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НА

СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

Том 100

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L'UNIVERSITE DE SOFIA “ST. KLIMENT OHRIDSKI”

FACULTE DE MATHEMATIQUES ET INFORMATIQUE

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Факултет по математика и информатика

Address for correspondence:

Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5, Blvd J. Bourchier, P.O. Box 48
BG-1164 Sofia, Bulgaria

Fax xx(359 2) 8687 180
Electronic mail:
annuaire@fmi.uni-sofia.bg

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FACTORIZATIONS OF SOME SIMPLE LINEAR GROUPS

ELENKA GENTCHEVA

In this paper we have considered finite simple groups G which can be represented as a product $G = AB$ of two of their proper non-Abelian simple subgroups A and B. Any such representation is called a (simple) factorization of G. Supposing that G belongs to the infinite series of linear groups with some restrictions to the dimension of the natural vector space onto which G acts we have determined all the factorizations of G.

Keywords: Finite simple groups, groups of Lie type, factorizations of groups

2000 MSC: main 20D06, 20D40, secondary 20G40

1. INTRODUCTION

Let G be a finite (simple) group. We are interested in the factorizations of G into the product of two simple subgroups. In the present work we suppose that G is the simple linear group $L_n(q)$ and start our investigation of this series of groups in case that n is at most 7. The results obtained are included in the following

Theorem. *Let $G = L_n(q)$ with $2 \leq n \leq 7$. Suppose $G = AB$ where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:*

(1) $n = 2,\ q = 9$ *and* $A \cong B \cong A_5$;

(2) $n = 4,\ q = 2$ *and* $A \cong L_3(2)$, $B \cong A_6$ *or* A_7;

(3) $n = 4,\ q > 2, q \not\equiv 1 \pmod 3$ *and* $A \cong L_3(q), B \cong PSp_4(q)$;

(4) $\quad n = 6,\ q \not\equiv 1 \pmod 5$ *and* $A \cong L_5(q), B \cong PSp_6(q)$*;*

(5) $\quad n = 6,\ q = 2^s > 2, s \not\equiv 0 \pmod 4$ *and* $A \cong L_5(q), B \cong G_2(q)$.

The factorizations of the groups $L_2(q)$ have been determined in [7]. This gives (1) in the theorem. The groups $L_3(q)$ have no factorizations, see [2]. The factorizations of the groups $L_4(2) \cong A_8$ and $L_4(q)$ (q odd, $q \not\equiv 1 \pmod 8$) have been determined in [10]. This leads to the (2) and (3) (q odd, $q \not\equiv 1 \pmod 8$) in the theorem. Some isolated linear groups as $L_4(4), L_5(2)$ and $L_6(2)$ have been treated in [1]and [3]. It has been proved that the groups $L_4(4)$ and $L_5(2)$ have no factorizations whilst the group $L_6(2)$ has one factorization listed in (4) (with $q = 2$) in the theorem.

The factorizations of all the classical simple groups into the product of two maximal subgroups (so called maximal factorizations) have been determined in [9]. Particularly, an explicit list of the maximal factorizations of the groups $L_n(q)$ have also been given in [9]. We shall make use of this result here.

Note that,using the result of the above theorem especially for $n = 4$ ($G = L_4(q)$) and the results in previously published papers [4], [5] and [6], we have finished determination of the factorizations (with two proper simple subgroups) of all the finite simple groups of Lie type of Lie rank three. Indeed, only the groups $PSU_7(q)$ and $P\Omega_8^-(q)$ of Lie rank three are not covered from the mentioned results; but according to [9] these groups have no maximal factorizations and so it follows they have no factorizations with any two proper factors A and B as well.

In our considerations we shall freely use the notation and basic information on the finite (simple) classical groups given in [8]. Let V be the n-dimensional vector space over the finite field $GF(q)$ on which $G = L_n(q)$ acts naturally, and let P_k be the stabilizer in G of a k-dimensional subspace of V. From Proposition 4.1.17 in [8] we can obtain the structure of P_k. In particular, it follows that $P_1 \cong P_{n-1} \cong \{[q^{n-1}] : GL_{n-1}(q)\}/Z_{(n,q-1)}$. From this it follows immediately that P_1 ($\cong P_{n-1}$) contains a subgroup isomorphic to $L_{n-1}(q)$ if and only if $(n-1, q-1) = 1$.

