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ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

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ON SOME APPLICATIONS OF THE SINGULARITY METHOD

ZAPRYAN ZAPRYANOV, NIKOLAY MARKOV

Запрян Запрянов, Николай Марков. О НЕКОТОРЫХ ПРИМЕНЕНИЯХ МЕТОДА ОСОБЕННОСТЕЙ

Гидродинамика при малых числах Рейнольдса играет важную роль в исследованиях механики суспензий, колоидной химии и физиологии мембран. В статье методом особенностей исследуются некоторые стационарные вязкие течения при наличие твёрдых или жидких частиц. Течения представляются фундаментальными решениями уравнений Стокса в виде стокслетов, вырожденных квадрополов, стоксонов и других мультиполов. Рассмотрены следующие задачи: 1) течение, порожденное трансляцией и ротацией твёрдой сферической частицы; 2) трансляция сферической капли в неподвижной вязкой частице, обтекаемой линейным градиентным потоком. Обсуждаются построение решений этих задач и основные характеристики такого вида решений.

Zapryan Zapryanov, Nikolay Markov. ON SOME APPLICATIONS OF THE SINGULARITY METHOD

The hydrodynamics of low Reynolds number flows plays an important role in the study of suspension machanics, colloid science and membrane physiology. In the present paper once again (now via the singularity method) some steady viscous flows in the presence of rigid or fluid particles are examined. The flows are represented in terms of fundamental solutions to the governing Stokes equations, including Stokeslets, degenerated quadrupoles, Stokesons and some other multipoles. The problems considered are: 1) flow due to the translation or rotation of a rigid spherical particle; 2) a translating spherical drop in a viscous quiescent fluid; 3) small deformations of a fluid particle in a general linear flow.

The construction of the solutions of these problems and the salient features of such kind of solutions are discussed.

1. INTRODUCTION

An approximate solution to the Navier — Stokes equations can be obtained for the case in which the Reynolds number, or the ratio of inertial to viscous forces, is very small. Then the inertial effects can be neglected, and the action of viscosity is considered to be controlling. We can imagine that the Reynolds number $\mathrm{Re} = \dfrac{LU\rho}{\mu}$ is small either because the fluid is very viscous ($\mu \to \infty$) or because the inertia, or density, is very small ($\rho \to 0$). These flows are frequently called "creeping" flows. This simplification is justified since many multiparticle systems do involve sufficiently slow motions for this assumption to be valid.

The creeping flow and continuity equations are

(1.1) $$\nabla . T = \mu \nabla^2 \vec{v} - \nabla p + \vec{F}^{(c)} \delta(\vec{r}) = 0,$$

(1.2) $$\nabla . \vec{v} = 0.$$

Here $\vec{F}^{(c)}$ denotes a point force applied to the fluid at $\vec{r} = 0$, $\delta(\vec{r})$ is the Dirac delta function, $\vec{v}$ is the velocity, $\vec{p}$ is the pressure and T is the stress tensor. The meaning of the first equation is:

i) For $\vec{r} \neq 0$, $\nabla . T = 0$;

ii) For any volume V that encloses the point $\vec{r} = 0$,

$$\iiint_V \nabla . T d\tau = -\vec{F}^{(c)}.$$

The linearity of the creeping flow equations allows the creation of a class of solution methods that is readily applied to various types of hydrodymanic problems. These methods for solving Stokes equations are based upon fundamental solutions, corresponding to the flow produced by a point force in a fluid space. If the boundary shapes of the problem under consideration are simple, then an analytic solution can be achieved by using the internal distributions of force and force multipole singularities.

In this connection it is useful to write down some of the derivatives of the Ossen — Burger's tensor

(1.3) $$B_{ij} = \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}$$

and degenerate quadrupole $\nabla^2 B_{ij}$:

i) The first derivative of the Ossen — Burger's tensor (the Stokes dipole)

(1.4) $$\nabla B_{ij} = \frac{\partial B_{ij}}{\partial x_k} = B_{ij,k} = \frac{1}{r^3}(-\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ki} x_j) - \frac{3}{r^5} x_i x_j x_k;$$

ii) The second derivative of the Ossen — Burger's tensor

(1.5) $$\frac{\partial^2 B_{ij}}{\partial x_k \partial x_k} = B_{ij,kk} = -\frac{\delta_{ij}}{r^3} + \frac{3\delta_{ij}}{r^3} + \frac{2}{r^3}\delta_{ik}\delta_{jk} - \frac{3}{r^5}(\delta_{ik} x_j + \delta_{jk} x_i) x_k$$

$$-\frac{3}{r^5}(\delta_{ik}x_jx_k + \delta_{jk}x_kx_i + x_ix_j) + \frac{15}{r^5}x_ix_j;$$

iii) The degenerate quadrupole

$$\nabla^2 B_{ij} = \nabla.(\nabla B_{ij}) = B_{ij,ii} = \frac{2\delta_{ij}}{r^3} - \frac{6}{r^5}x_ix_j; \tag{1.6}$$

iv) The derivative of the degenerate quadrupole (the degenerate octupole)

$$\nabla(\nabla^2 B_{ij}) = \nabla^2 B_{ij,k} = -\frac{6}{r^5}(\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ki}x_j) + \frac{30}{r^7}x_ix_jx_k. \tag{1.7}$$

(Here we shall note that in order to obtain (1.3)–(1.7) one has to use the following formulas:

$$\delta_{ij}x_i = x_j, \quad \delta_{ii} = 3, \quad \frac{\partial x_i}{\partial x_j} = \delta_{ij}, \quad x_ix_i = r^2, \quad \frac{\partial r}{\partial x_i} = \frac{x_i}{r}.)$$

For the general particle shape the multipole expansion representation [1–3] requires an infinite number of terms. If the particle shape is simple the multipole expansion representation may contain only a finite number of terms. For the case of a spherical particle, for example, the multipole expansion contains (as we shall see in section 2) only two terms. For the other simple shapes like ellipsoid a truncated expansion in just the lower order singularities is possible, provided that these singularities are distributed over a region. The other example is the solution for very elongated, slender particles where the integral representation can approximately be reduced to a line distribution of point forces along the centre line of the particle.

For interior flows (like the drop inside a flow) the velocity field of the Stokeson

$$v_i = H_{ij}U_j = (2r^2\delta_{ij} - x_ix_j)U_j \tag{1.8}$$

is used. Since the Stokeson is a linear with respect to a constant vector $\vec{U}$ (as we shall see in section 3), it enters into the solution for a translating drop. In view of the fact that the Stokeson is quadratic in r, its gradient is linear in r.

Other interior solutions are roton and stresson which are equal to the symmetric and antisymmetric derivatives of the Stokeson, respectively. Since the roton and stresson correspond to a rigid body rotation and a constant rate of strain field (which are not typical for fluid flows) they are used rarely.

A knowledge of the flow in and around a droplet submerged in an unbounded or bounded fluid is of considerable practicle interest. The submerged of the basic equation subject to the boundary conditions for such type of problems has to yield explicitly the flow fields interior to the droplet and exterior to it, and the general equation of the interface. However, the mathematical treatment of solving simultaneously the flow fields and the equation of the interface is excessively difficult. That is why an iterative procedure was adopted by Taylor [4].

First, the drop is postulated to be spherical and the flow fields are determined using the boundary conditions of continuity of the tangential velocity vectors, vanishing of the normal component of the velocity vectors, and continuity of the tangential components of the stress vectors inside and outside of the spherical drop.

Later, the function describing the deviation of the droplet from sphericity is determined using the relation between the outside and inside values of the normal components of the stress vectors. The newly determined interface may then be used for calculating the flow fields of the second iteration and so on.

2. FLOW DUE TO THE TRANSLATION OR ROTATION OF A RIGID SPHERICAL PARTICLE

The slow translation of a rigid spherical particle of radius a through a quiescent viscous fluid induces a flow which can be found by means of the singularity method. Since this flow produces a net force on the spherical particle in order to construct a solution via internal singularities, we require a Stokeslet $B.\vec{F}^{(c)}$ located at the sphere centre. However, the Stokeslet is most often accompanied by the degenerate quadrupole. Thus, we suggest trying to construct a solution that is a superposition of a Stokeslet and a degenerate quadrupole both located at the centre of the spherical particle, i. e.

(2.1) $$\vec{v}(\vec{r}) = \vec{p}.B(\vec{r}) + \vec{q}\nabla^2 B(\vec{r}).$$

Since each term in this expression satisfies the creeping flow equations (1.1) and (1.2), further we shall try to determine the unknown vectors $\vec{p}$ and $\vec{q}$ from the following boundary conditions:

(2.2.) $$\vec{v} = \vec{U} \quad \text{at} \quad r = a,$$

(2.3) $$\vec{v} \to 0 \quad \text{as} \quad r \to \infty,$$

where $\vec{U}$ is the velocity of the particle.

In fact, since $B(\vec{r})$ and $\nabla^2 B(\vec{r})$ tend to zero as $r \to \infty$, the boundary condition (2.3) is fulfilled automatically. If we succeed to do this then from the uniqueness theorem (cited in section 1) it follows that we have found the solution of the considered problem.

Introducing in (2.1) the explicit forms of the singularities we obtain

(2.4) $$\vec{v}(\vec{r}) = \vec{p}.\left(\frac{I}{r} + \frac{\vec{r}\vec{r}}{r^3}\right) + \vec{q}.\left(-\frac{2I}{r^3} + \frac{6\vec{r}\vec{r}}{r^5}\right).$$

If $\vec{n}$ is the unit normal vector to the spherical particle surface, we have $\vec{r} = a\vec{n}$ at $r = a$ and the boundary condition (2.3) gives

$$\vec{U} = \vec{p}.\left(\frac{I}{a} + \frac{\vec{n}\vec{n}}{a}\right) + \vec{q}.\left(\frac{2I}{a^3} - 6\frac{\vec{n}\vec{n}}{a^3}\right)$$

or

(2.5) $$a^3\vec{U} = a^2\vec{p} + 2\vec{q} + (a^2\vec{p} - 6\vec{q}).\vec{n}\vec{n}.$$

Since the problem considered is linear it is reasonable to assume that the unknown vectors $\vec{p}$ and $\vec{q}$ in the equation (2.1) are expressed linearly via the particle velocity $\vec{U}$, i. e.

(2.6) $$\vec{p} = C_0\vec{U}, \quad \vec{q} = C_2\vec{U},$$

where C_0 and C_2 are unknown constants. With (2.6) the equation (2.5) becomes

$$(-a^3 + C_0 a^2 + 2C_2)\vec{U} + (a^2 C_0 - 6C_2)\vec{U}.\vec{n}\vec{n} = 0$$

or

$$(-a^3 + C_0 a^2 + 2C_2)\vec{U} + (a^2 C_0 - 6C_2)U_n \vec{n} = 0. \tag{2.7}$$

Taking into account that $\vec{U}$ and $\vec{n}$ are independent vectors we obtain the following system for the constants C_0 and C_2:

$$\begin{vmatrix} a^2 C_0 + 2C_2 = a^3 \\ a^2 C_0 - 6C_2 = 0. \end{vmatrix}$$

Hence $C_0 = \dfrac{3}{4}a$ and $C_2 = \dfrac{a^3}{8}$, and (2.1) becomes

$$\begin{aligned} \vec{v}(\vec{r}) &= \frac{3a}{4}\vec{U}.B(\vec{r}) + \frac{a^3}{8}\vec{U}.\nabla^2 B(\vec{r}) \\ &= 6\pi\mu a\vec{U}.\left(1 + \frac{a^2}{6}\nabla^2\right)\frac{B(\vec{r})}{8\pi\mu}. \end{aligned} \tag{2.8}$$

It is easy to show that this expression is identical to the standard result in spherical co-ordinates given in elementary books on fluid mechanics. (See for example [5].)

Therefore, indeed the translating spherical particle in a Stokes flow requires a degenerate quadrupole, in addition to a monopole of strength $6\pi\mu aU$. Of special interest is the fact that we have derived the Stokes law, $\vec{F} = -6\pi\mu a\vec{U}$, for the drag on the spherical particle undergoing a steady translation, without an explicit computation of the surface stress vector $\vec{t_n} = T.\vec{n}|_{r=a}$. Here we have used the statement that the solutions expressed as a multipole expansion yield quantities of interest, such as the hydrodynamic force, in a straight-forward fashion.

Now let us consider the flow due to a rotating spherical particle through an unbounded quiescent viscous fluid. We suppose that the spherical particle rotates with an angular velocity $\vec{\omega}$ and that the radius of the sphere is equal to a. If we take a cartesian co-ordinate system with an origin that coincides with the sphere centre then the boundary conditions of the problem considered are as follows:

$$\vec{v} \to 0 \quad \text{as} \quad r \to \infty, \tag{2.9}$$

$$\vec{v} = \vec{\omega} \times a\vec{e_r} \quad \text{at} \quad r = a, \tag{2.10}$$

where $\vec{e_r}$ is a unit vector in direction $\vec{r}$. The equation (2.10) suggests that the produced flow may be represented merely in terms of a rotlet (couplet) with strength $C_1\vec{\omega}$, located at the centre of the sphere, i. e.

$$\vec{v} = C_1\vec{\omega} \times \frac{\vec{r}}{r^3}. \tag{2.11}$$

With boundary condition (2.10) the velocity field (2.11) gives

$$\vec{\omega} \times a\vec{e_r} = C_1\omega \times \frac{a\vec{e_r}}{a^3},$$

whence $C_1 = a^3$. Therefore

$$\vec{v} = \frac{a^3}{r^3}(\vec{\omega} \times \vec{r}). \tag{2.12}$$

Using (2.12) we can calculate the torque acting on the particle:

$$\vec{M} = \iint_S \vec{r} \times (T.\vec{n}) d\sigma = -8\pi\mu a^3 \vec{\omega}.$$

3. A TRANSLATING SPHERICAL DROP IN A VISCOUS QUIESCENT FLUID

Consider a spherical drop moving slowly with velocity $\vec{U}$ in a viscous quiescent fluid. We suppose that the fluids both outside and inside the drop surface are immiscible, and that the surface tension, σ, at the interface is sufficiently strong to keep the drop approximately spherical against any deforming effect of viscous forces. It is also assumed that the Reynolds number of the motion within the drop is small compared with the unity, like that of the motion outside the drop. The two fluid motions are discribed by the equations (1.1) and (1.2) with different values of the viscosity — μ and $\hat{\mu}$ (here the caret indicates a quantity relating to the internal fluid and its motions).

We choose the origin of the co-ordinate system to be at the instantaneous position of the centre of the drop with radius a. The velocity $\vec{v}$ and the difference $p - p_\infty$ must vanish at infinity, and $\hat{\vec{v}}$ and $\hat{p} - \hat{p}_o$ are finite everywhere within the fluid particle.

The boundary conditions at the interface of the droplet, $r = a$, are as follows:

(i) Vanishing of the normal component of the velocity vectors:

$$(\vec{v} - \vec{U}).\vec{n} = 0, \quad (\hat{\vec{v}} - \vec{U}).\vec{n} = 0;$$

(ii) Continuity of the tangential velocity vectors:

$$\vec{v}.(I - \vec{n}\vec{n}) = \hat{\vec{v}}.(I - \vec{n}\vec{n});$$

(iii) Continuity of the tangential components of the stress vectors:

$$(T.\vec{n}).(I - \vec{n}\vec{n}) = (\hat{T}.\vec{n}).(I - \vec{n}\vec{n}).$$

With the equation

$$T.\vec{n} = -p\vec{n} + 2\mu E.\vec{n}$$

the equation (iii) has the form

$$(E.\vec{n}).(I - \vec{n}\vec{n}) = \lambda(\hat{E}.\vec{n}).(I - \vec{n}\vec{n}), \tag{3.1}$$

where $\lambda = \frac{\hat{\mu}}{\mu}$, the ratio of the drop and the solvent viscosities, respectively. A cursory inspection of the menu of the available singularities results in the following selections: a Stokeslet and a degenerate quadrupole outside the drop, and a Stokeson and a uniform field $\vec{U}$ inside the drop. Then the velocity fields outside and inside the drop can be written, respectively, as

$$\vec{V} = \frac{3a}{4}\vec{U}.(C_0 + C_2 a^2 \nabla^2) B(\vec{r}), \tag{3.2}$$

$$\vec{v} = D_0 \vec{U} + D_2 a^{-2} \vec{U}.H(\vec{r}), \tag{3.3}$$

where the four unknown constants C_0, C_2, D_0 and D_2 are determined from the boundary conditions at the drop interface.

Taking into account formulas (1.3), (1.5), from (3.2) we obtain

$$\begin{aligned} v_i &= C_0 \frac{3a}{4} B_{ij} U_j + C_2 \frac{3}{4} a^3 \nabla^2 B_{ij} U_j \\ &= C_0 \frac{3a}{4} \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) U_j + C_2 \frac{3}{4} a^3 \left(\frac{2\delta_{ij}}{r^3} - \frac{6 x_i x_j}{r^3} \right) U_j . \end{aligned} \tag{3.4}$$

Since at $r = a$ we have $x_i = a n_i$, the equation (3.4) gives

$$v_i = \frac{3}{2} U_i \left(\frac{C_0}{2} + C_2 \right) + \frac{3}{2} n_i n_j U_j \left(\frac{C_0}{2} - 3C_2 \right) \tag{3.5}$$

and thus

$$\begin{aligned} v_i n_i|_{r=a} &= \frac{3}{2} n_i U_i \left(\frac{C_0}{2} + C_2 \right) + \frac{3}{2} n_i n_i n_j U_j \left(\frac{C_0}{2} - 3C_2 \right) \\ &= n_i U_i \left(\frac{3}{2} C_0 - 3C_2 \right). \end{aligned} \tag{3.6}$$

Therefore, from the first kinematic condition (i) we find

$$3C_0 - 6C_2 = 2. \tag{3.7}$$

Reverting to (3.4) we obtain

$$\hat{v}_i = D_0 U_i + D_2 \frac{1}{a^2} (2r^2 \delta_{ij} - x_i x_j) U_j;$$

$$\hat{v}_i|_{r=a} = (D_0 + 2D_2) U_i - n_i n_j U_j D_2,$$

and thus

$$\hat{v}_i n_i|_{r=a} = (D_0 + D_2) n_i U_i . \tag{3.8}$$

It follows from the second kinematic condition (i) that

$$D_0 + D_2 = 1. \tag{3.9}$$

The condition of continuity of the velocity at the surface of the drop (ii) requires to accomplish the following computations:

$$(I - \vec{n}\vec{n}).\vec{v}|_i = (\vec{v} - \vec{n}\vec{n}.\vec{v})_i = \left(\frac{3}{4} C_0 + \frac{3}{2} C_2 \right) (U_i - n_i n_j U_j)$$

and

$$(I - \vec{n}\vec{n}), \vec{v}|_i = (\vec{v} - \vec{n}\vec{n}.\vec{v})_i = (D_0 + 2D_2)(U_i - n_i n_j U_j).$$

Substituting these expressions into (ii) yields

$$3C_0 + 6C_2 = 4D_0 + 8D_2. \tag{3.10}$$

In order to apply the boundary condition (iii) we have need of the calculation of the rate of stress tensors E and $\hat{E}$. From (3.2) and (3.3) it follows that

$$\begin{aligned} e_{ij}|_{r=a} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \frac{3}{8}aC_0(B_{ik,j} + B_{jk,i})U_k \\ + \frac{3}{8}a^3 C_2(\nabla^2 B_{ik,j} + \nabla^2 B_{jk,i})U_k, \end{aligned} \tag{3.11}$$

$$\hat{e}_{ij}|_{r=a} = \frac{1}{2}\left(\frac{\partial \hat{v}_i}{\partial x_j} + \frac{\partial \hat{v}_j}{\partial x_i}\right)|_{r=a} = \frac{D_2}{2a}(3n_i U_j - 3n_j U_i - 2\delta_{ij} n_k U_k). \tag{3.12}$$

Using (1.5) it is easy to obtain the formula

$$(\nabla^2 B_{ik,j} + \nabla^2 B_{jk,i})|_{r=a} = -\frac{12}{a^4}(\delta_{jk} n_i + \delta_{ki} n_j + \delta_{ij} n_k) + \frac{60}{a^4} n_i n_j n_k,$$

and thus

$$\begin{aligned} e_{ij}|_{r=a} = \frac{3}{4}\frac{C_0}{a} n_k U_k(\delta_{ij} - 3n_i n_j) + \frac{9}{2}\frac{C_2}{a} n_k U_k(5n_i n_j + \delta_{ij}) \\ - \frac{9}{2a}C_2(n_i U_j + n_j U_i) \end{aligned}$$

and

$$e_{ij} n_j n_i|_{r=a} = \frac{n_k U_k}{a}\left(9C_2 - \frac{3}{2}C_0\right).$$

Hence

$$\begin{aligned} &(I - \vec{n}\vec{n}).(E.\vec{n})|_i = E.\vec{n} - \vec{n}\vec{n}.(E.\vec{n})|_i \\ &= -\frac{9}{2a}C_2 U_i + \frac{9C_2}{2a} n_i n_k U_k = \frac{9C_2}{2a}(n_i n_k U_k - U_i) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &(I - \vec{n}\vec{n}).(\hat{E}.\vec{n})|_i = \frac{D_2}{2a}(n_i n_k U_k + 3U_i) - \frac{2D_2}{a} n_i n_k U_k \\ &= \frac{3}{2}\frac{D_2}{a}(n_i n_k U_k - U_i). \end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (ii) we find the fourth equation for the constants C_0, C_2, D_0 and D_2:

$$\lambda D_2 + 3C_2 = 0. \tag{3.15}$$

Solving the system (3.7), (3.9), (3.10) and (3.15) for the coefficients C_0, C_2, D_0 and D_2 we find

$$C_0 = \frac{2+3\lambda}{3(1+\lambda)}, \quad C_2 = \frac{\lambda}{6(1+\lambda)},$$

$$D_0 = \frac{3+2\lambda}{2(1+\lambda)}, \quad D_2 = -\frac{1}{2(1+\lambda)}.$$

Consequently the equations (3.2) and (3.3) have the form

$$\vec{v} = 2\pi\mu a \frac{2+3\lambda}{1+\lambda}\left[1 + \frac{\lambda a^2 \nabla^2}{2(2+3\lambda)}\right] \frac{B(\vec{r})}{8\pi\mu}.\vec{U}, \tag{3.16}$$

$$\vec{\tilde{v}} = \frac{3+2\lambda}{2(1+\lambda)}\vec{U} - \frac{1}{2(1+\lambda)}\frac{1}{a^2}\vec{U}.H(\vec{r}). \tag{3.17}$$

The notable feature of (3.16) is that in the limit as $\lambda \to \infty$, C_0 and C_2 assume the values for the rigid particle given previously in (2.8) while in the limit as $\lambda \to 0$ the degenerate quadrupole vanishes and the Stokeslet alone provides the exact solution for a translating bubble.

In order to calculate the force on the fluid particle we have to integrate the surface stress vector over the drop surface:

$$\vec{F} = \iint_S (T.\vec{n})d\sigma = 4\pi\mu a\vec{U}\left(\frac{3\lambda+2}{2(\lambda+1)}\right). \tag{3.18}$$

In the limiting case, $\lambda \to \infty$, this expression for the drag becomes $\vec{F} = 6\pi\mu a\vec{U}$, which is simply the Stokes' law for the drag on a rigid spherical particle. In the limit $\lambda \to 0$ the expression (3.18) becomes

$$\vec{F} = 4\pi\mu a\vec{U}, \tag{3.19}$$

which is the drag on a spherical bubble at Re $\ll 1$.

