \beta) \text{ for any } \beta \in B^N.$$

φ_F may be thought as the function, computable by F, so it is reasonable to expect that the following proposition will be true.

Proposition 4.2. *If F is stable, then φ_F is computable by means of some recursively enumerable definitional scheme* (REDS) *of Shepherdson* [9].

Proof. It is a straightforward consequence of Proposition 4.2.

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ON THE DETECTION OF SOME LOOPS IN RECURSIVE COMPUTATIONS*

DIMITER SKORDEV

Димитр Скордев. ОБ ОБНАРУЖЕНИИ НЕКОТОРЫХ ЗАЦИКЛИВАНИЙ В РЕКУРСИВНЫХ ВЫЧИСЛЕНИЯХ

В настоящей работе рекурсивные вычисления трактуются как некоторая детерминированная форма переработки термов. Термы строятся обычным способом из атомов при помощи функциональных символов. Интуитивно атомы трактуются как константы. Простой неатомарный терм — это терм, получающийся, когда некоторый функциональный символ снабжается аргументами, являющимися атомами. Предполагается, что дано некоторое правило рекурсии в форме отображения множества простых неатомарных термов в множество всех термов. Рекурсивное вычисление — это процес конструирования термов, начиная с некоторого данного терма и заменяя, пока возможно, самое левое вхождение простого неатомарного терма в текущий терм на терм, соответствующий согласно данному правилу рекурсии. Предлагается метод в стиле Брента — Ван Гельдера для обнаружения случая, когда некоторый простой неатомарный терм воспроизводит себя снова в самом левом положении после положительного числа шагов в рекурсивном вычислении.

Dimiter Skordev. ON THE DETECTION OF SOME LOOPS IN RECURSIVE COMPUTATIONS

In the present paper recursive computations are treated as a certain deterministic kind of term processing. The terms are built up in the usual way from atoms by means of function symbols. The atoms are intuitively viewed as constants. Simple non-atomic terms are those ones

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constructed by supplying some function symbol with arguments which are atoms. A recursion rule is supposed to be given, considered as a mapping of the set of the simple non-atomic terms into the set of *all* terms. *A recursive computation is a process of constructing terms by starting with* some given term and, while possible, replacing the leftmost occurrence of a simple non-atomic term in the current term by the corresponding term according to the given recursion rule. A Brent—Van Gelder's style method is proposed for detection of the case when some simple non-atomic term reproduces itself again in a leftmost position after a positive number of steps in a recursive computation.

1. RECURSIVE COMPUTATIONS AND CYCLIC LOOPS IN THEM

We consider terms built up in the usual way from atoms by means of function symbols. The atoms are intuitively viewed as constants. The set of the atoms will be denoted by $\mathcal{A}$, and the set of the function symbols will be denoted by $\mathcal{F}$. Let $\mathcal{U}$ (the computational universe) be the set of all terms. Let $\mathcal{U}_1$ be the set of all terms of the form $f(\alpha_1, \ldots, \alpha_n)$, where f is some n-ary symbol from $\mathcal{F}$ and $\alpha_1, \ldots, \alpha_n$ are atoms. The elements of $\mathcal{U}_1$ will be called *simple non-atomic terms*. Each mapping of $\mathcal{U}_1$ into $\mathcal{U}$ will be called a *recursion rule*. An example follows which illustrates the intuition behind this convention.

Example 1. Let $\mathbb{N}$ be the set of all non-negative integers (to be called further *natural numbers*). Consider the least defined two-argument partial function φ in $\mathbb{N}$ satisfying the following conditions for all y and z in $\mathbb{N}$:

$$\varphi(2z, y) = z + 1, \quad \varphi(2z + 1, y) = \varphi(\varphi(z, y), \varphi(y, z)).$$

We shall relate to this function definition a recursion rule in the introduced sense. Let $\mathcal{A}$ consist of the usual decimal denotations of the individual natural numbers, and $\mathcal{F}$ consist of a two-argument function symbol f. Identifying the decimal denotations of the natural numbers with the numbers themselves, we define a mapping $\mathcal{D}$ of $\mathcal{U}_1$ into $\mathcal{U}$ in the following way[1]:

$$\mathcal{D}(\mathrm{f}(2z, y)) = z + 1, \quad \mathcal{D}(\mathrm{f}(2z + 1, y)) = \mathrm{f}(\mathrm{f}(z, y), \mathrm{f}(y, z)).$$

For example, we shall have

$$\mathcal{D}(\mathrm{f}(6, 5)) = 4, \quad \mathcal{D}(\mathrm{f}(13, 11)) = \mathrm{f}(\mathrm{f}(6, 11), \mathrm{f}(11, 6)).$$

Turning back to the general situation, we note that each element u of $\mathcal{U} \setminus \mathcal{A}$ has at least one subterm belonging to $\mathcal{U}_1$, and there is a uniquely determined leftmost occurrence of a term from $\mathcal{U}_1$ in u. Replacing this occurrence by any other term will produce again an element of $\mathcal{U}$. From now on, a recursion rule $\mathcal{D}$ will be supposed to be given. We extend $\mathcal{D}$ to $\mathcal{U} \setminus \mathcal{A}$ in the following way: if u is an arbitrary non-atomic term, then we set $\mathcal{D}(\mathrm{u})$ to be the result of replacing the leftmost occurrence of a term from $\mathcal{U}_1$ in u by the corresponding image under the original mapping $\mathcal{D}$. If u is an arbitrary term, then there is a maximal sequence (finite or infinite)

[1] The equality sign here and in all further occasions, where terms are concerned, denotes graphic equality of terms.

of the form u, $\mathcal{D}(u)$, $\mathcal{D}^2(u)$, $\mathcal{D}^3(u)$, ... This sequence will be called *the recursive computation of* u.

Example 2. In the situation from Example 1 the recursive computation of the simple non-atomic term f(13, 11) looks as follows:

$$f(13,11),\quad f(f(6,11),f(11,6)),\quad f(4,f(11,6)),\quad f(4,f(f(5,6),f(6,5))),$$
$$f(4,f(f(f(2,6),f(6,2)),f(6,5))),\quad f(4,f(f(2,f(6,2)),f(6,5))),$$
$$f(4,f(f(2,4),f(6,5))),\quad f(4,f(2,f(6,5))),\quad f(4,f(2,4)),\quad f(4,2),\quad 3.$$

Again in this situation the first five members of the recursive computation of f(1, 0) are the following ones:

$$f(1,0),\quad f(f(0,0),f(0,0)),\quad f(1,f(0,0)),\quad f(1,1),\quad f(f(0,1),f(1,0)).$$

This computation is infinite, since its member f(1, 0) turns out to be a subterm of a further one.

Remark 1. It can be proved that an ordered pair (x, y) belongs to the domain of the function φ from Example 1 iff the recursive computation of the simple non-atomic term $f(x, y)$ is finite, and in such a case the computation terminates with the value of $\varphi(x, y)$ (hence $\varphi(13, 11)$ is equal to 3, and $\varphi(1, 0)$ is not defined).

It is natural to be interested in the problem how to decide whether the recursive computation of a given term is finite. Unfortunately, this problem is algorithmically unsolvable even in some relatively simple concrete cases. Therefore we shall consider a certain special sort of infinite recursive computations, such that the infinity of the computation can be effectively detected after making an appropriate finite number of computational steps. We are going now to introduce some denotations and notions which are useful for describing the mentioned sort of infinite recursive computations.

Whenever u is a non-atomic term, we shall denote by $\mathcal{H}(u)$ the leftmost occurring term from $\mathcal{U}_1$ in u; this term will be called *the head of* u.

Example 3. In the situation from Example 1 the equality

$$\mathcal{H}(f(4,f(f(5,6),f(6,5)))) = f(5,6)$$

holds.

The following fact seems to be intuitively obvious and we shall postpone its proof (cf. Appendix 2):

Fact 1. For any term u and any natural number t, if $\mathcal{H}(\mathcal{D}^t(\mathcal{H}(u)))$ makes sense, then $\mathcal{H}(\mathcal{D}^t(u))$ also makes sense and the equality

$$\mathcal{H}(\mathcal{D}^t(\mathcal{H}(u))) = \mathcal{H}(\mathcal{D}^t(u))$$

holds.

If u is a term, v is a simple non-atomic term, and t is a natural number, then u will be said to *activate* v *after t steps* if u belongs to the domain of $\mathcal{D}^t$ and v is the head of $\mathcal{D}^t(u)$.

Example 4. In the situation from Example 1 f(1, 0) activates f(0, 1) after 4 steps (cf. Example 2).

Remark 2. Another related notion also deserves attention. This is the following interrelation between a term u, a simple non-atomic term v and a natural number t: the term u belongs to the domain of $\mathcal{D}^t$, and v is a subterm of $\mathcal{D}^t(\mathrm{u})$. This requirement is weaker than the requirement u to activate v after t steps (as seen from Example 2, the above requirement is satisfied in the situation from Example 1 for $\mathrm{u} = \mathrm{v} = \mathrm{f}(1,0)$, $t = 4$, but $\mathrm{f}(1,0)$ does not activate $\mathrm{f}(1,0)$ after 4 steps). The described other notion can be used more or less instead of the notion of activation. However, we prefer to use the notion of activation for reasons of technical convenience.

Fact 1 can be reformulated as follows: for any term u and any natural number t if $\mathcal{H}(\mathrm{u})$ makes sense and activates a certain simple non-atomic term w after t steps, then u also activates w after t steps.

Lemma 1. *Let u be a term, v, w be simple non-atomic terms, s and t be natural numbers, let u activate v after s steps, and v activate w after t steps. Then u activates w after $s+t$ steps.*

Proof. We have the equalities $\mathrm{v} = \mathcal{H}(\mathcal{D}^s(\mathrm{u}))$, $\mathrm{w} = \mathcal{H}(\mathcal{D}^t(\mathrm{v}))$. Applying Fact 1 with $\mathcal{D}^s(\mathrm{u})$ in the role of u, we conclude that $\mathcal{D}^s(\mathrm{u})$ belongs to the domain of $\mathcal{D}^t$, and the equality $\mathrm{w} = \mathcal{H}(\mathcal{D}^t(\mathcal{D}^s(\mathrm{u})))$ holds, i.e. u belongs to the domain of $\mathcal{D}^{s+t}$ and the equality $\mathrm{w} = \mathcal{H}(\mathcal{D}^{s+t}(\mathrm{u}))$ holds. ∎

A simple non-atomic term v will be said to be *self-reactivating* iff there is a positive integer r such that v activates v after r steps. The following lemma shows a possibility to apply this notion for establishing the infinity of certain recursive computations.

Lemma 2. *Let a term u activate a self-reactivating simple non-atomic term after some number of steps. Then the recursive computation of u is infinite.*

Proof. Let u activate the self-reactivating simple non-atomic term v after s steps, and let v activate v after r steps, where $r > 0$. Then, by Lemma 1, u activates v after $s+kr$ steps for all k in $\mathbb{N}$. Hence u belongs to the domain of $\mathcal{D}^{s+kr}$ for all k in $\mathbb{N}$, i.e. u belongs to the domain of $\mathcal{D}^r$ for arbitrarily large values of r. ∎

Example 5. In the situation from Example 1 the simple non-atomic term $f(1,0)$ is self-reactivating. Namely, it activates itself after 5 steps. Indeed, making use of Example 2 we see that

$$\mathcal{D}^5(\mathrm{f}(1,0)) = \mathcal{D}(\mathrm{f}(\mathrm{f}(0,1),\mathrm{f}(1,0))) = \mathrm{f}(1,\mathrm{f}(1,0)).$$

This fact, together with Lemma 2, yields another proof of the statement that the recursive computation of $f(1,0)$ is infinite.

Example 6. Again in the situation from Example 1 the first four members of the recursive computation of $\mathrm{f}(1,3)$ are:

$$\mathrm{f}(1,3),\quad \mathrm{f}(\mathrm{f}(0,3),\mathrm{f}(3,0)),\quad \mathrm{f}(1,\mathrm{f}(3,0)),\quad \mathrm{f}(1,\mathrm{f}(\mathrm{f}(1,0),\mathrm{f}(0,1))).$$

Hence the term $\mathrm{f}(1,3)$ activates the term $\mathrm{f}(1,0)$ after 3 steps. Therefore, by Example 5 and Lemma 2, the recursive computation of $\mathrm{f}(1,3)$ is infinite. However, the term f(1, 3) is not self-reactivating. Namely, one can easily show by induction that no member of the recursive computation of f(1, 3) except the first three ones can contain atoms different from 0 and 1.

Example 7. Let us modify the definition of φ and the corresponding recursion rule from Example 1 by writing "$z + 9$" instead of "$z + 1$". Then the recursion rule will be

$$\mathcal{D}(\mathrm{f}(2z, y)) = z + 9, \quad \mathcal{D}(\mathrm{f}(2z + 1, y)) = \mathrm{f}(\mathrm{f}(z, y), \mathrm{f}(y, z)).$$

Under this recursion rule the term f(9, 17) is self-reactivating, but it activates itself only after a relatively large number of steps. Namely, f(9, 17) activates itself after 126 steps and does not activate itself after a smaller positive number of steps. This can be seen by application of a suitable computer program. For example, making use of Turbo Pascal, Version 4.0 or later, we could apply the program from Figure 1[2].

```
{$S+}
var k,l,m,t:word;

function f(x,y:word):word;
var z:word;
begin
 if (x=k) and (y=l) and (t>0) then
  begin
   writeln('f(',k,',',l,') activates itself after ',t,' steps!');
   halt
  end;
 if t<65535 then t:=t+1
           else begin writeln('Too many steps!');halt end;
 z:=x div 2;if odd(x) then f:=f(f(z,y),f(y,z)) else f:=z+9
end;

begin
 t:=0;
 readln(k,l);m:=f(k,l);writeln('f(',k,',',l,')=',m)
end.
```

Figure 1. A program detecting self-reactivation (Example 7)

[2] The program is obtained by adding appropriate side effects to an ordinary recursive Pascal program for computing the value of φ. The side effects in question are carried out through introducing the step counting variable t and through the operators containing it (the fact is used that the compiler implements the recursion present in the program by means of steps corresponding to applications of the mapping $\mathcal{D}$). A more straight-forward approach would lead to explicitly using dynamic data structures for modelling the recursive computation of the considered term. The way of using side effects in the programs given in this paper is similar to a technique used in 1992 by Igor Đurđanović in Paderborn (he implemented then in Prolog certain loop detection methods designed by the present author and concerning Prolog programs).

Remark 3. Let us call a simple non-atomic term ν *self-reproducing* if there is a positive integer r such that ν belongs to the domain of $\mathcal{D}^r$, and ν is a subterm of $\mathcal{D}^r(\nu)$. In general, self-reproduction is a weaker property than self-reactivation, but can be used in a similar way. For example, the assumption of Lemma 2 means that the recursive computation of u contains some member whose head is a self-reactivating simple non-atomic term, and this assumption can be weakened in the following way: some member of the recursive computation of u has a subterm which is a self-reproducing simple non-atomic term. We shall not make use of such a strengthening of the lemma, since, for reasons of technical convenience, we prefer to use the notion of self-reactivation.

Let u be a term. We shall say that *a cyclic loop is present in the recursive computation of* u iff u satisfies the assumption of Lemma 2, i.e. iff u activates some self-reactivating simple non-atomic term after some number of steps.

Example 8. In the situation from Example 1 a cyclic loop is present in the recursive computation of $\mathrm{f}(1,0)$, as well as in the recursive computation of $\mathrm{f}(1,3)$ (cf. Examples 5 and 6).

Example 9. In the situation from Example 7 a cyclic loop is present in the recursive computation of $\mathrm{f}(9,17)$. A cyclic loop is present also in the recursive computation of $\mathrm{f}(1,3)$, since $\mathrm{f}(1,3)$ activates $\mathrm{f}(9,17)$ after 48 steps (this can be shown by using a suitable modification of the program from Figure 1).

By Lemma 2 the presence of a cyclic loop in the recursive computation of a given term implies the infinity of that computation. It is easy to see that the converse is not true in the general case, i.e. the recursive computation of a term could be infinite without the presence of a cyclic loop in that computation. Nevertheless, the presence of a cyclic loop is a frequently encountered cause for infinity of the recursive computation of a term (cf. for example Appendix 1 in this connection). Therefore it could be useful to have some efficient method for the detection of this kind of loops. Such a method will be presented in the next section. The method will combine some features of a loop detection method of R. P. Brent, described in [1, p. 7, Exercise 7] and generalized in [2][3], and of another one proposed by A. Van Gelder in [3, 4][4] (both mentioned methods are intended for the examination of other kinds of computational processes).

2. THE LOOP DETECTION METHOD

We shall describe the method under the same assumptions as in the previous section. In fact, there will be infinitely many variants of the method. Any concrete variant is determined by the choice of a concrete infinite strictly increasing sequence $\tau_0, \tau_1, \tau_2, \ldots$ of natural numbers such that there is no upper bound for the set of

[3] No citation of related work of other authors is given in [2], due to the lack of information in this respect at the moment of writing that paper.

[4] The presentation in [3] is not correct, as shown in [5], but [4] indicates a way for correcting that presentation.

the differences $\tau_{n+1} - \tau_n$, $n = 0, 1, 2, \ldots$ (such sequences are considered in [2]; concordance with the description of Brent's method in [1] can be achieved by setting $\tau_n = 2^n - 1$). To simplify the presentation of the method, we shall restrict ourselves to the case when the equality $\tau_0 = 0$ holds. From now on, a sequence $\tau_0, \tau_1, \tau_2, \ldots$ with the formulated properties is supposed to be given.

To be able to detect the presence of a cyclic loop in the recursive computation of a given term, we add some loop detection activities to the process of constructing consecutive members of the recursive computation of the term. We shall firstly give a somewhat intuitive description of these additional activities and after that we shall describe them in a more formal way.

Roughly speaking, the main additional activity consists in making a snapshot of the head of the current term at certain moments of the computation, the moment τ_0 being the first of them, and comparing the heads of some of the further arising terms with this snapshot. A detailed formulation (although not yet a thoroughly formal one) concerning the comparison will look as follows.

Suppose we study the recursive computation of a given term u. Let a member $w = \mathcal{D}^t(u)$ of the computation be obtained from u by t applications of $\mathcal{D}$ and a snapshot of $\mathcal{H}(\mathcal{D}^{\bar{t}}(u))$ made after $\bar{t}$ applications of $\mathcal{D}$ to u be available, where $\bar{t} < t$. We suppose that no snapshot is made at moments between $\bar{t}$ and t, and $\mathcal{H}(\mathcal{D}^{\bar{t}}(u))$ belongs to the domain of $\mathcal{D}^{t-\bar{t}}$. After w is obtained, one firstly checks whether w belongs to the domain of $\mathcal{D}$. If w does not belong to this domain (i.e. if w is an atom), then the recursive computation of u is completed and nothing more has to be done (of course, there is no cyclic loop in the computation in this case). If w belongs to the domain of $\mathcal{D}$, then certain loop detection activities must be carried out. First of all, one checks whether $\mathcal{D}^{t-\bar{t}}(\mathcal{H}(\mathcal{D}^{\bar{t}}(u)))$ is an atom or not. If it is an atom, then we shall say that *the available snapshot becomes obsolete at the moment t*. In this case one replaces the snapshot of $\mathcal{H}(\mathcal{D}^{\bar{t}}(u))$ by a snapshot of $\mathcal{H}(\mathcal{D}^t(u))$ and then constructs the next member $\mathcal{D}^{t+1}(u)$ of the computation. Otherwise $\mathcal{H}(w)$ and $\mathcal{H}(\mathcal{D}^{\bar{t}}(u))$ are compared. If they turn out to be equal, then a loop is detected in the computation at the moment t and nothing more has to be done. Otherwise again the next member $\mathcal{D}^{t+1}(u)$ of the computation must be constructed. But before doing this one must check whether t is a member of the sequence $\tau_0, \tau_1, \tau_2, \ldots$, and if t is such a member, to replace again the snapshot of $\mathcal{H}(\mathcal{D}^{\bar{t}}(u))$ by a snapshot of $\mathcal{H}(\mathcal{D}^t(u))$.

We hope the next two examples will make the above presentation of the method more intelligible.

Example 10. Suppose we carry out the computation of $f(13, 11)$ in the situation from Example 1 (cf. Example 2). Let the sequence $\tau_0, \tau_1, \tau_2, \ldots$ be determined by the equality $\tau_n = 2^n - 1$, $n = 0, 1, 2, \ldots$ Then the computation process accompanied with loop detection activities can be represented by Figure 2, where making a snapshot is indicated by an asterisk, the corresponding head is printed bold-faced, and the vertical and broken lines indicate the results of successive applications of $\mathcal{D}$ to such a head (do not pay attention to the rightmost two columns of numbers — they will not be used yet!). Note that snapshots at moments 2, 7, 8 and 9

Moment	Term	c	$c^{\downarrow}$
0	* f(13,11)		1
1	* f(f(6,11),f(11,6))	3	1
2	* f(4,f(11,6))	0	1
3	* f(4,f(f(5,6),f(6,5)))	3	1
4	f(4,f(f(f(2,6),f(6,2)),f(6,5)))	3	3
5	f(4,f(f(2,f(6,2)),f(6,5)))	2	2
6	f(4,f(f(2,4),f(6,5)))	1	1
7	* f(4,f(2,f(6,5)))	0	1
8	* f(4,f(2,4))	0	1
9	* f(4,2)	0	1
10	3	0	

Figure 2. Computation with loop detection activities (Example 10)

are made because the available snapshot becomes obsolete at these moments, and snapshots at moments 0, 1 and 3 are made because these moments are members of the sequence $\tau_0, \tau_1, \tau_2, \ldots$ At the moment 10 the computation is completed, thus its finiteness is seen. Of course, this finiteness has been established much easier in Example 2 without using the loop detection method. In the present example we only observed that the application of the loop detection method to the same computation did not change the final conclusion.

Example 11. Again in the situation from Example 1 let us examine the recursive computation of the term f(1,3) using the same sequence $\tau_0, \tau_1, \tau_2, \ldots$ as in the above example. Figure 3 represents what happens (the same conventions hold as above). At the moments 2 and 8 snapshots are made because the available snapshots become obsolete at these moments, and at the moments 0, 1, 3 and 7 snapshots are made because these moments are members of the sequence $\tau_0, \tau_1, \tau_2, \ldots$ At the moment 13 a cyclic loop is detected in the computation. Note that the loop would be detected earlier (namely, at the moment 8) if we had $\tau_3 \geq 8$ instead of $\tau_3 = 7$.

The most troublesome thing in the method seems to be the necessity not only to construct successive members of the examined recursive computation, but also to consider results of successive applications of $\mathcal{D}$ to the snapped term heads (in order to check whether the last of these results is an atom or not). In the above two examples the results in question have been visualized by means of some lines. A formal variant of this technique is surely possible, but fortunately there is an way to perform the mentioned check without explicitly indicating the concerned results. We shall explain now that easier way.

Moment	Term	c	c↓
0	* f(1,3)		1
1	* f(f(0,3),f(3,0))	3	1
2	* f(1,f(3,0))	0	1
3	* f(1,f(f(1,0),f(0,1)))	3	1
4	f(1,f(f(f(0,0),f(0,0)),f(0,1)))	3	3
5	f(1,f(f(1,f(0,0)),f(0,1)))	2	2
6	f(1,f(f(1,1),f(0,1)))	1	1
7	* f(1,f(f(f(0,1),f(1,0)),f(0,1)))	3	1
8	* f(1,f(f(1,f(1,0)),f(0,1)))	0	1
9	f(1,f(f(1,f(f(0,0),f(0,0))),f(0,1)))	3	3
10	f(1,f(f(1,f(1,f(0,0))),f(0,1)))	2	2
11	f(1,f(f(1,f(1,1)),f(0,1)))	1	1
12	f(1,f(f(1,f(f(0,1),f(1,0))),f(0,1)))	3	3
13	f(1,f(f(1,f(1,f(1,0))),f(0,1)))	2	

Figure 3. Computation with loop detection activities (Example 11)

For any term u let $|u|$ denote the number of occurrences of symbols from $\mathcal{F}$ in u; this number will be called *the complexity of* u. Clearly, the complexity of a term is equal to 0 iff the term is an atom.

We shall make use of the following intuitively clear statement (cf. Appendix 2 for a proof):

Fact 2. For any term u and any natural number t, if $\mathcal{D}^t(\mathcal{H}(u))$ makes sense, then $\mathcal{D}^t(u)$ also makes sense and the equality

$$|\mathcal{D}^t(u)| - |\mathcal{D}^t(\mathcal{H}(u))| = |u| - 1$$

holds.

The following statement is an immediate corollary of the particular case of Fact 2 when $t = 1$:

Lemma 3. *For any non-atomic term* u *the equality*

$$|\mathcal{D}(u)| - |u| = |\mathcal{D}(\mathcal{H}(u))| - 1$$

holds (*consequently* $|\mathcal{D}(u)| - |u| \geqq -1$).

We shall prove now a lemma which will give us the main tool for incorporating the introduced complexity notion into the loop detection method.

Lemma 4. *Let* u *be a term,* $\bar{t}$ *and* t *be natural numbers such that* $\bar{t} < t$. *Suppose the left-hand side of the equality*

$$|\mathcal{D}^{t-\bar{t}+1}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| = |\mathcal{D}^{t-\bar{t}}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| + |\mathcal{D}(\mathcal{H}(\mathcal{D}^{t}(\mathrm{u})))| - 1$$

makes sense. Then the right-hand one also makes sense and the equality is true.

Proof. The first addend in the right-hand side of the equality obviously makes sense. Let us apply Fact 2 twice with $\mathcal{D}^{\bar{t}}(\mathrm{u})$ in the role of u taking in the role of t the number $t - \bar{t}$ the first time and the number $t - \bar{t} + 1$ the second time. We conclude that $\mathcal{D}^{t}(\mathrm{u})$ and $\mathcal{D}^{t+1}(\mathrm{u})$ make sense and the following equalities hold:

$$|\mathcal{D}^{t}(\mathrm{u})| - |\mathcal{D}^{t-\bar{t}}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| = |\mathcal{D}^{\bar{t}}(\mathrm{u})| - 1,$$

$$|\mathcal{D}^{t+1}(\mathrm{u})| - |\mathcal{D}^{t-\bar{t}+1}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| = |\mathcal{D}^{\bar{t}}(\mathrm{u})| - 1.$$

These two equalities imply that

$$|\mathcal{D}^{t}(\mathrm{u})| - |\mathcal{D}^{t-\bar{t}}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| = |\mathcal{D}^{t+1}(\mathrm{u})| - |\mathcal{D}^{t-\bar{t}+1}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))|$$

and therefore

$$|\mathcal{D}^{t-\bar{t}+1}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| = |\mathcal{D}^{t-\bar{t}}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| + |\mathcal{D}^{t+1}(\mathrm{u})| - |\mathcal{D}^{t}(\mathrm{u})|.$$

On the other hand, by Lemma 3 we have also the equality

$$|\mathcal{D}^{t+1}(\mathrm{u})| - |\mathcal{D}^{t}(\mathrm{u})| = |\mathcal{D}(\mathcal{H}(\mathcal{D}^{t}(\mathrm{u}))| - 1. \blacksquare$$

Suppose now we examine the recursive computation of a given term u by the already described method. Suppose also the other things assumed in the description of the method, namely: a member $w = \mathcal{D}^{t}(\mathrm{u})$ of the computation is obtained from u by t applications of $\mathcal{D}$, and a snapshot of $\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u}))$ made after $\bar{t}$ applications of $\mathcal{D}$ to u is available, where $\bar{t} < t$, no snapshot is made at moments between $\bar{t}$ and t, and $\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u}))$ belongs to the domain of $\mathcal{D}^{t-\bar{t}}$. In such a situation let us say that $\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u}))$ *is the available snapped head at the moment* t and $\mathcal{D}^{t-\bar{t}}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))$ *is the available descendent term at the moment* t. For the case when the next member $\mathcal{D}^{t+1}(\mathrm{u})$ of the computation must be constructed, it is convenient to introduce also the notion of *descendent term inherited from the moment* t. If no change of the snapshot is made at the moment t, then, by definition, this is the available descendent term at the moment t. Otherwise the descendent term inherited from the moment t is $\mathcal{H}(\mathcal{D}^{t}(\mathrm{u}))$, i.e. the new snapped head. We adopt also the convention that the descendent term inherited from the moment 0 is $\mathcal{H}(\mathrm{u})$.

Let c be the complexity of the available descendent term at the moment t. Consider again what has to be done if w belongs to the domain of $\mathcal{D}$. If $c = 0$, then the available snapshot becomes obsolete at the moment t and must be replaced by a snapshot of $\mathcal{H}(\mathcal{D}^{t}(\mathrm{u}))$, and then the next member $\mathcal{D}^{t+1}(\mathrm{u})$ of the computation must be constructed. If $c > 0$, then one must compare $\mathcal{H}(w)$ and $\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u}))$. If they turn out to be equal, then a loop is detected in the computation at the moment t. Otherwise, one has again to construct the next member $\mathcal{D}^{t+1}(\mathrm{u})$ of the computation, possibly replacing the snapshot of $\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u}))$ by a snapshot of $\mathcal{H}(\mathcal{D}^{t}(\mathrm{u}))$ (if t is a member of the sequence $\tau_0, \tau_1, \tau_2, \ldots$). In any case when a construction of $\mathcal{D}^{t+1}(\mathrm{u})$ has to be done, it makes sense to consider the available snapped head at the moment $t+1$ and the descendent term at the moment $t+1$. If no change of the snapshot is done at the moment t, then they are $\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u}))$ and $\mathcal{D}^{t-\bar{t}+1}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))$, respectively. Otherwise, they are $\mathcal{H}(\mathcal{D}^{t}(\mathrm{u}))$ and $\mathcal{D}(\mathcal{H}(\mathcal{D}^{t}(\mathrm{u})))$. Let c' be the complexity of the available descendent term at the moment $t+1$. If no change of the snapshot is done at the moment $t+1$, then, applying Lemma 4, we conclude that

$$c' = c + |\mathcal{D}(\mathcal{H}(\mathcal{D}^{t}(\mathrm{u}))| - 1.$$

In the case when a change of the snapshot is done at the moment $t+1$, we, of course, have

$$c' = |\mathcal{D}(\mathcal{H}(\mathcal{D}^{t}(\mathrm{u}))|.$$

Both equalities may be unified as follows:

$$c' = c^{\iota} + |\mathcal{D}(\mathcal{H}(\mathcal{D}^{t}(\mathrm{u}))| - 1,$$

where c^{ι} is the complexity of the descendent term inherited from the moment t (*the inherited complexity from the moment* t, for short). The last equality obviously holds also in the case of $t=0$.

The above considerations suggest the following modification of the detection method: instead of maintaining all the time the available descendent term, one maintains only its complexity. Such a modification works thanks to the above expression for the complexity of the available descendent term at the moment $t+1$. In fact, the expression shows how to calculate the complexity in question if we know the head of the current term at the moment t and the inherited complexity from the moment t. On the other hand, the last complexity is equal to 1 if $t=0$, otherwise it depends in a very simple way on t and on the complexity of the available descendent term at the moment t (both complexities are equal except for the cases when the complexity of the available descendant term at the moment t is 0 or the moment t is a member of the sequence $\tau_0, \tau_1, \tau_2, \ldots$ — in these cases the inherited complexity from the moment t is equal to 1). Figures 2 and 3 illustrate also the application of this modification of the method. Namely, one must pay no attention to the vertical and broken lines used before, and must look at the two rightmost columns instead.

Example 12. The modified form of the loop detection method is convenient for program implementation. Figure 4 shows a Pascal program which implements the application of the method in the situation from Example 7 to recursive computations with initial member in $\mathcal{U}_1$ (the sequence $\tau_0, \tau_1, \tau_2, \ldots$ with $\tau_n = 2^n - 1$ is used). The program writes the arguments of the snapped head to the variables a and b. Complexity of descendent terms is written to the variable c. Figure 5 displays the output from the application of the same program to the recursive computation of $\mathrm{f}(1,3)$ mentioned in Example 9 (for the sake of saving space the output is displayed in two columns).

The description of the proposed loop detection method can be given in a more formal way, and this is especially advisable for a correctness and completeness proof. Until this moment our presentation used some intuitive ideas without complete definition of the terminology, and no correctness and completeness proof for the

```
{$S+}
var k,l,m,t,tau,a,b,c:word;

procedure snap(x,y:word);
begin
 a:=x;b:=y;c:=1;
 writeln('Snapshot at moment ',t,': f(',x,',',y,')')
end;

function f(x,y:word):word;
var z:word;
begin
 if c=0 then snap(x,y)
 else if (x=a) and (y=b) then
  begin writeln('Loop detected at moment ',t);halt end
 else if t=tau then snap(x,y);
 if t<65535 then begin if t=tau then tau:=2*tau+1;t:=t+1 end
 else begin writeln('Too many steps!');halt end;
 z:=x div 2;
 if odd(x) then begin c:=c+2;f:=f(f(z,y),f(y,z)) end
 else begin c:=c-1;f:=z+9 end
end;

begin
 t:=0;tau:=0;c:=0;
 readln(k,l);m:=f(k,l);writeln('f(',k,',',l,')=',m)
end.
```

Figure 4. A loop detecting program (Example 12)

method has been given. Now we shall give a description of the method by means of standard mathematical terminology. As to the proof, it will be given in the next section.

The process of application of the method (in its modified form) will be presented in the form of constructing some elements of the Cartesian product $\mathbb{N}\times\mathcal{U}\times\mathcal{U}_1\times\mathbb{N}$ (i.e. of some quadruples (t,w,v,c), where w is a term, v is a simple non-atomic term, t and c are natural numbers). An element (t,w,v,c) of $\mathbb{N}\times\mathcal{U}\times\mathcal{U}_1\times\mathbb{N}$ will be said to *detect a loop* iff the inequality $c>0$ and the equality $\mathcal{H}(w)=v$ hold (clearly, w cannot be an atom in this case). A quadruple from $\mathbb{N}\times\mathcal{U}\times\mathcal{U}_1\times\mathbb{N}$ will be called *terminal* iff this quadruple detects a loop or the second component of the quadruple is an atom. A non-terminal element (t,w,v,c) of $\mathbb{N}\times\mathcal{U}\times\mathcal{U}_1\times\mathbb{N}$ will be said to *invoke a snapshot* iff the equality $c=0$ holds or $t=\tau_n$ for some n in $\mathbb{N}$. *The snapshot information inherited from* a non-terminal element (t,w,v,c) of $\mathbb{N}\times\mathcal{U}\times\mathcal{U}_1\times\mathbb{N}$ is, by definition, the pair $(\mathcal{H}(w),1)$ if (t,w,v,c) invokes a snapshot, and it is the pair (v,c) otherwise. To each non-terminal element (t,w,v,c) of $\mathbb{N}\times\mathcal{U}\times\mathcal{U}_1\times\mathbb{N}$ we make to correspond another quadruple called its *successor*. By definition, this is the quadruple

$$(t+1,\mathcal{D}(w),v^{\iota},c^{\iota}+|\mathcal{D}(\mathcal{H}(w))|-1),$$

```
Snapshot at moment 0: f(1,3)        Snapshot at moment 48: f(9,17)
Snapshot at moment 1: f(0,3)        Snapshot at moment 63: f(11,15)
Snapshot at moment 2: f(3,0)        Snapshot at moment 127: f(4,16)
Snapshot at moment 3: f(1,0)        Snapshot at moment 128: f(16,4)
Snapshot at moment 7: f(4,9)        Snapshot at moment 129: f(11,17)
Snapshot at moment 8: f(9,4)        Snapshot at moment 151: f(14,1)
Snapshot at moment 15: f(5,2)       Snapshot at moment 152: f(16,16)
Snapshot at moment 19: f(2,5)       Snapshot at moment 153: f(14,3)
Snapshot at moment 20: f(14,10)     Snapshot at moment 154: f(17,16)
Snapshot at moment 21: f(10,16)     Snapshot at moment 167: f(14,7)
Snapshot at moment 22: f(11,5)      Snapshot at moment 168: f(15,16)
Snapshot at moment 31: f(2,5)       Snapshot at moment 255: f(11,17)
Snapshot at moment 32: f(5,2)       Snapshot at moment 277: f(14,1)
Snapshot at moment 36: f(10,14)     Snapshot at moment 278: f(16,16)
Snapshot at moment 37: f(14,14)     Snapshot at moment 279: f(14,3)
Snapshot at moment 38: f(14,16)     Snapshot at moment 280: f(17,16)
Snapshot at moment 39: f(11,16)     Snapshot at moment 293: f(14,7)
Snapshot at moment 46: f(0,1)       Snapshot at moment 294: f(15,16)
Snapshot at moment 47: f(16,9)      Loop detected at moment 420
```

Figure 5. Output from the loop detecting program (Example 12)

where (ν^{ι}, c^{ι}) is the snapshot information inherited from the quadruple (t, w, ν, c).

Suppose now a term u is given. The process of construction of consecutive members of the recursive computation of u with the addition of loop detection activities looks as follows. We start with the quadruple $(0, u, \nu_0, 0)$, where ν_0 is an arbitrarily chosen simple non-atomic term (the concrete choice of ν_0 will be unessential). If this quadruple is non-terminal, then we go to its successor, and if this successor is also non-terminal, we do the same with it, and so on until eventually a terminal quadruple is obtained. If such a quadruple is obtained in the course of the process and this quadruple detects a loop, then a cyclic loop is present in the recursive computation of u. Conversely, if a cyclic loop is present in the recursive computation of u, then some quadruple detecting a loop will be obtained in the course of the process. The first of these two statements (with a supplement concerning the other kind of terminal quadruples) is the correctness theorem for the presented method, and the second of them (with a similar supplement) is the completeness theorem for this method. Clearly, any of these two theorems needs a proof. Such proofs will be given in the next section.

3. CORRECTNESS AND COMPLETENESS THEOREMS

The correctness of the method means that in all cases of terminating application of the method the conclusion given by it corresponds to the actual state of affairs. To formulate this more precisely, we define a mapping $\mathcal{D}_\tau$ of the set of the non-terminal elements of $\mathbb{N} \times \mathcal{U} \times \mathcal{U}_1 \times \mathbb{N}$ into the whole $\mathbb{N} \times \mathcal{U} \times \mathcal{U}_1 \times \mathbb{N}$ by the convention that $\mathcal{D}_\tau$ transforms all element of its domain into their successors. We shall write simply $\mathcal{D}_\tau^t$ instead of $(\mathcal{D}_\tau)^t$. The images of a given element of $\mathbb{N} \times \mathcal{U} \times \mathcal{U}_1 \times \mathbb{N}$ under the mappings $\mathcal{D}_\tau^t$, $t = 0, 1, 2, \ldots$, will be called *accessible from* this element.

The following statement can be established by means of a trivial induction:

Lemma 5. *Let* u *be an element of* $\mathcal{U}$, *and* v_0 *be an element of* $\mathcal{U}_1$. *For any natural number* t *such that* $(0, \mathrm{u}, \mathrm{v}_0, , 0)$ *belongs to the domain of* $\mathcal{D}_\tau^t$, *the first and the second component of the quadruple* $\mathcal{D}_\tau^t(0, \mathrm{u}, \mathrm{v}_0, 0)$ *are* t *and* $\mathcal{D}^t(\mathrm{u})$, *respectively.*

The correctness theorem reads as follows.

Theorem 1. *Let* u *be an element of* $\mathcal{U}$, v_0 *be an element of* $\mathcal{U}_1$, *and a terminal element of* $\mathbb{N} \times \mathcal{U} \times \mathcal{U}_1 \times \mathbb{N}$ *be accessible from the quadruple* $(0, \mathrm{u}, \mathrm{v}_0, 0)$. *If this terminal element detects a loop, then a cyclic loop is present in the recursive computation of* u, *otherwise the computation is finite.*

Proof. If the mentioned terminal element does not detect a loop, i.e. if the second component of this element is an atom, then, by Lemma 5, $\mathcal{D}^t(\mathrm{u})$ will be an atom for some t and consequently the recursive computation of u will be finite.

Consider now the case when the terminal element in question detects a loop. We must show that a cyclic loop is present in the recursive computation of u in this case.

We shall firstly prove that any quadruple $(t, \mathrm{w}, \mathrm{v}, c)$ accessible from $(0, \mathrm{u}, \mathrm{v}_0, 0)$ satisfies the following condition: $t = 0$ or there is some natural number $\bar{t}$ such that $\bar{t} < t$, the term v belongs to the domain of $\mathcal{D}^{t-\bar{t}}$, and the equalities

$$(1) \qquad \mathrm{v} = \mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})), \quad c = |\mathcal{D}^{t-\bar{t}}(\mathrm{v})|$$

hold. The proof will be done by induction on t. If $t = 0$, then the condition is satisfied. Assume now that the statement is true for a certain natural number t, and suppose that a quadruple of the form $(t+1, \mathrm{w}', \mathrm{v}', c')$ is accessible from $(0, \mathrm{u}, \mathrm{v}_0, 0)$. This quadruple is the successor of some quadruple $(t, \mathrm{w}, \mathrm{v}, c)$, also accessible from $(0, \mathrm{u}, \mathrm{v}_0, 0)$. By the induction hypothesis and the definition of successor the above condition is satisfied for $(t, \mathrm{w}, \mathrm{v}, c)$, and we have the equalities

$$\mathrm{v}' = \mathrm{v}^\downarrow, \quad c' = c^\downarrow + |\mathcal{D}(\mathcal{H}(\mathrm{w}))| - 1,$$

where $(\mathrm{v}^\downarrow, c^\downarrow)$ is the snapshot information inherited from the quadruple $(t, \mathrm{w}, \mathrm{v}, c)$. By Lemma 5, also the equality $\mathrm{w} = \mathcal{D}^t(\mathrm{u})$ holds. Let us consider firstly the case when $(t, \mathrm{w}, \mathrm{v}, c)$ invokes a snapshot. Then $\mathrm{v}^\downarrow = \mathcal{H}(\mathrm{w})$, $c^\downarrow = 1$, hence $\mathrm{v}' = \mathcal{H}(\mathrm{w})$, $c' = |\mathcal{D}(\mathcal{H}(\mathrm{w}))|$, and therefore the equalities

$$\mathrm{v}' = \mathcal{H}(\mathcal{D}^t(\mathrm{u})), \quad c' = |\mathcal{D}(\mathrm{v}')|$$

hold. Thus the condition in question will be satisfied if we take v', c', $t+1$, t in the roles of v, c, t, $\bar{t}$, respectively. Suppose now that $(t, \mathrm{w}, \mathrm{v}, c)$ does not invoke a snapshot. Then $\mathrm{v}^\downarrow = \mathrm{v}$, $c^\downarrow = c$, hence $\mathrm{v}' = \mathrm{v}$, $c' = c + |\mathcal{D}(\mathcal{H}(\mathrm{w}))| - 1$. Since certainly $t > 0$ in this case, we may find a natural number $\bar{t}$ such that $\bar{t} < t$, the term v belongs to the domain of $\mathcal{D}^{t-\bar{t}}$, and the equalities (1) hold. Then we shall have the inequality $\bar{t} < t + 1$ and the equalities

$$\mathrm{v}' = \mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})), \quad c' = |\mathcal{D}^{t-\bar{t}}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| + |\mathcal{D}(\mathcal{H}(\mathcal{D}^t(\mathrm{u})))| - 1.$$

By Lemma 4 the second of these equalities can be represented in the form $c' = |\mathcal{D}^{t-\bar{t}+1}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))|$ (the application of the lemma is allowed, since $|\mathcal{D}^{t-\bar{t}}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| = c > 0$). Hence

$$c' = |\mathcal{D}^{t+1-\bar{t}}(\mathrm{v}')|,$$

therefore the condition in question will be satisfied with the same $\bar{t}$ if we take v', c', $t+1$ in the roles of v, c, t, respectively. This completes the induction step of the proof.

Let us take now the mentioned terminal element of $\mathbb{N} \times \mathcal{U} \times \mathcal{U}_1 \times \mathbb{N}$ as $(t, \mathrm{w}, \mathrm{v}, c)$. Since this element detects a loop, the inequality $c > 0$ and the equality $\mathcal{H}(\mathrm{w}) = \mathrm{v}$ hold. By the above considerations we may find a natural number $\bar{t}$ such that $\bar{t} < t$, the term v belongs to the domain of $\mathcal{D}^{t-\bar{t}}$, and the equalities (1) hold. The first of these equalities shows that u activates v after $\bar{t}$ steps. On the other hand, an application of Fact 1 with $\mathcal{D}^{\bar{t}}(\mathrm{u})$ and $t - \bar{t}$ in the roles of u and t, respectively, together with the same equality and the equality $\mathrm{w} = \mathcal{D}^{t}(\mathrm{u})$, shows that $\mathcal{H}(\mathcal{D}^{t-\bar{t}}(\mathrm{v})) = \mathcal{H}(\mathrm{w})$ (the application of Fact 1 is allowed, since $|\mathcal{D}^{t-\bar{t}}(\mathcal{H}(\mathcal{D}^{\bar{t}}(\mathrm{u})))| = c > 0$). Taking into account also the equality $\mathcal{H}(\mathrm{w}) = \mathrm{v}$, we see that v activates itself after $t - \bar{t}$ steps, and hence v is a self-reactivating term. Thus u activates a self-reactivating term after $\bar{t}$ steps. ∎

For the proof of the completeness theorem two lemmas more will be needed.

Lemma 6. *Let* u *be a term, and let* u *activate a self-reactivating simple non-atomic term* v *after* s *steps. Let* v *activate itself after* r *steps, where* $r > 0$, *and let* t *be an arbitrary integer satisfying the inequality* $t \geqq s$. *Then*

$$\mathcal{H}(\mathcal{D}^{t+r}(u)) = \mathcal{H}(\mathcal{D}^{t}(\mathrm{u})), \quad |\mathcal{D}^{t+r}(\mathrm{u})| \geqq |\mathcal{D}^{t}(\mathrm{u})|. \tag{2}$$

Proof. The above statements make sense, since, by Lemma 2, the recursive computation of u is infinite. By Lemma 1 u activates v also after $s + r$ steps, i.e.

$$\mathcal{H}(\mathcal{D}^{s}(\mathrm{u})) = \mathcal{H}(\mathcal{D}^{s+r}(\mathrm{u})) = \mathrm{v}.$$

Set $d = t - s$. Then $t = s + d$, $t + r = s + r + d$, hence

$$\mathcal{H}(\mathcal{D}^{t}(\mathrm{u})) = \mathcal{H}(\mathcal{D}^{d}(\mathcal{D}^{s}(\mathrm{u}))), \quad \mathcal{H}(\mathcal{D}^{t+r}(\mathrm{u})) = \mathcal{H}(\mathcal{D}^{d}(\mathcal{D}^{s+r}(\mathrm{u}))).$$

From here, making use of Fact 1 with d in the role of t and $\mathcal{D}^{s}(\mathrm{u})$, $\mathcal{D}^{s+r}(\mathrm{u})$ in the role of u, we get

$$\mathcal{H}(\mathcal{D}^{t}(\mathrm{u})) = \mathcal{H}(\mathcal{D}^{d}(\mathrm{v})), \quad \mathcal{H}(\mathcal{D}^{t+r}(\mathrm{u})) = \mathcal{H}(\mathcal{D}^{d}(\mathrm{v})),$$

thus the equality in (2) is established. For the proof of the inequality we apply Fact 2 in the same way. The result of its application looks as follows:

$$\mathcal{D}^{t}(\mathrm{u})| - |\mathcal{D}^{d}(\mathrm{v})| = |\mathcal{D}^{s}(\mathrm{u})| - 1,$$
$$|\mathcal{D}^{t+r}(\mathrm{u})| - |\mathcal{D}^{d}(\mathrm{v})| = |\mathcal{D}^{s+r}(\mathrm{u})| - 1.$$

One more application of Fact 2, this time with r in the role of t and $\mathcal{D}^{s}(\mathrm{u})$ in the role of u, shows that

$$|\mathcal{D}^{s+r}(\mathrm{u})| - |\mathcal{D}^{r}(\mathrm{v})| = |\mathcal{D}^{s}(\mathrm{u})| - 1.$$

From the last three equalities we get

$$|\mathcal{D}^{t+r}(u)| - |\mathcal{D}^{t}(u)| = |\mathcal{D}^{s+r}(u)| - |\mathcal{D}^{s}(u)| = |\mathcal{D}^{r}(v)| - 1,$$

and the validity of the inequality is clear, since $|\mathcal{D}^{r}(v)| \geqq 1$.

Lemma 7. *Let* u *be a term,* v_0 *be an element of* $\mathcal{U}_1$. *Let* $\bar{t}$ *and* $\tilde{t}$ *be natural numbers such that* $\bar{t} < \tilde{t}$, *no member of the sequence* $\tau_0, \tau_1, \tau_2, \dots$ *is strictly between* $\bar{t}$ *and* $\tilde{t}$, *the quadruple* $(0, u, v_0, 0)$ *belongs to the domain of* $\mathcal{D}_\tau^{\bar{t}}$, *the quadruple* $\mathcal{D}_\tau^{\bar{t}}(0, u, v_0, 0)$ *invokes a snapshot, and* $|\mathcal{D}^{t}(u)| \geqq |\mathcal{D}^{\bar{t}}(u)|$ *holds, whenever* $\bar{t} < t < \tilde{t}$. *Then*

$$\mathcal{D}_\tau^{\tilde{t}}(0, u, v_0, 0) = (\tilde{t}, \mathcal{D}^{\tilde{t}}(u), \mathcal{H}(\mathcal{D}^{\bar{t}}(u)), 1 + |\mathcal{D}^{\tilde{t}}(u)| - |\mathcal{D}^{\bar{t}}(u)|).$$

Proof. Induction on $\tilde{t}$ is used with the case of $\tilde{t} = \bar{t} + 1$ as induction basis. To settle that case and to do the induction step from $\tilde{t}$ to $\tilde{t} + 1$, one applies Lemma 3 with $\mathcal{D}^{\bar{t}}(u)$ and with $\mathcal{D}^{\tilde{t}}(u)$ in the role of u, respectively. ■

Now we shall formulate and prove the completeness theorem.

Theorem 2. *For any term* u *and any* v_0 *from* $\mathcal{U}_1$ *the following two statements hold:*

(i) *If a cyclic loop is present in the recursive computation of* u, *then some element of* $\mathbb{N} \times \mathcal{U} \times \mathcal{U}_1 \times \mathbb{N}$ *accessible from* $(0, u, v_0, 0)$ *detects a loop.*

(ii) *If the recursive computation of* u *is finite, then some element of* $\mathbb{N} \times \mathcal{A} \times \mathcal{U}_1 \times \mathbb{N}$ *is accessible from* $(0, u, v_0, 0)$.

Proof. The proof of (ii) is quite easy. In fact, suppose that the recursive computation of u is finite and consider a natural number s such that $\mathcal{D}^{s}(u)$ is an atom. Making use of Lemma 5 we conclude that $(0, u, v_0, 0)$ does not belong to the domain of $\mathcal{D}_\tau^{s+1}$. Therefore some terminal quadruple (with first component not greater than s) is accessible from $(0, u, v_0, 0)$. It is not possible that this terminal quadruple detects a loop, since, by Theorem 1, then a cyclic loop would be present in the recursive computation of u. Hence the terminal quadruple in question has an atomic second component (the first component of the quadruple will be s, as it is easy to see).

Consider now the case when a cyclic loop is present in the recursive computation of u (hence this computation is infinite). Let u activate a self-reactivating simple non-atomic term v after s steps, and let v activate itself after r steps, where $r > 0$. Making use of the properties of the sequence $\tau_0, \tau_1, \tau_2, \dots$ we find a natural number n such that

$$\tau_n \geqq s, \quad \tau_{n+1} - \tau_n \geqq 2r - 1.$$

We shall prove that some quadruple accessible from $(0, u, v_0, 0)$ and having first component less than $\tau_n + 2r$ detects a loop.

Let $w = \mathcal{D}^{\tau_n}(u)$. Of course, w belongs to the domain of $\mathcal{D}^{i}$ for any natural number i. Let e be the minimal one among the complexities of the terms $\mathcal{D}^{i}(w)$, $i = 0, 1, 2, \dots, r-1$. It is easily seen that a finite sequence of natural numbers $i_0 < i_1 < \dots < i_p$ can be found with the following three properties: (a) $i_0 = 0$; (b) for any natural number j less than p, i_{j+1} is the least one among the natural numbers

i satisfying the inequalities $i_j < i \leqq r-1$, $|\mathcal{D}^i(w)| < |\mathcal{D}^{i_j}(w)$; (c) $|\mathcal{D}^{i_p}(w)| = e$ (in case of $|w| = e$ we have $p = 0$ and the properties (b) and (c) become trivial). Clearly, $i_p \leqq r-1$.

We set $T = \tau_n + i_p + r$. Then $\tau_n < T \leqq \tau_n + 2r - 1 \leqq \tau_{n+1}$. If the quadruple $(0, u, v_0, 0)$ does not belong to the domain of $\mathcal{D}_\tau^T$, then some terminal quadruple with first component smaller than T will be accessible from $(0, u, v_0, 0)$. This quadruple must detect a loop, since otherwise the recursive computation of u would be finite according to Theorem 1. Thus it remains to study only the case when $(0, u, v_0, 0)$ belongs to the domain of $\mathcal{D}_\tau^T$. Then we consider firstly the quadruple $\mathcal{D}_\tau^{\tau_n}(0, u, v_0, 0)$. It is non-terminal and its first component is τ_n. Therefore this quadruple invokes a snapshot.

We shall show by induction that for $j = 0, 1, \ldots, p$ the quadruple $\mathcal{D}_\tau^{\tau_n + i_j}(0, u, v_0, 0)$ invokes a snapshot. For $j = 0$ the statement has been already established. Suppose now this statement is true for a certain natural number j less than p and apply Lemma 7 with $\tau_n + i_j$ and $\tau_n + i_{j+1}$ in the roles of $\bar{t}$ and $\tilde{t}$, respectively. We get the equality

$$\mathcal{D}_\tau^{\tau_n + i_{j+1}}(0, u, v_0, 0) = (\tau_n + i_{j+1}, \mathcal{D}^{i_{j+1}}(w), \mathcal{H}(\mathcal{D}^{i_j}(w)), 1 + |\mathcal{D}^{i_{j+1}}(w)| - |\mathcal{D}^{i_j}(w)|).$$

Thus the non-terminal quadruple $\mathcal{D}_\tau^{\tau_n + i_{j+1}}(0, u, v_0, 0)$ has a last component less than 1, hence equal to 0, and therefore the quadruple invokes a snapshot.

By applying the proved statement with $j = p$ we conclude that the quadruple $\mathcal{D}_\tau^{\tau_n + i_p}(0, u, v_0, 0)$ invokes a snapshot. Let w' be the second component of this quadruple, i.e. $w' = \mathcal{D}^{i_p}(w)$. We shall show that $|\mathcal{D}^h(w')| \geqq |w'|$ for $h = 1, 2, \ldots, r$. In fact, $\mathcal{D}^h(w') = \mathcal{D}^{i_p + h}(w)$ and $0 < i_p + h \leqq 2r - 1$ for the specified values of h. Since $|w'| = e$, the inequality in question is obviously satisfied when $i_p + h \leqq r - 1$. On the other hand, if $i_p + h > r - 1$, then $i_p + h = i + r$ for some i among $0, 1, \ldots, r-1$, hence $|\mathcal{D}^h(w')| = |\mathcal{D}^{i+r}(w)| \geqq |\mathcal{D}^i(w)|$ by the inequality in (2) with $\tau_n + i$ in the role of t. Thus $|\mathcal{D}^h(w')| \geqq |w'|$ holds again.

Now let us apply Lemma 7 with $\tau_n + i_p$ and T in the roles of $\bar{t}$ and $\tilde{t}$, respectively. We get the equality

$$\mathcal{D}_\tau^T(0, u, v_0, 0) = (T, \mathcal{D}^r(w'), \mathcal{H}(w'), 1 + |\mathcal{D}^r(w')| - |w'|).$$

We note that $\mathcal{H}(\mathcal{D}^r(w')) = \mathcal{H}(w')$, as seen from the equality in (2) with $\tau_n + i_p$ in the role of t. Taking into account also the inequality $|\mathcal{D}^r(w')| \geqq |w'|$, we conclude that $\mathcal{D}_\tau^T(0, u, v_0, 0)$ detects a loop. ∎

APPENDIX 1. ON INFINITE RECURSIVE COMPUTATIONS WITH FINITELY MANY ATOMS AND FUNCTION SYMBOLS

We add one lemma and one theorem more to the results proved in the preceding sections.

Lemma 8. *Let w be a term such that the recursive computation of w is infinite and no member of this computation has a smaller complexity than w. Then the recursive computation of the head of w is also infinite.*

Proof. We shall show by induction that $\mathcal{H}(w)$ belongs to the domain of $\mathcal{D}^t$ for any natural number t. The statement is trivial for $t = 0$. Suppose $\mathcal{D}^t(\mathcal{H}(w))$ makes sense for a certain natural number t. Then $|\mathcal{D}^t(\mathcal{H}(w))| = |\mathcal{D}^t(w)| - |w| + 1 \geqq 1$ by Fact 2 and the assumption of the lemma. Hence $\mathcal{D}^{t+1}(\mathcal{H}(w))$ also makes sense. ∎

Theorem 3. *Let* u *be a term such that the recursive computation of* u *is infinite, but there are only finitely many atoms and function symbols which occur in the members of this computation. Then a cyclic loop is present in the computation.*

Proof. Let us call a natural number t *remarkable* if for any greater natural number s the inequality $|\mathcal{D}^s(u)| \geqq |\mathcal{D}^t(u)|$ holds. It is easy to prove (by *reductio ad absurdum*) the existence of infinitely many remarkable numbers. On the other hand, the set of the terms of the form $\mathcal{H}(\mathcal{D}^t(u))$ is finite due to the made assumption. Hence, there are remarkable numbers t and t' such that $t < t'$ and $\mathcal{H}(\mathcal{D}^t(u)) = \mathcal{H}(\mathcal{D}^{t'}(u))$. By Lemma 8 the recursive computation of the head of $\mathcal{D}^t(u)$ is infinite. Then, by Fact 1, also the equality $\mathcal{H}(\mathcal{D}^{t'}(u)) = \mathcal{H}(\mathcal{D}^{t'-t}(\mathcal{H}(\mathcal{D}^t(u))))$ holds and therefore $\mathcal{H}(\mathcal{D}^t(u))$ is a self-reactivating term. ∎

Theorem 3 shows that there are many cases when the proposed loop detection method completely solves the problem whether the recursive computation of an arbitrarily chosen term is finite or infinite.

Example 13. Let $\mathcal{A}$ and $\mathcal{F}$ be the same as in Example 1, and let d be some given natural number. Consider the recursion rule

$$\mathcal{D}(f(2z, y)) = z + d, \quad \mathcal{D}(f(2z+1, y)) = f(f(z, y), f(y, z))$$

(it covers the rules from Example 1 and Example 7). Suppose some term u is given. Denote by h some natural number which is not less than $2d - 1$ and than the value of any of the atoms occurring in u. An easy induction shows that no atom occurring in some member of the recursive computation of u has a value greater than h. Thus only finitely many atoms and only one function symbol may occur in the members of the computation. By Theorem 3, if this computation is infinite, then a cyclic loop will be present in it.

APPENDIX 2. PROOFS OF FACT 1 AND FACT 2

Let us define a mapping $\mathcal{P}$ of $\mathcal{U} \setminus (\mathcal{U}_1 \cup \mathcal{A})$ into $\mathcal{U} \setminus \mathcal{A}$ as follows: whenever $u = f(u_1, \ldots, u_n)$, where f is some n-ary function symbol, $u_1, \ldots, u_n$ are terms, and at least one among these terms is non-atomic, then $\mathcal{P}(u)$ is the first non-atomic member of the sequence $u_1, \ldots, u_n$.

Clearly, $|\mathcal{P}(u)| < |u|$ for any term u from the domain of $\mathcal{P}$. For any such term the equality $\mathcal{H}(u) = \mathcal{H}(\mathcal{P}(u))$ holds. Of course, $\mathcal{H}(u) = u$ for any term in $\mathcal{U}_1$. Consequently, the head of any non-atomic term u is equal to $\mathcal{P}^n(u)$, where n is the greatest natural number i such that u belongs to the domain of $\mathcal{P}^i$.

We note that the domain and the range of $\mathcal{P}$ are contained in the domain of $\mathcal{D}$ and that the following interrelations between the two mappings are easily verifiable:

Fact A. Whenever u belongs to the domain of $\mathcal{P}$ and the inequality $|\mathcal{D}(\mathcal{P}(\mathrm{u}))| > 0$ holds, then $\mathcal{P}(\mathcal{D}(\mathrm{u}))$ makes sense and the equality $\mathcal{D}(\mathcal{P}(\mathrm{u})) = \mathcal{P}(\mathcal{D}(\mathrm{u}))$ holds.

Fact B. For any u in the domain of $\mathcal{P}$ the equality

$$|\mathcal{D}(\mathrm{u})| - |\mathcal{D}(\mathcal{P}(\mathrm{u}))| = |\mathrm{u}| - |\mathcal{P}(\mathrm{u})| \tag{3}$$

holds.

Now we shall prove some generalizations of Fact A and Fact B.

Lemma A. *Whenever n and t are natural numbers,* u *is a term such that $\mathcal{D}^t(\mathcal{P}^n(\mathrm{u}))$ makes sense and the inequality $|\mathcal{D}^t(\mathcal{P}^n(\mathrm{u}))| > 0$ holds, then $\mathcal{P}^n(\mathcal{D}^t(\mathrm{u}))$ also makes sense and the equality*

$$\mathcal{D}^t(\mathcal{P}^n(\mathrm{u})) = \mathcal{P}^n(\mathcal{D}^t(\mathrm{u})) \tag{4}$$

holds.

Proof. One firstly considers the case of $t = 1$ and proceeds by induction on n in this case. The general case can be obtained from this particular one by induction on t. ∎

Lemma B. *Whenever n and t are natural numbers and* u *is a term such that $\mathcal{D}^t(\mathcal{P}^n(\mathrm{u}))$ makes sense, then $\mathcal{D}^t(\mathrm{u})$ also makes sense and the equality*

$$|\mathcal{D}^t(\mathrm{u})| - |\mathcal{D}^t(\mathcal{P}^n(\mathrm{u}))| = |\mathrm{u}| - |\mathcal{P}^n(\mathrm{u})|$$

holds.