If a, b are positive integers and $(a, b) = 1$, then $Ord_a(b)$ denotes the multiplicative order of b modulo a (i.e. the least positive integer n with $b^n \equiv 1 \pmod a$).

The following lemma is needed in the proof of the theorem.

Lemma 1.1 (see [9]). *Let q be a prime power and n a positive integer. Then there exists a prime r such that $Ord_r(q)$ unless $n = 6$ and $q = 2$ or $n = 2$ and q a Mersenne prime.*

Such a prime r is called a *primitive prime divisor* of $q^n - 1$.

2. PROOF OF THE THEOREM

Let $G = L_n(q)$, where $q = p^s$ and p is a prime. In our assumptions here $2 \leq n \leq 7$, and $G = AB$ where A, B are proper non-Abelian simple subgroups of G. According to the information about known factorizations of G provided above, it remains to treat G in the cases $n = 4$, 5, 6 and 7. If $G = L_4(q)$ we only suppose $q > 2$ and no other restrictions on q will be applied, as for $G = L_5(q)$ or $L_6(q)$ we assume that $q > 2$ as well. The list of maximal factorizations of G is given in [9]. In case that $G = L_n(q)$ with $n = 5$ or $n = 7$ only one maximal factorization appears with one factor a (maximal) subgroup of G isomorphic to $\{Z_{q^n-1/q-1}.n\}/Z_{(n,q-1)}$. Obviously, there is no choice for one of the groups A and B to be a non-Abelian simple subgroup of G. Now we proceed with the group $G = L_n(q)$ where $n = 4$ or $n = 6$ and choose (by Lemma 1.1) a primitive prime divisor of $p^{sn} - 1$ (recall that if $n = 6$ then $q > 2$) to be a divisor of $|B|$. Using the list of maximal factorizations in [9], by order considerations, we come to the following possibilities:

1) $n = 4$ or $n = 6$ and $A \cong L_{n-1}(q)$ (in P_1), $B \cong PSp_n(q)$ with $(n-1, q-1) = 1$;

2) $n = 6$ and $A \cong L_5(q)$ (in P_1), $B \cong G_2(q)$ with $q = 2^s > 2$, $s \not\equiv 0 \pmod 4$;

3) $n = 6$ and $A \cong L_5(q)$ (in P_1), $B \cong L_3(q^2)$ with $(5, q-1) = 1$.

We consider these possibilities case by case.

Case 1. These are the factorizations in (3) and (4) of the theorem. It remains to show that these factorizations actually exist. From Proposition 3.3 in [10] we have

$$SL_n(q) = SL_{n-1}(q).Sp_n(q)$$

with natural embeddings of $SL_{n-1}(q)$ and $Sp_n(q)$ in $SL_n(q)$. Moreover, the intersection of these naturally embedded subgroups $SL_{n-1}(q)$ and $Sp_n(q)$ is a subgroup isomorphic to $Sp_{n-2}(q)$ with natural embedding in $SL_n(q)$, too. Factoring out by $Z(SL_n(q))$, we obtain the factorizations in (3) and (4), as $SL_{n-1}(q) \equiv L_{n-1}(q)$ (by the condition $(n-1, q-1) = 1$).

Case 2. Here $q = 2^s > 2$, $s \not\equiv 0 \pmod 4$, and from the previous case it follows that $G = A.B_1$ where $A \cong L_5(q)$, $B_1 \cong PSp_6(q)$, and $A \cap B_1 \cong PSp_4(q)$. In [5] we have proved that $B_1 = (A \cap B_1).B$ where $B \cong G_2(q)$ with an explicit construction in B_1; also $(A \cap B_1) \cap B$ $(= A \cap B) \cong L_2(q)$. This leads, by order considerations, to the factorization $G = A.B$ in (5) of the theorem.

Case 3. This case is similar to one of those considered in [4]. Denote $D = A \cap B$; then $|D| = q(q^4 - 1).(6, q-1)/(3, q^2-1)$ (recall $(5, q-1) = 1$). By the known subgroup structure of $L_3(q^2)$, it follows that D is contained in a subgroup of B isomorphic to

$$H = \left\{ \left(\begin{array}{c|cc} a & b & c \\ \hline 0 & & \\ & A & \\ 0 & & \end{array} \right) \mid a,b,c \in GF(q^2);\ A \in GL_2(q^2),\ a.\det A = 1 \right\} \Big/ \langle \omega E \rangle$$

where ω is an element of order $(3, q^2-1)$ in $GF(q^2)$. Further, $H = FK$ and $F \lhd H, F \cap K = 1$ where

$$F = \left\{ \left(\begin{array}{c|cc} 1 & b & c \\ \hline 0 & & \\ & E & \\ 0 & & \end{array} \right) \mid b,c \in GF(q^2) \right\} \cong E_{q^4},$$

$$K = \left\{ \left(\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & & \\ & A & \\ 0 & & \end{array} \right) \mid a \in GF(q^2);\ A \in GL_2(q^2),\ a.\det A = 1 \right\} \Big/ \langle \omega E \rangle$$