4. SMALL DEFORMATIONS OF A FLUID PARTICLE IN A GENERAL LINEAR FLOW

In section 3 we solved the problem of a spherical drop of radius a in uniform flow at zero Reynolds number. It is of interest to compute the small deformations of a drop in a general linear flow at zero Reynolds number.

We assume that at large distances from the fluid particle the fluid undergoes a general linear flow

$$\vec{v}^{\infty} = G(E^{\infty} + \Omega^{\infty}).\vec{r}', \quad r' \to \infty, \tag{4.1}$$

where $\vec{r}'$ is a position vector, e_{ij}^{∞} = const and Ω_{ij}^{∞} = const $(i, j = 1, 2, 3)$ are the dimensionless rates of strain and vorticity tensors, respectively, and G represents the magnitude of $\nabla\vec{v}$. We know from section 3 that a drop will deform in almost any viscous flow, other than in uniform (steady) translation through a stationary fluid. With characteristic velocity $V_c = Ga$ the dimensionless form of the equation (4.1) is

$$\vec{v}^{\infty} = (E^{\infty} + \Omega^{\infty}).\vec{r}, \quad r \to \infty. \tag{4.2}$$

We assume that the density of the fluid inside the drop is equal to that of the ambient fluid but the viscosities of the two fluids $\hat{\mu}$ and μ are different (here also the physical parameters pertaining to the interior of the drop will be distinguished from the corresponding exterior parameters by a caret). Further we assume that the surface tension σ is constant on the drop surface and the capillary number $\mathrm{Ca} = \dfrac{\mu V_c}{\sigma}$ is small, i. e. $\mathrm{Ca} \ll 1$.

The magnitude of the drop deformation depends on the capillary number and the ratio of the internal and external viscosity $\lambda \doteq \dfrac{\hat{\mu}}{\mu}$. For $\mathrm{Ca} \ll 1$ very small deviations from a spherical shape are possible. Taking into account this fact we express the drop shape in the form

$$F(\vec{r}, t) \equiv r - [1 + \mathrm{Ca} f(\vec{r})] \equiv 0. \tag{4.3}$$

So we assume that the deviation from sphericity is contained in the function $f(\vec{r})$ and that the magnitude of this deviation is proportional to Ca. It is also assumed that the Reynolds number of the motion within the drop is small compared with unit like that of the motion outside the drop.

In general, as one is solving for the motion inside and outside of the drop one has to determine the shape of the drop. It should be emphasized that the shape of a neutrally buoyant immiscible liquid drop immersed in a continuous liquid undergoing shear is not governed solely by the bulk and interfacial properties of the two phases, but also depends upon the rate of the shear. Despite the linearity of the Stokes equations governing the flow both inside and outside of the drop, in general the determination of the drop surface equation constitutes a non-linear problem, owing to the fact that the unknown shape has to be calculated simultaneously along with the solution of the equations of motion. In consequence of this non-linearity, the droplet shape has not yet been found in its full generality, but rather only for small departures from the spherical form. For small values of capillary number Ca the boundary conditions on the drop surface could be linearized about the boundary conditions for an exactly spherical drop and we shall see that for the above problem an approximate analytic solution could be obtained. The two fluid motions are described by the equations (1.1) and (1.2) with different values of the viscosity — μ and $\hat{\mu}$. We choose the origin of the co-ordinate system to be at the instantaneous position of the centre of the drop.

The boundary conditions in a dimensionless form at the interface of the drop are as follows:

(i) vanishing of the normal components of the velocity vectors, i. e.

$$\vec{v}.\vec{n} = 0, \tag{4.4}$$

$$\vec{\hat{v}}.\vec{n} = 0; \tag{4.5}$$

(ii) continuity of the tangential velocity vectors, i. e.

$$\vec{v}.(I - \vec{n}\vec{n}) = \hat{\vec{v}}.(I - \vec{n}.\vec{n}); \tag{4.6}$$

(iii) continuity of the tangential components of the stress vectors

$$(E.\vec{n}).(I - \vec{n}\vec{n}) = \lambda(\hat{E}.\vec{n}).(I - \vec{n}\vec{n}); \tag{4.7}$$

(iv) relation between the outside and the inside values of the normal components of the stress vectors

$$(T.\vec{n} - \lambda\hat{T}.\vec{n}).\vec{n} = \frac{\nabla_s.\vec{n}}{\text{Ca}}. \tag{4.8}$$

Here ∇_s is acting over the drop surface and $\vec{n}$ is the outer normal. In addition, there is a boundary condition at infinity, namely (4.2), and the requirement that the solution be finite everywhere.

Applying the method of domain pertubations (described in section 1) we first postulate that the drop is spherical with radius "a", and the fields inside and outside it are solved, using only the equations (4.2) and (4.4)–(4.7).

Later the function $f(\vec{r})$ describing the deviation of the droplet from sphericity is determined using the boundary condition (4.8). Therefore, the solution presented herein should be considered as a first approximation of a much more complex problem [6].

If we consider the disturbance flow $\vec{v}\,'$ due to the drop, then the total flow is $\vec{v} = \vec{v}^{\infty} + \vec{v}\,'$ and $\vec{v}\,' \to \infty$ as $r \to \infty$. Inspecting the functional form of the various singularities presented in section 1 we decided to represent the disturbance velocity field outside of the drop in terms of a Stokeslet dipole and a degenerate octupole:

$$\vec{v}' = 2\pi\mu a^3(E^{\infty}.\nabla).(C_1 + C_2 a^2\nabla^2)\frac{B(\vec{r})}{8\pi\mu}, \tag{4.9}$$

where C_1 and C_2 are unknown constants.

Further we shall prove that in order to model correctly the flow inside the drop we have to use the velocity field

$$\hat{\vec{v}} = d_1 E^{\infty}.\vec{r} + d_2\vec{\omega}^{\infty} \times \vec{r} + d_3[5r^2(E^{\infty}.\vec{r}) - 2\vec{r}(\vec{r}.E^{\infty}.\vec{r})], \tag{4.10}$$

where d_1, d_2 and d_3 are specified by the boundary conditions.

Since the Stokeson $H = 2r^2 I - \vec{r}\vec{r}$ is quadratic in r the gradient of H is a third-order tensor that is linear in r. It turned out that the symmetric and antisymmetric derivatives (which are known as roton and stresson) can be used as "building materials" in the construction of the interior flow field for a spherical drop in the linear field (4.2). That is so because roton and stresson simply correspond to a rigid body rotation and a constant rate of strain field. In (4.9) we also use a cubic field which is the less obvious portion of the solution. It should be emphasized that in order to avoid any singularity at the origin of the co-ordinate system we must use growing harmonics inside the drop.

The cubic field can be obtained by the appropriate linear combination of E^{∞} : $\vec{r}\vec{r}\vec{r}$ and $r^2 E^{\infty}.\vec{r}$. If we seek the velocity field $\hat{v}$ inside the drop in the form

$$\hat{\vec{v}} = d_1 E^{\infty}.\vec{r} + d_2\vec{\omega} \times \vec{r} + \hat{\vec{v}}\,',$$

then

$$\vec{\hat{v}}\,' = AE^{\infty} : \vec{r}\vec{r}\vec{r} + Br^2 E^{\infty}.\vec{r},$$

where A and B are unknown constants, but it is easy to show that $A = -\frac{2}{5}B$. From the equation of continuity (4.2) it follows that

$$\begin{aligned}\nabla.\vec{\hat{v}}\,' &= 5Ae_{kl}^{\infty}x_k x_l + 2Be_{kl}^{\infty}x_k x_l\\ &= (5A+2B)e_{kl}^{\infty}x_k x_l = 0,\end{aligned}$$

and thus

$$5A + 2B = 0.$$

Therefore we can write down

$$\vec{\hat{v}}\,' = d_3(5r^2E^{\infty}.\vec{r} - 2E^{\infty} : \vec{r}\vec{r}\vec{r}).$$

Further we shall calculate the constants C_1, C_2, d_1, d_2 and d_3 from the boundary conditions (4.4)–(4.7).

Taking into account formulas (1.3) and (1.5) from (4.9) we obtain

$$\begin{aligned}v_i = e_{ki}^{\infty}x_k + \frac{1}{2}\varepsilon_{kji}\omega_k^{\infty}x_j + \frac{1}{4}C_1 e_{kj}^{\infty}\Big(-\frac{\delta_{ij}x_k}{r^3} + \frac{\delta_{jk}x_i}{r^3} + \frac{\delta_{ik}x_j}{r^3}\\ -\frac{3}{r^5}x_i x_j x_k\Big) + \frac{30}{4}C_2\frac{x_i x_j x_k}{r^7}e_{kj}^{\infty} - \frac{6}{4}\frac{C_2}{r^5}e_{kj}^{\infty}(\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ik}x_j).\end{aligned} \tag{4.11}$$

Since at $r = 1$ we have $x_i = rn_i = n_i$ and (4.11) gives

$$v_i = e_{ki}^{\infty}n_k(1 - 3C_2) + e_{kj}^{\infty}n_i n_j n_k\left(-\frac{3}{4}C_1 + \frac{15}{2}C_2\right) + \frac{1}{2}\varepsilon_{kji}\omega_k^{\infty}. \tag{4.12}$$

Therefore

$$\vec{v}.\vec{n} = v_i n_i = e_{ki}^{\infty}n_i n_k\left(1 + \frac{9}{2}C_2 - \frac{3}{4}C_1\right) + \frac{1}{2}\varepsilon_{kji}\omega_k^{\infty}n_i n_j. \tag{4.13}$$

Similarly from (4.10) one gets

$$\hat{v}_i = d_1 e_{ki}^{\infty}n_k + d_2\varepsilon_{kji}\omega_k^{\infty}n_j + 5d_3 e_{ki}^{\infty}n_k - 2d_3 e_{kj}^{\infty}n_i n_j n, \tag{4.14}$$

$$\vec{\hat{v}}.\vec{n} = \hat{v}_i n_i = e_{ki}^{\infty}n_k n_i(d_1 + 5d_2 - 2d_3) + d_2\varepsilon_{kji}\omega_k^{\infty}n_j n. \tag{4.15}$$

According to (4.4), (4.5), (4.12) and (4.15) we have

$$3C_1 - 18C_2 = 4, \tag{4.16}$$

$$d_1 + 3d_3 = 0. \tag{4.17}$$

From the boundary condition (4.6) we obtain

$$\begin{aligned}&e_{ki}^{\infty}n_k(1 - 3C_2) - e_{kj}^{\infty}n_k n_j n_i(1 - 3C_2) + \frac{1}{2}\varepsilon_{kji}\omega_k^{\infty}n_j\\ &= e_{ki}^{\infty}n_k(d_1 + 5d_3) - e_{kj}^{\infty}.n_k n_j n_i(d_1 + 5d_3) + d_2\varepsilon_{kji}\omega_k^{\infty}n_i.\end{aligned}$$

Therefore $d_2 = \frac{1}{2}$ and

$$1 - 3C_2 = d_1 + 5d_3. \tag{4.18}$$

In order to apply the boundary condition (4.7) we have need of the following quantities:

$$e_{ip}n_p n_i|_{r=1} = e^{\infty}_{kj} n_k n_j \left(\frac{3}{2}C_1 - 18C_2 + 1\right), \tag{4.19}$$

$$\hat{e}_{ip}n_p n_i|_{r=1} = e^{\infty}_{kj} n_k n_j (d_1 + 9d_3). \tag{4.20}$$

After some algebra we find that the left and the right-hand side of (4.7) at $r = 1$ are equal to

$$e_{kj}n_i(\delta_{jk} - n_k n_j)|_{r=1} = \left(1 - \frac{3}{4}C_1 + 12C_2\right)(-e^{\infty}_{kj}n_k n_j n_i + e^{\infty}_{ki}n_k),$$

$$\hat{e}_{kj}n_i(\delta_{jk} - n_k n_j)|_{r=1} = (d_1 + 8d_3)(-e^{\infty}_{kj}n_k n_j n_i + e^{\infty}_{ki}n_k).$$

Substituting these expressions into (4.7) yields

$$1 - \frac{3}{4}C_1 + 12C_2 = \lambda(d_1 + 8d_3). \tag{4.21}$$

In this way we have four equations, namely (4.16)–(4.18) and (4.21) for four unknown constants C_1, C_2, d_1 and d_3.

Solving this system we get

$$C_1 = \frac{2}{3}\frac{5\lambda + 2}{\lambda + 1}, \quad C_2 = \frac{\lambda}{3(\lambda + 1)}, \quad d_1 = -\frac{3}{2(\lambda + 1)}, d_3 = \frac{1}{2(\lambda + 1)}. \tag{4.22}$$

Therefore

$$\vec{v} = E^{\infty}.\vec{r} + \frac{1}{2}\vec{\omega} \times \vec{r} + 2\pi\mu a^3(E^{\infty}.\vec{r})\left(\frac{2}{3}\frac{5\lambda + 2}{\lambda + 1} + \frac{\lambda a^2}{3(\lambda + 1)}\nabla^2\right)\frac{B(\vec{r})}{8\pi\mu}$$

$$\hat{\vec{v}} = -\frac{3}{2(\lambda + 1)}E^{\infty}.\vec{r} + \frac{1}{2}\vec{\omega} \times \vec{r} + \frac{1}{2(\lambda + 1)}[5r^2(E^{\infty}.\vec{r}) - 2E^{\infty} : \vec{r}\vec{r}\vec{r}].$$

The solution just obtained can now be used to calculate from (4.8) the deformation of the drop for small values of the capillary number Ca. If $\vec{e}_r$ is the unit vector in the radial direction of a spherical co-ordinate system, then a first approximation to the unit normal vector $\vec{n}$ for small Ca is just $\vec{e}_r$. According to the definition of $\vec{n}$ in terms of F and the equation $\nabla r = \frac{\vec{r}}{r} = \vec{e}_r$ it follows that

$$\vec{n} \equiv \frac{\nabla E}{|\nabla F|} = \frac{\nabla r - \mathrm{Ca}\nabla f}{|\nabla r - \mathrm{Ca}\nabla f|} = \frac{\vec{e}_r - \mathrm{Ca}\nabla f}{\sqrt{1 + \mathrm{Ca}[(\nabla f)^2 - 2(\vec{e}_r.\nabla f)]}}. \tag{4.23}$$

Next it is important to observe that

$$\vec{n} = \vec{e}_r - \mathrm{Ca}\nabla f + O(\mathrm{Ca}^2), \tag{4.24}$$

$$\nabla.\vec{n} = \nabla.\vec{e}_r - \mathrm{Ca}\nabla^2 f + O(\mathrm{Ca}^2), \tag{4.25}$$

$$\nabla.\vec{e}_r = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = \frac{\delta_{ii}}{r} - \frac{x_i x_i}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}. \tag{4.26}$$

From (4.24)–(4.26) it follows that the surface curvature $\nabla.\vec{n}$ can be expressed as

$$\begin{aligned}\nabla.\vec{n} &= \frac{2}{r} - \mathrm{Ca}\nabla^2 f + O(\mathrm{Ca}^2) = 2(1-\mathrm{Ca}.f) - \mathrm{Ca}\nabla^2 f + O(\mathrm{Ca}^2)\\ &= 2 - \mathrm{Ca}(2f + \nabla^2 f) + O(\mathrm{Ca}^2).\end{aligned} \tag{4.27}$$

We shall note that the surface curvature can be expressed as the sum of the inverse principle radii of curvature, that is

$$\nabla.\vec{n} = \frac{1}{R_1} + \frac{1}{R_2}.$$

With (4.27) the boundary condition (4.8) gives

$$\begin{aligned}&\mathrm{Ca}\{T.[e_r - \mathrm{Ca}\nabla f + O(\mathrm{Ca}^2)] - \lambda\hat{T}.[e_r - \mathrm{Ca}\nabla f + O(\mathrm{Ca}^2)]\}\\ &\quad\times[e_r - \mathrm{Ca}\nabla f + O(\mathrm{Ca}^2)] = 2 - \mathrm{Ca}(2f + \nabla^2 f).\end{aligned} \tag{4.28}$$

It is clear that to order $O(\mathrm{Ca}^2)$ the boundary condition (4.28) takes the form

$$(T.e_r - \lambda\hat{T}.e_r).e_r = \frac{1}{\mathrm{Ca}}[2 - \mathrm{Ca}(2f + \nabla^2 f)],$$

or

$$(E.e_r - \lambda\hat{E}.e_r).e_r = \frac{1}{\mathrm{Ca}}[2 - \mathrm{Ca}(2f + \nabla^2 f)]. \tag{4.29}$$

The shape function $f(\vec{r})$ is a true scalar and linearly related to the variables $\vec{v}$, $\vec{\hat{v}}$ and $(T.\vec{n} - \lambda\hat{T}.\vec{n})$. Therefore $f(\vec{r})$ must be expressible in invariant form as a linear function of E^∞ (or Ω^∞), i. e.

$$f(\vec{r}) = b\vec{r}.E^\infty.\vec{r}, \tag{4.30}$$

where b is an unknown constant.

From (4.27) and (4.30) it follows that the surface curvature $\nabla.\vec{n}$ is equal to

$$\nabla.\vec{n} = 2 + 4(\vec{r}.E^\infty.\vec{r})b\mathrm{Ca} + O(\mathrm{Ca}^2), \tag{4.31}$$

and thus (4.28) becomes

$$(E^\infty.e_r - \lambda\hat{E}^\infty.e_r).e_r = \frac{1}{\mathrm{Ca}}[2 + 4(\vec{r}.E^\infty.\vec{r})b\mathrm{Ca}]. \tag{4.32}$$

In order to apply the boundary condition (4.32) we have to calculate the pressures p and $\hat{p}$ from the Stokes equations inside and outside of the drop. After some algebra one obtains

$$\frac{\partial v_i'}{\partial x_j} = 5d_3 e^\infty_{ij} x_p x_p + 10 d_3 e^\infty_{ki} x_k x_j - 2d_3 e^\infty_{kl} x_k x_l \delta_{ij} - 4d_3 e^\infty_{kj} x_k x_i, \tag{4.33}$$

$$(\nabla^2 \vec{v}')_i = 42 d_3 e^\infty_{ki} x_k, \tag{4.34}$$

(4.35) $$(\nabla \hat{p})_i = \hat{\mu}(\nabla^2 \vec{v})_i = \hat{\mu} 42 d_3 e^{\infty}_{ki} x_k.$$

Integrating the equation (4.35) with respect to x_i we find that

(4.36) $$\hat{p} = \hat{\mu} 21 d_3 e^{\infty}_{ki} x_k x_i + \hat{p}_0 = 21 \vec{\mu} d_3 E^{\infty} : \vec{r}\vec{r} + \hat{\mu}\hat{p}_0,$$

where $\hat{p}_0 = \text{const}$.

Similarly, one obtains the pressure outside the drop, namely

(4.37) $$p = -\frac{3}{2}\mu C_1 E^{\infty} : \vec{r}\vec{r} + \mu p_0,$$

where $p_0 = \text{const}$.

We observe that the pressure field inside and outside of the droplet can be calculated from Stokes equations up to a constant. The constant p_0 involved in the outside field is determined from the known pressure for the droplet. As we shall see, the constant $\hat{p}_0$ involved in the interior pressure field can be determined from the boundary condition (4.8) for the normal components of the stress vectors. Using the equations (4.19), (4.20), (4.32), (4.36) and (4.37) we obtain the following equation for the constant:

$$-p_0 + \hat{p}_0 + 21 d_3 \hat{\mu} E^{\infty} : \vec{r}\vec{r} + \frac{3}{2}\mu C_1 E^{\infty} \vec{r}\vec{r}$$

(4.38) $$+2\mu\left[1 + 12C_2 - \frac{3}{4}C_1 + \frac{9}{4}C_1 - 30C_2 - \lambda(d_1 + 8d_3 + d_3)\right] E^{\infty} : \vec{r}\vec{r}$$

$$= \mu\left[\frac{2}{\mathrm{Ca}} + 4b E^{\infty} : \vec{r}\vec{r}\right].$$

It follows from the equation (4.38) that

$$\hat{p}_0 - p_0 = \frac{2}{\mathrm{Ca}}$$

and

(4.39) $$\frac{3}{4}C_1 + \frac{21}{2}\lambda d_3 + 1 + \frac{3}{2}C_1 - 18C_2 - \lambda(d_1 + 9d_3) = 2b.$$

Finally, substituting the constants C_1, C_2, d_1, d_2 and d_3 from (4.22) in the equation (4.39) we obtain

$$b = \frac{19\lambda + 16}{8(\lambda + 1)},$$

and thus the corresponding drop shape is

(4.40) $$r = 1 + \mathrm{Ca}\frac{19\lambda + 16}{8(\lambda + 1)}(\vec{r}.E^{\infty}.\vec{r}).$$

In order to illustrate the result that we have obtained in (4.40) we shall note that for a simple shear flow ($v_1 = x_2$, $v_2 = 0$, $v_3 = 0$)

$$E = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$r = 1 + \mathrm{Ca}\, x_1 x_2 \frac{19\lambda + 16}{8(\lambda + 1)},$$

whereas for an extensional flow ($v_1 = -\frac{1}{2}x_1$, $v_2 = -\frac{1}{2}x_2$, $v_3 = x_3$)

$$E = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$r = 1 + \mathrm{Ca}\frac{19\lambda + 16}{8(\lambda + 1)}(x_3^2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2)$$

These two flows are sketched in Fig. 1. One can see that although the deformation is small in all cases for the limit $\mathrm{Ca} \ll 1$, the slight difference in shape of the two flows shows that extensional flow is more efficient at stretching deformable particles than the simple shear flow.

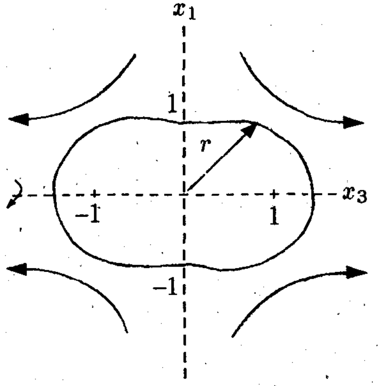

Fig. 1

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ВЪРХУ ДВИЖЕНИЕТО И ДЕФОРМАЦИЯТА НА ДВЕ ФЛУИДНИ ЧАСТИЦИ В ЕЛЕКТРИЧНО ПОЛЕ

ЕКАТЕРИНА ЧЕРВЕНИВАНОВА, ЯВОР ХРИСТОВ, ЗАПРЯН ЗАПРЯНОВ

Екатерина Червениванова, Явор Христов, Запрян Запрянов. О ДВИЖЕНИИ И ДЕФОРМАЦИИ ДВУХ ЖИДКИХ ЧАСТИЦ В ЭЛЕКТРИЧЕСКОМ ПОЛЕ

Рассмотрены медленные стационарные течения, порожденные электрическим полем вне и внутри двух жидких частиц. Течения осесимметричны, а жидкости — нютоновские, несжимаемые и обладающие постоянными, хотя и различными вязкостями. Получены полуаналитические решения как для электрического, так и для гидродинамического поля внутри и вне частиц.

Подсчитаны малые деформации частиц. Одновременно с этим определены порожденные электрическим полем движение и деформация составной жидкой частицы.