Proof. We firstly consider the case of $t = 1$ and proceed by the following induction on n in this case. For $n = 0$ the statement is trivial. Suppose the validity of the statement for a certain natural number n and let u be such a term that $\mathcal{D}(\mathcal{P}^{n+1}(\mathrm{u}))$ makes sense. Then an application of the induction hypothesis with $\mathcal{P}(\mathrm{u})$ in the role of u yields the equality

$$|\mathcal{D}(\mathcal{P}(\mathrm{u}))| - |\mathcal{D}(\mathcal{P}^{n+1}(\mathrm{u}))| = |\mathcal{P}(\mathrm{u})| - |\mathcal{P}^{n+1}(\mathrm{u})|.$$

This equality together with (3) implies the needed equality

$$|\mathcal{D}(\mathrm{u})| - |\mathcal{D}(\mathcal{P}^{n+1}(\mathrm{u}))| = |u| - |\mathcal{P}^{n+1}(\mathrm{u})|.$$

To obtain the general case from the particular one of $t = 1$, we proceed by induction on t. For $t = 0$ the general statement is trivial. Suppose the validity of the general statement for a certain natural number t, and let the term u and the natural number n be such that $\mathcal{D}^{t+1}(\mathcal{P}^n(\mathrm{u}))$ makes sense. Then $\mathcal{D}^t(\mathcal{P}^n(\mathrm{u}))$ also makes sense and the inequality $|\mathcal{D}^t(\mathcal{P}^n(\mathrm{u}))| > 0$ holds. Making use of Lemma A we conclude that $\mathcal{P}^n(\mathcal{D}^t(\mathrm{u}))$ also makes sense and the equality (4) holds. From here, applying consecutively the particular case of $n = 1$ with $\mathcal{D}^t(\mathrm{u})$ in the role of u and the induction hypothesis, we get

$$|\mathcal{D}^{t+1}(\mathrm{u})| - |\mathcal{D}^{t+1}(\mathcal{P}^n(\mathrm{u}))| = |\mathcal{D}^{t+1}(\mathrm{u})| - |\mathcal{D}(\mathcal{P}^n(\mathcal{D}^t(\mathrm{u})))| = |\mathcal{D}^t(\mathrm{u})| - |\mathcal{P}^n(\mathcal{D}^t(\mathrm{u}))|$$
$$= |\mathcal{D}^t(\mathrm{u})| - |\mathcal{D}^t(\mathcal{P}^n(\mathrm{u}))| = |\mathrm{u}| - |\mathcal{P}^n(\mathrm{u})|. \blacksquare$$

We shall present proofs of Fact 1 and Fact 2 using Lemma A and Lemma B, respectively.

Proof of Fact 1. Suppose that u is a term and t is a natural number such that $\mathcal{H}(\mathcal{D}^t(\mathcal{H}(\mathrm{u})))$ makes sense. Let n be a natural number such that $\mathcal{H}(\mathrm{u}) = \mathcal{P}^n(\mathrm{u})$. Then the expression $\mathcal{H}(\mathcal{D}^t(\mathcal{P}^n(\mathrm{u})))$ will make sense and therefore the premise of Lemma A will be satisfied. Hence, the conclusion of the lemma will be also satisfied. Thus $\mathcal{P}^n(\mathcal{D}^t(\mathrm{u}))$ will also make sense and the equality

$$\mathcal{D}^t(\mathcal{H}(\mathrm{u})) = \mathcal{P}^n(\mathcal{D}^t(\mathrm{u}))$$

will hold. Now it is clear that the terms $\mathcal{D}^t(\mathcal{H}(\mathrm{u}))$ and $\mathcal{D}^t(\mathrm{u})$ are non-atomic, and the term obtained by applicating $\mathcal{P}$ the maximal possible number of times will be one and the same if we start from either of these two terms.

Proof of Fact 2. Suppose that u is a term and t is a natural number such that $\mathcal{D}^t(\mathcal{H}(\mathrm{u}))$ makes sense. Let n be a natural number such that $\mathcal{H}(\mathrm{u}) = \mathcal{P}^n(\mathrm{u})$. Then the expression $\mathcal{D}^t(\mathcal{P}^n(\mathrm{u}))$ will make sense and therefore the premise of Lemma B will be satisfied. Hence, the conclusion of the lemma will be also satisfied. Thus $\mathcal{D}^t(\mathrm{u})$ also makes sense and the equality

$$|\mathcal{D}^t(\mathrm{u})| - |\mathcal{D}^t(\mathcal{H}(\mathrm{u}))| = |\mathrm{u}| - |\mathcal{H}(\mathrm{u})|$$

holds. It remains only to take into account that $|\mathcal{H}(\mathrm{u})| = 1$. ■

Added in proof: When writing the present paper, the author did not know about the paper of F. E. Fich "Lower bounds for the cycle detection problem", J. of Computer and System Sciences, **26**, 1983, 392–409. Thanks are due to Professor Donald Knuth who called the author's attention to that paper, essentially containing the results of [2] (as well as other ones).

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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AN EXAMPLE OF A FINITE NUMBER OF RECURSIVELY ENUMERABLE m-DEGREES CONTAINING AN INFINITE SEQUENCE OF RECURSIVELY ENUMERABLE SETS

ELKA BOJKOVA

Елка Божкова. ПРИМЕР КОНЕЧНОГО ЧИСЛА РЕКУРСИВНО ПЕРЕЧИСЛЯЕМЫХ m-СТЕПЕНЕЙ, СОДЕРЖАЩИХ БЕЗКОНЕЧНУЮ ПОСЛЕДОВАТЕЛЬНОСТЬ РЕКУРСИВНО ПЕРЕЧИСЛЯЕМЫХ МНОЖЕСТВ

Цель этой статьи построить для произвольного натурального числа n рекурсивно перечисляемое (р. п.) множество A, такого чтобы множества A, A^2, ..., A^n принадлежали разным m-степеням, а множества A^n, A^{n+1}, A^{n+2}, ... — одной и той же m-степени. Таким образом мы получаем пример для n разных р. п. m-степеней $d_m(A)$, $d_m(A^2)$, ..., $d_m(A^{n-1})$, $d_m(A^n)$, принадлежащих к одной btt-степени (точнее, c-степени — степень $d_c(A)$ множества A совпадающая с $d_c(A^2)$, ..., $d_c(A^n)$, ...). Для этого достаточно доказать, что множество A^{n+1} m-сводимо к множеству A^n, а множество A^n не m-сводимо к множеству A^{n-1}. Для этой цели создаем схему для построения р. п. множества A, сопоставимого последовательности р. п. множеств A_0, A_1, A_2, ..., так чтобы первое условие было выполнено независимо от выбора множеств A_0, A_1, A_2, ... После этого мы строим множества A_0, A_1, A_2, ... по этапам, пользуясь методом приоритета, чтобы выполнить второе условие.

Elka Bojkova. AN EXAMPLE OF A FINITE NUMBER OF RECURSIVELY ENUMERABLE m-DEGREES CONTAINING AN INFINITE SEQUENCE OF RECURSIVELY ENUMERABLE SETS

The aim of this paper is to construct for an arbitrary natural number n a recursively enumerable (r. e.) set A such that the sets A, A^2, ..., A^n belong to different m-degrees and the sets A^n, A^{n+1}, A^{n+2}, ... belong to the same m-degree. That will provide an example of n distinct r. e. m-degrees $d_m(A)$, $d_m(A^2)$, ..., $d_m(A^n)$ belonging to one r. e. btt-degree (more precisely, c-degree, the degree $d_c(A)$ of the set A coincidental with $d_c(A^2)$, ..., $d_c(A^n)$, ...). It suffices to prove that the set A^{n+1} is m-reducible to the set A^n, but the set A^n is not m-reducible to the set A^{n-1}. For this purpose we construct a scheme for building a r. e. set A, corresponding to a sequence of r. e. sets A_0, A_1, A_2, ..., such that the first condition holds regardless the choice of A_0, A_1, A_2, ... Further we build the sets A_0, A_1, A_2, ... by steps, using the priority argument to ensure the second condition.

In [1] Fischer proves that the reducibilities $\leqq_m$ and $\leqq_{btt}$ differ on the recursively enumerable (r.e.) non-recursive sets. He constructs an example of a r.e. btt-degree containing infinitely many distinct m-degrees. In [2] Odifreddi asks if every r.e. tt-degree contains one or infinitely many r.e. m-degrees. Downey [3] solves Odifreddi's question by constructing a r.e. tt-degree containing exactly 3 r.e. m-degrees. Rogers [4] and Odifreddi [5] summarize the obtained results about the structure of the different kinds of degrees.

In this paper we construct (for an arbitrary natural number n) a r.e. set A such that

$$A <_m A^2 <_m \cdots <_m A^n \equiv_m A^{n+1} \equiv_m A^{n+2} \equiv_m \cdots$$

That will provide an example of n distinct r.e. m-degrees ($d_m(A)$, $d_m(A^2)$, ..., $d_m(A^{n-1})$ and $d_m(A^n)$) belonging to one r.e. btt-degree (more precisely, c-degree — the degree $d_c(A)$ of the set A coincidental with $d_c(A^2)$, ..., $d_c(A^n)$, ...).

Obviously, all the recursive sets belong to one tt-degree and one m-degree, i.e. for any recursive set B the following equivalencies hold:

$$B \equiv_m B^2 \equiv_m B^3 \equiv_m \cdots$$

Now, we have to find a set A such that the sets A, A^2, ..., A^n belong to different m-degrees and the sets A^n, A^{n+1}, A^{n+2}, ... belong to the same m-degree. It suffices for this purpose to prove

$$A^{n+1} \leqq_m A^n \tag{1}$$

and

$$\neg A^n \leqq_m A^{n-1}. \tag{2}$$

The method elaborated by Ditchev in [6] is used to construct the set responding to both conditions. First, we construct a scheme for building a r.e. set A corresponding to a sequence of r.e. sets A_0, A_1, A_2, ... (answering to some conditions), such that (1) holds regardless the choice of A_0, A_1, A_2, ... After that we build the sets A_0, A_1, A_2, ... by steps, using the priority argument to ensure (2).

* * *

(1.1) **Definition.** Let A and B be sets of natural numbers:

a) We say that the set A is "m-reducible" to the set B ($A \leqq_m B$) if there exists a recursive function $f : N \to N$ such that

$$\forall x\ (x \in A \iff f(x) \in B);$$

b) We say that the sets A and B are "m-equivalent" ($A \equiv_m B$) if

$$A \leqq_m B \quad \text{and} \quad B \leqq_m A.$$

We shall denote by "$A <_m B$" the fact that $A \leqq_m B$ but $A \not\equiv_m B$.

Let for any natural number n the recursive functions J^n, J_1^n, J_2^n, ..., J_n^n be such that for any n and for any x there exists a single tuple $x_1, \ldots, x_n$ such that

$$x = J^n(x_1, \ldots, x_n) \tag{1.2}$$

and

$$\forall i_{1 \leqq i \leqq n}[J_i^n(J^n(x_1, \ldots, x_n)) = x_i]. \tag{1.3}$$

(1.4) **Definition.** Let A be an arbitrary set of natural numbers. For any natural number n we define the set A^n:

$$A^n = \{J^n(x_1, \ldots, x_n) \mid x_1 \in A \& \ldots \& x_n \in A\}.$$

Obviously, for any set A

$$A \leqq_m A^2 \leqq_m \cdots \leqq_m A^n \leqq_m A^{n+1} \leqq_m A^{n+2} \leqq_m \cdots,$$

i. e. the essential part of the problem is to find a set A such that:

(1) for every i, $1 \leqq i \leqq n-1$, $\neg A^{i+1} \leqq_m A^i$;

(2) for every i, $i \geqq n$, $A^{i+1} \leqq_m A^i$.

It is easy to prove the following two lemmas:

(1.5) **Lemma.** *Let A be an arbitrary subset of N and n be an arbitrary natural number. Then $A^{n+1} \leqq_m A^n$ iff there exist recursive functions $f_1 : N^{n+1} \to N$, $f_2 : N^{n+1} \to N$, ..., $f_n : N^{n+1} \to N$ such that*

$$x_1 \in A \& \ldots \& x_{n+1} \in A \tag{1.6}$$

$$\iff f_1(x_1, \ldots, x_{n+1}) \in A \& \ldots \& f_n(x_1, \ldots, x_{n+1}) \in A.$$

Proof. Using the fact that one unary function g reduces A^{n+1} to A^n iff for the $(n+1)$-ary functions $f_1, \ldots, f_n$, defined as follows:

$$f_i(x_1, x_2, \ldots, x_{n+1}) = J_i^n(g(J^{n+1}(x_1, x_2, \ldots, x_{n+1}))), \quad i = 1, 2, \ldots, n,$$

the condition (1.6) holds, one can easily verify that Lemma (1.5) is true.

(1.7) **Lemma.** a) *If $A^{n-1} <_m A^n$, then for any m, $2 \leqq m \leqq n-1$:*

$$A^{m-1} <_m A^m;$$

b) *If $A^n \equiv_m A^{n+1}$, then for any m, $m > n$:*

$$A^m \equiv_m A^{m+1}.$$

Hence to solve our problem, it is enough to prove

$$A^{n-1} <_m A^n \equiv_m A^{n+1}.$$

* * *

Let n be an arbitrary natural number*, $n > 1$. We shall find a set A such that

$$A <_m A^2 <_m \cdots <_m A^n \equiv_m A^{n+1} \equiv_m A^{n+2} \equiv_m \cdots$$

(2.1) **Definition.** Let L_n be a *language* with the following *alphabet*:

— $c_0, c_1, \ldots$ — infinite sequence of constants;

— $x_1, \ldots, x_n$ — n variables;

— $F_1, \ldots, F_n$ — $(n+1)$-ary functional symbols.

Terms are defined by means of the following inductive clauses:

a) c_i is a term, $i \in N$; x_i is a term, $1 \leqq i \leqq n$;

b) if $\tau^1, \ldots, \tau^{n+1}$ are terms, then $F_i(\tau^1, \ldots, \tau^{n+1})$ is a term, $1 \leqq i \leqq n$.

(2.2) **Definition.** For any term τ we define *subterms* of τ:

a) if $\tau = c_i$, $i \in N$, c_i is a subterm of τ; if $\tau = x_i$, $1 \leqq i \leqq n$, x_i is a subterm of τ;

b) if $\tau = F_i(\tau^1, \ldots, \tau^{n+1})$, $1 \leqq i \leqq n$, then the term τ and the subterms of $\tau^1, \ldots, \tau^{n+1}$ are subterms of τ.

(2.3) **Definition.** For any term τ we define *deepness* $\mathrm{dp}(\tau)$ *of* τ:

a) if $\tau = c_i$, $i \in N$ or $\tau = x_i$, $1 \leqq i \leqq n$, then $\mathrm{dp}(\tau) = 1$;

b) if $\tau = F_i(\tau^1, \ldots, \tau^{n+1})$, $1 \leqq i \leqq n$, and $\mathrm{dp}(\tau^1), \ldots, \mathrm{dp}(\tau^{n+1})$ are defined, then $\mathrm{dp}(\tau) = 1 + \max\limits_{1 \leqq j \leqq n+1} \mathrm{dp}(\tau^j)$.

(2.4) **Definition.** We will call a *partial structure* every ordered $(n+1)$-tuple $U = \langle N; \theta_1, \ldots, \theta_n \rangle$, where $\theta_1, \ldots, \theta_n$ are partial functions of $n+1$ variables.

(2.5) **Definition.** Let $U = \langle N; \theta_1, \ldots, \theta_n \rangle$ be a partial structure. We define the *value* τ_U *of the term* τ *in the partial structure* U:

a) if $\tau = c_i$, $\tau_U(t_1, \ldots, t_n) = i$ for any $t_1, \ldots, t_n$, $i \in N$;

if $\tau = x_i$, $\tau_U(t_1, \ldots, t_n) = t_i$ for any $t_1, \ldots, t_n$, $1 \leqq i \leqq n$;

b) if $\tau = F_i(\tau^1, \ldots, \tau^{n+1})$, $1 \leqq i \leqq n$, $t_1 \in N, \ldots, t_n \in N$ and $\tau_U^1, \ldots, \tau_U^{n+1}$ are defined, then

$$\tau_U(t_1, \ldots, t_n) \cong \theta_i(\tau_U^1(t_1, \ldots, t_n), \ldots, \tau_U^{n+1}(t_1, \ldots, t_n)).$$

(2.6) **Definition.** We define the *cod* $\mathrm{cd}(\tau)$ of the term τ:

a) if $\tau = c_i$, $i \in N$, $\mathrm{cd}(\tau) = J^2(0, i)$; if $\tau = x_i$, $1 \leqq i \leqq n$, $\mathrm{cd}(\tau) = J^2(1, i)$;

b) if $\tau = F_i(\tau^1, \ldots, \tau^{n+1})$, $1 \leqq i \leqq n$, $\mathrm{cd}(\tau) = J^{n+2}(i+1, \mathrm{cd}(\tau^1), \ldots, \mathrm{cd}(\tau^{n+1}))$.

That coding allows us to verify for any natural number whether it is a cod of any term and if so, to find this term.

Theorem 1. *Let* $M = \{2k \mid k \in N\}$. *There exists a partial structure* $U = \langle N, \theta_1, \ldots, \theta_n \rangle$ *such that the functions* $\theta_1, \ldots, \theta_n$ *are recursive and*:

* n is fixed till the end of this paper.

1) $\forall t_1 \in N \ldots \forall t_{n+1} \in N$,

$$(*) \qquad (t_1 \in M \,\&\, \ldots \,\&\, t_{n+1} \in M \iff \theta_1(t_1, \ldots, t_{n+1}) \in M \,\&\, \ldots \,\&\, \theta_n(t_1, \ldots, t_{n+1}) \in M);$$

2) $\forall (\tau^1, \ldots, \tau^{n-1} - \text{terms})\ \exists t_1 \in N, \ldots, t_n \in N$,

$$(**) \qquad \neg\,(t_1 \in M \,\&\, \ldots \,\&\, t_n \in M \iff \tau^1_U(t_1, \ldots, t_n) \in M \,\&\, \ldots \,\&\, \tau^{n-1}_U(t_1, \ldots, t_n) \in M)).$$

Proof. We shall say that for the partial functions $\omega_1, \ldots, \omega_n$ the condition $(\tilde{*})$ holds if $\operatorname{dom}\omega_1 = \cdots = \operatorname{dom}\omega_n$ and the condition $(*)$ holds for $\operatorname{dom}\omega_1$.

We build the functions $\theta_1, \ldots, \theta_n$ by steps: at any step $s = 0, 1, 2, \ldots$ we build the functions $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ as finite extensions of $\theta_1^{(s-1)}, \ldots, \theta_n^{(s-1)}$ such that for $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ the condition $(\tilde{*})$ holds. For $s = 0$ we accept $\theta_1^{(-1)} = \cdots = \theta_n^{(-1)} = \varnothing$. (The domain of $\varnothing$ is empty.)

Finally, $\theta_i = \bigcup_{s \in N} \theta_i^{(s)}, \quad i = 1, \ldots, n.$

I. By the even steps $s = 2p$ we shall ensure that the functions $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ are defined in $(J_1^{n+1}(p), \ldots, J_{n+1}^{n+1}(p))$ and therefore the functions $\theta_1, \ldots, \theta_n$ will be total.

1) If $(J_1^{n+1}(p), \ldots, J_{n+1}^{n+1}(p)) \notin \operatorname{dom}\theta_1^{(s-1)}$, then for all l, $1 \leqq l \leqq n$,

$$\theta_l^{(s)}(J_1^{n+1}(p), \ldots, J_{n+1}^{n+1}(p)) = \begin{cases} 2, & \text{if } J_1^{n+1}(p) \in M \,\&\ldots\&\, J_{n+1}^{n+1}(p) \in M, \\ 3, & \text{otherwise,} \end{cases}$$

and

$$\theta_l^{(s)}(y_1, \ldots, y_{n+1}) \cong \theta_l^{(s-1)}(y_1, \ldots, y_{n+1}), \quad \text{if } J^{n+1}(y_1, \ldots, y_{n+1}) \neq p.$$

2) If $(J_1^{n+1}(p), \ldots, J_{n+1}^{n+1}(p)) \in \operatorname{dom}\theta_1^{(s-1)}$, then $\forall l_{1 \leqq l \leqq n} \left[\theta_l^{(s)} = \theta_l^{(s-1)}\right]$.

II. By the odd steps $s = 2p + 1$, $p \in N$, if there are not terms $\tau^1, \ldots, \tau^{n-1}$ such that $p = J^{n-1}(\operatorname{cd}(\tau^1) \ldots, \operatorname{cd}(\tau^{n-1}))$, we take $\forall l_{1 \leqq l \leqq n} \left[\theta_l^{(s)} = \theta_l^{(s-1)}\right]$. If such terms exist, we shall find $t_1, \ldots, t_n$ such that the condition $(**)$ holds. In addition we obtain that the condition $(**)$ holds for any terms $\tau^1, \ldots, \tau^{n-1}$.

In the second case we first verify whether some of the terms $\tau^1, \ldots, \tau^{n-1}$ is a constant with a value — an odd number. If so, for any n-tuple of even numbers $t_1, \ldots, t_n$ the condition $(**)$ holds and we do nothing; i. e. we take $\theta_l^{(s)} = \theta_l^{(s-1)}$ for all l, $1 \leqq l \leqq n$. If all the terms $\tau^1, \ldots, \tau^{n-1}$, which are constants, have the values — even numbers, we shall ensure $(**)$ by finding $t_1, \ldots, t_n$ and building $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ such that:

— $\tau^1_{U^{(s)}}(t_1, \ldots, t_n) \in M \,\&\ldots\&\, \tau^{n-1}_{U^{(s)}}(t_1, \ldots, t_n) \in M$;

— $t_1 \notin M \vee \ldots \vee t_n \notin M$.

Every term is a constant, a variable, or its deepness is bigger than 1.

(1) For all the terms $\tau^1, \ldots, \tau^{n-1}$, which are constants, we have that their values belong to M.

(2) For these ones, which are variables, we shall choose the corresponding numbers $t_1, \ldots, t_n$ (which are values of such terms) to be even and these terms also will satisfy the condition. Because there is at least one number among $t_1, \ldots, t_n$ which is not a value of any term, let this number be odd and by this way $t_1, \ldots, t_n$ also will satisfy the condition. In both cases we shall take $t_1, \ldots, t_n$ big enough not to belong to the domains of $\theta_1^{(s-1)}, \ldots, \theta_n^{(s-1)}$ and thus to the values of the terms with deepness bigger than 1.

(3) For the terms with deepness bigger than 1 we need an auxiliary lemma — Lemma 1. It will be applied for the partial structure

$$U^{(s-1)} = \langle N, \theta_1^{(s-1)}, \ldots, \theta_n^{(s-1)} \rangle \text{ for } \tau^1, \ldots, \tau^{n-1}, \ \ t_1, \ldots, t_n$$

(obtained from (2) or, if there are no terms with deepness 1, we take $t_1, \ldots, t_n$ big enough and such that $t_1 \notin M \vee \ldots \vee t_n \notin M$) and for one term τ among the terms $\tau^1, \ldots, \tau^{n-1}$, $\mathrm{dp}(\tau) > 1$. Using this lemma we shall obtain the functions $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ such that $\theta_i^{(s-1)} \leqq \theta_i^{(s)}$, $1 \leqq i \leqq n$, the condition $(\tilde{*})$ holds and the value of the term τ for $(t_1, \ldots, t_n)$ is an even number. If at the same time $(t_1, \ldots, t_n)$ enters in the domain of the value of some other term among $\tau^1, \ldots, \tau^{n-1}$ (except τ), we have to ensure that the value of the other term in $(t_1, \ldots, t_n)$ is also an even number.

Lemma 1. *Let the terms $\tau^1, \ldots, \tau^{n-1}$ be given. For any partial structure $U = \langle N; \theta_1, \ldots, \theta_n \rangle$ with finite functions satisfying $(\tilde{*})$, for any term τ with $\mathrm{dp}(\tau) \geqq 2$, for any natural number m, and for any n-tuple of natural numbers $(t_1, \ldots, t_n)$ such that*

$$(t_1, \ldots, t_n) \notin \mathrm{dom}\ \tau_U \quad \textit{and} \quad t_1 \notin M \vee \ldots \vee t_{n+1} \notin M$$

there exists a partial structure $U' = \langle N, \theta_1', \ldots, \theta_n' \rangle$ with finite recursive functions satisfying $(\tilde{*})$, such that:

(i) $\theta_i' \geqq \theta_i$ *for every* i, $1 \leqq i \leqq n$;

(ii) $(t_1, \ldots, t_n) \in \mathrm{dom}\ \tau_{U'}$ *and* $\tau_{U'}, (t_1, \ldots, t_n) > z$,

where $z = \max(\{m, \max(I_1(\mathrm{dom}\ \theta_1')), \ldots, \max(I_{n+1}(\mathrm{dom}\ \theta_1'))\})$;

(iii) *if* $(t_1, \ldots, t_n) \in (\mathrm{dom}\ \tau^j_{U'} \setminus \mathrm{dom}\ \tau^j_U)$, *then* $\tau^j_{U'}(t_1, \ldots, t_n) \in M$, $j = 1, \ldots, n-1$.

We shall apply this lemma successively for all the terms among $\tau^1, \ldots, \tau^{n-1}$ with deepness bigger than 1 and we shall obtain

$$\tau^1_{U^{(s)}}(t_1, \ldots, t_n) \in M \ \& \ \ldots \ \& \ \tau^{n-1}_{U^{(s)}}(t_1, \ldots, t_n) \in M;$$

$$t_1 \notin M \vee \ldots \vee t_n \notin M.$$

The idea to prove Lemma 1 is to find for the term $\tau = F_i(\sigma^1, \ldots, \sigma^{n+1})$ a partial structure $V = \langle N, \theta_1'', \ldots, \theta_{n+1}'' \rangle$ such that $\forall j$, $1 \leqq j \leqq n+1$, $(t_1, \ldots, t_n) \in \mathrm{dom}\ \sigma^j_V$, $z_j = \sigma^j_V(t_1, \ldots, t_n)$ and $(z_1, \ldots, z_{n+1}) \notin \mathrm{dom}\ \theta''_{n+1}$. In this case we shall build the functions $\theta_1', \ldots, \theta_{n+1}'$ such that $\theta_1' \geqq \theta_1'', \ldots, \theta_{n+1}' \geqq \theta''_{n+1}$, $\mathrm{dom}\ \theta'_{n+1} =$

dom $\theta''_{n+1} \cup \{(z_1, \ldots, z_{n+1})\}$, and the values of the functions in $(z_1, \ldots, z_{n+1})$ are defined as follows (z is big enough):

1) If $z_1 \in M \,\&\ldots\&\, z_{n+1} \in M$, the values of all the functions are $2z$;

2) If $z_1 \notin M \vee\ldots\vee z_{n+1} \notin M$, and for some term τ^j among $\tau^1, \ldots, \tau^{n-1}$ there exist a number l, $1 \leqq l \leqq n$, and terms $\varepsilon^1, \ldots, \varepsilon^{n+1}$ such that $\tau^j = F_l(\varepsilon^1, \ldots, \varepsilon^{n+1})$, then the value of the function θ'_l is also $2z$;

3) In the other cases the values of the functions are $2z + 1$.

So we ensure that ($\tilde{*}$) is true for the new partial structure (if $z_1 \notin M \vee \ldots \vee z_{n+1} \notin M$, because the terms $\tau^1, \ldots, \tau^{n-1}$ are $n-1$ and the symbols $F_1, \ldots, F_n$ are n, at least one of the functions is in the case 3) and its value is $2z+1$). Except that we ensure $\tau_{U'}(t_1, \ldots, t_n)$ to be big enough and for any of the terms τ^j, $j = 1, \ldots, n-1$, such that $(t_1, \ldots, t_n) \in (\operatorname{dom} \tau^j_{U'} \setminus \operatorname{dom} \tau^j_U)$, $\tau^j_{U'}(t_1, \ldots, t_n) \in M$ is true. The last follows from 2), but not evidently. So we need an auxiliary lemma — Lemma (2.7).

With this lemma we have to prove that for some l, $1 \leqq l \leqq n$, $\tau^j_{U'}(t_1, \ldots, t_n) = \theta_1(z_1, \ldots, z_{n+1})$ and θ_1 is in case 2), and therefore $\tau^j_{U'}(t_1, \ldots, t_n) = 2z \in M$. In the general case the lemma can not be proved for the term τ^j, it is only true that there exists a subterm of τ^j for which the condition holds, but in this concrete application of Lemma (2.7) in Lemma 1 we can prove that this subterm may be only the term τ^j.

(2.7) **Lemma.** *For any two partial structures $U = \langle N, \theta_1, \ldots, \theta_n\rangle$ and $U' = \langle N, \theta'_1, \ldots, \theta'_n\rangle$, for any term σ and for any natural numbers $y_1, \ldots, y_{n+1}$; $t_1, \ldots, t_n$, such that:*

a) dom $\theta_1 \equiv \cdots \equiv$ dom θ_n *and that is a finite set;*

b) dom $\theta'_i =$ dom $\theta_i \cup \{(y_1, \ldots, y_{n+1})\}$, $i = 1, \ldots, n$, *and* $(y_1, \ldots, y_{n+1}) \notin$ dom θ_1;

c) $\theta_1 \leqq \theta'_1, \ldots, \theta_n \leqq \theta'_n$;

d) $(t_1, \ldots, t_n) \in (\operatorname{dom} \sigma_{U'} \setminus \operatorname{dom} \sigma_U)$,

there exist a subterm σ' of σ, $i \in \{1, \ldots, n\}$, and terms $\varepsilon^1, \ldots, \varepsilon^{n+1}$ such that:

$$\text{(i)}\quad \sigma' = F_i(\varepsilon^1, \ldots, \varepsilon^{n+1});$$

$$\text{(ii)}\quad \varepsilon^1_{U'}(t_1, \ldots, t_n) = y_1, \ldots, \varepsilon^{n+1}_{U'}(t_1, \ldots, t_n) = y_{n+1}.$$

Proof. By the definitions (2.3) and (2.5) we have

(2.8) $\forall \tau$-term, $\forall U$-partial structure $[\operatorname{dp}(\tau) = 1 \Rightarrow \tau_U$ is a total function$]$.

From condition d) we have $(t_1, \ldots, t_n) \notin \operatorname{dom} \sigma_U$. Therefore $\operatorname{dp}(\tau) \geqq 2$ and from (2.3) it follows that there exist k, $1 \leqq k \leqq n$, and terms $\sigma^1, \ldots, \sigma^{n+1}$ such that $F_k(\sigma^1, \ldots, \sigma^{n+1}) = \sigma$.

We have two cases:

C a s e I. $\forall j_{1 \leqq j \leqq n+1}[(t_1, \ldots, t_n) \in \operatorname{dom} \sigma^j_U]$. Let $z_j = \sigma^j_U(t_1, \ldots, t_n)$, $j = 1, \ldots, n+1$. From d) it follows that $(z_1, \ldots, z_{n+1}) \notin$ dom θ_k and $(z_1, \ldots, z_{n+1}) \in$ dom θ'_k and from b) we obtain $z_1 = y_1, \ldots, z_{n+1} = y_{n+1}$. From c) it follows that

$$\forall j_{1 \leqq j \leqq n+1}[\sigma^j_{U'}(t_1, \ldots, t_n) = \sigma^j_U\,(t_1, \ldots, t_n)].$$

We obtained $\sigma^1_{U'}(t_1, \ldots, t_n) = y_1, \ldots, \sigma^{n+1}_{U'}(t_1, \ldots, t_n) = y_{n+1}$, i.e. the conditions (i) and (ii) hold for $\sigma' = \sigma$ and $\varepsilon^j = \sigma^j$, $j = 1, \ldots, n+1$.

C a s e II. We apply induction on $\mathrm{dp}(\sigma)$:

1) $\mathrm{dp}(\sigma) = 2$. Then $\mathrm{dp}(\sigma^1) = \cdots = \mathrm{dp}(\sigma^{n+1}) = 1$ and the lemma follows from (2.8) and Case I;

2) Let it be true for the terms with deepness less than $\mathrm{dp}(\sigma)$;

3) We shall prove it for $\sigma = F_k(\sigma^1, \ldots, \sigma^{n+1})$. The terms $\sigma^1, \ldots, \sigma^{n+1}$ have smaller deepness. We have $(t_1, \ldots, t_n) \notin \mathrm{dom}\ \sigma_U$. Two cases are possible:

Case 1. $\exists j_{1 \leqq j \leqq n+1}[(t_1, \ldots, t_n) \notin \mathrm{dom}\ \sigma^j_U]$. But $(t_1, \ldots, t_n) \in \mathrm{dom}\ \sigma_{U'} \Rightarrow (t_1, \ldots, t_n) \in \mathrm{dom}\ \sigma^j_{U'}$, and by the induction hypothesis for σ^j we obtain that there exists a σ'-subterm of σ^j, i, $\varepsilon^1, \ldots, \varepsilon^{n+1}$ such that $\sigma' = F_i(\varepsilon^1, \ldots, \varepsilon^{n+1})$ and $\forall m_{1 \leqq m \leqq n+1}[\varepsilon^m_{U'}(t_1, \ldots, t_n) = y_m]$. But σ' is a subterm of σ^j, so σ' is a subterm of σ also and in this case the lemma is proved.

Case 2. $\forall j_{1 \leqq j \leqq n+1}[(t_1, \ldots, t_n) \in \mathrm{dom}\ \sigma^j_U]$. Then Lemma (2.7) follows from Case I.

* * *

(3.1) **Definition.** The total functions $\varphi_1, \ldots, \varphi_n$ of $n+1$ variables are defined as follows:

$$\varphi_i(t_1, \ldots, t_{n+1}) = J^{n+2}(i, t_1, \ldots, t_{n+1}),\ i = 1, \ldots, n;\ t_1 \in N, \ldots, t_{n+1} \in N.$$

(3.2) **Definition.** Let $N_0 = N \setminus (\bigcup_{i=1}^{n} \text{Range}(\varphi_i))$.

(3.3) **Definition.** Let $\{A_i\}_{i \in N}$ be a sequence of disjoint subsets of N_0. The sequence $\{[A_i]\}_{i \in N}$ of disjoint subsets of N and the set A are defined as follows:

a) if $p \in A_i$, then $p \in [A_i]$, $\quad p \in N_0$, $i \in N$;

b) if $p_1 \in [A_{i_1}] \& \ldots \& p_{n+1} \in [A_{i_{n+1}}] \& \theta_k(i_1, \ldots, i_{n+1}) = m$, $1 \leqq k \leqq n$, then $\varphi_k(p_1, \ldots, p_{n+1}) \in [A_m]$;

c) $A = \bigcup_{i \in M} [A_i]$. (We remind that $M = \{2k \mid k \in N\}$.)

Note. If the set $\{(p, i) \mid p \in A_i\}$ is r.e., then A is also r.e. In this case we say that we have a r.e. sequence of r.e. sets.

We can prove the following lemma:

(3.4) **Lemma.** *Let $\{A_i\}_{i \in N}$ be a sequence of disjoint subsets of N_0 and the set A be obtained by Definition* (3.3). *Then for any $n+1$ natural numbers $p_1, \ldots, p_{n+1}$ it is true that*

$$p_1 \in A \& \ldots \& p_{n+1} \in A \iff \varphi_1(p_1, \ldots, p_{n+1}) \in A \& \ldots \& \varphi_n(p_1, \ldots, p_{n+1}) \in A.$$

(3.5) **Corollary.** *Let $\{A_i\}_{i \in N}$ be a sequence of disjoint subsets of N_0 and the set A be obtained by Definition* (3.3). *Then $A^{n+1} \leqq_m A^n$.*

(3.6) **Definition.** We define a correspondence between two terms τ and σ as follows:

a) if τ has not a subterm that is a constant, then τ corresponds with $\sigma = \tau$;

b) if $c_{i_1}, \dots, c_{i_k}$ are all the subterms of τ which are constants, $i_1 \in [A_{r_1}], \dots, i_k \in [A_{r_k}]$, then τ corresponds with the term σ, where σ is obtained from τ by replacing c_{i_j} with c_{r_j} for any $j \in \{1, \dots, k\}$.

We need this correspondence to have the following

(3.7) **Lemma.** *If $\{A_i\}_{i \in N}$ is a sequence of disjoint subsets of N_0, τ is a term, $p_1, \dots, p_n$ are arbitrary natural numbers, and:*

a) *for any constant c_j which is a subterm of τ we have a number $i \in N$ such that $j \in [A_i]$;*

b) *τ corresponds with σ by Definition* (3.6);

and

c) $p_1 \in [A_{i_1}] \,\&\, \dots \,\&\, p_n \in [A_{i_n}] \,\&\, \sigma_U(i_1, \dots, i_n) = m$,

then $\tau_V(p_1, \dots, p_n) \in [A_m]$.

(3.8) **Lemma.** *Let $V = \langle N, \varphi_1, \dots, \varphi_n \rangle$. For any natural number x there is an effective way to find a term τ that have not subterms which are variables (for $x \notin N_0$, $\tau \neq$ constant c_x) such that $x = \tau_V(x_1, \dots, x_n)$ and the values of all the constants — subterms of τ, belong to N_0.*

We prove this lemma by defining the function $||z||$ for any $z \in N$:

$$(3.9) \qquad \begin{aligned} &1) \text{ if } z \in N_0, \quad \text{then } \ ||z|| = 0; \\ &2) \text{ if } z = \varphi_1(z_1, \dots, z_{n+1}), \quad \text{then } \ ||z|| = 1 + \max_{1 \leqq j \leqq n+1} ||z_j||, \ 1 \leqq i \leqq n, \end{aligned}$$

and applying induction on $||x||$.

We shall build a r.e. sequence of disjoint r.e. subsets $A_0, A_1, A_2, \dots$ of N_0 such that for the set A obtained by Definition (3.3) it holds

$$(3.10) \qquad A^n \not\equiv_m A^{n-1}.$$

Then it follows from Lemma (1.7) and Corollary (3.5) that A is the set we need.

We shall build the sets $\{A_i\}_{i \in N}$ by steps — at any step s we build $\{A_i^{(s)}\}_{i \in N}$, ensuring that $(A^{(s)})^n$ is not m-reducible to $(A^{(s)})^{n-1}$ by the e-th recursive function, $e = J_1^2(s)$. ($A^{(s)}$ is obtained from $\{A_i^{(s)}\}_{i \in N}$ by Definition (3.3).)

At the end we take $A_i = \bigcup_{s=0}^{\infty} A_i^{(s)}$, $i \in N$. For this purpose at any step s we shall find numbers $x_1, \dots, x_n$ such that if φ_e is the e-th partial recursive function, $e = J_1^2(s)$, $J^{(n)}(x_1, \dots, x_n) \in \operatorname{dom} \varphi_e$ and $\varphi_e(J^{(n)}(x_1, \dots, x_n)) = J^{(n-1)}(z_1, \dots, z_{n-1})$, one of both conditions holds:

$$\text{(i)} \quad x_1 \in A \,\&\, \dots \,\&\, x_n \in A \,\&\, \exists i_{1 \leqq i \leqq n-1}(z_i \notin A);$$

$$\text{(ii)} \quad (x_1 \notin A \vee \dots \vee x_n \notin A) \,\&\, z_1 \in A \,\&\, \dots \,\&\, z_{n-1} \in A.$$

If for this purpose we put the numbers $x_1, \dots, x_n$ in some sets $A_{i_1}, \dots, A_{i_n}$, we create a positive e-requirement $\{x_1, \dots, x_n\}$, and if some numbers $y_1, \dots, y_k$

must not belong to some set, we create a negative e-requirement $\{y_1, \ldots, y_k\}$. We shall use the priority argument: if at the step s we need one number x to belong to some set and at a step t — not to belong, the smaller between $J_1^2(s)$ and $J_1^2(t)$ has a priority. So, when we choose $x_1, \ldots, x_n$ at the step s, they must not belong to any negative requirements created at some steps $t < s$ such that $J_1^2(t) < J_1^2(s)$, but they may belong to negative requirements created at steps $r < s$ such that $J_1^2(r) > J_1^2(s)$. In the second case the $J_1^2(r)$-requirement is injured and we need at some later step r', $J_1^2(r') = J_1^2(r)$, to create a new $J_1^2(r)$-requirement.

If one $J_1^2(s)$-requirement is not injured at a step $r > s$, it is called active at this step. If it is active at every step $r > s$, it is called constant. At the end we shall prove that for any e the condition ((i) $\vee$ (ii)) is injured only finite times.

Now we shall describe the construction of the sets $\{A_i\}_{i \in N}$.

(3.11) Step $s = 0$. Let $N_0 = N_1 \cup N_2$, where N_1 and N_2 are infinite disjoint recursive sets and $N_2 = \{a_0 < a_1 < \cdots\}$. Let r' be a monotonically increasing function such that $\mathrm{Ran}\,(r') = N_1$ and $r(x) = r'(n.2^x + x)$. Let

$$\varphi_{e,s}(x) \cong \begin{cases} \varphi_e(x), \text{ if } x \in \operatorname{dom}\varphi_e \text{ and } \varphi_e(x) \text{ is countable for less than } s \text{ steps}, \\ \text{not defined, otherwise.} \end{cases}$$

Let $A_i^{(0)} = \{a_i\}$, $i \in N$. So, all the sets are not empty.

Step $s > 0$. $e = J_1^2(s)$. First we verify whether there exists an active e-requirement. If such a requirement exists, then we do nothing, i.e. we take $A_i^{(s)} = A_i^{(s-1)}$, $i \in N$. Otherwise we verify whether there exist $x_1 \in N_1, \ldots, x_n \in N_1$, $x_1 > r(e) \,\&\ldots\&\, x_n > r(e)$, $J^n(x_1, \ldots, x_n) \in \operatorname{dom}\varphi_{e,s}$, belonging neither to $\bigcup_{i \in N} A_i^{(s-1)}$ nor to any active negative requirement, created at a step $t < s$ such that $J_1^2(t) < J_1^2(s)$. If such numbers do not exist, we do nothing. If there are such numbers $x_1, \ldots, x_n$, we take the smallest — $x_1^{(e)}, \ldots, x_n^{(e)}$. From the choice of J^n there exist $z_1, \ldots, z_{n-1}$ such that

$$\varphi_e(J^n(x_1^{(e)}, \ldots, x_n^{(e)})) = J^{n-1}(z_1, \ldots, z_{n-1}).$$

It follows from Lemma (3.8), applied for $z_1, \ldots, z_{n-1}$, that there exist terms $\psi^1, \ldots, \psi^{n-1}$ such that

$$z_i = \psi_V^i(x_1, \ldots, x_n), \quad i = 1, \ldots, n-1. \tag{3.12}$$

We consider these numbers among $z_1, \ldots, z_{n-1}$ for which the constants-subterms of the corresponding terms $\psi^1, \ldots, \psi^{n-1}$ already belong to some sets $A_i^{(s-1)}$, $i \in N$. Let that be $z_1, \ldots, z_q$.

Case I. $\exists i_{1 \leq i \leq q}\, \exists k \in N\,(z_i \in [A_{2k+1}^{(s-1)}])$, i.e. $z_i \notin A$ and in this case we satisfy (i). Let $A_{2j}^{(s)} = A_{2j}^{(s-1)} \cup \{x_j^{(e)}\}$, $j = 1, \ldots, n$, and $A_l^{(s)} = A_l^{(s-1)}$ for $l \notin \{2, 4, 6, \ldots, 2n\}$. We create a positive e-requirement $\{x_1^{(e)}, \ldots, x_n^{(e)}\}$, which is also constant.

Case II. $z_1 \in A^{(s)} \,\&\ldots\&\, z_q \in A^{(s)}$.

II.1. $q = n - 1$. Then we satisfy (ii).

Let $A_j^{(s)} = A_j^{(s-1)} \cup \{x_j^{(e)}\}$, $j = 1, \ldots, n$, and $A_l^{(s)} = A_l^{(s-1)}$ for $l \notin \{1, \ldots, n\}$. We create a positive e-requirement $\{x_1^{(e)}, \ldots, x_n^{(e)}\}$, which is also constant.

II.2. $q < n - 1$.

II.2.1. $\exists p_{1 \leqq p \leqq n-1}$ (ψ_p has a subterm-constant with a value y_j^p,

$$y_j^p \notin (\bigcup_{i \in N} A_i^{(s-1)} \cup \{x_1^{(e)}, \ldots, x_n^{(e)}\})).$$

For simplicity let $p = n - 1$. We satisfy (i) — we put $x_1^{(e)}, \ldots, x_n^{(e)}$ into A and create a negative requirement such that z_{n-1} stays always out of A.

We take $A_{2j}^{(s)} = A_{2j}^{(s-1)} \cup \{x_j^{(e)}\}$, $j = 1, \ldots, n$, and $A_l^{(s)} = A_l^{(s-1)}$ for $l \notin \{2, 4, 6, \ldots, 2n\}$. We create a positive e-requirement $\{x_1^{(e)}, \ldots, x_n^{(e)}\}$ and a negative e-requirement $\{y_j^{n-1}\}$;

II.2.2. $\forall p_{1 \leqq p \leqq n-1}$ (ψ_p has a subterm-constant with value y_j^p, $y_j^p \notin \bigcup_{i \in N} A_i^{(s-1)}$ $\Rightarrow y_j^p \in \{x_1^{(e)}, \ldots, x_n^{(e)}\}$). We find the terms $\psi^1, \ldots, \psi^{n-1}$ corresponding to the terms $\tau^1, \ldots, \tau^{n-1}$ according to Definition (3.6). From Theorem 1 for the terms $\tau^1, \ldots, \tau^{n-1}$ there exist natural numbers $i_1, \ldots, i_n$ such that

$$i_1 \in M \,\&\, \ldots \,\&\, i_n \in M \iff \tau_U^1(i_1, \ldots, i_n) \in M \,\&\, \ldots \,\&\, \tau_U^{n-1}(i_1, \ldots, i_n) \in M$$

does not hold, i.e. one of the following holds:

(3.13) $i_1 \in M \,\&\, \ldots \,\&\, i_n \in M \,\&\, \left(\tau_U^1(i_1, \ldots, i_n) \notin M \vee \ldots \vee \tau_U^{n-1}(i_1, \ldots, i_n) \notin M\right)$,

(3.14) $\left(i_1 \notin M \vee \ldots \vee i_n \notin M\right) \,\&\, \tau_U^1(i_1, \ldots, i_n) \in M \,\&\, \ldots \,\&\, \tau_U^{n-1}(i_1, \ldots, i_n) \in M$.

Let $m_j = \tau_U^j(i_1, \ldots, i_n)$, $1 \leqq j \leqq n - 1$. We take: $A_j^{(s)} = A_j^{(s-1)} \cup \{x_j^{(e)}\}$, $j \in \{i_1, \ldots, i_n\}$, and $A_l^{(s)} = A_l^{(s-1)}$ for $l \notin \{i_1, \ldots, i_n\}$.

If (3.13) holds, then by Definition (3.3c) we have

$$x_1^{(e)} \in A^{(s)} \,\&\, \ldots \,\&\, x_n^{(e)} \in A^{(s)}$$

and from Lemma (3.7):

$$z_i = \psi_V^i(x_1^{(e)}, \ldots, x_n^{(e)}) \in [A_{m_j}^{(s)}],$$

and therefore $z_1 \notin A^{(s)} \vee \ldots \vee z_{n-1} \notin A$, so in this case (i) is true.

If (3.14) holds, then $m_1 \in M \,\&\, \ldots \,\&\, m_{n-1} \in M$ and therefore $z_1 \in A^{(s)} \,\&\, \ldots \,\&\, z_{n-1} \in A^{(s)}$. We have also $i_1 \notin M \vee \ldots \vee i_n \notin M$ and then $x_1^{(e)} \notin A^{(s)} \vee \ldots \vee x_n^{(e)} \notin A^{(s)}$, i.e. (ii) is true.

We create a positive e-requirement $\{x_1^{(e)}, \ldots, x_n^{(e)}\}$, which is also constant.

At the end we take $A_i = \bigcup_{s=0}^{\infty} A_i^{(s)}$, $i \in N$. Now we have to prove that the set A satisfies the condition of our problem for n.

(3.15) **Lemma.** *For any e-number of a p.r. function the condition* ((i) $\vee$ (ii)) *is injured only* $2^e - 1$ *times, i.e. we create not more than* 2^e *e-requirements.*

Proof. We shall use induction on e.

1) For $e = 0$ we have that the e-requirement can not be injured, because there is no requirement with higher priority and $2^0 - 1 = 0$.

For $e = 1$ we can injure the e-requirement only once when we create the single 0-requirement (if it exists) and $2^1 - 1 = 1$.

2) Let the statement hold for all the numbers smaller than e.

3) Let e be a number of a p.r. function. The condition ((i) $\vee$ (ii)) for e is injured when a requirement with number between 0 and $e-1$ is created, i.e. not more than $2^0 + 2^1 + \cdots + 2^{e-1} = 2^e - 1$ times.

The lemma is proved.

(3.16) **Lemma.** *The set $N_1 \setminus A$ is infinite.*

Proof. Let $(N_1)_x = \{y \mid y \in N_1 \,\&\, y < x\}$. We shall prove that the set $(N_1)_{r(x)} \cap (N_1 \setminus A)$ contains at least x elements.

$$(N_1)_{r(x)} = \{y \mid y \in N_1 \,\&\, y < r(x)\} = \{y \mid y \in N_1 \,\&\, y < r'(n.2^x + x)\}$$

and because $r'(0) < r'(1) < \cdots < r'(n.2^x + x - 1) < r'(n.2^x + x) < \cdots$, the elements of the set $(N_1)_{r(x)}$ are $n.2^x + x$. Between them only 0-, 1-, ..., $x-1$-requirements may be elements of A (the others are bigger than $r(x)$), i.e. not more than $n.2^0 + n.2^1 + \cdots + n.2^{x-1} = 2^x$. Therefore, the elements of $(N_1)_{r(x)} \cap (N_1 \setminus A)$ are at least $n.2^x + x - n.2^x = x$ and the lemma is proved.

(3.17) **Lemma.** *For any natural number e such that $N_1 \subseteq \operatorname{dom}\varphi_e$ (and especially for any e which is a number of recursive function) there exists a constant e-requirement.*

Proof. Let $e \in N$ and $N_1^n \subseteq \operatorname{dom}\varphi_e$. Let us assume that there is not a constant e-requirement. We find s_0 such that at the step s_0 all e_1-requirements for $e_1 < e$ are already built.

From Lemma (3.16) we have that there exist $x_1 \in N_1 \setminus A, \ldots, x_n \in N_1 \setminus A$ such that $x_1 > r(e) \,\&\ldots\&\, x_n > r(e)$. Let $s > s_0$ and $J^n(x_1, \ldots, x_n) \in \operatorname{dom}\varphi_{e,s}$. There at the step s a constant e-requirement is created.

The lemma is proved.

Let A be obtained from $\{A_i\}_{i \in N}$ according to Definition (3.3). According the construction, A is a r.e. set, $A^n \not\equiv_m A^{n-1}$, and $A^{n+1} \leq_m A^n$, i.e. the needed set is built.

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A NOTE ON INTERSECTION OF MODALITIES

TINKO TINCHEV

Тинко Тинчев. ЗАМЕТКА О ПЕРЕСЕЧЕНИЯ МОДАЛЬНОСТЕЙ

Рассматривается метод семантических таблиц для пропозициональной полимодальной логике, когда некоторые модальностей интерпретируются в моделей Крипке как теоретико-множественное пересечение других.

Tinko Tinchev. A NOTE ON INTERSECTION OF MODALITIES

In this paper a variant of the tableaux method for a polymodal language is considered, where some of the modalities are interpreted in the Kripke models as the set theoretic intersection of the interpretations of some of the remaining ones. As a consequence the completeness theorem with respect to the class of finite tree-like frames and decidability are obtained.

0. INTRODUCTION

The development of modal logic applications in theoretical computer science (viz. propositional dynamic logic) motivates many new directions of investigations in the (poly)modal logic. For example, raised in 1979 by Vakarelov, the question of axiomatizing the logic of intersection of modalities has led to enriched modal languages with names and universal modality (cf. [3, 4]). The main reason for this enrichement is to avoid the modal undefinability of intersection known from Gargov (1984, unpublished) and Van der Hoek ([5]). More precisely, there is no set Γ of formulae from the propositional modal language with three modalities $\langle R_1\rangle$, $\langle R_2\rangle$, $\langle R_3\rangle$ such that for each Kripke frame $\mathcal{F} = \langle W, R_1, R_2, R_3\rangle$, $\mathcal{F} \models \Gamma$ iff $R_3 = R_1 \cap R_2$. However, in the simplest case of tri-modal language one can axiomatize the logic of intersection — the set of modal formulae valid in all Kripke frames $\mathcal{F} = \langle W, R_1, R_2, R_3 \rangle$ with $R_3 = R_1 \cap R_2$ — joining the axiom scheme $[R_1]\varphi \vee [R_2]\varphi \Longrightarrow [R_3]\varphi$ with the minimal normal tri-modal logic. This is shown by Tinchev and Vakarelov in an unpublished manuscript (February 1986) and in [1], where the Vakarelov's copying method is used; another proof is contained in [5]. All these proofs make use of the standard canonical construction followed by some p-morphic pre-image.

Here we shall give an elementary proof — without using the axiom of choice — of the above mentioned completeness theorem, finite model property and decidability (the last two not mentioned, but also known). The semantic tableaux method — in some sense "reversed" version of that from [2, 6] — is used. The "prix" is completeness with respect to the smaller class of Kripke frames — the so-called tree-like frames — and an easily obtainable refutation system.

1. SYNTAX AND SEMANTICS

We shall consider the poly-modal language L with a denumerable set of propositional letters Φ and a set $\mathcal{O}$ of unary modal operators. The set of well-formed formulas over Φ and $\mathcal{O}$ is build up using propositional letters $p \in \Phi$ and modal operators $O \in \mathcal{O}$, as usual, according to the following inductive rules:

— every propositional letter is a formula;

— if φ_1 and φ_2 are formulas, then so are $\neg\varphi_1$ and $(\varphi_1 \& \varphi_2)$;

— if φ is a formula and O is a modal operator, then $[O]\varphi$ is a formula.

We adopt the usual abbreviations: $(\varphi \vee \psi)$, $(\varphi \Longrightarrow \psi)$, $(\varphi \Longleftrightarrow \psi)$ and $\langle O\rangle\varphi$ for $\neg(\neg\varphi \& \neg\psi)$, $(\neg\varphi \vee \psi)$, $((\varphi \Longrightarrow \psi) \& (\psi \Longrightarrow \varphi))$ and $\neg[O]\neg\varphi$, respectively.

For the rest of this paper we assume that the set of modal operators $\mathcal{O}$ is structured — there exists a set B containing at least two elements such that $\mathcal{O}$ is a set of finite subsets of B, $\varnothing \notin \mathcal{O}$ and $\{a\} \in \mathcal{O}$ for any $a \in B$. If $O = \{a_1, \ldots, a_n\}$ is a modal operator, we shall write $[a_1, \ldots, a_n]\varphi$ and $\langle a_1, \ldots, a_n\rangle\varphi$ instead of $[\{a_1, \ldots, a_n\}]\varphi$ and $\langle\{a_1, \ldots, a_n\}\rangle\varphi$, respectively.

The *modal depth* of a formula φ — the number of nested modalities in φ — we define, as usual, inductively:

depth$(p) = 0$ for any propositional letter p,

depth$(\neg\varphi) =$ depth(φ), depth$(\varphi_1 \& \varphi_2) = \max($depth$(\varphi_1),$ depth$(\varphi_2))$,

depth$([O]\varphi) =$ depth$(\varphi) + 1$.

The semantics of the language L is based on the *Kripke structures* $\mathcal{F} = \langle W, R\rangle$, where $W \neq \varnothing$, $R : \mathcal{O} \to 2^{W \times W}$ and $R(O) = \bigcap_{a \in O} R(\{a\})$ for any $O \in \mathcal{O}$, which are called *frames*. A *model* $\mathfrak{M}$ *over a frame* $\mathcal{F}$ is a tuple $\langle \mathcal{F}, V\rangle$, where V is an evaluation assigning subsets of W to the propositional letters in Φ, i.e. $V : \Phi \to 2^W$. The truth conditions are

$\mathfrak{M}, x \models p$ iff $x \in V(p)$,

$\mathfrak{M}, x \models \neg\varphi$ iff $\mathfrak{M}, x \not\models \varphi$,

$\mathfrak{M}, x \models \varphi \& \psi$ iff $\mathfrak{M}, x \models \varphi$ and $\mathfrak{M}, x \models \psi$,

$\mathfrak{M}, x \models [O]\varphi$ iff $\forall y(xR(O)y \to \mathfrak{M}, y \models \varphi)$.

One can immediately verify $\mathfrak{M}, x \models \varphi \vee \psi$ iff $\mathfrak{M}, x \models \varphi$ or $\mathfrak{M}, x \models \psi$, and $\mathfrak{M}, x \models \langle O\rangle\varphi$ iff $\exists y(xR(O)y$ and $\mathfrak{M}, y \models \varphi)$.

A formula φ is *valid on* $\mathfrak{M}$, $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, x \models \varphi$ for all $x \in W$. A formula φ is *valid on* a frame $\mathfrak{F}$, $\mathfrak{F} \models \varphi$, if φ is valid on any model over $\mathfrak{F}$. If φ is valid on every frame, then it is called *valid*. A formula φ is *refutable* if $\neg\varphi$ is not valid, i.e. if there are a model $\mathfrak{M}$ and a world x in $\mathfrak{M}$ such that $\mathfrak{M}, x \models \neg\varphi$. A formula φ is called *satisfiable* if $\neg\varphi$ is refutable, i.e. if there are a model $\mathfrak{M}$ and a world x in $\mathfrak{M}$ such that $\mathfrak{M}, x \models \varphi$.

The *logic of intersection*, $K^{\cap}(\mathbb{O})$, is the set of all valid formulas.

A frame $\mathfrak{F} = \langle W, R\rangle$ is called *tree-like* if $\langle W, \bigcup_{O\in\mathbb{O}} R(O)\rangle$ is a tree, the root of this tree is called the root of $\mathfrak{F}$.

If a frame $\mathfrak{F} = \langle W, R\rangle$ is tree-like and $R^{\mathrm{ref}}(O) = R(O)\cup\{\langle x, x\rangle \mid x \in \mathrm{dom}R(O)\cup \mathrm{range}R(O)\}$ for all $O \in \mathbb{O}$, then $\langle W, R^{\mathrm{ref}}\rangle$ is called *reflexive tree-like* frame. A frame $\langle W, R^{\mathrm{tr}}\rangle$ is called *transitive tree-like* frame if there is a frame $\langle W, R\rangle$ such that $R^{\mathrm{tr}}(O)$ is the transitive closure of $R(O)$ for all $O \in \mathbb{O}$.

2. THE LOGIC OF INTERSECTION

It is well-known that the logic of intersection $K^{\cap}(\mathbb{O})$ is axiomatizible by the following

Axioms: Ax0. All (or enough) boolean tautologies;

Ax1. $[O](\varphi \Longrightarrow \psi) \Longrightarrow ([O]\varphi \Longrightarrow [O]\psi)$;

Ax2. $[O_1]\varphi \Longrightarrow [O]\varphi,\ O_1 \subseteq O,\ O, O_1 \in \mathbb{O}$; and

Rules: (MP) If $\vdash \varphi$ and $\vdash \varphi \Longrightarrow \psi$, then $\vdash \psi$;

(Nec) If $\vdash \varphi$, then $\vdash [O]\varphi$.

We say that a formula φ is in *normal form* if φ is a disjunction of *basic conjunctions*, i.e. conjunctions of the form

$$(*)\qquad \begin{aligned}&\lambda_1 p_1 \,\&\, \ldots \,\&\, \lambda_s p_s\\ &\&\, \langle O_1\rangle\varphi_1^1 \,\&\, \ldots \,\&\, \langle O_1\rangle\varphi_{n_1}^1 \,\&\, \langle O_2\rangle\varphi_1^2 \,\&\, \ldots \,\&\, \langle O_k\rangle\varphi_{n_k}^k\\ &\&\, [O_1]\psi_1 \,\&\, \ldots \,\&\, [O_l]\psi_l,\end{aligned}$$

where λ_i is $\neg$ or the empty word, and $O_1, \ldots, O_l$ are different modal operators, $l \geqq k$.

Proposition. *For any formula φ one can effectively find a formula φ' which is in normal form, $\vdash \varphi \Longleftrightarrow \varphi'$ and* $\mathrm{depth}(\chi) \leqq \mathrm{depth}(\varphi)$ *for any basic conjunction χ from φ'.*

Proof. The proof is carried out by an easy induction on the construction of φ.

Throughout we assume obvious conditions for effectiveness of $\mathbb{O}$.

Theorem 1. *There is an algorithm $\mathfrak{A}$ such that for any formula φ after a finite number of steps $\mathfrak{A}$ gives a result $\mathfrak{A}(\varphi)$, which is a finite tree-like model such that in its root φ is true or a proof of $\neg\varphi$ in $K^{\cap}(\mathcal{O})$.*

Proof. We construct $\mathfrak{A}$ by induction on the modal depth of formulas. Let we assume that $\mathfrak{A}$ is defined for any formula θ with $\text{depth}(\theta) < \text{depth}(\varphi)$.

First of all we assume without a loss of generality that φ is in normal form. If any basic conjunction from φ contains conjunctive terms $\lambda_i p_i$ and $\lambda_j p_j$ such that one of them is the negation of the other, then we obtain a proof of $\neg\varphi$ by boolean arguments. This proof is $\mathfrak{A}(\varphi)$. Therefore we can suppose that every disjunctive term from φ has a satisfiable boolean part.

If there is a basic conjunction from φ which does not contain a term of the form $\langle O\rangle\theta$, then we set $\mathfrak{A}(\varphi)$ to be the model $\mathfrak{M}$ with $W = \{w\}$, $R(O) = \varnothing$ and $V(p) = \{w\}$ iff the occurrance of p in this term is positive, and $V(p) = \varnothing$ otherwise.

Therefore we have to consider only the case when every basic conjunction from φ contains a term of the form $\langle O\rangle\theta$.

Let χ be the basic conjunction (*) and define $n_1+\cdots+n_k$ formulas χ_{ij}, $1 \leqq i \leqq k$, $1 \leqq j \leqq n_i$, in the following way: If $O_{x_1}, \ldots, O_{x_{\alpha(i)}}$ are all modal operators among $O_1, \ldots, O_l$, which are subsets of O_i, then

$$\chi_{ij} = \varphi^i_{n_j} \,\&\, \psi_{x_1} \,\&\, \ldots \,\&\, \psi_{x_{\alpha(i)}}.$$

From $\text{depth}(\chi_{ij}) < \text{depth}(\chi) \leqq \text{depth}(\varphi)$ for $1 \leqq i \leqq k$ and $1 \leqq j \leqq n_i$ we conclude that the algorithm $\mathfrak{A}$ is defined for them.