$\cong GL_2(q^2)/Z_{(3,q^2-1)}$.

Suppose that $T = D \cap F \neq 1$. Then $T \lhd D$ and $T \cong E_{p^k}$ where $p \leq p^k \leq q$. The centralizer of any non-identity p-element in $L_3(q^2)$ has order dividing $q^6(q^2-1)$. Hence $|C_D(T)|$ divides $q(q^2-1).(6, q-1)/(3, q^2-1)$. Then $|D/C_D(T)|$ is divisible by q^2+1. However, $D/C_D(T)$ is a subgroup of $Aut(T) \cong GL_k(p)$, so

$$|GL_k(p)| = p^{k(k-1)/2}.(p-1)\cdots(p^k-1)$$

must be divisible by q^2+1 which (in view of $p^k \leq q$) contradicts Lemma 1.1. Indeed,using this lemma we can choose a primitive prime divisor of $p^{4s}-1$ dividing q^2+1 but not dividing the order of $GL_k(p)$, which is impossible.

Thus $D \cap F = 1$ and hence D is isomorphic to a subgroup of $H/F \cong K$. Of course, K contains a subgroup $L \cong SL_2(q^2)$ of index $(q^2-1)/(3, q^2-1)$ and then $D \cap L$ is a proper subgroup of L of order divisible by $q(q^2+1).(6, q-1)$. It follows that $L_2(q^2)$ has a proper subgroup of order divisible by $q(q^2+1)$ which (for $q \geq 3$) contradicts the structure of $L_2(q^2)$.

This completes the proof.

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Department of Mathematics
Technical University
Varna
BULGARIA
E-mail: elenkag@abv.bg

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A NOTE ON THE SECTIONAL CURVATURE

STEFKA T. HINEVA

The type of the matrices of the second fundamental form of a submanifold M^n in a Riemannian manifold N^{n+p} is given, when the equalities in the estimates of the sectional curvature $K_M(\sigma)$ of M^n by means of its mean curvature H and length S of the second fundamental form hold. It is shown that the equality in the upper estimate of the sectional curvature $K_M(\sigma)$ of M^n in a space form $N^{n+p}(c)$ is achieved when the normal bundle of M^n is flat and M^n is a product submanifold of the type $M^2 \times M^{n-2}$ or $M^2 \times E^{n-2}$ (cylinder), where M^2, M^{n-2} are umbilical manifolds, E^{n-2} — Euclidean. It is also shown that among all the submanifolds in $N^{n+p}(c)$ which pass through its point x and have at this point the same $S(x)$, the product submanifold $M^n = M^2 \times E^{n-2}$ has at x the biggest sectional curvature $K_M(\sigma)(x) = c + \frac{1}{2}S(x)$.

Keywords: Sectional curvature, length of the second fundamental form, mean curvature, product submanifold, eigenvalues

2000 MSC: 53B25, 53C40

1. PRELIMINARIES

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We choose a local frame of orthonormal fields $e_1, \dots, e_{n+p}$ in N^{n+p} such that, restricted to M^n, the vectors $e_1, \dots, e_n$ are tangent to M^n and the remaining vectors $e_{n+1}, \dots, e_{n+p}$ are normal to M^n.

We shall use the following convention on the ranges of the indices:

$$1 \leq i, j, k, \dots \leq n; \qquad 1 \leq \alpha, \beta, \gamma, \dots \leq p.$$

We denote the second fundamental form h: $T_xM^n \times T_xM^n \to T_x^\perp M^n$ on M^n for $x \in M^n$ where T_xM^n is the tangent space of M^n at x and $T_x^\perp M^n$ is the normal space to M^n at x, by its components h_{ij}^α with respect to the frame $e_1, \ldots, e_{n+p}$.