Ekaterina Chervenivanova, Yavor Christov, Zapryan Zapryanov. ON THE MOTION AND DEFORMATION OF TWO FLUID PARTICLES IN AN ELECTRIC FIELD

The steady creepind flows, produced by an electric field, in and around two fluid particles are considered. The flows are assumed to be axisymmetric and the fluids are considered to be Newtonian, incompressible, and possessing constant, though different viscosities. Semianalytical solutions for both the electric and the flow fields in and around the fluid particles are obtained. Further, small deformations of the two fluid particles are calculated. At the same time the motion and deformation of a compound droplet produced by an electric field are determined.

Ще разгледаме ососиметричната задача за влиянието върху хидродинамичното поле на електричното поле $E = -\mathrm{grad}\Phi$, приложено успоредно на оста z, във флуид, съдържащ в себе си две капки, поставени по тази ос, или една съставна капка. Ще покажем как може да се намери деформацията на междуфазовите граници. В предложения теоретичен модел е избягнато използването на напреженията, породени от електричното поле, които са нелинейни (квадратични) и се отнасят до второ приближение. Досега такава постановка на електрохидродинамичната задача почти не е използвана, поради което теоретичният модел не е разработен в дълбочина. В статиите на *Sozou* [1] и *Oguz, Sadhal* [2] е намерено второто приближение на движението на флуида, породено от електричното поле, без да е намерено първото! Флуидът започва да се движи поради две причини — вследствие на напреженията, породени от електричното поле върху междуфазовите граници, но преди това в първо приближение се получава тангенциална скорост U_s вследствие възникване и запазване на зарядите върху тези граници, $\mathrm{div}_s \vec{v}_s = k\vec{j}n$ [*Левич* [3]:(95.7), с. 476; (69.7), с. 392]. При така определената повърхнинна тангенциална скорост за хидродинамичната задача се използват уравненията на Скривън за тангенциалните и нормалните напрежения.

1. ФОРМУЛИРАНЕ НА ЗАДАЧАТА

Предполагаме, че флуидът е вискозен, несвиваем и електрически проводим. Използваме бисферична координатна система

$$z = \frac{c.\mathrm{sh}\eta}{\mathrm{ch}\eta - \cos\xi}, \quad r = \frac{c.\sin\xi}{\mathrm{ch}\eta - \cos\xi}, \quad \varphi = \varphi,$$

$$0 \leqq \xi \leqq \pi, \quad -\infty < \eta < \infty, \quad 0 \leqq \varphi < \infty.$$

При две външни капки междуфазовите повърхнини имат уравнения

$$\eta_1 = \mathrm{const} > 0, \quad \eta_2 = \mathrm{const} < 0,$$

а при съставна капка —

$$\eta_1 = \mathrm{const} > 0, \quad \eta_2 = \mathrm{const} > 0,$$

като $\eta_1 > \eta_2$ и η_1 е вътрешната междуфазова граница.

Предполага се, че извън междуфазовите граници електричното поле е електростатично — *Reed, Morrison* [4], и флуидите не са нито перфектни проводници, нито перфектни изолатори, т. е. имат малка електропроводимост ε_i и диелектрични константи k_i, вследствие на което се получава тангенциална повърхностна скорост. Може да се направи сравнение с двойния слой върху твърдата сфера. При тези условия електростатичната задача се отделя от хидродинамичната.

1.1. Определяне на електростатичния потенциал $\Phi(E = -\mathrm{grad}\Phi)$. Във всяка от флуидните области трябва да се удовлетворява уравнението на Лаплас

$$\nabla^2\Phi_i = 0. \tag{1}$$

В безкрайната външна област

$$\Phi_3 \to -E_0 z \quad \text{при} \quad z^2 + z^2 \to \infty. \tag{2}$$

Върху всяка от междуфазовите граници трябва да има непрекъснатост на Φ (или непрекъснатост на тангенциалната компонента на E), непрекъснатост на нормалната компонента на потока на зарядите, т. е. непрекъснатост на нормалната компонента на $\varepsilon' E$, където ε' е електричната проводимост на флуида, т. е. върху η_i за две капки са изпълнени

$$(3)\qquad \begin{aligned} &\eta=\eta_1, \quad \Phi_1=\Phi_3, \quad \frac{\partial \Phi_3}{\partial \eta}=\varepsilon_1 \frac{\partial \Phi_1}{\partial \eta}, \quad \varepsilon_1=\frac{\varepsilon_1'}{\varepsilon_3'}, \\ &\eta=\eta_2, \quad \Phi_2=\Phi_3, \quad \frac{\partial \Phi_3}{\partial \eta}=\varepsilon_2 \frac{\partial \Phi_2}{\partial \eta}, \quad \varepsilon_2=\frac{\varepsilon_2'}{\varepsilon_3'}, \end{aligned}$$

а върху междуфазовите граници за съставна капка:

$$(4)\qquad \begin{aligned} &\eta=\eta_1, \quad \Phi_1=\Phi_2, \quad \frac{\partial \Phi_2}{\partial \eta}=\varepsilon_1 \frac{\partial \Phi_1}{\partial \eta}, \quad \varepsilon_1=\frac{\varepsilon_1'}{\varepsilon_2'}, \\ &\eta=\eta_2, \quad \Phi_2=\Phi_3, \quad \frac{\partial \Phi_2}{\partial \eta}=\varepsilon_2 \frac{\partial \Phi_3}{\partial \eta}, \quad \varepsilon_2=\frac{\varepsilon_3'}{\varepsilon_2'}. \end{aligned}$$

1.2. Хидродинамична задача. Тъй като задачите са ососиметрични, можем да въведем функция на тока Ψ. Както и в *Taylor* [5] предполагаме, че около проводящата капка в проводящия флуид елекричното поле е статично, и пренебрегваме пренасянето от хидродинамичния поток на електростатичните повърхнинни заряди, т. е. предполагаме, че електростатичният потенциал не се влияе от индуцираното движение. Но за разлика от досегашните разработки няма да пренебрегнем напълно повърхнинните заряди, които предизвикват движението на флуида, а ще ги включим в граничните условия.

Във всички области функцията на тока Ψ_i удовлетворява уравнението на Стокс

$$E^4\Psi_i=0, \quad E^2 \equiv \frac{\mathrm{ch}\eta-\beta}{c^2}\left\{\frac{\partial}{\partial\eta}(\mathrm{ch}\eta-\beta)\frac{\partial}{\partial\eta}+(1-\beta^2)\frac{\partial}{\partial\beta}(\mathrm{ch}\eta-\beta)\frac{\partial}{\partial\beta}\right\}.$$

Връзката на функцията на тока Ψ_i с компонентите на скоростта има вида

$$u_\xi=-\frac{(\mathrm{ch}\eta-\beta)^2}{c^2\sin\xi}\frac{\partial\Psi}{\partial\eta}, \quad u_\eta=\frac{(\mathrm{ch}\eta-\beta)^2}{c^2}\frac{\partial\Psi}{\partial\beta}, \quad \beta=\cos\xi.$$

Далеч от капките векторът на скоростта $\vec{u}_3$ трябва да клони към нула:

$$\vec{u}_3 \to 0 \quad \text{при} \quad z^2+z^2 \to 0.$$

Върху междуфазовите граници допускаме, че се появяват дрейфови скорости $\vec{U}_i$ на създадения поток от електричното поле. Тези скорости ще бъдат изчислени при баланса на силите, действащи върху капките, както това се прави при температурно хидродинамичните задачи.

Задачите са линейни. Затова функцията на тока Ψ се търси като сбор от функции на тока Ψ^{u_i}, които са пропорционални съответно на u_i и на Ψ^E, дължаща се на електричното поле, т. е.

$$\Psi=u_1\Psi^{u_1}+u_2\Psi^{u_2}+\Psi^E.$$

Следователно от непрекъснатостта на нормалните и тангенциалните компоненти на скоростта следват граничните условия:

1. *При две капки:*

(5)
$$\begin{aligned} \eta=\eta_1, &\quad \vec{u}_1.\vec{n}_1=\vec{u}_3.\vec{n}_1=\vec{U}^{(1)}.\vec{n}_1, \quad \vec{u}_1.\vec{\tau}_1=\vec{u}_3.\vec{\tau}_1,\\ \eta=\eta_2, &\quad \vec{u}_2.\vec{n}_2=\vec{u}_3.\vec{n}_2=\vec{U}^{(2)}.\vec{n}_2, \quad \vec{u}_2.\vec{\tau}_2=\vec{u}_3.\vec{\tau}_2. \end{aligned}$$

2. *При съставна капка:*

(6)
$$\begin{aligned} &\vec{u}_1.\vec{n}_1=\vec{u}_2.\vec{n}_1=\vec{U}^1.\vec{n}_1,\\ &\eta=\eta_1,\\ &\qquad \vec{u}_1.\vec{\tau}_1=\vec{u}_2.\vec{\tau}_1,\\ &\vec{u}_2.\vec{n}_2=\vec{u}_3.\vec{n}_2=\vec{U}^2.\vec{n}_2,\\ &\eta=\eta_2,\\ &\qquad \vec{u}_2.\vec{\tau}_2=\vec{u}_3.\vec{\tau}_2. \end{aligned}$$

Тук $\vec{n}_i$ и $\vec{\tau_i}$ са съответно единичните нормални и тангенциални вектори към междуфазовите граници.

От уравненията на Скривън за тангенциалните напрежения имаме

(7)
$$T^{(l)}_{nt}-T^{(i)}_{nt}=(\gamma+\varepsilon)\mathrm{grad}_s(\mathrm{div}_s v^{s_i})$$

върху всяка от междуфазовите граници. Тук (l) означава външния флуид, а i — вътрешния, $\vec{v}^{(s)}$ е тангенциалната скорост върху повърхността, породена вследствие запазването на повърхнинните заряди, т. е.

(8)
$$\mathrm{div}_s v^{(s)}=-4\pi\rho=\frac{\mathrm{ch}\eta-\beta}{c}\left[k^{(l)}\frac{\partial\Phi^{(l)}}{\partial\eta}-k^{(i)}\frac{\partial\Phi^{(i)}}{\partial\eta}\right],$$

където k е диелектрична константа.

С помощта на потенциалите $\Phi^{(l)}$ и $\Phi^{(i)}$ можем да намерим $div_s v^{(s)}$, а оттам определяме и тангенциалните напрежения в хидродинамичната задача. Следователно скоростното поле на флуидния поток се определя от потенциала Φ в първо приближение.

След като определим скоростното поле на флуидния поток, можем да намерим и деформацията на междуфазовите граници по метода, разработен в дисертацията на *Е. Черв̀ениванова* [6] и публикациите [7 – 10], но в случая нормалните напрежения се записват, като се използват уравненията на *Скривън* [11], съдържащи допълнителна дясна част $(\gamma+\varepsilon)\mathrm{div_s}\vec{v}^{(s)}$, т. е.

$$T^{(l)}_{nn}-T^{(i)}_{nn}=2H\sigma+(\gamma+\varepsilon)\mathrm{div}_s\vec{v}^{(s)},$$

където $2H=\dfrac{1}{R_1}+\dfrac{1}{R_2}$ е главната кривина.

2. РЕШАВАНЕ НА ЗАДАЧАТА

2.1. Намиране на решението за електрично поле. Решението на уравнението на Лаплас (1) в бисферични координати се дава от формулата

$$(9)\qquad \Phi = (\mathrm{ch}\eta - \beta)^{1/2} E_0 . c . \sum_{n=0}^{\infty} \left[A_n \mathrm{sh}\left(n+\frac{1}{2}\right)\eta + B_n \mathrm{ch}\left(n+\frac{1}{2}\right)\eta \right] P_n(\beta),$$

където $\Phi \to 0$ при $r^2 + z^2 \to \infty$.

За потенциалите в различните флуидни области имаме:

1. *При две капки*

$$\Phi_1 = E_0 c(\mathrm{ch}\eta - \beta)^{1/2} \sum_{n=0}^{\infty} a_n e^{-(n+\frac{1}{2})\eta} P_n(\beta) - E_0 z,$$

където $P_n(\beta)$ са полиномите на Льожандър, $\eta > \eta_1 > 0$;

$$\Phi_2 = E_0 c(\mathrm{ch}\eta - \beta)^{1/2} \sum_{n=0}^{\infty} b_n e^{(n+\frac{1}{2})\eta} P_n(\beta) - E_0 z, \quad \eta < \eta_2 < 0;$$

$$\Phi_3 = E_0 c(\mathrm{ch}\eta - \beta)^{1/2} \sum_{n=0}^{\infty} \left[A_n \mathrm{sh}\left(n+\frac{1}{2}\right)\eta + B_n \mathrm{ch}\left(n+\frac{1}{2}\right)\eta \right] P_n(\beta) - E_0 z,$$

където

$$z = \frac{c\mathrm{ch}\eta}{\mathrm{ch}\eta - \beta} = \mp\sqrt{2}(\mathrm{ch}\eta - \beta)^{1/2} \sum_{n=0}^{\infty} (2n+1) e^{\pm(n+\frac{1}{2})\eta} . P_n(\beta).$$

2. *При съставена капка*

$$\Phi_1 = E_0 c(\mathrm{ch}\eta - \beta)^{1/2} \sum_{n=0}^{\infty} a_n e^{-(n+\frac{1}{2})\eta} - E_0 z, \quad \eta > \eta_1 > 0;$$

$$\Phi_2 = E_0 c(\mathrm{ch}\eta - \beta)^{1/2} \sum_{n=0}^{\infty} \left[A_n \mathrm{sh}\left(n+\frac{1}{2}\right)\eta + B_n \mathrm{ch}\left(n+\frac{1}{2}\right)\eta \right] P_n(\beta) - E_0 z,$$

$$\eta_2 < \eta < \eta_1;$$

$$\Phi_3 = E_0 c(\mathrm{ch}\eta - \beta)^{1/2} \sum_{n=0}^{\infty} b_n e^{(n+\frac{1}{2})\eta} P_n(\beta) - E_0 z, \quad \eta < \eta_2.$$

Навсякъде в разглежданите области решението е ограничено, а във външната област $\Phi_3 \to \mathrm{E}_0 \mathrm{z}$ при $r^2 + z^2 \to \infty$ и този потенциал се появява и вътре в капките поради непрекъснатостта. При поставените гранични условия (3) или (4) намираме неизвестните коефициенти A_n, B_n, a_n, b_n съответно за всеки от потенциалите.

По-конкретно получаваме следната рекурентна система:
а) при две външни капки

$$a_n e^{-(n+\frac{1}{2})\eta_1} = A_n \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_2 + B_n \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_1,$$

$$b_n e^{(n+\frac{1}{2})\eta_2} = A_n \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_2 + B_n \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_2,$$

$$\begin{aligned}
&\left(n+\frac{1}{2}\right)\Big\{A_{n+1}\left[\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1 + \varepsilon_1 \mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1\right] \\
&\quad + B_{n+1}\left[\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1 + \varepsilon_1 \mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1\right]\Big\} \\
&\quad + n\Big\{A_{n-1}\left[\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1 + \varepsilon_1 \mathrm{sh}\left(n-\frac{1}{2}\right)\eta_1\right] \\
&\quad + B_{n-1}\left[\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1 + \varepsilon_1 \mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1\right]\Big\} \\
&-A_n\Big\{(1-\varepsilon_1)\mathrm{sh}\eta_1 \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_1 + (2n+1)\mathrm{ch}\eta_1\left[\mathrm{ch}\left(n+\frac{1}{2}\right)\eta_1 + \varepsilon_1 \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_1\right]\Big\} \\
&-B_n\Big\{(1-\varepsilon_1)\mathrm{sh}\eta_1 \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_1 + (2n+1)\mathrm{ch}\eta_1\left[\mathrm{ch}\left(n+\frac{1}{2}\right)\eta_1 + \varepsilon_1 \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_1\right]\Big\} \\
&\quad = -2\sqrt{2}(1-\varepsilon_1)e^{-(n+\frac{1}{2})\eta_1}[\mathrm{ch}\eta_1 - (2n+1)\mathrm{sh}\eta_1];
\end{aligned}$$

$$\begin{aligned}
&(n+1)\Big\{A_{n+1}\left[\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2 - \varepsilon_2 \mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1\right] \\
&\quad + B_{n+1}\left[\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2 - \varepsilon_2 \mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2\right]\Big\} \\
&\quad + n\Big\{A_{n-1}\left[\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2 - \varepsilon_2 \mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2\right] \\
&\quad + B_{n-1}\left[\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2 - \varepsilon_2 \mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2\right]\Big\} \\
&-A_n\Big\{(1-\varepsilon_2)\mathrm{sh}\eta_2 \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_2 + (2n+1)\mathrm{ch}\eta_1\left[\mathrm{ch}\left(n+\frac{1}{2}\right)\eta_2 - \varepsilon_2 \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_2\right]\Big\} \\
&-B_n\Big\{(1-\varepsilon_2)\mathrm{sh}\eta_2 \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_2 + (2n+1)\mathrm{ch}\eta_2\left[\mathrm{sh}\left(n+\frac{1}{2}\right)\eta_2 - \varepsilon_2 \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_2\right]\Big\} \\
&\quad = -2\sqrt{2}(1-\varepsilon_2)e^{(n+\frac{1}{2})\eta_2}[\mathrm{ch}\eta_2 + (2n+1)\mathrm{sh}\eta_2].
\end{aligned}$$

б) при съставна капка

$$a_n e^{-(n+\frac{1}{2})\eta_1} = A_n \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_1 + B_n \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_1,$$

$$b_n e^{(n+\frac{1}{2})\eta_2} = A_n \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_2 + B_n \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_2,$$

$$(n+1)\Big\{A_{n+1}\left[\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1 + \varepsilon \mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1\right]$$
$$+B_{n+1}\left[\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1 + \varepsilon_1 \mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1\right]\Big\}$$
$$+n\Big\{A_{n-1}\left[\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1 + \varepsilon_1 \mathrm{sh}\left(n-\frac{1}{2}\right)\eta_1\right]$$
$$+B_{n-1}\left[\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_1 + \varepsilon_1 \mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1\right]\Big\}$$
$$-A_n\left\{(1-\varepsilon_1)\mathrm{sh}\eta_1 \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_1 + (2n+1)\mathrm{ch}\eta_1\left[\mathrm{ch}\left(n+\frac{1}{2}\right)\eta_1 + \varepsilon_1 \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_1\right]\right\}$$
$$-B_n\left\{\mathrm{sh}\eta_1(1-\varepsilon_1)\mathrm{ch}\left(n+\frac{1}{2}\right)\eta_1 + \mathrm{ch}\eta_1(2n+1)\left[\mathrm{sh}\left(n+\frac{1}{2}\right)\eta_1 + \varepsilon_1 \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_1\right]\right\}$$
$$= -2\sqrt{2}(1-\varepsilon_1)e^{-(n+\frac{1}{2})\eta_1}[\mathrm{ch}\eta_1 - (2n+1)\mathrm{sh}\eta_1];$$

$$(n+1)\Big\{A_{n+1}\left[\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2 - \varepsilon_2 \mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2\right]$$
$$+B_{n+1}\left[\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2 - \varepsilon_2 \mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2\right]\Big\}$$
$$+n\Big\{A_{n-1}\left[\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2 - \varepsilon_2 \mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2\right]$$
$$+B_{n-1}\left[\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2 - \varepsilon_2 \mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2\right]\Big\}$$
$$-A_n\left\{(1-\varepsilon_2)\mathrm{sh}\eta_2 \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_2 + (2n+1)\mathrm{ch}\eta_2\left[\mathrm{ch}\left(n+\frac{1}{2}\right)\eta_2 - \varepsilon_2 \mathrm{sh}\left(n+\frac{1}{2}\right)\eta_2\right]\right\}$$
$$-B_n\left\{(1-\varepsilon_2)\mathrm{sh}\eta_2 \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_2 + (2n+1)\mathrm{ch}\eta_2\left[\mathrm{sh}\left(n+\frac{1}{2}\right)\eta_2 - \varepsilon_2 \mathrm{ch}\left(n+\frac{1}{2}\right)\eta_2\right]\right\}$$
$$= -2\sqrt{2}(1-\varepsilon_2)e^{-(n+\frac{1}{2})\eta_2}[\mathrm{ch}\eta_2 - (2n+1)\mathrm{sh}\eta_2].$$

При намиране на уравненията в горните две рекурентни системи са използвани следните две формули:

$$\frac{\mathrm{ch}\eta}{(\mathrm{ch}\eta-\beta)^2}-\frac{\mathrm{sh}^2\eta}{(\mathrm{ch}\eta-\beta)^4}=\frac{1}{\sqrt{2}(\mathrm{ch}\eta-\beta)^{1/2}}\sum_{n=0}^{\infty}[\mathrm{ch}\eta+(2n+1)\mathrm{sh}|\eta|]e^{-(n+\frac{1}{2})|\eta|}.P_n(\beta),$$

$$\frac{\mathrm{sh}\eta}{\mathrm{ch}\eta-\beta}=\sqrt{2}(\mathrm{ch}\eta-\beta)^{1/2}\sum_{n=0}^{\infty}(2n+1)e^{\pm(n+\frac{1}{2})\eta}.P_n(\beta).$$

Двете рекурентни системи се решават числено чрез триточкова матрична прогонка.

След като се определят потенциалите от двете страни на междуфазовите граници, може да се намери повърхнинният заряд, който се дължи на съществуването на потенциалния градиент от вътрешната и външната страна на границата между диелектрици, като полученият заряд трябва да се запази (*Taylor* [5]).