Let we consider the set $\{\mathfrak{A}(\chi_{ij}) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\}$. If one of its elements $\mathfrak{A}(\chi_{ij})$ is a proof (of $\neg\chi_{ij}$), then this proof can be extended to the proof of $\neg\chi$ in the following way:

$$\vdash \psi_{x_1} \,\&\, \ldots \,\&\, \psi_{x_{\alpha(i)}} \Longrightarrow \neg\varphi^i_{n_j},$$

$$\vdash [O_i]\psi_{x_1} \,\&\, \ldots \,\&\, [O_i]\psi_{x_{\alpha(i)}} \Longrightarrow [O_i]\neg\varphi^i_{n_j} \quad \text{(by Ax0, (Nec), Ax1, (MP), Ax0)},$$

$$\vdash [O_{x_1}]\psi_{x_1} \Longrightarrow [O_i]\psi_{x_1}, \quad \ldots, \quad \vdash [O_{x_{\alpha(i)}}]\psi_{x_{\alpha(i)}} \Longrightarrow [O_i]\psi_{x_{\alpha(i)}} \qquad \text{(by Ax2)}.$$

From here we obtain $\vdash \neg\chi$ by boolean arguments.

C a s e 1. Let for every disjunctive term χ of φ the corresponding set $\{\mathfrak{A}(\chi_{ij}) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\}$ contain a proof. Then we have a proof of $\neg\chi$ by the above considerations. Then one can easily obtain a proof of $\neg\varphi$ and it is $\mathfrak{A}(\varphi)$.

C a s e 2. Let there exist a disjunctive term χ of φ such that the corresponding set $\{\mathfrak{A}(\chi_{ij}) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\}$ does not contain a proof. Then $\mathfrak{A}(\chi_{ij})$ is a finite tree-like model $\mathfrak{M}_{ij}$, in which root w_{ij} the formula χ_{ij} is true. Let $\mathfrak{M}_{ij} = \langle \mathcal{F}_{ij}, V_{ij}\rangle$, where $\mathcal{F}_{ij} = \langle W_{ij}, R_{ij}\rangle$, and w_{ij} be the root of $\mathfrak{M}_{ij}$. Without a loss of generality we can suppose that the sets W_{ij} are disjoint. We are ready to define a finite tree-like model $\mathfrak{M}$ with a root w such that $\mathfrak{M}, w \models \varphi$, and we set $\mathfrak{A}(\varphi) = \mathfrak{M}$. Let $W' = \bigcup\{W_{ij} \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\}$, $w \notin W'$ and $W = W' \cup \{w\}$. We set

$$R(\{a\}) = \bigcup\{R_{ij}(\{a\}) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\} \cup \{\langle w, w_{ij}\rangle \mid a \in O_i\} \quad \text{for } a \in B,$$

$$R(O) = \bigcap_{a \in O} R(\{a\}) \quad \text{for } O \in \mathcal{O},$$

$$V'(p) = \bigcup \{V_{ij}(p) \mid 1 \leqq i \leqq k,\ 1 \leqq j \leqq n_i\},$$

and

$$V(p) = \begin{cases} V'(p) & \text{if } p \text{ has not a positive occurrance in the boolean part of } \chi, \\ V'(p) \cup \{w\} & \text{otherwise.} \end{cases}$$

One can immediately verify $\mathfrak{M}, w \models \varphi$.

Thus $\mathfrak{A}(\varphi)$ is defined and has the desired properties.

Theorem 1 is proven.

Corollary 1. *The logic of intersection is decidable and complete with respect to the finite tree-like frames.*

3. SOME SIMPLE EXTENSIONS OF THE LOGIC OF INTERSECTION

If we put some additional axioms to the considered in the previous section formal system, we obtain the so-called *simple extensions* of $K^{\cap}(\mathcal{O})$. For the sake of notational simplicity let assume that $\mathcal{O} = \{\{a\}, \{b\}, \{a, b\}\}$. In many cases adding axiom schemes only for the modalities $[a]$ and $[b]$, which have corresponding first order conditions, leads to the obvious modification of the tableaux used in the proof of Theorem 1. For example, we shall mention only the seriality axiom $\langle a \rangle$**true**, $\langle b \rangle$**true**, $\langle a, b \rangle$**true**. We can add an arbitrary subset of these axioms and find an appropriate modification of the construction to obtain decidability and completeness with respect to the class of finite frames much like to the tree-like frames (some leafs must be reflexive).

A bit more complicated case is when we add axioms that guarantee the reflexivity:

(Ref^a)	$[a]\varphi \Longrightarrow \varphi,$
(Ref^b)	$[b]\varphi \Longrightarrow \varphi,$
$(\mathrm{Ref}^{a,b})$	$[a, b]\varphi \Longrightarrow \varphi.$

If we add only (Ref^a) and (Ref^b), the obtained system is incomplete as one can easily see. The reason is that the reflexivity of $R(a)$ and $R(b)$ implies the reflexivity of $R(a) \cap R(b) = R(\{a, b\})$. Let $T^{\cap}(\mathcal{O})$ be the logic $K^{\cap}(\mathcal{O}) + (\mathrm{Ref}^{a,b})$.

Theorem 2. *There is an algorithm $\mathfrak{A}$ such that for any formula φ after a finite number of steps $\mathfrak{A}$ gives a result $\mathfrak{A}(\varphi)$, which is a finite reflexive tree-like model such that in its root φ is true or a proof of $\neg\varphi$ in $T^{\cap}(\mathcal{O})$.*

Proof. It is enough to make small modifications in the proof of Theorem 1. When the case χ does not contain a term of the form $\langle O \rangle \theta$ is considered, we take the formula which is the conjunction of the boolean part of χ and the formulas after $[O]$. And when χ_{ij} is formed, we have to consider a formula obtained as just we mentioned, write it in a normal form and so on.

More complicated case is when we add axioms for transitivity:

$$[O]\varphi \Longrightarrow [O][O]\varphi \quad \text{for any } O \in \mathcal{O}.$$

Let denote this simple extension by $K4^{\cap}(\mathcal{O})$.

Theorem 3. *There is an algorithm $\mathfrak{A}$ such that for any formula φ after a finite number of steps $\mathfrak{A}$ gives a result $\mathfrak{A}(\varphi)$, which is a transitive tree-like model such that in its root φ is true or a proof of $\neg\varphi$ in $K4^{\cap}(\mathcal{O})$.*

Sketch of a proof. It is enough to destroy a formula in a systematic way guaranteeing transitivity. The condition for finishing and starting the construction of the model is some periodicity of the considered finite number of potential conjunctive terms.

Let we mention that we can produce finite models in the last theorem, but we lose its property to be a tree.

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REPRESENTATION PAR DIFFERENCES DIVISEES DES MOMENTS RECIPROQUES DE COMBINAISONS LINEAIRES DE VARIABLES ALEATOIRES INDEPENDANTES EXPONENTIELLEMENT REPARTIES

MODY DIALLO, TZVETAN IGNATOV

Моди Диало, Цветан Игнатов. ПРЕДСТАВЛЕНИЕ РЕЦИПРОЧНЫХ МОМЕНТОВ ЛИНЕЙНЫХ КОМБИНАЦИЙ НЕЗАВИСИМЫХ ЕКСПОНЕНЦИАЛЬНО РАСПРЕДЕЛЕННЫХ СЛУЧАЙНЫХ ВЕЛИЧИН ЧЕРЕЗ РАЗДЕЛЕННЫЕ РАЗНОСТИ

Для реципрочных моментов линейных комбинациях составленны независимых експоненциально распределенных случайних величин найдено представление (формула (3)) посредством разделенных разностей подходящей функции.

Mody Diallo, Tzvetan Ignatov. REPRESENTATION OF THE RECIPROCAL MOMENTS OF LINEAR COMBINATIONS OF INDEPENDENT EXPONENTIAL VARIATES

For the reciprocal moments of linear combinations of independent exponential variates is given the representation (formula (3)) through divided differences of a suitable function.

1. INTRODUCTION

Dans beaucoup de problèmes de statistique surgit la nécessité de la détermination des moments réciproques de la variable aléatoire

$$\eta_n = \frac{1}{\lambda_1}\xi_1 + \frac{1}{\lambda_2}\xi_2 + \cdots + \frac{1}{\lambda_n}\xi_n, \tag{1}$$

où $\xi_1, \xi_2, \ldots, \xi_n$ sont des variables aléatoires indépendantes exponentiellement réparties de moyenne unité. Comme exemple, nous pouvons donner les résultats de Cox et Lewis (1966) relatifs à l'intensité moyenne du flux des naissances dans le processus des naissances. Plus précisément,

$$E\left(\frac{n-m}{\eta_n-\eta_m}\right)=(n-m)E\left(\frac{1}{\frac{1}{\lambda_{m+1}}\xi_{m+1}+\cdots+\frac{1}{\lambda_n}\xi_n}\right),$$

qui se réduit jusqu'à l'obtention du premier moment réciproque $E\left(\frac{1}{\eta_n}\right)$. On peut voir d'autres exemples dans l'article de Thomas (1976). Dans le même article est donnée l'expression du moment réciproque d'ordre r de η_n, mais dans laquelle figurent des grandeurs qui ne s'expriment pas explicitement. Cependant, il faut les chercher à partir de la décomposition en fractions élémentaires de la fonction caractéristique de η_n.

Le but principal du présent travail est la présentation des moments réciproques de η_n par différences divisées de fonctions convenables.

2. REPRESENTATION DES MOMENTS RECIPROQUES DE η_n PAR DIFFERENCES DIVISEES

Rappelons (cf. [4], p. 47) que les différences divisées d'ordre n de la fonction $\varphi(u)$ suffisamment lisse sur les points (nœuds) de la droite réelle $\ldots \leqq t_i \leqq t_{i+1} \leqq \cdots \leqq t_{i+n} \leqq \ldots$ doivent être définies comme il suit:

$$(2)\qquad [t_i,\ldots,t_{i+n}]_u\varphi(u)=\frac{[t_{i+1},\ldots,t_{i+n}]_u\varphi(u)-[t_i,\ldots,t_{i+n-1}]_u\varphi(u)}{t_{i+n}-t_i}$$

avec la supposition que $t_i \neq t_{i+n}$. Si $t_i=t_{i+1}=\cdots=t_{i+n}$ alors

$$[t_i,\ldots,t_{i+n}]_u\varphi(u)=\frac{(D^n\varphi(t_i))}{n!},$$

où avec $D^n\varphi(t_i)$ est désignée la n^e dérivée de la fonction $\varphi(u)$ au point t_i, $n \geqq 0$, et $D^0\varphi(t_i)$ est désignée par $\varphi(t_i)$.

A l'aide des différences divisées nous pouvons représenter les moments réciproques de η_n par le théorème suivant:

Théorème. *Soit un nombre naturel r, $r<n$, et $\lambda_1,\ldots,\lambda_n$ des nombres réels positifs. Alors il existe le moment réciproque $E\left(\frac{1}{\eta_n^r}\right)$ d'ordre r, qui est égal à*

$$(3)\qquad E\left(\frac{1}{\eta_n^r}\right)=(-1)^{n-r+1}\frac{1}{(r-1)!}\lambda_1\ldots\lambda_n\left([\lambda_1,\ldots,\lambda_n]_\lambda\lambda^{r-1}\ln\lambda\right).$$

Démonstration. Dans l'article de Ignatov et Stateva (1989) (formule 9) la densité de η_n est représentée à l'aide des différences divisées par

$$(4)\qquad f_{\eta_n}(x)=(-1)^{n-1}\mathrm{sgn}(x)\lambda_1\lambda_2\ldots\lambda_n[\lambda_1,\lambda_2,\ldots,\lambda_n]_\lambda(\mathrm{sgn}(\lambda x))_+\exp(-\lambda x),$$

$$\text{où } (u)_+ = \max(0,u) \text{ et } \operatorname{sgn}(x) = \begin{cases} 1, & \text{si } x > 0, \\ 0, & \text{si } x = 0, \\ -1, & \text{si } x < 0. \end{cases}$$

Utilisant la formule (4), pour le moment réciproque d'ordre r de η_n nous avons

$$(5) \qquad E\left(\frac{1}{\eta_n^r}\right) = \int_0^\infty \frac{1}{x^r}(-1)^{n-1}\lambda_1 \ldots \lambda_n [\lambda_1, \ldots, \lambda_n]_\lambda e^{-\lambda x} dx$$

$$= (-1)^{n-1}\lambda_1 \ldots \lambda_n \lim_{\varepsilon \to 0} \int_\varepsilon^\infty [\lambda_1, \ldots, \lambda_n]_\lambda \frac{e^{-\lambda x}}{x^r}\, dx$$

$$= (-1)^{n-1}\lambda_1 \ldots \lambda_n \lim_{\varepsilon \to 0} [\lambda_1, \ldots, \lambda_n]_\lambda \left\{ - \sum_{k=1}^{r-1} \frac{(-1)^{k-1} e^{-\lambda x}}{(r-1)\ldots(r-k)x^{r-k}} \Bigg|_\varepsilon^{+\infty} \right.$$

$$\left. + (-\lambda)^{r-1} \frac{1}{(r-1)!} \int_\varepsilon^{+\infty} \frac{e^{-\lambda x}}{x}\, dx \right\}$$

$$= (-1)^{n-1}\lambda_1 \ldots \lambda_n \lim_{\varepsilon \to 0} [\lambda_1, \ldots, \lambda_n]_\lambda \left\{ \sum_{k=1}^{r-1} \frac{(-\lambda)^{k-1} e^{-\lambda \varepsilon}}{(r-1)\ldots(r-k)\varepsilon^{r-k}} \right.$$

$$\left. + (-\lambda)^{r-1} \frac{1}{(r-1)!} \int_\varepsilon^{+\infty} \frac{e^{-\lambda x}}{x}\, dx \right\}.$$

En considérant que $1 \leqq k \leqq r-1 \leqq n-1$, on a

$$(6) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{r-k}} [\lambda_1, \ldots, \lambda_n]_\lambda \lambda^{k-1} e^{-\lambda \varepsilon} = 0.$$

En effet, de la formule de Leibnitz (voir Schumaker (1981), p. 50, formule (2.96))

$$[t_i, t_{i+1}, \ldots, t_{i+n}]_u f(u) g(u)$$

$$= \sum_{j=0}^{n} \big([t_i, \ldots, t_{i+j}]_u f(u)\big).\big([t_{i+j}, t_{i+j+1}, \ldots, t_{i+n}]_u g(u)\big)$$

nous avons

$$(7) \qquad [\lambda_1, \ldots, \lambda_n]_\lambda \lambda^{k-1} e^{-\lambda \varepsilon}$$

$$= \sum_{j=1}^{n} \big([\lambda_1, \ldots, \lambda_j]_\lambda \lambda^{k-1}\big).\big([\lambda_j, \lambda_{j+1}, \ldots, \lambda_n]_\lambda e^{-\lambda \varepsilon}\big)$$

$$= \sum_{j=1}^{k} \big([\lambda_1, \ldots, \lambda_j]_\lambda \lambda^{k-1}\big).\big([\lambda_j, \ldots, \lambda_n]_\lambda e^{-\lambda \varepsilon}\big).$$

Du développement

$$e^{-\lambda \varepsilon} = 1 - \frac{\lambda \varepsilon}{1!} + \frac{(\lambda \varepsilon)^2}{2!} - \cdots + (-1)^{n-j} \frac{(\lambda \varepsilon)^{n-j}}{(n-j)!} + \varepsilon^{n-j+1}.\psi(\lambda, \varepsilon, n-j)$$

et de la formule (7) nous obtenons

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{r-k}}[\lambda_1,\dots,\lambda_n]_\lambda \lambda^{k-1}e^{-\lambda\varepsilon}$$

$$=\sum_{j=1}^{k}([\lambda_1,\dots,\lambda_j]_\lambda\lambda^{k-1})\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{r-k}}[\lambda_j,\dots,\lambda_n]_\lambda\Big(1-\frac{\lambda\varepsilon}{1!}+\frac{(\lambda\varepsilon)^2}{2!}-\dots$$

$$+(-1)^{n-j}\frac{(\lambda\varepsilon)^{n-j}}{(n-j)!}+\varepsilon^{n-j+1}\psi(\lambda,\varepsilon,n-j)\Big)$$

$$=\sum_{j=1}^{k}([\lambda_1,\dots,\lambda_j]_\lambda\lambda^{k-1})\left(\lim_{\varepsilon\to 0}\frac{\varepsilon^{n-j+1}}{\varepsilon^{r-k}}[\lambda_j,\dots,\lambda_n]_\lambda\psi(\lambda,\varepsilon,n-j)\right)=0.$$

Il s'en suit la relation (6).

De (5) et (6) nous avons

$$(8)\quad E\left(\frac{1}{\eta_n^r}\right)=(-1)^{n-1}\lambda_1\dots\lambda_n\lim_{\varepsilon\to 0}[\lambda_1,\dots,\lambda_n]_\lambda\left((-\lambda)^{r-1}\frac{1}{(r-1)!}\int_\varepsilon^{+\infty}\frac{e^{-\lambda x}}{x}\,dx\right).$$

On sait que (cf. [1])

$$\int_0^\infty(\ln x)\lambda e^{-\lambda x}dx=-c-\ln\lambda,$$

où c est la constante d'Euler; donc

$$(9)\qquad \lim_{\varepsilon\to 0}\int_\varepsilon^\infty(\ln x)\lambda e^{-\lambda x}dx=\lim_{\varepsilon\to 0}\left(-e^{-\lambda x}\ln x\Big|_\varepsilon^{+\infty}+\int_\varepsilon^{+\infty}\frac{e^{-\lambda x}}{x}\,dx\right)$$

$$=\lim_{\varepsilon\to 0}\left(e^{-\lambda\varepsilon}\ln\varepsilon+\int_\varepsilon^{+\infty}\frac{e^{-\lambda x}}{x}\,dx\right)=-c-\ln\lambda.$$

De (9) nous pouvons écrire pour $\lambda>0$

$$(10)\qquad \int_\varepsilon^{+\infty}\frac{e^{-\lambda x}}{x}\,dx=-c-\ln\lambda-e^{-\lambda\varepsilon}\ln\varepsilon+o(\varepsilon,\lambda),$$

où $\lim_{\varepsilon\to 0}o(\varepsilon,\lambda)=0$.

De (8) et (10) nous obtenons

$$(11)\qquad E\left(\frac{1}{\eta_n^r}\right)$$

$$=(-1)^{n-1}\lambda_1\dots\lambda_n(-1)^{r-1}\frac{1}{(r-1)!}\lim_{\varepsilon\to 0}[\lambda_1,\dots,\lambda_n]_\lambda\lambda^{r-1}\int_\varepsilon^{+\infty}\frac{e^{-\lambda x}}{x}\,dx$$

$$= (-1)^{n-r}\lambda_1\dots\lambda_n\frac{1}{(r-1)!}\Big(\lim_{\varepsilon\to 0}[\lambda_1,\dots,\lambda_n]_\lambda(-c\lambda^{r-1})$$
$$-\lim_{\varepsilon\to 0}(\ln\varepsilon)[\lambda_1,\dots,\lambda_n]_\lambda(\lambda^{r-1}e^{-\lambda\varepsilon})+\lim_{\varepsilon\to 0}[\lambda_1,\dots,\lambda_n]_\lambda(\lambda^{r-1}o(\varepsilon,\lambda))$$
$$-\lim_{\varepsilon\to 0}[\lambda_1,\dots,\lambda_n]_\lambda(\lambda^{r-1}\ln\lambda)\Big)$$
$$= (-1)^{n-r+1}\frac{\lambda_1\dots\lambda_n}{(r-1)!}[\lambda_1,\dots,\lambda_n]_\lambda(\lambda^{r-1}\ln\lambda).$$

Thomas (1976) a donné d'une façon explicite le premier moment réciproque pour la variable aléatoire η_n de (1) quand

$$\lambda_1=m-j+1,\quad \lambda_2=m-j+2,\quad \dots,\quad \lambda_{j-i}=m-i, \tag{12}$$

où m, i et j sont des nombres entiers fixés tels que $i<j<m$. L'expression donnée par Thomas a la forme

$$E\left(\left(\frac{1}{m-j+1}\xi_1+\dots+\frac{1}{m-i}\xi_{j-i}\right)^{-r}\right) \tag{13}$$
$$=\frac{(-1)^{j-i+1}}{\Gamma(r)}\binom{m-i}{j-i}\sum_{k=1}^{j-i}k\binom{j-i}{k}(-1)^k(m-i-k+1)^{r-1}\ln(m-i-k+1),\ r<j-i.$$

Nous déduisons (13) comme cas particulier de (3) avec les suppositions (12). Conformement à (3) nous avons

$$E\left(\left(\frac{1}{m-j+1}\xi_1+\dots+\frac{1}{m-i}\xi_{j-i}\right)^{-r}\right)$$
$$=\frac{(-1)^{j-i-r+1}}{(r-1)!}\left(\prod_{k=1}^{j-i}(m-i-k+1)\right)[m-j+1,\dots,m-i]_\lambda(\lambda^{r-1}\ln\lambda)$$
$$=\frac{(-1)^{j-i-r+1}}{\Gamma(r)}\left(\prod_{k=1}^{j-i}(m-i-k+1)\right)\sum_{k=1}^{j-i}\frac{(m-i-k+1)^{r-1}\ln(m-i-k+1)}{(m-i-k+1)\prod\limits_{s=1,\,s\neq k}^{j-i}(s-k)}$$
$$=(-1)^{j-i-r+1}\frac{1}{\Gamma(r)}\frac{(m-i)!}{(m-j)!}\sum_{k=1}^{j-i}\frac{(m-i-k+1)^{r-2}\ln(m-i-k+1)}{(-1)^{k-1}(k-1)!(j-i-k)!}$$
$$=\frac{(-1)^{j-i-r+1}}{\Gamma(r)}\binom{m-i}{j-i}\sum_{k=1}^{j-i}k\frac{(j-i)!(-1)^k}{k!(j-i-k)!}(m-i-k+1)^{r-2}\ln(m-i-k+1)$$
$$=\frac{(-1)^{j-i-r+1}}{\Gamma(r)}\sum_{k=1}^{j-i}k\binom{j-i}{k}(-1)^k(m-i-k+1)^{r-2}\ln(m-i-k+1),\ r<j-i.$$

Enfin, remarquons que si entre les coefficients de la variable aléatoire η_n de (1) il y a au moins deux indices différents, alors il n'existe pas un seul moment réciproque de

η_n. Si tous les coefficients de η_n ont le même indice, alors son moment réciproque d'ordre r existe à condition que $r < n$.

En effet, si $r \geqq n$, pour $\lambda_i > 0$, $i = 1, 2, \ldots, n$,

$$E(\eta_n^{-r}) \geqq \left(\max_{1 \leqq i \leqq n} \lambda_i\right) E\big((\xi_1 + \xi_2 + \cdots + \xi_n)^{-r}\big).$$

Puisque $\xi_1 + \cdots + \xi_n$ est une variable aléatoire répartie suivant la loi gamma de paramètres n et 1, alors

$$E\big((\xi_1 + \cdots + \xi_n)^{-r}\big) = \int_0^\infty x^{-r} \frac{x^{n-1}}{\Gamma(n)} e^{-x}\, dx = +\infty,$$

par conséquent le $r^{ième}$ moment réciproque de η_n n'existe pas. La formule (3) même "donne une indication" que le $r^{ième}$ moment réciproque de $\eta_n = \xi_1 + \xi_2 + \cdots + \xi_n$ pour $r \geqq n$ n'existe pas. En effet, pour $\varepsilon > 0$ arbitraire, nous avons pour $r \geqq n$:

$$\begin{aligned} E(\eta_n^{-r}) &= \varepsilon^{-r} E\left(\left(\frac{\eta_n}{\varepsilon}\right)^{-r}\right) = \varepsilon^{-r} \frac{(-1)^{n-r+1}}{(r-1)!} \varepsilon^n \frac{1}{(n-1)!} D^{n-1} \lambda^{r-1} \ln \lambda \bigg|_{\lambda=\varepsilon} \\ &= \frac{(-1)^{n-r+1}}{(r-1)!(n-1)!} (\ln \varepsilon + \cdots). \end{aligned}$$

Le premier élément, $\ln \varepsilon$, dans la précédente somme tend vers l'infini si $\varepsilon \to 0$ et cette situation ne s'est pas produite pour $r < n$, puisque dans ce cas l'élément $\ln \varepsilon$ manquerait.

Enfin, donnons explicitement la formule pour le $r^{ième}$ moment réciproque de η_n de (1). Sans restriction à la généralité nous pouvons supposer que $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$. Puisque quelques λ_i peuvent être égaux, soit alors

$$\{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{\overbrace{\tau_1, \ldots, \tau_1}^{l_1}, \ldots, \overbrace{\tau_d, \ldots, \tau_d}^{l_d}\},$$

où $\tau_1 < \tau_2 < \ldots < \tau_d$ et $l_1 + l_2 + \cdots + l_d = n$.

Les différences divisées d'une fonction $f(\lambda)$ suffisamment lisse peuvent s'exprimer par (cf. [4], p. 45)

(14)

$$[\lambda_1, \ldots, \lambda_n]_\lambda f(\lambda) = [\overbrace{\tau_1, \ldots, \tau_1}^{l_1}, \ldots, \overbrace{\tau_d, \ldots, \tau_d}^{l_d}]_\lambda f(\lambda) = \begin{bmatrix} \tau_1 & \tau_2 & \ldots & \tau_d \\ l_1 & l_2 & \ldots & l_d \end{bmatrix}_\lambda f(\lambda)$$

$$= \frac{\det \begin{pmatrix} \overbrace{\tau_1 & \ldots & \tau_1}^{l_1} & \ldots & \overbrace{\tau_d & \ldots & \tau_d & \tau_d}^{l_d} \\ u_1(\lambda) & \ldots & u_{l_1}(\lambda) & \ldots & u_{l_1+\cdots+l_{d-1}+1}(\lambda) & \ldots & u_{n-1}(\lambda) & f(\lambda) \end{pmatrix}}{\det \begin{pmatrix} \overbrace{\tau_1 & \ldots & \tau_1}^{l_1} & \ldots & \overbrace{\tau_d & \ldots & \tau_d}^{l_d} \\ u_1(\lambda) & \ldots & u_{l_1}(\lambda) & \ldots & \ldots & \ldots & u_n(\lambda) \end{pmatrix}},$$

où le déterminant du dénominateur est défini par

$$(15) \qquad \det\begin{pmatrix} \overbrace{\tau_1 \quad \dots \quad \tau_1}^{l_1} & \dots & \overbrace{\tau_d \quad \dots \quad \tau_d}^{l_d} \\ u_1(\lambda) \quad \dots \quad u_{l_1}(\lambda) & \dots & \dots \quad \dots \quad u_n(\lambda) \end{pmatrix}$$

$$= \begin{vmatrix} u_1(\tau_1) & u_2(\tau_1) & \dots & u_n(\tau_1) \\ Du_1(\tau_1) & Du_2(\tau_1) & \dots & Du_n(\tau_1) \\ \dots & \dots & \dots & \dots \\ D^{l_1-1}u_1(\tau_1) & D^{l_1-1}u_2(\tau_1) & \dots & D^{l_1-1}u_n(\tau_1) \\ \dots & \dots & \dots & \dots \\ u_1(\tau_d) & u_2(\tau_d) & \dots & u_n(\tau_d) \\ Du_1(\tau_d) & Du_2(\tau_d) & \dots & Du_n(\tau_d) \\ \dots & \dots & \dots & \dots \\ D^{l_d-1}u_1(\tau_d) & D^{l_d-1}u_2(\tau_d) & \dots & D^{l_d-1}u_n(\tau_d) \end{vmatrix},$$

où les fonctions $u_i(\lambda)$ sont données par $u_i(\lambda) = \lambda^{i-1}$, $i = 1, \dots, n$, et $D^k g(\lambda)$ est la $k^{ième}$ dérivée d'une certaine fonction g au point λ ($D^0 g(\lambda) = g(\lambda)$), c'est-à-dire pour les éléments de la matrice dans (15) nous avons

$$D^k(u_i(\tau_j)) \equiv D^k(\tau_j^{i-1}) = \begin{cases} \dfrac{(i-1)!}{(i-k-1)!}\,\tau_j^{i-1-k}, & \text{si } k \leqq i-1, \\ 0 & \text{dans le cas contraire.} \end{cases}$$

Le déterminant du dénominateur de (14) s'obtient de façon explicite par (cf. [4], p. 30)

$$\det\begin{pmatrix} \overbrace{\tau_1 \quad \tau_1 \quad \dots \quad \tau_1}^{l_1} & \dots & \overbrace{\tau_d \quad \dots \quad \tau_d}^{l_d} \\ 1 \quad \lambda \quad \dots \quad \dots & \dots & \dots \quad \dots \quad \lambda^{n-1} \end{pmatrix} = \prod_{1 \leqq i<j \leqq d} (\tau_j - \tau_i)^{l_i l_j} \prod_{i=1}^{d} \prod_{\nu=1}^{l_i-1} \nu!.$$

Le déterminant du numérateur de (14) se calcule analogiquement en remplaçant la fonction $u_n(\lambda)$ par $f(\lambda) = \lambda^{r-1}\ln\lambda$.

Remarquons que pour trouver le déterminant du numérateur, on considère que

$$D^k(f(\tau_j)) \equiv D^k(\tau_j^{r-1}\ln\tau_j)$$

$$= \begin{cases} \displaystyle\sum_{i=0}^{k-1} \frac{(-1)^{k-1-i}k!}{(k-1)i!}\frac{(r-1)!}{(r-1-i)!}\tau_j^{r-1-k} + \ln\tau_j \frac{(r-1)!}{(r-1-k)!}\tau_j^{r-1-k}, & \text{si } k \leqq r-1, \\ \displaystyle\sum_{i=0}^{r-1} \frac{(-1)^{k-i-1}k!(r-1)!}{(k-i)i!(r-1-i)!}\,\tau_j^{r-k-1}, & \text{si } k > r-1. \end{cases}$$

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COHOMOLOGIES OF COUNTABLE UNIONS OF CLOSED SETS WITH APPLICATIONS TO CANTOR MANIFOLDS

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Н. Хаджииванов, Е. Щепин. КОГОМОЛОГИИ СЧЕТНОГО ОБЪЕДИНЕНИЯ ЗАМКНУТЫХ МНОЖЕСТВ С ПРИЛОЖЕНИЯМИ ДЛЯ КАНТОРОВЫХ МНОГООБРАЗИЯХ

Основной результат: Пусть X — компакт, A — замкнутое подмножество компакта X, и $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, где F_i — замкнутые в $X \setminus A$ множества, такие что $\dim(F_i \cap F_j) \leqq n-1$ для $i \neq j$. Тогда естественный гомоморфизм $H^r(X, A; G)$ в прямую сумму $\prod_{i=1}^{\infty} H^r(A \cup F_i, A; G)$ является мономорфизмом для $r \geqq n+1$. Получены некоторые применения этого результата для сильных канторовых многообразиях (относительно группы G).

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The main result: Let X be a compact space, A be its closed subset, and $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, where F_i are closed subsets of $X \setminus A$ such that $\dim(F_i \cap F_j) \leqq n-1$ for $i \neq j$. Then the natural homomorphism of $H^r(X, A; G)$ into the direct sum $\prod_{i=1}^{\infty} H^r(A \cup F_i, A; G)$ is a monomorphism for $r \geqq n+1$. Some applications of this result to strong Cantor manifolds (with respect to a group G) are obtained.

Let X be a compact topological space, let A be a closed subset of X and let $X \setminus A = \bigcup_{i=1}^{m} F_i$, where F_i are closed subsets of $X \setminus A$ such that $\dim(F_i \cap F_j) \leqq n-1$ for $i \neq j$. Then by the Meyer–Vietoris sequence we may conclude that for $r \geqq n+1$ there exists a natural isomorphism of $H^r(X, A; G)$ into the direct sum $\prod_{i=1}^{m} H^r(F_i \cup A, A; G)$. (Here we denote by $H^r(X, A; G)$ the r-th relative cohomology group in the sense of Alexandroff–Čech with coefficients in G.).

The same is true under the assumption that the cohomological dimension of $F_i \cap F_j$ with respect to G is less or equal to $n-1$: $\dim_G(F_i \cap F_j) \leqq n-1$.

In case $X \setminus A$ is a countable union $X \setminus A = \bigcup_{i=1}^{\infty} F_i$ such that $\dim_G(F_i \cap F_j) \leqq n-1$ for $i \neq j$, there is a natural homomorphism of $H^r(X, A; G)$ into the direct sum $\prod_{i=1}^{\infty} H^r(F_i \cup A, A; G)$. Generally speaking, this homomorphism is not an isomorphism, but it remains a monomorphism for $r \geqq n+1$. The purpose of this paper is to prove the last result.

In fact we shall prove the following result about extensions of continuous maps:

Theorem 1. *Let X be a compact space, let A be a closed subset of X, and let $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, where F_i are closed in $X \setminus A$. Let furthermore Y be an n-connected CW-complex, i.e. all the homotopy groups of Y up to the n-th are trivial: $\pi_1(Y) = \pi_2(Y) = \cdots = \pi_n(Y) = 0$. Suppose that the inequality $\dim_{G_k}(F_i \cap F_j) \leqq n$ for $i \neq j$, where $G_k = \pi_k(Y)$, holds for any $k \geqq n+1$. Then a continuous map $f : A \to Y$, which is extendable over $A \cup F_i$ for any i, can be extended over X.*

Let us show now that Theorem 1 implies the above result about cohomologies.

It follows from the characteristic property of the Eilenberg–McLane complex $K(G, r)$ that there is an one-to-one correspondence between the cohomology group $H^r(X, A; G)$ and the homotopy classes of maps of X into $K(G, r)$ which are constant on A (cf. [1], p. 550). The natural homomorphism of $H^r(X, A; G)$ into $\prod_{i=1}^{\infty} H^r(F_i \cup A, A; G)$ is a monomorphism if (and only if) each map $f : (X, A) \to (K(G, r), p_0)$ with homotopically trivial restrictions on $(F_i \cup A, A)$ for any i is homotopically trivial globally.

Let $I = [0, 1]$, and let set

$$X_1 = X \times I, \quad A_1 = (A \times I) \cup (X \times \{0\}) \cup (X \times \{1\}), \quad F_i' = F_i \times I,$$

and define $f_1 : A_1 \to K(G, r)$ by $f_1|_{X \times \{0\}} = f$ and $f_1|_{(X \times \{1\}) \cup (A \times I)} = p_0 = \text{const}$. Then applying Theorem 1 to the case X_1, A_1, F_i' and f_1, we get the desired result.

Indeed, the condition $\dim_G(F_i \cap F_j) \leqq n-1$ implies $\dim_G(F_i' \cap F_j') \leqq n$. Then $\dim_{G_k}(F_i' \cap F_j') \leqq n$, where $G_k = \pi_k[K(G, r)]$, since

$$\pi_k[K(G, r)] = \begin{cases} G & \text{for } k = r, \\ 0 & \text{for } k \neq r \end{cases}$$

by definition of the Eilenberg–McLane complexes. Thus we may refer to Theorem 1 and get the following

Theorem 1′. *Let X be a compact space, let A be closed in X, and let* $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, *where F_i are closed subsets of $X \setminus A$ such that* $\dim_G(F_i \cap F_j) \leqq n-1$ *for* $i \neq j$. *Then the natural homomorphism of $H^r(X, A; G)$ into* $\prod_{i=1}^{\infty} H^r(F_i \cup A, A; G)$ *is a monomorphism for* $r \geqq n+1$.

Let us recall that $\dim_{\mathbb{Z}} X = \dim X$ for a finite-dimensional X. Then, by Hu's theorem for obstructions (cf. [2]), it is possible to deduce Theorem 1 from Theorem 1′ as well, in the situation $G = \mathbb{Z}$, $\dim X < \infty$.

Hereafter we shall obtain, by means of Theorem 1′, some results about strong Cantor manifolds.

Let us recall the definition of a strong Cantor n-manifold (see [3]).

The space C is called a *strong Cantor n-manifold* if for an arbitrary representation $C = \bigcup_{i=1}^{\infty} F_i$, where F_i are proper closed subsets of C, we have $\dim(F_i \cap F_j) \geqq n-1$ for some $i \neq j$.

C is called a *strong Cantor n-manifold with respect to a group G* if for any of the above mentioned representations we have $\dim_G(F_i \cap F_j) \geqq n-1$ for some $i \neq j$.

Clearly, if C is a strong Cantor n-manifold with respect to G, then it is a strong Cantor n-manifold as well. The first author has achieved some development of the theory of strong Cantor manifolds (cf. [4]).

Now we shall prove that Theorem 1′ implies the following results:

Theorem 2. *Each compact space X with* $\dim_G X = n$ *contains a strong Cantor n-manifold (with respect to G).*

Theorem 3. *Let the k-dimensional cycle z^k* $(\mathrm{mod}\, G)$ *be irreducibly linked with the compact space X in some n-ball $\mathbb{B}^n$. Then X is a strong Cantor $(n-k-1)$-manifold with respect to G.*

Theorem 4. *The ball $\mathbb{B}^n$ is a strong Cantor n-manifold with respect to any group G.*

Theorem 5. *Each absolute boundary in $\mathbb{R}^n$ is a strong Cantor $(n-1)$-manifold with respect to any G. (Recall that C is an absolute boundary in $\mathbb{R}^n$ if it is a common boundary of at least two open domains in $\mathbb{R}^n$.)*

Proof of Theorem 2. The equality $\dim_G X = n$ means that there is a closed subset $A \subset X$ such that $H^n(X, A; G) \neq 0$, where n is the greatest number with this property (cf. [5]). By Zorn's lemma we may find a minimal closed subset $F \subset X$ such that $H^n(F, A \cap F; G) \neq 0$.

We shall show that F is a strong Cantor n-manifold with respect to G. Suppose this is not true, i.e. $F = \bigcup_{i=1}^{\infty} F_i$, where F_i are proper closed subsets of F such that $\dim_G(F_i \cap F_j) \leqq n-2$ for $i \neq j$. Then $H^n(F_i, A \cap F_i; G) = 0$ by the minimal property of F. According to Theorem 1′ the natural homomorphism

$$H^n(F, A \cap F; G) \to \prod_{i=1}^{\infty} H^n(F_i, A \cap F_i; G)$$

is a monomorphism, which is a contradiction. (Here we make use of the fact that $H^n(F, A \cap F; G) = H^n(F \cup A, A; G)$ for $n > 0$.)

Remark. Using the fact that the covering dimension "dim" equals "$\dim_{\mathbb{Z}}$" in the finite-dimensional case (cf. [5]), we obtain a result of the first author about strong Cantor manifolds (cf. [3]).

Proof of Theorem 3. Recall that the k-cycle z^k, lying in $\mathbb{B}^n \setminus X$, is irreducibly linked with X in $\mathbb{B}^n$ if z^k is not homologous to zero in $\mathbb{B}^n \setminus X$, but for any proper closed subset $X' \subset X$ z^k is homologous to zero in $\mathbb{B}^n \setminus X'$.

Let $p : \mathbb{B}^n \to S^n$ be a map sending $\partial\mathbb{B}^n$ into a point p_0 and the interior of $\mathbb{B}^n$ homeomorphically onto $S^n \setminus \{p_0\}$. Then it is easy to see that

$$H_k(\mathbb{B}^n \setminus X, \partial\mathbb{B}^n \setminus X) = H_k(S^n \setminus p(X))$$

for $k > 0$ and

$$H_0(\mathbb{B}^n \setminus X, \partial\mathbb{B}^n \setminus X) = \widetilde{H}_0(S^n \setminus p(X)),$$

where $\widetilde{H}_0$ is the reduced homology group.

Suppose the assertion of the theorem is not true, i.e. $X = \bigcup_{i=1}^{\infty} X_i$, where $\dim_G(X_i \cap X_j) \leqq n-k-3$ (for $i \neq j$). Then $\dim_G(p(X_i) \cap p(X_j)) \leqq n-k-3$ as well. Consider the commutative diagram

$$\begin{array}{ccccc} H_k(\mathbb{B}^n \setminus X) & \longrightarrow & H_k(\mathbb{B}^n \setminus X, \partial\mathbb{B}^n \setminus X) = H_k(S^n \setminus p(X)) & \longrightarrow & \prod_{i=1}^{\infty} H_k(S^n \setminus p(X_i)) \\ & & \downarrow & & \downarrow \\ & & H^{n-k-1}(p(X)) & \xrightarrow{q} & \prod_{i=1}^{\infty} H^{n-k-1}(p(X_i)), \end{array}$$

where the vertical maps are the isomorphisms furnished by Alexander duality (cf. [1], p. 381).

Then, analyzing the image of the element $[z^k] \in H_k(\mathbb{B}^n \setminus X)$ and taking into account that q is a monomorphism by Theorem 1′, and having in view the minimal property of X, we arrive to a contradiction as above. (If $k = 0$, we have to consider the reduced groups $\widetilde{H}_0(S^n \setminus p(X))$ at the first row of the diagram.)

Theorems 4 and 5 follow immediately from Theorem 3.

Let us note that Theorem 1 implies directly that $\mathbb{B}^n$ is a strong Cantor n-manifold. To prove this, one has to suppose the contrary and to apply Theorem 1 to the situation $X = \mathbb{B}^n$, $A = \partial\mathbb{B}^n$, $f = \mathrm{id} : \partial\mathbb{B}^n \to \partial\mathbb{B}^n$.

Further the paper is aimed at the proof of Theorem 1.

Lemma 1. *Let X be a compact space, $A \subset X$ be a closed subset, and let $f : A \to Y$ map A into the CW-complex Y. Suppose that f is extendable over both $A \cup F_1$ and $A \cup F_2$ for some closed F_1, F_2. Then there exist a neighbourhood $N(A)$ of A and an extension $f' : N(A) \to Y$ of f, which is still extendable over $N(A) \cup F_1$ and $N(A) \cup F_2$.*

This technical lemma is quite elementary and follows immediately from Borsuk's lemma about extensions of homotopies (cf. [6], p. 231). It remains valid for any Y which is ANE (Absolute Neighbourhood Extensor in the class of normal spaces).

Lemma 2. *Let X be a compact space and let $A \subset X$ be a closed subset such that $X \setminus A = \bigcup\limits_{i=1}^{m} F_i$, where F_i are closed in $X \setminus A$. Let furthermore Y be an n-connected CW-complex and suppose that $\dim_{G_k}(F_i \cap F_j) \leqq n$, where $G_k = \pi_k(Y)$ for any $k \geqq n+1$. Then a map $f : A \to Y$, which is extendable over $A \cup F_i$ for any i, can be extended over X.*

Proof. The first obstruction for extending the map f lies in $H^{n+2}(X, A; \pi_{n+1}(Y))$ (cf. [1], p. 574). The image of this first obstruction in $H^{n+2}(F_i \cup A, A; \pi_{n+1}(Y))$ is the first obstruction for extending f over $F_i \cup A$, which is trivial, since f can be extended over $F_i \cup A$ by hypothesis. But, as we have already noticed, the group $H^{n+2}(X, A; \pi_{n+1}(Y))$, in virtue of the Meyer–Vietoris sequence, is naturally isomorphic to $\prod\limits_{i=1}^{m} H^{n+2}(F_i \cup A, A; \pi_{n+1}(Y))$. Hence, this first obstruction is trivial. We have the same situation for the second, third and higher obstructions. Therefore, there is no obstruction to the extension of f over X.

To go further, we need the following construction.

Let X be a locally compact space and let $\sigma = \{F_i\}_{i=1}^{\infty}$ be a covering of X by closed sets. For any $A \subset X$ let us set

$$A(\sigma) = A \setminus \bigcup_{i=1}^{\infty} \mathrm{Int}_A(F_i \cap A).$$

It follows from Baire's theorem that $A \neq A(\sigma)$ for any non-empty closed set A. We may define by transfinite induction a decreasing transfinite sequence of closed sets B_α as follows:

$$B_1 = X, \quad B_\alpha = B_{\alpha-1}(\sigma) \text{ for a non-limit ordinal } \alpha,$$
$$B_\alpha = \bigcap_{\beta<\alpha} B_\beta \text{ for a limit ordinal } \alpha.$$

We call the family $\{B_\alpha\}$ *filtration of X generated by σ*. Furthermore we shall have to manage with the following situation: X is a compact space, A is its closed subset, and $\sigma = \{F_i\}_{i=1}^{\infty}$ is a covering of $X \setminus A$ by closed in $X \setminus A$ sets F_i. The main property of the filtration of $X \setminus A$ generated by σ is the following:

(P) For any neighbourhood N of $A \cup B_{\alpha+1}$ in X there exists such an m that $A \cup B_\alpha$ is contained in the union $N \cup F_1 \cup \ldots \cup F_m$.

Indeed, $B_\alpha \setminus N$ is a compact space covered by the interiors of F_i with respect to B_α, so we may choose a finite subcover and take m greater than the maximal index of elements of this subcover.

Lemma 3. *Let* $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, *where* A *is closed in* X *and* F_i *are closed in* $X \setminus A$, *and let* $\dim_{G_k}(F_i \cap F_j) \leqq n$ *for* $i \neq j$, $k \geqq n+1$, *where* $G_k = \pi_k(Y)$ *for a given* n*-connected CW-complex* Y. *Suppose that the map* $f : A \to Y$ *is extendable over both* $A \cup F_1$ *and* $\left[A \cup \bigcup_{i=2}^{\infty} F_i\right]$. *Then* f *is extendable over* X.

Proof. Let $\{B_\alpha\}$ be the filtration of $X \setminus A$ generated by $\{F_i\}_{i=1}^{\infty}$. Let α be the smallest ordinal such that it is still possible to construct a continuous map $f_\alpha : A \cup B_\alpha \to Y$ which is extendable over both F_1 and $\left[\bigcup_{i=2}^{\infty} F_i\right]$. It follows from the compactness of X and from Lemma 1 that α cannot be a limit ordinal. Hence there exists $\alpha - 1$, or $\alpha = 1$. The second case concludes the proof. It is sufficient now to lead the first case to a contradiction. Lemma 1 provides us with an extension $f'_\alpha : N(A \cup B_\alpha) \to Y$ over some neighbourhood of $A \cup B_\alpha$, which is still extendable over both F_1 and $\left[\bigcup_{i=2}^{\infty} F_i\right]$. By property (P) of the filtration we have $B_{\alpha-1} \subset N(A \cup B_\alpha) \cup \bigcup_{i=2}^{m} F_i$ for some m.

To obtain the needed contradiction, it suffices to prove that f'_α is extendable over $\bigcup_{i=1}^{m} F_i$. According to Lemma 2 it is sufficient to prove that f'_α is extendable over F_i for any i. But this is true by the hypothesis.

Proof of Theorem 1. Let $X \setminus A = \bigcup_{i=1}^{\infty} F_i$ and $\{B_\alpha\}$ be the filtration generated by $\{F_i\}_{i=1}^{\infty}$. Suppose that α is the smallest ordinal such that the extension of f on $A \cup B_\alpha$ is possible. It is possible to extend f on some neighbourhood $N(A \cup B_\alpha)$. If we assume that α is a limit ordinal, then $B_\alpha = \bigcap_{\beta<\alpha} B_\beta$ and in virtue of the compactness of X one may conclude that for some $\beta < \alpha$ we have $A \cup B_\beta \subset N(A \cup B_\alpha)$ in contradiction with the minimal property of α. If $\alpha = 1$, the theorem is proved. Suppose that $\alpha \neq 1$. Then $\alpha - 1$ exists and we have some extension $f' : N(A \cup B_\alpha) \to Y$. For any i we may extend f over F_i by hypothesis. According to Lemma 3 we may extend f over $F_i \cup B_\alpha$, and by Lemma 2 we may extend f over $\bigcup_{i=1}^{m} F_i \cup B_\alpha$ for any m. Therefore by the property (P) of the filtration we may extend f over $A \cup B_{\alpha-1}$ *in contradiction with the minimal property of* α.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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CLASSIFICATION OF DECOMPOSABLE MANIN TRIPLES AND SOLUTIONS OF THE CLASSICAL YANG-BAXTER EQUATION

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Тони Пантев, Васил Цанов. КЛАССИФИКАЦИЯ РАЗЛОЖИМЫХ ТРОЙНЫХ СИСТЕМ МАНИНА И РЕШЕНИЯ КЛАССИЧЕСКИХ УРАВНЕНИЙ ЯНГА-БАКСТЕРА

Получена классификация разложимых тройных систем Манина алгебр Ли над произвольной гладкой алгебраической кривой. Дается описание соответсвующих решений классических уравнений Янга-Бакстера. Приводятся примеры (и контрапримеры), выясняющие природу соответствия между общими тройными системами Манина и решениями классических уравнений Янга-Бакстера.

Tony Pantev, Vasil Tsanov. CLASSIFICATION OF DECOMPOSABLE MANIN TRIPLES AND SOLUTIONS OF THE CLASSICAL YANG-BAXTER EQUATION

We classify decomposable triple Manin systems over an arbitrary smooth algebraic curve and describe the corresponding solutions of the classical Yang-Baxter equation. We also give some examples and counter-examples clarifying the nature of the correspondence between general triple Manin systems and solutions of the classical Yang–Baxter equation.

1. INTRODUCTION

In the present paper we discuss solutions of the classical Yang-Baxter equation

$$\langle [r,r] \rangle \stackrel{\mathrm{def}}{=} [r^{1,2}(u_1,u_2),\ r^{1,3}(u_1,u_3)] + [r^{1,3}(u_1,u_3),\ r^{2,3}(u_2,u_3)] + [r^{1,2}(u_1,u_2),\ r^{2,3}(u_2,u_3)] = 0, \tag{1}$$

$$r^{1,2}(u_1, u_2) = -r^{2,1}(u_2, u_1),$$

where $r(u_1,\ u_2)$ is a rational function of the cartesian square $X \times X$ of an algebraic curve X, which takes values in the tensor square of a simple finite dimensional Lie algebra $\mathfrak{p}$, and, e. g., $r^{1,3}(u_1, u_3)$ is the superposition of the functions $r : X \times X \to \mathfrak{p} \otimes \mathfrak{p}$ and $\phi^{1,3} : p \otimes \mathfrak{p} \to U(\mathfrak{p})^{\otimes 3}$, defined by $\phi^{1,3}(a \otimes b) = a \otimes 1 \otimes b$ (see, e. g., [11] for details). After the works of Drinfeld and Cherednik (see [5–8] and [11]) it is general wisdom that there is a certain correspondence between solutions of the Yang-Baxter equations and systems of relevant infinite dimensional Lie algebras called **triple Manin systems** (see the definition bellow). We treat the triple Manin systems which are related to Lie algebras of types $\mathcal{A}(X) \otimes \mathfrak{p}$ and $\mathcal{R}(X) \otimes \mathfrak{p}$, where $\mathcal{A}(X)$, $\mathcal{R}(X)$ are respectively the ring of adels and the field of rational functions on X. The study of the relevant solutions of the equation (1) is extremely important for the classification of completely integrable systems of non-linear equations representable as Lax pairs by the method of Adler-Konstant-Simms (see, e. g., [10]).

Solutions of (1) which are meromorphic functions on $\mathbb{C} \times \mathbb{C}$ of type $r(u_1 - u_2)$ are classified in [1, 2]. It is also proved in [1, 3] that any solution of (1) meromorphic on the cartesian product of two discs is equivalent (on the germ level) to a solution of type $r(u_1 - u_2)$, where the function $r(u)$ can be extended to a meromorphic function on $\mathbb{C}$. This gives a complete classification of the local solutions of the equation (1). However, the classification of global solutions of (1) on an arbitrary algebraic curve X is an open problem. We discuss this problem from the viewpoint of triple systems of Manin.

In Sect. 3 we define and study the natural class of **decomposable Manin triple systems** to obtain their complete classification (compare with [5], where a very close class of triple Manin systems is defined and discussed). It turns out that they produce essentially only one solution of the equation (1).

In Sect. 4 we discuss other important examples. The systems of Example 1 are known ([8]), but treated in the present scheme they produce **rational** solutions of the Yang-Baxter equation which seem to have been overlooked. These solutions are **not** of type $r(u_1 - u_2)$, but can be reduced to this type by a local (trigonometric) change of variables to obtain the well-known trigonometric solutions. Example 2 is a Manin triple which does **not correspond** to any solution of the equation (1). Example 3 is known (but Remark 3 might be interesting).

2. BASIC CONCEPTS

Let X be a smooth algebraic curve (over the field of complex numbers $\mathbb{C}$), let $\mathcal{R}(X)$, $\mathcal{A}(X)$ be respectively the field of rational functions and the ring of adels on X (for general information on adels see, e. g., [4, Ch. VII, § 2]), and let $\mathfrak{p}$ be a simple Lie algebra (finite dimensional).

Definition 1. A **triple Manin system** is an ordered triple $(A, B, \mathfrak{g} \otimes \mathfrak{h})$, where A is a Lie algebra, $\mathfrak{g}$, $h \subseteq A$ are Lie subalgebras, B is a non-degenerate, symmetric, bilinear, ad-invariant form on A, $\mathfrak{g}$ and $\mathfrak{h}$ are isotopic for B, i. e.

$$B(\xi, \eta) = 0 \ \text{ for all } \ \xi, \eta \in \mathfrak{g} \ (\text{or } \xi, \eta \in \mathfrak{h}),$$

and $A = \mathfrak{g} \oplus \mathfrak{h}$ as a linear space.

We shall discuss Manin triples for which $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathcal{R}(X) \otimes \mathfrak{p}$. There is a general method due to Cherednik (see [5, 6, 11]) to construct Manin triples of this type, which we summarize briefly here. Denote by $\mathcal{A}^+(X)$ the ring of regular adels on the curve X, let $(\ ,\ )$ denote the Killing form of the algebra $\mathfrak{p}$, let $S \subseteq X$ be a non-empty subset of X and ω be a meromorphic 1-form on X. Let $\chi_S \in \mathcal{A}(X)$ be the characteristic function of the set S. For each subalgebra $A \subseteq \mathcal{A}(X)$ we denote by A_S the subalgebra $\chi_S.A$ of $A(X)$. Let A_S denote the Lie algebra $\mathcal{A}_S(X) \otimes \mathfrak{p}$ and H_S be the bilinear form on A_S defined by

$$H_S(f,g) = \sum_S \mathrm{Res}_x(f,g).\omega. \tag{2}$$

Obviously, H_S satisfies all the conditions required by the definition of a triple system.

The problem of constructing triple systems in this context amounts to finding a couple of H_S-isotropic subalgebras $\mathfrak{g}$, $\mathfrak{h}$ of A_S with trivial intersection such that the direct sum $\mathfrak{F} = \mathfrak{g} \oplus \mathfrak{h}$ is a subalgebra and the restriction of H_S is still non-degenerate on $\mathfrak{F}$.

Cherednik proposes that the algebra $\mathfrak{h}$ of the triple is chosen to be the subalgebra $A_S^+(X) \otimes \mathfrak{p}$, which is obviously isotropic, whence the problem reduces to finding a suitable complement $\mathfrak{g}$. It is essential (and convenient) for our purpose of looking for solutions

$$r \in \mathcal{R}(X \times X) \otimes (\mathfrak{p} \otimes \mathfrak{p})$$

of the equation (1) that the algebra $\mathfrak{g}$ be a subalgebra of $\mathcal{R}(X) \otimes \mathfrak{p}$. For each triple Manin system $(A, B, \mathfrak{g} \otimes \mathfrak{h})$ we define a map $\tilde{p} : \mathfrak{g} \to (\Lambda^2\mathfrak{h})^*$ with the formula

$$\tilde{p}(x)\,(\xi\Lambda\eta) = B(x, [\eta, \xi]). \tag{3}$$

A map $p : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ such that

$$B(p(x), \xi\Lambda\eta) = \tilde{p}(x)\,(\xi\Lambda\eta)$$

is called a **cocommutator** of $\mathfrak{g}$ (if it exists). One can check by a straightforward computation that p is a cocycle of $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $\Lambda^2\mathfrak{g}$ and that p satisfies the equation $p_2(p(x)) = 0$ for all $x \in \mathfrak{g}$, where

$$p_2(x\Lambda y) \stackrel{\text{def}}{=} p(x)\Lambda y - x\Lambda p(y)$$

(this is exactly the adjoin of the Jacobi identity, see, e. g., [11]). If it happens that the cocycle p is a coboundary, i. e. if

$$\partial(r) = p \tag{4}$$

for some $r \in C^0(\mathfrak{g}, \Lambda^2\mathfrak{g}) = \Lambda^2\mathfrak{g}$, then $\langle [r,r] \rangle$ is ad-variant, and with some luck we may expect that

$$\langle [r,r] \rangle = 0,$$

i. e. that r is a solution of the equation (1).

It is a rare occasion that the equation (4) be satisfied by some $r \in \Lambda^2\mathfrak{g}$ (this module is "too small"). In the situation we are treating, one looks for a solution of the equation (4) (and hence of the equation (1)) as a 0-chain in the $\mathfrak{g}$-module

$$\mathcal{B}(X) = \mathcal{R}(X \times X) \otimes (\mathfrak{p} \otimes \mathfrak{p})$$

of rational functions on the Cartesian square of X with values in $\mathfrak{p} \otimes \mathfrak{p}$.

From this moment on we treat only subalgebras of A_S as described above.

Definition 2. A Manin triple $(A_S, H_S, \mathfrak{g}\otimes\mathfrak{h})$ will be called **decomposable** iff

$$\mathfrak{g} = I \otimes \mathfrak{p}, \quad \mathfrak{h} = J \otimes \mathfrak{p},$$

where I, J are subalgebras of $\mathcal{R}(X)_S$, $\mathcal{A}^+(X)_S$, respectively.

Remark 1. It is easy to see that the classification of **decomposable** Manin triples amounts to the classification of triples (I, J, ω), where I, J are as above, $I \otimes J = A_S(X)$ and ω is a meromorphic differential on X such that the bilinear form

$$B(a,b) = \sum_S \mathrm{Res}_x a.b.\omega \tag{5}$$

is non-degenerate on $A_S(X)$ and vanishes identically on I and J. Indeed, as the Killing form on $\mathfrak{p}$ is non-degenerate, by suitable choice of coefficients the vanishing of (2) is reduced to the vanishing of (5).

Proposition 1. *Let (I, J, ω) be as in Definition 2. Then either*

i) $\mathbb{C} \subseteq J$ *(implying $J = \Pi_S \widehat{\mathcal{O}}_X = \mathcal{A}^+_S(X)$) or*

ii) $\mathbb{C} \subseteq I$ *(implying $J = (\Pi_{S\setminus\{y\}}\widehat{\mathcal{O}}_X) \times \mathfrak{M}_y$ for some $y \in S$), where $\mathfrak{M}_y$ is the maximal ideal of the local ring $\widehat{\mathcal{O}}_y$.*

Proof. We need two lemmas.

Lemma 1. $I \cap \mathcal{A}^+_S(X) \subseteq \mathbb{C}$.

Proof. Assume that there exists an $f \in I \cap A^+_S(X)$ which is not constant. Let $x_1, \ldots, x_n \in X$ be the poles of the differential ω and let k_i be the multiplicity of the pole x_i. The function

$$g \overset{\text{def}}{=} \Pi_i (f^2 - f(x_i) f)$$

belongs to I, because $f \in I$, and I is an algebra over $\mathbb{C}$. Obviously, $g(x_i) = 0$ for each i, but g does not vanish identically, because f is not constant. Thus one can conclude that there exists a $k \in \mathbf{N}$ such that $\mathrm{mult}_{X_i}(g^k) \geqq k_i$ for each i. For any $a \in \mathcal{A}^+_S(X)$ we have

$$B(a, g^k) = \sum_S \mathrm{Res}_x(a \cdot g^k \cdot \omega).$$

For each i $\mathrm{mult}_{X_i}(a \cdot g^k \cdot \omega) \geqq 0$, and hence $B(a, g^k) = 0$, because for each $x \in S \setminus \{x_1, \ldots, x_n\}$ we have $\mathrm{mult}_X(a \cdot g^k \cdot \omega) \geqq 0$ (as a, g and ω have no poles in $S \setminus \{x_1, \ldots, x_n\}$).

On the other hand, $g^k \in I$ and $g \neq 0$, whence there exists (Remark 1) an element $b \in J$ such that $B(b, g^k) \neq 0$, which is a contradiction. Thus we have $I \cap \mathcal{A}^+_S(X) \subseteq \mathbb{C}$.

Lemma 2. *J is an ideal of the ring $\mathcal{A}_S^+(X)$.*

Proof. Let $a \in \mathcal{A}_S^+(X)$, $b \in J$. By definition $\mathcal{A}_S(X) = I \oplus J$, so $a = f + a_1$ with unique $f \in I$ and $a_1 \in J$. But $a - a_1 = f \in \mathcal{A}_S^+(X)$, because a and a_1 are elements of $\mathcal{A}_S^+(X)$. Thus $f \in I \cap \mathcal{A}_S^+(X)$ and by Lemma 1 f is a constant. As J is an algebra over $\mathbb{C}$, we have $f.b \in J$ and $a.b = f.b + a_1.b$ is an element of J, i. e. J is an ideal of $\mathcal{A}_S^+(X)$.

Obviously, the field $\mathbb{C}$ may belong to only one of the algebras I, J.

i) Let $\mathbb{C} \subseteq J$. Then $J = \mathcal{A}_S^+(X)$, because $J \subseteq \mathcal{A}_S^+(X)$ is an ideal by Lemma 2.

ii) Let $\mathbb{C} \subseteq I$. We have $\mathcal{A}_S^+(X) \subseteq \mathcal{A}_S(X) = I \oplus J$ and by Lemma 1 we get $\mathcal{A}_S^+(X) = \mathbb{C} + J$. Because of $\mathbb{C} \subseteq I$ we have $\mathbb{C} \cap J = \{0\}$ and hence $\mathcal{A}_S^+(X) = \mathbb{C} \oplus J$. By Lemma 2 J is an ideal in $\mathcal{A}_S^+(X)$ and $\mathcal{A}_S^+(X)_{/J} \cong \mathbb{C}$, whence we conclude that J is a maximal ideal. Let $x \in S$, then $J_x \lhd \widehat{\mathcal{O}}_x$. But $\widehat{\mathcal{O}}_x$ is a local ring, hence we have two possibilities:

$$\text{either} \quad J_x \subseteq \mathfrak{M}_x \quad \text{or} \quad J_x = (\mathcal{A}_S^+(X))_x = \widehat{\mathcal{O}}_x.$$

If $J_x = \widehat{\mathcal{O}}_x$ for all $x \in S$, then $J = \mathcal{A}_S^+(X)$, which contradicts $\mathcal{A}_S^+(X) = \mathbb{C} \oplus J$. Thus there exists an $y \in S$ such that $J_y \subseteq \mathfrak{M}_y$, whence $J \subseteq \Pi_{S\setminus\{y\}} \widehat{\mathcal{O}}_x \times \mathfrak{M}_y \lhd \mathcal{A}_S^+(X)$. But J is a maximal ideal, i.e. $J = \Pi_{S\setminus\{y\}} \widehat{\mathcal{O}}_x \times \mathfrak{M}_y$, which concludes the proof of Proposition 1.

In the following theorem we keep the notation of Proposition 1.