Следователно върху междуфазовата граница имаме

$$\frac{\mathrm{ch}\eta-\beta}{c}\left[k^{(e)}\frac{\partial\Phi^{(e)}}{\partial\eta}-k^{(i)}\frac{\partial\Phi^{(i)}}{\partial\eta}\right]=-4\pi\rho.$$

Ще преобразуваме тази формула в удобен за по-нататъшните ни пресмятания вид. Като използваме граничното условие

$$\frac{\partial\Phi^{(i)}}{\partial\eta}=\frac{\varepsilon^{(e)}}{\varepsilon^{(i)}}\frac{\partial\Phi^{(e)}}{\partial\eta},$$

получаваме

$$\begin{aligned}k^{(e)}\frac{\partial\Phi^{(e)}}{\partial\eta}-k^{(i)}\frac{\partial\Phi^{(i)}}{\partial\eta}&=k^{(e)}\frac{\partial\Phi^{(e)}}{\partial\eta}-k^{(i)}\frac{\varepsilon^{(e)}}{\varepsilon^{(i)}}\frac{\partial\Phi^{(e)}}{\partial\eta}\\&=(k^{(e)}-k^{(i)}\frac{\varepsilon^{(e)}}{\varepsilon^{(i)}}\frac{\partial\Phi^{(e)}}{\partial\eta}.\end{aligned}\tag{10}$$

Целта на тези преобразувания е да се създадат условия за разделяне на променливите. По-конкретно получаваме:

а) за две капки

$$\left.\frac{\partial\Phi_3}{\partial\eta}\right|_{\eta=\eta_1}=-\frac{\varepsilon_1.E_0c}{1-\varepsilon_1}(\mathrm{ch}\eta_1-\beta)^{1/2}\sum_{n=0}^{\infty}(2n+1)(A_n+B_n)l^{(n+\frac{1}{2})\eta_1}P_n(\beta),$$

$$\left.\frac{\partial\Phi_3}{\partial\eta}\right|_{\eta=\eta_2}=-\frac{\varepsilon_2E_0c}{1-\varepsilon_2}(\mathrm{ch}\eta_2-\beta)^{1/2}\sum_{n=0}^{\infty}(2n+1)(A_n-B_n)e^{-(n+1)\eta_2}P_n(\beta).$$

б) при съставна капка

$$\left.\frac{\partial\Phi_2}{\partial\eta}\right|_{\eta=\eta_1}=-\frac{\varepsilon_1.E_0c}{1=\varepsilon_1}(\mathrm{ch}\eta_1-\beta)^{1/2}\sum_{n=0}^{\infty}(2n+1)(A_n+B_n)e^{(n+\frac{1}{2})\eta_1}.P_n(\beta),$$

$$\left.\frac{\partial\Phi_2}{\partial\eta}\right|_{\eta=\eta_2} = -\frac{\varepsilon_2.E_0c}{1-\varepsilon_2}(\mathrm{ch}\eta_2-\beta)^{1/2}\sum_{n=0}^{\infty}(2n+1)(A_n-B_n)e^{-(n+\frac{1}{2})\eta_2}.P_n(\beta).$$

За краткост тези формули могат да се запишат така:

$$\frac{\partial\Phi}{\partial\eta} = cWE(\mathrm{ch}\eta-\beta)^{1/2}\sum_{n=0}^{\infty}Q_n(\eta)P_n(\beta),$$

където

$$WE = \frac{\varepsilon}{1-\varepsilon}E_0.$$

2.2. Намиране полето на скоростите на флуидния поток, породен от електричното поле. Функциите на тока имат следния вид:

1. При две капки

$$\Psi_1^{(i)} = (\mathrm{ch}\eta-\beta)^{-3/2}\sum_{n=1}^{\infty}\left[C_n^1 e^{-(n-\frac{1}{2})\eta} + d_n^1 e^{-(n+\frac{3}{2})\eta}\right]v_n(\beta), \quad \eta > \eta_1 > 0;$$

$$\Psi_2^{(i)} = (\mathrm{ch}\eta-\beta)^{-3/2}\sum_{n=1}^{\infty}\left[C_n^{(2)} e^{-(n-\frac{1}{2})\eta} + d_n^2 e^{(n+\frac{3}{2})\eta}\right]v_n(\beta), \quad \eta < \eta_2 < 0;$$

$$\Psi^{(l)} = (\mathrm{ch}\eta-\beta)^{-\frac{3}{2}}$$

$$\times\sum_{n=1}^{\infty}\left[A_n\mathrm{ch}\left(n-\frac{1}{2}\right)\eta+B_n\mathrm{sh}\left(n-\frac{1}{2}\right)\eta+C_n\mathrm{ch}\left(n+\frac{1}{2}\right)\eta+D_n\mathrm{sh}\left(n+\frac{3}{2}\right)\eta\right]v_n(\beta),$$

където $v_n(\beta) = P_{n-1}(\beta) - P_{n+1}(\beta)$, $P_n(\beta)$ са полиномите на Льожандър.

2. При съставна капка

$$\Psi_1^{(i)} = (\mathrm{ch}\eta-\beta)^{-3/2}\sum_{n=1}^{\infty}\left[C_n^{(1)} e^{-(n-\frac{1}{2})\eta} + d_n^{(1)} e^{-(n+\frac{1}{2})\eta}\right]v_n(\beta), \quad \eta > \eta_1 > 0;$$

$$\Psi_2 = (\mathrm{ch}\eta-\beta)^{-3/2}$$

$$\times\sum_{n=1}^{\infty}\left[A_n\mathrm{ch}\left(n-\frac{1}{2}\right)\eta+B_n\mathrm{sh}\left(n-\frac{1}{2}\right)\eta+C_n\mathrm{ch}\left(n+\frac{3}{2}\right)\eta+D_n\mathrm{sh}\left(n+\frac{3}{2}\right)\eta\right]v_n(\beta),$$

$$\eta_1 > \eta > \eta_2 > 0;$$

$$\Psi_3 = (\mathrm{ch}\eta-\beta)^{-3/2}\sum_{n=1}^{\infty}\left[C_n^{(3)} e^{(n-\frac{1}{2})} + d_n^{(3)} e^{(n+\frac{3}{2})\eta}\right]v_n(\beta), \quad \eta < \eta_2 > 0.$$

За по-кратко записване ще използваме означението

$$\Psi = (\mathrm{ch}\eta-\beta)^{-3/2}\sum_{n=1}^{\infty}U_n(\eta)v_n(\beta).$$

Неизвестните константи се определят от граничните условия (5) – (7).

Ще запишем по-подробно условието (7) за тангенциалните напрежения, чрез които се индуцира движението на флуида вследствие на повърхнинния заряд.

От закона за запазване на зарядите имаме

$$\operatorname{div}_s v^{(s)} = -4\pi\rho = \frac{\operatorname{ch}\eta - \beta}{c}\left(k^{(l)} - k^{(i)}\frac{\varepsilon^{(l)}}{\varepsilon^{(i)}}\right)\frac{\partial\Phi^{(l)}}{\partial\eta}.$$

Ако означим $(k^{(e)} - k^{(i)})(\gamma + \varepsilon) = \chi$, условията за напреженията добиват вида

(11) $$T_{nt}^{(e)} - T_{nt}^{(i)} = \chi \operatorname{grad}_s\left(\frac{\operatorname{ch}\eta - \beta}{c}\frac{\partial\Phi^{(e)}}{\partial\eta}\right),$$

(12) $$T_{nn}^{(e)} - T_{nn}^{(i)} = \left[\chi\cdot\frac{\operatorname{ch}\eta - \beta}{c}\cdot\frac{\partial\Phi^{(e)}}{\partial\eta} + \sigma\right]2H.$$

Изразени чрез функцията на тока и скоростта U_i върху повърхнините на междуфазовите граници, напреженията са

$$T_{\xi\eta}\Big|_{\eta=\eta_i} = \mu\left\{\frac{\partial}{\partial\eta}\left[\frac{(\operatorname{ch}\eta - \beta)^3}{c^3\sin\xi}\frac{\partial\Psi}{\partial\eta}\right] + \frac{U_i}{c}\operatorname{ch}\eta\sin\xi\right\}\Big|_{\eta=\eta_i},$$

където U_i е скоростта на междуфазовата граница.

За скока на нормалните напрежения имаме

$$\Delta T_{\xi\eta} = \frac{(\operatorname{ch}\eta - \beta)^{3/2}}{c^3\sin\xi}\left\{\sum_{n=0}^{\infty}\left[\frac{\partial^2 U_n^{(e)}}{\partial\eta^2} - \lambda_i\frac{\partial^2 U_n^{(i)}}{\partial\eta^2}\right]v_n(\beta)\right.$$
$$\left. -\frac{1}{2}(1-\lambda_i)U_i c^2(1-\beta^2)\frac{\partial^2}{\partial\eta^2}(\operatorname{ch}\eta - \beta)\right\}\Big|_{\eta},$$

$$\operatorname{grad}_s\left(\frac{\operatorname{ch}\eta - \beta}{c}\frac{\partial\Phi^{(e)}}{\partial\eta}\right) = -\sin\xi\frac{\operatorname{ch}\eta - \beta}{c^2}\frac{\partial}{\partial\beta}(\operatorname{ch}\eta - \beta)\frac{\partial\Phi^{(l)}}{\partial\eta}$$
$$= -\sin\xi\frac{\operatorname{ch}\eta - \beta}{c^2}\frac{\partial}{\partial\beta}WE\frac{\partial}{\partial\beta}(\operatorname{ch}\eta - \beta)^{3/2}\sum_{n=0}^{\infty}Q_n(\eta)v_n(\beta),$$

където $\lambda_i = \dfrac{\mu^{(i)}}{\mu^{(e)}}$.

След преобразуване получаваме

$$\operatorname{grad}_s\left(\frac{\operatorname{ch}\eta - \beta}{c}\frac{\partial\Phi^{(l)}}{\partial\eta}\right) = WE\frac{(\operatorname{ch}\eta - \beta)^{3/2}}{c\sin\xi}\left[\sum_{n=0}^{\infty}\frac{n(n+1)}{2(2n-1)}Q_{n-1}\right.$$
$$\left. -\frac{n(n+1)}{2n+1}\operatorname{ch}\eta Q_n + \frac{(n+1)(n+2)}{2(2n+1)}Q_{n+1}\right]v_n(\beta).$$

Сега ще вземем предвид, че $\dfrac{1-\beta^2}{(\text{ch}\eta-\beta)^{1/2}}$

$$= \sqrt{2}\sum_{n=0}^{\infty}\frac{n(n+1)}{2n+1}\left[\frac{e^{\pm(n-\frac{1}{2})\eta}}{2n-1}-\frac{e^{\pm(n+\frac{3}{2})\eta}}{2n+3}\right]v_n(\beta)=\sum_{n=0}^{\infty}W_n^{(1)}(\eta).v_n(\beta),$$

$$\frac{\partial^2(1-\beta^2)}{\partial^2\eta^2(\text{ch}\eta-\beta)^2}=\sum_{n=0}^{\infty}W_n^{(3)}(\eta).v_n(\beta),\qquad W_n^{(3)}=\frac{\partial^2 W_n^{(1)}(\eta)}{\partial\eta^2},$$

и за баланса на тангенциалните напрежения върху междуфазовите граници имаме

$$\frac{\partial^2 U_n^{(l)}(\eta)}{\partial\eta^2}-\lambda_i\frac{\partial^2 U_n^{(i)}(\eta)}{\partial\eta^2}=-\frac{1}{2}(1-\lambda_i)U_ic^2W_n^{(3)}(\eta)$$

$$+GWc^2\frac{n(n+1)}{2(2n-1)}Q_{n-1}-\frac{n(n+1)}{2n+1}\text{ch}\eta.Q_n+\frac{(n+1)(n+2)}{2(2n+1)}Q_{n+1}$$

(тук няма β!) при $\eta=\eta_i$.

Това означава, че задачата може да се реши чрез разделяне на променливите и използване на вече решените хидродинамични задачи (напр. вж. Червениванова — канд. дисертация и публикациите, свързани с тази тематика), като се прибави допълнителният член от електричното поле в дясната страна. Задачата е линейна и скоростите на флуидните частици U_i се намират от баланса на силите, действащи върху междуфазовите граници, а функцията на тока се разглежда като сбор от вида

$$\Psi=U_1\Psi^{U_1}+U_2\Psi^{U_2}+\Psi^E,$$

където всяка функция на тока е породена от съответното поле на скоростите.

Така за коефициентите на функциите на така получаваме следната безкрайна система от уравнения:

1. При външни капки

$$A_n\text{ch}\left(n-\frac{1}{2}\right)\eta_1+B_n\text{sh}\left(n-\frac{1}{2}\right)\eta_1+C_n\text{ch}\left(n+\frac{3}{4}\right)\eta_1+D_n\text{sh}\left(n+\frac{3}{2}\right)\eta_1$$

$$=-\frac{1}{2}U_1c^2W_n^{(1)}(\eta_1)$$

$$A_n\text{ch}\left(n-\frac{1}{2}\right)\eta_2+B_n\text{sh}\left(n-\frac{1}{2}\right)\eta_2+C_n\text{ch}\left(n+\frac{3}{2}\right)\eta_2+D_n\text{sh}\left(n+\frac{3}{2}\right)\eta_2$$

$$=-\frac{1}{2}U_2c^2W_n^{(1)}(\eta_2),$$

$$(2n-1)\left\{A_n\left[(2n-1)\text{ch}\left(n-\frac{1}{2}\right).\eta_1+2\lambda_1(2n+1)\text{sh}\left(n-\frac{1}{2}\right)\eta_1\right]\right.$$

$$+B_n\left[(2n-1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_1+2\lambda_1(2n+1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2\right]\Bigg\}$$

$$+(2n+3)\Bigg\{Cn\left[(2n+3)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1+2\lambda_1(2n+1)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1\right]$$

$$+D_n\left[(2n+3)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1+2\lambda_1(2n+1)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1\right]\Bigg\}$$

$$=U_1c^2\left[\frac{1}{2}\lambda_1(2n-1)(2n+3)W_n^{(1)}(\eta_1)-2(1-\lambda_1)W_n^{(3)}(\eta_1)\right]$$

$$+4c^2GW^1\left[\frac{n(n+1)}{2(2n-1)}Q_{n-1}^1(\eta_1)-\frac{n(n+1)}{2n+1}\mathrm{ch}\eta_1Q_n^2(\eta_1)+\frac{(n+1)(n+2)}{2(2n+1)}Q_{n+1}^1(\eta_1)\right],$$

$$(2n-1)\Bigg\{A_n\left[(2n-1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2-2\lambda_2(2n+1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2\right]$$

$$+B_n\left[(2n-1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2-2\lambda_2(2n+1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2\right]\Bigg\}$$

$$+(2n+3)\Bigg\{C_n\left[(2n+3)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2-2\lambda_1(2n+1)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2\right]$$

$$+D_n\left[(2n+3)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2-2\lambda_2(2n+1)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2\right]\Bigg\}$$

$$=U_2c^2\left[\frac{1}{2}\lambda_2(2n-1)(2n+3)W_n^{(1)}(\eta_2)-2(1-\lambda_2)W_n^{(3)}(\eta_2)\right]$$

$$+4c^2GW^2\left[\frac{n(n+1)}{2(2n+1)}Q_{n-1}^2(\eta_2)-\frac{n(n+1)}{2n+1}\mathrm{ch}\eta_2Q_n^{(2)}(\eta_2)+\frac{(n+1)(n+2)}{2(2n+1)}Q_{n+1}^2(\eta_2)\right],$$

$$d_n^1=\left[\frac{1}{2}U_1c^2(2n-1)W_n^{(1)}(\eta_1)-A_n(2n-1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_1-B_n(2n-1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1\right.$$

$$\left.-C_n(2n+3)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1-D_n(2n+3)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1\right]\cdot\frac{1}{4e^{-(n+3/2)\eta_1}},$$

$$C_n^1=-\left[\frac{1}{2}U_1c^2W_n^{(1)}(\eta_1)+d_n^{(1)}e^{-(n+3/2)\eta_1}\right].e^{(n-1/2)\eta_1},$$

$$d_n^2=\left[\frac{1}{2}U_2c^2(2n-1)W_n^{(1)}(\eta_2)+A_n(2n-1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2+B_n(2n-1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2\right.$$

$$\left.+C_n(2n+3)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2+D_n(2n+3)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2\right]\frac{e^{-(n+3/2)\eta_2}}{4},$$

$$C_n^2 = -e^{-(n-\frac{1}{2})\eta_2}\left[\frac{1}{2}U_2c^2W_n^{(1)}(\eta_2) + d_n^2 e^{(n+\frac{3}{2})\eta_2}\right]$$

2. При съставна капка

$$A_n\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1 + B_n\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_1 + C_n\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1 + D_n\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1$$
$$= -\frac{1}{2}U_1c^2W_n^{(1)}(\eta_1),$$

$$C_n^1 = -\left[\frac{1}{2}U_1c^2W_n^{(1)}(\eta_1) + d_n^1e^{-(n+3/2)\eta_1}\right]e^{(n-\frac{1}{2})\eta_1},$$

$$d_n^2 = \left[\frac{1}{2}U_2c^2(2n-1)W_n^{(1)}(\eta_2)+A_n(2n-1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2+B_n(2n-1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2\right.$$
$$\left.+C_n(2n+3)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2 + D_n(2n+3)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2\right]\frac{e^{-(n+\frac{3}{2})\eta_2}}{4},$$

$$C_n^2 = -e^{-(n-\frac{1}{2})\eta_2}\left[\frac{1}{2}U_2c^2W_n(\eta_2) + d_n^2e^{(n+\frac{3}{2})\eta_2}\right].$$

Освен неизвестните A_n, B_n, C_n и т. н. неизвестни в тази система са и скоростите U_1 и U_2. Те се определят от баланса на силите, действащи върху капките.

Системите се решават на два етапа. Задачата е линейна, поради което можем да ги разглеждаме като сбор от движения и сили, пропорционални съответно на U_1, U_2 и GW^1 (елекрично поле), т. е. за коефициентите можем да запишем

$$A_n = U_1A_n^{U_1} + U_2A_n^{U_2} + A_n^E,$$
$$B_n = U_1B_n^{U_1} + U_2B_n^{U_2} + B_n^E,$$
$$C_n = U_1C_n^{U_1} + U_2C_n^{U_2} + C_n^E,$$
$$D_n = U_1D_n^{U_1} + U_2D_n^{U_2} + D_n^E,$$

$$A_n\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2 + B_n\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2 + C_n\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2 + D_n\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2$$
$$= -\frac{1}{2}U_2c^2W_n^{(1)}(\eta_2),$$

$$(2n-1)\left\{A_n\left[(2n-1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1 + 2\lambda_1(2n+1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_1\right]\right.$$

$$+B_n\left[(2n-1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_1+2\lambda_1(2n+1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1\right]\Bigg\}$$

$$+(2n+3)\Bigg\{C_n\left[(2n+3)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1+2\lambda_1(2n+1)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1\right]$$

$$+D_n\left[(2n+3)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1+2\lambda_1(2n+1)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1\right]\Bigg\}$$

$$=U_1c^2\left[\frac{1}{2}\lambda_1(2n-1)(2n+3)W_n^{(1)}(\eta_1)-2(1-\lambda_1)W_n^{(3)}(\eta_1)\right]$$

$$+4c^2GW^1\left[\frac{n(n+1)}{2(2n-1)}Q_{n-1}^1(\eta_1)-\frac{n(n+1)}{2n+1}\mathrm{th}\eta_1Q_n^1(\eta_1)+\frac{(n+1)(n+2)}{2(2n+1)}Q_{n+1}^1(\eta_1)\right],$$

$$(2n-1)\Bigg\{A_n\left[\lambda_2(2n-1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2+2(2n+1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2\right]$$

$$+B_n\left[\lambda_2(2n-1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_2+2(2n+1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_2\right]\Bigg\}$$

$$+(2n+3)\Bigg\{C_n\left[\lambda_2(2n+3)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_2+2(2n+1)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2\right]$$

$$+D_n\left[\lambda_2(2n+3)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_2+2(2n+1)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1\right]\Bigg\}$$

$$-U_2c^2\left[\frac{1}{2}(2n-1)(2n+3)W_n^{(1)}(\eta_2)-2(1-\lambda_2)W_n^{(3)}(\eta_2)\right]$$

$$-4c^2GW^2\left[\frac{n(n+1)}{2(2n-1)}Q_{n-1}^2(\eta_2)-\frac{n(n+1)}{2n+1}\mathrm{ch}\eta_2Q_n^2(\eta_2)+\frac{(n+1)(n+2)}{2(2n+1)}Q_{n+1}^2(\eta_2)\right],$$

$$d_n^1=\frac{e^{(n+\frac{3}{2})\eta_1}}{4}\left[\frac{1}{2}U_1c^2(2n-1)W_n^{(1)}(\eta_1)-A_n(2n-1)\mathrm{sh}\left(n-\frac{1}{2}\right)\eta_1-B_n(2n-1)\mathrm{ch}\left(n-\frac{1}{2}\right)\eta_1\right.$$

$$\left.-C_n(2n+3)\mathrm{sh}\left(n+\frac{3}{2}\right)\eta_1-D_n(2n+3)\mathrm{ch}\left(n+\frac{3}{2}\right)\eta_1\right].$$

Допускаме, че задачата се решава, без да вземаме предвид гравитационното поле. Хидродинамичната сила върху капките в бисферични координати се дава от формулата

$$F_D=\frac{2\sqrt{2}\pi}{c}\sum_{n=1}^{\infty}(2n+1)(A_n\pm B_n+C_n\pm D_n),$$

където силата е обезразмерена с $a_1\mu_3 U$, а U е характерната скорост. За случаите на две външни една на друга капки и на съставна капка имаме съответно:

1. *Две външни капки*

$$F_{DU_1}^1 = \frac{2\sqrt{2}\pi}{c} U_i \sum_{n=1}^{\infty} (2n+1)(A_n^{U_i} + B_n^{U_1} + C_n^{U_1} + D_n^{U_1}) = U_i f_{DU_i}^1, \quad i = 1,\, 2;$$

$$F_{DU_i}^2 = \frac{2\sqrt{2}\pi}{c} U_i \sum_{n=1}^{\infty} (2n+1)(A_n^{U_i} - B_n^{U_i} + C_n^{U_i} - D_n^{U_i}) = U_i f_{DU_i}^2,$$

$$F_D^{12} = \frac{2\sqrt{2}\pi}{c} \sum_{n=1}^{\infty} (A_n^E \pm B_n^E + C_n^E \pm D_n^E).$$

От баланса

$$F_{DU_1}^{(1)} + F_{DU_2}^{(1)} + F_{DE}^{(1)} = 0, \quad \eta = \eta_1,$$

$$F_{DU_1}^{(2)} + F_{DU_2}^{(2)} + F_{DE}^{(2)} = 0, \quad \eta = \eta_2,$$

получаваме система за U_1 и U_2, която решаваме и определяме U_1 и U_2. След това определяме коефициентите на функциите на тока A_n, B_n, C_n, D_n, C_n^1, d_n^1, C_n^2 и d_n^2.

2. *Съставна капка*

При съставна капка постъпваме по същия начин, но там силите се определят така:

а) вътрешна капка

$$F_D^1 = \frac{2\sqrt{2}\pi}{c} \sum_{n=1}^{\infty} (2n+1)(\mathrm{A}_n + \mathrm{B}_n + \mathrm{C}_n + \mathrm{D}_n), \quad \eta = \eta_1.$$

б) външна капка

$$F_D^2 = \frac{2\sqrt{2}\pi}{c} \sum_{n=1}^{\infty} (2n+1)\left[2(d_n + C_n) - \lambda_2(A_n + B_n + C_n + D_n)\right], \quad \eta = \eta_2.$$

За тези сили се прилага описаният подход и се определят U_1 и U_2. След това се определят $\mathrm{A_n}$, $\mathrm{B_n}$ и т.н. за функциите на тока.

3. ДЕФОРМАЦИЯ НА МЕЖДУФАЗОВИТЕ ГРАНИЦИ

Деформацията на междуфазовите граници се определя от баланса на нормалните напрежения, където средната кривина при недеформирана сфера е

$$2\overline{H} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{c}\mathrm{sh}\eta_i,$$

а при деформируема —

$$2\overline{H} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{c}\Bigg\{(\text{ch}\eta_i - \beta)^3 \frac{\partial}{\partial\beta}\left[\frac{1-\beta^2}{(\text{ch}\eta_i - \beta)^2}\frac{\partial H}{\partial\beta}\right]$$

$$+2H(\beta)\text{ch}\eta + 2\text{sh}\eta_i\Bigg\},$$

$$T_{nn}^{(l)} - T_{nn}^{(i)} = \left(\chi\frac{\text{ch}\eta - \beta}{c}\frac{\partial\Phi^{(l)}}{\partial\eta} + \sigma\right)2H,$$

т. е. деформираната повърхнина се задава с уравнението

(13) $$\pm\frac{\sigma}{c}\left\{(\text{ch}\eta_i - \beta)^3\frac{\partial}{\partial\beta}\left[\frac{1-\beta^2}{(\text{ch}\eta_i-\beta)^2}\frac{\partial H(\beta)}{\partial\beta}\right] + 2H(\beta)\text{ch}\eta_i + 2\text{sh}\eta_i\right\}$$

$$= T_{nn}^{(l)} - T_{nn}^{(i)} \mp \chi\frac{\text{ch}\eta_i - \beta}{c}.\frac{\partial\Phi^{(l)}}{\partial\eta}.\frac{2\text{sh}\eta_i}{c}$$

(„+“ при $\eta_i < 0$ и „−“ при $\eta_i > 0$).

Предполагаме, че деформациите са малки и функцията $\eta H(\beta)$, описваща формата на деформираната сфера, има вида

$$\eta^H(\eta) = \eta_0 + H(\beta),$$

където

$$\max_\beta |H(\beta)| < 1, \quad \eta_0 = \eta_i, \quad i = 1, 2.$$

Тук за $T_{\eta\eta}^{(l)}$ и $T_{\eta\eta}^{(i)}$ имаме

$$T_{\eta\eta}^{(l)} = -p^{(l)} - 2\frac{\text{ch}\eta - \beta}{c}\left\{\frac{\partial}{\partial\eta}\left[\frac{(\text{ch}\eta-\beta)^2}{c^2}\frac{\partial\Psi^{(l)}}{\partial\beta}\right] - \frac{\text{ch}\eta-\beta}{c^2}\frac{\partial\Psi^{(l)}}{\partial\eta}\right\},$$

$$T_{\eta\eta}^{(i)} = -p^{(i)} - 2\lambda\frac{\text{ch}\eta - \beta}{c}\left\{\frac{\partial}{\partial\eta}\left[\frac{(\text{ch}\eta-\beta)^2}{c^2}\frac{\partial\Psi^{(i)}}{\partial\beta}\right] - \frac{\text{ch}\eta-\beta}{c^2}\frac{\partial\Psi^{(i)}}{\partial\eta}\right\},$$

където $p^{(l)}$ и $p^{(i)}$ са съответно външното и вътрешното налягане. Тъй като задачата се решава в стоксово приближение, налягането удовлетворява уравнението на Лаплас $\nabla^2 p = 0$. Тогава

$$\frac{\text{ch}\eta - \beta}{c}\frac{\partial p^{(l)}}{\partial\eta} = WE(\text{ch}\eta - \beta)^{3/2}\sum_{n=0}^{\infty} Q_n(\eta)P_n(\beta)$$

$$= WE(\text{ch}\eta-\beta)^{1/2}\sum_{n=0}^{n}\left(\frac{n}{2n-1}Q_{n-1} - \text{ch}\eta_i Q_n + \frac{n+1}{2n+3}Q_{n+1}\right)P_n(\beta).$$

Функцията $\text{H}(\beta)$, която се определя от уравнение (13), трябва да удовлетворява следните две интегрални условия:

(14) $$\int_{-1}^{1}\frac{H(\beta)}{(\text{ch}\eta_1 - \beta)^3}d\beta = 0, \qquad \int_{-1}^{1}\frac{H(\beta)d\beta}{(\text{ch}\eta_i-\beta)^4}, \qquad \max_\beta H(\beta) < 1.$$

Те са условия за запазване на обема и на центъра на тежестта на капката.

Решението на (13) ще търсим във вида

$$H(\beta) = C_a \frac{1}{c^2}(\mathrm{ch}\eta_i - \beta)^{3/2}\left[A\frac{1-\beta\mathrm{ch}\eta_i}{(\mathrm{ch}\eta_i-\beta)^{3/2}} + \sum_{n=0}^{\infty} H_n P_n(\beta)\right]. \tag{15}$$

Тук първото събираемо е решение на хомогенното уравнение, а второто (сумата)

$$C_a\frac{(\mathrm{ch}\eta_i-\beta)^{3/2}}{c^2}\sum_{n=0}^{\infty} H_n.P_n(\beta)$$

е частно решение на нехомогенното уравнение.

Константите A и π (от налягането) се определят от условията за запазване на обема и центъра на масата на капката.

Като заместим функцията (2) в уравнението за деформацията, получаваме

$$\frac{c^2}{C_a}\left\{(\mathrm{ch}\eta_0-\beta)^3\frac{d}{d\beta}\left[\frac{(1-\beta^2)}{(\mathrm{ch}\eta_0-\beta)^2}\frac{\partial H}{\partial\beta}\right] + H(\beta)\mathrm{ch}\eta_0\right\}$$

$$= \sqrt{\mathrm{ch}\eta_0-\beta}\sum_{n=0}^{\infty}\left\{-\frac{(n+2)(n+1)}{4}H_{n+2} + (n+1)^2\mathrm{ch}\eta_0 H_{n+1}\right. \tag{16}$$

$$\left. -\left[\frac{n^2+(n+1)^2+9}{4} + (n-1)(n+2)\mathrm{ch}^2\eta_0\right]H_n + n^2\mathrm{ch}\eta_0 H_{n-1}\frac{-n(n-1)}{4}H_{n-2}\right\}.$$

Определяне на знака в нормалните напрежения

Налягането p в различните области има вида

$$p = \frac{\sqrt{\mathrm{ch}\eta-\beta}}{c^3}\sum_{n=0}^{\infty}\left[\alpha_n \mathrm{ch}\left(n+\frac{1}{2}\right)\eta + \beta_n \mathrm{sh}\left(n+\frac{1}{2}\right)\eta\right]P_n(\beta) + \pi,$$

където π е произволна константа, а коефициентите α_n и β_n се определят така, че да са изпълнени уравненията на Стокс за функцията на тока:

$$\frac{\partial P}{\partial\eta} = -\frac{\mathrm{ch}\eta-\beta}{c}\frac{\partial[E^2\Psi]}{\partial\beta}; \qquad \frac{\partial P}{\partial\beta} = \frac{\mathrm{ch}\eta-\beta}{c(1-\beta^2)}\frac{\partial[E^2\Psi]}{\partial\eta}.$$

Така за α_u и β_u получаваме

$$\alpha_u = \sum_{m=1}^{n-1}\frac{2m+1}{m(m+1)}q_m + \frac{2n+1}{n}q_n + \alpha_0,$$

$$\beta_u = \sum_{m=1}^{n-1}\frac{2m+1}{m(m+1)}r_m + \frac{2n+1}{n}r_n + \beta_0,$$

където

$$r_n = -(2n-1)A_n + (2n+3)C_n + \frac{2n(2n+3)}{2n+1}A_{n+1} + \frac{2(n+1)(2n-1)}{2n-1}C_{n-1},$$

$$q_n = (2n-1)B_n + (2n+3)D_n - \frac{2n(2n+3)}{2n+1}B_{n+1} - \frac{2(n+1)(2n-1)}{2n+1}D_{n-1},$$

$$\alpha_0 = -\sum_{m=0}^{\infty}\frac{2m+1}{m(m+1)}q_m; \quad \beta_0 = -\sum_{m=0}^{\infty}\frac{2m+1}{m(m+1)}r_m.$$

Конкретните изрази за налягането в различните фази при различните задачи са:

а) *При две капки*

1.Във външната област

$$p^{(l)} = \frac{\sqrt{\mathrm{ch}\eta - \beta}}{c^3}\sum_{n=0}^{\infty}\left[\alpha_n^{(l)}\mathrm{ch}\left(n+\frac{1}{2}\right)\eta + \beta_n^{(l)}\mathrm{sh}\left(n+\frac{1}{2}\right)\eta\right]P_n(\beta) + \pi^{(l)}.$$

2. Вътре в капките

$$P_1^{(i)} = \frac{\sqrt{\mathrm{ch}\eta - \beta}}{c^3}\lambda_1\sum_{n=0}^{\infty}\alpha_n^{(i_1)}e^{-(n+\frac{1}{2})\eta}P_n(\beta) + \pi_1^{(i_1)},$$

$$P_2^{(i)} = \frac{\sqrt{\mathrm{ch}\eta - \beta}}{c^3}\lambda_2\sum_{n=0}^{\infty}\alpha_n^{(i_2)}e^{(n+\frac{1}{2})\eta}P_n(\beta) + \pi_2^{(i_2)};$$

б) *При съставна капка*

1) Между двете междуфазови граници

$$P_3 = \frac{\sqrt{\mathrm{ch}\eta - \beta}}{c^3}\lambda_2\sum_{n=0}^{\infty}\left[\alpha_n^3\mathrm{ch}\left(n+\frac{1}{2}\right)\eta + \beta_n^3\mathrm{sh}\left(n+\frac{1}{2}\right)\eta\right]P_n(\beta) + \pi.$$

2) Вън от съставната капка

$$P_2 = \frac{\sqrt{\mathrm{ch}\eta - \beta}}{c^3}\sum_{n=0}^{\infty}\alpha_n^2 e^{(n+\frac{1}{2})\eta}P_n(\beta) + \pi_2.$$

3) В ядрото на съставната капка

$$P_1 = \frac{\sqrt{\mathrm{ch}\eta - \beta}}{c^3}\lambda_1\sum_{n=0}^{\infty}\alpha_n^1 e^{-(n+\frac{1}{2})\eta}P_n(\beta) + \pi_1$$

$$\lambda_1' = \frac{\mu_1}{\mu_2} = \frac{\mu_1}{\mu_3}.\frac{\mu_3}{\mu_2} = \frac{\lambda_1}{\lambda_2}.$$

За да намерим разликата $T_{\eta\eta}^{(l)} - T_{\eta\eta}^{(i)}$, заместваме функциите $\Psi^{(l)}$, $\Psi^{(i)}$, $p^{(l)}$ и $p^{(i)}$ в изразите за напреженията и получените изрази преобразуваме така, че да се разделят променливите. Заместваме също $\frac{\partial\Phi^{(l)}}{\partial\eta}$ и от (16) за функцията $H(\beta)$ получаваме уравнението

$$\frac{1}{C_a}c^2\left\{(\mathrm{ch}\eta_0-\beta)^3\frac{d}{d\beta}\left[\frac{1-\beta^2}{(\mathrm{ch}\eta_0-\beta)^2}\frac{\partial H}{\partial\beta}\right]+2H(\beta)\mathrm{ch}\eta_0\right\}$$

$$=\mp\sqrt{\mathrm{ch}\eta_0-\beta}\left\{\sum_{n=0}^{\infty}\left[\lambda\alpha_n^{(i)}e^{\pm(n+\frac{1}{2})\eta}-\alpha_n^{(l)}\mathrm{ch}\left(n+\frac{1}{2}\right)\eta-\beta_n^l\mathrm{sh}\left(n+\frac{1}{2}\right)\eta\right]P_n(\beta)\right.$$

$$(17)\qquad +2(1-\lambda)\left[\sum_{n=1}^{\infty}(2n+1)\left(\frac{1}{2}\mathrm{sh}\eta U_n^{(l)}+\mathrm{ch}\eta\frac{dU_n^{(l)}}{d\eta}\right)P_n(\beta)\right]$$

$$+\sum_{n=2}^{\infty}\left[-\frac{2n-1}{2}\frac{dU_{n-1}^{(l)}}{\eta}\pm Uc^2\frac{n(n+1)}{2n-1}e^{\pm(n-\frac{3}{2})\eta}\right]P_n(\beta)$$

$$+\sum_{n=0}^{\infty}\left[-\frac{2n+3}{2}\frac{dU_{n+1}^{(l)}}{d\eta}\pm\frac{3}{4\sqrt{2}}Uc^2\left(e^{\pm(n+\frac{5}{2})\eta}\frac{(n+2)(n+1)}{2n+3}\right.\right.$$

$$\left.\left.\left.-\frac{2(2n+1)(n^2+n-1)}{(2n-1)(2n+3)}\right)e^{\pm(n+\frac{1}{2})\eta}\right]P_n(\beta)\right\},$$

$$WE.\chi=\left[k^{(l)}-k^{(i)}\frac{\varepsilon^{(l)}}{\varepsilon^{(i)}}\right](\gamma+\varepsilon)\frac{\varepsilon}{1-\varepsilon}E_0,\qquad \varepsilon=\frac{\varepsilon^{(i)}}{\varepsilon^{(l)}}.$$

Като вземем предвид (16) и приравним коефициентите пред полиномите на Льожандър, получаваме рекурентни връзки за неизвестните коефициенти $\{H_n\}_{n=0}^{\infty}(H_n=\mathrm{H}_\mathrm{n}^{(1)}+\pi\mathrm{H}_\mathrm{n}^{(2)})$.

Да означим с $F\Psi$ израза в дясната страна на (17), който зависи от известните функции Ψ и $\Phi^{(l)}$, а с $F\pi$ — израза, който съдържа неизвестната константа π. Тогава рекурентните връзки за $\{H_n\}_{n=0}^{\infty}$ имат вида

$$n=0:\quad -\frac{1}{2}H_2+\mathrm{ch}\eta_0H_1-\left(\frac{5}{2}-2\mathrm{ch}^2\eta_0\right)H_0=F\Psi_0+F\pi_0,$$

$$n=1:\quad -\frac{3}{2}H_3+4\mathrm{ch}\eta_0.H_2-\frac{7}{2}H_1+\mathrm{ch}\eta_0H_0=F\Psi_1+F\pi_1,$$

$$n\geqq 2:\quad -\frac{(n+2)(n+1)}{4}H_{n+2}+(n+1)^2\mathrm{ch}\eta_0H_{n+1}$$

$$-\left[\frac{n^2+(n+1)^2+9}{4}+(n+1)(n+2)\mathrm{ch}^2\eta_0\right]H_n+n^2\mathrm{ch}\eta_0H_{n+1}\frac{-n(n-1)}{4}H_{n-2}$$

$$=F\Psi_n+F\pi_n.$$

Тази система се решава по различни начини. Един от тях е петточкова прогонка.

Направената програма и съответните подпрограми на „Фортран“ се отнасят за две външни флуидни частици. Главната програма е направена така, че чрез нея да могат да се решават три типа задачи:

1) Чисто хидродинамичната задача — при зададени скорости върху частиците намираме деформациите.

2) Електричната задача, при която капките не се движат. Архимедовата сила уравновесява другите сили.

3) Движение и деформация на капките при наличие на електричен потенциал.

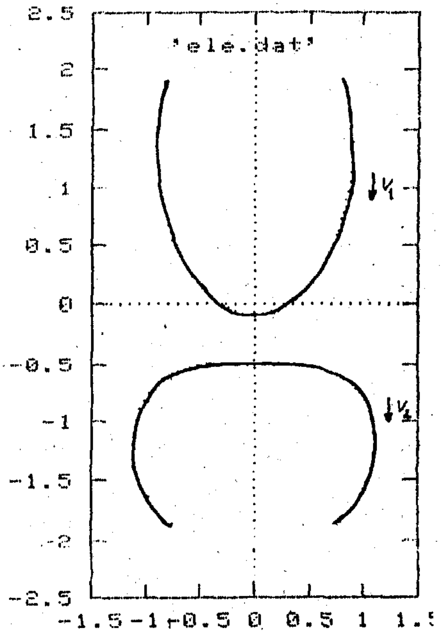
Програмата се състои от три блока. Първо се намира разпределението на електричното поле, след това — функцията на тока и скоростите, и накрая — налягането, напреженията и деформациите.

Числени резултати:

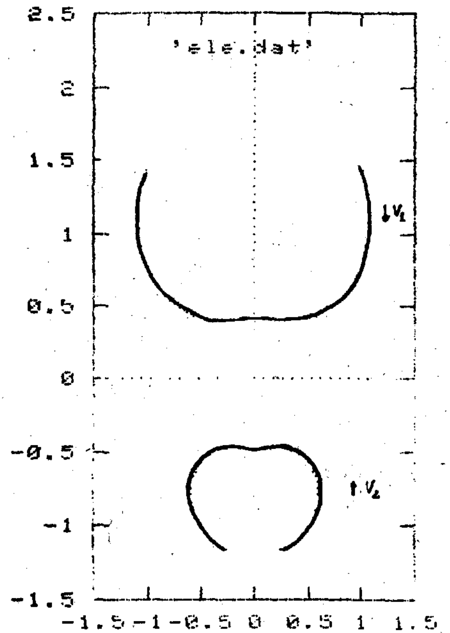
Най-напред ще отбележим, че скоростите, които получават капките вследствие на електричния потенциал, се дължат на флуида, който ги заобикаля. Външният флуид предизвиква хидродинамичната сила, а тя придава скоростите. Въпреки наличието на електрично поле, ако няма външен вискозен флуид, капките няма да се задвижват. Ето защо ще дадем някои резултати за деформацията на капките, като от електричната и хидродинамичната задача ще получим скоростите, с които трябва да се движат флуидните частици (в случая мехурите), а след това при тези скорости решаваме чисто хидродинамичната задача и намираме деформациите.

Получените резултати са дадени при $\lambda_1 = \lambda_2 = 0$, $\varepsilon_1 = \varepsilon_2 = 0,5$ и $(We)_1 = (We)_2 = 0,5$. Другите параметри са подбрани така, че да се види как съотношението между скоростите на двата мехура влияе на тяхното деформиране. На фиг. 1 и 4 мехурите се движат по посока на електричния потенциал. Когато мехурите са еднакви, техните скорости също са еднакви и водещата капка се сплесква и засмуква следващата я капка (фиг. 1). На фиг. 4 и двата мехура се засмукват, водещият по-голям мехур ($r_2 = 1,5$) се движи с по-голяма скорост и изоставането на следващия мехур го засмуква, като по този начин и двете флуидни частици получават удължена (продълговата) форма. Подобна форма се получава и когато двете флуидни частици се движат в противоположни посоки (фиг. 3), като се раздалечават. На фиг. 2 двата мехура се сплескват, защото се движат един срещу друг.

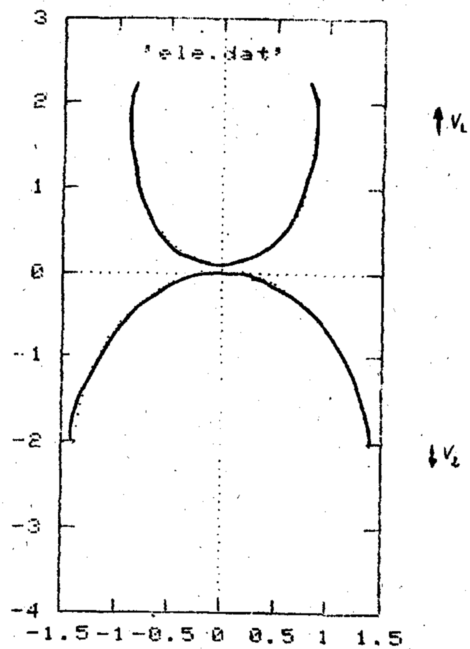
Това изследване е финансирано от фонд „Научни изследвания“ при МОНТ под номер ММ – 419/94 г.



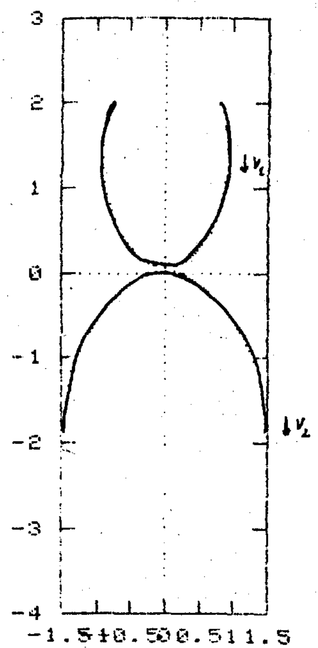
Фиг. 1



Фиг. 2



Фиг. 3



Фиг. 4

Фиг. 1: $r_1 = 1$, $r_2 = 1$, $d = 0,5$, $v_1 = v_2 = -0,075$, $Ca_1 = Ca_2 = 16$.
Фиг. 2: $r_1 = 1$, $r_2 = 0,5$, $d = 0,5$, $v_1 = 0,11$, $v_2 = 0,01$, $Ca_1 = 8,6$, $Ca_2 = 9,8$.
Фиг. 3: $r_1 = 1$, $r_2 = 1,5$, $d = 1$, $v_1 = 0,006$, $v_2 = -0,063$, $Ca_1 = 16$, $Ca_2 = 17$.
Фиг. 4: $r_1 = 1$, $r_2 = 1,5$, $d = 0,5$, $v_1 = -0,006$, $v_2 = -0,014$, $Ca_1 = 3$, $Ca_2 = 7$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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QUELQUES MODELES DE TEMPERATURE D'ASPERITES PAR FROTTEMENT

VASSIL DIAMANDIEV

Васил Диамандиев. НЕКОТОРЫЕ МОДЕЛИ ТЕМПЕРАТУРЫ ВЫСТУПОВ ПРИ ТРЕНИИ

В настоящей работе обобщается модель температуры выступов, данная *Чичинадзе, Гинсбурге* [2]. В работе [2] рассмотрена проблема о температуре при двух гипотезах: 1) размеры тел, перпендикулярные поверхности трения, бесконечные; 2) тепловой поток трения постоянный. В данной статье эти две гипотезы обобщаются и тепловой поток рассмотрен как линейная функция времени. Представлена ещё отдельная модель, в которой делается тепловой баланс одного выступа и получается соответствующая температура. Полученные результаты могут найти применение в машиностроении.

Vassil Diamandiev. ON SOME MODELS OF TEMPERATURE OF ASPERITIES AT FRICTION

In this paper we generalize the model of temperature of asperities given by *Chichinadze, Ginzburg* [2]. In the paper [2] the problem of temperature is considered at the assumption of two hypotheses: 1) The dimensions of the bodies which are perpendicular to the friction surface are infinite; 2) The termal flux of the friction is constant. In the present paper both restrictions are generalized and the termal flux is taken to be a linear function of time. A model of a single asperity for which the termal balance is given and its temperature is directly determined is developed.

Il est notoire que les surfaces des corps flottants sont rugueses avec un nombre des aspérités. La température des surfaces flottantes peut se mesurer avec un instrument convenable. Mais pour les aspérités cette méthode expérimentale n'est pas possible à cause de leurs dimensions microscopiques [1]. Voilà pourquoi le calcul théorique de la température des aspérités a une grande signification pratique. Un grand nombre d'auteurs comme *Tchitchinadze, Ginzbourg* [2], *Block, Jaeger* [10], *Holm* etc. considèrent ce problème et obtiennent des formules respectives pour la température.

Dans l'ouvrage [2] on détermine la température des aspérités par deux hypothèses : 1) les dimensions des corps perpendiculairement du plan du frottement sont infinies; 2) le flux thermique du frottement est constant. Dans notre article on considère le problème quand les dimensions des corps sont finies et d'autre part quand le flux thermique est variable, en particulier il est une fonction linéaire du temps. Des formules dans notre article obtiennent en particulier les résultats de l'ouvrage [2]. Le modèle considéré ne prend pas en considération la pluralité des aspérités [11] et la question de l'usure.