Theorem 1. *Let (I, J, ω) be as in Definition 2. Then:*

i) *If $\mathbb{C} \subseteq I$, then there exists a function $u \in I$ which is injective on the set S with the following properties:*

a) *$\omega = dz$, where $z = u^{-1}$;*

b) *The multiplicity of z at all points of S is 1;*

c) *Let $u_x = (z - z(x))^{-1}$. The algebra I is generated by the functions u_x for all x of S.*

ii) *If $\mathbb{C} \subseteq J$, then there exists a function $u \in I$ which is injective on the set S with a pole at the point y such that:*

a) *$\omega = du$;*

b) *The multiplicity of u at all points of $S \setminus \{y\}$ is 1;*

c) *The algebra I is generated by the functions 1, u, $(u - u(x))^{-1}$ for all $x \in S \setminus \{y\}$.*

Proof. Case ii). The fact that $\mathcal{A}_S(X) = I \oplus J$ yields for each $x \in S$ the existence of a function

$$\phi_x \in I$$

which has one simple pole in S at the point x.

By Lemma 1 each function $f \in I$ has at least one pole in S. Thus each x determines ϕ_x up to a constant. Obviously, the algebra I is generated by the functions 1, ϕ_x for all $x \in S$. By Lemma 1 for a fixed $x \in S$ the function

$$(\phi_x - \phi_x(Y)) \cdot (\phi_y - \phi_y(X)) \in I$$

is a non-zero constant, whence ϕ_x is a non-degenerate Möbius transformation of ϕ_y. We set

$$u \stackrel{\text{def}}{=} \phi_y,$$

so now the algebra I is generated by

$$1,\ u,\ (u - u(x))^{-1}$$

for all $x \in S \setminus \{y\}$. We expand the differential ω at each point $x \in S$ in power series. At each point $x \in S \setminus \{y\}$ the differential ω is regular. Indeed, fix $x \in S \setminus \{y\}$ and denote by z_x the meromorphic function $u - u(x)$. The function z_x is a local parameter in a neighbourhood of the point x.

Let the Taylor expansion of ω in the parameter z_x be

$$\omega = (\alpha_k . z_x^{-k} + \cdots) . dz_x, \quad k \geqq 1,\ \alpha_k \neq 0.$$

Define the adels a and b by

$$a_p \stackrel{\text{def}}{=} \chi_x(p) . z_x^{k-1}, \quad b_p \stackrel{\text{def}}{=} \chi_x(p)$$

for a point $p \in S$, where $\chi_x(\cdot)$ is the characteristic function of the set $\{x\}$. As $J_x = \widehat{\mathcal{O}}_x$, we know that $a,\ b \in J$. By Remark 1 the bilinear form B is isotopic on J, i. e. $B(a, b) = \alpha_k = 0$, which is a contradiction.

To estimate the order of the pole of ω at the point y, consider the Taylor expansion of ω at y in the local parameter $z = u^{-1}$:

$$\omega = (\alpha_{-k} . z^{-k} + \cdots + \alpha_{-1} . z^{-1} + \alpha_0 + \cdots) . dz.$$

The adel a defined by $a_p = \chi_x(p) . z$ obviously belongs to the algebra $J = \mathfrak{M}_y \times \Pi_{S \setminus \{y\}} \widehat{\mathcal{O}}_x$. Also by Remark 1 we obtain $B(a, a^s) = 0$ for each $s \geqq 1$. Direct computation of $B(a, a^s)$ gives

$$B(a, a^s) = \alpha_{-(s+2)} = 0,$$

whence the order k of the pole of ω at y is estimated by $k \leqq 2$.

Assume that $k < 2$. Then the Taylor expansion is

$$\omega = (\alpha_0 + \cdots) . dz$$

as $1 \in I$, using Remark 1, we obtain

$$\alpha_{-1} = B(1, 1) = 0.$$

Let α_l be the first non-vanishing coefficient. We compute $B(u^{l+1}, 1)$ and obtain by Remark 1 (note that $1, u \in I$) that

$$\alpha_l = B(u^{l+1}, 1) = 0,$$

which is a contradiction. Thus $\alpha_{-2} \neq 0$ and normalizing u we may presume that $\alpha_{-2} = -1$. The same argument as for α_{-1} gives $\alpha_n = 0$ for all $n \geqq 0$, whence

$$\omega = -z^{-2} . dz = du,$$

which settles case ii).

Case i). Let $x \in S$, $\phi_x \in I$, and z_x be as above. The differential ω has no poles in S. Indeed, if the Taylor expansion of ω at some point $x \in S$ were

$$\omega = (\alpha_{-k}.z_x^{-k} + \cdots).\, dz_x \text{ with } k \geqq 1 \text{ and } \alpha_{-k} \neq 0,$$

then for the adel $a \in \mathcal{A}_S^+(X) = J$ defined by $a_y = \chi_x(y).z_x^{k-1}$ we would have $\alpha_{-k} = B(a, 1) = 0$, which is a contradiction. So, for each $x \in S$ we have

$$\omega = (\alpha_0^x + \alpha_1^x.z_x + \cdots + \alpha_k^x.z_x^k + \cdots).\, dz_x.$$

But $\phi_x \in I$ and it has no pole in $S \setminus \{x\}$. Hence

$$\alpha_s^x = B(\phi_x, \phi_x^s) = 0$$

for all $s \geqq 1$. Thus in a neighbourhood of x we have

$$\omega = dz_x$$

(normalizing ϕ_x suitably). Let $y \in S$ be an arbitrary point. Obviously, the functions $\phi_x = z_x^{-1}$ $(x \in S)$ generate I. Denote $u = z_y$. The differentials ω and du are meromorphic sections of $\Omega^1(X)$, whence $f = \frac{\omega}{du}$ is a global meromorphic function on X. But ω and du coincide in an open neighbourhood of y, hence $f = 1$ and

$$\omega = du.$$

Similarly, $\omega = dz_x$ for each $x \in S$. So $dz_x = du$ for each x, i. e. $z_x = u + a_x$, where a_x is a constant. But $z_x(x) = 0$, whence $a_x = u(x)$ and the algebra I is generated by $u_x = (u - u(x))^{-1}$ $(x \in S)$.

Remark. Repeating the above arguments at each point $x \in S$, one can prove under natural restrictions that if we assume the differential ω to be adel (not a global meromorphic differential on X), no real generalization is obtained.

Corollary 1. *If $\mathfrak{F} = (A_S, H_S, \mathfrak{g} \oplus \mathfrak{h})$ is a decomposable Manin triple on a curve X, then there exists a decomposable Manin triple $\mathfrak{F}' = (A', H', \mathfrak{g}' \oplus \mathfrak{h}')$ on $\mathbb{P}^1$ such that $\mathfrak{F}$ is the pullback of $\mathfrak{F}'$ by the function $u : X \to \mathbb{P}^1$ defined in Theorem* 1.

Proof. In the notations of Theorem 1 we have: the algebra I is a direct sum $I = \bigoplus_S u_x.(\mathbb{C}.u_x)$ (as a linear space) in the case i), and $I = \mathbb{C} \oplus \mathbb{C}.u \oplus (\bigoplus_{S \setminus \{y\}} \mathbb{C}.(u - u(x))^{-1})$ in the case ii). This remark makes the corollary obvious.

The Killing form on $\mathfrak{p}$ is represented by a symmetric, non-degenerate, ad-invariant element $k \in \mathfrak{p}^* \otimes \mathfrak{p}^*$. Let K be the element of $\mathrm{Hom}(\mathfrak{p}, \mathfrak{p}^*)$, canonically corresponding to k. If $K^{-1} \in \mathrm{Hom}(\mathfrak{p}^*, \mathfrak{p})$ is the inverse linear map of K, then there is a non-degenerate, symmetric, ad-invariant element $k^{-1} \in \mathfrak{p} \otimes \mathfrak{p}$, canonically corresponding to K^{-1}. For an arbitrary function $f \in \mathcal{R}(X)$ we denote by r_f the $\mathfrak{p} \otimes \mathfrak{p}$ valued function on $X \times X$ defined by the formula

$$r_f(x, y) = (f(x) - f(y))^{-1}.k^{-1}. \tag{6}$$

Let ∂ be the differential of the cochain complex $C^*(\mathfrak{g}, \mathcal{B})$. In the following theorem we describe the cocommutators $p : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$ of the Lie algebras appearing in the current context.

Theorem 2. *Let* $(A_S, H_S, \mathfrak{g} \oplus \mathfrak{h})$ *be a decomposable Manin triple, and let* (I, J, ω) *be the corresponding triple as in Remark* 1. *Then*:

i) *In the case* i) *of Theorem* 1 *we have* $p(x) = \partial(r_z)(x)$ *for all* $x \in \mathfrak{g}$;

ii) *In the case* ii) *of Theorem* 1 *we have* $p(x) = -\partial(r_u)(x)$ *for all* $x \in \mathfrak{g}$, *where* $u,\ z \in \mathcal{R}(X)$ *are determined in Theorem* 1.

Proof. We shall prove only the case i). The case ii) can be proved similarly.

We have $I = \bigoplus_S u_x.(\mathbb{C}.u_x)$, hence the functions of the set

$$\{u_x^k \otimes a \mid k \geqq 1,\ x \in S,\ a \in \mathfrak{p}\}$$

span $\mathfrak{g}$ as a linear space. Denote by $a.z_x^1$ the adel $(a.z_x^1)_y = \delta_y(x).(a \otimes z_x^1)$. It is obvious that

$$a.z_x^1 \in \mathcal{A}_S^+(X) \otimes \mathfrak{p} = \mathfrak{h}.$$

The form H_S is a non-degenerate on $\mathcal{A}_S(X)$, so it is sufficient to prove that

$$H_S(p(u_x^k \otimes a),\ b.z_x^n \otimes c.z_x^m) = H_S(\partial(r_z)(u_x^k \otimes a),\ b.z_x^n \otimes c.z_x^m) \text{ for each } x \in S,$$

$a, b, c \in \mathfrak{p}$, $k \geqq 1$, $n, m \geqq 0$. Let us define for each $x \in S$ the functions v_x, w_x, α_x, $\beta_x \in M(X \times X)$:

$$\begin{array}{cc} v_x : X \times X \to \mathbb{C} & w_x : X \times X \to \mathbb{C} \\ (p, q) \to u_x(p) & (p, q) \to u_x(q) \\ \alpha_x : X \times X \to \mathbb{C} & \beta_x : X \times X \to \mathbb{C} \\ (p, q) \to z_x(p) & (p, q) \to z_x(q). \end{array}$$

In this notations we have $r_z = (\alpha - \beta).k^{-1}$ as a meromorphic function on $X \times X$ and if $k^{-1} = \sum_i a_i \otimes b_i \in \mathfrak{p} \otimes \mathfrak{p}$, then we compute $\partial(r_z)(u_x^k \otimes a)$ as follows:

$$\begin{aligned} \partial(r_z)(u_x^k \otimes a) &= \sum_i \partial((\alpha - \beta)^{-1}.a_i \otimes b_i)(u_x^k \otimes a) \\ &= \sum_i (v_x^k.(\alpha - \beta)^{-1}.\mathrm{ad}_a a_i \otimes b_i + w_x^k.(\alpha - \beta)^{-1}.a_i \otimes b_i). \end{aligned}$$

But k^{-1} is an ad-invariant element of $\mathfrak{p} \otimes \mathfrak{p}$, whence $\mathrm{ad}_a k^{-1} = 0$, and whence $\sum_i \mathrm{ad}_a a_i \otimes b_i = -\sum_i a_i \otimes \mathrm{ad}_a b_i$.

Thus

$$\begin{aligned} \partial(r_z)(u_x^k \otimes a) &= (v_x^k - w_x^k).(\alpha - \beta)^{-1}.\sum_i \mathrm{ad}_a a_i \otimes b_i \\ &= (v_x - w_x).(v_x^{k-1} + \cdots + w_x^{k-1}).(\alpha - \beta)^{-1}.\sum_i \mathrm{ad}_a a_i \otimes b_i. \end{aligned}$$

By definition we have $\alpha = v^{-1}$, $\beta = w^{-1}$, and $z_x = z - z(x)$, hence $u.u_x^{-1} = (z - z(x)).z^{-1} = 1 - z(x).u$, and

$$(v_x - w_x).(\alpha - \beta)^{-1} = (v_x - w_x).(w - v)^{-1}.wv = -v_x w_x.vw.(vw)^{-1} = -v_x w_x.$$

Finally, we obtain

$$\partial(r_z)(u_x^k \otimes a) = -(v_x^k w_x + \cdots + w_x^k v_x).\sum_i \mathrm{ad}_a a_i \otimes b_i.$$

Let us compute

$$H_S(\partial(r_z)(u_x^k\otimes a),\ b.z_x^n\Lambda c.z_x^m) = -H_S(\sum_{i,s} v_x^s w_x^{k+1-s}.\mathrm{ad}_a a_i\otimes b_i,\ b.z_x^n\Lambda c.z_x^m)$$

$$= -\frac{1}{2}.\sum_{i,\ s}(\mathrm{Res}_x(z_x^{n-s}.dz_x).\mathrm{Res}_x(z_x^{m-(k+1)+s}.dz_x)$$

$$\times(\mathrm{ad}_a a_i,\mathfrak{b}).(b_i,c) - \mathrm{Res}_x(z_x^{m-s}.dz_x).\mathrm{Res}_x(z_x^{n-(k+1)+s}.dz_x)\times(\mathrm{ad}_a a_i,c).(b_i,b))$$

$$= -\frac{1}{2}.\sum_{i,\ s}(\delta_{n,\ s+1}.\delta_{m,\ k+2-s}.(\mathrm{ad}_a a_i,b).(b_i,c) - \delta_{m,\ s+1}.\delta_{n,\ k+2-s}.(\mathrm{ad}_a a_i,c).(b_i,b))$$

$$= -\frac{1}{2}\delta_{n+m,k+1}.\sum_i((\mathrm{ad}_a a_i,b).(b_i,c) - (\mathrm{ad}_a a_i,c).(b_i,b)).$$

Using that $\mathrm{ad}_a k^{-1}=0,\ \forall a\in\mathfrak{p}$, we write

$$H_S(\partial(r_z)(u_x^k\otimes a),b.z_x^n\Lambda c.z_x^m)$$

$$= -\frac{1}{2}.\delta_{n+m,k+1}.\sum_i((\mathrm{ad}_a a_i,b).(b_i,c)+(a_i,c).(\mathrm{ad}_a b_i,b)).$$

As k^{-1} is a symmetric tensor and the Killing form (,) is an ad-invariant, we can write the identities

$$-\sum_i((\mathrm{ad}_a a_i,b).(b_i,c)+(a_i,c).(\mathrm{ad}_a b_i,b))$$

$$=\sum_i((a_i,\mathrm{ad}_a b).(b_i,c)+(a_i,c).(b_i,\mathrm{ad}_a b))$$

$$=2.(k^{-1},\mathrm{ad}_a b\otimes c).$$

In such a way we obtain

$$H_S(\partial(r_z)(u_x^k\otimes a),b.z_x^n\Lambda c.z_x^m) = \delta_{n+m,k+1}.(k^{-1},\mathrm{ad}_a b\otimes c).$$

For $p(u_x^k\otimes a)$ we have by definition

$$H_S(p(u_x^k\otimes a),\ b.z_x^n\Lambda c.z_x^m) = H_S([u_x^k\otimes a,b.z_x^n],c.z_x^m) = H_S(z_x^{n-k}\otimes[a,b],c.z_x^m)$$

$$=\delta_{n+m,k+1}.(\mathrm{ad}_a b,c).$$

But $(x\otimes y,a\otimes b)\overset{\mathrm{def}}{=}(x,a).(y,b)=k\,(x,a).k\,(y,b)$, consequently

$$(k^{-1},\ \mathrm{ad}_a b\otimes c) = \langle k^{-1}\llcorner(k\llcorner\mathrm{ad}_a b),\ K(c)\rangle$$

$$=\langle K^{-1}\circ K(\mathrm{ad}_a b),K(c)\rangle = \langle\mathrm{ad}_a b,\ K(c)\rangle = k\,(\mathrm{ad}_a b,c) = (\mathrm{ad}_a b,c)$$

(here $\langle\ ,\ \rangle$ is the natural pairing of $\mathfrak{p}$ and $\mathfrak{p}^*$).
Thus

$$H_S(\partial(r_z)(u_x^k\otimes a),b.z_x^n\Lambda c.z_x^m) = \delta_{n+m,k+1}.(k^{-1},\ \mathrm{ad}_a b\otimes c)$$

$$=\delta_{n+m,k+1}.(\mathrm{ad}_a b,c) = H_S(p(u_x^k\otimes a),\ b.z_x^n\Lambda c.z_x^m),$$

i.e. $p(x)=\partial(r_z)(x)$ for each $x\in\mathfrak{g}$.

It is obvious that any two solutions of the equation (4) with a fixed cocommutator p differ by an element c of the module $\mathcal{B}(X)$ such that $\partial(c) = 0$, i.e. by elements of $H^0(\mathfrak{g}, \mathcal{B})$. Thus the problem of uniqueness of solutions for the equation (4) is solved by the next proposition.

Proposition 2. *In the notations of Theorem* 1 *we have*

$$H^0(\mathfrak{p}, \mathcal{B}(X)) = 0. \tag{7}$$

Proof. Let $\mathfrak{p}_0$ be a Cartan subalgebra of $\mathfrak{p}$ and let $\Pi = \{\alpha_1, \ldots, \alpha_\Gamma\}$ be a fixed system of simple roots for $\mathfrak{p}_0$. Let $\{e_\alpha^-\}_{\alpha\in\Delta^-}$, $\{e_i^0\}_{i=1,\ldots,\Gamma}$, $\{e_\beta^+\}_{\beta\in\Delta^+}$ be the corresponding Weyl base. We know that it satisfies the following relations:

$$(*) \qquad \begin{gathered} [e_i^0, e_\alpha^-] = \alpha(e_i^0).e_\alpha^-, \quad \forall\alpha\in\Delta^-, \\ [e_i^0, e_\mu^+] = \mu(e_i^0).e_\mu^+, \quad \forall\mu\in\Delta^+, \\ [e_i^0, e_j^0] = 0, \quad \forall i,j = 1,\ldots,r, \\ [e_{-\alpha}^-, e_\alpha^+] = \sum_i \alpha(e_i^0).e_i^0, \quad \forall\alpha\in\Delta^+, \\ [e_\alpha^\pm, e_\beta^\pm] = \begin{cases} N_{\alpha\beta}.e_{\alpha+\beta}^\pm, & \alpha+\beta\neq 0\in\Delta, \\ 0, & \alpha+\beta\neq 0\in\Delta. \end{cases} \end{gathered}$$

Hence the base of $\mathcal{B}(X)$ over the field $\mathcal{R}(X\times X)$ is

$$\begin{array}{lll} e_\alpha^-\otimes e_\beta^-, & e_\alpha^-\otimes e_i^0, & e_\alpha^-\otimes e_\mu^+, \\ e_i^0\otimes e_\alpha^-, & e_i^0\otimes e_j^0, & e_i^0\otimes e_\mu^+, \\ e_\lambda^+\otimes e_\alpha^-, & e_\lambda^+\otimes e_j^0, & e_\lambda^+\otimes e_\mu^+ \end{array}$$

for each $\alpha,\ \beta\in\Delta^-,\ i,j = 1,\ldots,r,\ \lambda,\ \mu\in\Delta^+$.

Let $A\in\mathcal{B}(X)$ be an arbitrary element and $f_{\alpha\beta}^{--},\ldots,f_{\lambda\mu}^{++}\in\mathcal{R}(X\times X)$ be the coefficients of A of the corresponding base element (e. g., $e_\alpha^-\otimes e_\beta^-$ and so on). Assume that $A\in Z^0(\mathfrak{g},\mathcal{B}(X)) = H^0(\mathfrak{g},\mathcal{B}(X))\subseteq\mathcal{B}(X)$. Let $u\in\mathfrak{g}$ be non-constant (there exists such an element u by Theorem 1). We have

$$\partial(A) = 0, \quad \partial(A)(u^k\otimes a) = 0.$$

By the relations $(*)$ we obtain for the coefficients of $\partial(A)(u^k\otimes e_t^0)$ the following expressions:

— for the coefficient in front of $e_\alpha^-\otimes e_\beta^-$:

$$\tilde{f}_{\alpha\beta}^{--}(p,q) = \alpha(e_t^0).f_{\alpha\beta}^{--}(p,q).u^k(p) + \beta(e_t^0).f_{\alpha\beta}^{--}(p,q).u^k(q), \quad \forall\alpha,\beta\in\Delta^-;$$

— for the coefficient in front of $e_\alpha^-\otimes e_i^0$:

$$\tilde{f}_{\alpha i}^{-0}(p,q) = \alpha(e_t^0).f_{\alpha i}^{-0}(p,q).u^k(p), \quad \forall\alpha\in\Delta^-,\ \forall i = 1,\ldots,r;$$

— the coefficient in front of $e_i^0\otimes e_j^0$ vanishes identically, and so on.

If $\partial(A) = 0$, then all the coefficients of $\partial(A)(u^k\otimes e_t^0)$ vanish identically on $X\times X$. Consider for instance $\alpha(e_t^0).u^k(p).f_{\alpha\beta}^{-0}(p,q) = 0$ for each $k\geqq 1,\ t = 1,\ldots,r$.

For each α, $\exists t : \alpha(e_t^0) \neq 0$ (because α is a non-zero root of $\mathfrak{p}_0$ and $e_1^0, \dots, e_r^0$ are a base of $\mathfrak{p}_0$). Then for a suitable t we have $u(p).f_{\alpha i}^{-0}(p,q) = 0$, $\forall (p,q) \in X \times X$, i. e. $f_{\alpha i}^{-0} \equiv 0$. Similarly, $f_{i\beta}^{0-} = f_{i\mu}^{0+} = f_{\lambda i}^{+0} = 0$, $\forall i = 1, \dots, r$; $\forall \alpha \in \Delta^-$; $\forall \lambda, \mu \in \Delta^+$. Also $f_{\alpha\beta}^{--}(p,q).(\alpha(e_t^0).u^k(p) + \beta(e_t^0).u^k(q)) = 0$, $\forall k \geqq 1$, $t = 1, \dots, r$; $\forall (p,q) \in X \times X$.

Let

$$D(u) = \{(p,q) \in X \times X \mid u(p) = u(q)\} \cup \operatorname{supp}(\operatorname{div}_{X\times X}(u)).$$

If we assume that $f_{\alpha\beta}^{--}(p,q) \neq 0$ for some point, then $f_{\alpha\beta}^{--}(p,q) \neq 0$ in an open subset of $X \times X$. The set $D(u)$ is closed, hence $f_{\alpha\beta}^{--}(p,q) \neq 0$ in an open subset of $X \times X \setminus D(u)$. Setting $k = 1, 2$, we obtain for each (p,q) in an open subset of $X \times X \setminus D(u)$:

$$\left\| \begin{aligned} &\alpha(e_t^0).u(p) + \beta(e_t^0).u(q) = 0, \\ &\alpha(e_t^0).u^2(p) + \beta(e_t^0).u^2(q) = 0, \quad \forall t = 1, \dots, r. \end{aligned} \right.$$

The determinant of this linear system is $u(p).u(q).(u(p) - u(q))$ and it does not vanish for $(p,q) \in X \times X \setminus D(u)$. Hence this system has only the zero solution, i. e. $\alpha(e_t^0) = 0$, $\forall t = 1, \dots, r$, which is a contradiction, because α is a non-zero root and $e_1^0, \dots, e_r^0$ are a base of $\mathfrak{p}_0$. So we have $f_{\alpha\beta}^{--} = f_{\alpha\beta}^{-+} = f_{\lambda\beta}^{+-} = f_{\lambda\mu}^{++} = 0$. The above argument implies that for $A \in \operatorname{Ker}\partial$ we have

$$A = \sum_{i,j} e_i^0 \otimes e_j^0 \otimes f_{ij}^{00}.$$

Consequently, for $\alpha \in \Delta^-$, using the fact that $\partial(A)(u^k \otimes e_\alpha^-) = 0$, $\forall k \geqq 1$, we derive $\alpha(e_i^0).f_{ij}^{00}(p,q).u^k(p) = 0$ for each $i, j = 1, \dots, r$, $\alpha \in \Delta^-$ and $k \geqq 1$.

But for each $i = 1, \dots, r$ there exists $\alpha \in \Delta^-$ such that $\alpha(e_i^0) \neq 0$, hence setting $k = 1$ we obtain $u(p).f_{ij}^{00}(p,q) = 0$, $\forall (p,q) \in X \times X$, i. e. $f_{ij}^{00} = 0$.

It turns out that for all triple systems described in Theorem 1 there exists a solution r_u of the equation (4) which coincides (up to pullback from $\mathbb{P}^1$ to the curve X) with the well-known "rational" solution (6) of the classical Yang-Baxter equation (compare with [8, 11]).

Remark 2. It is easy to construct Manin triple systems on an arbitrary curve X, which are "subtriples" of those described in Theorem 1 and which determine the respective curve X (by taking the intersections of A_S and $\mathfrak{h}$ with $\mathcal{R}(X) \otimes \mathfrak{p}$). However, one can check that we get no essentially new solutions of the Yang-Baxter equation in this way.

4. OTHER EXAMPLES

Apart from the solution (6) there are two well-known classes of solutions of the Yang-Baxter equation — the elliptic and the trigonometric solutions, described, e. g., in [1, 2, 8, 10]. While (as known) the elliptic solutions are directly obtained from (**non-decomposable**) triple Manin systems which we shall describe briefly later, we show here that the trigonometric solutions are obtained from rational solutions of the equation (1) (and (4)) corresponding to suitable triple Manin systems, which we proceed to describe.

Example 1. Let $X = \mathbb{P}^1$, $S = \{0, \infty\}$, and let t be the co-ordinate function on $\mathbb{P}^1$. Let $\mathfrak{u}(\mathbb{P}^1)$ be the ring of polynomial adels on $\mathbb{P}^1$. Denote $A = \mathfrak{u}(\mathbb{P}^1)_S \otimes \mathfrak{p}$. One may interpret A as the cartesian square $L(\mathfrak{p}) \times L(\mathfrak{p})$, where

$$L(\mathfrak{p}) \stackrel{\mathrm{def}}{=} \mathbb{C}[t, t^{-1}] \otimes \mathfrak{p}$$

is the non-twisted affine (Kac-Moody without central extensions) Lie algebra of $\mathfrak{p}$ (see, e. g., [9]). Indeed,

$$A = \mathfrak{u}(\mathbb{P}^1)_S \otimes \mathfrak{p} = (\mathbb{C}[t, t^{-1}])_{\{0\}} \otimes \mathfrak{p} \times (\mathbb{C}[t, t^{-1}])_{\{\infty\}} \otimes \mathfrak{p}.$$

Thus the first factor $L(\mathfrak{p})$ is the polynomial algebra $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{p}$ interpreted as a subalgebra of the adels at the point $0 \in \mathbb{P}^1$, likewise the second factor at ∞. Denote

$$\mathfrak{g} = (\mathbb{C}[t, t^{-1}])_S \otimes \mathfrak{p} \cong L(\mathfrak{p}).$$

We choose a Cartan decomposition of $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{n}_0 \oplus \mathfrak{n}_+,$$

and hence of $L(\mathfrak{p})$:

$$L(\mathfrak{p}) = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{n}}_0 \oplus \tilde{\mathfrak{n}}_+,$$

where

$$\tilde{\mathfrak{n}}_- = \mathfrak{n}_- \oplus (t^{-1}.\mathbb{C}[t^{-1}] \oplus \mathfrak{p}), \quad \tilde{\mathfrak{n}}_0 = \mathfrak{n}_0, \quad \tilde{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus (t.\mathbb{C}[t] \oplus \mathfrak{p})$$

(see [9, p. 78]. For each $a \in A$ denote by a^0 the projection of $(a)_0$ on $\tilde{\mathfrak{n}}_0$ and with a^∞ the projection of $(a)_\infty$ on $\tilde{\mathfrak{n}}_0$ (note that $A_{\{0\}} = A_{\{\infty\}} = L(\mathfrak{p})$). Define

$$\mathfrak{h} = \{a \in (\tilde{\mathfrak{b}}_-)_{\{0\}} \times (\tilde{\mathfrak{b}}_+)_{\{\infty\}} \mid a^0 + a^\infty = 0\}.$$

Let ω be the meromorphic differential $t^{-1}.dt$ on $\mathbb{P}^1$. Then $(\mathcal{A}, H_S, \mathfrak{g} \otimes \mathfrak{h})$ is a triple Manin system. The **cocommutator** p of $\mathfrak{g}$ is described as follows:

$$p(H_i) = 0,$$
$$p(X_i^\pm) = \pm X_i^\pm \wedge H_i,$$

where $X_i^\pm$, H_i are the canonic generators of the Kac-Moody algebra $\mathfrak{g} \cong L(\mathfrak{p})$ (see [11]). A straightforward computation gives an element $r \in \mathcal{B}(\mathbb{P}^1)$ such that r is a solution to both equations (1) and (4). Thus we get a **rational** solution of the classical Yang-Baxter equation, which corresponds to the Manin triple $(\mathcal{A}, H_S, \mathfrak{g} \oplus \mathfrak{h})$. If we substitute $\exp(u)$ for the co-ordinate function t, we obtain the so-called trigonometric solution of the classical Yang-Baxter equation, which is studied in detail in [1, 2].

The solution r for the case $\mathfrak{p} = \mathfrak{sl}(2, \mathbb{C})$ is the following:

$$r = (t - s)^{-1}.(t.e_- \otimes e_+ + \frac{1}{4}(t + s).e_0 \otimes e_0 + s.e \otimes e_-),$$

where t, s are co-ordinate functions of the first and the second copy of $\mathbb{P}^1$ in $\mathcal{B}(\mathbb{P}^1) = \mathcal{R}(\mathbb{P}^1 \times \mathbb{P}^1) \otimes (\mathfrak{p} \otimes \mathfrak{p})$, and $\{e_-, e_0, e_+)$ is the cànonical Cartan-Wayl basis of the algebra $\mathfrak{p} = \mathfrak{sl}(2, \mathbb{C})$.

Example 2. We preserve the notations of Example 1. Define

$$\tilde{\mathfrak{h}} = \{a \in (b_- \otimes \mathbb{C}[t, t^{-1}])_{\{0\}} \times (b_+ \otimes \mathbb{C}[t, t^{-1}])_{\{\infty\}} \mid$$

$$\pi_0((a)_0) + \pi_0((a)_\infty) = 0\},$$

where π_0 is the projection on the subalgebra $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{n}_0 \subseteq L(\mathfrak{p})$. Again $(A, H_S, \mathfrak{g} \oplus \tilde{\mathfrak{h}})$ is a triple Manin system, but this time there exists **no** cocommutator of the algebra $\mathfrak{g}$ determined by it. More explicitly, the functional defined by formula (3) is an infinite series and does not correspond to any function $p : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$. Presumably, this pathology is due to the choice of the "bad" definition of the "Cartan subalgebra" of $L(\mathfrak{p})$, contrary to the "good" definition in Example 1. Observe that this drastic change in the cocommutator situation of $\mathfrak{g}$ is achieved by changing only the "$\mathfrak{h}$" part of the triple system.

Example 3. For completeness we include a brief description of the triple Manin system generating the elliptic solutions of the classical Yang-Baxter equation. For more details see [1, 10]. Let I_1, I_2 be the internal automorphisms of $\mathfrak{sl}(n, \mathbb{C})$, defined by the matrices

$$M_1 = \begin{Vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & . & \dots & . \\ 1 & 0 & \dots & 1 \end{Vmatrix}; \quad M_2 = \begin{Vmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & \varepsilon^{n-1} \end{Vmatrix},$$

where ε is a primitive n-th root of unity. Let $X = \mathbb{C}/\{\mathbb{Z} \oplus \mathbb{Z}\tau\}$ be an elliptic curve. We choose $S = \{0\}$; $\omega = dz$ (z being the co-ordinate on $\mathbb{C}$); $\mathfrak{p} = \mathfrak{sl}(n, \mathbb{C})$. Define

$$A_S = \mathcal{A}_S(X) \otimes \mathfrak{p}, \quad \mathfrak{h} = \mathcal{A}_S^+(X) = \mathbb{C}[z] \otimes \mathfrak{p},$$

$$\mathfrak{g} = \{f \in \mathcal{R}(X) \otimes \mathfrak{p} \mid f\left(z + \frac{k + l.\tau}{n}\right) = I_1^k(I_2^l(f(z))),$$

and f has no poles outside the set $\left\{\frac{k + l.\tau}{n}\right\}_S$.

The well-known elliptic solutions of Belavin (see, e. g., [10]) correspond to the cocommutator of the algebra $\mathfrak{g}$ determined by the Manin triple $(A_S, H_S, \mathfrak{g} \oplus \mathfrak{h})$.

Remark 3. The elliptic solutions of Belavin are the only essentially non-rational solutions known (to us). It is curious to observe that the triple of Example 3 generates also a (local) solution $\tilde{r}$ of the equations (1) and (4) determined by the cocommutator of the algebra $\mathfrak{h}$. As a function of the local parameter z, $\tilde{r}$ coincides exactly with the "rational" solution r_z (see formula (6)).

Remark 4. One can check that the analog of Proposition 2 is valid for all examples treated above.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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О K_4'-ГРАФАХ

НИКОЛАЙ ХАДЖИИВАНОВ

Николай Хаджииванов. О K_4'-ГРАФАХ

G называется K_4'-графом, если при произвольной 2-раскраске его ребер появляются монохроматические треугольники с общей стороной. Через $R^k(K_4')$ обозначаем минимальное число вершин K_4'-графов с кликовым числом k. Доказаны неравенства $R^5(K_4') \leqq 29$ и $R^4(K_4') \leqq 61$.

Nikolaj Khadzhiivanov. ON K_4'-GRAPHS

G is called K_4'-graph if for every 2-coloring of its edges there are monochromatic triangles with common edge. By $R^k(K_4')$ we denote the minimum of vertex number of K_4'-graphs with clique number k. The inequalities $R^5(K_4') \leqq 29$ and $R^4(K_4') \leqq 61$ are proved.

1. ВВЕДЕНИЕ

Пусть фиксирован некоторый граф Г. Для графа G будем говорить, что он является Г-*графом*, если для любой 2-раскраски ребер графа G имеется хотя бы один подграф этого графа, который одновременно монохроматичен и изоморфен графу Г. Согласно теореме Рамсея [1] имеются Г-графы.

Минимальное n, которое является числом вершин Г-графа, обозначается через $R(\Gamma)$ и называется *числом Рамсея графа* Г.

Через K_n обозначается полный n-вершинный граф, а через K_n' — его подграф, полученный удалением некоторого ребра из K_n.

В настоящей заметке докажем несколько предложений, относящихся к Г-графам в специальном случае $\Gamma = K_4'$. В [2] (см. также [4]) доказано, что K_{10} является K_4'-графом, однако K_9 нет, так что $R(K_4') = 10$.

В [3] доказано, что для любого графа Г существует Г-граф с тем же кликовым числом. Следовательно имеются K_4'-графы с кликовым числом 3. Для данного k с $R^k(\Gamma)$ обозначим минимальное n, для которого имеется Г-граф с кликовым числом k.

Здесь найдем оценки сверху для чисел $R^5(K_4')$ и $R^4(K_4')$.

2. СУЩЕСТВЕННО 3-СТЕПЕННЫЕ ГРАФЫ

Граф G будем называть *существенно 3-степенным*, если для любого разбиения множества его вершин в объединение двух подмножеств A_i, хотя бы для одного A_i верно следующее: как бы ни выбирали вершину v в A_i, остальные вершины множества A_i порождают подграф графа G, который имеет степень хотя бы 3, т. е. $\deg\langle A_i \setminus v\rangle_G \geqq 3$. Совершенно легко указать примеры существенно 3-степенных графов. Для нас представляют интерес такие из них, которые имеют кликовое число 2.

ПРИМЕР СУЩЕСТВЕННО 3-СТЕПЕННОГО ГРАФА С КЛИКОВЫМ ЧИСЛОМ 2

Пусть V_i, $1 \leqq i \leqq 5$, — дизъюнктные 5-элементные множества. Положим $V = \cup V_i$ и объявим V множеством вершин графа H, в котором две вершины смежны тогда и только тогда, когда принадлежат соответственно двум V_i и V_{i+1} , где $1 \leqq i \leqq 5$ и $V_6 = V_1$. Очевидно в графе H нет треугольников, так что эго кликовое число 2. Граф H получается из простого цикла C_5 расщеплением каждой из его вершин в 5 долей, причем, конечно, любое ребро расщепляется в 25 долей.

Лемма 1. *Только что построенный граф H является существенно 3-степенным графом.*

Доказательство. Пусть $V = A_1 \cup A_2$. Так как множество V_i — 5-элементное и $V_i = (V_i \cap A_1) \cup (V_i \cap A_2)$, то хотя бы одно из множеств $V_i \cap A_1$ или $V_i \cap A_2$ имеет хотя 3 элементов. Число множеств V_i равняется 5 и поэтому для одного из индексов $j \in \{1, 2\}$ имеются три индекса i, для которых $|V_i \cap A_j| \geqq 3$. Следовательно найдутся такие i и j, что $|V_i \cap A_j| \geqq 3$ и $|V_{i+1} \cap A_j| \geqq 3$. Пусть $V_i \cap A_j \supset \{a_1, a_2, a_3\}$ и $V_{i+1} \cap A_j \supset \{a_4, a_5, a_6\}$. Эти 6 вершин порождают в H полный бихроматический граф $K_{3,3}$ и, очевидно, удаляя любую из них, получим подграф $K_{2,3}$. Теперь ясно, что удаляя произвольную вершину v из A_j, остальные вершины этого множества порождают подграф графа H, который имеет степень хотя бы 3, что и требовалось доказать.

Проблема. Найти минимальное n, для которого существует n-вершинный существенно 3-степенный граф с кликовым числом 2.

3. СОЕДИНЕНИЕ СВЯЗНОГО И СУЩЕСТВЕННО 3-СТЕПЕННОГО ГРАФА

Пусть L — связный граф, а M — существенно 3-степенный граф и эти два графа не имеют общих вершин. Через G обозначим соединение графов L и M, $G = L + M$.

Лемма 2. *Если при некоторой 2-раскраске ребер графа $G = L + M$ нет монохроматических K_4', тогда подграф L монохроматичен.*

Доказательство. Очевидно, достаточно в качестве L иметь ввиду связный 2-реберный граф, т. е. считать, что L состоит из двух ребер $[a_0, a_1]$ и $[a_0, a_2]$ с общей вершиной a_0. Допустим, противно тому, что требуется доказать, что имеется 2-раскраска ребер графа G, при которой ребро $[a_0, a_1]$ — черное, а ребро $[a_0, a_2]$ — белое, причем нет монохроматических K_4'.

Через A_1 обозначим множество всех вершин подграфа M, любая из которых связана с вершиной a_0 черным ребром. Через A_2 обозначим множество всех остальных вершин подграфа M, так что вершина a_0 связана с любой из вершин множества A_2 белым ребром.

Установим, что в A_i имеется такая вершина c_i, что $\deg\langle A_i \setminus c_i\rangle_M \leqq 2$, что противоречит предположению, что M является существенно 3-степенным графом. Разумеется, из-за симметрии, это достаточно доказать только для множества A_1.

Все вершины множества A_1, за исключением самое более одной, связаны вершиной a_1 белым ребром, потому что иначе существовали бы два черные треугольника с общей стороной $[a_0, a_1]$, т. ė. имели бы черный K_4'.

Итак, в A_1 имеется вершина c_1, такая что все вершины множества $A_1 \setminus c_1$ связаны белыми ребрами с вершиной a_1.

Теперь уже нетрудно сообразить, что в порожденном подграфе $\langle A_1 \setminus c_1\rangle_M$ нет смежных черных ребер и нет смежных белых ребер. Действительно, если имеются в $\langle A_1 \setminus c_1\rangle_M$ черные ребра с общей вершиной v, тогда мы имели бы два черные треугольника с общей стороной $[a_0, v]$, а если имеются две белые ребра с общей вершиной v, тогда $[a_1, v]$ будет общей стороной двух белых треугольников.

Окончательно в графе $\langle A_1 \setminus c_1\rangle_M$ нет смежных ребер одинокового цвета и следовательно нет вершин степени 3. Таким образом доказано, что в A_1 имеется такая вершина c_1, что $\deg\langle A_1 \setminus c_1\rangle_M \leqq 2$. Как мы уже отметили, таким же образом доказывается, что в A_2 имеется вершина c_2, для которой $\deg\langle A_2 \setminus c_2\rangle_M \leqq 2$. Оказывается, граф M не существенно 3-степенен, что является противоречием.

Доказательство леммы 2 завершено.

4. K'_4-ГРАФ С КЛИКОВЫМ ЧИСЛОМ 5

Из леммы 2 элементарно вытекает:

Лемма 3. *Если M — существенно 3-степенный граф, тогда соединение $G = K'_4 + M$ является K'_4-графом.*

Доказательство. Если допустить, что утверждение не верно, тогда найдется 2-раскраска ребер графа G без монохроматических подграфов, изоморфных графу K'_4. Тогда, согласно лемме 2, подграф K'_4 будет монохроматическим, что является противоречием.

Пусть H — граф, определенный в п. 2. Из леммы 3 тривиально следует

Теорема 1. *Соединение $G_1 = K'_4 + H$ является K'_4-графом с кликовым числом* 5.

Действительно, из леммы 1 и леммы 3 следует, что соединение G_1 является K'_4-графом. Кликовое число этого соединения равняется 5, потому что слагаемое K'_4 имеет кликовое число 3, а слагаемое H — 2.

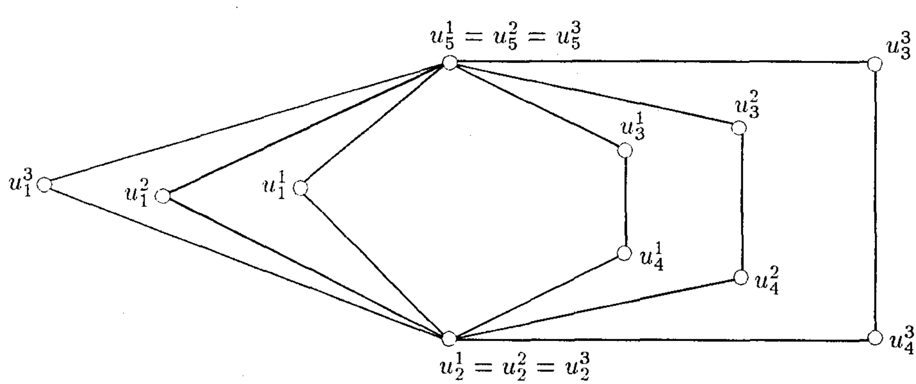
Так как число вершин соединения G_1 равняется 29, то из доказанной теоремы следует, что

$$R^5(K'_4) \leqq 29.$$

Проблема. Найти $R^5(K'_4)$.

5. K'_4-ГРАФ С КЛИКОВЫМ ЧИСЛОМ 4

Рассмотрим граф на фиг. 1, состоящий из трех 5-циклов с двумя общими несмежными вершинами; i-ый 5-цикл нумерован u^i_1, u^i_2, u^i_3, u^i_4, u^i_5, причем $u^i_2 = a$, $u^i_5 = b$, $i = 1, 2, 3$.



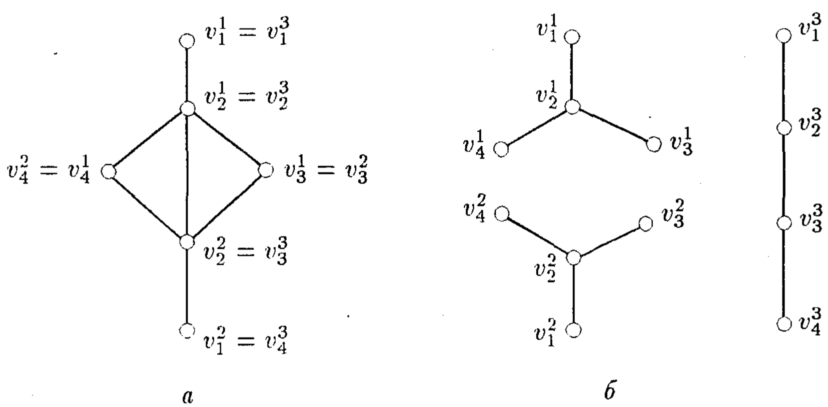
Фиг. 1

Из этого графа получим новый граф $\widehat{H}$ следующим образом: заменим каждую из 11 вершин u_j^i с 5-антикликой U_j^i, а каждое из 15 ребер $[u_j^i, u_{j+1}^i]$ (будем считать $u_6^i = u_1^i$) заменим группой из 25 ребер, соединяющих любую вершину из U_j^i с любой вершиной из U_{j+1}^i; ясно, что $U_2^1 = U_2^2 = U_2^3$ и $U_5^1 = U_5^2 = U_5^3$.

Таким образом граф $\widehat{H}$ получается из трех экземпляров графа H (определенного в п. 2) — H^1, H^2 и H^3 — посредством отождествления некоторых вершин.

Граф $\widehat{H}$ содержит в качестве подграфа граф H и поэтому является существенно 3-степенным графом (см. лемму 1).

Рассмотрим еще граф L, изображенный на фиг. 2*а*. Он получен объединением трех графов L^i, изображенных на фиг. 2*б*, причем отождествлены некоторые вершины и ребра.



Фиг. 2

Наконец, сконструируем граф G_2 следующим образом: G_2 является дизъюнктным объединением графов L и $\widehat{H}$, к которому присоединены в качестве ребер всевозможные отрезки, соединяющие любую из вершин графа L^i с любой из вершин графа H^i, где $i = 1, 2, 3$.

Таким образом, граф G_2 получен из трех соединений $L^i + H^i$ отождествлением некоторых вершин и ребер этих соединений.

Теорема 2. *Только-что построенный граф G_2 является K_4'-графом с кликовым числом 4.*

Доказательство. Докажем сначала, что G_2 — K_4'-граф. Допустим противное. Тогда имеется 2-раскраска ребер графа G_2 без монохроматических подграфов, изоморфных графу K_4'. Так как каждый H^i — существенно 3-степенный граф, при помощи леммы 2 заключаем, что L^i — монохроматический при рассматриваемой раскраске. С другой стороны,

L^1 и L^2 имеют общее ребро с L^3, так что все ребра графа L должны быть окрашены одинаково.

Таким образом, при рассматриваемой раскраске ребер графа G_2, лишенной монохроматических K_4', подграф L — монохроматический. Однако L содержит в качестве подграфа граф K_4'. Окончательно, мы достигли до заключения, что при рассматриваемой раскраске ребер графа G_2 все таки имеется монохроматический K_4', что является противоречием.

Итак, доказано, что G_2 — K_4'-граф. Остается показать, что G_2 имеет кликовое число 4. То что граф G_2 содержит 4-клики ясно — любой из подграфов $L^i + H^i$ очевидно имеет это свойство. Поэтому надо установить, что граф G_2 не имеет 5-клик. Во первых, подграф $\widehat{H}$ очевидно не содержит 3-клик, а если $[u, v]$ — 2-клика этого подграфа, тогда хотя бы одна из ее вершин — будем считать, что это u — не содержится в объединении $U_2^i \cup U_5^i$. Это показывает, что u является вершиной ровно одного из подграфов H^i; обозначим его через H^{i_0}. Из определения графа G_2 следует, что вершина u смежна в L только вершинам подграфа L^{i_0}, который не содержит 3-клик. Следовательно в G_2 нет 5-клик.

Доказательство теоремы 2 завершено.

Число вершин графа G_2 равняется 61. Поэтому из только-что доказанной теоремы следует, что

$$R^4(K_4') \leqq 61.$$

Проблема. Найти $R^4(K_4')$.

Как мы уже упомянули в п. 1, существуют K_4'-графы с кликовым числом 3.

Проблема. Построить K_4'-граф с кликовым числом 3.

6. ПРИБАВЛЕНИЕ

В [5] построены K_3-графы с кликовым числом 5 и 4. В идейном отношении [5] имеет некоторое сходство с настоящей статей. Здесь хотим исправить несколько допущенных в [5] опечаток на стр. 213.

1) На строчке 1 дополнить определения графа M следующим образом: $M = \bigcup_{i=1}^{3}(R_i + S_i) \cup \{[c', c''], [c'', c'''], [c''', c']\}$.

2) На строчке 5 вместо $U_i \cup V_i$ поставить $U_i + V_i$.

3) На строчке 9 приписать в конце: „... без монохроматических треугольников“.

4) На рис. $2a$ соединить вершины c' и c'' ребром.

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A BORSUK–ULAM TYPE THEOREM FOR $\mathbb{Z}_4$–ACTIONS

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Симеон Стефанов. ОДНА ТЕОРЕМА ТИПА ТЕОРЕМЫ БОРСУКА-УЛАМА ДЛЯ $\mathbb{Z}_4$-ДЕЙСТВИЙ

Пусть $n = 2k + 1$ и сфера S^n представлена в виде

$$S^n = \left\{ z = (z_1, \ldots, z_{k+1}) \in \mathbb{C}^{k+1} \mid \|z\| = 1 \right\}.$$

Рассмотрим каноническое действие группы $\mathbb{Z}_4 = \{1, i, -1, -i\}$ в S^n, определенное умножением. Основным результатом работы является следующая теорема типа теоремы Борсука–Улама:

Для каждой непрерывной функции $f : S^n \to \mathbb{R}^1$ рассмотрим множество

$$A(f) = \{z \in S^n \mid f(z) = f(iz) = f(-z) = f(-iz)\}.$$

Тогда $\dim A(f) \geqq n - 3$.

Основное следствие: Для каждой непрерывной функции $f : S^3 \to \mathbb{R}^1$ существует $z \in S^3$ такое, что

$$f(z) = f(iz) = f(-z) = f(-iz).$$

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Let $n = 2k + 1$ and the sphere S^n be represented as

$$S^n = \left\{ z = (z_1, \ldots, z_{k+1}) \in \mathbb{C}^{k+1} \mid \|z\| = 1 \right\}.$$

Consider the canonical action of the group $\mathbb{Z}_4 = \{1, i, -1, -i\}$ in S^n defined by multiplication. The main result in the article is the following Borsuk-Ulam type theorem:

For any continuous function $f : S^n \to \mathbb{R}^1$ consider the set

$$A(f) = \{z \in S^n \mid f(z) = f(iz) = f(-z) = f(-iz)\}.$$

Then $\dim A(f) \geqq n - 3$.

The main corollary: For any continuous function $f : S^3 \to \mathbb{R}^1$ there exists $z \in S^3$ such that

$$f(z) = f(iz) = f(-z) = f(-iz).$$

There exist various generalizations of the classical Borsuk-Ulam theorem, where the antipodal map in the n-sphere S^n is replaced by a free $\mathbb{Z}_p$-action for p prime. Then some coincidence point theorems are obtained ([3–6, 10], etc.). The case of a composite p is more complicated and no result of this kind is known to us. The purpose of this note is to get some Borsuk-Ulam type result for free $\mathbb{Z}_4$-actions in S^n (n is odd). More precisely, we consider the canonical $\mathbb{Z}_4$-action on S^n defined by $z \to iz$ and prove that for any continuous function $f : S^n \to \mathbb{R}^1$ the covering dimension of the set

$$A(f) = \{z \in S^n \mid f(z) = f(iz) = f(-z) = f(-iz)\}$$

is $\geqq n - 3$. (The proof works for arbitrary $\mathbb{Z}_4$-actions). In particular, for any $f : S^3 \to \mathbb{R}^1$ there exists $z_0 \in S^3$ such that $f(z_0) = f(iz_0) = f(-z_0) = f(-iz_0)$. It is easy to see that the estimate $\dim A(f) \geqq n - 3$ cannot be strengthened in general, since the set $A(f)$ is defined by 3 equalities.

1. PRELIMINARIES

Recall first some basic definitions. Let G be a group. A G-action in X is a continuous map $\mu : G \times X \to X$, $\mu(g, x) = gx$ such that: i) $1x = x$ and ii) $g_1(g_2x) = (g_1g_2)x$. Then X is called G-space. The subset $A \subset X$ is invariant if $gA = A$ for any $g \in G$. The orbit of a point $x \in X$ is the set orbit $x = \{gx \mid g \in G\}$. The orbit space is the factor $\widetilde{X} = X/\sim$, where $x \sim y$ iff $x \in$ orbit y.

A G-action is said to be free if $gx \neq x$ for any $g \neq 1$, $x \in X$. Let X and Y be G-spaces. A map $\varphi : X \to Y$ is equivariant if $\varphi(gx) = g\,\varphi(x)$. The join of X and Y is the factor $X * Y = X \times Y \times [0, 1]/\sim$, where $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for any $x, x' \in X$, $y, y' \in Y$. As usual, we write

$$X * Y = \{(t_1x, t_2y) \mid t_1 + t_2 = 1;\ t_1,\ t_2 \geqq 0;\ x \in X,\ y \in Y\}.$$

Note that if X and Y are G-spaces, their join is also a G-space with respect to the action

$$g(t_1x, t_2y) = (t_1gx, t_2gy).$$

Proposition 1. *Let the (metric) G-space X be a sum of two closed invariant subsets $X = X_1 \cup X_2$ and there exist equivariant maps $\varphi_i : X_i \to K_i$, $i = 1, 2$ into some polyhedra K_i. Then there exists an equivariant map $\varphi : X \to K_1 * K_2$.*

Proof. Since K_i are equivariant ANR's, there exist equivariant extensions $\widetilde{\varphi}_i : U_i \to K_i$ for some invariant open neighbourhoods $U_i \supset X_i$. Take

$$\lambda_i(x) = \frac{\operatorname{dist}(x, X \setminus U_i)}{\operatorname{dist}(x, X \setminus U_1) + \operatorname{dist}(x, X \setminus U_2)}.$$

Then $\varphi(x) = (\lambda_1(x)\,\widetilde{\varphi}_1(x), \lambda_2(x)\,\widetilde{\varphi}_2(x))$ is an equivariant map $\varphi : X \to K_1 * K_2$.

We shall define and list some properties of the so-called "B-index" introduced for a space with a fixed point free involution (free $\mathbb{Z}_2$-action) by C. T. Yang [11].

Definition. Let X be a compact space with a fixed point free involution $T : X \to X$. We say that the B-index of X (with respect to T) is not greater than n if there exists a map $\varphi : X \to S^n$ such that

$$\varphi(Tx) = -\varphi(x)$$

for any $x \in X$. Then we write $B(X;\, T) \leqq n$. The equality $B(X;\, T) = n$ means that $B(X;\, T) \leqq n$ and $B(X;\, T) \nleqq n - 1$.

We shall make use only of the following properties of this index:

i) $B(X;T) \geqq 1$ iff X contains an invariant (with respect to T) continuum;

ii) $B(S^n;T) = n$ for any fixed point free involution T in S^n;

iii) Let U be an open connected T_0-invariant subset of S^n. Then

$$B(S^n \setminus U; T_0) \leqq n - 2,$$

where $T_0(x) = -x$.

The properties i) and ii) (and many others) may be found for example in [8, 9, 11]; iii) follows from a theorem of J. W. Jaworowski [2].

We shall give now a definition, which is important for the following.

Definition. Let X be a G-space. A closed invariant subset $F \subset X$ is said to be *equivariant partition* in X if for any $x \in X$, $g \neq 1$, the points x, gx lie in different components of $X \setminus F$.

It is easy to see that if A is an invariant closed subset of X, then every equivariant partition in A may be extended to an equivariant partition in X.

Proposition 2. *Let X be a compact space with free $\mathbb{Z}_4$-action and F be a closed invariant subset with* $\dim F \leqq k$. *Then there exist equivariant partitions in X — $\Phi_1, \ldots, \Phi_{k+1}$, such that*

$$\left(\bigcap_{i=1}^{k+1} \Phi_i\right) \cap F = \varnothing.$$

Proof. Apply an induction on k. For $k = 0$ take some sufficiently small (finite) invariant covering ω of X with open sets U such that $F \cap Fr\,U = \varnothing$. Then $\Phi_1 = \cup\{Fr\,U \mid U \in \omega\}$ is an equivariant partition in X and $\Phi_1 \cap F = \varnothing$.

Suppose the proposition is valid for $k - 1$ and $\dim F \leqq k$. There exists in F an equivariant partition F_0 with $\dim F_0 \leqq k - 1$, hence, there are equivariant partitions $F_1, \ldots, F_k$ in F such that $\left(\bigcap\limits_{i=1}^{k} F_i\right) \cap F_0 = \varnothing$. The partitions F_i may be extended to equivariant ones $\widetilde{F}_i$ in X. Write $\Phi_i = \widetilde{F}_i$, $i = 1, \ldots, k$; $\Phi_{k+1} = \widetilde{F}_0$. Then $\left(\bigcap\limits_{i=1}^{k+1} \Phi_i\right) \cap F = \left(\bigcap\limits_{i=1}^{k} F_i\right) \cap F_0 = \varnothing$.

2. $\mathbb{Z}_4$–ACTIONS IN S^n

Let $n = 2k + 1$ and

$$S^n = \{z = (z_1, \ldots, z_{k+1} \mid \|z\| = 1\}.$$

Then the group $\mathbb{Z}_4 = \{1, i, -1, -i\}$ acts freely in S^n as usual:

$$iz = (iz_1, \ldots, iz_{k+1}).$$

By $2\,S^n$ we denote the space $S^n \times \{-1, 1\}$, where $\mathbb{Z}_4$ acts as follows:

$$i(z \times \{\varepsilon\}) = i\,z \times \{-\varepsilon\}.$$

Proposition 3. *Let X be a compact $\mathbb{Z}_4$-space with equivariant partitions $\Phi_1, \ldots, \Phi_{n+1}$ $(n = 2k+1)$ such that $\bigcap_{j=1}^{n+1} \Phi_j = \varnothing$. Then there exists a $\mathbb{Z}_4$-equivariant map $\varphi : X \to S^n$.*

Proof. Apply an induction on n. Let $n = 1$ and $O\Phi_j$ be open invariant neighbourhoods of Φ_j such that $\bigcap_{j=1}^{2} \overline{O\Phi_j} = \varnothing$. Since Φ_j is an equivariant partition, $X \setminus O\Phi_j = A_j \cup (i\,A_j) \cup (-A_j) \cup (-i\,A_j)$, where $\varepsilon\,A_j$ are non-intersecting closed sets ($\varepsilon \in \mathbb{Z}_4$). Define $\varphi : X \setminus O\Phi_1 \to S^1$ by $\varphi(\varepsilon\,A_1) = \varepsilon$. Then φ may be extended to a map $\varphi : (X \setminus O\Phi_1) \cup A_2 \to S^1$ and finally to an equivariant $\varphi : X \to S^1$ by the formula $\varphi(\varepsilon z) = \varepsilon\,\varphi(z)$.

Suppose the proposition is valid for $n-1$ and consider $\Phi_1, \ldots, \Phi_{n+1}$ with $\bigcap_{j=1}^{n+1} \Phi_j = \varnothing$. Take as above $O\Phi_j$ with $\bigcap_{j=1}^{n+1} O\Phi_j = \varnothing$. Put $X_1 = \bigcup_{j=1}^{n-1} X \setminus O\Phi_j$, $X_2 = \bigcup_{j=n}^{n+1} X \setminus O\Phi_j$, so $X = X_1 \cup X_2$. Then, evidently, the sets $X_1 \cap \Phi_j$, $j = 1, \ldots, n-1$, are equivariant partitions in X_1 with an empty intersection, as well as $X_2 \cap \Phi_j$, $j = n, n+1$, are non-intersecting equivariant partitions in X_2. Then by the induction hypothesis we have equivariant maps $\varphi_1 : X_1 \to S^{n-2}$, $\varphi_2 : X_2 \to S^1$ which induce (by Proposition 1) an equivariant $\varphi : X \to S^{n-2} * S^1$. Note, finally, that $S^{n-2} * S^1 = S^n$ as $\mathbb{Z}_4$-spaces.

Lemma. *Let $f : S^n \to \mathbb{R}^1$ be a continuous map (n is odd). Consider the set*

$$A(f) = \{z \in S^n \mid f(z) = f(iz) = f(-z) = f(-iz)\}$$

and suppose that $\dim A(f) \leqq n-4$. *Then there exists a $\mathbb{Z}_4$-equivariant map*

$$\psi : S^n \to 2\,S^{n-2} * S^{n-2}.$$

Proof. Since $A(f)$ is a closed invariant set with $\dim A(f) \leqq n-4$, by Proposition 2 we can find in S^n equivariant partitions $\Phi_1, \ldots, \Phi_{n-3}$ such that $\left(\bigcap_{i=1}^{n-3} \Phi_i\right) \cap A(f) = \varnothing$. Write $\Phi = \bigcap_{i=1}^{n-3} \Phi_i$, then for any $z \in \Phi$ $f(\text{orbit}\,z) \neq \text{const}$. Consider the sets

$$M = \{z \in \Phi \mid f(z) = f(-z) \text{ or } f(iz) = f(-iz)\},$$
$$N = \{z \in \Phi \mid f(z) = f(-z) \text{ and } f(iz) = f(-iz)\}.$$

Evidently, these are closed invariant sets and $N \subset M$. Moreover, M is an equivariant partition in Φ, and N is an equivariant partition in M. Really, suppose, first, that M is not an equivariant partition in Φ. Then for some $z \in \Phi \setminus M$, $g \neq 1$, the points z, gz lie in the same component K of $\Phi \setminus M$. If $g = i$ or $g = -i$, we have $iK = K$; if $y = -1$, then $-K = K$. In both cases $-K = K$, so we must have $f(z_0) = f(-z_0)$ for some $z_0 \in K \subset \Phi \setminus M$, which means $z_0 \in M$ — a contradiction.

Suppose now N is not an equivariant partition in M. Proceeding as above, we find a component K of $M \setminus N$ such that $-K = K$. Set

$$K_+ = \{z \in K \mid f(z) = f(-z)\},$$
$$K_- = \{z \in K \mid f(iz) = f(-iz)\}.$$

Then $K = K_+ \cup K_-$ and $K_+ \cap K_- = \varnothing$, since $K \cap N = \varnothing$. Hence $K \equiv K_+$ (par example). Considering f on iK, we get some $z_0 \in iK$ such that $f(z_0) = f(-z_0)$. On the other hand, iz_0, $-iz_0 \in K$; therefore $f(iz_0) = f(-iz_0)$, which means that $z_0 \in N$ — a contradiction.