1. TEMPÉRATURE D'ASPÉRITÉS PAR DIMENSIONS FINIES DES CORPS ET FLUX THERMIQUE CONSTANT

On prend le schéma de l'ouvrage [2]. On considère le prolongement de l'aspérité comme une perche avec une longueur l_1 (non infinie). L'aspérité se frotte sur la surface de l'autre corps qui a une longueur l_2. On accepte que la diffusion de la chaleur est seulement verticalement de la surface du frottement. Par ces conditions le problème thermique se pose ainsi: il faut résoudre les équations de Fourier

$$\frac{\partial\theta_1}{\partial t} = a_1\frac{\partial^2\theta_1}{\partial z^2}, \quad \frac{\partial\theta_2}{\partial t} = a_2\frac{\partial^2\theta_2}{\partial z^2} \tag{1}$$

aux conditions suivantes:

$$\begin{aligned} &\theta_1(z, 0) = 0, \quad \lambda_1\frac{\partial\theta_1}{\partial z}(0, t) = -q_1, \quad \theta_1(l_1, t) = 0, \\ &\theta_2(z, 0) = 0, \quad \lambda_2\frac{\partial\theta_2}{\partial z}(0, t) = q_2, \quad \theta_2(-l_2, t) = 0 \end{aligned} \tag{2}$$

où $\theta_1(z, t)$, $\theta_2(z, t)$ sont les champs thermiques en deux corps, λ_1, λ_2 — les coefficients de la conductivité thermique, q_1, q_2 — les flux thermiques qui s'obtiennent de la puissance du frottement q par une unité de sa surface.

Par une manière connue [9] on obtient les formules suivantes pour la température:

$$\begin{aligned} \theta_1(z, t) &= \frac{q_1}{\lambda_1}\left[l_1 - z - \frac{8l_1}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos\frac{\pi z(2m+1)}{2l_1}}{(2m+1)^2}e^{-\frac{\pi^2 a_1(2m+1)^2}{4l_1^2}t}\right], \quad z \geqq 0, \\ \theta_2(z, t) &= \frac{q_2}{\lambda_2}\left[l_2 + z - \frac{8l_2}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos\frac{\pi z(2m+1)}{2l_2}}{(2m+1)^2}e^{-\frac{\pi^2 a_2(2m+1)^2}{4l_2^2}t}\right], \quad z \leqq 0. \end{aligned} \tag{3}$$

Maintenant nous trouverons une formule approximative pour les séries en (4). Il est évident que la série

$$\sum_{m=0}^{\infty}\frac{e^{-k(2m+1)^2}}{(2m+1)^2}, \quad k > 0, \tag{4}$$

est convergente uniformément et par conséquent on peut différencier leur somme par rapport a k, c.-à.-d. on a

$$F(k) = \sum_{m=0}^{\infty} \frac{e^{-k(2m+1)^2}}{(2m+1)^2}, \tag{5}$$

$$F'(k) = -\sum_{m=0}^{\infty} e^{-k(2m+1)^2}. \tag{6}$$

On exprime la somme $\sum\limits_{m=0}^{\infty} e^{-k(2m+1)^2}$ approximativement par une intégrale définie, c.-à.-d. on obtient

$$\sum_{m=0}^{\infty} e^{-k(2m+1)^2} \approx \frac{1}{4}\sqrt{\frac{\pi}{k}} - \frac{1}{2}. \tag{7}$$

Des relations (6) et (7) on trouve

$$F(k) = C - \frac{1}{2}\sqrt{\pi k} + \frac{1}{2}k \tag{8}$$

où C est une constante indéfinie. Mais selon (5) on obtient [3]

$$F(0) = C = \frac{\pi^2}{8}.$$

Par cette manière on trouve la formule approximative

$$\sum_{m=0}^{\infty} \frac{e^{-k(2m+1)^2}}{(2m+1)^2} = \frac{\pi^2}{8} - \frac{1}{2}\sqrt{\pi k} + \frac{1}{2}k. \tag{9}$$

On applique (9) pour les équations (4) et on trouve les formules approximatives

$$\begin{aligned} \theta_1(0,\, t) &= \frac{q_1}{\lambda_1}\left[2\sqrt{\frac{a_1 t}{\pi}} - \frac{a_1 t}{l_1}\right], \\ \theta_2(0,\, t) &= \frac{q_2}{\lambda_2}\left[2\sqrt{\frac{a_2 t}{\pi}} - \frac{a_2 t}{l_2}\right]. \end{aligned} \tag{10}$$

Selon le schéma de l'ouvrage [2] on prend la température pour un petit intervalle du temps. Pour l'aspérité on prend $0 \leqq t \leqq \frac{L_r}{v}$ où L_r est la route du frottement de l'aspérité jusqu'à leur existence et v — la vitesse du glissement. Pour la surface immobile on prend $0 \leqq t \leqq \frac{d_r}{v}$ où d_r est la dimension moyenne de l'aspérité. Pour la température maximale on obtient de (10)

$$\begin{aligned} \theta_{1\,\max} &= \frac{q_1}{\lambda_1}\left[2\sqrt{\frac{a_1 L_r}{\pi v}} - \frac{a_1 L_r}{l_1 v}\right], \\ \theta_{2\,\max} &= \frac{q_2}{\lambda_2}\left[2\sqrt{\frac{a_2 d_r}{\pi v}} - \frac{a_2 d_r}{l_2 v}\right]. \end{aligned} \tag{11}$$

Les flux thermiques se déterminent des relations

(12) $$q_1 = (1-\alpha)q, \quad q_2 = \alpha q$$

où α est une constante. Puisque il n'y a pas un saut aux températures, c.-à.-d. $\theta_{1\,\max} = \theta_{2\,\max}$, de (11) on obtient

(13) $$\alpha = \frac{\lambda_2\left[2\sqrt{\frac{a_1 L_r}{\pi v}} - \frac{a_1 L_r}{l_1 v}\right]}{\lambda_1\left[2\sqrt{\frac{a_2 d_r}{\pi v}} - \frac{a_2 d_r}{l_2 v}\right] + \lambda_2\left[2\sqrt{\frac{a_1 L_r}{\pi v}} - \frac{a_1 L_r}{l_1 v}\right]}.$$

On remplace (12), (13) en (11) et on trouve pour la température maximale

(14) $$\theta_{\max} = \frac{q\left[2\sqrt{\frac{a_1 L_r}{\pi v}} - \frac{a_1 L_r}{l_1 v}\right]\left[2\sqrt{\frac{a_2 d_r}{\pi v}} - \frac{a_2 d_r}{l_2 v}\right]}{\lambda_1\left[2\sqrt{\frac{a_2 d_r}{\pi v}} - \frac{a_2 d_r}{l_2 v}\right] + \lambda_2\left[2\sqrt{\frac{a_1 L_r}{\pi v}} - \frac{a_1 L_r}{l_1 v}\right]}.$$

Quand les dimensions des corps sont infinies, c.-à.-d. $l_1 = l_2 = \infty$, on obtient de (14) la formule de *Tchitchinadze, Ginzbourg* [2]:

(15) $$\theta_{\max} = \frac{2q\sqrt{a_1 a_2 L_r d_r}}{\sqrt{\pi v}[\lambda_1\sqrt{a_2 d_r} + \lambda_2\sqrt{a_1 L_r}]}.$$

On détermine la puissance du frottement par une unité de la surface q qui se forme sur une aspérité. La grandeur q s'obtient comme une transformation de l'énergie mécanique du frottement en une énergie thermique par l'équivalent respectif de la chaleur. Ayant en vue cette conception on trouve

(16) $$q = \frac{J f \Delta N . v}{d_r^2}.$$

où J est l'équivalent thermique de l'énergie mécanique, f — le coefficient du frottement, ΔN — la charge sur une aspérité; d_r^2 est la surface d'une aspérité que nous acceptons pour un carré.

On suppose que la charge totale N se distribue uniformément à toutes les aspérités, c.-à.-d.

(17) $$\Delta N = \frac{N}{n},$$

où n est le nombre des aspérités sur la surface totale. On note que la relation (17) est approximative. Evidemment la surface réelle du contact A_r se donne par la formule

(18) $$A_r = n d_r^2.$$

Selon (16) — (18) on obtient définitivement

(19) $$q = \frac{J f N v}{A_r} \frac{\text{cal}}{\text{s cm}^2}.$$

On prend un exemple numérique. On a les données suivantes:

$$J = \frac{10^{-4}}{4.27}\frac{\text{cal}}{\text{kg cm}}, \quad f = 0.1, \quad N = 500 \text{ kg}, \quad v = 2500 \ \frac{\text{cm}}{\text{s}},$$

$$\lambda_1 = \lambda_2 = 1.08 \times 10^{-9} \ \frac{\text{cal}}{\text{cm s grad}}, \quad a_1 = a_2 = 0.125 \frac{\text{cm}^2}{\text{s}} \ \text{(pour acier)},$$

$$l_1 = l_2 = 100 \text{ cm}, \quad L_r = 2.2 \times 10^{-2}\text{cm}, \quad d_r = 2.2 \times 10^{-3}\text{cm}.$$

De la formule (19) on trouve

$$q = \frac{2.927}{A_r}\frac{\text{cal}}{\text{s cm}^2}. \tag{20}$$

On remplace (20) en (14) pour les données et on obtient

$$\theta_{\max} = \frac{6.9}{A_r}. \tag{20'}$$

La relation (20′) montre que la température d'une aspérité dépend de la surface réelle du contact, c.-à.-d. de la surface nominale.

Quand $A_r = 2.10^{-2}$ cm^2 de (20′) on trouve $\theta_{max} = 345°$.

Ce résultat est proche de la température d'une brûlure de l'aspérité, c.-à.-d. les données pour L_r et d_r sont choisies d'une manière convenable pour cet exemple.

2. TEMPÉRATURE D'ASPÉRITÉS PAR DIMENSIONS FINIES DES CORPS ET PAR FLUX THERMIQUE VARIABLE

Par le freinage le flux thermique peut devenir une fonction du temps quand la charge totale N est variable. En particulier on accepte que N est une fonction linéaire du temps et selon la formule (19) q est la même fonction du temps, c.-à.-d. on a

$$q = \frac{Jfv(N_0 + N_1 t)}{A_r}, \tag{21}$$

ou on a

$$q = q' + q''t,$$

où q' et q'' selon (21) sont

$$q' = \frac{JfvN_0}{A_r}, \quad q'' = \frac{JfvN_1}{A_r}. \tag{22}$$

Maintenant le problème thermique des corps frottants se pose ainsi: déterminer les champs thermiques par rapport les équations (1) aux conditions suivantes:

$$\begin{aligned} &\theta_1(z,\,0) = 0, \quad \lambda_1 \frac{\partial \theta_1}{\partial z}(0,\,t) = -(1-\alpha)(q' + q''t), \quad \theta_1(l_1,\,t) = 0, \\ &\theta_2(z,\,0) = 0, \quad \lambda_2 \frac{\partial \theta_2}{\partial z}(0,\,t) = \alpha(q' + q''t), \quad \theta_2(-l_2,\,t) = 0. \end{aligned} \tag{23}$$

Ici on accepte que la température de l'environnement est zéro.

On applique la méthode de Heaviside sur ce problème mathématique. Nous considérons seulement les formules pour la température $\theta_1(z, t)$; pour $\theta_2(z, t)$ ils sont analogiques. De (1) on obtient [4]

$$\theta_{1L}(z, s) = B_1(s)e^{-\sqrt{\frac{s}{a_1}}z} + B_2(s)e^{\sqrt{\frac{s}{a_1}}z}, \quad z \geqq 0, \tag{24}$$

où

$$\theta_{1L}(z, s) = \int_0^\infty e^{-st}\theta_1(z, t)dt. \tag{25}$$

On applique la dernière condition en (23) à (24) et on trouve

$$B_1(s)e^{-\sqrt{\frac{s}{a_1}}l_1} + B_2(s)e^{\sqrt{\frac{s}{a_1}}l_1} = 0. \tag{26}$$

De (24) et (26) on obtient

$$\theta_1(z, t) = L^{-1}\left[B_1(s)\left\{e^{-\sqrt{\frac{s}{a_1}}z} - e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}\right\}\right] \tag{27}$$

où L^{-1} est l'opérateur inverse de Laplace. De (27) et la deuxième condition de (23) on trouve

$$L^{-1}\left\{B_1(s)\sqrt{\frac{s}{a_1}}\left(1 + e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)\right\} = \frac{(1-\alpha)(q' + q''t)}{\lambda_1}. \tag{28}$$

On prend l'opérateur de Laplace à (28) et on obtient

$$B_1(s) = \frac{(1-\alpha)\sqrt{a_1}}{\lambda_1}\left[\frac{q'}{s^{3/2}\left(1 + e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)} + \frac{q''}{s^{5/2}\left(1 + e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}\right]. \tag{29}$$

De (26) et (29) on trouve

$$B_2(s) = -\frac{(1-\alpha)\sqrt{a_1}}{\lambda_1}\left[\frac{q'e^{-2l_1\sqrt{\frac{s}{a_1}}}}{s^{3/2}\left(1 + e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)} + \frac{q''e^{-2l_1\sqrt{\frac{s}{a_1}}}}{s^{5/2}\left(1 + e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}\right]. \tag{30}$$

On remplace (29) et (30) en (24) et on obtient

$$\theta_{1L}(z,\ s)=\frac{(1-\alpha)\sqrt{a_1}}{\lambda_1}\left[\frac{q'\left(e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}\right)}{s^{3/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}+\frac{q''\left(e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}\right)}{s^{5/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}\right]. \tag{31}$$

On prend l'opérateur inverse de Laplace a (31) et on trouve

$$\theta_1(z,\ t)=\frac{(1-\alpha)\sqrt{a_1}}{\lambda_1}\left[q'L^{-1}\left\{\frac{e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}}{s^{3/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}\right\}+q''L^{-1}\left\{\frac{e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}}{s^{5/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}\right\}\right]. \tag{32}$$

On applique le théorème du retournement [5] à (32), c.-à.-d. on a

$$L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}}{s^{3/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}\right]=\frac{1}{2\pi i}\int\limits_{\substack{\gamma-i\omega\\ \omega\to\infty}}^{\gamma+i\omega} e^{st}\frac{e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}}{s^{3/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}ds, \tag{33}$$

$$(34)\quad L^{-1}\left[\frac{e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}}{s^{5/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}\right]=\frac{1}{2\pi i}\int\limits_{\substack{\gamma-i\omega\\ \omega\to\infty}}^{\gamma+i\omega} e^{st}\frac{e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}}{s^{5/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}ds,$$

où γ et ω sont des nombres réels et positifs. Pour le calcul de l'intégrales en (33), (34) on fait une intégration dans un domaine complexe sur un contour représentant le droit $x=\gamma$ et la circonférence (c) avec un centre $O(0,\,0)$ et un rayon $R\to\infty$ [5]. On applique le théorème des résidus [6] sur le contour donné.

Les pôles des fonctions en (33), (34) sont

$$(35)\qquad s=0,\quad s_m=-\frac{\pi^2 a_1}{4l_1^2}(2m+1)^2\quad (m=0,\,1,\,2,\,\ldots).$$

Pour les résidus de ces fonctions on a respectivement

$$(36)\qquad \mathrm{Res}(0)=\frac{l_1-z}{\sqrt{a_1}},\quad \mathrm{Res}(s_m)=-\frac{8l_1\cos\frac{\pi z}{2l_1}(2m+1)}{\pi^2\sqrt{a_1}(2m+1)^2}e^{-\frac{\pi^2 a_1(2m+1)^2}{4l_1^2}t}$$

pour la fonction en (33),

$$(37)\qquad \begin{aligned}\mathrm{Res}(0)&=\frac{1}{a_1^{3/2}}\left(-\frac{1}{3}l_1^3+\frac{1}{2}l_1z^2-\frac{1}{6}z^3\right),\\ \mathrm{Res}(s_m)&=\frac{32l_1^3\cos\frac{\pi z}{2l_1}(2m+1)}{\pi^4a_1^{3/2}(2m+1)^4}e^{-\frac{\pi^2 a_1(2m+1)^2}{4l_1^2}t}\end{aligned}$$

pour la fonction en (34).

Pour les intégrales dans le domaine complexe on obtient

$$(38)\qquad \begin{aligned}&\frac{1}{2\pi i}\int\limits_{\substack{\gamma-i\omega\\ \omega\to\infty}}^{\gamma+i\omega} e^{st}\frac{e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}}{s^{3/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}ds\\ &=\frac{1}{\sqrt{a_1}}\left[l_1-z-\frac{8l_1}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos\frac{\pi z}{2l_1}(2m+1)}{(2m+1)^2}e^{-\frac{\pi^2 a_1(2m+1)^2}{4l_1^2}t}\right]\end{aligned}$$

et respectivement

$$\frac{1}{2\pi i}\int\limits_{\substack{\gamma-i\omega\\ \omega\to\infty}}^{\gamma+i\omega} e^{st}\frac{e^{-\sqrt{\frac{s}{a_1}}z}-e^{-\sqrt{\frac{s}{a_1}}(2l_1-z)}}{s^{5/2}\left(1+e^{-2l_1\sqrt{\frac{s}{a_1}}}\right)}ds \tag{39}$$

$$=\frac{1}{a_1^{3/2}}\left[-\frac{1}{3}l_1^3+\frac{1}{2}l_1z^2-\frac{1}{6}z^3+\frac{32l_1^3}{\pi^4}\sum_{m=0}^{\infty}\frac{\cos\frac{\pi z}{2l_1}(2m+1)}{(2m+1)^4}e^{-\frac{\pi^2a_1(2m+1)^2}{4l_1^2}t}\right].$$

Selon (32)–(34), (38) et (39) on trouve définitivement

$$\theta_1(z,\,t)=\frac{1-\alpha}{\lambda_1}\left[q'\left\{l_1-z-\frac{8l_1}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos\frac{\pi z}{2l_1}(2m+1)}{(2m+1)^2}e^{-\frac{\pi^2a_1(2m+1)^2}{4l_1^2}t}\right\}\right. \tag{40}$$

$$\left.-\frac{q''}{a_1}\left\{\frac{1}{3}l_1^3-\frac{1}{2}l_1z^2+\frac{1}{6}z^3-\frac{32l_1^3}{\pi^4}\sum_{m=0}^{\infty}\frac{\cos\frac{\pi z}{2l_1}(2m+1)}{(2m+1)^4}e^{-\frac{\pi^2a_1(2m+1)^2}{4l_1^2}t}\right\}\right].$$

La température de la surface frottante s'obtient par $z=0$ dans l'équation (40), c.-à.-d. on a

$$\theta_1(0,\,t)=\frac{(1-\alpha)l_1}{\lambda_1}\left[q'\left\{1-\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{e^{-\frac{\pi^2a_1(2m+1)^2}{4l_1^2}t}}{(2m+1)^2}\right\}\right. \tag{41}$$

$$\left.-\frac{q''l_1^2}{3a_1}\left\{1-\frac{96}{\pi^4}\sum_{m=0}^{\infty}\frac{e^{-\frac{\pi^2a_1(2m+1)^2}{4l_1^2}t}}{(2m+1)^4}\right\}\right].$$

Analogiquement pour la température de la surface immobile on a

$$\theta_2(0,\,t)=\frac{\alpha l_2}{\lambda_2}\left[q'\left\{1-\frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{e^{-\frac{\pi^2a_2(2m+1)^2}{4l_2^2}t}}{(2m+1)^2}\right\}\right. \tag{42}$$

$$\left.-\frac{q''l_2^2}{3a_2}\left\{1-\frac{96}{\pi^4}\sum_{m=0}^{\infty}\frac{e^{-\frac{\pi^2a_2(2m+1)^2}{4l_2^2}t}}{(2m+1)^4}\right\}\right].$$

On considére la somme

$$\Phi(k)=\sum_{m=0}^{\infty}\frac{e^{-k(2m+1)^2}}{(2m+1)^4} \tag{43}$$

qui est convergente uniformément. On différencie la fonction (43) par rapport à k et on trouve

$$\Phi'(k)=-\sum_{m=0}^{\infty}\frac{e^{-k(2m+1)^2}}{(2m+1)^2}. \tag{44}$$

On utilise la formule approximative (9) et on obtient de (44)

$$\Phi'(k)=-\frac{\pi^2}{8}+\frac{1}{2}\sqrt{\pi k}-\frac{1}{2}k. \tag{45}$$

On intègre (45) et on trouve

$$\Phi(k)=C-\frac{\pi^2}{8}k+\frac{1}{3}\sqrt{\pi}.k^{3/2}-\frac{1}{4}k^2. \tag{46}$$

Evidemment [3]

$$\Phi(0)=C=\frac{\pi^4}{96}.$$

De cette manière nous obtenons une nouvelle formule approximative

$$\sum_{m=0}^{\infty}\frac{e^{-k(2m+1)^2}}{(2m+1)^4}=\frac{\pi^4}{96}-\frac{\pi^2}{8}k+\frac{\sqrt{\pi}}{3}k^{3/2}-\frac{1}{4}k^2. \tag{47}$$

Selon les formules (9) et (47) nous obtenons de (41) la formule suivante:

$$\theta_1(0,\,t)=\frac{1-\alpha}{\lambda_1}\left[q'\left\{2\sqrt{\frac{a_1t}{\pi}}-\frac{a_1t}{l_1}\right\}-q''\left\{l_1t-\frac{4\sqrt{a_1}}{3\sqrt{\pi}}.t^{3/2}+\frac{1}{2}\frac{a_1t^2}{l_1}\right\}\right]. \tag{48}$$

Analogiquement on a

$$\theta_2(0,\,t)=\frac{\alpha}{\lambda_2}\left[q'\left\{2\sqrt{\frac{a_2t}{\pi}}-\frac{a_2t}{l_2}\right\}-q''\left\{l_2t-\frac{4\sqrt{a_2}}{3\sqrt{\pi}}t^{3/2}+\frac{1}{2}\frac{a_2t^2}{l_2}\right\}\right]. \tag{49}$$

Selon le schéma de l'ouvrage [2] on prend $t=\dfrac{L_r}{v}$ pour la température $\theta_1(0,\,t)$ et $t=\dfrac{d_r}{v}$ pour $\theta_2(0,\,t)$. Ainsi on obtient pour les températures maximales:

(50)

$$\theta_{1\,\max}=\frac{1-\alpha}{\lambda_1}\left[q'\left\{2\sqrt{\frac{a_1L_r}{\pi v}}-\frac{a_1L_r}{l_1v}\right\}-q''\left\{\frac{l_1L_r}{v}-\frac{4\sqrt{a_1}}{3\sqrt{\pi}}\left(\frac{L_r}{v}\right)^{3/2}+\frac{1}{2}\frac{a_1}{l_1}\left(\frac{L_r}{v}\right)^2\right\}\right],$$

$$\theta_{2\,\max}=\frac{\alpha}{\lambda_2}\left[q'\left\{2\sqrt{\frac{a_2d_r}{\pi v}}-\frac{a_2d_r}{l_2v}\right\}-q''\left\{\frac{l_2d_r}{v}-\frac{4\sqrt{a_2}}{3\sqrt{\pi}}\left(\frac{d_r}{v}\right)^{3/2}+\frac{1}{2}\frac{a_2}{l_2}\left(\frac{d_r}{v}\right)^2\right\}\right].$$

Puisqu'il n'y a pas un saut aux températures, c.-à.-d. $\theta_{1\max} = \theta_{2\max}$ pour la grandeur α on trouve

$$\alpha = \frac{\lambda_2 A}{\lambda_1 B + \lambda_2 A} \tag{51}$$

où les grandeurs A et B sont

$$\begin{aligned} A &= q'\left\{2\sqrt{\frac{a_1 L_r}{\pi v}} - \frac{a_1 L_r}{l_1 v}\right\} - q''\left\{\frac{l_1 L_r}{v} - \frac{4\sqrt{a_1}}{3\sqrt{\pi}}\left(\frac{L_r}{v}\right)^{3/2} + \frac{1}{2}\frac{a_1}{l_1}\left(\frac{L_r}{v}\right)^2\right\}, \\ B &= q'\left\{2\sqrt{\frac{a_2 d_r}{\pi v}} - \frac{a_2 d_r}{l_2 v}\right\} - q''\left\{\frac{l_2 d_r}{v} - \frac{4\sqrt{a_2}}{3\sqrt{\pi}}\left(\frac{d_r}{v}\right)^{3/2} + \frac{1}{2}\frac{a_2}{l_2}\left(\frac{d_r}{v}\right)^2\right\}. \end{aligned} \tag{52}$$

On remplace (51) en (49) et on trouve pour la température maximale de l'aspérité

$$\theta_{\max} = \frac{AB}{\lambda_1 B + \lambda_2 A}. \tag{53}$$

Quand le flux thermique est constant, c.-à.-d. $q'' = 0$, $q' = q$, la formule (53) coïncide avec (14) selon (52).