Let Φ_{n-2}, Φ_{n-1} be equivariant partitions in S^n such that $\Phi_{n-2} \cap \Phi = M$, $\Phi_{n-1} \cap M = N$. Then we have

$$\bigcap_{i=1}^{n-1} \Phi_i = N.$$

Put $N_+ = \{z \in N \mid f(z) < f(iz)\}$; $N_- = iN_+$. Then $N = N_+ \cup N_-$, $N_+ \cap N_- = \varnothing$ and $-N_+ = N_+$, $-N_- = N_-$. Consider the set N_+ together with the antipodal involution $T_0(z) = -z$. For its B-index we must have

$$B(N_+;\, T_0) \leqq n - 2.$$

Indeed, if $B(N_+;\, T_0) \geqq n-1$, then $B(N_+;\, T_0) \geqq 1$; it means (see i)) that N_+ contains a T_0-invariant continuum and hence by iii) $B(N_-;\, T_0) \leqq n-2$, which contradicts $B(N_-;\, T_0) = B(N_+;\, T_0) \geqq n-1$. Therefore, by the definition of B-index we have a T_0-equivariant map $\varphi_+ : N_+ \to S^{n-2}$. Define $\varphi_- : N_- \to S^{n-2}$ by $\varphi_-(iz) = i\varphi_+(z)$. Then we get a $\mathbb{Z}_4$-equivariant map $\varphi : N \to 2\,S^{n-2}$ defined by

$$\varphi(z) = \begin{cases} \varphi_+(z) \times \{1\}, & z \in N_+, \\ \varphi_-(z) \times \{-1\}, & z \in N_-. \end{cases}$$

Extend φ to some closed invariant neighbourhood $\varphi : \overline{ON} \to 2\,S^{n-2}$. The sets $\Phi_i \setminus ON$ are equivariant partitions in $S^n \setminus ON$ with $\bigcap_{i=1}^{n-1} \Phi_i \setminus ON = \varnothing$. Then by Proposition 3 there exists an equivariant map of $S^n \setminus ON$ into S^{n-2}. Therefore, by Proposition 1 we get some equivariant

$$\psi : S^n \to 2\,S^{n-2} * S^{n-2}.$$

The lemma is proved.

3. INFORMATION ABOUT THE COHOMOLOGIES OF THE LENS SPACES

Let n be an odd number. The lens space L_4^n is the orbit space of the $\mathbb{Z}_4$-space S^n (with the canonical $\mathbb{Z}_4$-action $z \to iz$), so we write $L_4^n = \widetilde{S}^n$. The space $L_4^\infty = \bigcup_{n=1}^{\infty} L_4^n$ is the classifying space for all principal $\mathbb{Z}_4$-bundles. For their cohomology rings with coefficients modulo 2 we have (see for example [1])

$$H^*(L_4^\infty;\, \mathbb{Z}_2) = \Lambda[u] \otimes \mathbb{Z}_2[v],$$

where $\deg u = 1,\ \deg v = 2$,

$$H^*(L_4^n;\, \mathbb{Z}_2) = \Lambda[u_0] \otimes \mathbb{Z}_2[v_0] \ /\ (v_0^{k+1} = 0),$$

where $\deg u_0 = 1,\ \deg v_0 = 2$ and $n = 2k + 1$.

The natural inclusion $i : L_4^n \to L_4^\infty$ induces an epimorphism i^* in the cohomologies, such that $i^*(u) = u_0,\ i^*(v) = v_0$.

Let X be a compact (free) $\mathbb{Z}_4$-space with an orbit space $\widetilde{X}$. Consider the principal $\mathbb{Z}_4$-bundle $\xi = (X, \widetilde{X}, p)$. Let $f : \widetilde{X} \to L_4^\infty$ be a classifying map for ξ. Then $f^*(u)$ and $f^*(v)$ are some characteristic classes in $H^*(\widetilde{X};\, \mathbb{Z}_2)$ that we shall denote by $u(\widetilde{X})$ and $v(\widetilde{X})$, respectively. (So we have $u_0 = u(L_4^n),\ v_0 = v(L_4^n)$.) Note, that every $\mathbb{Z}_4$-equivariant map $\varphi : X \to Y$ induces a map $\widetilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ and if $f : \widetilde{Y} \to L_4^\infty$ is a classifying map for $\eta = (Y, \widetilde{Y}, \pi)$, then $f\widetilde{\varphi}$ is a classifying map for $\xi = (X, \widetilde{X}, p)$, therefore $u(\widetilde{X}) = \widetilde{\varphi}^*\, u(\widetilde{Y})$ and $v(\widetilde{X}) = \widetilde{\varphi}^*\, v(\widetilde{Y})$.

4. THE MAIN RESULT

Theorem. *Let $n = 2k + 1$ and $f : S^n \to \mathbb{R}^1$ be a continuous map. Consider the set*

$$A(f) = \{z \in S^n \mid f(z) = f(iz) = f(-z) = f(-iz)\}.$$

Then $\dim A(f) \geqq n - 3$.

Proof. Suppose the contrary. Then by the lemma we have some $\mathbb{Z}_4$-equivariant $\psi : S^n \to 2\,S^{n-2} * S^{n-2}$. We shall show that such a map does not exist.

The orbit space of S^n is L_4^n; denote by M the orbit space of $2\,S^{n-2} * S^{n-2}$. Let $u_1 = u(M),\ v_1 = v(M)$. It is enough to prove that

$$u_1\, v_1^k = 0$$

in $H^n(M;\, \mathbb{Z}_2)$. Really, then $u_0 = u(L_4^n) = \widetilde{\psi}^*(u_1),\ v_0 = v(L_4^n) = \widetilde{\psi}^*(v_1)$, hence $u_0 v_0^k = \widetilde{\psi}^*(u_1 v_1^k) = 0$ in $H^n(L_4^n;\, \mathbb{Z}_2)$, which is a contradiction.

Decompose $2\,S^{n-2} * S^{n-2} = A_1 \cup A_2$, where

$$A_1 = \{(t_1 x,\, t_2 y) \mid t_1 \geqq \tfrac{1}{2}\},\quad A_2 = \{(t_1 x,\, t_2 y) \mid t_1 \leqq \tfrac{1}{2}\}.$$

Let $M_1 = \widetilde{A}_1,\ M_2 = \widetilde{A}_2$ be the corresponding orbit spaces. Then $M = M_1 \cup M_2$. Define the homotopy equivalences $r_1 : A_1 \to 2\,S^{n-2},\ r_2 : A_2 \to S^{n-2}$

by $r_1(t_1x, t_2y) = x$, $r_2(t_1x, t_2y) = y$. Since r_i are equivariant, they induce maps between the orbit spaces, which are also homotopy equivalences. Hence

$$H^*(M_1) = H^*(\widetilde{2S}^{n-2}), \quad H^*(M_2) = H^*(\widetilde{S}^{n-2}) = H^*(L_4^{n-2}).$$

(All cohomologies are taken with $\mathbb{Z}_2$-coefficients.)

Consider the Meyer–Vietoris sequence

$$\longrightarrow H^{n-2}(M_1 \cap M_2) \xrightarrow{\delta} H^{n-1}(M) \xrightarrow{(i_1^*, i_2^*)} H^{n-1}(M_1) \oplus H^{n-1}(M_2) \longrightarrow .$$

It is clear that $i_1^*(v_1^k) = i_2^*(v_1^k) = 0$, since $H^{n-1}(M_1) = H^{n-1}(M_2) = \{0\}$. Therefore $v_1^k = \delta\omega$ for some $\omega \in H^{n-2}(M_1 \cap M_2)$. Then $u_1 v_1^k = u_1\,\delta\omega = \delta(i^*(u_1)\,\omega)$, where $i : M_1 \cap M_2 \to M$ is the inclusion map (see for example [7]). To prove $u_1 v_1^k = 0$ it is enough to show that $i^*(u_1) = 0$. Since $i = i_1 j_1$, where $j_1 : M_1 \cap M_2 \to M_1$, $i_1 : M_1 \to M$ are the inclusions, we have $i^*(u_1) = j_1^* i_1^*(u_1) = j_1^*(u(M_1))$. We shall prove that $u(M_1) = 0$. Note that $\widetilde{2S}^{n-2}$ is a deformation retract of M_1, therefore $u(M_1) = u(\widetilde{2S}^{n-2})$. We have to show that

$$u(\widetilde{2S}^{n-2}) = 0.$$

Suppose, first, $n > 3$. Clearly, $\widetilde{2S}^{n-2} = \mathbb{R}P^{n-2}$. Let $f : \mathbb{R}P^{n-2} \to L_4^\infty$ be a classifying map for $\xi = (2\,S^{n-2}, \mathbb{R}P^{n-2}, p)$, then $u(\widetilde{2S}^{n-2}) = f^*(u)$. Suppose that $f^*(u) \neq 0$. Then in the cohomology ring of $\mathbb{R}P^{n-2}$ we have $[f^*(u)]^2 \neq 0$ (see [7]). On the other hand, $[f^*(u)]^2 = f^*(u^2) = f^*(0) = 0$, which is a contradiction.

Let now $n = 3$. Then we directly see that a classifying map for the $\mathbb{Z}_4$-bundle $\xi = (2S^1, \widetilde{2S}^1, p)$ is the map $f : \widetilde{2S}^1 \to L_4^1 \subset L_4^\infty$ which is a double covering (both $\widetilde{2S}^1$ and L_4^1 are homeomorphic to S^1). Therefore $u(\widetilde{2S}^1) = f^*(u) = 0$.

The theorem is proved.

Corollary 1. *For any continuous function $f : S^3 \to \mathbb{R}^1$ there exists $z \in S^3$ such that*

$$f(z) = f(iz) = f(-z) = f(-iz).$$

Corollary 2. *Let S^3 be a sum of two closed (non-invariant) subsets $S^3 = A \cup B$. Then some of them contains a whole* orbit $z = \{z, iz, -z, -iz\}$.

Proof. Take $f(z) = \mathrm{dist}(z, A)$ in Corollary 1.

The 3-sphere is a group with respect to the multiplication induced by the quaternion structure in $\mathbb{C}^2$.

Corollary 3. *Let F and A be closed subsets of S^3. Then there exists $z \in S^3$ such that*

$$\mathrm{vol}(F \cap zA) = \mathrm{vol}(F \cap izA) = \mathrm{vol}(F \cap -zA) = \mathrm{vol}(F \cap -izA),$$

where "vol" is the 3-volume in S^3.

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MAPPING THEOREMS FOR COHOMOLOGICALLY TRIVIAL MAPS

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Симеон Стефанов. ТЕОРЕМЫ О СОВПАДЕНИИ ДЛЯ КОГОМОЛОГИЧЕСКИ ТРИВИАЛЬНЫХ ОТОБРАЖЕНИЙ

Получены некоторые теоремы о совпадении для отображений n-сферы S^n. Следствием показано, что каждое когомологически тривиальное отображение $f : S^n \xrightarrow{\text{на}} Y$ сферы S^n на некоторое Y склеивает пару точек $x_1,\ x_2 \in S^n$, расстояние между которыми не меньше диаметра правильного $(n+1)$-симплекса вписанного в S^n:

$$f(x_1) = f(x_2), \quad \|x_1 - x_2\| \geqq \sqrt{\frac{2(n+2)}{n+1}}.$$

Дальше доказано, что для каждого разложения S^n на n замкнутые подмножества, некоторое из них содержит континуум K с $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$. Показано также, что каждое понижающее размерность отображение $f : S^n \to Y$ постоянно на континуум K с $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$. Наконец получена теорема о совпадении для отображений S^n в k-мерные стягиваемые полиэдры (для $k < n$).

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Some mapping theorems for maps of the n-sphere S^n are obtained. As a corollary, it is shown that every cohomologically trivial map $f : S^n \xrightarrow{\text{on}} Y$ of S^n onto some Y identifies a pair of points $x_1,\ x_2 \in S^n$ such that the distance between them is not less than the diameter of

the regular $(n+1)$-simplex inscribed in S^n:

$$f(x_1) = f(x_2), \quad \|x_1 - x_2\| \geqq \sqrt{\frac{2(n+2)}{n+1}}.$$

Furthermore, it is proved that for any decomposition of S^n into n closed subsets some of them contains a continuum K with $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$. Also it is shown that every lowering dimension map $f : S^n \to Y$ is constant on a continuum K with $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$. Finally, a mapping theorem for maps of S^n into k-dimensional contractible polyhedra is obtained (for $k < n$).

The Borsuk-Ulam theorem states that every map of the n-dimensional unit sphere S^n into $\mathbb{R}^n$ identifies a pair of antipodal points. This theorem ceases to be true if we replace $\mathbb{R}^n$ by an arbitrary n-dimensional contractible polyhedron P_n. However, it is easy to see that there exists a positive α such that any map of S^n into some P_n identifies a pair of points $x_1,\ x_2 \in S^n$ with $\|x_1 - x_2\| \geqq \alpha$. We shall find the greatest α with this property and we shall prove the corresponding mapping theorem in a more general situation (Theorem 1). The fact is that the greatest α with the above-mentioned property is the diameter of the regular $(n+1)$-simplex inscibed in S^n. Corollary 1 of Theorem 1 gives a generalization of a theorem due to J. Väisälä [1]. Furthermore, we obtain a theorem for decomposition of S^n into n closed subsets (Theorem 2), a mapping theorem for lowering dimension maps of S^n (Theorem 3), and finally a mapping theorem for maps of S^n into lower dimensional contractible polyhedra (Theorem 4).

Lemma 1. *Let $e_1,\ e_2,\ \ldots,\ e_{n+1}$ be unit vectors in $\mathbb{R}^n$ such that $\sum_1^{n+1} \lambda_i\, e_i = O$ for some $\lambda_i \geqq 0$ with $\sum_1^{n+1} \lambda_i = 1$.*

Then $(e_i,\ e_j) \leqq -\frac{1}{n}$ for some $i,\ j$.

Proof. Suppose the contrary — $(e_i,\ e_j) > -\frac{1}{n}$ for every $i,\ j$. Then

$$1 = \left(\sum_1^{n+1} \lambda_i\right)^2 = \sum_1^{n+1} \lambda_i^2 + 2\sum_{i<j} \lambda_i \lambda_j$$

and

$$0 = \left(\sum_1^{n+1} \lambda_i\, e_i,\ \sum_1^{n+1} \lambda_i\, e_i\right) = \sum_1^{n+1} \lambda_i^2 + 2\sum_{i<j} \lambda_i \lambda_j (e_i,\ e_j) > \sum_1^{n+1} \lambda_i^2 - \frac{2}{n}\sum_{i<j} \lambda_i \lambda_j,$$

hence

$$0 > \sum_1^{n+1} \lambda_i^2 - \frac{1}{n}\left(1 - \sum_1^{n+1} \lambda_i^2\right),$$

i. e. $\sum_1^{n+1} \lambda_i^2 < \frac{1}{n+1}$. On the other hand, a well-known inequality gives

$$\sum_1^{n+1} \lambda_i^2 \geqq \frac{\left(\sum_1^{n+1} \lambda_i\right)^2}{n+1} = \frac{1}{n+1},$$

which is a contradiction.

Lemma 2. *Let F be a closed subset of S^n with* $\operatorname{diam} F < \sqrt{\frac{2(n+2)}{n+1}}$. *Then F is contained in some open semisphere of S^n.*

Proof. Denote by $\operatorname{co} F$ the convex hull of F in $\mathbb{R}^{n+1}$ and suppose that $\operatorname{co} F$ does not contain the origin O. Then there is a hyperplane T in $\mathbb{R}^{n+1}$ such that $O \in T$ and $F \cap T = \varnothing$, hence one of the components of $S^n \setminus T$ is an open semisphere containing F.

Suppose now that $O \in \operatorname{co} F$, then, according to the theorem of Caratheodory, O is a convex linear combination of $n+2$ points of F : $O = \sum_{i=1}^{n+2} \lambda_i x_i$, where $\lambda_i \geqq 0$, $\sum_{i=1}^{n+2} \lambda_i = 1$ and $x_1, \ldots, x_{n+2} \in F$. Consider the unit vectors $e_i = \overrightarrow{Ox_i}$. According to Lemma 1, we have $(e_i, e_j) \leqq -\frac{1}{n+1}$ for some i, j. Let φ denote the angle between e_i and e_j, then $\cos\varphi = (e_i, e_j) \leqq -\frac{1}{n+1}$ and we have $\|x_i - x_j\| = 2\sin\frac{\varphi}{2} = 2\sqrt{\frac{1-\cos\varphi}{2}} \geqq 2\sqrt{\frac{1}{2}\left(1 + \frac{1}{n+1}\right)} = \sqrt{\frac{2(n+2)}{n+1}}$, which contradicts the condition $\operatorname{diam} F < \sqrt{\frac{2(n+2)}{n+1}}$.

Note that the number $\sqrt{\frac{2(n+2)}{n+1}}$ is exactly the diameter of the regular $(n+1)$-simplex inscribed in S^n.

All cohomologies in this note are Čech cohomologies with integral coefficients.

Lemma 3. *Let ω be a finite open covering of the compact space X and $\pi : X \to N_\omega$ be the canonical projection of X into the nerve of ω. Suppose $f : X \xrightarrow{\text{on}} Y$ is a map of X on Y and there exists $\xi \in H^n(X)$ such that $\xi \in \operatorname{Im}\pi^* \setminus \operatorname{Im} f^*$. Then for some $y_0 \in Y$ the set $f^{-1}(y_0)$ is not contained in any element of ω.*

This proposition is proved by the author in [4, Lemma 2].

Given a map $f : X \to Y$ we shall say that f is *trivial in dimension n* if $f^*(H^n(Y)) = 0$.

Theorem 1. *Let X be a compact metric space and $\varphi : X \xrightarrow{\text{on}} S^n$ be a map, which is non-trivial in dimension n. Suppose $f : X \longrightarrow Y$ is a map of X on Y and there is $\xi \in H^n(X)$ such that $\xi \in \operatorname{Im}\varphi^* \setminus \operatorname{Im} f^*$. Then there exist $x_1,\ x_2 \in X$ such that $f(x_1) = f(x_2)$ and $\|\varphi(x_1) - \varphi(x_2)\| \geqq \sqrt{\frac{2(n+2)}{n+1}}$.*

Proof. It is enough to prove that there exists $y_0 \in Y$ such that the set $\varphi(f^{-1}(y_0))$ is not contained in any open semisphere of S^n. Really, in this case Lemma 2 implies diam $\varphi(f^{-1}(y_0)) \geqq \sqrt{\frac{2(n+2)}{n+1}}$ and the theorem is proved.

Suppose that for any $y \in Y$ the set $\varphi(f^{-1}(y))$ is contained in some open semisphere O_y. Choose an open $V_y \ni y$ with the property $\varphi(f^{-1}(V_y)) \subset O_y$. The covering $\{V_y \mid y \in Y\}$ has a finite subcovering $\{V_{y_i} \mid i = 1,\ 2,\ \ldots,\ k\}$. Put $\omega = \{O_{y_i} \mid i = 1,\ 2,\ \ldots,\ k\}$. Since $\varphi(X) = S^n$, ω is a finite open covering of S^n such that for any $y \in Y$ the set $\varphi(f^{-1}(y))$ is contained in some element of ω. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & S^n \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_0} \\ N_{\varphi^{-1}(\omega)} & \underset{\rho}{\approx} & N_\omega \end{array}$$

where π and π_0 are the canonical projections and ρ is the natural simplicial isomorphism. By the same reasoning as in Lemma 3 the diagram

$$\begin{array}{ccc} H^n(X) & \xleftarrow{\ \varphi^*\ } & H^n(S^n) \\ {\scriptstyle\pi^*}\uparrow & & \uparrow{\scriptstyle\pi_0^*} \\ H^n(N_{\varphi^{-1}(\omega)}) & \underset{\rho^*}{\approx} & H^n(N_\omega) \end{array}$$

is commutative. Note that π_0^* is an isomorphism, since the elements of ω are open semispheres and the intersection of each finite system of semispheres is cohomologically trivial. Then $\xi \in \operatorname{Im}\varphi^*$ implies $\xi \in \operatorname{Im}\pi^*$, so that $\xi \in \operatorname{Im}\pi^* \setminus \operatorname{Im} f^*$. Hence, by Lemma 3, there exists $y_0 \in Y$ such that $f^{-1}(y_0)$ is not contained in any element of $\varphi^{-1}(\omega)$. It means that $\varphi f^{-1}(y_0)$ is not contained in any element of ω, but $y_0 \in V_{y_i}$ for some i, thus $\varphi f^{-1}(y_0) \subset \varphi f^{-1}(V_{y_i}) \subset O_{y_i} \in \omega$, which is a contradiction.

The theorem is proved.

Note that in the case $\varphi^* \not\equiv 0$, $f^* \equiv 0$ the existance of $\xi \in \operatorname{Im}\varphi^* \setminus \operatorname{Im} f^*$ is guaranteed and the theorem is valid.

Let X and A be disjoint closed subsets of $\mathbb{R}^N$. We say that X is *n-linked* with A in $\mathbb{R}^N$ if the inclusion map $i : X \to \mathbb{R}^N \setminus A$ is non-trivial in dimension n.

Assume for convenience that $\mathbb{R}^k \subset \mathbb{R}^n$ for $k < n$.

Corollary 1. *Let the compact space X be n-linked with $\mathbb{R}^k$ in $\mathbb{R}^{n+k+1}$ and $\pi : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+1}$ be the projection of $\mathbb{R}^{n+k+1}$ on the orthogonal complement* *of* $\mathbb{R}^k$ $(\mathbb{R}^{n+1} \cap \mathbb{R}^k = \{O\})$. *Put* $\alpha_0 = \inf_{x \in X} \|\pi(x)\|$. *Then for any map* $f : X \xrightarrow{\text{on}} Y$ *trivial in dimension* n *there exist* $x_1,\ x_2 \in X$ *such that* $f(x_1) = f(x_2)$ *and* $\|x_1 - x_2\| \geqq \alpha_0 \sqrt{\frac{2(n+2)}{n+1}}$.

Proof. Put $\varphi = \frac{\alpha_0 \pi}{\|\pi\|} i$, where $i : X \to \mathbb{R}^{n+k+1} \setminus \mathbb{R}^k$ is the inclusion map. Then $\frac{\alpha_0 \pi}{\|\pi\|} : \mathbb{R}^{n+k+1} \setminus \mathbb{R}^k \to S_0^n$, where $S_0^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = \alpha_0\}$. We have $\varphi^* = i^* \left(\frac{\alpha_0 \pi}{\|\pi\|}\right)^*$, therefore φ is non-trivial in dimension n, since $\left(\frac{\alpha_0 \pi}{\|\pi\|}\right)^*$ is an isomorphism and i is non-trivial in dimension n. Then the conditions of Theorem 1 are fulfiled, hence there exist $x_1,\ x_2 \in X$ such that $f(x_1) = f(x_2)$ and $\|\varphi(x_1) - \varphi(x_2)\| \geqq \alpha_0 \sqrt{\frac{2(n+2)}{n+1}}$ (φ maps X into S_0^n). Finally, we have

$$\|x_1 - x_2\| \geqq \|\pi(x_1) - \pi(x_2)\| \geqq \left\| \alpha_0 \frac{\pi(x_1)}{\|\pi(x_1)\|} - \alpha_0 \frac{\pi(x_2)}{\|\pi(x_2)\|} \right\|$$
$$= \|\varphi(x_1) - \varphi(x_2)\| \geqq \alpha_0 \sqrt{\frac{2(n+2)}{n+1}}.$$

The first inequality is obvious, the second one holds by the definition of α_0.

In his paper [1] J. Väisälä has proved that if X is a partition in $\mathbb{R}^{n+1}$ between O and ∞, then for any map $f : X \xrightarrow{\text{on}} Y$ trivial in dimension n there exists $y_0 \in Y$ such that the set $f^{-1}(y_0)$ is not contained in any open halfspace W with $\partial W \ni O$. Clearly, this theorem may be obtained by the non-metrical variant of Theorem 1 and Corollary 1 — we only have to replace the condition "there exist $x_1,\ x_2 \in X$ such that $f(x_1) = f(x_2)$ and $\|\varphi(x_1) - \varphi(x_2)\| \geqq \sqrt{\frac{2(n+2)}{n+1}}$" by "there exists $y_0 \in Y$ such that $\varphi f^{-1}(y_0)$ is not contained in any open semisphere". To obtain now the theorem of Väisälä, it is enough to take Corollary 1 in the case $k = 0$; really, a compact space X is n-linked with O in $\mathbb{R}^{n+1}$ iff X is a partition between O and ∞.

Corollary 2. *For any map* $f : S^n \xrightarrow{\text{on}} Y$ *trivial in dimension* n *there exist* $x_1,\ x_2 \in S^n$ *such that* $f(x_1) = f(x_2)$ *and* $\|x_1 - x_2\| \geqq \sqrt{\frac{2(n+2)}{n+1}}$.

This corollary may be immediately obtained by the theorem of Väisälä and Lemma 2.

We may ask whether f identifies a pair of points $x_1,\ x_2 \in S^n$ with $\|x_1 - x_2\| = \sqrt{\frac{2(n+2)}{n+1}}$. It is not difficult to show that the answer is "no" — there exists a map $f : S^1 \xrightarrow{\text{on}} T$ from S^1 onto the letter T such that $\|x_1 - x_2\| = \sqrt{3}$ implies $f(x_1) \neq f(x_2)$.

Another question is whether $\sqrt{\dfrac{2(n+2)}{n+1}}$ is the greatest number with this property. The answer is "yes" and the corresponding example may be constructed as follows:

Let P be the regular $(n+1)$-simplex inscribed in S^n and $P^{(n-1)}$ be its $(n-1)$-dimensional squeleton. Put $Y = C\,P^{(n-1)}$, where $C\,P^{(n-1)}$ is the cone over $P^{(n-1)}$ with a vertex O. The obvious deformation $f : S^n \to Y$ has the property $\operatorname{diam} f^{-1}(y) \leqq \sqrt{\dfrac{2(n+2)}{n+1}}$ for any $y \in Y$. Moreover, $\operatorname{diam} f^{-1}(y) = \sqrt{\dfrac{2(n+2)}{n+1}}$ iff $y = O$.

Theorem 2. *Let S^n be the union of n closed subsets $S^n = \bigcup_{i=1}^{n} F_i$. Then some F_i contains a continuum K with* $\operatorname{diam} K \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$.

Proof. Suppose the contrary — then there exist $\varepsilon > 0$ such that none of the sets $\overline{O_\varepsilon F_i}$ contains such a continuum. Let ω_i be the finite family of all components of $\overline{O_\varepsilon F_i}$. Set $\omega = \bigcup_{i=1}^{n} \omega_i$. Then ω is a covering of S^n with $\operatorname{ord} \omega \leqq n$, since every ω_i is a disjoint family. Thus, the nerve N_ω is an $(n-1)$-dimensional polyhedron and the cone $C\,N_\omega$ is an n-dimensional contractible one. Consider the map $f = i\pi$, where $\pi : S^n \to N_\omega$ is the canonical projection and $i : N_\omega \to C\,N_\omega$ is the inclusion map. Then $f : S^n \to C\,N_\omega$ is trivial in each dimension and according to Corollary 2 there exist $x_1,\ x_2 \in S^n$ such that $f(x_1) = f(x_2)$ and $\|x_1 - x_2\| \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$. But $f(x_1) = f(x_2)$ implies $\pi(x_1) = \pi(x_2)$, i. e. x_1 and x_2 belong to one and the same element K of ω, therefore $\operatorname{diam} K \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$, which contradicts the assumption.

Evidently, n is the greatest number with this property. In the case $n = 2$ we may even prove that for arbitrary decomposition $S^2 = F_1 \cup F_2$ some F_i contains a continuum K with $\operatorname{diam} K = 2$ (i. e. containing a pair of antipodal points). Really, if we suppose the contrary, we find as above a map $\pi : S^2 \to N_\omega$, where N_ω is an 1-dimensional polyhedron such that $\pi(x) \neq \pi(-x)$ for any $x \in S^2$, which contradicts a theorem of E. V. Schepin [3]. For $n \geq 3$ this is not true. Väisälä [1] has constructed a map $f : S^n \to P_k$ of S^n into a k-dimensional polyhedron, where $k = \left[\dfrac{n+1}{2}\right]$, such that $f(x) \neq f(-x)$ for any $x \in S^n$. Let $P_k = \bigcup_{i=1}^{k+1} F_i$ be the representation of P_k from Lemma 4, where $\varepsilon = \dfrac{1}{2} \min_{x \in S^n} \|f(x) - f(-x)\|$.

Then $S^n = \bigcup_{i=1}^{k+1} f^{-1}(F_i)$ and none of the $f^{-1}(F_i)$ contains a continuum K with $\operatorname{diam} K = 2$. Nevertheless, we do not know whether $\sqrt{\frac{2(n+2)}{n+1}}$ is the greatest number with this property.

Lemma 4. *Given $\varepsilon > 0$, every n-dimensional polyhedron P_n may be represented as the union $P_n = \bigcup_{i=1}^{n+1} F_i$ of $n+1$ closed subsets such that the components of each F_i have a diameter $< \varepsilon$.*

To prove it, one has to carry out induction on n taking some sufficiently small subdivision of P_n and considering its $(n-1)$-squeleton.

Theorem 3. *Let $f : S^n \to Y$ be a lowering dimension map. Then for some $y_0 \in Y$ the set $f^{-1}(y_0)$ contains a continuum K with* $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$.

Proof. Assume that $Y = f(X)$, so that Y is a compact space with $\dim Y \leqq n-1$. Suppose the contrary. Then we can find a closed finite covering ω of Y with ord $\omega \leqq n$ such that for any $\Phi \in \omega$ the set $f^{-1}(\Phi)$ does not contain such a continuum. Since $\dim Y \leqq n-1$, there is an ω-map $h : Y \to P_{n-1}$ of Y into some $(n-1)$-dimensional polyhedron. Let γ be a closed covering of P_{n-1} such that $h^{-1}(\gamma)$ is inscribed in ω. According to Lemma 4, P_{n-1} may be represented as $P_{n-1} = \bigcup_{i=1}^{n} F_i$, where each component of the sets F_i is contained in some element of γ. Consider the representation of S^n

$$S^n = \bigcup_{i=1}^{n} f^{-1}\, h^{-1}(F_i).$$

According to Theorem 2, some $f^{-1}\, h^{-1}(F_i)$ contains a continuum K with $\operatorname{diam} K \geqq \sqrt{\frac{2(n+2)}{n+1}}$. Then $h\, f(K)$ is contained in some component of F_i and thus in some element of γ. Since $f(K) \subset h^{-1}\, h\, f(K)$, then $f(K)$ is contained in an element of $h^{-1}(\gamma)$ and hence in some element Φ of ω. But then we have $f^{-1}(\Phi) \supset K$, which contradicts the definition of ω.

Our remarks on Theorem 2 remain valid here — for any lowering dimension map $f : S^2 \to Y$ some $f^{-1}(y_0)$ contains a continuum K with $\operatorname{diam} K = 2$ (this is proved by the author in [2] for arbitrary maps $f : S^{2k} \to Y_k$, where $\dim Y_k \leqq k$). On the other hand, for $n \geqq 3$ it is not true, as following from the Väisälä's example.

Let $\varphi : X \to S^n$ be a map non-trivial in dimension n and $f : X \to P_n$ maps X into an n-dimensional contractible polyhedron P_n. We shall prove that for some $x_0 \in X$ the set $\varphi f^{-1} f(x_0)$ is not contained in any open semisphere of S^n. It is enough to show that $H^n(f(X)) = \{0\}$. Really, consider in this case f as a map

$f : X \to f(X)$, then $f^* \equiv 0$ and the existence of $x_0 \in X$ with this property follows by the proof of Theorem 1. Form the exact sequence

$$H^n(P_n) \xrightarrow{i^*} H^n(f(X)) \xrightarrow{\partial^*} H^{n+1}(P_n,\, f(X)).$$

Here $H^n(P_n) = \{0\}$, since P_n is contractible and $H^{n+1}(P_n,\, f(X)) = \{0\}$, consequently $H^n(f(X)) = \{0\}$.

Then Lemma 2 implies diam $\varphi f^{-1} f(x_0) \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$, i. e. there exists $x_1 \in X$ such that $f(x_0) = f(x_1)$ and $\|\varphi(x_0) - \varphi(x_1)\| \geqq \sqrt{\dfrac{2(n+2)}{n+1}}$.

Suppose now $f : X \to P_k$ maps X into a k-dimensional polyhedron and $k < n$. At the close of this note we shall answer the question how many $x_0 \in X$ do there exist with the above-mentioned property.

Let $\varphi : X \to S^n$ maps the compact space X into the n-sphere S^n. We shall write

$$\gamma(X,\, \varphi) \leqq k$$

if there exists a map $f : X \to P_{k+1}$ of X into a $(k+1)$-dimensional contractible polyhedron, such that for any $x \in X$ the set $\varphi f^{-1} f(X)$ is contained in some open semisphere of S^n.

The previous reasoning shows, that if φ is non-trivial in dimension n, then $\gamma(X,\, \varphi) \geqq n$.

Lemma 5. $\gamma(X,\, \varphi) \leqq \dim X$ *for any compact space* X.

Proof. Suppose $\dim X = k$. There is a finite open covering ω of X such that for every $U \in \omega$ the set $\varphi(U)$ is contained in some open semisphere of S^n. There exists an ω-map $f : X \to P_k$ of X into a k-dimensional polyhedron P_k. Then for any $x \in X$ the set $\varphi f^{-1} f(x)$ is contained in some open semisphere. Denote by $C\,P_k$ the cone over P_k. It is clear that $C\,P_k$ is a contractible $(k+1)$-dimensional polyhedron, and if we consider f as a map $f : X \to C\,P_k$, then f has the required property.

Consequently, $\gamma(X,\, \varphi) \leqq k$.

Lemma 6. *Let* $\gamma(X,\, \varphi) \geqq n$ *and* $f : X \to P_k$ *maps* X *into a contractible* k*-dimensional polyhedron. Consider the set*

(1) $A(f) = \left\{x \in X \mid \varphi f^{-1} f(x) \text{ is not contained in any open semisphere of } S^n\right\}$.

Then $\gamma(A(f),\, \varphi|_{A(f)}) \geqq n - k$.

Proof. Suppose that $\gamma(A(f),\, \varphi|_{A(f)}) \leqq n-k-1$, i. e. that there exists a map $g : A(f) \to Q$ of $A(f)$ into the $(n-k)$-dimensional contractible polyhedron Q such that for every $x \in A(f)$ the set $\varphi g^{-1} g(x)$ is contained in some open semisphere of S^n. Since Q is contractible, g has an extension $\tilde{g} : X \to Q$. Form the map

$$h = f \times \tilde{g} : X \to P_k \times Q.$$

Clearly, $P_k \times Q$ is an n-dimensional contractible polyhedron. We shall prove that for any $x \in X$ the set $\varphi\, h^{-1}\, h(x)$ is contained in some open semisphere. Note, that

$$\varphi\, h^{-1}\, h(x) \subset \varphi\, f^{-1}\, f(x) \cap \varphi\, \tilde{g}^{-1}\, \tilde{g}(x).$$

In the case $x \notin A(f)$ the set $\varphi\, f^{-1}\, f(x)$ is contained in some open semisphere, thus $\varphi\, h^{-1}\, h(x)$ is contained in the same semisphere.

Suppose now that $x \in A(f)$. Then $f^{-1}\, f(x) \subset A(f)$ and $h^{-1}\, h(x) = f^{-1}\, f(x) \cap \tilde{g}^{-1}\, \tilde{g}(x) \subset A(f) \cap \tilde{g}^{-1}\, \tilde{g}(x)$. But $A(f) \cap \tilde{g}^{-1}\, \tilde{g}(x) \subset g^{-1}\, g(x)$, really, if $y \in A(f) \cap \tilde{g}^{-1}\, \tilde{g}(x)$, then $\tilde{g}(y) = \tilde{g}(x)$ and $\tilde{g}(y) = g(y)$, $\tilde{g}(x) = g(x)$, so that $g(y) = g(x)$, thus $y \in g^{-1}\, g(x)$. Consequently, $h^{-1}\, h(x) \subset g^{-1}\, g(x)$, hence $\varphi\, h^{-1}\, h(x) \subset \varphi\, g^{-1}\, g(x)$. The set $\varphi\, g^{-1}\, g(x)$ is contained in some open semisphere, therefore $\varphi\, h^{-1}\, h(x)$ is contained in the same one.

All this reasoning implies that $\gamma(X, \varphi) \leqq n-1$, which contradicts the condition $\gamma(X, \varphi) \geqq n$.

Theorem 4. *Let X be a compact metric space and the map $\varphi : X \to S^n$ be non-trivial in dimension n. Let $f : X \to P_k$ map X into a contractible k-dimensional polyhedron. Consider the set*

$$B(f) = \left\{ x \in X \mid \operatorname{diam} \varphi\, f^{-1}\, f(x) \geqq \sqrt{\frac{2(n+2)}{n+1}} \right\}.$$

Then $\dim B(f) \geqq n - k$.

Proof. As it is shown above, $\gamma(X, \varphi) \geqq n$. If we consider the set $A(f)$ defined by (1), then the inequality $\gamma(A(f), \varphi|_{A(f)}) \geqq n - k$ holds by Lemma 6, hence $\dim A(f) \geqq n - k$ by Lemma 5. Obviously, $A(f) \subset B(f)$, therefore $\dim B(f) \geqq n - k$.

Corollary. *Let $f : S^n \to P_k$ map S^n into a k-dimensional contractible polyhedron P_k. Consider the set*

$$B(f) = \left\{ x \in S^n \mid \operatorname{diam} f^{-1}\, f(x) \geqq \sqrt{\frac{2(n+2)}{n+1}} \right\}.$$

Then $\dim B(f) \geqq n - k$.

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SOME SITUATION THEORETICAL NOTIONS*

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Русанка Луканова, Робин Купр. НЕКОТОРЫЕ ПОНЯТИЯ СИТУАЦИОННОЙ ТЕОРИИ

Ситуационная теория ставит себе целью предложить адекватные математические средства для семантики естественных языков. За последние десять лет она пережила ряд перемен и продолжает развиваться. Со своей стороны ситуационная теория побуждает появление формализмов, подходящих для строения моделей некоторых ее объектов, которые не всегда фундированные [1, 2].

В этой статье мы напоминаем о некоторых понятиях ситуационной теории, введеных в [5, 11, 12], и предлагаем другие как например сильная/слабая информативность ситуации s' по отношению к другой ситуации s'', следование и эквивалентность суждений и типов. Они и их свойства необходимы для „исчисления" интерпретаций выражений (фраз), порожденных грамматикой GR2, исследована в [14]. Лучше было бы рассматривать эту статью как ставящую задачу о построении модели введеных понятий с помощью аппарата из [2], чем как вклад в теории моделей о ситуациях.

Roussanka Loukanova, Robin Cooper. SOME SITUATION THEORETICAL NOTIONS

Situation Theory is meant to provide an adequate mathematical tool for Semantics of Natural Languages. In the last ten years it has passed through a lot of changes and is still developing. On its part Situation Theory has motivated the appearance of formalisms appropriate for modelling situation theoretical objects that are not necessarily well founded ([1, 2]).

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* After having been used in [14], the notions introduced in this paper have undergone quite a lot of revisions in result of discussions between the authors during the stay of one of them, R. Loukanova, at the Center for Cognitive Science, Edinburgh University, Feb.–Mar., 1994. The stay was supported by a TEMPUS grant.

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In this article we remind some situation theoretical notions, introduced in [5, 11, 12], and suggest some others such as *strong/weak informativeness of a situation s' with respect to another situation s''*, *envolving* and *equivalence* for propositions and types. All of them and their properties are needed for the "calculations" of the interpretations of phrases generated by the grammar GR2 elaborated in [14]. One of the possibilities for representing the information transferred by uttering of natural language phrases is to accept that the models of the real situations including the situations described by the utterances are *informative with respect to* the situations used in them for describing some objects. Something more, in the situation semantics, proposed by the GR2, we assume that the described situations are *strongly informative with respect to* the speaker's resource situations. That gives a way to conclude whether what the speaker claims by an utterance is true. An important means for representing an equivalent but differently structured information in Situation Theory is the operation *absorption* of parameters (abstraction) over infons and propositions. By this operation complex relations and types are received in addition to the primitive relations and types. In this article we propose to use complex types as special kind relations — "situated" relations that could be prescribed to objects.

The article should be considered as setting a task for modelling the introduced notions with the apparatus developed in [2] rather than a contribution to the Model Theory of Situations.

1. INTRODUCTION

Let introduce informally some notions from Naive Situation Theory used in Situation Semantics. Short introductions into the terminology are also [5] and [11]. Building a formal model of Situation Theory presupposes some familiarity with the books [1] and [2]. For building models of Situation Theory see [9] and [10].

Situation theoretical objects are objects built up from the next primitives:

A — the collection of primitive individuals;

R — the collection of primitive relations; each relation comes with a set of argument roles associated with conditions for appropriate filling;

$\mathbb{T}$ — the collection of primitive types; each type comes with a set of argument roles associated with conditions for appropriate filling;

$\mathbb{P}$ — the collection of primitive parameters.

Let γ be a relation or a type (primitive or complex) with a set of argument roles $\text{Arg}(\gamma)$. An *assignment (filling) for γ* is any partial function θ with $\text{Dom}(\theta) \subseteq \text{Arg}(\gamma)$, the values of which are situation theoretical objects satisfying the conditions for appropriateness for γ.

Basic *infons* (the term *infon* is an abbreviation for *information*) are objects of the kind

$$\ll \gamma, \theta; i \gg,$$

where γ is a relation or a type (primitive or complex), θ is an *assignment for* γ, and $i \in \{0, 1\}$. No order over the argument roles is assumed, but the following notations for infons $\ll \gamma, \theta; i \gg$ are used often in the literature on Situation Semantics:

$\ll \gamma, \text{arg}_1 : \theta(\text{arg}_1), \ldots, \text{arg}_n : \theta(\text{arg}_n); i \gg$,

$\ll \gamma, \theta(\text{arg}_1), \ldots, \theta(\text{arg}_n); i \gg$ (in this notation argument roles are implied and their revelation is left to the reader),

where γ is a relation, $\{\text{arg}_1, \ldots, \text{arg}_n\}$ is the set of argument roles of γ, θ is an assignment for γ, and $i \in \{0, 1\}$. Usually, it is said that $\theta(\text{arg}_j)$ fills the argument role arg_j.

For example, if we have the primitive relations *chair* and *sit* and $a, b \in A$, then

$$\ll \text{chair}, \text{arg} : b; 1 \gg$$

and

$$\ll \text{sit}, \text{subj} - \text{arg} : a, \text{obj} - \text{arg} : b; 1 \gg$$

are infons. In the alternative notation these infons are written

$$\ll \text{chair}, b; 1 \gg \quad , \qquad \ll \text{sit}, a, b; 1 \gg .$$

When θ is not defined for some argument role arg_k, $k \in \{1, \ldots, n\}$, the following notation is used:

$$\ll \gamma, \text{arg}_1 : \theta(\text{arg}_1), \ldots, \text{arg}_k : \_\ , \ldots, \text{arg}_n : \theta(\text{arg}_n); i \gg .$$

For example,

$$\ll \text{sit}, \text{subj} - \text{arg} : a, \text{obj} - \text{arg} : \_\ ; 1 \gg$$

is an infon that does not specify the object a is sitting on. Such infons are called unsaturated and they are interpreted existentially, i.e. there is an object μ such that

$$\ll \text{sit}, \text{subj} - \text{arg} : a, \text{obj} - \text{arg} : \mu; 1 \gg .$$

Complex infons are obtained out of infons by the traditional operations like $\vee$, $\wedge$ and by quantification.

There is an operation over infons building complex relations — *absorption* of some of the parameters occurred in an infon (basic or complex).

Because of simplicity of the representation we assume that different occurrences of the constituent relations in an infon σ have different argument roles. We write $\sigma[\theta]$ when $\theta = \{\theta_1, \ldots, \theta_n\}$ is a list of the assignments occurred in σ. Let $\theta' = \{\theta'_1, \ldots, \theta'_n\}$ be another list of assignments for the argument roles in σ. We also write $\sigma[\theta']$ for the result of replacement in σ of the occurrences of the assignments $\theta_1, \ldots, \theta_n$ correspondingly with $\theta'_1, \ldots, \theta'_n$.

Let $\sigma(\xi_1, \ldots, \xi_n)$ be a parametric infon (basic or complex), where $\xi_1, \ldots, \xi_n$ is a list of some of the parameters in σ. The result of application of the operation *absorption of the parameters* $\xi_1, \ldots, \xi_n$ over the infon $\sigma(\xi_1, \ldots, \xi_n)$ is a *complex relation*, written

$$[\xi_1, \ldots, \xi_n / \sigma(\xi_1, \ldots, \xi_n)].$$

The argument roles of this relation are noted as $[\xi_1], \ldots, [\xi_n]$.

For example, the objects

$$[\xi / \ll \text{chair}, \xi; 1 \gg] \quad \text{and} \quad [\xi / \ll \text{chair}, \xi; 1 \gg \wedge \ll \text{sit}, a, \xi; 1 \gg]$$

are complex relations representing correspondingly the property to be a chair and the property to be a chair the individual a is sitting on.

The *propositions* are objects of the form $(\theta : T)$, where T is a type (primitive or complex) and θ is an assignment for argument roles of T.

A proposition $(\theta : T)$ is true just in the case when the objects that are values of the assignment θ are of type T.

In this article we are concerned mainly with a special kind of propositions, modelling the claims that in a situation s some objects are in or are not in some relations. For this purpose there is a primitive type $\models$ with two argument roles — the *situation* argument role and the *infon* role.

The proposition $(\theta : \models)$, where

$$\theta(\text{situation}) = s \text{ for some situation } s$$

and

$$\theta(\text{infon}) = \sigma \text{ for some infon } \sigma,$$

is written usually as

$$(s \models \sigma).$$

We write $s \models \sigma$ when the proposition $(s \models \sigma)$ is true, and $s \not\models \sigma$ when the proposition $(s \models \sigma)$ is false.

The situations are sets of infons and it is required that:

$$s \models \sigma \text{ iff } \sigma \in s \quad \text{for every basic infon } \sigma;$$

and for any infons σ_1 and σ_2

$$s \models \sigma_1 \wedge \sigma_2 \text{ iff } s \models \sigma_1 \text{ and } s \models \sigma_2,$$

$$\text{if } s \models \sigma_1 \vee \sigma_2, \text{ then } s \models \sigma_1 \text{ or } s \models \sigma_2.$$

Complex types are formed from propositions by the operation absorption. Let $p(\xi_1, \ldots, \xi_n)$ be a parametric proposition, where $\xi_1, \ldots, \xi_n$ is a list of some of the parameters occurred in p. The result of application of the operation *absorption of parameters* $\xi_1, \ldots, \xi_n$ over the proposition $p(\xi_1, \ldots, \xi_n)$ is a *complex type*, written

$$[\xi_1, \ldots, \xi_n / p(\xi_1, \ldots, \xi_n)].$$

The argument roles of this *type* are noted by $[\xi_1], \ldots, [\xi_n]$.

For example, we could form the type

$$[\xi, \zeta / (s \models \ll \text{chair}, \xi; 1 \gg \wedge \ll \text{sit}, \zeta, \xi; 1 \gg)].$$

The proposition

$$(\theta : [\xi, \zeta / (s \models \ll \text{chair}, \xi; 1 \gg \wedge \ll \text{sit}, \zeta, \xi; 1 \gg)]),$$

where $\theta([\xi]) = a$ and $\theta([\zeta]) = b$, represents the claim that the objects a and b are of type

$$[\xi, \zeta / (s \models \ll \text{chair}, \xi; 1 \gg \wedge \ll \text{sit}, \zeta, \xi; 1 \gg)].$$

This proposition is true just in the case when

$$s \models \ll \text{chair}, b; 1 \gg \wedge \ll \text{sit}, a, b; 1 \gg .$$

The absorption is a binding operator that binds the absorbed parameters, i.e. the parameters $\xi_1, \ldots, \xi_n$ are not already among the parameters of the object $[\xi_1, \ldots, \xi_n / \mu(\xi_1, \ldots, \xi_n)]$. The set of the parameters of a situation theoretical object μ, $\text{Par}(\mu)$, is the set of the "free" parameters that occur in it. More precisely, we could define the set $\text{Par}(\mu)$ inductively:

1. If $\mu \in A \cup R \cup \mathbb{T}$, then $\text{Par}(\mu) = \varnothing$;
2. If $\mu \in \mathbb{P}$, then $\text{Par}(\mu) = \{\mu\}$;

3. If $\mu = \ll \gamma, \theta; i \gg$, where γ is a relation or type, θ is an assignment for γ and $i \in \{0,1\} \cup \mathbb{P}$, then

$$\mathrm{Par}(\mu) = \mathrm{Par}(\gamma) \cup \mathrm{Par}(i) \cup \bigcup_{\mathrm{arg} \in \mathrm{Arg}(\gamma)} \mathrm{Par}(\theta(\mathrm{arg}));$$

4. If $\mu = \mu_1 \vee \mu_2$, where μ_1 and μ_2 are infons (or types), then

$$\mathrm{Par}(\mu) = \mathrm{Par}(\mu_1) \cup \mathrm{Par}(\mu_2);$$

5. If $\mu = \mu_1 \wedge \mu_2$, where μ_1 and μ_2 are infons (or types), then

$$\mathrm{Par}(\mu) = \mathrm{Par}(\mu_1) \cup \mathrm{Par}(\mu_2);$$

6. If $\mu = [\xi/\nu(\xi)]$, where ξ is a set of parameters and ν is an infon or a proposition, then

$$\mathrm{Par}(\mu) = \mathrm{Par}(\nu) - \xi.$$

Let μ be a parametric situation theoretical object. Let c be a function such that $\mathrm{Dom}(c) \subseteq \mathbb{P}$, $\mathrm{Par}(\mu) \subseteq \mathrm{Dom}(c)$, and its values are situation theoretical objects that are not parametric. Let $\mu(c)$ be the object obtained by replacing each "free" occurrence of any parameter $\zeta \in \mathrm{Dom}(c) \cap \mathrm{Par}(\mu)$ with $c(\zeta)$. The function c is called *anchor* for μ when $\mu(c)$ is a situation theoretical object (i.e. all conditions for appropriateness are satisfied).

2. INFORMATIVENESS IN SITUATION SEMANTICS

Let go through some of the notions and their properties used for Situation Semantics provided by the grammar GR2 in [14]. Everywhere to the end of these notes we deal only with assignments which are total in sense that an assignment for a relation or a type is defined for all of its argument roles.

We accept that it is possible for a parametric proposition $p(\zeta_1, \ldots, \zeta_n)$ to be true. Truth parametric propositions have the existential interpretation:

Definition 1. A parametric proposition $p(\zeta)$ is true with respect to an anchor c for p if $p(c)$ is true.

Definition 2. A parametric proposition $p(\zeta)$ is *true* if there exists an anchor c for the parameters of $p(\zeta)$ such that $p(c)$ is true.

Definition 3. Let $p_1(\zeta)$ and $p_2(\xi)$ be parametric propositions, where ζ and ξ are correspondingly the lists of the parameters of the propositions p_1 and p_2. The proposition $p_1(\zeta)$ *involves* the proposition $p_2(\xi)$, written $p_1(\zeta) \Rightarrow p_2(\xi)$, if for any anchor c_1 for $p_1(\zeta)$, such that $p_1(c_1)$ is true, there exists an anchor c_2 for $p_2(\xi)$ that is an extension of c_1 and such that $p_2(c_2)$ is true too.

Definition 4. The propositions $p_1(\zeta)$ and $p_2(\xi)$ are *equivalent*, written $p_1(\zeta) \Leftrightarrow p_2(\xi)$, iff $p_1(\zeta) \Rightarrow p_2(\xi)$ and $p_2(\xi) \Rightarrow p_1(\zeta)$.

Let $\sigma(\xi)$ be a parametric infon (basic or complex), where $\xi = \xi_1, \ldots, \xi_n$ is a list of some of the parameters in σ. Let Arg_j be the set of the argument roles in σ, filled by the parameter ξ_j, $j \in \{1, \ldots, n\}$ (a parameter could fill more than one argument role in σ). We assume that for each $i, j \in \{1, \ldots, n\}$, if $i \neq j$, then $\mathrm{Arg}_i \cap \mathrm{Arg}_j = \varnothing$.

Every assignment θ of the argument roles of the relation $[\xi/\sigma(\xi)]$ in the infon $\ll [\xi/\sigma(\xi)], \theta; 1 \gg$ (or of the type $[\xi/(s \models \sigma(\xi))]$ in the proposition $(\theta : [\xi/(s \models \sigma(\xi))]))$ could be used to define an assignment θ', filling the argument roles in σ. The assignment θ' is the same as the existing already assignment in σ, possibly except for the argument roles in $\mathrm{Arg}_1, \ldots, \mathrm{Arg}_n$, and

$$(1) \qquad \theta'(\mathrm{arg}) = \theta([\xi_j]) \quad \text{for each } \mathrm{arg} \in \mathrm{Arg}_j,\ j \in \{1, \ldots, n\}.$$

Property 1 ([12, p. 233]). For every situation s

$$(s \models \ll [\xi/\sigma(\xi)], \theta; 1 \gg) \Leftrightarrow (s \models \sigma[\theta']).$$

Property 2. For every type $[\xi/(s \models \sigma(\xi))]$ and its assignment θ

$$(\theta : [\xi/(s \models \sigma(\xi))]) \Leftrightarrow (s \models \sigma[\theta']).$$

Definition 5a. A situation s is *(weakly) propositionally informative*[3] if for any proposition p

$$(s \models \ll \mathrm{true}, p; 1 \gg) \Rightarrow p.$$

Definition 5b. A situation s is *strongly propositionally informative* if for any proposition p

$$(s \models \ll \mathrm{true}, p; 1 \gg) \Leftrightarrow p.$$

Corollary 1. Let s_1 be a propositionally informative situation. Then for any situation s, any parametric infon $\sigma(\xi)$ and any assignment θ of the argument roles of the type $[\xi/(s \models \sigma(\xi))]$

$$(s_1 \models \ll \mathrm{true}, (\theta : [\xi/(s \models \sigma(\xi))]); 1 \gg) \Rightarrow (s \models \sigma[\theta']),$$

where θ' is defined as in (1).

Property 3. If s is a model of a real situation (i.e. s is in a set of situations modelling parts of the real world), then it is (weakly) propositionally informative.

We would like to model limited, partial parts of the world, that is why we do not accept that the real situations are strongly propositionally informative. A special case of the notion of strong propositional informativeness looks more appropriate for representing the cases when some situations are "truth–tellers" with respect to some situations.

Definition 5c. A situation s_1 is *strongly propositionally informative with respect to a situation* s if for any parametric infon $\sigma(\xi)$ and for any assignment θ of the argument roles of the type $[\xi/(s \models \sigma(\xi))]$

$$(s_1 \models \ll \mathrm{true}, (\theta : [\xi/(s \models \sigma(\xi))]); 1 \gg) \Leftrightarrow (\theta : [\xi/(s \models \sigma(\xi))]).$$

If we accept a version of Situation Theory in which types are relations, a special kind of *situated relations*, then we could use them to build infons $\ll T, \theta; 1 \gg$, where T is a type and θ is an assignment for T. In such a way we could build two different propositions: $(\theta : T)$ and $(s \models \ll T, \theta; 1 \gg)$. If somebody insists on keeping a strong

[3] See the T-schema in [10].

distinction between the two kinds of objects — relations and types, a special two place primitive relation *be-of-type* could be used. Then the object $\ll T, \theta; i \gg$ could be introduced as a shorten record for the infon

$$\ll \text{be-of-type}, \theta, T; i \gg,$$

where T is a type, θ is an assignment for T, and $i \in \{0,1\}$.

The intuition behind accepting the types to be used as relations and for building infons is that the proposition $(s \models \ll T, \theta; 1 \gg)$ carries different information than $(\theta : T)$. The proposition $(\theta : T)$ just says that the objects in θ are of type T, while the proposition $(s \models \ll T, \theta; 1 \gg)$ claims something else — situation s supports the information that the objects in θ are of type T. In particular, we could build the following propositions:

(2) $$(\theta : [\xi/(s \models \sigma(\xi))])$$

— the proposition that the objects in the assignment θ are of type $[\xi/(s \models \sigma(\xi))]$;

(3) $$(s_1 \models \ll \text{true}, (\theta : [\xi/(s \models \sigma(\xi))]); 1 \gg)$$

— the proposition that the situation s_1 contains the information that the proposition $(\theta : [\xi/(s \models \sigma(\xi))])$ is true;

(4) $$(s_1 \models \ll [\xi/(s \models \sigma(\xi))], \theta; 1 \gg)$$

— the proposition that the situation s_1 contains the information that the objects in the assignment θ are of type $[\xi/(s \models \sigma(\xi))]$.

Definition 6. A situation s_1 is (*weakly*) *informative with respect to a situation* s if for any parametric infon $\sigma(\xi)$ and for any assignment θ of the argument roles of the type $[\xi/(s \models \sigma(\xi))]$

$$(s_1 \models \ll [\xi/(s \models \sigma(\xi))], \theta; 1 \gg) \Rightarrow (s \models \sigma[\theta']),$$

where the assignment θ' is defined as in (1).

The difference between these notions of informativeness is that in the propositional variant (Definition 5a), when the proposition (3) is true, we could involve the information that the objects in θ have the property σ in the situation s: $s \models \sigma[\theta']$ (Corollary 1). In the other variant, given by Definition 6, the proposition (4) says that the situation s_1 contains the information that the objects θ have some "situated" properties. We could involve the information $s \models \sigma[\theta']$ directly by the fact $\ll [\xi/(s \models \sigma(\xi))], \theta; 1 \gg$ that is supported by the situation s_1. The proposition (3) says "too much", while the information given by the proposition (2) is not "enough" — it does not say where the information that the objects θ are of type $[\xi/(s \models \sigma(\xi))]$ comes from, i.e. where the information that $s \models \sigma[\theta']$ comes from.

The most suitable formalism for modelling situation theoretical notions is given by Aczel's universes of structured objects, [1], and by Lunnun's generalized λ-universes which contain structured objects supplied by a component function and a replacement operation, [2]. Lunnun's λ-universes come with parameters and abstraction (i.e. absorption), where alphabetic variants are identified. The notion of many sorted λ-universe is used to build a situation theoretical universe of structured objects. In such an universe we need Property 1 and Property 2 to be held.

These properties insist on adding an appropriate application operation within the λ-universes. In [2] the application of an abstract is at the meta-level (see [2, p. 18]). Something more, accepting the types as relations legalized for building infons opens new questions about a special application operation that enables the replacement "in depth" of the informative situations (see Definition 6, Corollary 2, Corollary 3 and Property 6).

Property 4a. If s_1 is a model of a real situation, then it is propositionally informative with respect to every situation s.

Property 4b. If s_1 is a model of a real situation, then it is informative with respect to every situation s.

Definition 7. A situation s' is *strongly informative with respect to a situation* s'' if for any parametric infon $\sigma(\xi)$ and for any assignment θ, filling argument roles of the type $[\xi/(s'' \models \sigma(\xi))]$,

$$(s'' \models \sigma[\theta']) \Leftrightarrow (s' \models \ll [\xi/(s'' \models \sigma(\xi))], \theta; 1 \gg),$$

where the function θ' is defined as in (1).

We do not insist that the situations modelling parts of the world are strongly informative with respect to all situations. But for the calculations of the linguistic meanings in Gr2, [14], it is the case that some situations (described situations) are strongly informative with respect to others (resource situations).

Corollary 2. Let $\sigma(\xi)$ be a parametric infon, where $\xi = \xi_1, \ldots, \xi_n$ is a list of some of the parameters in σ. Let s and s_1 be situations such that s is strongly informative with respect to s_1. Then

$$(s \models \ll [\xi/(s_1 \models \sigma(\xi))], \theta; 1 \gg) \Leftrightarrow (s_1 \models \sigma(\xi)),$$

where θ is the assignment such that $\theta([\xi_j]) = \xi_j$ for each $j \in \{1, \ldots, n\}$.

Definition 8. A type T_1 *involves* a type T_2 with respect to a situation s, written $T_1 \underset{s}{\Rightarrow} T_2$, if for any assignment θ_1 for T_1 exists an assignment θ_2 for T_2 such that

$$(s \models \ll T_1, \theta_1; 1 \gg) \Rightarrow (s \models \ll T_2, \theta_2; 1 \gg).$$

Definition 9. Types T_1 and T_2 are *equivalent with respect to a situation s*, $T_1 \underset{s}{\Leftrightarrow} T_2$, if $T_1 \underset{s}{\Rightarrow} T_2$ and $T_2 \underset{s}{\Rightarrow} T_1$.

Definition 10. A type T_1 involves a type T_2, written $T_1 \Rightarrow T_2$, if for any assignment θ_1 for T_1 and for any situation s_1 exist an assignment θ_2 for T_2 and a situation s_2 such that

$$(s_1 \models \ll T_1, \theta_1; 1 \gg) \Rightarrow (s_2 \models \ll T_2, \theta_2; 1 \gg).$$

Definition 11. Types T_1 and T_2 are *equivalent*, written $T_1 \Leftrightarrow T_2$, if $T_1 \Rightarrow T_2$ and $T_2 \Rightarrow T_1$.

Proposition 1. Let

$$T_1 = [\xi/(s_1 \models \sigma_1(\xi))], \quad T_2 = [\xi/(s_2 \models \sigma_2(\xi))], \quad (s_1 \models \sigma_1(\xi)) \Rightarrow (s_2 \models \sigma_2(\xi)),$$

where s_1 and s_2 are situations, $\sigma_1(\xi)$ and $\sigma_2(\xi)$ are parametric infons, and $\xi = \xi_1, \ldots, \xi_n$ is a list of some of the common parameters in σ_1 and σ_2, i.e. $\{\xi_1, \ldots, \xi_n\} \subseteq \mathrm{Par}(\sigma_1) \cap \mathrm{Par}(\sigma_2)$ (i.e. there is an one-to-one function from $\mathrm{Arg}(T_1)$ onto $\mathrm{Arg}(T_2)$).

Let s be a situation that is informative with respect to s_1 and strongly informative with respect to s_2. Then $T_1 \underset{s}{\Rightarrow} T_2$.

Proof. Let Arg_j be the set of the argument roles in σ_1, filled by the parameter ξ_j, $j \in \{1, \ldots, n\}$. Let Arg'_j be the set of the argument roles in σ_2, filled by the parameter ξ_j, $j \in \{1, \ldots, n\}$. Let θ be an assignment for T_1 such that

$$s \models \ll T_1, \theta; 1 \gg, \quad \text{i.e. } s \models \ll [\xi/(s_1 \models \sigma_1(\xi))], \theta; 1 \gg.$$

Then by Definition 6

$$s_1 \models \sigma_1[\theta'],$$

where θ' is the assignment that is the same as the assignment in σ_1, possibly except for the argument roles in $\mathrm{Arg}_1, \ldots, \mathrm{Arg}_n$, and for each $\mathrm{arg} \in \mathrm{Arg}_j$, $j \in \{1, \ldots, n\}$,

$$\theta'(\mathrm{arg}) = \theta([\xi_j]).$$

By Definition 2 there is an anchor c for $(s_1 \models \sigma_1[\theta'])$ such that $s_1 \models \sigma_1[\theta'](c)$. Let define the anchor c' for $(s_1 \models \sigma_1(\xi))$ that is the same as c, possibly except for the parameters in ξ, and for them it is defined in the following way:

$$c'(\xi_j) = \theta'(\mathrm{arg})(c) \quad \text{for each } \mathrm{arg} \in \mathrm{Arg}_j,\ j \in \{1, \ldots, n\}.$$

Then $\sigma_1[\theta'](c)$ and $\sigma_1(c')$ are one and the same infon[4] and $s_1 \models \sigma_1(c')$. Hence, $s_2 \models \sigma_2(c'')$, where c'' is an extension of c'. Then $s_2 \models \sigma_2(\xi)$.