On prend un exemple numérique par les données suivantes:

$$J = \frac{10^{-4}}{4.27}\frac{\text{cal}}{\text{kg cm}}, \quad f = 0.1, \quad v = 2500\frac{\text{cm}}{\text{s}}, \quad N_0 = 1000 \text{ kg}, \quad N_1 = 200 \text{ kg},$$

$$d_r = 2.2 \times 10^{-3} \text{ cm}, \quad L_r = 2.2 \times 10^{-2} \text{ cm}, \quad \lambda_1 = \lambda_2 = 1.08 \times 10^{-4}\frac{\text{cal}}{\text{cm s g}},$$

$$l_1 = l_2 = 100 \text{ cm}, \quad a_1 = a_2 = 0.125\frac{\text{cm}^2}{\text{s}}.$$

Des formules (22) on trouve

$$q' = \frac{5.32}{A_r}\frac{\text{cal}}{\text{s cm}^2}, \quad q'' = \frac{1.06}{A_r}\frac{\text{cal}}{\text{s cm}^2}. \tag{54}$$

On remplace (54) en (52) et (53) et on trouve

$$\theta_{\max} = \frac{7.48}{A_r}. \tag{55}$$

On voit que la température dépend de la surface réelle du contact, c.-à.-d. de la surface nominale. Quand $A_r = 2.10^{-2}$ cm^2 on obtient de (55)

$$\theta_{\max} = 374^\circ.$$

3. CALCUL DE TEMPÉRATURE D'ASPÉRITÉS PAR AUTRE MÉTHODE

Ici nous exposerons une autre méthode pour le calcul de la température de l'aspérité. Par cette méthode on étudie la balance de la chaleur d'une aspérité au cours du frottement pour un petit intervalle du temps [7]. Ici on prend la microgéométrie des aspérités qui rend compte des mollesses [8].

On prend l'équation de Fourier pour une aspérité qu'on peut considérer comme une petite perche, c.-à.-d.

$$\frac{\partial \theta_1}{\partial t} = a_1 \frac{\partial^2 \theta_1}{\partial z^2}. \tag{56}$$

Selon la méthode de Heaviside de (56) on trouve [4]

$$\theta_1(z,\, t) = L^{-1}\left[B_1(s)e^{-\sqrt{\frac{s}{a_1}}z} + B_2(s)e^{\sqrt{\frac{s}{a_1}}z}\right] \tag{57}$$

où z change à l'intervalle

$$0 \leqq z \leqq H_0. \tag{58}$$

Ici H_0 est la hauteur de la couche rugueuse. Les aspérités sont les segments sphériques avec les rayons r_2 et r_1 de la base supérieure, respectivement inférieure.

Selon la loi de Fourier la quantité de la chaleur qui entre en l'aspérité par le frottement se donne par l'expression

$$d\theta = -\lambda_1 \pi r_2^2 \frac{\partial \theta_1}{\partial z}(0,\, t)dt. \tag{59}$$

D'autre part la même quantité se calcule directement de la puissance du frottement, c.-à.-d.

$$dQ = \frac{1}{2} J f N_1 v dt \tag{60}$$

où N_1 est la charge sur une aspérité. Pour N_1 on a [8]

$$N_1 = \frac{E\sqrt{R}\delta^{3/2}}{1.55^{3/2}} = 0.518 E\sqrt{R}\delta^{3/2}. \tag{61}$$

Ici E est le module de Ung, R — le rayon du courbement d'une aspérité et δ — la déformation à Hertz.

De formules (59) et (60) on trouve la relation

$$L^{-1}[\sqrt{s}(B_1 - B_2)] = \frac{J f N_1 v \sqrt{a_1}}{2\pi \lambda_1 r_2^2}. \tag{62}$$

La quantité dQ qui entre en l'aspérité se distribue ainsi: une part dQ' quitte par la base inférieure, une autre part dQ'' s'absorbe de la masse de l'aspérité et une troisième part dQ''' s'emet en l'environnement, c.-à.-d. on a la relation

$$dQ = dQ' + dQ'' + dQ'''. \tag{63}$$

Selon la définition des quantités dQ', dQ'', dQ''' on a les relations suivantes:

$$dQ' = -\pi \lambda_1 r_1^2 \frac{\partial \theta_1}{\partial z}(H_0,\, t)dt, \tag{64}$$

$$dQ'' = c_1 \rho_1 V \frac{\partial \theta_1}{\partial t} dt, \tag{65}$$

(66) $$dQ''' = \alpha S\theta_1(z,\, t)dt$$

où V et S sont respectivement le volume et la surface environnante de l'aspérité; ces grandeurs se donnent par les expressions

(67) $$V = \frac{\pi r_1 H_0}{3}\frac{3-k}{2-k},$$

(68) $$S = 2\pi R H_0 = \frac{2\pi r_1^2}{2-k}$$

où $k = \frac{H_0}{R}$ est le nombre qui caractérise la façon des surfaces frottantes.

Selon (57) pour la grandeur $\frac{\partial\theta_1}{\partial z}(H_0,\, t)$ on a la relation

(69) $$\frac{\partial\theta_1}{\partial z}(H_0,\, t) = -\frac{1}{\sqrt{a_1}}L^{-1}[\sqrt{s}(B_1 - B_2)] + \frac{H_0}{a_1}L^{-1}[s(B_1 + B_2)]$$

qui s'obtient quand on prend en considération que la grandeur H_0 est petite, c.-à.-d. $H_0^2,\, H_0^3 \approx 0$. De (62) et (69) on trouve

(70) $$\frac{\partial\theta_1}{\partial z}(H_0,\, t) = -\frac{JfN_1v}{2\pi\lambda_1 r_2^2} + \frac{H_0}{a_1}L^{-1}[s(B_1 + B_2)].$$

Selon (63)–(68) et (70) on obtient la relation

(71) $$\frac{H_0}{a_1}L^{-1}[s(B_1 + B_1)] = \frac{JfN_1v}{2\pi\lambda_1 r_2^2} - \frac{JfN_1v}{2\pi\lambda_1 r_1^2} + \frac{H_0}{3a_1}\frac{3-k}{2-k}\frac{\partial\theta_1}{\partial t} + \frac{2\alpha\theta_1(z,\, t)}{\lambda_1(2-k)}.$$

On prend l'opérateur de Laplace à (71) et après quelques calculs on obtient

(72) $$\frac{H_0}{a_1}s(B_1 + B_2) = \frac{1}{s}\frac{JfN_1v}{2\pi\lambda_1}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) + \frac{H_0}{3a_1}\frac{3-k}{2-k}s\theta_{1L} + \frac{2\alpha\theta_{1L}}{\lambda_1(2-k)}.$$

De (57) on a

$$Q_{1L}(z,\, s) = B_1(s)e^{-\sqrt{\frac{s}{a_1}}z} + B_2(s)e^{\sqrt{\frac{s}{a_1}}z}.$$

Selon (58) on peut écrire approximativement

(73) $$\theta_{1L} = B_1(s) + B_2(s).$$

De (72) et (73) on trouve l'égalité

(74) $$\theta_{1L} = \frac{3JfN_1va_1(2-k)\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)}{2\pi s[\lambda_1 H_0(3-2k)s - 6\alpha a_1]}.$$

L'expression en (74) se développe aux fractions élémentaires ainsi

(75) $$\frac{1}{s[\lambda_1 H_0(3-2k)s - 6\alpha a_1]} = \frac{1}{6\alpha a_1}\left[\frac{1}{s - \frac{6\alpha a_1}{H_0\lambda_1(3-2k)}} - \frac{1}{s}\right].$$

On applique l'opérateur inverse de Laplace et selon (75) on trouve définitivement

$$(76) \qquad \theta_1(t) = \frac{JfN_1v(2-k)\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)}{4\pi\alpha}\left[e^{\frac{6\alpha a_1 t}{H_0\lambda_1(3-2k)}} - 1\right].$$

On prend un exemple numérique par les données suivantes:

$$f = 0.1, \quad v = 2500\frac{\text{cm}}{\text{s}}, \quad \delta = 1.1\times10^{-4}\text{cm}, \quad H_0 = 10^{-3}\ \text{cm}, \ R = 10^{-2}\ \text{cm},$$

$$k = 0.1, \quad N = 500\ \text{kg}, \quad a_1 = 0.125\frac{\text{cm}^2}{\text{s}}, \quad \lambda_1 = 1.08\times10^{-4}, \quad E = 2.1\times10^6\frac{\text{kg}}{\text{cm}^2},$$

$$\alpha = 10^{-6}\frac{\text{cal}}{\text{cm}^2\ \text{s grad}}, \quad r_2 = 1.48\times10^{-3}\ \text{cm}, \quad r_1 = 2.55\times10^{-3}\ \text{cm}.$$

Pour cet exemple on obtient de (76)

$$(77) \qquad \theta_1(t) = 3.35\times10^7(e^{2.48t} - 1).$$

Selon (77) pour le temps 10^{-6} s $< t < 10^{-5}$ s la température varie respectivement

$$83^\circ < \theta_1 < 830^\circ.$$

Ayant en vue que la température d'une brûlure est environ 400°, il est clair que la formule (77) est valide pour $t < 10^{-5}$ s par ces données numériques.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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ON THE NON-INTEGRABILITY OF A HAMILTONIAN SYSTEM RESULTING FROM A PROBLEM FOR ELASTIC STRING

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Христо Илиев. О НЕИНТЕГРИРУЕМОСТИ ГАМИЛЬТОНОВОЙ СИСТЕМЫ В ЗАДАЧЕ О ВИБРАЦИИ УПРУГОЙ ОПОРЫ

В этой работе рассмотрена задача о нелинейной вибрации струны. Задача сводится к одной системе обыкновенных дифференциальных уравнений Гамильтоного типа. Установлена аналитическая неинтегрируемость Гамильтоновой системы с двумя степенями свободы.

Christo Iliev. ON THE NON-INTEGRABILITY OF A HAMILTONIAN SYSTEM RESULTING FROM A PROBLEM FOR ELASTIC STRING

In this paper the problem of nonlinear vibration of an elastic string is considered. The problem is reduced to a system of ordinary differential equations of Hamiltonian type. The analytical non-integrability of the corresponding Hamiltonian system in the case of two degrees of freedom is proved.

1. INTRODUCTION AND MAIN RESULTS

The equation governing the free lateral vibrations of an elastic string which ends are restricted to remain a fixed distance apart is given by

$$(1) \qquad \rho h \frac{\partial^2 \omega}{\partial t^2} = \left[P_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial \omega}{\partial x} \right)^2 dx \right] \frac{\partial^2 \omega}{\partial x^2}$$

with initial and boundary conditions

$$\omega(0,\ x) = \omega_0(x), \quad \frac{\partial \omega(0,\ x)}{\partial t} = \omega_1(x),$$
(2)
$$\omega(t,\ 0) = \frac{\partial^2 \omega(t,\ 0)}{\partial x^2} = \omega(t,\ L) = \frac{\partial^2 \omega(t,\ L)}{\partial x^2} = 0,$$

where ω is the measure of the lateral deflection of the string, x is the space co-ordinate, t is the time, E is the Young's modulus, ρ is the mass density, h is the thickness of the string, L is its length, and P_0 is the initial axial tension. The Cauchy problem (1)–(2) was considered by Nishida [9] under the essential assumption that the initial data do not contain infinitely higher harmonics, i. e. there exists a natural N such that the functions ω and ω_1 can be represented as

$$\omega_0(x) = \sum_{k=1}^{N} a_k \sin\left(\frac{k\pi}{L}x\right), \quad \omega_1(x) = \sum_{k=1}^{n} b_n \sin\left(\frac{k\pi}{L}x\right),$$

where $a_k,\ b_k,\ k = 1, \ldots, N$, are real constants. The solution of this problem is easily proved to exist uniquely in the large in time, see [9] and references therein. Besides, if certain harmonics are not presented in the initial data, then they do not appear in the solution in the course of time, i. e. it is natural to search for the solution of the kind

$$\omega(t,\ x) = \sum_{k=1}^{N} u_k(t) \sin\left(\frac{k\pi}{L}x\right). \tag{3}$$

Substituting (3) in the integro-differential equation (1) leads to a Hamiltonian system of differential equations for the functions $u_k(t)$. By means of the Birghoff transformation and KAM theory Nishida [9] obtained a result for conservation near the equilibrium of the conditionally periodic motion.

Here we shall consider the lateral vibrations of an elastic string subjected to an external volume forcing caused by the medium. In this case the equation of motion is given by

$$\frac{\partial^2 \omega}{\partial t^2} - \left[c_1 + h_1 \int_0^{\pi} \left(\frac{\partial \omega}{\partial x}\right)^2 dx\right] \frac{\partial^2 \omega}{\partial x^2} = \left[c_2 + h_2 \int_0^{\pi} \omega^2 dx\right] \omega, \tag{4}$$

where $c_1,\ c_2,\ h_1,\ h_2$ are some real constants. The right hand side term in (4) stands for that additional effect. Suppose, however, that the initial and boundary conditions are given in the form

$$\omega(0,\ x) = \sum_{k=1}^{N} a_k \sin\left(\frac{k\pi}{L}x\right), \quad \frac{\partial \omega(0,\ x)}{\partial t} = \sum_{k=1}^{N} a_k \sin\left(\frac{k\pi}{L}x\right),$$
(5)
$$\omega(t,\ 0) = \frac{\partial^2 \omega(t,\ 0)}{\partial x^2} = \omega(t,\ L) = \frac{\partial^2 \omega(t,\ L)}{\partial x^2} = 0.$$

The existence and uniqueness of the solution of (4)–(5) may be attained in the framework of the nonlinear perturbation theory for linear evolution equations again, see [9] and references therein. Thus, the solution of the Cauchy problem (4) — (5) has a similar to (3) structure

$$\omega(t,\, x) = \sum_{k=1}^{N} u_k(t)\sin(kx). \tag{6}$$

Substituting (6) in the integro-differential equation (4) we get after sampling the system of differential equations

$$\begin{aligned} &\ddot{u}_k(t) + \left[c_1 + \frac{h_1}{2}\sum_{l=1}^{N} l^2 u_l^2(t)\right] k^2 u_k(t) = \left[c_2 + \frac{h_2}{2}\sum_{l=1}^{N} l^2 u_l^2(t)\right] u_k(t),\\ &\dot{u}_k(0) = b_k, \quad u_k(0) = a_k,\\ &k = 1,\, \ldots,\, N. \end{aligned} \tag{7}$$

It is clear that the system (7) is equivalent to the *Hamiltonian system*

$$\begin{aligned} \dot{u}_n &= \frac{\partial H}{\partial v_n},\\ \dot{v}_n &= -\frac{\partial H}{\partial u_n},\\ n &= 1, \ldots, N \end{aligned} \tag{8}$$

with *Hamiltonian function*

$$H = \frac{1}{2}\sum_{n=1}^{N} v_n^2 + \frac{c_1}{2}\sum_{n=1}^{N} n^2 u_n^2 - \frac{c_2}{2}\sum_{n=1}^{N} u_n^2 + \frac{h_1}{8}\left(\sum_{n=1}^{N} n^2 u_n^2\right)^2 - \frac{h_2}{8}\left(\sum_{n=1}^{N} u_n^2\right)^2,$$

where the terms $\frac{c_1}{2}\sum_{n=1}^{N} n^2 u_n^2 - \frac{c_2}{2}\sum_{n=1}^{N} u_n^2 + \frac{h_1}{8}\left(\sum_{n=1}^{n} n^2 u_n^2\right)^2 - \frac{h_2}{8}\left(\sum_{n=1}^{N} u_n^2\right)^2$ and $\frac{1}{2}\sum_{n=1}^{N} v_n^2$ stand for potential and kinetic energy, correspondingly.

Our main result concerns analytical non-integrability of (8) in the case of two degrees of freedom.

Theorem 1.1. *Suppose the constants c_1, c_2, h_1, and h_2 satisfy*

$$\text{(I)}: \frac{c_2 - 4c_1}{c_2 - c_1} < 0 \qquad \textit{and} \qquad \text{(II)}: \sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}} \quad \textit{is not odd};$$

then the system

$$\begin{aligned} &\dot{u}_1 = v_1, \quad \dot{v}_1 = \left(\frac{h_2 - h_1}{2}\right) u_1^3 + \left(\frac{h_2}{2} - 2h_1\right) u_1 u_2^2 + (c_2 - c_1)u_1,\\ &\dot{u}_2 = v_2 \quad \dot{v}_2 = \left(\frac{h_2}{2} - 8h_1\right) u_2^3 + \left(\frac{h_2}{2} - 2h_1\right) u_1^2 u_2 + (c_2 - 4c_1)u_2 \end{aligned} \tag{9}$$

obtained from (8) *for* $N = 2$ *does not possess an additional holomorphic, functionally independent of* H *first integral.*

As it is easy to see, the theorem does not result in the substantial case of $c_2 = h_2 = 0$, which corresponds to the considered by Nishida case of free vibration of an elastic string, presented by (1). For that reason it is a subject of the next theorem.

Theorem 1.2. *If* $h_1 \neq 0$ *then the system*

$$\begin{aligned} \dot{u}_1 &= v_1, \quad \dot{v}_1 = -\frac{h_1}{2}u_1^3 - 2h_1u_1u_2^2 - c_1u_1, \\ \dot{u}_2 &= v_2, \quad \dot{v}_2 = -8h_1u_2^3 - 2h_1u_1^2u_2 - 4c_1u_2 \end{aligned} \tag{10}$$

does not possess holomorphic, functionally independent of the Hamiltonian H *first integral.*

Since the theorems above are obtained by means of the algebraic Ziglin's method, we shall present in the next section a brief summary of his technique. In section 3 the proof of Theorem 1.1 will be given in details. As an intermediate result, the solution of the Cauchy problem (5)–(6) will be found in the case $N = 1$. The proof of the Theorem 1.2 will be explained in short in section 4, where additional assertions concerning integrability of the system (9) will be stated. In the last section we shall derive also a conclusion for the algebraic non-integrability of (9) in the framework of the definition given there.

2. DESCRIPTION OF THE ZIGLIN'S METHOD

We shall state the two main Ziglin's theorems as they are originally formulated and proved in [11], nevertheless that for the proof of our main results we need quite a weak one than their versions which may be found in [7].

Let us consider the analytic Hamiltonian system

$$\dot{z} = v(z), \tag{11}$$

defined by Hamiltonian $H : M^{2n} \to C$. Let $\varphi(t)$ be a non-trivial solution of (11) and Γ be its phase curve. Consider the restrictions to $T_\Gamma M$ of equations in variations for equation (11):

$$\dot{\xi} = T(v)\xi, \quad \xi \in T_\Gamma M. \tag{12}$$

Let $F = T_\Gamma M/T\Gamma$ be the normal bundle of Γ, and $\pi : T_\Gamma M \to F$ be its projection. Equations

$$\dot{\eta} = \pi_*(T(v))(\pi^{-1}\eta), \quad \eta \in F, \tag{13}$$

induced by (12) are called *equations in normal variations.* These are Hamiltonian equations defined by the linear Hamiltonian $dH \circ \pi^{-1}$, which is induced by H. The level set $F_p = \{\eta \in F \mid dH \circ \pi^{-1} = p\}$, $p \in C$, of the integral $dH \circ \pi^{-1}$ is called *reduced phase space* for (13).

Consider the reduced equations in variations

$$\dot{\eta} = \pi * (T(v)(\pi^{-1}\eta)), \quad \eta \in F_p. \tag{14}$$

Let $x_0, x_1 \in \Gamma$. Then to each continuous path $\alpha: [0, 1] \to \Gamma$, $\alpha(0) = x_0$, $\alpha(1) = x_1$, corresponds symplectic transformations $g(\alpha): F_{p|_{x_1}} \to F_{p|_{x_2}}$, defined as follows.

Let $\Omega = \{(t, \varphi(t)\} \subset C \times M$ be the integral curve of the solution $z = \varphi(t)$ and the maps $P_\Gamma: (t, \varphi(t)) \to \varphi(t)$, $P_C: (t, \varphi(t)) \to t$ are the projections in M and C, respectively. Let $\hat{\alpha}: [0, 1] \to \Omega$ be the lift of α with respect to P_Γ, i. e. $P_\Gamma \hat{\alpha} = \alpha$. Then $g(\alpha): F_{p|_{x_1}} \to F_{p|_{x_2}}$ is the map in virtue of (14) at time $T = P_C \circ \hat{\alpha}: [0,1] \to C$. On account of the local single valuedness of solution of (14), the map $g(\alpha)$ does not change under the homotopy of α with fixed end points. When $x_0 = x_1$ we get an antihomomorphism $g: \pi_1(\Gamma) \to Aff(F_{p|_{x_0}})$ from the fundamental droup $\pi_1(\Gamma)$ of the phase curve Γ into the group $Aff(F_{p|_{x_0}})$ of affine transformations of the fiber $F_{p|_{x_0}}$. The image $G = g(\pi_1(\Gamma))$ of this antihomomorphism is called *reduced monodromy group*.

Definition 2.1 [2]. *A symplectic linear transformation $A: C^{2k} \to C^{2k}$ is called resonant if its eigenvalues $\lambda_1, \ldots, \lambda_k, \lambda_1^{-1}, \ldots, \lambda_k^{-1}$ satisfy an equation of the kind $\lambda_1^{m_1} \ldots \lambda_k^{m_k} = 1$, where $m_1, \ldots, m_k$ are integers for which $\sum_{i=1}^{k} m_i^2 \neq 0$.*

Theorem 2.1 [11]. *Suppose the monodromy group of Γ contains a nonresonant symplectic transformation g. The number of meromorphic first integrals of* (11) *in a connected neighbourhood of the curve Γ and which are functionally independent together with Hamiltonian, does not exceed the order of integrability of the monodromy group.*

The next theorem provides restrictive conditions in order the system (11) to be completely integrable.

Theorem 2.2 [11]. *Suppose that the monodromy group of the curve Γ contains a nonresonant transformation g. In order that the Hamiltonian system* (11) *has $n-1$ meromorphic first integrals in a connected neighbourhood of Γ, and which are functionally independent together with the Hamiltonian, it is necessary that any other transformation g' from the monodromy group has the same fixed point and transforms the set of eigendirections of g into itself. If none of the eigenvalues of g' form a regular polygon in the complex plane centred at the origin, then g and g' commute.*

3. PROOF OF THEOREM 1.1

To apply the Ziglin's method we have to find an elliptic solution of the Hamiltonian system (9). It is easy to see that such family of curves for (9) is given by

$$(15) \qquad \Gamma(c): \; v_1^2 = \left(\frac{h_2 - h_1}{4}\right) u_1^4 + (c_2 - c_1) u_1^2 + c, \quad u_2 = v_2 = 0.$$

We shall solve explicitly (15), and we shall point at a nonresonant transformation from the monodromy group associated to that solution. Furthermore, if the system (9) possesses an independent of its Hamiltonian first integral, then any other element of the monodromy group preserves its fixed point and the set of eigendirections. Our goal will be to establish that it does not match with the assumptions (I) and (II).