Let define the assignment θ'', filling the argument roles of the type $T_2 = [\xi/(s_2 \models \sigma_2(\xi))]$ in the following way:

$$\theta''([\xi_j]) = \xi_j \quad \text{for each } j \in \{1, \ldots, n\}.$$

The situation s is strongly informative with respect to s_2, so by Corollary 2

$$s \models \ll [\xi/(s_2 \models \sigma_2(\xi))], \theta''; 1 \gg, \quad \text{i.e. } s \models \ll T_2, \theta''; 1 \gg.$$

Proposition 2. Let

$$T_1 = [\xi/(s_1 \models \sigma_1(\xi))], \quad T_2 = [\xi/(s_2 \models \sigma_2(\xi))], \quad (s_1 \models \sigma_1(\xi)) \Leftrightarrow (s_2 \models \sigma_2(\xi)),$$

where s_1 and s_2 are situations, $\sigma_1(\xi)$ and $\sigma_2(\xi)$ are parametric infons. Let s be a situation that is strongly informative with respect to s_1 and s_2. Then $T_1 \underset{s}{\Leftrightarrow} T_2$.

Corollary 3. Let

$$T_1 = [\xi/(s \models \ll [\gamma/\sigma(\gamma, \xi)], \theta; 1 \gg)] \quad \text{and} \quad T_2 = [\xi/(s \models \sigma(\theta'))],$$

where $\sigma(\gamma, \xi)$ is a parametric infon such that $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n$ and $\xi = \xi_1, \ldots, \xi_k$ are some of its parameters, each parameter γ_j fills the argument roles of σ that are in the set Arg_j, $j \in \{1, \ldots, n\}$, θ' is the assignment for the argument roles in σ that is the same as the assignment in $\sigma(\gamma, \xi)$, possibly except for the argument roles in Arg_j, and $\theta'(\mathrm{arg}) = \theta([\gamma_j])$ for each $\mathrm{arg} \in \mathrm{Arg}_j$, $j \in \{1, \ldots, n\}$. Then $T_1 \underset{s'}{\Leftrightarrow} T_2$ for any situation s' that is strongly informative with respect to s.

Corollary 4. Let s_1 and s_2 be situations such that s_1 is strongly informative with respect to s_2. Let

$$T_1 = [\xi/(s_1 \models \ll [\gamma/(s_2 \models \sigma(\gamma, \xi))], \theta; 1 \gg)] \quad \text{and} \quad T_2 = [\xi/(s_2 \models \sigma(\theta'))],$$

[4] More precisely, in a model of Situation Theory, like [9], that could be proved by induction with respect to building the infon σ_1.

where $\sigma(\gamma, \xi)$ is a parametric infon such that $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n$ and $\xi = \xi_1, \ldots, \xi_k$ are some of its parameters, each parameter γ_j fills the argument roles of σ that are in the set Arg_j, $j \in \{1, \ldots, n\}$, θ' is the assignment for the argument roles in σ that is the same as the assignment in $\sigma(\gamma, \xi)$, possibly except for the argument roles in Arg_j, and $\theta'(\text{arg}) = \theta([\gamma_j])$ for each $\text{arg} \in \text{Arg}_j$, $j \in \{1, \ldots, n\}$.

Then $T_1 \underset{s}{\Leftrightarrow} T_2$ for any situation s that is strongly informative with respect to s_1 and s_2.

It is accepted in Situation Semantics that the speakers make claims $(s \models \sigma)$ describing some situation s via the utterances of natural language sentences. The situation s is called *described situation* for the utterance, and $(s \models \sigma)$ — *propositional content of the utterance.* Resource situations are used by the speakers to provide some individuals or information about some individuals needed for describing a situation s as being of certain type, i.e. for the propositional content of the utterance. Which situations are resource situations for an utterance is up to the speaker references. In Gr2, [14], it is accepted that if s is the described situation by an utterance, then it is strongly informative with respect to any resource situation s' for this utterance.

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CODE EVALUATION IN OPERATIVE SPACES WITH STORAGE OPERATION

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Йордан Зашев. КОДОВО ОЦЕНИВАНИЕ В ОПЕРАТОРНЫХ ПРОСТРАНСТВАХ С ОПЕРАТОРОМ СКЛАДИРОВАНИЯ

Рассматривается понятие оператора складирования в операторных пространствах Иванова, родственное соответствующему понятию Иванова. Для операторных пространств с оператором складирования доказана теорема кодового оценивания, из которой легко следуют почти все основные результаты алгебраической теории рекурсии для таких пространств. В качестве применений получаются основные результаты теории комбинаторных пространств Скордева в обобщенном варианте, свободном от использования констант.

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A concept of storage operation in an operative space is considered, which is closely related to Ivanov's concept of storing operation in such spaces. For operative spaces with storage operation a *code evaluation theorem*, implying almost all principal results of algebraic recursion theory for such spaces, is proved. As a special case these results are obtained for a generalized version of Skordev's theory of combinatory spaces, free from using constants.

0. INTRODUCTION

One of the methods in algebraic recursion theory is based on a principle which we call "the code evaluation theorem". This theorem is a fundamental result in the sense that all principal facts of algebraic recursion theory usually follow easily from it. For instance, in operative spaces in the sense of [1] the first recursion theorem, the normal form theorem, and the universal element theorem are near consequences of the code evaluation theorem. On the other hand, the last theorem in operative spaces requires suppositions which differ from those needed for other methods, especially the method of Ivanov [1]. The principal advantages of the suppositions needed for the method of code evaluation are connected with the possibility for further generalization of the theory. For instance, in the context of categorial generalizations [7], suppositions called "axioms μA_1, μA_2, μA_3, $t\mu A$" etc. in [1] become rather gross; there are examples of DM-categories in which to prove the analogue of these suppositions is almost as much difficult as to prove the analogue of the recursion theorem. Another important advantage is in the fact that the method of code evaluation is not crucially dependent upon the totality of the operation of iteration and gives interesting consequences for non-iterative spaces, as suggested in [8].

The code evaluation theorem was proved for various kinds of algebraic systems [4]. For operative spaces in the sense of [1] it was not published in its original form, but only for generalizations in different directions, as in [5], [7] and [8]. It was not clear, however, how this method will do in the case of the theory of combinatory spaces in the sense of [2].

The purpose of the present paper is to prove the code evaluation theorem for operative spaces with a storage operation and to show how it applies for combinatory spaces, providing in this way a basis for some generalizations of the theory of last spaces. The operative spaces with a storage operation were essentially introduced in [1], but the notion of storage operation in the sense of the present paper is not a special case of the notion of t-operation in [1]. The last spaces are interesting by themselves, but being a generalization of combinatory spaces, the code evaluation theorem in them gives as a consequence all principal results of recursion theory in last spaces, except the theorem of representation of partial recursive functions, however in suppositions which differ from those in [2], the difference being but of secondary significance. It provides also an elimination of constants from the theory of combinatory spaces, i. e. a generalization of the last theory, which is free from using "points", the elements of the set $\mathcal{C}$ in the original theory of combinatory spaces [2] playing a similar role.

1. PRELIMINARIES

In order to avoid confusion with notations in [2], our notations for operative spaces will differ from those of Ivanov [1], especially multiplication will be denoted in reverse order. Thus in the present paper by an operative space we shall mean a partially ordered algebra $\mathcal{F}$ with two binary operations: multiplication $\varphi\psi$ and pairing (φ, ψ) (for $\varphi, \psi \in \mathcal{F}$), and three constants I, T_+, F_+ such that $\mathcal{F}$ is a semigroup with an unit I with respect to multiplication, and the following equalities hold for all $\varphi, \psi, \chi \in \mathcal{F}$: $(\varphi, \psi)T_+ = \varphi$, $(\varphi, \psi)F_+ = \psi$, $\chi(\varphi, \psi) = (\chi\varphi, \chi\psi)$. Note that the last definition includes also the supposition that the both operations are increasing on both arguments. By *storage operation* in $\mathcal{F}$ we shall mean an unary operation S in $\mathcal{F}$ which is increasing and satisfies for suitable constants D, A_0, $A_1 \in \mathcal{F}$ and all $\varphi, \psi, \chi \in \mathcal{F}$ the following three equalities:

(S1) $$S(\varphi\psi) = S(\varphi)S(\psi);$$

(S2) $$S((\varphi, \psi)) = (S(\varphi), S(\psi))D;$$

(S3) $$S(S(\varphi)) = A_0 S(\varphi) A_1.$$

A partially ordered algebra $\mathcal{F}$ consisting of an operative space, an operation S and constants $D, A_0, A_1 \in \mathcal{F}$ satisfying (S1)–(S3) will be called an *operative space with storage*, or shortly OSS. An OSS will be called *regular*, iff the inequality

(S4) $$(T_+ S(I), F_+ S(I))D \leq DS(I)$$

is fulfilled in it.

Now let us fix an operative space $\mathcal{F}$ with a storage S; we shall write $\varphi^\frown$ for $S(\varphi)$. Let $\mathcal{K}$ be an arbitrary subset of $\mathcal{F}$; then by a *simple $\mathcal{K}$-admissible initial segment* we shall mean a subset $\mathcal{A} \subseteq \mathcal{F}$ of one of the following three forms:

1) $\mathcal{A} = \{\xi \in \mathcal{F} \mid \xi \leq \psi\}$, where $\psi \in \mathcal{F}$;

2) $\mathcal{A} = \{\xi \in \mathcal{F} \mid \varphi\xi\varkappa \leq \psi\}$, where $\varphi, \psi \in \mathcal{F}$ and $\varkappa \in \mathcal{K}$;

3) $\mathcal{A} = \{\xi \in \mathcal{F} \mid (\xi\varkappa)^\frown \leq \psi I^\frown\}$, where $\psi \in \mathcal{F}$ and $\varkappa \in \mathcal{K}$.

A *$\mathcal{K}$-admissible initial segment* is by definition a countable intersection of simple $\mathcal{K}$-admissible initial segments. An element $\vartheta \in \mathcal{F}$ will be called *$\mathcal{K}$-iteration* of another element $\varphi \in \mathcal{F}$, iff $(I, \vartheta)\varphi \leq \vartheta$ and for every $\mathcal{K}$-admissible initial segment $\mathcal{A} \subseteq \mathcal{F}$ such that

$$(I, \mathcal{A})\varphi = \{(I, \xi)\varphi \mid \xi \in \mathcal{A}\} \subseteq \mathcal{A}$$

we have $\vartheta \in \mathcal{A}$. We shall fix the set $\mathcal{K}$ and we shall write simply "iteration" instead of "$\mathcal{K}$-iteration". If ϑ is an iteration of φ, then ϑ is the least solution of $(I, \xi)\varphi \leq \xi$ with respect to ξ in $\mathcal{F}$, since the sets of the above form 1) are $\mathcal{K}$-admissible initial segments. Therefore the iteration of φ, if it exists, is unique, and in this case we shall denote it by $\mathbb{I}(\varphi)$. (Note the difference between our iteration $\mathbb{I}(\varphi)$ and that used by Ivanov $[\varphi]$; however, both iterations are easily expressible by each other.)

Next, suppose we are given an infinite list of formal symbols called variables and denoted by x, y, z with or without indexes; and suppose we have another list of symbols $c_0, \ldots, c_{l-1}$ called parameter symbols. We shall fix an interpretation assigning to each parameter symbol c_i a *parameter*, i. e. an element $\gamma_i \in \mathcal{F}$, called also *value* of c_i. We shall have also symbols for the elements I, T_+, F_+, D, A_0, A_1 of $\mathcal{F}$ which we shall denote by the same letters, so each of these elements is the value of the corresponding symbol denoted by the same letter. We shall call the last symbols basic constants; both parameter symbols and basic constants together will be called *constants*, and constants and variables together will be called *prime terms*. Now *terms* are defined inductively as it follows: all prime terms are terms; if t and s are terms, then (ts), (t, s) and $S(t)$ (or shortly $t^\frown$) are terms. We adopt usual conventions of dropping external brackets in multiplication (ts) of terms etc. Terms of the following two kinds: p, $p^\frown$, where p is a prime term, will be called *simple terms.* If no otherwise indicated, the letters t, s, p, q, r with or without indexes will denote the terms below. Terms of the form

$$(\ldots((tt_0)t_1)\ldots t_{n-1})t_n$$

will be written shortly as $tt_0\ldots t_{n-1}t_n$. An *evaluation* ϑ is a function with a finite domain $\mathrm{Dom}(\vartheta)$ consisting of variables and values in $\mathcal{F}$. The *value* $\widetilde{\vartheta}(t) \in \mathcal{F}$ of a term t under an evaluation ϑ defined for all variables occurring in t is defined inductively as it follows: $\widetilde{\vartheta}(t) = \vartheta(t)$ if t is a variable; $\widetilde{\vartheta}(t)$ is the value of t if t is a constant; $\widetilde{\vartheta}(ts) = \widetilde{\vartheta}(t)\widetilde{\vartheta}(s)$; $\widetilde{\vartheta}((t,s)) = (\widetilde{\vartheta}(t), \widetilde{\vartheta}(s))$; and $\widetilde{\vartheta}(t^\frown) = (\widetilde{\vartheta}(t))^\frown$. For some purposes it will be convenient to consider the empty word Λ as a special term with value I, and, accordingly, $\widetilde{\vartheta}(\Lambda) = I$ for any evaluation ϑ. By an *extraterm* we shall mean a word which is either a term or empty, and the letter P below will always range over extraterms. Thus we have $\Lambda P = P = P\Lambda$ for all extraterms P and we define $\Lambda^\frown = \Lambda$.

2. REDUCTIONS AND NORMAL FORMS OF TERMS

Expressions of the following five kinds:

(R1) $$(ts)^\frown \to t^\frown s^\frown;$$

(R2) $$((t,s))^\frown \to (t^\frown, s^\frown)D;$$

(R3) $$(t^\frown)^\frown \to A_0 t^\frown A_1;$$

(R4) $$t(sr) \to tsr;$$

(R5) $$t(s,r) \to (ts, tr),$$

will be called *contractions*. As usual, the notion of contraction gives rise to a reduction notion: we shall write $t \Vdash\!\!\to_1 s$ for "s is obtained from t by contracting of a redex in t", where by redex we mean an occurrence of a left side of a contraction, and contracting of a redex means replacing it by a corresponding occurrence of the right side of the same contraction; by $\Vdash\!\!\to$ we shall denote the reflexive transitive closure of the relation $\Vdash\!\!\to_1$. A term is *normal*, iff it does not contain redexes. An *S-redex* is a redex of one of the first three kinds, i. e. an occurrence of left side of (R1), (R2) or (R3). For each term t we define another one t^N by the following equalities:

(1) $$s^N = s, \quad \text{if } s \text{ is a simple term};$$

(2) $$(ts)^N = t^N s, \quad \text{if } s \text{ is a simple term};$$

(3) $$(t(sr))^N = (tsr)^N;$$

(4) $$(t(s,r))^N = ((ts)^N, (tr)^N);$$

(5) $$(ts^\frown)^N = (t(s^\frown)^N)^N;$$

(6) $$((t,s))^N = (t^N, s^N);$$

(7) $$((ts)^\frown)^N = (t^\frown, s^\frown)^N;$$

(8) $$((t,s)^\frown)^N = ((t^\frown)^N, (s^\frown)^N)D;$$

(9) $$((t^\frown)^\frown)^N = (A_0 t^\frown)^N A_1.$$

To see that this is indeed a correct definition, consider the ordinal number

$$\mu(t) = \alpha(t)\omega^3 + \beta(t)\omega^2 + \gamma(t)\omega + \delta(t),$$

where $\alpha(t)$ is the maximal length of S-redexes in t; $\beta(t)$ is the number of S-redexes in t; $\gamma(t)$ is the length of t; and $\delta(t) = \sum_{i<k} \gamma(t_i)$, where $t = pt_0 \dots t_{k-1}$ and the term p is not of the form p_0p_1. The equalities (1)–(9) are obviously defining at least a partial function t^N on terms t, but this function is total since an induction on $\mu(t)$ shows that t^N is defined and t^N is normal for every term t. Moreover, we have

Lemma 1. *For all terms t and s:*
(a) *$t \Vdash\!\!\to t^N$; and*
(b) *if $t \Vdash\!\!\to s$, then $t^N = s^N$.*

Consequently, $s^N = s$ for every normal term s and for an arbitrary term t t^N is the unique normal term s for which $t \Vdash\!\!\to s$.

Proof. (a) is obvious by an induction on $\mu(t)$; to prove (b) it is enough to show that $t \Vdash\!\!\to_1 s$ implies $t^N = s^N$. This is done also by induction on $\mu(t)$. It is convenient to write $t \Vdash\!\!\to_0 s$ for $t = s$. Suppose the hypothesis of the induction and consider nine cases for t as in the definition (1)–(9) of t^N. We shall consider the case corresponding to (3) only, the rest ones being similar or simpler (we are leaving them to the reader). This is the case when t has the form $r(pq)$. Let $t \Vdash\!\!\to_1 s$. Then two subcases are possible:

Subcase 1) $s = rpq$. Then $t^N = s^N$ by (3).

Subcase 2) $s = r_0(p_0q_0)$, where $r \Vdash\!\!\to_i r_0$, $p \Vdash\!\!\to_j p_0$, $q \Vdash\!\!\to_k q_0$, and i, j, k are natural numbers such that $i + j + k = 1$. Then $\mu(rpq) < \mu(t)$ and $rpq \Vdash\!\!\to_1 r_0p_0q_0$, whence, using the induction hypothesis, we have $t^N = (rpq)^N = (r_0p_0q_0)^N = s^N$. ∎

Lemma 2. *The function t^N on terms t is primitive recursive.*

Proof. This is not obvious since an induction on a higher ordinal was used in the definition of t^N. But the normal form function t^N can be represented as a composition of two primitive recursive functions R and B on terms defined below.

First define a function F on terms by the following equality:

$$F(t) = \begin{cases} t^\frown & \text{if } t \text{ is a prime term,} \\ F(s)F(r) & \text{if } t = sr, \\ (F(s), F(r))D & \text{if } t = (s, r), \\ A_0F(s)A_1 & \text{if } t = s^\frown. \end{cases}$$

This is a definition by induction on complexity of t, so F is primitive recursive and by a similar induction we see that $t^\frown \Vdash\!\!\to F(t)$ and $F(t)$ does not contain S-redexes.

Next, define by the same induction the function R as it follows:

$$R(t) = \begin{cases} t & \text{if } t \text{ is a simple term,} \\ R(s)R(r) & \text{if } t = sr, \\ (R(s), R(r)) & \text{if } t = (s, r), \\ F(s) & \text{if } t = s^\frown. \end{cases}$$

In the same way we see that R is primitive recursive, $t \Vdash\!\!\rightarrow R(t)$ and $R(t)$ does not contain S-redexes.

Finally, define the function B on terms containing no S-redexes by the following equality:

$$B(t) = \begin{cases} t & \text{if } t \text{ is a simple term,} \\ B(p)s & \text{if } t = ps \text{ and } s \text{ is a simple term,} \\ B(pqr) & \text{if } t = p(qr), \\ (B(pq), B(pr)) & \text{if } t = p(q, r), \\ (B(q), B(r)) & \text{if } t = (q, r). \end{cases}$$

The last definition proceeds by induction on the number $\varepsilon(t)$, defined by

$$\varepsilon(t) = \begin{cases} 0 & \text{if } t \text{ is a simple term,} \\ \varepsilon(r) + \gamma(s) & \text{if } t = rs, \\ \varepsilon(r) + \varepsilon(s) + 1 & \text{if } t = (r, s), \end{cases}$$

whence B is primitive recursive. Moreover, by induction on $\varepsilon(t)$ we see that for every term t containing no S-redexes $t \Vdash\!\!\rightarrow B(t)$ and $B(t)$ is normal. Therefore, for an arbitrary t we have $t \Vdash\!\!\rightarrow B(R(t))$ and $B(R(t))$ is normal, and by Lemma 1 $t^N = B(R(t))$. ∎

Finally, let us mention that the extraterm Λ will be considered as normal and, accordingly, Λ^N is Λ by definition; the function P^N on extraterms P is obviously primitive recursive.

3. THE CODE EVALUATION THEOREM

As in Section 1 we shall have fixed an OSS $\mathcal{F}$, an interpretation of parameter symbols in $\mathcal{F}$, and a subset $\mathcal{K} \subseteq \mathcal{F}$. Consider a formal system Σ of inequalities of the form

$$(10) \qquad s_i \le x_i, \quad i < n,$$

where $n \neq 0$ and $s_0, \ldots, s_{n-1}$ are normal terms containing no other variables than $x_0, \ldots, x_{n-1}$. For every extraterm P containing no other variables than $x_0, \ldots, x_{n-1}$ we shall write $\widetilde{P}(\xi_0, \ldots, \xi_{n-1})$ for the value $\widetilde{\vartheta}(P)$ of P under the evaluation $\vartheta : \{x_0, \ldots, x_{n-1}\} \to \mathcal{F}$ defined by $\vartheta(x_i) = \xi_i$, $i < n$. We shall write shortly $\overline{x}$ for $(x_0, \ldots, x_{n-1})$, $\overline{s}$ for $(s_0, \ldots, s_{n-1})$, and $\overline{\xi}$ for $(\xi_0, \ldots, \xi_{n-1})$. As usual, terms without variables will be called closed terms and the value of a closed term t will be denoted by $\widetilde{t}$. We shall call a term t a *fit* term, iff every occurring in t simple term of the form $x^\frown$ occurs in t only through occurrences in a subterm of the form $Px^\frown I^\frown$, as explicitly indicated in the last subterm; the extraterm Λ is also considered as fit. Obviously, for every term t there is a fit term t' containing the same variables and with the same value as t for every evaluation defined for the variables in t. A solution of (10) is defined as an n-tuple $\overline{\xi} \in \mathcal{F}^n$ such that $\widetilde{s}_i(\overline{\xi}) \le \xi_i$ for all $i < n$. *Least solution* of (10) is a solution $\mu = (\mu_0, \ldots, \mu_{n-1})$ of (10) such that for every solution $\overline{\xi} = (\xi_0, \ldots, \xi_{n-1})$ of (10) we have $\mu_i \le \xi_i$ for all $i < n$. It is obvious that any system Σ of the form (10) is equivalent to such one for which the left sides s_i are fit normal terms. We shall write $\underline{N}(\Sigma)$ for the set of all normal extraterms containing no other variables than those occurring in Σ, and $\underline{N}$ will be the set of all normal extraterms.

A set $K \subseteq \underline{N}(\Sigma)$ will be called *closed* with respect to the system Σ of the form (10) iff the following five conditions are fulfilled:

(C1) $x_i \in K$ for all $i < n$;

(C2) if $Ps \in K$ and s is a simple closed term, then $P \in K$;

(C3) if $(t_0, t_1) \in K$, then both $t_0, t_1 \in K$;

(C4) if $Px_i \in K$, then $(Ps_i)^N \in K$ and $P \in K$;

(C5) if $P(x_i)^\frown Q \in K$, where Q is either Λ or A_1, then

$$(P(s_i)^\frown)^N Q \in K \text{ and } P \in K.$$

Now by *coding* for the system Σ of the form (10) we shall mean a triple $\langle K, k, \sigma \rangle$, where $K \subseteq \underline{N}(\Sigma)$ is closed with respect to Σ, $k : K \to \mathcal{F}$ is a function, and σ is an element of $\mathcal{F}$ such that the following five conditions are fulfilled:

(K1) $\sigma k(t) = T_+$ if $t = \Lambda$;

(K2) $\sigma k(t) = F_+ k(P)\widetilde{s}$ if $t = Ps$ and s is a simple closed term;

(K3) $\sigma k(t) = F_+(k(t_0), k(t_1))$ if $t = (t_0, t_1)$;

(K4) $\sigma k(t) = F_+ k((Ps_i)^N)$ if $t = Px_i$, $i < n$;

(K5) $\sigma k(t) = F_+ k((P(s_i)^\frown)^N)$ if $t = P(x_i)^\frown$, $i < n$.

Theorem 1. *Suppose the OSS $\mathcal{F}$ is regular. Let $\langle K, k, \sigma \rangle$ be a coding for the system* (10) *with fit left sides and let $\omega \in \mathcal{F}$ be an iteration of σ. Then the n-tuple*

$$\omega k(\overline{x}) = (\omega k(x_0), \dots, \omega k(x_{n-1}))$$

is the least solution of the system (10) *in $\mathcal{F}$.*

Proof. Since ω is an iteration of σ, it satisfies the equality

$$(I, \omega)\sigma = \omega,$$

whence by multiplication from right by $k(t)$, $t \in K$, and using (K1)–(K5) we obtain the following equalities:

(11) $\omega k(t) = I$ if $t = \Lambda$;

(12) $\omega k(t) = \omega k(P)\widetilde{s}$ if $t = Ps$ and s is a simple closed term;

(13) $\omega k(t) = (\omega k(p), \omega k(q))$ if $t = (p, q)$;

(14) $\omega k(t) = \omega k((Ps_i)^N)$ if $t = Px_i$, $i < n$;

(15) $\omega k(t) = \omega k((P(s_i)^\frown)^N)$ if $t = P(x_i)^\frown$, $i < n$.

We shall prove the inequality

(16) $$\omega k(t)\omega k(s) \leq \omega k((ts)^N)$$

for all $t, s \in K$ such that $(st)^N \in K$. For that fix $t \in K$ and denote by K' the set $\{s \in k \mid (ts)^N \in K\}$. Then the subset $\mathcal{A} \subseteq \mathcal{F}$, defined by

$$\mathcal{A} = \left\{\vartheta \in \mathcal{F} \mid \forall s \in K' \left(\omega k(t)\vartheta k(s) \leq \omega k((ts)^N)\right)\right\},$$

is a $\mathcal{K}$-admissible initial segment. To show that $(I, \mathcal{A}) \subseteq \mathcal{A}$, suppose $\vartheta \in \mathcal{A}$ and consider cases for $s \in K'$ as it follows:

Case 1) $s = \Lambda$. Then by (K1) we have

$$\omega k(t)(I, \vartheta)\sigma k(s) = \omega k(t)(I, \vartheta)T_+ = \omega k(t) = \omega k((ts)^N).$$

Case 2) $s = Pq$ and q is a simple closed term. Then using (K2) and (12) we have

$$\begin{aligned}\omega k(t)(I, \vartheta)\sigma k(s) &= \omega k(t)(I, \vartheta)F_+k(P)\widetilde{q} = \omega k(t)\vartheta k(P)\widetilde{q}\\ &\leq \omega k((tP)^N)\widetilde{q} = \omega k((tP)^N q) = \omega k((ts)^N).\end{aligned}$$

Case 3) $s = (p, q)$. Then, similarly, by (K3) and (13)

$$\begin{aligned}\omega k(t)(I, \vartheta)\sigma k(s) &= \omega k(t)\vartheta(k(p), k(q)) = (\omega k(t)\vartheta k(p), \omega k(t)\vartheta k(q))\\ &\leq (\omega k((tp)^N), \omega k((tq)^N) = \omega k(((tp)^N, (tq)^N)) = \omega k((ts)^N).\end{aligned}$$

Case 4) $s = Px_i$, $i < n$. Then, similarly, using (K4), (13) and Lemma 1

$$\begin{aligned}\omega k(t)(I, \vartheta)\sigma k(s) &= \omega k(t)\vartheta k((Ps_i)^N) \leq \omega k((t(Ps_i)^N)^N) = \omega k((tPs_i)^N)\\ &= \omega k(((tP)^N s_i)^N) = \omega k((tP)^N x_i) = \omega k((ts)^N).\end{aligned}$$

Case 5) $s = P(x_i)\widehat{\ }$, $i < n$. Similarly,

$$\begin{aligned}\omega k(t)(I, \vartheta)\sigma k(s) &= \omega k(t)\vartheta k((P(s_i)\widehat{\ })^N) \leq \omega k((t(P(s_i)\widehat{\ })^N)^N) = \omega k(((tP)^N(s_i)\widehat{\ })^N)\\ &= \omega k((tP)^N(x_i)\widehat{\ }) = \omega k((ts)^N).\end{aligned}$$

Note that in cases 2)–5) we used also conditions (C1)–(C5) to ensure that the involved terms belong to K'. These cases exhaust all possible cases for $s \in K'$ since s is normal. Therefore, for all $s \in K'$ we have

$$\omega k(t)(I, \vartheta)\sigma k(s) \leq \omega k((ts)^N),$$

which means that $(I, \vartheta)\sigma \in \mathcal{A}$, and the inclusion $(I, \mathcal{A})\sigma \subseteq \mathcal{A}$ is proved. Since ω is an iteration of σ, we conclude that $\omega \in \mathcal{A}$, whence we get (16).

Next we shall prove that for all $t \in K \setminus \{\Lambda\}$ such that $(t\widehat{\ })^N \in K$ we have

$$(\omega k(t))\widehat{\ } \leq \omega k((t\widehat{\ })^N)I\widehat{\ }. \tag{17}$$

Indeed, let K'' be the set $\left\{t \in K \mid t \neq \Lambda \,\&\, (t\widehat{\ })^N \in K\right\}$ and consider the $\mathcal{K}$-admissible initial segment

$$\mathcal{B} = \left\{\vartheta \in \mathcal{F} \mid \vartheta k(\Lambda) \leq I \,\&\, \forall t \in K'' \left((\vartheta k(t))\widehat{\ } \leq \omega k((t\widehat{\ })^N)I\widehat{\ }\right)\right\}.$$

To prove that $(I, \mathcal{B})\sigma \subseteq \mathcal{B}$, take $\vartheta \in \mathcal{B}$. Then $(I, \vartheta)\sigma k(\Lambda) = (I, \vartheta)T_+ = I$ and to prove

$$((I, \vartheta)\sigma k(t))\widehat{\ } \leq \omega k((t\widehat{\ })^N) \tag{18}$$

for all $t \in K''$, consider cases for t as it follows:

Case 1) $t = \Lambda$. Impossible, since $t \in K''$.

Case 2) $t = Ps$, where s is a simple closed term. Then if s is a constant, we have, using (K2), (11) and (12),

$$((I,\vartheta)\sigma k(t))^\frown = (\vartheta k(P)\widetilde{s})^\frown \\ \leq \omega k((P^\frown)^N)I^\frown(\widetilde{s})^\frown = \omega k((P^\frown)^N)(\widetilde{s})^\frown I^\frown = \omega k((t^\frown)^N)I^\frown,$$

and if $s = c^\frown$, where c is a constant, we have, similarly,

$$((I,\vartheta)\sigma k(t))^\frown = (\vartheta k(P)\widetilde{s})^\frown \leq \omega k((P^\frown)^N)((\widetilde{c})^\frown)^\frown I^\frown = \omega k((P^\frown)^N)A_0(\widetilde{c})^\frown A_1 I^\frown \\ = \omega k((P^\frown)^N)A_0 c^\frown A_1)I^\frown = \omega k((t^\frown)^N)I^\frown.$$

Case 3) $t = (p,q)$. Using conditions (C2) and (C3) we see that $p, q \in K''$, hence by (K3), (S4), (12) and (13) we have

$$((I,\vartheta)\sigma k(t))^\frown = ((\vartheta k(p), \vartheta k(q)))^\frown = ((\vartheta k(p))^\frown, (\vartheta k(q))^\frown)D \\ \leq (\omega k((p^\frown)^N)I^\frown, \omega k((q^\frown)^N)I^\frown)D = (\omega k((p^\frown)^N), \omega k((q^\frown)^N))(T_+ I^\frown, F_+ I^\frown)D \\ \leq (\omega k((p^\frown)^N), \omega k((q^\frown)^N))DI^\frown = \omega k(((p^\frown)^N, (q^\frown)^N)D)I^\frown = \omega k((t^\frown)^N)I^\frown.$$

Case 4) $t = Px_i$, $i < n$. Then using (15) we get

$$((I,\vartheta)\sigma k(t))^\frown = (\vartheta k((Ps_i)^N))^\frown \leq \omega k(((Ps_i)^\frown)^N)I^\frown = \omega k((P^\frown(s_i)^\frown)^N)I^\frown \\ = \omega k((P^\frown)^N(x_i)^\frown)I^\frown = \omega k((t^\frown)^N)I^\frown.$$

Case 5) $t = P(x_i)^\frown$, $i < n$. Then $t = P(x_i)^\frown \in K$, whence by (C5) $(P(s_i)^\frown)^N \in K$. On the other hand, $((P(s_i)^\frown)^\frown)^N = (P^\frown A_0(s_i)^\frown)^N A_1$, and since $(t^\frown)^N = (P^\frown A_0)^N(x_i)^\frown A_1 \in K$, we see by (C5) again that $((P(s_i)^\frown)^\frown)^N \in K$, i. e. $(P(s_i)^\frown)^N \in K''$. Therefore, using Lemma 1 and (15) we have

$$((I,\vartheta)\sigma k(t)))^\frown = (\vartheta k((P(s_i)^\frown)^N))^\frown \leq \omega k((((P(s_i)^\frown)^N)^\frown)^N)I^\frown \\ = \omega k(((P(s_i)^\frown)^\frown)^N)I^\frown = \omega k(((P)^\frown A_0(s_i)^\frown)^N)A_1 I^\frown \\ = \omega k(((P)^\frown A_0)^N(x_i)^\frown)A_1 I^\frown = \omega k(((P(x_i)^\frown)^\frown)^N)I^\frown = \omega k((t^\frown)^N)I^\frown.$$

Thus (18) is proved and thence $(I,\vartheta)\sigma \in \mathcal{B}$. Since ω is an iteration of σ, by the inclusion $(I,\mathcal{B})\sigma \subseteq \mathcal{B}$ we have $\omega \in \mathcal{B}$, whence we obtain (17). Using (16) and (17) we are able to show by induction on t that

$$\widetilde{t}(\omega k(\overline{x})) \leq \omega k(t) \tag{19}$$

for all **fit** $t \in K$. For that suppose t is fit and consider the cases for t as above:

Case 1) $t = \Lambda$. Then (19) is obvious from (11).

Case 2) $t = Ps$, where s is a simple closed term. Then if P is fit we have by the induction hypothesis

$$\widetilde{t}(\omega k(\overline{x})) = \widetilde{P}(\omega k(\overline{x}))\widetilde{s} \leq \omega k(P)\widetilde{s} = \omega k(t),$$

and if P is not fit, then, obviously, $s = I^\frown$ and $P = P'x_i{}^\frown$ for some $i < n$ and extraterm P', and using (17), (16) and the induction hypothesis we have

$$\widetilde{t}(\omega k(\overline{x})) = \widetilde{P}'(\omega k(\overline{x}))(\omega k(x_i))^\frown I^\frown \leq \omega k(P')\omega k(x_i)I^\frown I^\frown = \omega k(P'x_i)I^\frown = \omega k(t).$$

Case 3) $t = (p,q)$. Then using the induction hypothesis and (13) we get

$$\widetilde{t}(\omega k(\overline{x})) = (\widetilde{p}\omega k(\overline{x}), \widetilde{q}\omega k(\overline{x})) \leq (\omega k(p), \omega k(q)) = \omega k(t).$$

Case 4) $t = Px_i$. Using the induction hypothesis, (16) and (C5), we have

$$\widetilde{t}(\omega k(\overline{x})) = \widetilde{P}(\omega k(\overline{x}))\omega k(x_i) \leq \omega k(P)\omega k(x_i) \leq \omega k(t).$$

Case 5) $t = P(x_i)\hat{}$. Impossible, since t is a fit term.

This completes the proof of (19). Since the terms s_i are supposed to be fit ones, (19) implies that $\omega k(\overline{x})$ is a solution of (10):

$$\widetilde{s}_i(\omega k(\overline{x})) \leq \omega k(s_i) = \omega k(x_i).$$

Now let $\overline{\xi} = (\xi_0, \ldots, \xi_{n-1})$ be an arbitrary solution of (10) in $\mathcal{F}$. We shall show that for each $t \in K$

$$\omega k(t) \leq \widetilde{t}(\overline{\xi}). \tag{20}$$

For that consider the $\mathcal{K}$-admissible initial segment

$$\mathcal{B}_1 = \left\{\vartheta \in \mathcal{F} \mid \forall t \in K \left(\vartheta k(t) \leq \widetilde{t}(\overline{\xi})\right)\right\}.$$

We shall show that $(I, \mathcal{B}_1)\sigma \subseteq \mathcal{B}_1$, whence (20) will follow immediately. For that suppose $\vartheta \in \mathcal{B}_1$ and prove for all $t \in K$ that

$$(I, \vartheta)\sigma k(t) \leq \widetilde{t}(\overline{\xi}), \tag{21}$$

considering cases for t as it follows:

Case 1) $t = \Lambda$. Then

$$(I, \vartheta)\sigma k(t) = (I, \vartheta)T_+ = I = \widetilde{t}(\overline{\xi}).$$

Case 2) $t = Pq$, where q is a simple closed term. Then

$$(I, \vartheta)\sigma k(t) = \vartheta k(P)\widetilde{q} \leq \widetilde{P}(\overline{\xi})\widetilde{q} = \widetilde{t}(\overline{\xi}).$$

Case 3) $t = (p, q)$. Then

$$(I, \vartheta)\sigma k(t) = (\vartheta k(p), \vartheta k(q)) \leq (\widetilde{p}(\overline{\xi}), \widetilde{q}(\overline{\xi})) = \widetilde{t}(\overline{\xi}).$$

Case 4) $t = Px_i$, $i < n$. Let $s = Ps_i$. Then, since reductions do not change the value of terms and $\overline{\xi}$ is a solution of (10), we have

$$(I, \vartheta)\sigma k(t) = \vartheta k((Ps_i)^N) \leq \widetilde{s}(\overline{\xi}) = \widetilde{P}(\overline{\xi})\widetilde{s}_i(\overline{\xi}) \leq \widetilde{P}(\overline{\xi})\xi_i = \widetilde{t}(\overline{\xi}).$$

Case 5) $t = P(x_i)\hat{}$, $i < n$. Similarly,

$$(I, \vartheta)\sigma k(t) = \vartheta k((P(s_i)\hat{})^N) \leq \widetilde{P}(\overline{\xi})(\widetilde{s}_i(\overline{\xi}))\hat{} \leq \widetilde{P}(\overline{\xi})(\xi_i)\hat{} = \widetilde{t}(\overline{\xi}).$$

This completes the proof of (21) and, therefore, of (20). Then for each $i < n$ we have

$$\omega k(x_i) \leq \widetilde{x}_i(\overline{\xi}) = \xi_i. \blacksquare$$

Remark. As it may be noticed by the reader, the previous theorem holds with the following variation: we leave the supposition that $\mathcal{F}$ is regular and terms s_i are fit, and replace the supposition that ω is a $\mathcal{K}$-iteration of σ by a (possibly) stronger one, which is obtained from the definition of $\mathcal{K}$-iteration by erasing the occurrence of $I\hat{}$ in the definition of simple $\mathcal{K}$-admissible initial segment. The proof is the same with corresponding simplifications, namely we need not to pay attention to fit terms and we prove (19) for all $t \in K$. The preference of the presented above version was made for purposes of applications to combinatory spaces, but the version, mentioned in the present remark, is interesting as well.

To apply Theorem 1 one needs to construct a coding which in many cases is more or less a straightforward work. We shall describe a general situation when codings exist always. Namely, let $\mathcal{F}$ be an OSS and let as before an interpretation of parameter symbols in $\mathcal{F}$ and a set $\mathcal{K} \subseteq \mathcal{F}$ be fixed. Denote by $\underline{S}$ the set of all pairs (Σ, t), where Σ is a system of the form (10) and $t \in \underline{N}$. Then by *universal coding* in $\mathcal{F}$ we shall mean a pair $\langle k, \sigma \rangle$, where $\sigma \in \mathcal{F}$ and $k : \underline{S} \to \mathcal{K}$ is a function such that for every fixed system Σ of the form (10) the triple $\langle \underline{N}, k_\Sigma, \sigma \rangle$ is a coding for Σ, where $k_\Sigma(t) = k(\Sigma, t)$ for all $t \in \underline{N}$. By *proper representation of natural numbers* in $\mathcal{F}$ we shall mean a function assigning to each natural number n an element $n^+ \in \mathcal{F}$ and satisfying the following condition: for every natural number m there is a mapping $R_m : \mathcal{F}^m \to \mathcal{F}$ such that

$$R_m(\varphi_0, \ldots, \varphi_{m-1})n^+ = \varphi_n$$

for every m-tuple $(\varphi_0, \ldots, \varphi_{m-1}) \in \mathcal{F}^m$ and all $n < m$. If these mappings R_m are of the form

$$R_m(\varphi_0, \ldots, \varphi_{m-1}) = (\varphi_0, (\varphi_1, \ldots, (\varphi_{m-1}, I)\rho \ldots)\rho)\rho,$$

where $\rho \in \mathcal{F}$, we shall say that the representation in question is *normal* and ρ is its specific element. We shall say also for a proper representation of natural numbers in $\mathcal{F}$ that it is *primitive recursive*, iff unary primitive recursive functions are representable with respect to this representation, i. e. for every primitive recursive function f of one argument there is $\varphi \in \mathcal{F}$ such that $\varphi n^+ = (f(n))^+$ for all natural n. We shall call an element $\varphi \in \mathcal{F}$ *elementary* in a set $\mathcal{B} \subseteq \mathcal{F}$, iff φ may be expressed through basic constants and elements of $\mathcal{B}$ by means of multiplication, pairing and storage operations, i. e. iff φ is the value of a closed term with parameters in $\mathcal{B}$. *Elementary mappings* $f : \mathcal{F}^n \to \mathcal{F}$ are similarly defined as mappings of the form $f(\bar{\xi}) = \tilde{t}(\bar{\xi})$ for suitable term t with parameters in $\mathcal{B}$.

Proposition 1. *Let a primitive recursive representation n^+ of natural numbers n be given in $\mathcal{F}$, and suppose $n^+ \in \mathcal{K}$ for all n, and let the set $\mathcal{K}$ satisfy the following three conditions:*
(a) $\varkappa\varphi = \varphi^\frown\varkappa$ *for all* $\varphi \in \mathcal{F}$ *and* $\varkappa \in \mathcal{K}$;
(b) *there is* $\delta \in \mathcal{F}$ *such that* $\delta\varkappa = \varkappa\varkappa$ *for all* $\varkappa \in \mathcal{K}$;
(c) *there is* $\pi \in \mathcal{F}$ *such that* $\pi\varkappa = (T_+\varkappa, F_+\varkappa)$ *for all* $\varkappa \in \mathcal{K}$.

Then there is an universal coding $\langle k, \sigma \rangle$ in $\mathcal{F}$, and if the representation n^+ is normal, then such a coding may be found with σ of the following special form:

$$\sigma = (\delta_0\gamma_0{}^\frown, (\delta_1\gamma_1{}^\frown, \ldots (\delta_{2l-1}\gamma_{2l-1}{}^\frown, I) \ldots))\beta, \tag{22}$$

where $\gamma_0, \ldots, \gamma_{l-1}$ are the parameters, $\gamma_{l+i} = \gamma_i$ for all $i < l$, $\delta_0 = \cdots = \delta_{l-1} = F_+$, $\delta_l = \cdots = \delta_{2l-1} = F_+A_0$, and β is elementary in a fixed finite set $\mathcal{B} \subseteq \mathcal{F}$.

Proof. Take a primitive recursive numeration of elements of $\underline{S}$ and define $k(\Sigma, t) = (\textit{the number of}\,(\Sigma, t))^+$. Using the representability of primitive recursive functions we see that there is $\sigma' \in \mathcal{F}$ such that

$$\sigma' k(\Sigma, t) = \begin{cases} 0^+ & \text{if } t \text{ is a simple closed term,} \\ 1^+ & \text{if } t = ps, \text{ where } s \text{ is a simple closed term,} \\ 2^+ & \text{if } t = (p, q) \text{ for suitable } p,\ q, \\ 3^+ & \text{if } t = Px \text{ for a suitable variable } x, \\ 4^+ & \text{if } t = Px\hat{}\ \text{for a suitable variable } x. \end{cases}$$

Next we construct $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathcal{F}$ such that

$$\sigma_0 k(\Sigma, t) = T_+ \quad \text{if } t = \Lambda;$$
$$\sigma_1 k(\Sigma, t) = F_+ k(\Sigma, P)\tilde{s} \quad \text{if } t = Ps \text{ and } s \text{ is a simple closed term;}$$
$$\sigma_2 k(\Sigma, t) = F_+(k(\Sigma, p), k(\Sigma, q)) \quad \text{if } t = (p, q);$$
$$\sigma_3 k(\Sigma, t) = F_+(k(\Sigma, (P\Sigma(x))^N) \quad \text{if } t = Px;$$
$$\sigma_4 k(\Sigma, t) = F_+(k(\Sigma, (P(\Sigma(x))\hat{})^N) \quad \text{if } t = Px\hat{},$$

where $\Sigma(x)$ is the left side of the inequality in Σ with right side x, if such inequality exists, and $\Sigma(x) = x$ otherwise. The existence of σ_3 and σ_4 follows from the supposition of primitive recursiveness of the given representation of natural numbers in $\mathcal{F}$; the existence of σ_0 is obvious, since the last representation is a proper one. The crucial point is the construction of σ_1. It may be done by making use of (a) and (b) as it follows:

$$\begin{aligned} F_+ k(\Sigma, P)\tilde{s} &= F_+ k(\Sigma, P)\alpha k(\Sigma, s) = F_+\alpha\hat{}k(\Sigma, P)k(\Sigma, s) \\ &= F_+\alpha\hat{}\sigma_{10}k(\Sigma, Ps)\sigma_{11}k(\Sigma, Ps) = F_+\alpha\hat{}\sigma_{10}(\sigma_{11})\hat{}\delta k(\Sigma, Ps), \end{aligned}$$

and define $\sigma_1 = F_+\alpha\hat{}\sigma_{10}(\sigma_{11})\hat{}\delta$, where $\alpha \in \mathcal{F}$ is such that $\alpha k(\Sigma, t) = \tilde{t}$ for all simple closed terms t (the existence of α follows from the fact that simple closed terms are finite in number), s is a simple closed term, and σ_{10} and σ_{11} are constructed using the representability of primitive recursive functions. The construction of σ_2 is based on the use of (c): for $t = (p, q)$ we have

$$\begin{aligned} F_+(k(\Sigma, p), k(\Sigma, q)) &= F_+\sigma_{20}k(\Sigma, t), \sigma_{21}k(\Sigma, t)) \\ &= F_+(\sigma_{20}, \sigma_{21})(T_+ k(\Sigma, t), F_+ k(\Sigma, t)) = F_+(\sigma_{20}, \sigma_{21})\pi k(\Sigma, t), \end{aligned}$$

and we define $\sigma_2 = F_+(\sigma_{20}, \sigma_{21})\pi$; σ_{20} and σ_{21} are constructed using the representability of primitive recursive functions. Finally, taking $\sigma'' \in \mathcal{F}$ such that $\sigma'' i^+ = \sigma_i$ for all $i < 5$, and defining $\sigma = \sigma''\sigma'\delta$ we see directly that $\langle k, \sigma \rangle$ is an universal coding. The form (22) of σ in the case of normal representation n^+ follows easily from the above construction by some simple transformations using basic equalities in the definitions of operative space and storage operation and the set $\{\delta, \pi, \sigma_{10}, \sigma_{11}, \sigma_{20}, \sigma_{21}, \sigma_3, \sigma_4, \sigma', \rho\}$ for $\mathcal{B}_0$, where ρ is the specific element of the representation n^+. ∎

Corollary 1. *Suppose for every $\varphi \in \mathcal{F}$ there is a solution $\mathbb{I}(\varphi)$ of the equality $(I, \xi)\varphi = \xi$ with respect to ξ in $\mathcal{F}$, and there are two elements $M, Q \in \mathcal{F}$ such that for all $\varphi \in \mathcal{F}$ the following equalities hold:*
(i) $\varphi\hat{}MF_+ = MF_+\varphi\hat{}$;

(ii) $\varphi^\frown MT_+ = MT_+\varphi^\frown$;
(iii) $\varphi^\frown T_+ = T_+\varphi$;
(iv) $QMF_+M = F_+M$;
(v) $QMT_+T_+ = T_+$.

Then there is an universal coding $\langle k, \sigma\rangle$ *in* $\mathfrak{F}$ *such that* σ *has the form* (22) *with respect to a set* $\mathfrak{B}_0$ *consisting of elements explicitely expressible by means of basic constants,* Q*,* M*, and operations of multiplication, pairing, storage and iteration* $\mathbb{I}$.

Indeed, define $n^+ = (MF_+)^n MT_+T_+$, and let $\mathfrak{K}$ be the set of all elements of the form n^+ for a natural n. Then (i)–(iii) imply the condition (a) in Proposition 1. Using (iv) and (v), we see that the operation $\mathbb{R}_0$ defined by $\mathbb{R}_0(\varphi,\psi) = (\varphi,\psi)Q$ satisfies the equalities

$$\mathbb{R}_0(\varphi,\psi)0^+ = \varphi \quad \text{and} \quad \mathbb{R}_0(\varphi,\psi)(n+1)^+ = \psi n^+ \tag{23}$$

for all natural n and all $\varphi, \psi \in \mathfrak{F}$; and for the element $\tau = \mathbb{I}((T_+, F_+\psi^\frown)Q)$ we see by induction on n that $\tau n^+ = \psi^n$, whence the operation $\mathbb{R}_1$ defined by $\mathbb{R}_1(\varphi,\psi) = \mathbb{I}((T_+, F_+\psi^\frown)Q)\varphi^\frown$ satisfies

$$\mathbb{R}_1(\varphi,\psi)0^+ = \varphi \quad \text{and} \quad \mathbb{R}_1(\varphi,\psi)(n+1)^+ = \psi\mathbb{R}_1(\varphi,\psi)n^+ \tag{24}$$

for all natural n and all $\varphi, \psi \in \mathfrak{F}$. Using Theorem 1 in [6] we conclude that n^+ is a normal primitive recursive representation with specific element Q. Conditions (b) and (c) in Proposition 1 are satisfied with $\delta = \mathbb{R}_1(0^+0^+, MF_+M^\frown F_+^\frown)$ and $\pi = \mathbb{R}_1((T_+0^+, F_+0^+), (T_+MF_+, F_+MF_+))$. Applying Proposition 1 we obtain the corollary. ■

A system Σ of the form (10) will be called *finitely codable* iff there is a finite set $K \subseteq \underline{N}(\Sigma)$ which is closed with respect to Σ. For finitely codable systems we can easily find codings with a special simple form of the third component σ.

Proposition 2. *Let a normal representation* n^+ *of natural numbers* n *with a specific element* ρ *be given in* $\mathfrak{F}$*. Suppose* $n^+ \in \mathfrak{K}$ *for all* n *and the set* $\mathfrak{K}$ *satisfies the condition* (a) *in Proposition* 1. *Then for every finitely codable system* Σ *there is a coding* $\langle K, k, \sigma\rangle$ *for* Σ *such that* K *is finite and* σ *has the form* (22) *with* β *elementary in* $\{\rho\}$ *and the set of representations* n^+ *of natural numbers* n*. Moreover, if the system* Σ *contains occurences of a parameter symbol* c_i *only through occurences of* $c_i^\frown$*, then the part concerning the correspondent parameter* γ_i *may be erased from the form* (22)*, i. e.* σ *may be supposed of the form*

$$\sigma = (F_+\gamma_0^\frown, \ldots, (F_+\gamma_{i-1}^\frown, (\delta_{i+1}\gamma_{i+1}^\frown \ldots (F_+A_0\gamma_{2l-1}^\frown, I)\ldots))\beta.$$

Indeed, if $K \subseteq \underline{N}(\Sigma)$ is finite and closed with respect to Σ, then we can enumerate elements of K and define $k(t)$ as $(\ulcorner(t))^+$, where $\ulcorner(t)$ is the number of $t \in K$. Then we may construct σ satisfying (K1)–(K5) in a way which is obvious enough and obtain the necessary form (22) of σ by some elementary transformations using (a) of Proposition 1. To obtain the last form of σ in the Proposition 2, we have to notice also that if the system Σ possesses the property in question, namely that it contains occurrences of the parameter symbols c_i only through such of $c_i^\frown$, then all extraterms in the least closed with respect to Σ set $K \subseteq \underline{N}(\Sigma)$ possess the same property, which is clear from the closure conditions (C1)–(C5). ∎

An element $\varphi \in \mathcal{F}$ will be called *finitely recursive* in a set $\mathcal{B} \subseteq \mathcal{F}$, iff it is definable by a finitely codable system with parameters in $\mathcal{B}$, i. e. φ is a member of the least solution of a finitely codable system of the form (10) with respect to an interpretation of parameter symbols in $\mathcal{B}$. Similarly, by varying one or several parameters, finitely recursive in $\mathcal{B}$ mappings of one or several arguments are defined.

Proposition 3. *The set of finitely recursive elements (and mappings as well) is closed with respect to the operations multiplication, pairing and storage. If for every $\varphi \in \mathcal{F}$ the least solution $\mathbb{I}(\varphi)$ of the inequality $(I,\xi)\varphi \leq \xi$ with respect to ξ exists in $\mathcal{F}$, then the set of finitely recursive elements (and mappings as well) is closed also under the operation $\mathbb{I}$.*

Proof. For the operations multiplication, pairing and $\mathbb{I}$ this is easy. For instance, if φ is the member φ_0 of the least solution $(\varphi_0, \ldots, \varphi_{n-1})$ of a system Σ of the form (10), then $(\varphi_0, \ldots, \varphi_{n-1}, \mathbb{I}(\varphi))$ is the least solution of the system $\Sigma' = \Sigma$, $(I,y)x_0 \leq y$ obtained from Σ by adding the inequality $(I,y)x_0 \leq y$, where y is a new variable. If $K \subseteq \underline{N}(\Sigma)$ is finite and closed with respect to Σ, then the set

$$K' = K \cup \{y,\ (I,y),\ I\} \cup \left\{((I,y)t)^N \mid t \in K\right\}$$

is finite and closed with respect to Σ'. This is the proof for the operation $\mathbb{I}$, and the cases with multiplication and pairing are similar or simpler and are left to the reader. The case with the operation storage offers a little bit more difficulties.

Lemma. *Let Σ be a system of the form (10) and let $U \subseteq \underline{N}(\Sigma)$ be finite and closed with respect to Σ. Then there is a set $U' \subseteq \underline{N}(\Sigma)$ such that $U \subseteq U'$, U' is finite and closed with respect to Σ, and $x^\frown \in U'$ for all variables x in Σ.*

Proof (a sketch). Define:

$$U_1 = \left\{s \in \underline{N}(\Sigma) \mid \exists t \in U \left(s = (t^\frown)^N \text{ or } sA_1 = (t^\frown)^N\right)\right\};$$

$$U_2 = \left\{(P^\frown)^N A_0 \mid P \in \underline{N}(\Sigma),\ Pq^\frown \in U, \text{ and } q \text{ is a prime term}\right\};$$

$$U_3 = \left\{((p^\frown)^N, (q^\frown)^N) \mid (p,q) \in U\right\}.$$

Then the set $U' = U \cup U_1 \cup U_2 \cup U_3$ satisfies the conditions of the lemma. We leave to the reader to check this in detail. ∎

Now, if φ is the member φ_0 of the least solution $(\varphi_0, \ldots, \varphi_{n-1})$ of a system Σ of the form (10), then $(\varphi_0, \ldots, \varphi_{n-1}, \varphi^\frown)$ is the least solution of the system $\Sigma' = \Sigma$, $x_0{}^\frown \leq y$, where y is a new variable. If $U \subseteq \underline{N}(\Sigma)$ is finite and closed with respect to Σ, then the set $U' \cup \{y\}$, where $U' \subseteq \underline{N}(\Sigma)$ satisfies the conclusions of the Lemma, is finite and closed with respect to Σ'. ∎

5. RECURSION THEORY IN NORMAL OPERATIVE SPACES WITH STORAGE

The existence of codings combined with Theorem 1 implies basic facts of the recursion theory. We shall illustrate this in the present section with the case of regular OSS with constants M and Q satisfying the conditions (i)–(v) in Corollary 1. The last structures will be called *normal* OSS (shortly NOSS). For an arbitrary NOSS $\mathcal{F}$ we shall fix the representation n^+ of natural numbers n and the set $\mathcal{K}$ defined in the proof of Corollary 1. A NOSS $\mathcal{F}$ will be called *iterative*, iff the $\mathcal{K}$-iteration $\mathbb{I}(\varphi)$ exists for every $\varphi \in \mathcal{F}$. By Theorem 1 and Corollary 1 we have immediately

Corollary 2. *If a NOSS $\mathcal{F}$ is iterative, then every system of the form* (10) *has a least solution in $\mathcal{F}$, which members are explicitely expressible by means of parameters, basic constants including M and Q, and operations of multiplication, pairing, storage and iteration $\mathbb{I}$.*

Calling *recursive in parameters* those elements (respectively mappings) of an arbitrary OSS which are members of least solutions of systems of the form (10) (respectively, of a system of the same form with respect to the set of parameters enlarged with such for the arguments of the mapping), we have as well

Corollary 3. *In the iterative NOSS $\mathcal{F}$ there is an element $\omega \in \mathcal{F}$ which is recursive in parameters and universal in the following sense: for every recursive in parameters mapping $\Gamma : \mathcal{F}^{n+1} \to \mathcal{F}$ there is a primitive recursive function f of m arguments such that for all natural $m_0, \ldots, m_{n-1}$ we have*

$$\Gamma(\omega, m_0^+, \ldots, m_{n-1}^+) = \omega(f(m_0, \ldots, m_{n-1}))^+. \tag{25}$$

Indeed, by Corollary 1 there is an universal coding $\langle k, \sigma \rangle$, and let ω be the iteration of σ. It is obvious that we can find a system Σ of the form (10) such that for all $m_0, \ldots, m_{n-1}$ the element $\Gamma(\omega, m_0^+, \ldots, m_{n-1}^+)$ is a member, corresponding to a variable x of the least solution of the system $\Sigma(m_0, \ldots, m_{n-1})$, obtained from Σ by replacing the parameters, corresponding to the last n arguments of the mapping Γ, with $m_0^+, \ldots, m_{n-1}^+$, respectively. Then taking the function f for which $(f(m_0, \ldots, m_{n-1}))^+ = k(\Sigma(m_0, \ldots, m_{n-1}), x)$ in the notations of the proof of Proposition 1, we obtain the equality (25) from Theorem 1. ■

Corollary 4. *Let $\mathcal{F}$ be an iterative NOSS, and let $\Gamma : \mathcal{F} \to \mathcal{F}$ be a recursive mapping. Then there is an elementary in parameters $\beta \in \mathcal{F}$ such that*

$$\Gamma(\xi) = \mathbb{I}((I, F_+ A_0 \xi^\frown)\beta) M T_+ T_+ \tag{26}$$

for all $\xi \in \mathcal{F}$.