In the proof we shall consider that $h_2-h_1>0$ and $c_2-c_1>0$. This assumption is not restrictive. If it is not fulfilled, the solution has to be slightly modified. In any case, the solution is expressed by elliptic Jacobi's functions [6].

Lemma 3.1. *For $c>0$*

$$u_1(t)=\sqrt{\lambda_1}.\mathrm{sn}\left(\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t\right),$$

(16)
$$v_1(t)=\sqrt{c}.\mathrm{cn}\left(\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t\right)\mathrm{dn}\left(\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t\right),$$

$$u_2(t)=v_2(t)=0$$

is a non-trivial particular solution of (9), *where we denote by λ_1 and λ_2 the roots of $\left(\frac{h_2-h_1}{4}\right)\lambda^2+(c_2-c_1)\lambda+c=0$, $0<-\lambda_1\leqq-\lambda_2$. The elliptic constant k which is involved in the construction of the elliptic functions is given by $k=\sqrt{\frac{\lambda_1}{\lambda_2}}$.*

Remark 3.1. *For the elliptic constant k is required $0\leqq k\leqq 1$. Since in the proof we consider c close to 0, it may be assumed that λ_1 and λ_2 are real, which implies $k\in[0,\ 1]$.*

Proof. Equation (15) may be written as

$$\left(\frac{d\frac{u_1}{\sqrt{\lambda_1}}}{d\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t}\right)^2=\left(1-\left(\frac{u_1}{\sqrt{\lambda_1}}\right)^2\right)\left(1-k^2\left(\frac{u_1}{\sqrt{\lambda_1}}\right)^2\right).$$

For a fixed $k\in[0,\ 1]$ the function which solves the differential equation

$$\left(\frac{df(t)}{dt}\right)^2=(1-f^2(t))(1-k^2f^2(t))$$

is precisely defined. It is the Jacobi's function $\mathrm{sn}(t-t_0)$, where $t_0\in C$ is arbitrary [6]. Hence, (16) presents a particular solution as the lemma states.

Corollary 3.1. *For $N=1$*

$$\omega(t,\ x)=\sqrt{\lambda_1}\mathrm{sn}\left(\frac{\sqrt{\lambda_2(h_2-h_1)}}{2}t-t_0\right)\sin(x)$$

is the solution of the Cauchy problem (5)–(6). *The constant c — yielding λ_1 and λ_2 which of their turn are involved in the construction of* $\mathrm{sn}(\tau)$, *and the constant t_0 have to be evaluated from the initial conditions.*

Since $\mathrm{sn}(\tau)$ is a double periodic meromorphic function [6], the double periodic meromorphic function $u_1(t)$, with periods $T_1=\frac{8K(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ and $T_2=\frac{i4K'(k)}{\sqrt{\lambda_2(h_2-h_1)}}$, has simple poles $\frac{i2K'(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ and $\frac{i2K'(k)+4k(\varkappa)}{\sqrt{\lambda_2(h_2-h_1)}}$ in the parallelogram of periods. So, the domain of the family of solutions (16) is mapped as complex tori with two points removed. Furthermore, in order to reduce the domain of solution (16), we shall consider the involution $R: (u_1, v_1, u_2, v_2) \to (-u_1, -v_1, u_2, v_2)$. Then factorizing $\hat{\Gamma}(c) = \Gamma(c)/R$ and keeping in mind that $\mathrm{sn}(\tau + 2K(k)) = -\mathrm{sn}(\tau)$ for each $\tau \in C$ [6], we obtain that the domain of the family of curves is mapped as tori with one point removed. Let denote by M the phase space of (9), and by F_R the set of fixed points of the involution R, i. e. $F_R = \{(0, 0, u_2, v_2) | (u_2, v_2) \in C^2\}$. Then factorizing $M \setminus F_R$ in R we get the smooth symplectic manifold $\hat{M} = (M \setminus F_R)/R$. By that way, Hamiltonian H is mapped in Hamiltonian function $\hat{H}$ for the same system, but in the reduced phase space $\hat{M}$. It is obvious that if there exist two functionally independent holomorphic integrals for the system (9), these integrals are mapped in holomorphic functionally independent first integrals for the same system, which is considered yet in $\hat{M}$.

Due to the Ziglin's approach we found non-trivial solution defined over smooth symplectic manifold $\hat{M}$. Now we shall introduce local co-ordinates in $T_{z_0}\hat{M}$ fibers in such a way, that the obtaining of the reduced variational equations in a convenient for further investigation form will be assured. It is easy to see that

$$\xi_1 = v_1, \quad \eta_1 = -\left(\left(\frac{h_1 - h_2}{2}\right) u_1^3 + (c_1 - c_2)\right) u_1, \quad \xi_2 = 0, \quad \eta_2 = 0$$

is a tangent vector to $\hat{\Gamma}(c)$. Then for local co-ordinates in $T_{z_0}\hat{M}$ we may chose $\xi_1, \eta_1, \xi_2, \eta_2$. Since the restriction of differential $d\hat{H}$ over $T_{z_0}\hat{M}$ is $d\hat{H} = v_1 dv_1 + \left(\left(\frac{h_1 - h_2}{2}\right) u_1^3 + (c_1 - c_2)u_1\right) du_1$, i. e. it does not depend on ξ_2 and η_2, we choose ξ_2 and η_2 for local co-ordinates in the reduced phase space

$$F_p = \left\{(\xi_1, \eta_1, \xi_2, \eta_2) \in C^4 | d\hat{H}(\xi_1, \eta_1, \xi_2, \eta_2) = v_1 dv_1 + \left(\left(\frac{h_1 - h_2}{2}\right) u_1^3 + (c_1 - c_2)u_1\right) du_1 = p\right\}$$

along $\hat{\Gamma}(c)$. So, the following lemma is almost argued.

Lemma 3.2. *In the co-ordinates introduced above the reduced system in variations is written by*

$$\dot{\xi}_2 = \eta_2, \quad \dot{\eta}_2 = \left(c_2 - 4c_1 + \frac{h_2 - 4h_1}{2} u_1^2(t)\right) \xi_2. \tag{17}$$

Proof. In $(\xi_1, \eta_1, \xi_2, \eta_2)$ local co-ordinates in $T_{z_0}\hat{M}$ the equations in variations associated to the solution (16) are given by

$$\begin{aligned} \dot{\xi}_1 &= \eta_1, \quad \dot{\eta}_1 = \left(c_2 - c_1 + 3\frac{h_2 - h_1}{2} u_1^2(t)\right) \xi_1, \\ \dot{\xi}_2 &= \eta_2, \quad \dot{\eta}_2 = \left(c_2 - 4c_1 + \frac{h_2 - 4h_1}{2} u_1^2(t)\right) \xi_2. \end{aligned} \tag{18}$$

Since for local co-ordinates of the restricted over F_p normal bundle to $\hat{\Gamma}_{z_0}\hat{M}$ were chosen ξ_2 and η_2, the reduced equations in variations are just the ones the lemma states.

Now we shall find a nonresonant transformation from the monodromy group of (17). Let α_1 be a path over $\hat{\Gamma}(c)$ which corresponds to adding of the imaginary period $\dfrac{4K(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ of $u_1^2(t)$, and α_2 be a path over $\hat{\Gamma}(c)$ which corresponds to adding of the real period $\dfrac{i4K'(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ of $u_1^2(t)$. Let $g(\alpha_1)$ and $g(\alpha_2)$ be the transformations of monodromy which correspond to the closed paths α_1 and α_2 on $\hat{\Gamma}(c)$, respectively.

Lemma 3.3. *The transformation $g(\alpha_1)$ is nonresonant for c close to* 0.

Proof. In order to show that $g(\alpha_1)$ is a nonresonant transformation we shall begin with computing its eigenvalues for $c = 0$. Let consider the phase curve $\hat{\Gamma}(0)$ in $\hat{M}$. It is easy to see that the partial solution (16) for $c = 0$ degenerates in

$$
(19)\qquad \begin{aligned}
u_1(t) &= 2\sqrt{\frac{c_2-c_1}{h_2-h_1}}.\sinh^{-1}(\sqrt{c_2-c_1}.t),\\
v_1(t) &= \dot{u}_1(t),\\
u_2(t) &= v_2(t) = 0.
\end{aligned}
$$

Since we consider the solution (16) in its reduced range of values $\hat{M}$, its real period goes to infinity, and its imaginary period changes to $\dfrac{\pi i}{\sqrt{c_2-c_1}}$, which is the period of the degenerated solution (19) [6]. It makes sense to say that the domain of the solution (16) degenerates in the domain of the solution (19), which is a closed at infinity cylinder. Now, for $c = 0$, $g(\alpha_1)$ is the transformation of monodromy for the system

$$
(20)\qquad \dot{\xi}_2 = \eta_2, \quad \dot{\eta}_2 = \left(c_2 - 4c_1 + 2(c_2-c_1)\frac{h_2-4h_1}{h_2-h_1}\sinh^{-2}(\sqrt{c_2-c_1}.t)\right)\xi_2.
$$

The computation of the eigenvalues of $g(\alpha_1)$ will be done along a closed trajectory from the same homotopic class to which α_1 belongs. Let decompose $t = s+T$, where T is real and s is purely imaginary, and let denote for convenience the function $\sinh^{-2}(\sqrt{c_2-c_1}.(s+T))$ by $p(s,\,T)$. It is easy to see that $p(s,\,T)$ is periodic in s with a period $\dfrac{\pi i}{\sqrt{c_2-c_1}}$, and also $\lim_{T\to\infty} p(s,\,T) = 0$.

Let $\hat{\alpha}_1(T)$ be a trajectory over $\hat{\Gamma}(0)$ defined by (19) with argument t for which $t = i\tau + T$, $T \in R$ is fixed, and τ changes from 0 to $\dfrac{\pi}{\sqrt{c_2-c_1}}$. Now we can compute the eigenvalues of $g(\alpha_1)$ along $\hat{\alpha}_1(T)$. For $T \to \infty$ the system (20) reduces to

$$
\dot{\xi}_2 = \eta_2, \quad \dot{\eta}_2 = (c_2 - 4c_1)\xi_2.
$$

Since the eigenvalues of the corresponding matrice are $\pm\sqrt{c_2-4c_1}$, the principal matrix solution is written as

$$
\Phi(s) = \begin{bmatrix} \exp(\sqrt{c_2-4c_1}.s & 0 \\ 0 & \exp(-\sqrt{c_2-4c_1}.s) \end{bmatrix}.
$$

The transformation of $g_{|T\to\infty}(\hat{\alpha}_1(T))$ is determined by the equality

$$\Phi\left(s+\frac{i\pi}{\sqrt{c_2-c_1}}\right)=g_{|T\to\infty}(\hat{\alpha}_1(T))\Phi(s).$$

Hence, for $c=0$ the eigenvalues of $g(\alpha_1)$ are

$$\exp\left(\pm i\pi\sqrt{\frac{c_2-4c_1}{c_2-c_1}}\right). \tag{21}$$

When c is close to 0, the eigenvalues of $g(\alpha_1)$ will be close to (21). Hence, they can not lie on the unit circle, because by (I) $\pm i\pi\sqrt{\frac{c_2-4c_1}{c_2-c_1}}$ is a non-zero real quantity. Therefore, the eigenvalues of $g(\alpha_1)$ are not roots of unity and $g(\alpha_1)$ is a nonresonant transformation.

Now we shall compute the eigenvalues of the commutator of $g(\alpha_1)$ and $g(\alpha_2)$.

Lemma 3. 4. *For each complex c the eigenvalues of the commutator* $[g(\alpha_1), g(\alpha_2)]$ of $g(\alpha_1)$ and $g(\alpha_2)$ *are just*

$$\exp\left(i\pi\left(1\pm\sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}}\right)\right). \tag{22}$$

Proof. The commutator $[g(\alpha_1),\, g(\alpha_2)] = g(\alpha_1)g(\alpha_2)g^{-1}(\alpha_1)g^{-1}(\alpha_2)$ corresponds to one winding around the regular-singular point $a(c)=\dfrac{iK'(k)}{\sqrt{\lambda_2(h_2-h_1)}}$ of the second order Fuchsian equation

$$\ddot{\xi}_2+f(t)\xi_2=0,\quad f(t)=-(c_2-4c_1+\frac{h_2-4h_1}{2}u_1^2(t)). \tag{23}$$

For the equation $\ddot{\xi}(t)+\dfrac{p(t)}{t-t_0}\dot{\xi}(t)+\dfrac{q(t)}{(t-t_0)^2}\xi(t)=0$, where $p(t)$ and $q(t)$ are holomorphic near $t_0\in C$ functions, the eigenvalues of the transformation of monodromy, which corresponds to a loop around t_0, are just $\exp(i2\pi\rho_{1,2})$ [3], where $\rho_{1,2}$ are the roots of the indicial equation

$$\rho(\rho-1)+p(t_0)\rho+q(t_0)=0. \tag{24}$$

Recall that

$$\mathrm{sn}(\tau)=\frac{1}{k(\tau-iK'(k))}+O(1)$$

[6], so we get easily that

$$f(t)=-2\frac{h_2-4h_1}{h_2-h_1}\frac{1}{(t-a(c))^2}+O\left(\frac{1}{t-a(c)}\right),$$

and a simple computation gives the eigenvalues.

We are now at the point to prove Theorem 1.1.

Assume that the system (9) has an additional functionally independent of H first integral, which is holomorphic in a neighbourhood of $\Gamma(c)$. Then the corresponding system on $\hat{M}$ has an additional, functionally independent of $\hat{H}$, holomorphic in a neighbourhood of $\hat{\Gamma}(c)$ first integral. By Lemma 3.3 $g(\alpha_1)$ is a nonresonant element of the monodromy group associated to the phase curve $\hat{\Gamma}(c)$. Therefore, by Theorem 2.2, the other element of the monodromy group preserves the fixed point and the set of eigendirections of $g(\alpha_1)$. Hence, $g(\alpha_2)$ either keeps or interchanges the two eigendirections of $g(\alpha_1)$. We shall show that neither of these opportunities takes place.

If we suppose that $g(\alpha_2)$ keeps the eigendirections of $g(\alpha_1)$, i. e. $g(\alpha_1)$ and $g(\alpha_2)$ commute, it follows that $[g(\alpha_1), g(\alpha_2)] = \mathrm{id}$, which is a contradiction because the eigenvalues of the commutator are not equal to unity by Lemma 3.4 and the assumption (II).

Let suppose that $g(\alpha_2)$ interchanges the eigendirections of $g(\alpha_1)$. In an appropriate basis $g(\alpha_1)$ is written as

$$g(\alpha_1) = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{bmatrix}.$$

Hence, in the same basis $g(\alpha_2)$ looks like

$$g(\alpha_2) = \begin{bmatrix} 0 & \delta \\ \nu & 0 \end{bmatrix}.$$

Since $g(\alpha_2)$ is a symplectic transformation, $\delta = -\nu^{-1}$. Therefore,

$$[g(\alpha_1), g(\alpha_2)] = \begin{bmatrix} \gamma^2 & 0 \\ 0 & \gamma^{-2} \end{bmatrix}$$

and the quotient

$$\gamma^{\pm 2} = \exp\left(i\pi\left(1 \pm \sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}}\right)\right) \tag{25}$$

for the eigenvalues γ and γ^{-1} of $g(\alpha_1)$ must be met. For $\sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}}$ being real (25) can not be fulfilled because $\gamma^{\pm 2}$ does not lie on the unit circle as we saw in the proof of Lemma 3.3. For $\sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}}$ being purely imaginary (25) can not be fulfilled also, because $\exp\left(i\pi\left(1 \pm \sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}}\right)\right) < 0$, whereas the real parts of $\gamma^{\pm 2}$ are close to $\exp\left(\pm i2\pi\sqrt{\frac{c_2 - 4c_1}{c_2 - c_1}}\right) > 0$. Hence, the quotient (25) can not be met, and that contradicts the requirement for $g(\alpha_2)$ to keep the set of eigendirections of $g(\alpha_1)$, which was inferred by the assumption that there exists an additional holomorphic integral for (9).

This concludes the proof of Theorem 1.1.

4. PROOF OF THEOREM 1.2 AND SOME ADDITIONAL REMARKS FOR NON-INTEGRABILITY

We shall prove Theorem 1.2 and state additional assertions for analytical non-integrability of (9). Also, we shall show the algebraic non-integrability of (9).

Proof of Theorem 1.2.

Since the proof is going in a similar to the proof of Theorem 1.1 manner, we shall refer to some of the results obtained in section 3.

The partial solution (16) is now slightly modified to look as

$$u_1(t) = \sqrt{\lambda_1}.\mathrm{sn}\left(\frac{\sqrt{-\lambda_2 h_1}}{2}.t\right),$$
$$v_1(t) = \sqrt{c}.\mathrm{cn}\left(\frac{\sqrt{-\lambda_2 h_1}}{2}.t\right).\mathrm{dn}\left(\frac{\sqrt{-\lambda_2 h_1}}{2}.t\right), \tag{26}$$
$$u_2(t) = v_2(t) = 0.$$

Its phase curve, in virtue of the involution R, will be considered in $\hat{M}$. Having in mind Lemma 3.2, the reduced equations in variations have to be written as

$$\dot{\xi}_2 = \eta_2, \quad \dot{\eta}_2 = -(4c_1 + 2h_1 u_1^2(t))\xi_2, \tag{27}$$

where $u_1(t)$ is defined in (26). The elements $g(\alpha_1)$ and $g(\alpha_2)$ of the monodromy group associated with (27) are defined in a similar way also, i. e. as transformations which correspond to adding the imaginary and real periods of the solution (26), respectively. By Lemma 3.4 g determined as a commutator of $g(\alpha_1)$ and $g(\alpha_2)$ is nonresonant.

Let assume now that (10) has two functionally independent holomorphic integrals in a neighbourhood of the complex curve $H(u_1, v_1, u_2, v_2) = 2c$ for c close to 0. By Theorem 2.1 the reduced system in variations has a non-trivial rational first integral. Since g is nonresonant, $g(\alpha_1)$ preserves its fixed point and keeps or exchanges its eigendirections. We shall prove that $g(\alpha_1)$ can not keep the set of eigendirections of g. In the proof of Lemma 3.3 we saw that for c close to 0 the eigenvalues of $g(\alpha_1)$ were close to $\exp\left(\pm i\pi\sqrt{\frac{c_2 - 4c_1}{c_2 - c_1}}\right)$. Since now $c_2 = 0$, they are close to $\exp(\pm i2\pi) = 1$. Therefore, they can not form a regular polygon centred at the origin. Then, having in mind Theorem 2.2, it follows that $g(\alpha_1)$ must keep the eigendirections of g. Let consider now $g(\alpha_2)$. It can not keep the eigendirections of g, otherwise it commutes with g and, therefore, with $g(\alpha_1)$, which is a contradiction with $g = g(\alpha_1)g(\alpha_2)g^{-1}(\alpha_1)g^{-1}(\alpha_2) \neq \mathrm{id}$. Hence, $g(\alpha_2)$ exchanges the eigendirections of g and $g(\alpha_1)$. Then, as in the proof of Theorem 1.1, we easily obtain that the eigenvalues of g are squares of the eigenvalues of $g(\alpha_1)$. But it is not the case, because for c close to 0 the squares of the eigenvalues of $g(\alpha_1)$ are close to $\exp(\pm i4\pi)$, whereas the eigenvalues of g are just $\exp(i\pi(1 \pm \sqrt{33}) \neq 1$. This contradiction proves Treorem 1.2.

The results from section 3 allow a simple criterion [4] for algebraic complete integrability to be applied for the system (9). We shall examine for an algebraic integrability in the sense of the definition given by Adler and Moerbeke in [1].

Definition 4.1. *A Hamiltonian system is called algebraically completely integrable, if its first integrals are all rational functions and their level sets in the complex domain are complex tori.*

Proposition 4.1. *Hamiltonian system (9) is algebraically non-integrable, unless*

$$\sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}}$$

is odd.

Proof. For the proof of this proposition it is enough to find a phase curve, along which the equations in variations have a multi-valued solution [4]. Such phase curve is represented by the partial solution (16) and the corresponding equations in variations are (18). One partial solution of (18) is given by

$$(\xi_1(t),\ \eta_1(t),\ \xi_2(t),\ \eta_2(t)), \tag{28}$$

where $\xi_1(t)=\eta_1(t)=0$ and $(\xi_2(t),\eta_2(t))$ is defined as a solution of (17). The multi-valuedness of (28) is established by examining the roots of the indicial equations (24) [3]. Since the roots

$$\rho_{1,2}=\frac{1\pm\sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}}}{2}$$

are not integers, the partial solution (28) is multi-valued, which proves the proposition.

At the end, we state an amplified version of the Theorem 1.1. Following Ziglin's analysis, demonstrated in the proof of Theorem 1.1 in which a similar technic to that developed in [11], [5], [10] was used, it is easy to obtain a resembling statement for the analytical non-integrability of the system (9). One way to do it, is to find another partial solution for the system (9), or to consider such a level set for the Hamiltonian, in a neighbourhood of which the presence of another nonresonant element of the associated monodromy group is assured.

Theorem 4.1. *If*

(i) $$\frac{c_2-4c_1}{c_2-c_1}<0 \quad \textit{and} \quad \sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}} \quad \textit{is not odd,}$$

or

(ii) $$\frac{c_2-c_1}{c_2-4c_1}<0 \quad \textit{and} \quad \sqrt{1+8\frac{h_2-4h_1}{h_2-16h_1}} \quad \textit{is not odd,}$$

or

(iii) $$\frac{c_2-4c_1}{c_2-c_1}-\frac{h_2-4h_1}{h_2-h_1}>0 \quad \textit{and} \quad \sqrt{1+8\frac{h_2-4h_1}{h_2-h_1}} \quad \textit{is not odd,}$$

or

(iv) $$\frac{c_2-4c_1}{c_2-c_1}-\frac{h_2-4h_1}{h_2-16h_1}>0 \quad \textit{and} \quad \sqrt{1+8\frac{h_2-4h_1}{h_2-16h_1}} \quad \textit{is not odd,}$$

than the Hamiltonian system (9) *does not possess an additional integral which is holomorphic and functionally independent of the Hamiltonian.*

An empirical method to test for analytical integrability of a system (9) is to compute Poincare surface of section [8]. It is done by keeping track on the successive points of intersections of a given trajectory of the system with a fixed plane. If the points of intersection form a regular curve, a conjecture for integrability is implied. The other kind of behavior is observed when the orbit is not quasi periodic: the points on the surface of section fill an area, which implies for non-integrability. The numerical investigation for integrability of (9) carried out by the standard 4–5th order Runge-Kutta method revealed chaotic regions which matches with the results obtained analytically. The non-integrability of the system under consideration has as consequences the strong dependence on the initial conditions of the orbit of the Cauchy problem (4)–(5) and its chaotic behavior in the course of time.

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