Indeed, by Corollary 2 $\Gamma(\xi)$ is explicitly expressible through ξ, the constants and the operations, mentioned in Corollary 2, whence by Proposition 3 Γ is a finitely recursive mapping. Applying Proposition 2, we conclude that there is a system Σ of the form (10) (containing a parameter symbol for ξ) such that $\Gamma(\xi)$ is a member corresponding to a variable x of the least solution of Σ; and there is a coding $\langle K, k, \sigma \rangle$ for Σ such that K is finite and σ has the form $(F_+\xi^\frown, (F_+ A_0 \xi^\frown, I))\beta$, where β is elementary in parameters (and M and Q as well, but the last are treated now as basic constants). Then by Theorem 1

$$\Gamma(\xi) = \mathbb{I}((F_+\xi^\frown, (F_+ A_0 \xi^\frown, I))\beta) k(x). \tag{27}$$

Obviously, we may suppose that $k(x) = 0^+$. Moreover, by the last representation (27) it is clear that the element $\Gamma(\xi)$ can be defined by a finitely codable system containing occurrences of the parameter ξ only through such of $\xi^\frown$. Then by applying for the second time Proposition 2, we get the representation (26). ∎

Corollary 5. *Let $\mathcal{F}$ be an iterative NOSS and define for all $\varphi, \psi \in \mathcal{F}$*

$$\mathrm{App}(\varphi, \psi) = \mathbb{I}((I, F_+ A_0 \psi^\frown)\varphi) M T_+ T_+.$$

Then $\mathcal{F}$ is a combinatory algebra with respect to the operation App *as an aplication operation.*

Indeed, writing $\mathrm{App}(\varphi, \varphi_0, \ldots, \varphi_n)$ for $\mathrm{App}(\ldots \mathrm{App}(\mathrm{App}(\varphi, \varphi_0), \varphi_1) \ldots \varphi_n)$, we prove by induction on n that for every recursive in parameters mapping $\Gamma : \mathcal{F}^{n+1} \to \mathcal{F}$ there is an elementary in parameters $\gamma \in \mathcal{F}$ such that for all $\xi_0, \ldots, \xi_n \in \mathcal{F}$ we have

$$\Gamma(\xi_0, \ldots, \xi_n) = \mathrm{App}(\gamma, \xi_0, \ldots, \xi_n).$$

Using the representation (26) and the induction hypothesis we have for suitable β and γ elementary in parameters:

$$\begin{aligned}
\Gamma(\xi_0, \ldots, \xi_n) &= \mathbb{I}((I, F_+ A_0(\xi_n{}^\frown, (\xi_0, \ldots, (\xi_{n-2}, \xi_{n-1}) \ldots)^\frown) D)\beta) M T_+ T_+ \\
&= \mathbb{I}((I, F_+ A_0 \xi_n{}^\frown)(T_+, (F_+, T_+ F_+ A_0(\xi_0, \ldots, (\xi_{n-2}, \xi_{n-1}) \ldots)^\frown) D)\beta) M T_+ T_+ \\
&= \mathrm{App}((T_+, (F_+, T_+ F_+ A_0(\xi_0, \ldots, (\xi_{n-2}, \xi_{n-1}) \ldots)^\frown) D)\beta, \xi_n) \\
&= \mathrm{App}(\mathrm{App}(\gamma, \xi_0, \ldots, \xi_{n-1}), \xi_n) = \mathrm{App}(\gamma, \xi_0, \ldots, \xi_n).
\end{aligned}$$

Hence, $\mathcal{F}$ is a combinatory algebra with respect to the operation App, because the last operation being recursive in $\varnothing$, every mapping defined by explicit expression in terms of this operation is recursive in $\varnothing$. ∎

6. APPLICATIONS TO COMBINATORY SPACES

Let $\mathcal{S} = \langle \mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F \rangle$ be a combinatory space in the sense of [2] (unexplained terminology and notations concerning combinatory spaces, mentioned in this section, may be found in [2]). We shall write $\langle \varphi, \psi \rangle$ for $\Pi(\varphi, \psi)$ and we shall suppose that T and F belong to $\mathcal{C}$, which is not a loss of generality. Consider a corresponding companion operative space $\mathcal{S}_*$. We shall denote the basic constants and operations in $\mathcal{S}_*$ in the same way as we have done above for an arbitrary operative space, especially $T_+ = \langle T, I \rangle$ and $F_+ = \langle F, I \rangle$. There is a storage operation S in $\mathcal{S}_*$ defined by $\mathcal{S}(\varphi) = \langle L, \varphi R \rangle$ (see [2], exercises 7, 10 and 11, pp. 55, 56); the corresponding constants are defined as follows:

$$D = (LR \to T_+ \langle L, R^2 \rangle,\ F_+ \langle L, R^2 \rangle) \langle L, R \rangle = (LR \to T_+ R^\frown, F_+ R^\frown) I^\frown,$$

$A_0 = \langle L^2, \langle RL, R \rangle \rangle$ and $A_1 = M I^\frown = M \langle L, R \rangle$, where $M = \langle \langle L, LR \rangle, R^2 \rangle$; condition (S4) is obviously satisfied: $(T_+ I^\frown, F_+ I^\frown) D = D I^\frown$. Moreover, $\mathcal{S}_*$ is a NOSS with respect to M, defined as above, and $Q = (L^2 \to T_+ R, F_+ \langle RL, R \rangle)$. Indeed, for arbitrary $c \in \mathcal{C}$ and $\varphi \in \mathcal{F}$ we have

$$\begin{aligned}
M F_+ \langle c, I \rangle \varphi &= \langle \langle F, L \rangle, R \rangle \langle c, I \rangle \varphi = \langle \langle F, c \rangle, I \rangle \varphi \\
&= \langle \langle F, c \rangle, \varphi \rangle = \langle L, \varphi R \rangle \langle \langle F, c \rangle, I \rangle = \varphi^\frown M F_+ \langle c, I \rangle,
\end{aligned}$$

whence (by [2], exercise 9, p. 55) $MF_+\varphi^\frown = \varphi^\frown MF_+ I^\frown$; but $MF_+ I^\frown = MF_+$ since

$$MF_+ I^\frown c = \langle\langle F, L\rangle, R\rangle\langle L, R\rangle c = \langle\langle F, L\rangle, R\rangle\langle I, Rc\rangle Lc,$$

and for an arbitrary $b \in \mathcal{C}$

$$\langle\langle F, L\rangle, R\rangle\langle I, Rc\rangle b = \langle\langle F, L\rangle, R\rangle\langle b, I\rangle Rc = \langle\langle F, b\rangle, I\rangle Rc = \langle\langle F, b\rangle, Rc\rangle\rangle = \langle\langle F, I\rangle, Rc\rangle b,$$

whence

$$MF_+ I^\frown c = \langle\langle F, I\rangle, Rc\rangle Lc = \langle\langle F, Lc\rangle, Rc\rangle = MF_+ c.$$

Therefore the condition (i) in Corollary 1 holds and similarly we have (ii), and (iii) is obvious. To see (iv), consider for an arbitrary $c \in \mathcal{C}$ the equality

$$QMF_+\langle c, I\rangle = Q\langle\langle F, c\rangle, I\rangle = F_+\langle RL, R\rangle\langle\langle F, c\rangle, I\rangle = F_+\langle c, I\rangle.$$

It implies by the exercise 9, mentioned above, that $QMF_+ I^\frown = F_+ I^\frown$, but $MF_+ I^\frown = MF_+$, whence $QMF_+ = F_+ I^\frown$ and $QMF_+ M = F_+ I^\frown M$. On the other hand, $I^\frown M = M$, whence we get the condition (iv). The last equality follows from $I^\frown\langle\varphi, \psi\rangle = \langle\varphi, \psi\rangle$ for all $\varphi, \psi \in \mathcal{F}$, which can be proved by the same method as above: take arbitrary $b, c \in \mathcal{F}$ and from

$$\langle L, R\rangle\langle I, \psi c\rangle b = \langle L, R\rangle\langle b, I\rangle\psi c = \langle b, I\rangle\psi c = \langle I, \psi c\rangle b$$

conclude that $\langle L, R\rangle\langle I, \psi c\rangle = \langle I, \psi c\rangle$, whence

$$I^\frown\langle\varphi, \psi\rangle c = \langle L, R\rangle\langle\varphi, \psi\rangle c = \langle L, R\rangle\langle I, \psi c\rangle\varphi c = \langle I, \psi c\rangle\varphi c = \langle\varphi, \psi\rangle c.$$

Finally,

$$QMT_+T_+ = Q\langle\langle T, T\rangle, I\rangle = T_+ R\langle\langle T, T\rangle, I\rangle = T_+,$$

and the condition (v) in Corollary 1 holds as well.

Remark. To apply Corollaries 2–5, we need to suppose that the space $\mathcal{S}_*$ is iterative with respect to the fixed set $\mathcal{K}$. We should comment a little upon the connection of the last supposition and that of the iterativity of $\mathcal{S}$ in the sense of [2] and [3]. The condition of iterativity, used in [2], is possibly more general than that in [3] (no proof is mentioned that it really is), but the former condition employs the set $\mathcal{C}$ of "points" of the space $\mathcal{S}$ and can not be stated in point-free generalizations of the theory of combinatory spaces, the last being one of our objectives. That is why the condition in [3] has to be regarded as natural for such generalizations. It may be said that up to secondary details the suppositions of iterativity of the NOSS $\mathcal{S}_*$ with respect to $\mathcal{K}$ and that of iterativity of $\mathcal{S}$ in the sense of [3] are equivalent. The first of them possibly does not imply the second for two reasons: the set $\mathcal{K}$ is too small and only countable intersections of simple $\mathcal{K}$-admissible initial segments are allowed as $\mathcal{K}$-admissible ones. But if we admit arbitrary intersections of that kind and take $\mathcal{K}' = \mathcal{C} \cup \{\langle c, I\rangle \mid c \in \mathcal{C}\}$ instead of $\mathcal{K}$, then it does. The condition of iterativity of the NOSS $\mathcal{S}_*$ thus strengthened, namely that a solution $\mathbb{I}(\varphi)$ of

$$(I, \xi)\varphi \leq \xi \tag{28}$$

with respect to ξ belongs to every intersection $\mathcal{A}$ of simple $\mathcal{K}'$-admissible initial segments satisfying $(I, \mathcal{A})\varphi \subseteq \mathcal{A}$ (let call this condition "non-countable $\mathcal{K}'$-iterativity"), is equivalent to the following one for the space $\mathcal{S}$:

(I) For every $\varphi \in \mathcal{F}$ there is a solution $\mathbb{I}(\varphi)$ of (28), which belongs to every intersection $\mathcal{A}$ of subsets of $\mathcal{F}$ of the form

$$(29) \qquad \{\xi \in \mathcal{F} \mid \chi\xi\zeta \leq \psi\},$$

where $\chi, \psi \in \mathcal{F}$ are arbitrary and ζ is a normal element of $\mathcal{F}$ such that $(I, \mathcal{A})\varphi \subseteq \mathcal{A}$.

Indeed, if $\mathcal{S}$ satisfies (I), then $\mathcal{S}_*$ is a non-countably $\mathcal{K}'$-iterative, because every simple $\mathcal{K}'$-admissible initial segment is an intersection of sets of the form (29): for such segments of the first two kinds this is obvious and for segments $\mathcal{A}$ of the form $\{\xi \in \mathcal{F} \mid (\xi\varkappa)^\frown \leq \psi I^\frown\}$, where $\varkappa \in \mathcal{K}'$, it follows from the equivalence

$$(\xi\varkappa)^\frown \leq \psi I^\frown \iff \forall c \in \mathcal{C}(\langle c, I\rangle\xi\varkappa \leq \psi\langle c, I\rangle).$$

Conversely, if $\mathcal{S}$ is a non-countably $\mathcal{K}'$-iterative, then $\mathcal{S}$ satisfies (I), because every set of the form (29) is an intersection of simple $\mathcal{K}'$-admissible initial segments of the same form with $\zeta \in \mathcal{C}$. On the other hand, the condition of iterativity of $\mathcal{S}$ in the sense of [3] possibly does not imply the condition (I), because the first one uses a solution $[\varphi, \psi]$ of

$$(30) \qquad (\psi \to I,\ \xi\varphi) \leq \xi$$

instead of (28). (Actually, $\mathcal{S}$ is iterative in the sense of [3] iff for all $\varphi, \psi \in \mathcal{F}$ there is a solution $[\varphi, \psi]$ of (30) in $\mathcal{F}$ which belongs to every intersection $\mathcal{A}$ of sets of the form (29) such that $(\psi \to I,\ \mathcal{A}\varphi) \subseteq \mathcal{A}$.) It should be noticed that the existences of least solutions of (28) and (30) are equivalent and both solutions are easily expressible by each other, namely: $\mathbb{I}(\varphi) = R[\varphi R, L]F_+$ and $[\varphi, \psi] = \mathbb{I}((T_+, F_+\varphi)\langle\psi, I\rangle)$. But in the case of least solutions in the stronger sense as in conditions (I) and that of iterativity of $\mathcal{S}$ in the sense of [3], this equivalence is not obvious, and that is why (I) is possibly less general than iterativity of $\mathcal{S}$. This loss of generality is, however, rather insignificant (condition (I) holds in any case when general criteria of iterativity of $\mathcal{S}$, given in [3], are applicable). And it may be completely avoided by some simple complications in the proof of Proposition 1, which are valid for the present kind of NOSS, namely companion operative spaces $\mathcal{S}_*$ of combinatory ones $\mathcal{S}$. (These complications consist of modifying the definition of $k(\Sigma, t)$ and the rest of the proof of the proposition, so that $k(\Sigma, t) = 0^+$ for $t = \Lambda$ and $k(\Sigma, t) = (n+1)^+$ for $t \neq \Lambda$, where n is the number of the pair (Σ, t), and the element σ of the universal coding $\langle k, \sigma\rangle$, constructed by the proof, is of the form $(L^2 \to T_+R,\ F_+\tau\langle RL, R\rangle)$ for some τ of a certain form, similar to (22), but with erased F_+. Then, if $\mathcal{S}$ is iterative in the sense of [3], the element $R[L^2, \tau\langle RL, R\rangle]$ is a $\mathcal{K}''$-iteration of σ, where $\mathcal{K}'' = \{\langle c, I\rangle \mid c \in \mathcal{C}\}$, and applying Proposition 1 instead of Corollary 1 with the set $\mathcal{K}''$ instead of $\mathcal{K}$, we obtain corollaries analogous to the above Corollaries 2–5 for the present kind of NOSS.) Thus it may be finally said that by the method of code evaluation, based on Theorem 1 or its variants, the principal results of the theory of combinatory spaces may be obtained even in a little bit better suppositions in comparison with [3], but this improvement is at most of a secondary significance.

Now, when the NOSS $\mathcal{S}_*$ is iterative, the Corollaries 2–5 hold for it and they consist the principal facts of the theory of combinatory spaces, excluding the theorem of representation of partially recursive functions. Thus we obtain a generalization of the last theory which uses no "points". We should note that Corollary 2 for this case is equivalent to the first recursion theorem in $\mathcal{S}$, since operation II is expressible by means of the storage S, namely $\langle\varphi,\psi\rangle = S(\psi)\langle R,L\rangle S(\varphi)\langle I,I\rangle$ (see [2], p. 55). Note also the normal form of computable mappings Γ obtained from Corollary 4:

$$\Gamma(\xi) = R[(I, F_+ A_0 \xi^\frown)\beta,\ L]\langle F, \langle\langle T,T,I\rangle\rangle,$$

where β is elementary in parameters. We may also obtain the existence of universal elements ω of the kind considered in [2, III.7], namely: for every recursive in parameters mapping $\Gamma : \mathcal{F}^{n+1} \to \mathcal{F}$ there is an absolutely normal in the sense of [2] element γ such that for all $b_0, \ldots, b_{n-1} \in \mathcal{C}$ we have

$$\Gamma(\omega, b_0, \ldots, b_{n-1}) = \omega\langle\gamma\langle b_0, \ldots, b_{n-1}\rangle, I\rangle,$$

where $\langle b_0, \ldots, b_{n-1}\rangle = \langle b_0, \langle b_1, \ldots \langle b_{n-2}, b_{n-1}\rangle \ldots\rangle\rangle$. For that purpose the proof of Corollary 3 has to be applied to a modified version of that of Proposition 1, which is valid for companion spaces $\mathcal{S}_*$ of combinatory spaces $\mathcal{S}$ and uses new parameter symbols for the parameters $b_0, \ldots, b_{n-1}$, and another code function k' instead of the old one k:

$$k'(\Sigma, t) = \langle\langle Lk(\Sigma, t), \langle b_0, \ldots, b_{n-1}\rangle\rangle, I\rangle.$$

7. FINAL REMARKS

The Corollary 5 for combinatory spaces (and its obvious analogue for an arbitrary operative space) is an important corollary which was not mentioned in monographs [2] and [1]. Its principal significance is in the fact that it shows that the recursiveness in combinatory spaces (respectively, operative ones) is a special case of explicit expressibility in combinatory algebras, thus confirming the view that combinatory algebras (or their equivalents like C-monoids of Lambek and Scott) are, perhaps, the best abstract system for the recursion theory. But the principal questions, arising in this connection about structures like combinatory spaces, NOSS etc., have not been investigated. Especially, it is not known whether the analogue of the Park's theorem holds, i. e. whether the Curry combinator in the algebra in Corollary 5 provides the least fixed point of the corresponding recursive mapping. And many interesting questions for concretes examples of NOSS about properties like extensionality and weak extensionality of corresponding combinatory algebras are open. An interesting perspective is connected as well with non-iterative NOSS for which the operation App in Corollary 5 may define a partial combinatory algebra. There are examples in this respect, which suggest interesting applications (for instance examples 2 in [8]). We are leaving these topics for possible further publications.

Finally, let us note that the theory of OSS, as exposed above, holds (without big complications in the proofs) also for a generalized kind of storage operation S, for which there are two constants D_0, D_1 such that the equality

(S2a) $$S((\varphi,\psi)) = (D_0 S(\varphi), D_1 S(\psi))D$$

is satisfied instead of (S2). Operative spaces with such generalized storage operation have interesting models arising from some category theoretic considerations. Namely, let C be a monoidal category in which binary co-products $X \oplus Y$ exist for all $X, Y \in C$ and satisfy the isomorphism $Z \otimes (X \oplus Y) \cong (Z \otimes X) \oplus (Z \otimes Y)$ naturally in X, Y, Z. Then any object V of C, which satisfies the isomorphisms

$$V \cong V \oplus V \cong V \otimes V,$$

provides such a model — the semigroup $C(V, V)$ of arrows from V to V with an operation S defined by $S(\varphi) = \tau^{-1} \circ (1_V \otimes \varphi) \circ \tau$, where $\tau : V \to V \otimes V$ is the given isomorphism. These models suggest connection with "recursion categories" of Di Paola — Heller [9] and deserve further examination in a separate paper.

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ГОДИШНИК

НА

СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

КНИГА 2 — МЕХАНИКА

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Livre 2 — Mécanique
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MOUVEMENT AVEC FROTTEMENT D'UNE SPHERE HOMOGENE DANS UN CYLINDRE HORIZONTAL

SONIA DENEVA

Соня Денева. КАЧЕНИЕ ШАРА ПО ГОРИЗОНТАЛЬНОМУ ЦИЛИНДРУ

В работе рассматривается задача о движении однородного шара по горизонтальному неподвижному цилиндру под действием сили трения.

Sonia Deneva. MOVEMENT OF A ROLLING SPHERE ON A HORIZONTAL CYLINDER

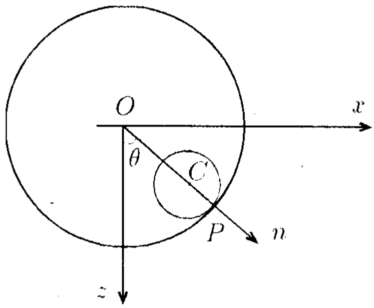
Some aspects of the classical problem about rolling sphere on a homogeneous horizontal cylinder are considered in this paper.

Le mouvement d'une sphère dans un cylindre horizontal immobile est un problème classique que nous considerons ayant vu la force du frottement entre deux corps selon la loi de Coulon. Pour simplification du problème nous supposons que la force du frottement est située dans un plan perpendiculaire de l'axe du cylindre.

Par ailleurs nous montrons comment calculer la chaleur qui se sépare par suite du frottement ayant vu que le travail et la puissance de la force du frottement sont equvalents à l'énérgie thermique.

Le cylindre est circulaire de rayon R; la sphère de centre C, de masse m et de rayon r se roule sur la part inférieur du cylindre. La section verticale du cylindre et de la sphère est representée au dess. 1; ici P est le point de contacte de deux corps. Nous introduisons l'angle $\theta = (\widehat{Oz, OP})$ qui determine la position du point P. Le vecteur unique n qui est normal à deux surfaces s'exprime par la formule

$$n = \sin\theta\, i + \cos\theta\, k, \tag{1}$$



Dess. 1

où i, j, k sont orts de système $Oxyz$ (l'axe Oy détermine la direction de l'axe du cylindre). Le vecteur unique t qui est tangentiel à la section normal du cylindre s'exprime par la formule

$$t = -\cos\theta\, i + \sin\theta\, k. \tag{2}$$

Evidement t est à sens du mouvement du point P. La vitesse du glissement de P s'exprime par la formule

$$v_P = v(-\cos\theta\, i + \sin\theta\, k), \tag{3}$$

où v est la grandeur de la vitesse.

Selon la loi de Coulon la force du frottement est

$$T = fR_n(\cos\theta\, i - \sin\theta\, k), \tag{4}$$

où f est coefficient du frottement et R_n — la pression normale du cylindre sur la sphère.

Puisque le plan équatorial de la sphère reste toujours sur le plan vertical Oxz la vitesse de rotation de la sphère est

$$\omega = \omega j, \quad \omega > 0 \tag{5}$$

parce que la sphère décendant se tourne en sens inverse des aiguilles d'une montre. La sphère est un solide, c'est-à-dire nous avons la relation cinématique

$$v_P = v_C + \omega \times CP. \tag{6}$$

Ayant vu (1), (3) et (5) la relation (6) se réduit à l'équation

$$v = -\delta\dot{\theta} - r\omega \quad \text{où} \quad \delta = R - r. \tag{7}$$

La formule (7) donne la vitesse du glissement du point P où $\theta < 0$; les grandeurs $\dot{\theta}$ et ω doivent être déterminées. Le théorème de la resultante cinétique et le théorème du moment cinétique appliqué au point C se traduisent par les équations

$$\frac{d}{dt}(mv_C) = mg + R_n + T, \tag{8}$$

$$\frac{2}{5}mr^2\frac{d\omega}{dt} = CP \times T, \tag{9}$$

où selon (1) on a

$$(10) \qquad R_n = -R_n(\sin\theta\, i + \cos\theta\, k), \quad R_n > 0.$$

Ayant vu que

$$x_C = \delta\sin\theta, \quad z_C = \delta\cos\theta, \quad y_C = 0$$

nous obtenons de (8), (10) et (4)

$$(11) \qquad \begin{aligned} m\delta\cos\theta\,\ddot{\theta} - m\delta\sin\theta\,\dot{\theta}^2 &= -R_n\sin\theta + fR_n\cos\theta, \\ -m\delta\sin\theta\,\ddot{\theta} - m\delta\cos\theta\,\dot{\theta}^2 &= mg - R_n\cos\theta - fR_n\sin\theta, \end{aligned}$$

où

$$m\delta\ddot{\theta} = -mg\sin\theta + fR_n, \qquad m\delta\dot{\theta}^2 = R_n - mg\cos\theta.$$

Des formules (11) on obtient

$$(12) \qquad R_n = mg\cos\theta + m\delta\dot{\theta}^2.$$

Remplaçons (12) en (11) et on trouve l'équation

$$(13) \qquad \delta\ddot{\theta} - f\delta\dot{\theta}^2 - fg\cos\theta + g\sin\theta = 0.$$

On remarquera que si f a des valeurs petites (13) se réduit à l'équation du pendule.

Il n'est pas difficile de voir que l'équation (13) exprime le mouvement d'un point qui se mouvoit sur une circonférence matérielle avec frottement. De l'équation (9) ayant vu (4), (5) et (12) on obtient

$$(14) \qquad \frac{d\omega}{dt} = \frac{5f}{2r}\left(g\cos\theta + \delta\dot{\theta}^2\right).$$

Dernière équation montre que $\omega(t)$ est une fonction croissant du temps mais si le coefficient du frottement est très petit ω devient une constante en particulier zéro (si $\omega_0 = 0$), c'est à dire si le frottement est très faible nous pouvons avoir (pour la sphère) seulement glissement sans roulement.

Retournons a l'équation (13); on a

$$\ddot{\theta} = \frac{1}{2}\frac{d}{d\theta}\left(\dot{\theta}^2\right).$$

Posons $\dot{\theta}^2 = u$ et nous obtenons de (13)

$$(15) \qquad \frac{1}{2}\delta\frac{du}{d\theta} - f\delta u + g\sin\theta - fg\cos\theta = 0.$$

La solution générale de l'équation (15) s'exprime par la formule

$$(16) \qquad u = \dot{\theta}^2 = Ce^{2f\theta} + \lambda\sin\theta + \mu\cos\theta,$$

où C est une constante d'integration et les constantes λ, μ sont

$$(17) \qquad \lambda = \frac{6fg}{\delta(1+4f)}, \qquad \mu = \frac{2g(1-2f^2)}{\delta(1+4f^2)}.$$

La constante C s'exprime par les conditions imitiables du mouvement

$$(18) \qquad C = e^{-2f\theta_0}\left(\dot{\theta}_0^2 - \lambda\sin\theta_0 - \mu\cos\theta_0\right).$$

Ayant vu que $\theta(t)$ est une fonction décroissante du temps on obtient de (16) et (18)

$$\dot\theta = -\sqrt{e^{-2f(\theta-\theta_0)}A + \lambda\sin\theta + \mu\cos\theta}, \tag{19}$$

où

$$A = \dot\theta_0^2 - \lambda\sin\theta_0 - \mu\cos\theta_0 \tag{20}$$

et λ, μ sont données par (17).

Nous donnerons une solution approximative de l'équation (19) supposant que le coefficient f est un nombre petit, c'est-à-dire

$$f^2 = 0, \quad f^3 = 0 \quad \text{etc.} \tag{21}$$

D'autre part nous supposons que l'angle θ varie en telles limites que nous pouvons accepter

$$\sin\theta = \theta, \quad \cos\theta = 1 - \frac{\theta^2}{2}, \quad e^{2f(\theta-\theta_0)} = 1 + 2f(\theta - \theta_0), \tag{22}$$

c'est-à-dire nous acceptons par exemple que $\theta_0 \leqq \frac{\pi}{4}$.

Ayant en vue ces raisons nous obtenons de (17) et (19)

$$\lambda = 6\frac{fg}{\delta}, \quad \mu = \frac{2g}{\delta}, \tag{23}$$

$$\frac{d\theta}{dt} = -\sqrt{\frac{2g}{\delta} + a(1 - 2f\theta_0) + f(2a + 6g)\theta\frac{g}{\delta}\theta^2}. \tag{24}$$

Posons

$$\theta = \varphi + f\left(3 + \frac{a\delta}{\theta}\right) \tag{25}$$

et remplaçons dans (24); on obtient

$$\frac{d\varphi}{dt} = -\sqrt{\frac{2g}{\delta} + a(1 - 2f\theta_0) - \frac{g}{\delta}\varphi^2}. \tag{26}$$

Nous acceptons que

$$f < \frac{1}{2\theta_0} \tag{27}$$

qui n'est pas une grande restriction pour f et encore que

$$\dot\theta_0^2 > 6\frac{fg}{\delta}\sin\theta_0 + \frac{2g}{\delta}\cos\theta_0. \tag{28}$$

C'est-à-dire nous avons une impulsion initiable. Alors il est évidement que l'expression $\frac{2g}{\delta} + (1 - 2f\theta_0)a$ est positive. Posons

$$b^2 = 2 + \frac{\delta}{g}a(1 - 2f\theta_0). \tag{29}$$

De l'équation (26) nous obtenons

$$\frac{d\varphi}{dt} = -\sqrt{\frac{g}{\delta}}\sqrt{b^2-\varphi^2}. \tag{30}$$

Apres l'integration de (30) et conformat (25) nous recevons

$$\theta = f\left(3+\frac{a\delta}{g}\right) + C_1\cos\sqrt{\frac{g}{\delta}}\,t - C_2\sin\sqrt{\frac{g}{\delta}}\,t, \tag{31}$$

où a se détermine par (20) et C_1, C_2 sont les constantes suivantes:

$$C_1 = \theta_0 - f\left(3+\frac{a\delta}{g}\right), \qquad C_2 = \sqrt{b^2-C_1^2}. \tag{32}$$

Ayant en vue (20), (23), (29) nous obtenons

$$C = -\sqrt{\frac{\delta}{g}}\,\dot{\theta}, \quad \dot{\theta} < 0. \tag{33}$$

Retournons à l'équation (14) d'où nous déterminons la vitesse de la rotation $\omega(t)$ de la sphère ayant en vue (22) et (31); on obtient

$$\omega(t) = \omega_0 + \frac{5fg}{2r}\left(t\left(1+\frac{1}{4}b\right)^2 + \frac{3}{8}\sqrt{\frac{\delta}{g}}\left(C_2^2-C_1^2\right)\sin 2\sqrt{\frac{g}{\delta}}\,t \right.$$
$$\left. + \frac{3}{2}C_1C_2\sqrt{\frac{\delta}{g}}\sin^2\sqrt{\frac{g}{\delta}}\,t\right), \tag{34}$$

où ω_0 est la vitesse de la rotation initiale.

De la formule (7) nous déterminons la vitesse du glissement du point de contact P.

Remplaçons dans (7), (31́) et (34) et nous obtenons

$$v(t) = C_1\sqrt{g\delta}\sin\sqrt{\frac{g}{\delta}}t + C_2\sqrt{g\delta}\cos\sqrt{\frac{g}{\delta}}t - r\omega_0$$
$$-\frac{5f}{2}\left(g\left(1+\frac{1}{4}b^2\right)t + \frac{3}{8}\sqrt{g\delta}\left(C_2^2-C_1^2\right)\sin 2\sqrt{\frac{g}{\delta}}t + \frac{3}{2}C_1C_2\sqrt{g\delta}\sin^2\sqrt{\frac{g}{\delta}}t\right). \tag{35}$$

Posons $t=0$ dans (35) et selon (33) on obtient

$$v_0 = -\delta\dot{\theta}_0 - r\omega_0 \tag{36}$$

qui coordonne avec formule (7).

Puisque $v_0 \neq 0$ pour avoir un mouvement frottement de glissement il faut de (36)

$$\omega_0 < \frac{\delta|\dot{\theta}_0|}{r}, \tag{37}$$

c'est-à-dire ω_0 est une grandeur bornée.

La formule (35) est valide aux valeurs suivantes de t, $0 \leqq t \leqq t_1$, où

$$t_1 = \frac{\pi}{2}\sqrt{\frac{\delta}{g}} + \frac{C_1\sqrt{g\delta} - r\omega_0}{\sqrt{g\delta}|\dot{\theta}_0|}. \tag{38}$$

Au moment t_1 la vitesse v devient zéro, c'est-à-dire $v(t_1) = 0$. Après l'instant t_1 la vitesse du point de contacte P reste toujours zèro, c'est-à-dire nous avons un mouvement nonholonome de la sphère dans le cylindre.

Remplaçons t_1 dans la formule (31) et nous obtenons

$$\theta_1 = f\left(3 + \frac{a\delta}{g}\right) - C_1 \cos\frac{\delta|\dot{\theta}_0| + r\omega_0}{\dot{\theta}_0\sqrt{g\delta}} - C_2 \sin\frac{\delta\dot{\theta}_0 + r\omega_0}{\dot{\theta}_0\sqrt{g\delta}}. \tag{39}$$

Si le coefficient f est très petit $C_1 > 0$ et la formule (39) montre que $\theta_1 < 0$, c'est-à-dire selon le dess. 1 la sphère a passé le plus bas point du cylindre à gauche de l'axe Oz. Après le point θ_1 la sphère se retourne sans glissement sur le cylindre.

A la fin nous calculerons le travail de la force du frottement T qui se transforme en chaleur, c'est-à-dire en énergie thèrmique. Ayant en vue (3) et (4) nous obtenons pour le travail élémentaire de T

$$dA = -fR_n v\, dt, \tag{40}$$

où R_n est la pression normale et v est la vitesse du glissement de la sphère. Le signe moins montre que le travail de T est négatif parcequ'il est dirigé au contraire du mouvement. Remplaçons (12) et (35) dans (40) et en conformant (21), (22), (31) nous obtenons

$$\begin{aligned} dA = & -fmg\,dt\Big(C_1\sqrt{g\delta}\sin\sqrt{\frac{g}{\delta}}t + C_2\sqrt{g\delta}\cos\sqrt{\frac{g}{\delta}}t - r\omega_0 \\ & + C_1\sqrt{g\delta}\left(C_2^2 - \frac{1}{2}C_1^2\right)\sin\sqrt{\frac{g}{\delta}}t\cos^2\sqrt{\frac{g}{\delta}}t + C_2\sqrt{g\delta}\left(C_2 - \frac{1}{2}C_1^2\right)\cos^3\sqrt{\frac{g}{\delta}}t \\ & - r\omega_0\left(C_2^2 - \frac{1}{2}C_1^2\right)\cos^2\sqrt{\frac{g}{\delta}}t + C_1\sqrt{g\delta}\left(C_1^2 - \frac{1}{2}C_2^2\right)\sin^3\sqrt{\frac{g}{\delta}}t \\ & + C_2\sqrt{g\delta}\left(C_1^2 - \frac{1}{2}C_2^2\right)\sin^2\sqrt{\frac{g}{\delta}}t\cos\sqrt{\frac{g}{\delta}}t - r\omega_0\left(C_1^2 - \frac{1}{2}C_2^2\right)\sin^2\sqrt{\frac{g}{\delta}}t \\ & + 3C_1^2C_2\sqrt{g\delta}\sin^2\sqrt{\frac{g}{\delta}}t\cos\sqrt{\frac{g}{\delta}}t + 3C_1C_2^2\sqrt{g\delta}\sin\sqrt{\frac{g}{\delta}}t\cos^2\sqrt{\frac{g}{\delta}}t \\ & - 3r\omega_0C_1C_2\sin\sqrt{\frac{g}{\delta}}t\cos\sqrt{\frac{g}{\delta}}t\Big). \end{aligned} \tag{41}$$

Après l'intégration de (41) nous obtenons pour le travail de T:

$$(42)\quad A(t) = -fmg\delta\Big(C_1 + C_1C_2^2 + \frac{1}{2}C_1^3 - \frac{r\omega_0}{\delta}\left(1+\frac{1}{4}b^2\right)t$$
$$+\left(C_2 + C_2^3 - \frac{1}{2}C_1^2C_2\right)\sin\sqrt{\frac{g}{\delta}}t - \left(C_1 + C_1^3 - \frac{1}{2}C_1C_2^2\right)\cos\sqrt{\frac{g}{\delta}}t$$
$$+\frac{3r\omega_0}{8\sqrt{g\delta}}(C_1^2 - C_2^2)\sin\left(2\sqrt{\frac{g}{\delta}}t\right) - \frac{3r\omega_0}{2\sqrt{g\delta}}C_1C_2\sin^2\sqrt{\frac{g}{\delta}}t$$
$$+\frac{1}{2}(3C_1^2C_2 - C_2^3)\sin^3\sqrt{\frac{g}{\delta}}t - \frac{1}{2}(3C_1C_2^2 - C_1^3)\cos^3\sqrt{\frac{g}{\delta}}t\Big).$$

Quand $\omega_0 = 0$ on obtient de (42)

$$(43)\quad A(t) = -fm\delta\Big(C_1 + C_1C_2^2 + \frac{1}{2}C_1^3 + \left(C_2 + C_2^3 - \frac{1}{2}C_1^2C_2\right)\sin\sqrt{\frac{g}{\delta}}t$$
$$-\left(C_1 + C_1^3 - \frac{1}{2}C_1C_2^2\right)\cos\sqrt{\frac{g}{\delta}}t + \frac{1}{2}(3C_1^2C_2 - C_2^3)\sin^3\sqrt{\frac{g}{\delta}}t$$
$$-\frac{1}{2}(3C_1C_2^2 - C_1^3)\cos^3\sqrt{\frac{g}{\delta}}t\Big).$$

Les constantes b, C_1, C_2 se donnent par (29), (32), (33). Tout le travail de la force du frottement se détermine par (42) et (43) pour $t = t_1$ où t_1 se donne de la formule (38).

Nous donnerons un exemple numérique. Prenons $R = 100$ ten, $r = 10$ ten, $\delta = 90$ ten, $\theta_0 = \frac{\pi}{4} = 0{,}785$, $\dot{\theta}_0 = 5\,\mathrm{s}^{-1}$, $f = 0{,}1$, $\omega_0 = 0$.

Des formules (23), (20) nous obtenons

$$\lambda = 6{,}54\,\mathrm{s}^{-1},\quad \mu = 21{,}8\,\mathrm{s}^{-2},\quad a = 5{,}02\,\mathrm{s}^{-2}.$$

Si la sphère est construite de fer nous avons

$$mg = 32{,}67\,\mathrm{kg},\quad fmg\delta = 294{,}03\,\mathrm{kg\,ten}.$$

Remplaçons les données numériques dans la formule (43) et nous obtenons

$$|A| = 1477{,}2\,\mathrm{kg\,ten}.$$

C'est le travail que la sphère a achevée avec la force du frottement pendant leur mouvement. Ce travail se transforme en chaleur par la formule

$$(44)\qquad Q = IA,$$

où I est l'équivalent thermique du travail.

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УСТОЙЧИВОСТ НА СТАЦИОНАРНИТЕ ДВИЖЕНИЯ НА СИСТЕМИ ОТ СИМЕТРИЧНИ ТЕЛА, СВЪРЗАНИ СЪС СФЕРИЧНИ ШАРНИРИ

НИКОЛИНА ВАСИЛЕВА

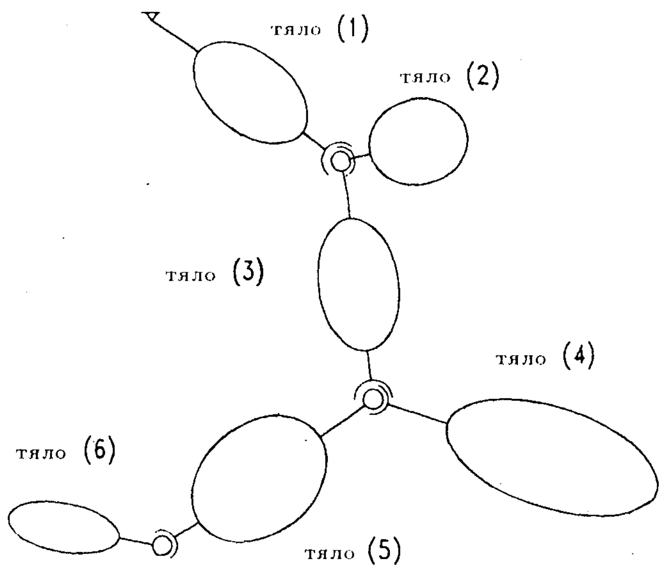
Николина Василева. УСТОЙЧИВОСТЬ СТАЦИОНАРНЫХ ДВИЖЕНИЙ СИСТЕМЫ ОСЕСИММЕТРИЧЕСКИХ ТЕЛ, СВЯЗАННЫХ СФЕРИЧЕСКИМИ ШАРНИРАМИ

Работа посвящена устойчивости стационарных движений систем абсолютно твердых тел со структурой дерева. В концах динамических осей симметрии тел расположены идеальные сферические шарниры. Одно из тел имеет неподвижную точку. Стационарными являются движения, при которым оси тел вращаются как твердое тело вокруг вертикальной оси с постоянной угловой скорости. Притом каждое тело системы совершает вращение вокруг своей оси симметрии. Достаточные условия выведены из теоремы Рауса об устойчивости приведенной системы.

Nikolina Vasileva. STABILITY OF STEADY-STATE MOTIONS OF SYSTEMS OF SYMMETRIC RIGID BODIES WITH BALL-AND-SOCKET JOINTS

The paper is developed to the study of stability of steady-state motions of a tree-like system. Heavy symmetric rigid bodies are connected at the ends of their symmetry axes with ball-and-socket joints. One of the bodies is fixed. The steady-state motions are obtained when the symmetry axes and rods move as one rigid body, rotating with a constant angular velocity around the vertical and at the same time the bodies rotate uniformly around their symmerty axes. The sufficient conditions for stability of steady-state motions are derived from Routh's theorem for stability of the reduced system.

Да разгледаме система абсолютно твърди тела, за която са изпълнени следните условия: системата има структура на дърво и се намира под действието на силата на тежестта, тяло (0) е неподвижно, телата от системата имат динамични оси на симетрия и са свързани помежду си в краищата на осите си на симетрия с помощта на сферични шарнири. Допускаме също, че кинематичните връзки, реализирани в съчлененията между телата, са идеални, т. е. силите на реакциите не извършват работа за виртуални премествания на системата. На дадената система съпоставяме ориентиран граф, в който дъгите $u_\alpha (\alpha = 1, \ldots, n)$ съответстват на съчлененията, а върховете s_i $(i = 1, \ldots, n)$ — на телата от системата ([1]). Предполагаме, че номерацията на дъгите и върховете в графа е правилна, т. е. всеки връх и предшестващата го дъга носят един и същ номер. Предполагаме също, че дъгите са насочени от връх с по-малък номер към връх с по-голям номер (фиг. 1). Тогава матрицата на инцидентност T ([1]) има елементи T_{ij}, които приемат стойности -1 и 0, а именно $T_{ij} = -1$, ако дъгата u_i (върхът s_i) принадлежи на пътя от върха s_0 до върха s_i, и $T_{ij} = 0$ в противен случай.



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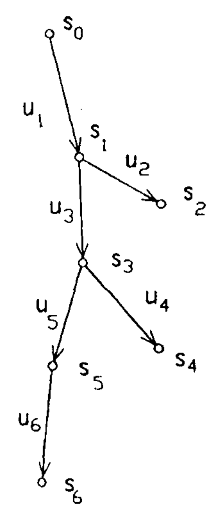
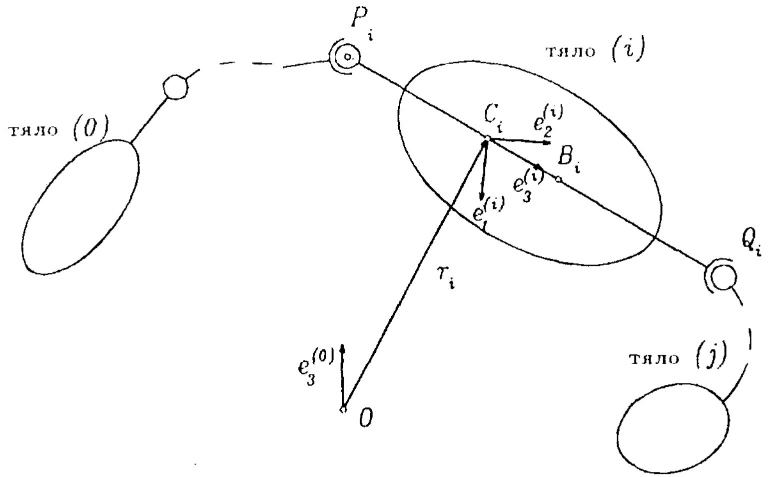


б

Фиг. 1

Разглеждаме тяло с номер i. Тъй като пътят между върха s_i и произволен друг връх s_j от графа е единствен, то само една от шарнирните точки на тяло (i) води към тяло (j). Точката P_i, водеща към тяло (0), се нарича предшестваща точка. Към шарнирните точки P_i и Q_i, съвпадащи с центровете на сферичните шарнири на тяло (i) (фиг. 2), добавяме точкова маса, равна на сумата от масите на всички тела, към които води тази точка. Полученото тяло с маса M се нарича допълнено тяло (i), а неговият масов център B_i — барицентър на тяло (i). Тъй като телата са симетрични, масовият център C_i и барицентърът B_i на тяло (i) лежат на оста на симетрия P_iQ_i на тялото. Ако $\mathbf{e}_i$ е единичният вектор, насочен по оста на симетрия на тяло (i) в посока от шарнирна точка P_i към масовия център C_i, то

$$(1)\quad \overline{P_iC_i}=c_i\mathbf{e}_i,\ c_i \geqq 0,\ \overline{P_iQ_i}=l_i\mathbf{e}_i,\ \overline{P_iB_i}=b_i\mathbf{e}_i,\ b_i=\frac{1}{M}\left[(c_i-l_i)m_i-l_i\sum_{j=1}^{n}T_{ij}m_j\right].$$



Фиг. 2

Въвеждаме неизменно свързан с тяло (i) базис

$$\mathbf{e}^{(i)} = \left(\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}, \mathbf{e}_3^{(i)}\right)^T \qquad (i = 1, \ldots, n),$$

за който координатният вектор $\mathbf{e}_3^{(i)} = \mathbf{e}_i$. Тензорът на инерцията на допълненото тяло (i) относно точката P_i може да се представи във вида $\mathbf{J}_i = J_1^{(i)}\left(\mathbf{e}_1^{(i)}\mathbf{e}_1^{(i)} + \mathbf{e}_2^{(i)}\mathbf{e}_2^{(i)}\right) + J_3^{(i)}\mathbf{e}_3^{(i)}\mathbf{e}_3^{(i)}$. Инерчните моменти $J_1^{(i)}, J_3^{(i)}$ и

$$\varepsilon_{ij} = -M(T_{ij}l_ib_j + T_{ji}l_jb_i) = \varepsilon_{ji}, i \neq j, \varepsilon_{ii} = J_1^{(i)}$$

са постоянни величини и чрез тях кинетичната енергия на системата се записва във вида

$$T = \sum_{i=1}^{n} J_3^{(i)}(\mathbf{e}_i.\omega_i)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_{ij}\dot{\mathbf{e}}_i \cdot \dot{\mathbf{e}}_j,$$

където ω_i и $\dot{\mathbf{e}}_i$ са съответно абсолютната ъглова скорост на тяло (i) и абсолютната производна на вектора $\mathbf{e}_i$.

Инерциалната координатна система, за която предполагаме, че векторът $\mathbf{e}_3^{(0)}$ има посока, противоположна на силата на тежестта, означаваме с $Oe_1^{(0)}e_2^{(0)}e_3^{(0)}$. Тензорът Γ_i, задаващ положението на базиса $e^{(i)}$ относно базиса $e^{(0)}$, може да се определи с помощта на три скаларни величини. Най-често това са ъглите на три последователни ротации около съответни оси, привеждащи базиса $e^{(0)}$ в базиса $e^{(i)}$. Обикновено последната ос на въртене е определена от третата координатна ос e_i на базиса $e^{(i)}$. Ако означим с ϕ_i ъгъла на завъртане около тази ос, то абсолютната ъглова скорост ω_i може да се запише във вида $\omega_i = \omega_i' + \dot\phi_i e_i$, където ω_i' и e_i не зависят от величините $\phi_1, \ldots, \phi_n$. Като обобщени координати на системата избираме ъглите $\phi_1, \ldots, \phi_n$ и величини, определящи положението на векторите e_i в инерциалното пространство. Тъй като кинетичната енергия на системата не зависи от $\phi_1, \ldots, \phi_n$, то тези величини се явяват циклични *координати, а останалите обобщени координати, определящи векторите* e_i — позиционни.

От уравненията на движението, получени в [2], следва съществуването на n първи интеграла

$$\rho_i = \frac{\partial L}{\partial \dot\phi_i} = J_3^{(i)}(\dot\phi_i + \omega_i' \cdot e_i) = \rho_{i0} \quad (i = 1, \ldots, n), \tag{2}$$

където константите $\rho_{10}, \ldots, \rho_{n0}$ са началните стойности на импулсите ρ_i.

Уравненията на движението имат още един първи интеграл, който изразява, че проекцията на кинетичния момент на системата върху вертикалата остава постоянна. Това позволява да се въведе още една циклична координата Ψ, определяща въртенето с постоянна ъглова скорост $\dot\Psi e_3^{(0)}$ на една нова координатна система $\zeta = (\zeta_1, \zeta_2, \zeta_3)$, $\zeta_3 = e_3^{(0)}$.

Стационарни движения на произволна механична система са движенията, за които позиционните координати и цикличните скорости остават постоянни. Следователно при стационарните движения на системи от сферични тела цикличните координати ϕ_i, определящи ъгъла на въртене на телата около осите им на симетрия, и ъгълът Ψ на въртене на базиса ζ се изменят линейно във времето, а позиционните координати e_i остават неизменни относно базиса ζ. Това означава, че осите на телата и съединяващите ги шарнири се движат като едно твърдо тяло, въртящо се с постоянна ъглова скорост $\dot\Psi$ около вертикалната ос. От своя страна всяко от телата също се върти около оста си на симетрия с определена постоянна ъглова скорост $\dot\phi_i$.

Стационарните движения се определят от полагането

$$\dot e_3^{(i)} = \dot\psi e_3^{(0)} \times e_i, \quad \omega_i = \dot\psi e_3^{(0)} + \dot\phi_i e_i, \quad \dot\psi = \text{const}, \dot\phi_i = \dot\phi_{i0} = \text{const}$$

в уравненията на движението. Получаваме равенствата

$$\dot\psi^2 e_3^{(0)} \times (e_3^{(0)} \times \sum_{j=1}^{n} \varepsilon_{ij} e_j) + A_i e_3^{(0)} = x_i e_i, \tag{3}$$

където $A_i = M\varepsilon b_i - \dot\psi \rho_{i0}$, а x_i са константи. В [2] е получено необходимото и достатъчно условие за съществуване на стационарни движения на разглеждания тип системи тела, което тук ще формулираме в следния вид: Ако съществуват скаларни величини $x_1, \ldots, x_n$ и $y_1, \ldots, y_m$ и единични вектори $\mathbf{e}_3^{(1)}, \ldots, \mathbf{e}_3^{(n)}$, за които

$$\sum_{j=1}^{n} v_{ij}\mathbf{e}_j = y_i \mathbf{e}_3^{(0)}, \quad v_{ij} = \begin{cases} \dot{\psi}^2 \varepsilon_{ij} & \text{при } i \neq j, \\ j_1^{(i)} + x_i & \text{при } i = j, \end{cases}$$

то движението, определено от началните условия

$$\mathbf{e}_i(t_0) = \mathbf{e}_i, \phi_i(t_0) = \phi_{i0}, \dot{\phi}_{i0}(t_0) = \dot{\phi}_{i0} = \frac{Mb_i\varepsilon_{ij} - (\dot{\Psi}^2 J_3^{(i)} + x_i)\mathbf{e}_i\mathbf{e}_3^{(0)}}{J_3^{(i)}\dot{\Psi}},$$

е стационарно и обратно. Ще отбележим, че стационарните движения притежават едно общо свойство, което следва от равенство (3). Нека рангът на матрицата $V = (v_{ij})$ е $r = \operatorname{rank}(V)$ и $V_{i_1}, \ldots, V_{i_r}$ са r нейни базисни стълба. Ако небазисните стълбове $V_{j_1}, \ldots, V_{j_{n-m}}$ са представени във вида

$$V_{j_m} = -\sum_{s=1}^{r} r_{ms} V_{i_s} \quad (m = 1, \ldots, n-r),$$

то могат да се намерят константи $\varepsilon_{i_1}, \ldots, \varepsilon_{i_r}$, с помощта на които векторите $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_r}$ се изразяват като линейните комбинации

$$\mathbf{e}_{i_s} = -\sum_{m=1}^{n-r} r_{ms}\mathbf{e}_{j_m} + \varepsilon_{i_s}\mathbf{e}_s^{(0)} \quad (s = 1, \ldots, r).$$

Очевидно положенията на осите на телата в пространството при стационарните движения зависят от ранга на матрицата V. При $n = r$ векторите $\mathbf{e}_1, \ldots, \mathbf{e}_n$ са успоредни на вектора $\mathbf{e}_3^{(0)}$. Ако $r = n - 1$, то $\mathbf{e}_1, \ldots, \mathbf{e}_n$ се изразяват като линейни комбинации на векторите $\mathbf{e}_3^{(0)}$ и $\mathbf{e}_{j_1}$, откъдето следва, че в този случай осите на всички тела са разположени в една вертикална равнина. При $r \leqq n - 2$ съществуват стационарни движения с пространствена конфигурация, т. е. не всички оси лежат в една вертикална равнина.

Нека за дадена система тела е намерено едно нейно стационарно движение и нека стационарните стойности на векторите $\mathbf{e}_i$, импулсите ρ_i и ъгловата скорост $\dot{\Psi}$ са означени съответно с $\eta_i \rho_{i0} (i = 1, \ldots, n)$ и $\dot{\Psi}$. Тогава пресмятаме константите

$$A_i = M_\varepsilon b_i - \dot{\Psi}\rho_{i0}, (\eta_i \mathbf{e}_3^{(0)})x_i = A_i \quad (i = 1, \ldots, n) \tag{4}$$

и диагоналните елементи на матрицата $V = (v_{ij})$

$$v_{ii} = \dot{\Psi}^2 \varepsilon_i + x_{ii} \quad (i = 1, \ldots n) \tag{5}$$

(останалите ѝ елементи са константи, зависещи от геометричните и механичните свойства на системата тела). Като имаме предвид условията (3) за съществуване на стационарни движения, ще отбележим, че векторите $\eta_1, \ldots, \eta_n$ удовлетворяват получената по този начин линейна система

$$\sum_{j=1}^{n} v_{ij}\eta_j = y_i \mathbf{e}_3^{(0)} \quad (i = 1, \ldots, n)$$

за някакви стойности на $y_1, \ldots, y_n$.

Нека рангът на матрицата V е r и нека $i_1 < \ldots < i_r$ са номерата на r нейни базисни стълба. Останалите стълбове с номера $j_1 < \ldots < j_{n-r}$ се изразяват от линейните зависимости

$$V_{j_m} = -\sum_{s=1}^{r} r_{ms} V_{i_s} \quad (m = 1, \ldots, n-r).$$

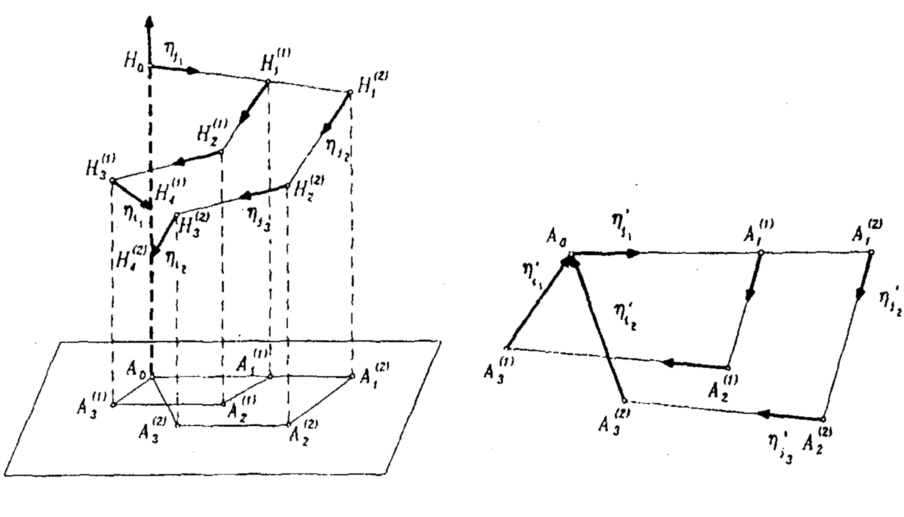
Тогава могат да се намерят такива константи ε_{i_s} $(s = 1, 2, \ldots, r)$, че между векторите $\eta_1, \ldots, \eta_r$ да съществува зависимостта

$$\eta_{i_s} = -\sum_{m=1}^{n-r} r_{ms}\eta_{j_m} + \varepsilon_{i_s}\mathbf{e}_3^{(0)} \quad (s = 1, \ldots, r). \tag{6}$$

Линейна зависимост от този вид има проста геометрична интерпретация (фиг. 3а) — начупените линии $L^{(s)} = H_0^{(s)} H_1^{(s)} \ldots H_{n-r+1}^{(s)}$ $(s = 1, \ldots, r)$, съставени от отсечките

$$\overline{H_{m-1}^{(s)} H_m^{(s)}} = r_{ms} k_m \quad (m = 1, \ldots, n-r), \qquad \overline{H_{n-r}^{(s)} H_{n-r+1}^{(s)}} = \mathbf{e}_{i_s}, \qquad H_0^{(s)} = H_0,$$

определят вектори $\overline{H_0^{(s)} H_{n-r+1}^{(s)}}$, успоредни на вектора $\mathbf{e}_3^{(0)}$.



а　　　　　　　　　б

Фиг. 3

На базата на този геометричен модел ще изследваме съществуването на други стационарни движения, получени при същите стойности $\rho_{10},\ldots,\rho_{n0}$ на импулсите, но при различни единични вектори $\mathbf{e}_1,\ldots,\mathbf{e}_n$.

От формули (4) и (5) получаваме, че ако ъглите между векторите $\mathbf{e}_1, \ldots, \mathbf{e}_n$ и вертикалната ос са съответно равни на ъглите, които векторите $\eta_1,\ldots,\eta_n$ сключват с вертикалната ос, то матрицата V и началните стойности на импулсите $\rho_{10},\ldots,\rho_{n0}$ няма да се променят. Следователно намирането на друго стационарно движение, определено от векторите $\mathbf{e}_1,\ldots,\mathbf{e}_n$ и същите импулси $\rho_{10},\ldots,\rho_{n0}$, е свързано с техните проекции върху хоризонтална равнина.

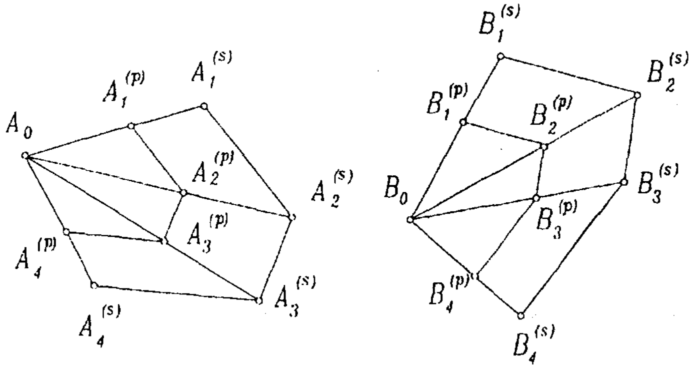
Да разгледаме проекциите на начупените линии $L^{(1)},\ldots,L^{(r)}$ върху такава равнина (фиг. 3б). Тези проекции се състоят от r многоъгълника, $A_0^{(s)}A_1^{(s)}\ldots A_{n-r}^{(s)}A_0^{(s)}$, $A_0^{(s)}=A_0$ $(s=1,\ldots,r)$, чиито страни са дефинирани от векторите

$$A_{m-1}^{(s)}A_m^{(s)}=r_{ms}\eta'_{j_m},\qquad A_{n-r}^{(s)}A_0=\eta'_{i_s}\qquad (m=1,\ldots,n-r;\ s=1,\ldots,r).$$

Тук η'_{j_m} и η'_{i_s} са проекциите на векторите η_{j_m} и η_{i_s}. Очевидно съответните страни на многоъгълниците с изключение на страните $A_{n-r}^{(s)}A_0^{(s)}$ са успоредни помежду си. Оттук следва, че за да получим друго стационарно движение, запазващо импулсите, е необходимо да съществуват различни многоъгълници $A_0^{(s)}A_1^{(s)}\ldots A_{n-r}^{(s)}A_0^{(s)}$ и $B_0^{(s)}B_1^{(s)}\ldots B_{n-r}^{(s)}B_0^{(s)}$ с еднакви дължини на съответните страни. Например такъв случай е възможен, ако проекциите $\eta'_{i_1},\ldots,\eta'_{i_r}$ са колинеарни и всяка двойка от многоъгълниците $A_0^{(s)}A_1^{(s)}\ldots A_{n-r}^{(s)}A_0^{(s)}$ и $A_0^{(p)}A_1^{(p)}\ldots A_{n-r}^{(p)}A_0^{(p)}$ са подобни, т. е.

$$(7)\qquad \frac{|\eta'_{i_s}|}{|\eta'_{i_p}|}=\frac{|r_{1s}|}{|r_{1p}|}=\ldots=\frac{|r_{n-r,s}|}{|r_{n-r,p}|}\qquad (s,p=1,\ldots,r).$$

Един пример на такъв случай е изобразен на фиг. 4.



Фиг. 4

Ако разглежданото стационарно движение е определено от такива вектори $\eta_1, \ldots, \eta_n$, че многоъгълниците $A_0^{(s)} A_1^{(s)} \ldots A_{n-r}^{(s)} A_0^{(s)}$ $(s = 1, \ldots, r)$ могат да се построят по единствен начин, то от равенствата $\sphericalangle(\mathbf{e}_i, \mathbf{e}_3^{(0)}) = \sphericalangle(\eta_i, \mathbf{e}_3^{(0)})$ и $\sphericalangle(\mathbf{e}_i', \mathbf{e}_j') = \sphericalangle(\eta_i', \eta_j')$ $(i, j = 1, \ldots, n)$ следва, че $\mathbf{e}_i = \eta_i$, т. е. стационарното движение, определено от векторите $\eta_1, \ldots, \eta_n$ и импулсите $\rho_{10}, \ldots, \rho_{n0}$, е единствено. Например, ако $n - r = 2$ и не всички вектори $\eta_{i_1}, \ldots, \eta_{i_r}$ са успоредни помежду си, то проекциите на начупените линии — триъгълниците $A_0^{(s)} A_1^{(s)} A_2^{(s)}$ $(s = 1, \ldots, r)$, са с фиксирани дължини и очевидно са построими по единствен начин. Този пример показва, че едно достатъчно условие за единственост на стационарните движения е за дадените стойности на величините $\rho_{10}, \ldots, \rho_{n0}$ да бъде изпълнено неравенството $\operatorname{rank} V \geqq n - 2$.

Ще изследваме устойчивостта на стационарното движение, определено от векторите $\eta_1, \ldots, \eta_n$ и импулсите $\rho_{10}, \ldots, \rho_{n0}$. Тъй като координатите $\phi_1, \ldots, \phi_n$ и Ψ са циклични, то можем да говорим за устойчивост само относно позиционните координати $\mathbf{e}_i$, техните скорости $\dot{\mathbf{e}}_i$ и импулсите $\rho_i = \dfrac{\partial T}{\partial \phi_i}$ ([3]).

Устойчивостта относно импулсите ρ_i следва от съществуването на цикличните първи интеграли (2), които изразяват постоянството на смущенията ε_i на обобщените импулси ρ_i.

Цикличните координати $\phi_1, \phi_2, \ldots, \phi_n$ се елиминират с въвеждането на функцията на Раус

$$R = T - \Pi - \sum_{i=1}^{n} \rho_i \dot{\phi}_i = T - \Pi - \sum_{i=1}^{n} \rho_i \left(\frac{1}{J_3^{(i)}} \rho_i - (\dot{\Psi} \mathbf{e}_3^{(i)} + \Omega_i) \cdot \mathbf{e}_3^{(i)} \right) = R_2 + R_1 - W.$$

Функциите R_2, R_1 и W обединяват съответно членовете от втора, първа и нулева степен на позиционните скорости. Както е известно [3], всяко стационарно движение може да се разглежда като състояние на равновесие на една нова механична система, наречена приведена, за която функцията W представя нейна потенциална енергия. За да изследваме устойчивостта, ще използваме теоремата на Раус [3]: ако функцията W има изолиран минимум за дадени стойности $\rho_{10}, \ldots, \rho_{n0}$ на импулсите, то стационарното движение е устойчиво по отношение на позиционните координати на всички обобщени скорости поне за смущения, които не изменят константите $\rho_{10}, \ldots, \rho_{n0}$.

Функцията W е образувана от всички членове на R, които не съдържат позиционни скорости, и освен от позиционните координати зависи и от импулсите ρ_i на цикличните координати. Нейният вид може да се получи, ако в израза за R приравним на нула всички позиционни скорости. Тъй като позиционните координати определят положенията на струните и осите на симетрия на телата относно базиса $\zeta = (\zeta_1, \zeta_2, \zeta_3)$, трябва да положим $\omega_i = \dot\Psi \mathbf{e}_3^{(0)} + \dot\phi_i \mathbf{e}_i$ $(i = 1, \dots, n)$ в израза за функцията на Раус. По такъв начин получаваме

$$W = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{J_3^{(i)}}\rho_i^2 + \sum_{i=1}^{n} A_i \mathbf{e}_3^{(0)} \cdot \mathbf{e}_i - \frac{\dot\Psi^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon_{ij}(\mathbf{e}_3^{(0)} \times \mathbf{e}_i)\cdot(\mathbf{e}_3^{(0)} \times \mathbf{e}_j).$$

Нека

$$W_0 = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{J_3^{(i)}}\rho_{i0}^2 + \sum_{i=1}^{n} A_i \mathbf{e}_3^{(0)} \cdot \eta_i - \frac{\dot\Psi^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon_{ij}(\mathbf{e}_3^{(0)} \times \eta_i)\cdot(\mathbf{e}_3^{(0)} \times \eta_j)$$

е стойността на приведената потенциална енергия W за дадено стационарно движение. За да определим условията, при които W има изолиран минимум, ще изследваме разликата $W - W_0$ в околност на стационарната точка на приведената система, определена от векторите $\eta_1, \dots, \eta_n$.

Смутеното положение $\mathbf{e}_i$ на вектора η_i ще представим във вида $\mathbf{e}_i = \eta_i + \xi_i$, където ξ_i е вектор с достатъчно малка дължина. Поради равенствата $|\eta_i| = |\mathbf{e}_i| = 1$ векторът ξ_i е подчинен на условието

$$\xi_i^2 = -2\xi_i \cdot \eta_i. \tag{8}$$

За да изключим варирането на цикличната координата $\dot\Psi$, можем да изберем базиса $\zeta = (\zeta_1, \zeta_2, \zeta_3)^T, \zeta_3 = \mathbf{e}_3^{(0)}$ по следния начин: Нека например векторът η_{j_1} не е вертикален и $\zeta_3 = \mathbf{e}_3^{(0)}, \zeta_1 = \mathbf{e}_3^{(0)} \times \eta_{j_1}, \zeta_2 = \zeta_3 \times \zeta_1$. Тогава допускането, че ъгловата скорост $\dot\Psi$ не се смущава, означава, че $\eta_{j_1} = \mathbf{e}_{j_1}$, т. е. смущението на вектора η_{j_1} е $\xi_{j_1} = 0$.

Изчисляваме разликата

$$W - W_0 = \sum_{i=1}^{n} A_i \mathbf{e}_3^{(0)} \cdot (e_i - \eta_i) - \frac{\dot\Psi^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon_{ij}[\mathbf{e}_3^{(0)} \times (\eta_i + \xi_i)]\cdot[\mathbf{e}_3^{(0)} \times (\eta_j + \xi_j)]$$

$$+\frac{\dot\Psi^2}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon_{ij}(\mathbf{e}_3^{(0)} \times \eta_i)\cdot(\mathbf{e}_3^{(0)} \times \eta_j).$$

Като използваме равенствата (3), (5) и (8), получаваме

$$W - W_0 = \frac{\dot\Psi^2}{2}\sum_{i=1}^{n} g_{ij}(\xi_i \cdot \mathbf{e}_3^{(0)})(\xi_j \cdot \mathbf{e}_3^{(0)}) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}\xi_i \cdot \xi_j.$$

По този начин $W - W_0$ е представена като сума от две квадратични форми. Матрицата $(\varepsilon_{ij})_{i,j=1}^n$ участва в израза за кинетичната енергия, така че първата форма е положително дефинитна. Както беше отбелязано, в нетривиалния случай (когато не всички оси на симетрия и струни са вертикални) матрицата $(v_{ij})_{i,j=1}^n$ е особена. Тъй като $(-v_{i_s i_p})_{s,p=1}^r$ е базисна матрица на матрицата $-V$, то $W - W_0$ се представя във вида

$$W - W_0 = \frac{1}{2}\sum_{i=1}^{n} \varepsilon_{ij}(\xi_i \cdot \mathbf{e}_3^{(0)})(\xi_j \cdot \mathbf{e}_3^{(0)}) \tag{9}$$

$$-\frac{1}{2}\sum_{s=1}^{r}\sum_{p=1}^{r} v_{i_s i_p}(\xi_{i_s}+\sum_{m=1}^{n-r} r_{ms}\xi_{j_m})\cdot(\xi_{i_p}+\sum_{i=1}^{n-r} r_{ip}\xi_{j_i}).$$

Като предполагаме, че матриците $(\varepsilon_{ij})_{i,j=1}^{n}$ и $(-v_{i_s i_p})_{s,p=1}^{r}$ са положително дефинитни, виждаме, че $W-W_0 \geqq 0$ и че $W-W_0=0$ за $\xi_i=0$, $i=1,2,\ldots,n$. Следователно важно е да се намерят случаите, когато уравнението $W-W_0=0$ има и ненулеви решения. Очевидно в този случай уравнението $W-W_0=0$ е в сила за смущенията ξ_i, удовлетворяващи уравненията

$$\xi_i\cdot \mathbf{e}_3^{(0)}=0 \quad (i=1,2,\ldots,n),$$
$$\xi_i^2+2\xi_i\cdot\eta_i=0 \quad (i=1,2,\ldots,n),$$
$$\xi_{i_s}+\sum_{m=1}^{n-r} r_{ms}\xi_{j_m}=0 \quad (s=1,2,\ldots,r).$$

За дадено i първото равенство изразява, че смущението на вектора η_i е перпендикулярно на $\mathbf{e}_3^{(0)}$, а второто — че дължината на смутения вектор $\mathbf{e}_i=\eta_i+\xi_i$ е равна на единица. Оттук следва, че смутеният вектор $\mathbf{e}_i$ се получава от вектора η_i чрез въртене около оста $\mathbf{e}_3^{(0)}$ на някакъв малък ъгъл χ_i. Тогава ξ_i може да се запише във вида

$$(10)\quad \xi_i=\mathbf{e}_i-\eta_i=(1-\cos\chi_i)\mathbf{e}_3^{(0)}\times(\mathbf{e}_3^{(0)}\times\eta_i)+\sin\chi_i\mathbf{e}_3^{(0)}\times\eta_i \quad (i=1,2,\ldots,n).$$

Заместваме в третото равенство

$$(11)\quad -\cos\chi_{i_s}\mathbf{e}_3^{(0)}\times(\mathbf{e}_3^{(0)}\times\eta_{i_s})+\sin\chi_{i_s}\mathbf{e}_3^{(0)}\times\eta_{i_s}$$
$$+\sum_{m=1}^{n-r} r_{ms}\{-\cos\chi_{j_m}\mathbf{e}_3^{(0)}\times(\mathbf{e}_3^{(0)}\times\eta_{j_m})+\sin\chi_{j_m}\mathbf{e}_3^{(0)}\times\eta_{j_m}\}=0 \quad (s=1,\ldots,r),$$

което записваме във вида

$$\sum_{m=1}^{n-r} r_{ms}(\cos\chi_{i_s}-\cos\chi_{j_m})\mathbf{e}_3^{(0)}\times(\mathbf{e}_3^{(0)}\times\eta_{j_m})+\sum_{m=1}^{n-r} r_{ms}(\sin\chi_{i_s}-\sin\chi_{j_m})\mathbf{e}_3^{(0)}\times\eta_{j_m}=0.$$

Очевидно това равенство е изпълнено за $\chi_{i_s}=\chi_{j_m}=\chi$, т. е. за смущения на системата, при които осите на телата се завъртат около оста $\mathbf{e}_3^{(0)}$ на един и същ ъгъл χ. Но тъй като $\xi_{j_1}=0$, то $\chi_{j_1}=0$, тогава $\chi_1=\ldots=\chi_n=0$ и това е нулевото решение. Сега трябва да определим условията, при които за малки смущения системата (11) има единствено нулевото решение.

Изразът във формула (10)

$$-\cos\chi_i\mathbf{e}_3^{(0)}\times(\mathbf{e}_3^{(0)}\times\eta_i)+\sin\chi_i\mathbf{e}_3^{(0)}\times\eta_i=\cos\chi_i\eta_i'+\sin\chi_i\mathbf{e}_3^{(0)}\times\eta_i$$

дефинира вектор, получен от ортогоналната проекция $\eta_i'=-\mathbf{e}_3^{(0)}\times(\mathbf{e}_3^{(0)}\times\eta_i)$ на вектора η_i чрез ротация на ъгъл χ_i около оста $\mathbf{e}_3^{(0)}$. От друга страна, ако векторът ξ_i е представен във вида (10), то проекцията e'_i на смутения вектор $e_i = \eta_i + \xi_i$ се изразява по формулата

$$e'_i = -e_3^{(0)} \times [e_3^{(0)} \times (\eta_i + \xi_i)] = \cos\chi_i \eta'_i + \sin\chi_i e_3^{(0)} \times \eta_i,$$

при това е очевидно, че $|e'_i| = |\eta'_i|$. Следователно равенствата (11) се записват във вида

$$e'_{i_s} = -\sum_{m=1}^{n-r} r_{ms} e'_{j_m} \quad (s = 1, 2, \dots, r).$$

Като сравняваме последните равенства със съществуващата между векторите η_i зависимост (6), заключаваме, че $W = W_0$ за вектори $e_i = \eta_i + \xi_i$, които също са решение на система (6), запазващо дължините на проекциите върху хоризонтална равнина, т. е. $e_i.e_3^{(0)} = \eta_i.e_3^{(0)}$. Следователно, когато стационарното движение е определено от такава съвкупност от вектори $\eta_1, \dots, \eta_n$, че построението на многоъгълниците $A_0^{(s)} A_1^{(s)} \dots A_{n-r}^{(s)} A_0^{(s)}$ е единствено, системата (11) има единствено нулевото решение. В този случай, ако матрицата $\left(-v_{i_s i_p}\right)_{s,p=1}^{r}$ е положително дефинитна, то разликата $W - W_0$ се анулира за даденото стационарно движение и приема положителни стойности в достатъчно малка околност, т. е. това стационарно движение е точка на изолиран минимум на приведената потенциална енергия. Съгласно теоремата на Раус стационарното движение е устойчиво по отношение на позиционните координати и всички обобщени скорости поне за смущения, запазващи началните стойности на импулсите.

Едно достатъчно условие за устойчивост е $\operatorname{rank} V \geqq n - 2$ и базисната подматрица на матрицата $-V$ да бъде положително дефинитна. Ще отбележим, че положителната дефинитност на $(-v_{i_s i_p})_{s,p=1}^{r}$ зависи изключително от геометричните характеристики на системата.

Ако за разглежданото стационарно движение многоъгълниците $A_0^{(s)} A_1^{(s)} \dots A_{n-r}^{(s)} A_0^{(s)}$ $(s = 1, \dots, r)$ нямат единствено построение, то това движение принадлежи към семейство от стационарни движения, които имат еднакви начални стойности на импулсите, но определящите ги вектори $\eta_1, \dots, \eta_n$ са различни. За всяко от тези решения функцията $W - W_0$ се анулира и следователно приведената система няма изолиран минимум. Като имаме предвид равенство (9), можем да заключим, че $W - W_0$ е положително дефинитна функция относно проекциите $e_3^{(0)} \cdot \xi_i$ на смущенията ξ_i върху вертикалната ос и можем да приложим допълнението на Румянцев ([3]) за устойчивост относно част от позиционните координати. Съгласно тази теорема движенията, принадлежащи към това семейство, са устойчиви по отношение на позиционните координати $\gamma_i = e_3^{(0)} \cdot e_i$ $(i = 1, \dots, n)$ и всички обобщени скорости. Броят на позиционните координати в този случай е изследван по-точно в [4].

Пример. Ще изследваме някои от стационарните движения на система от шест симетрични тела, представена на фиг. 1. Съответната ѝ матрица V има следния вид:

$$V = M \begin{pmatrix} x_1 & l_1b_2 & l_1b_3 & l_1b_4 & l_1b_5 & l_1b_6 \\ l_1b_2 & x_2 & 0 & 0 & 0 & 0 \\ l_1b_3 & 0 & x_3 & l_3b_4 & l_3b_5 & l_3b_6 \\ l_1b_4 & 0 & l_3b_4 & x_4 & 0 & 0 \\ l_1b_5 & 0 & l_3b_5 & 0 & x_5 & l_5b_6 \\ l_1b_6 & 0 & l_3b_6 & 0 & l_5b_6 & x_6 \end{pmatrix}.$$

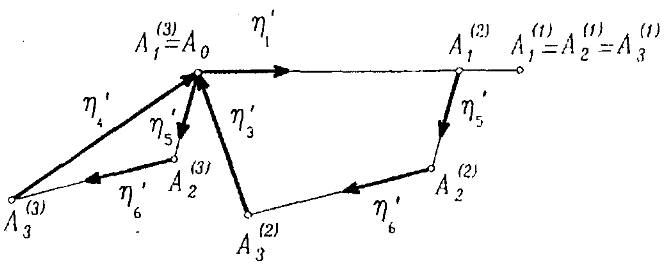
Едно стационарно движение на тази система е получено в [4] за случая $r = 3$ и базисни стълбове V_2, V_3 и V_4. За стационарните стойности на векторите $\mathbf{e}_1, \ldots, \mathbf{e}_6$ имаме

$$\eta_2 = \varepsilon_1 \mathbf{e}_3^{(0)} - \sigma\eta_1,$$

$$\eta_3 = \varepsilon_2 \mathbf{e}_3^{(0)} - \frac{l_1}{l_3}\eta_1 - \frac{l_5}{l_3}\left(\eta_5 + \frac{b_6}{b_5}\eta_6\right),$$

$$\eta_4 = \varepsilon_3 \mathbf{e}_3^{(0)} - \frac{l_3b_5 - l_5b_3}{l_3b_4}\left(\eta_5 + \frac{b_6}{b_5}\eta_6\right),$$

където σ е параметър. Проекциите на многоъгълниците $A_0^{(s)}A_1^{(s)}A_2^{(s)}A_3^{(s)}A_0^{(s)}$ $(s = 1, 2, 3)$ са представени на фиг. 5 и при дадени дължини на страните са единствени.



Фиг. 5

Базисната подматрица на $(-V)$ се представя във вида

$$\Delta = -M \begin{pmatrix} \frac{l_1b_2}{\sigma} & 0 & 0 \\ 0 & l_3b_3 & l_3b_4 \\ 0 & l_3b_4 & \frac{l_3l_5b_4^2}{l_5b_3 - l_3b_5} \end{pmatrix}$$

и е положително дефинитна, ако са изпълнени неравенствата

$$-\frac{l_1b_2}{\sigma} > 0, \quad -l_3b_3 > 0, \quad \frac{l_3^3b_5b_4^2}{l_5b_3 - l_3b_5} > 0.$$

От формула (1) следва, че константите b_i и l_i могат да имат различни знаци само в случая $l_i < 0$, т. е. когато предшестващата точка P_i се намира между барицентъра B_i и другия край Q_i на оста на симетрия. Като имаме предвид това, получаваме, че Δ е положително дефинитна, ако константите σ, l_3, l_5 са отрицателни и $l_5 b_3 < l_3 b_5$. Това означава, че телата (3) и (5) трябва да имат подходящо разпределение на масите.

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SUR LA DISTRIBUTION DE FLUX THERMIQUES PAR FROTTEMENT POUR QUELQUES SYSTEMES TRIBOMECANIQUES

VASSIL DIAMANDIEV

Васил Диамандиев. О РАСПРЕДЕЛЕНИИ ТЕПЛОВЫХ ПОТОКОВ ПРИ ТРЕНИИ ДЛЯ НЕКОТОРЫХ ТРИБОМЕХАНИЧЕСКИХ СИСТЕМАХ

В этой работе рассмотрена общая термическая проблема при трении для конкретных трибомеханических систем. В работе найдены температурные поля этих систем, через которых определено распределение тепловых потоков возникающих при трении.

Vassil Diamandiev. ON THE DISTRIBUTION OF THERMAL FLUXES AT FRICTION TO SOME TRIBOMECHANICAL SYSTEMS

A general thermal problem at friction to some concrete tribomechanical systems is considered in this paper. The temperature fields of these systems are found, on the base of which the distribution of thermal fluxes arisen at the friction is determined.

Il est notoire que quand deux corps se frottent dans un plan quelconque on surgit une chaleur qui se distribue par des flux entre deux corps. Ces flux forment des champs thermiques en deux corps qui déterminent la température du frottement. Ce problème thermique a une grande signification pratique pour les propriétés d'exploitation des machines. Le problème thermique selon la physique mathématique [1] peut se formuler de la manière suivante [2]. Il faut résoudre l'équation de la conductivité thermique

$$\frac{1}{a_i}\frac{\partial \theta_i}{\partial t} = \frac{\partial^2 \theta_i}{\partial x^2} + \frac{\partial^2 \theta_i}{\partial y^2} + \frac{\partial^2 \theta_i}{\partial z^2} \quad (i = 1, 2) \tag{1}$$

par les conditions suivantes initiales et limitées:

$$\theta_i(x, y, z, 0) = T_0 \quad (i = 1, 2), \tag{2}$$

$$\theta_i(x, y, z, t)/\Sigma_i = T_0 \quad (i = 1, 2), \tag{3}$$

$$\theta_1(0, y, z, t) = \theta_2(0, y, z, t), \tag{4}$$

$$\lambda_2 \frac{\partial \theta_2}{\partial x}(0, y, z, t) - \lambda_1 \frac{\partial \theta_1}{\partial x}(0, y, z, t) = Jfpv. \tag{5}$$

Ici $\theta_1(x, y, z, t)$, $\theta_2(x, y, z, t)$ sont les champs thermiques en deux corps qui sont en général non stationnaires, c'est-à-dire dépendent du temps. On suppose que le plan dans lequel on a le frottement est le plan $x = 0$. Les surfaces Σ_1 et Σ_2 limitent les corps. Les coefficients a_i s'expriment par les coefficients thermiques λ_i, c_i, ρ_i par la formule

$$a_i = \frac{\lambda_i}{c_i\, \rho_i} \quad (i = 1, 2). \tag{6}$$

L'équation (2) exprime que la température des corps au moment initial coïncide avec celle de l'environnement. L'équation (4) exprime la température du contact de deux corps. L'équation (5) exprime que la somme des flux thermiques selon la loi de Fourier est égale à la chaleur qui se forme par le frottement.

On applique ce schéma général du problème thermique pour des systèmes concrets tribomécaniques. On trouve les champs thermiques pour deux corps et après on résout le problème pour la distribution de la chaleur à des flux thermiques.

Dans cet article on considère un système tribomécanique suivant: deux vilebrequins se frottent dans leur section frontale et transversale. L'un des vilebrequins est immobile tandis que l'autre se tourne avec une vitesse angulaire constante ω. On veut trouver les champs thermiques de deux corps qui sont en général non stationnaires. On suppose que la section transversale a des dimensions minimales.

Ce système tribomécanique est examiné par Klémentev [3] mais en *un aspect très borné*. Klémentev suppose que la température ne dépend pas du temps. Cette proposition simplifie le problème jusqu'à une équation différentielle ordinaire tandis que le cas général *amène* jusqu'à une équation différentielle partielle.

Ici on *considère* des variantes différentes selon les longueurs des vilebrequins.

1. CAS DE VILEBREQUINS AVEC LONGUEUR INFINIE ET COEFFICIENTS THERMIQUES DIFFÉRENTS

Le système tribomécanique est représenté à Fig. 1. On suppose que les deux vilebrequins ont une longueur infinie et que leurs coefficients thermiques λ_i, c_i, ρ_i

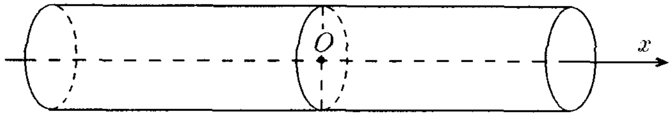

Fig. 1

sont différents. On suppose aussi que les corps ont une isolation thermique, c'est-à-dire il n'y a pas un échange thermique avec l'environnement.

A cause des dimensions minimales de la section transversale la température dépendra seulement de x et t, c'est-à-dire l'équation (1) se réduit aux équations

$$\frac{1}{a_1}\frac{\partial \theta_1}{\partial t} = \frac{\partial^2 \theta_1}{\partial x^2}, \tag{7}$$

$$\frac{1}{a_2}\frac{\partial \theta_2}{\partial t} = \frac{\partial^2 \theta_2}{\partial x^2}, \tag{8}$$

où a_1 et a_2 sont données de (6).

Respectivement les conditions initiales et limitées sont données par les formules

$$\theta_1(x,0) = \theta_2(x,0) = T_0, \tag{9}$$

$$\theta_1(\infty,t) = T_0, \tag{10}$$

$$\theta_2(-\infty,t) = T_0, \tag{11}$$

$$\theta_1(0,t) = \theta_2(0,t). \tag{12}$$

On déduidra en détail la condition (5) pour le cas concret. On suppose que l'énergie mécanique du frottement se transforme entièrement en chaleur. La quantité de la chaleur qui se forme pour un temps dt de la surface dS est égale à la grandeur

$$dQ_{(dS)} = J\,\frac{f\,N}{S}\,dS\,v\,dt, \tag{13}$$

où N est la charge normale entre les vilebrequins. Ici on a

$$\begin{aligned} dS &= r\,dr\,d\varphi, \\ v &= r\omega \end{aligned} \tag{14}$$

à cause de la rotation uniforme ω. On remplace (14) en (13) et on obtient

$$dQ_{(dS)} = J\,f\,p\omega r^2\,dr\,d\varphi\,dt, \tag{15}$$

où $p = \dfrac{N}{S}$ est la pression nominale.

La quantité de la chaleur qui se forme dans toute section transversale se trouve de (15), c'est-à-dire

$$dQ = J\,f\,p\omega \int_0^{r_0} r^2\,dr \int_0^{2\pi} d\varphi\,dt,$$

où r_0 est le rayon du vilebrequin. On trouve après l'intégration

$$dQ = \frac{2\pi}{3} J\, f\, p\omega\, r_0^3\, dt. \tag{16}$$

On note avec dQ_1 et dQ_2 les flux thermiques des deux vilebrequins. Selon la loi de Fourier on a

$$\begin{aligned} dQ_1 &= -\lambda_1 \frac{\partial \theta_1}{\partial x}(0,t)\, S\, dt, \\ dQ_2 &= \lambda_2 \frac{\partial \theta_2}{\partial x}(0,t)\, S\, dt, \end{aligned} \tag{17}$$

où $S = \pi r_0^2$ et λ_1, λ_2 sont les coefficients de la conductivité thermique. Evidemment

$$dQ_1 + dQ_2 = dQ. \tag{18}$$

On remplace (17) et (16) en (18) et on trouve la condition limitée

$$\lambda_2 \frac{\partial \theta_2}{\partial x}(0,t) - \lambda_1 \frac{\partial \theta_1}{\partial x}(0,t) = \frac{2}{3} J\, f\, p\omega r_0. \tag{19}$$

Le problème de la détermination des champs thermiques des deux corps se réduit à la solution des équations (7) et (8) par les conditions (9) — (12) et (19). On utilise la méthode symbolique de Heaviside [4]. Pour ce but on applique l'opération de Laplace vers (7) et on obtient selon (9)

$$\frac{\partial^2 \theta_{L_1}}{\partial x^2} - \frac{s}{a_1}\theta_{L_1} = -\frac{T_0}{a_1}, \tag{20}$$

ou

$$\theta_{L_1}(x,\, s) = \int_0^\infty e^{-st}\, \theta_1(x,\, t)\, dt. \tag{21}$$

La solution générale de (20) a la forme

$$\theta_{L_1}(x,\, s) = \frac{T_0}{s} + B_1(s)\, e^{-\sqrt{\frac{s}{a_1}}\,x} + B_2(s)\, e^{\sqrt{\frac{s}{a_1}}\,x}, \quad x \geqq 0, \tag{22}$$

où $B_1(s)$ et $B_2(s)$ sont des coefficients inconnus. Analogiquement de (8) on trouve

$$\theta_{L_2}(x,\, s) = \frac{T_0}{s} + B_3(s)\, e^{\sqrt{\frac{s}{a_2}}\,x} + B_4(s)\, e^{-\sqrt{\frac{s}{a_2}}\,x}, \quad x \leqq 0. \tag{23}$$

Les conditions limitées (10) et (11) selon (22) et (23) s'expriment par les dépendances

$$B_2(s) = B_4(s) = 0. \tag{24}$$

On applique l'opération inverse de Laplace vers (22) et (23) et on obtient selon (24)

$$\begin{aligned} \theta_1(x,\, t) &= T_0 + L^{-1}\left[B_1(s)\, e^{-\sqrt{\frac{s}{a_1}}\,x}\right], \quad x \geqq 0, \\ \theta_2(x,\, t) &= T_0 + L^{-1}\left[B_3(s)\, e^{\sqrt{\frac{s}{a_2}}\,x}\right], \quad x \leqq 0. \end{aligned} \tag{25}$$

De la condition (12) on trouve

$$B_1(s) = B_3(s). \tag{26}$$

La condition (19) selon (25) et (26) se réduit jusqu'à la dépendance

$$\frac{\lambda_2}{\sqrt{a_2}} L^{-1} \left[\sqrt{s}\, B_1(s)\right] + \frac{\lambda_1}{\sqrt{a_1}} L^{-1} \left[\sqrt{s}\, B_1(s)\right] = \frac{2}{3} J f p \omega\, r_0. \tag{27}$$

On applique l'opération de Laplace vers (27) et on obtient

$$B_1(s) = B_3(s) = \frac{2}{3} J f p \omega r_0 \frac{\sqrt{a_1 a_2}}{s^{\frac{3}{2}} \left(\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2}\right)}. \tag{28}$$

On remplace (28) en (25) et on trouve les formules suivantes pour les champs thermiques:

$$\theta_1(x,\, t) = T_0 + \frac{2}{3} J f p \omega\, r_0 \frac{\sqrt{a_1 a_2}}{\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2}} L^{-1} \left[\frac{e^{-\sqrt{\frac{s}{a_1}}\, x}}{s^{\frac{3}{2}}}\right], \quad x \geqq 0, \tag{29}$$

$$\theta_2(x,\, t) = T_0 + \frac{2}{3} J f p \omega\, r_0 \frac{\sqrt{a_1 a_2}}{\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2}} L^{-1} \left[\frac{e^{\sqrt{\frac{s}{a_1}}\, x}}{s^{\frac{3}{2}}}\right], \quad x \leqq 0. \tag{30}$$

Pour calculer l'opérateur inverse de Laplace on prend une formule connue de la théorie de la conductivité thermique [4]:

$$L^{-1} \left[\frac{1}{\sqrt{s}} e^{-k\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}} e^{-\frac{k^2}{4t}} \quad (k > 0). \tag{31}$$

On peut prouver la formule

$$\int_{k_1}^{k_2} L^{-1} \left[\Phi(k,\, s)\right] dk = L^{-1} \left[\int_{k_1}^{k_2} \Phi(k,\, s)\, dk\right], \tag{32}$$

où $\Phi(k,\, s)$ est une fonction intégrable arbitraire. On applique deux fois la formule (32) vers (31) et après des calculs respectifs on obtient la dépendance

$$L^{-1} \left[\frac{1}{s^{\frac{3}{2}}} e^{-k\sqrt{s}}\right] = \frac{k}{\sqrt{\pi t}} \int_0^k e^{-\frac{u^2}{4t}}\, du + 2\sqrt{\frac{t}{\pi}}\, e^{-\frac{k^2}{4t}} - k, \tag{33}$$

où k est un paramètre positif arbitraire. Selon (33) on obtient par $k = \dfrac{x}{\sqrt{a_1}}$

$$L^{-1} \left[\frac{e^{-\sqrt{\frac{s}{a_1}}\, x}}{s^{\frac{3}{2}}}\right] = \frac{x}{\sqrt{a_1}\sqrt{\pi t}} \int_0^{\frac{x}{\sqrt{a_1}}} e^{-\frac{u^2}{4t}}\, du + 2\sqrt{\frac{t}{\pi}}\, e^{-\frac{x^2}{4 a_1 t}} - \frac{x}{\sqrt{a_1}}, \quad x > 0. \tag{34}$$

L'expression (34) peut être inscrite encore de la manière suivante:

$$(35)\quad L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{a_1}}\,x}}{s^{\frac{3}{2}}}\right]=\frac{1}{\sqrt{a_1}}\left[\frac{2x}{\sqrt{\pi}}\int\limits_0^{\frac{x}{2\sqrt{a_1 t}}} e^{-\alpha^2}\,d\alpha+2\sqrt{\frac{a_1 t}{\pi}}\,e^{-\frac{x^2}{4a_1 t}}-x\right],\quad x>0.$$

Analogiquement on trouve

$$(36)\quad L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{a_2}}\,x}}{s^{\frac{3}{2}}}\right]=\frac{1}{\sqrt{a_2}}\left[-\frac{2x}{\sqrt{\pi}}\int\limits_0^{-\frac{x}{2\sqrt{a_2 t}}} e^{-\alpha^2}\,d\alpha+2\sqrt{\frac{a_2 t}{\pi}}\,e^{-\frac{x^2}{4a_2 t}}+x\right],\quad x<0.$$

On remplace (35) et (36) en (29) et (30) et on obtient les champs thermiques de deux vilebrequins

$$(37)\quad \begin{aligned}\theta_1(x,t)=&T_0+\frac{2}{3}\,J\,f\,p\omega\,r_0\,\frac{\sqrt{a_2}}{\lambda_2\sqrt{a_1}+\lambda_1\sqrt{a_2}}\\ &\times\left[\frac{2x}{\sqrt{\pi}}\int\limits_0^{\frac{x}{2\sqrt{a_1 t}}} e^{-\alpha^2}\,d\alpha+2\sqrt{\frac{a_1 t}{\pi}}\,e^{-\frac{x^2}{4a_1 t}}-x\right],\ x\geqq 0,\\ \theta_2(x,t)=&T_0+\frac{2}{3}\,J\,f\,p\omega\,r_0\,\frac{\sqrt{a_1}}{\lambda_2\sqrt{a_1}+\lambda_1\sqrt{a_2}}\\ &\times\left[-\frac{2x}{\sqrt{\pi}}\int\limits_0^{-\frac{x}{2\sqrt{a_2 t}}} e^{-\alpha^2}\,d\alpha+2\sqrt{\frac{a_2 t}{\pi}}\,e^{-\frac{x^2}{4a_2 t}}+x\right],\ x\leqq 0.\end{aligned}$$

La température du frottement se détermine de (37) par $x=0$, c'est-à-dire on a

$$(38)\quad \theta_1(0,\,t)=\theta_2(0,\,t)=T_0+\frac{4}{3}\,J\,f\,p\,\omega\,r_0\,\frac{\sqrt{a_1 a_2}\,\sqrt{t}}{\sqrt{\pi}\,(\lambda_2\sqrt{a_1}+\lambda_1\sqrt{a_2})}.$$

On voit de (38) que la température du frottement grandit sans restriction avec le temps.

De (17) et (37) on peut déterminer la distribution des flux thermiques. On a

$$(39)\quad \begin{aligned}dQ_1&=\frac{2}{3}\,J\,f\,p\,\omega\,r_0\,\frac{\lambda_1\sqrt{a_2}}{\lambda_2\sqrt{a_1}+\lambda_1\sqrt{a_2}}\,S\,dt,\\ dQ_2&=\frac{2}{3}\,J\,f\,p\,\omega\,r_0\,\frac{\lambda_2\sqrt{a_1}}{\lambda_2\sqrt{a_1}+\lambda_1\sqrt{a_2}}\,S\,dt.\end{aligned}$$

De (39) on trouve le rapport

$$\frac{dQ_2}{dQ_1}=\frac{\lambda_2\sqrt{a_1}}{\lambda_1\sqrt{a_2}},$$

où selon (6) on trouve la dépendance

$$\frac{dQ_2}{dQ_1} = \frac{\sqrt{\lambda_2 c_2 \rho_2}}{\sqrt{\lambda_1 c_1 \rho_1}}. \tag{40}$$

Le rapport (40) est connu dans la littérature comme la formule de Charon [6]. Cette formule n'est pas valide pour des autres systèmes tribologiques.

2. CAS DE VILEBREQUINS AVEC UNE LONGUEUR FINIE ET COEFFICIENTS THERMIQUES DIFFÉRENTS

Le système tribologique est représenté aussi à Fig. 1 mais les deux vilebrequins ont une longueur finie respectivement l_1 et l_2. Pour une simplicité dans les considérations on suppose que les longueurs sont liées avec la dépendance

$$\frac{l_2}{l_1} = \sqrt{\frac{a_2}{a_1}}. \tag{41}$$

Il faut résoudre aussi les équations (7) et (8) mais par les conditions suivantes:

$$\theta_1(x,0) = \theta_2(x,0) = T_0, \tag{42}$$

$$\theta_1(l_1,t) = T_0, \tag{43}$$

$$\theta_2(-l_2,t) = T_0 \tag{44}$$

et la condition (12). La condition (19) est la même.

De nouveau on applique la méthode de Heaviside vers les équations (7) et (8), c'est-à-dire on trouve

$$\theta_{L_1}(x,s) = \frac{T_0}{s} + B_1(s)\, e^{-\sqrt{\frac{s}{a_1}}\,x} + B_2(s)\, e^{\sqrt{\frac{s}{a_1}}\,x}, \quad x \geqq 0, \tag{45}$$

$$\theta_{L_2}(x,s) = \frac{T_0}{s} + B_3(s)\, e^{\sqrt{\frac{s}{a_2}}\,x} + B_4(s)\, e^{-\sqrt{\frac{s}{a_2}}\,x}, \quad x \leqq 0, \tag{46}$$

où θ_{L_i} se détermine de la dépendance

$$\theta_{L_i}(x,s) = \int_0^\infty e^{-st}\, \theta_i(x,\, t)\, dt \quad (i = 1,2). \tag{47}$$

On applique les conditions (43) et (44) vers (45) et (46) et on obtient selon (47)

$$\begin{aligned} B_1(s)\, e^{-\sqrt{\frac{s}{a_1}}\,l_1} + B_2(s)\, e^{\sqrt{\frac{s}{a_1}}\,l_1} &= 0, \\ B_3(s)\, e^{-\sqrt{\frac{s}{a_2}}\,l_2} + B_4(s)\, e^{\sqrt{\frac{s}{a_2}}\,l_2} &= 0. \end{aligned} \tag{48}$$

La condition (12) s'exprime par la dépendance

$$B_1(s) + B_2(s) = B_3(s) + B_4(s). \tag{49}$$

De (45) et (46) on obtient les fonctions originales

(50) $$\theta_1(x,\, t) = T_0 + L^{-1}\left[B_1(s)\, e^{-\sqrt{\frac{s}{a_1}}\, x} + B_2(s)\, e^{\sqrt{\frac{s}{a_1}}\, x}\right], \quad x \geqq 0,$$

(51) $$\theta_2(x,t) = T_0 + L^{-1}\left[B_3(s)\, e^{\sqrt{\frac{s}{a_2}}\, x} + B_4(s)\, e^{-\sqrt{\frac{s}{a_2}}\, x}\right], \quad x \leqq 0.$$

On applique la condition (19) vers (50) et (51) et après quelques calculs on trouve

(52) $$\lambda_2 \sqrt{a_1}\,(B_3 - B_4) + \lambda_1 \sqrt{a_2}\,(B_1 - B_2) = \frac{2}{3}\, \frac{J\, f\, p\, \omega\, r_0\, \sqrt{a_1 a_2}}{s^{\frac{3}{2}}}.$$

Les équations (48), (49) et (52) forment un système de quatre équations relatives à $B_1(s),\ B_2(s),\ B_3(s),\ B_4(s)$. La solution de ce système se détermine par les formules

(53) $$B_1(s) =$$

$$\frac{\frac{2}{3} J f p \omega r_0 \sqrt{a_1 a_2}\,(1 - e^{-2\sqrt{\frac{s}{a_2}} l_2})}{s^{\frac{3}{2}}\left[\lambda_2\sqrt{a_1}(1+e^{-2\sqrt{\frac{s}{a_2}} l_2})(1-e^{-2\sqrt{\frac{s}{a_1}} l_1}) + \lambda_1\sqrt{a_2}(1+e^{-2\sqrt{\frac{s}{a_1}} l_1})(1-e^{-2\sqrt{\frac{s}{a_2}} l_2})\right]},$$

$$B_2(s) =$$

$$\frac{-\frac{2}{3} J f p \omega r_0 \sqrt{a_1 a_2}\, e^{-2\sqrt{\frac{s}{a_1}} l_1}(1 - e^{-2\sqrt{\frac{s}{a_2}} l_2})}{s^{\frac{3}{2}}\left[\lambda_2\sqrt{a_1}(1+e^{-2\sqrt{\frac{s}{a_2}} l_2})(1-e^{-2\sqrt{\frac{s}{a_1}} l_1}) + \lambda_1\sqrt{a_2}(1+e^{-2\sqrt{\frac{s}{a_1}} l_1})(1-e^{-2\sqrt{\frac{s}{a_2}} l_2})\right]},$$

$$B_3(s) =$$

$$\frac{\frac{2}{3} J f p \omega r_0 \sqrt{a_1 a_2}\,(1 - e^{-2\sqrt{\frac{s}{a_1}} l_1})}{s^{\frac{3}{2}}\left[\lambda_2\sqrt{a_1}(1+e^{-2\sqrt{\frac{s}{a_2}} l_2})(1-e^{-2\sqrt{\frac{s}{a_1}} l_1}) + \lambda_1\sqrt{a_2}(1+e^{-2\sqrt{\frac{s}{a_1}} l_1})(1-e^{-2\sqrt{\frac{s}{a_2}} l_2})\right]},$$

$$B_4(s) =$$

$$\frac{-\frac{2}{3} J f p \omega r_0 \sqrt{a_1 a_2}\, e^{-2\sqrt{\frac{s}{a_2}} l_2}(1 - e^{-2\sqrt{\frac{s}{a_1}} l_1})}{s^{\frac{3}{2}}\left[\lambda_2\sqrt{a_1}(1+e^{-2\sqrt{\frac{s}{a_2}} l_2})(1-e^{-2\sqrt{\frac{s}{a_1}} l_1}) + \lambda_1\sqrt{a_2}(1+e^{-2\sqrt{\frac{s}{a_1}} l_1})(1-e^{-2\sqrt{\frac{s}{a_2}} l_2})\right]}.$$

Selon la dépendance (41) de (53) on trouve

$$
(54) \qquad
\begin{aligned}
B_1(s) &= \frac{\frac{2}{3}\, J f p \omega r_0 \sqrt{a_1 a_2}}{s^{\frac{3}{2}} (\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2})(1 + e^{-2\sqrt{\frac{s}{a_1}} l_1})}, \\
B_2(s) &= -\frac{\frac{2}{3}\, J f p \omega r_0 \sqrt{a_1 a_2} e^{-2\sqrt{\frac{s}{a_1}} l_1}}{s^{\frac{3}{2}} (\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2})(1 + e^{-2\sqrt{\frac{s}{a_1}} l_1})}, \\
B_3(s) &= \frac{\frac{2}{3}\, J f p \omega r_0 \sqrt{a_1 a_2}}{s^{\frac{3}{2}} (\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2})(1 + e^{-2\sqrt{\frac{s}{a_2}} l_2})}, \\
B_4(s) &= -\frac{\frac{2}{3}\, J f p \omega r_0 \sqrt{a_1 a_2} e^{-2\sqrt{\frac{s}{a_2}} l_2}}{s^{\frac{3}{2}} (\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2})(1 + e^{-2\sqrt{\frac{s}{a_2}} l_2})}.
\end{aligned}
$$

On remplace les coefficients (54) en (50) et (51) et on obtient

$$
(55) \quad \theta_1(x,t) = T_0 + \frac{\frac{2}{3} J f p \omega r_0 \sqrt{a_1 a_2}}{\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2}} L^{-1} \left[\frac{e^{-\sqrt{\frac{s}{a_1}} x} - e^{-\sqrt{\frac{s}{a_1}} (2l_1 - x)}}{s^{\frac{3}{2}} (1 + e^{-2\sqrt{\frac{s}{a_1}} l_1})} \right], \quad x \geqq 0,
$$

$$
(56) \quad \theta_2(x,\, t) = T_0 + \frac{\frac{2}{3} J f p \omega r_0 \sqrt{a_1 a_2}}{\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2}} L^{-1} \left[\frac{e^{\sqrt{\frac{s}{a_2}} x} - e^{-\sqrt{\frac{s}{a_2}} (l_2 - x)}}{s^{\frac{3}{2}} (1 + e^{-2\sqrt{\frac{s}{a_2}} l_2})} \right], \quad x \leqq 0.
$$

Pour le calcul de l'opérateurs inverses de Laplace on utilise le théorème du retournement [7], c'est-à-dire on a

$$
(57) \quad L^{-1} \left[\frac{e^{-\sqrt{\frac{s}{a_1}} x} - e^{-\sqrt{\frac{s}{a_1}} (2l_1 - x)}}{s^{\frac{3}{2}} (1 + e^{-2\sqrt{\frac{s}{a_1}} l_1})} \right] = \frac{1}{2\pi i} \int\limits_{\substack{\gamma - i\omega \\ \omega \to \infty}}^{\gamma + i\omega} e^{st} \frac{e^{-\sqrt{\frac{s}{a_1}} x} - e^{-\sqrt{\frac{s}{a_1}} (2l_1 - x)}}{s^{\frac{3}{2}} (1 + e^{-2\sqrt{\frac{s}{a_1}} l_1})} ds,
$$

où γ et ω sont des nombres réels et positifs. Pour le calcul de l'intégrale en (57) on construit un contour représentant le droit $x = \gamma$ et la circonférence (C) avec un centre O et un rayon R. On fait une intégration dans un domaine complexe de la fonction sous l'intégrale sur le contour donné. On applique le théorème des résidus [10] sur ce contour et on obtient

$$
(58) \quad
\begin{aligned}
&\frac{1}{2\pi i} \int\limits_{\gamma - i\omega}^{\gamma + i\omega} e^{st} \frac{e^{-\sqrt{\frac{s}{a_1}} x} - e^{-\sqrt{\frac{s}{a_1}} (2l_1 - x)}}{s^{\frac{3}{2}} (1 + e^{-2\sqrt{\frac{s}{a_1}} l_1})} ds \\
&+ \frac{1}{2\pi i} \int\limits_{(C)} e^{st} \frac{e^{-\sqrt{\frac{s}{a_1}} x} - e^{-\sqrt{\frac{s}{a_1}} (2l_1 - x)}}{s^{\frac{3}{2}} (1 + e^{-2\sqrt{\frac{s}{a_1}} l_1})} ds = \mathrm{Res}(0) + \sum_m \mathrm{Res}(s_m).
\end{aligned}
$$

On peut prouver que

$$
(59) \quad \lim_{R \to \infty} \frac{1}{2\pi i} \int\limits_{(C)} e^{st} \frac{e^{-\sqrt{\frac{s}{a_1}} x} - e^{-\sqrt{\frac{s}{a_1}} (2l_1 - x)}}{s^{\frac{3}{2}} (1 + e^{-2\sqrt{\frac{s}{a_1}} l_1})} ds = 0.
$$

Les résidus de l'intégrale en (58) sont pour les grandeurs $s = 0$ et $s = s_m$ qui sont les racines de l'équation

$$1 + e^{-2\sqrt{\frac{s}{a_1}}\, l_1} = 0. \tag{60}$$

Dans le domaine complexe pour (60) on a des racines qui se déterminent par l'expression

$$\sqrt{s_m} = -i\,\frac{\pi\sqrt{a_1}}{2\,l_1}(2m+1) \quad (m = 0, 1, 2, \ldots), \tag{61}$$

ou par les grandeurs

$$s_m = -\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2 \quad (m = 0, 1, 2, \ldots). \tag{61'}$$

Pour calculer Res(0) on représente la fonction sous l'intégrale de la manière suivante:

$$e^{st}\,\frac{2\sqrt{\frac{s}{a_1}}(l_1 - x) + s\,\Phi(s,\, x)}{s\sqrt{s}\,(1 + e^{-2\sqrt{\frac{s}{a_1}}\, l_1})}. \tag{62}$$

De (62) on suivit que

$$\operatorname{Res}(0) = \frac{l_1 - x}{\sqrt{a_1}}. \tag{63}$$

On prend les résidus pour les pôles (61′) et on trouve

$$\operatorname{Res}(s_m) = -e^{-\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2 t}\,\frac{8l_1\,\cos\frac{\pi x}{2l_1}(2m+1)}{\pi^2\sqrt{a_1}\,(2m+1)^2}. \tag{64}$$

Selon (57) — (59), (63) et (64) on obtient la valeur suivante pour l'opérateur inverse de Laplace:

$$\begin{aligned} &L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{a_1}}x} - e^{-\sqrt{\frac{s}{a_1}}(2l_1 - x)}}{s^{\frac{3}{2}}(1 + e^{-2\sqrt{\frac{s}{a_1}}l_1})}\right] \\ &\quad = \frac{l_1 - x}{\sqrt{a_1}} - \frac{8l_1}{\sqrt{a_1}}\sum_{m=0}^{\infty}\frac{\cos\frac{\pi x}{2l_1}(2m+1)}{(2m+1)^2}\,e^{-\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2 t}. \end{aligned} \tag{65}$$

La série infinie en (65) est absolument et uniformément convergente à cause de la convergence de la série $\sum\limits_{m=0}^{\infty}\frac{1}{(2m+1)^2}$.

On remplace (65) en (55) et on trouve

$$\begin{aligned} \theta_1(x,\, t) = {} & T_0 + \frac{\frac{2}{3}J\,f p\omega r_0\sqrt{a_2}}{\lambda_2\sqrt{a_1} + \lambda_1\sqrt{a_2}} \\ & \times\left[l_1 - x - \frac{8l_1}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos\frac{\pi x}{2l_1}(2m+1)}{(2m+1)^2}\,e^{-\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2 t}\right]. \end{aligned} \tag{66}$$

La formule (66) a une structure compliquée. On peut vérifier que $\theta_1(x, t)$ satisfait l'équation (7) et la condition (43). La condition (42) est vérifiée par l'existence de l'identité

$$\sum_{m=0}^{\infty} \frac{\cos \frac{\pi x}{2l_1}(2m+1)}{(2m+1)^2} = \frac{\pi^2}{8}(l_1 - x).$$

Cette identité se constate facilement par les séries de Fourier [9].

Analogiquement pour le champ thermique de vilebrequin immobile on obtient

$$\begin{aligned} \theta_2(x,t) = & T_0 + \frac{\frac{2}{3} J p\omega r_0 \sqrt{a_1}}{\lambda_2\sqrt{a_1} + \lambda_1\sqrt{a_2}} \\ & \times \left[l_2 + x - \frac{8l_2}{\pi^2} \sum_{m=0}^{\infty} \frac{\cos \frac{\pi x}{2l_2}(2m+1)}{(2m+1)^2} e^{-\frac{\pi^2 a_2}{4l_2^2}(2m+1)^2 t} \right]. \end{aligned} \tag{67}$$

La température du frottement pour les deux vilebrequins est donnée de la formule

$$\theta_1(0,t) = \theta_2(0,t) = T_0 + \frac{\frac{2}{3} J f p\omega r_0 l_1 \sqrt{a_2}}{\lambda_2\sqrt{a_1} + \lambda_1\sqrt{a_2}} \left[1 - \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{e^{-\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2 t}}{(2m+1)^2} \right]. \tag{68}$$

L'égalité de $\theta_1(0, t)$ et $\theta_2(0, t)$ suivit de (41). Après un temps infini on obtient de (68) la formule stationnaire pour la température

$$\theta_1 = \theta_2 = T_0 + \frac{\frac{2}{3} J f p\omega r_0 l_1 \sqrt{a_2}}{\lambda_2 \sqrt{a_1} + \lambda_1 \sqrt{a_2}}. \tag{69}$$

La distribution des flux thermiques on trouve de (17), (66), (67). On a

$$\begin{aligned} dQ_1 &= \frac{\frac{2}{3} J f p\omega r_0 \lambda_1 \sqrt{a_2}}{\lambda_2\sqrt{a_1} + \lambda_1\sqrt{a_2}} S\, dt, \\ dQ_2 &= \frac{\frac{2}{3} J f p\omega r_0 \lambda_2 \sqrt{a_1}}{\lambda_2\sqrt{a_1} + \lambda_1\sqrt{a_2}} S\, dt. \end{aligned} \tag{70}$$

Selon (6) de (70) on obtient

$$\frac{dQ_2}{dQ_1} = \frac{\sqrt{\lambda_2 c_2 \rho_2}}{\sqrt{\lambda_1 c_1 \rho_1}}, \tag{71}$$

c'est-à-dire de nouveau la dépendance de Charon [6].

Comme un cas particulier considérons des vilebrequins avec une longueur identique et des coefficients thermiques identiques. Alors le rapport (41) est vrai. De (71) on trouve

$$\frac{dQ_2}{dQ_1} = 1,$$

c'est-à-dire la quantité de la chaleur du frottement se distribue en parties égales en deux vilebrequins.

Le champ thermique a la forme

$$(72)\quad \theta(x,t) = T_0 + \frac{Jfp\omega r_0}{3\lambda}\left[l - x - \frac{8l}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos\frac{\pi x}{2l}(2m+1)}{(2m+1)^2}e^{-\frac{\pi^2 a}{4l^2}(2m+1)^2 t}\right],$$

où l est la longueur du vilebrequin. La température du frottement s'obtient de (72) par $x = 0$, c'est-à-dire on a

$$(73)\qquad \theta(0,t) = T_0 + \frac{Jfp\omega r_0 l}{3\lambda}\left[1 - \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}, e^{-\frac{\pi^2 a}{4l^2}(2m+1)^2 t}\right].$$

Après un temps infini de (73) s'obtient la température stationnaire

$$(74)\qquad \theta_{st} = T_0 + \frac{J\,f\,p\,\omega\,r_0\,l}{3\lambda}.$$

La formule (74) exprime la température maximale du système des corps.

On prend un exemple numérique pour (74). On a les données numériques suivantes: $T_0 = 0$, $N = 100$ kg, $\omega = 100\mathrm{s}^{-1}$, $f = 0.1$, $r_0 = 5$ cm, $l = 100$ cm, $\lambda = 1.08 \times 10^{-4}\frac{\mathrm{cal}}{\mathrm{cm\,s\,grad}}$, $J = \frac{10^{-4}}{4.27}$. On trouve $p = 1.24\,\frac{\mathrm{kg}}{\mathrm{cm}^2}$. De (74) on obtient

$$(75)\qquad \theta_{\max} = 448\,^{\circ}\mathrm{C}.$$

On note que nos considérations supposent que toute la surface nominale participe au frottement. Si on considère un frottement des surfaces rugueux on obtient une température plus petite de (75).

3. CAS DE VILEBREQUINS INFINI ET FINI AVEC COEFFICIENTS THERMIQUES IDENTIQUES

On considère le problème thermique quand le vilebrequin tournant a une longueur finie l mais le vilebrequin immobile a une longueur infinie. Les deux corps ont des coefficients thermiques identiques, c'est-à-dire

$$\lambda_1 = \lambda_2,\quad c_1 = c_2,\quad \rho_1 = \rho_2.$$

Les équations de la conductivité thermique sont

$$(76)\qquad \frac{1}{a}\frac{\partial\theta_1}{\partial t} = \frac{\partial^2\theta_1}{\partial x^2},\quad x \geqq 0,$$

$$(77)\qquad \frac{1}{a}\frac{\partial\theta_2}{\partial t} = \frac{\partial^2\theta_2}{\partial x^2},\quad x \leqq 0.$$

Les conditions initiales et limitées sont

$$(78)\qquad \theta_1(x,0) = \theta_2(x,\,0) = T_0,$$

$$(79)\qquad \theta_1(l,t) = T_0,$$

$$(80)\qquad \theta_2(-\infty,t) = T_0,$$

(81) $$\theta_1(0,t) = \theta_2(0,t),$$

(82) $$\frac{\partial\theta_2}{\partial x}(0,t) - \frac{\partial\theta_1}{\partial x}(0,\,t) = \frac{2}{3}\,\frac{J\,f\,p\,\omega\,r_0}{\lambda}.$$

Selon la méthode de Heaviside on a

(83) $$\theta_{L_1}(x,\,s) = \frac{T_0}{s} + B_1(s)\,e^{-\sqrt{\frac{s}{a}}\,x} + B_2(s)\,e^{\sqrt{\frac{s}{a}}\,x}, \quad x \geqq 0,$$

(84) $$\theta_{L_2}(x,\,s) = \frac{T_0}{s} + B_3(s)\,e^{\sqrt{\frac{s}{a}}\,x} + B_4(s)\,e^{-\sqrt{\frac{s}{a}}\,x}, \quad x \leqq 0,$$

où $\theta_{L_i}(x,\,s)$ se détermine de (47). On applique les conditions (79) — (82) et on obtient

(85) $$\begin{aligned} &B_1(s)\,e^{-\sqrt{\frac{s}{a}}\,l} + B_2(s)\,e^{\sqrt{\frac{s}{a}}\,l} = 0,\\ &B_4(s) = 0,\\ &B_1(s) + B_2(s) = B_3(s),\\ &B_1(s) - B_2(s) + B_3(s) = \frac{2}{3}\,\frac{J\,f\,p\,\omega\,r_o\sqrt{a}}{s^{\frac{3}{2}}\,\lambda}. \end{aligned}$$

Le système (85) a la solution suivante:

(86) $$\begin{aligned} &B_1(s) = \frac{Jfp\omega r_0\sqrt{a}}{3\lambda\,s^{\frac{3}{2}}},\\ &B_2(s) = -\frac{Jfp\omega r_0\sqrt{a}}{3\lambda\,s^{\frac{3}{2}}}\,e^{-2\sqrt{\frac{s}{a}}\,l},\\ &B_3(s) = \frac{Jfp\omega r_0\sqrt{a}}{3\lambda\,s^{\frac{3}{2}}}\,(1 - e^{-2\sqrt{\frac{s}{a}}\,l}),\\ &B_4(s) = 0. \end{aligned}$$

On prend l'opérateur inverse de Laplace de (83) et (84) et selon (86) on obtient

(87) $$\theta_1(x,t) = T_0 + \frac{Jfp\omega r_0\sqrt{a}}{3\lambda}\,L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{a}}\,x}}{s^{\frac{3}{2}}} - \frac{e^{-\sqrt{\frac{s}{a}}\,(2l-x)}}{s^{\frac{3}{2}}}\right], \quad x \geqq 0,$$

(88) $$\theta_2(x,t) = T_0 + \frac{Jfp\omega r_0\sqrt{a}}{3\lambda}\,L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{a}}\,|x|}}{s^{\frac{3}{2}}} - \frac{e^{-\sqrt{\frac{s}{a}}\,(2l+|x|)}}{s^{\frac{3}{2}}}\right], \quad x \leqq 0.$$

On applique la formule (33) à (87) et (88) et après des calculs respectifs on trouve

(89) $$\begin{aligned} \theta_1(x,t) = &T_0 + \frac{2Jfp\omega r_0}{3\lambda}\Big[\frac{x}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{at}}} e^{-\alpha^2}\,d\alpha - \frac{2l-x}{\sqrt{\pi}}\int_0^{\frac{2l-x}{2\sqrt{at}}} e^{-\alpha^2}\,d\alpha\\ &+ \sqrt{\frac{at}{\pi}}\left(e^{-\frac{x^2}{4at}} - e^{-\frac{(2l-x)^2}{4at}}\right) + l - x\Big], \quad x \geqq 0, \end{aligned}$$

$$(90)\quad \theta_2(x,t) = T_0 + \frac{2Jfp\omega r_0}{3\lambda}\Big[-\frac{x}{\sqrt{\pi}}\int_0^{-\frac{x}{2\sqrt{at}}} e^{-\alpha^2}\,d\alpha - \frac{2l-x}{\sqrt{\pi}}\int_0^{\frac{2l-x}{2\sqrt{at}}} e^{-\alpha^2}\,d\alpha$$
$$+\sqrt{\frac{at}{\pi}}\left(e^{-\frac{x^2}{rat}} - e^{-\frac{(2l-x)^2}{rat}}\right) + l\Big], \quad x \leqq 0.$$

La température du frottement s'obtient par $x = 0$, c'est-à-dire on a

$$\theta_1(0,t) = \theta_2(0,t)$$

$$(91)\quad = T_0 + \frac{2\,J\,f\,p\,\omega\,r_0}{3\lambda}\left[l + \sqrt{\frac{at}{\pi}}\,(1 - e^{-\frac{l^2}{at}}) - \frac{2l}{\sqrt{\pi}}\int_0^{\frac{l}{\sqrt{at}}} e^{-\alpha^2}\,d\alpha\right].$$

Après un temps infini ($t \to \infty$) on obtient de (91) la température stationnaire

$$(92)\quad \theta_{st} = T_0 + \frac{2\,J\,f\,p\,\omega\,r_0\,l}{3\lambda}.$$

La formule (92) est pareille à (74) quand les deux corps ont une longueur finie. On trouve la distribution des flux thermiques de (17), (89) et (90). On a

$$(93)\quad \begin{aligned} dQ_1 &= \frac{2Jfp\omega r_0}{3}\left[1 - \frac{1}{\sqrt{\pi}}\int_0^{l/\sqrt{at}} e^{-\alpha^2}\,d\alpha\right] S\,dt, \\ dQ_2 &= \frac{2Jfp\omega r_0}{3\sqrt{\pi}}\int_0^{l/\sqrt{at}} e^{-\alpha^2}\,d\alpha \,.\, S\,dt. \end{aligned}$$

On obtient de (93) le rapport

$$(94)\quad \frac{dQ_2}{dQ_1} = \frac{\int_0^{l/\sqrt{at}} e^{-\alpha^2}\,d\alpha}{\sqrt{\pi} - \int_0^{l/\sqrt{at}} e^{-\alpha^2}\,d\alpha}.$$

La formule (94) montre que le rapport des flux thermiques n'est pas une constante mais qu'il est une fonction du temps. Au moment initial on a

$$\frac{dQ_2}{dQ_1} = 1,$$

c.-à.-d. les flux thermiques sont égales. Après un temps infini on a de (93)

$$(95)\quad \begin{aligned} dQ_1 &\to \frac{2Jfp\omega r_0}{3} S\,dt, \\ dQ_2 &\to 0. \end{aligned}$$

Le résultat (95) montre que par des conditions stationnaires le flux thermique dans le vilebrequin avec une longueur infinie diminue et converge vers zéro. Le flux thermique du vilebrequin avec une longueur finie grandit et obtient toute la chaleur du frottement avec le temps.

4. SYSTÈME TRIBOMÉCANIQUE EN ASPECT THERMODYNAMIQUE

Ici nous considérons les systèmes tribomécaniques étudiés en un aspect thermodynamique. On détermine l'entropie du même système en zone du frottement selon le second principe de la thermodynamique. Il est notoire [11] que la méthode de l'entropie a une grande signification pour la valeur de l'usure d'un système tribomecanique.

La quantité de la chaleur qui se forme pour le temps dt dans la zone du frottement à la système considéré se détermine ainsi

$$dQ = \frac{2\pi}{3} J f p \omega r_0^3 \, dt. \tag{96}$$

Quand les vilebrequins ont une longueur infinie nous avons obtenu pour le champ thermique la formule suivante:

$$\begin{aligned} \theta(x,t) = T_0 + \frac{2}{3} J f p \omega r_0 \frac{\sqrt{a_2}}{\lambda_1 \sqrt{a_2} + \lambda_2 \sqrt{a_1}} \\ \times \left[\frac{2x}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{a_1 t}}} e^{-\alpha^2} \, d\alpha + 2 \sqrt{\frac{a_1 t}{\pi}} e^{-\frac{x^2}{4 a_1 t}} - x \right], \quad x \geqq 0. \end{aligned} \tag{97}$$

La température du frottement se détermine par $x = 0$, c.-à.-d. on a

$$\theta(0,t) = T_0 + \frac{4}{3} J f p \omega r_0 \frac{\sqrt{a_1 a_2 t}}{\sqrt{\pi}[\lambda_1 \sqrt{a_2} + \lambda_2 \sqrt{a_1}]}. \tag{98}$$

Selon le second principe de la thermodynamique on a

$$dU = T dS - p dV, \tag{99}$$

où U est l'énergie intérieure, S — l'entropie, et V — le volume du système. Dans le cas considéré $dV = 0$ et $U = Q$, où Q est la quantité de la chaleur, c.-à.-d. de (99) on obtient

$$dQ = T \, dS.$$

D'ici pour l'entropie on trouve

$$dS = \frac{dQ}{T}. \tag{100}$$

Pour le système donné selon (96), (98) on obtient de (100)

$$dS = \frac{\frac{2\pi}{3} J f p \omega r_0^3 \, dt}{T_0 + \frac{4}{3} J f p \omega r_0 \frac{\sqrt{a_1 a_2 t}}{\sqrt{\pi}(\lambda_1 \sqrt{a_2} + \lambda_2 \sqrt{a_1})}}, \tag{101}$$

ou on trouve pour l'entropie

$$(102)\qquad S = S_0 + \frac{2\pi}{3} J f p \omega r_0^3 \int\limits_0^t \frac{dt}{T_0 + \frac{4}{3} J f p \omega r_0 \frac{\sqrt{a_1 a_2}\sqrt{t}}{\sqrt{\pi}(\lambda_1\sqrt{a_2}+\lambda_2\sqrt{a_1})}},$$

où S_0 est l'entropie initiale. Après l'intégration on obtient

$$(103)\qquad \begin{aligned} S = S_0 + \pi r_0^2 \frac{\lambda_1\sqrt{a_2}+\lambda_2\sqrt{a_1}}{\sqrt{a_1 a_2}}\sqrt{t} - \frac{3\pi r_0(\lambda_1\sqrt{a_2}+\lambda_2\sqrt{a_1})^2 T_0}{4fpJ\omega a_1 a_2} \\ \times \ln\left(1 + \frac{4J f p \omega r_0 \sqrt{a_1 a_2 t}}{3T_0(\lambda_1\sqrt{a_2}+\lambda_2\sqrt{a_1})}\right). \end{aligned}$$

La formule (103) nous donne le développement de l'entropie dans la zone du frottement pour le système tribomécanique donné.

Maintenant on considère une autre variante du même système, c.-à.-d. quand les vilebrequins ont une longueur finie. Pour la température du frottement on a

$$(104)\qquad \theta(t) = T_0 + \frac{\frac{2}{3} J f p \omega r_0 l_1 \sqrt{a_2}}{\lambda_1\sqrt{a_2}+\lambda_2\sqrt{a_1}}\left[1 - \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{e^{-\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2 t}}{(2m+1)^2}\right], \quad t \geqq 0.$$

La grandeur dQ se conserve comme en (96). Pour l'entropie on trouve de (100) et (104)

$$dS = \frac{\frac{2\pi}{3} J f p \omega r_0^3\, dt}{T_0 + \frac{\frac{2}{3} J f p \omega r_0 l_1 \sqrt{a_2}}{\lambda_2\sqrt{a_1}+\lambda_1\sqrt{a_2}}\left(1 - \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{e^{-\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2 t}}{(2m+1)^2}\right)}.$$

D'ici pour le développement de l'entropie on obtient

(105)

$$S(t) = S_0 + \frac{2\pi}{3} J f p \omega r_0^3 \int\limits_0^t \frac{dt}{T_0 + \frac{\frac{2}{3} J f p \omega l_1 \sqrt{a_2}}{\lambda_2\sqrt{a_1}+\lambda_1\sqrt{a_2}}\left(1 - \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{e^{-\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2 t}}{(2m+1)^2}\right)}.$$

On applique le théorème des valeurs moyennes vers (105) et on trouve

$$(106)\quad S(t) = S_0 + \frac{2\pi J f p \omega r_0^3}{3}\frac{t}{T_0 + \frac{\frac{2}{3} J f p \omega r_0 l_1 \sqrt{a_2}}{\lambda_1\sqrt{a_2}+\lambda_2\sqrt{a_1}}\left(1 - \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{e^{-\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2 t^*}}{(2m+1)^2}\right)},$$

où $0 < t^* < t$. Si on considère le processus du développement dans une grande intervalle du temps t^* est une grandeur grande et le facteur exponentiel peut s'éliminer. Ainsi on trouve pour l'entropie

$$S(t) = S_0 + \frac{2\pi J f p \omega r_0^3 t}{3T_0 + \frac{2J f p \omega r_0 l_1 \sqrt{a_2}}{\lambda_1\sqrt{a_2}+\lambda_2\sqrt{a_1}}}.$$

Evidemment elle grandit sans restriction avec le temps.

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