We call

$$H = \sum_\alpha \frac{1}{n} h^\alpha e_\alpha, \qquad H^2 = \frac{1}{n^2} \sum (h^\alpha)^2, \quad \text{where} \quad h^\alpha = \sum_i h_{ii}^\alpha \tag{1.1}$$

the *mean curvature vector* of M^n.

The square S of the length of the second fundamental form is given by:

$$S = \sum_\alpha \left[\sum_{i,j} (h_{ij}^\alpha)^2 \right] \tag{1.2}$$

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, i.e. $N(A) = \operatorname{trace} A.A^t = \sum_{i,j} (a_{ij})^2$ and

$$|\operatorname{trace} A| \leq \sqrt{n.N(A)}. \tag{1.3}$$

S and h^α are independent of our choice of orthonormal basis.

Let X and Y be a pair of orthonormal vectors tangent to M^n at a point $x \in M^n$, and let us suppose that the local frame $e_1, \ldots, e_{n+p}$ $(*)$ is so chosen that X and Y coincide with two arbitrary vectors of that frame. Let $X = e_{n-1}$, $Y = e_n$. Then the sectional curvature $K_M(\sigma)$ of M^n at the point x for the plane σ spanned by X and Y is written as follows:

$$K_M(\sigma) = \overline{K}_N(\sigma) + \sum_\alpha \left[h_{n-1,n-1}^\alpha h_{nn}^\alpha - (h_{n-1,n}^\alpha)^2 \right] \tag{1.4}$$

where $\overline{K}_N(\sigma)$ is the sectional curvature of N^{n+p}.

This paper is a continuation of the papers [1] and [2] where we proved that the sectional curvature $K_M(\sigma)$ of a submanifold M^n in a Riemannian manifold N^{n+p} at a point $x \in M^n$ satisfies the following inequalities:

$$K_M(\sigma) \leq K_N(\sigma) + \frac{4-n}{2} H^2 + \frac{n-2}{2n} S + \sqrt{\frac{2(n-2)}{n} H^2 (S - nH^2)}, \tag{1.5}$$

$$K_M(\sigma) \geq K_N(\sigma) + \frac{n^2}{2(n-1)} H^2 - \frac{1}{2} S \quad \text{when} \quad \frac{n^2}{n-1} H^2 - S < 0, \tag{1.6$_1$}$$

$$K_M(\sigma) \geq K_N(\sigma) \qquad \text{when} \quad \frac{n^2}{n-1} H^2 - S \geq 0. \tag{1.6$_2$}$$

The purpose of this paper is to show for which submanifolds the equalities in (1.5), $(1.6)_1$ and $(1.6)_2$ are fulfilled. For this purpose we will formulate Theorem 1.1 from [2] more precisely describing the types of the matrices (h_{ij}^α) of the second fundamental form of M^n with respect to the suitably chosen orthonormal basis $e_1, \dots, e_n, \dots, e_{n+p}$ (*), when these equalities are achieved:

Theorem 1.1. *Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. For the sectional curvature $K_M(\sigma)$ of the 2-plane section σ spanned by the two orthonormal vectors X and Y tangent to M^n at a non-totally geodesic point $x \in M^n$ we have (1.5), $(1.6)_1$ and $(1.6)_2$, where $K_N(\sigma)$ denotes the sectional curvature of N^{n+p}.*

The equality in (1.5) hold only when either $n = 2$ or if $n \geq 3$ all the matrices (h^α_{ij}) of the second fundamental form with respect to the orthonormal basis $e_1, \dots, e_{n-1} = X, e_n = Y, \dots, e_{n+p}$ (*) are of the form

$$\begin{pmatrix} \lambda_1^\alpha & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \lambda_1^\alpha & 0 & 0 \\ 0 & \dots & 0 & \lambda_n^\alpha & 0 \\ 0 & \dots & 0 & 0 & \lambda_n^\alpha \end{pmatrix} \tag{1.7}$$

where

$$\lambda_1^\alpha = \frac{h^\alpha}{n} \mp \frac{1}{n}\sqrt{\frac{2[nS^\alpha - (h^\alpha)^2]}{n-2}}; \quad \lambda_n^\alpha = \frac{h^\alpha}{n} \pm \frac{1}{n}\sqrt{\frac{(n-2)[nS^\alpha - (h^\alpha)^2]}{2}}.$$

The equalities in $(1.6)_1$ and $(1.6)_2$ are fulfilled if and only if either $n = 2$ or when $n \geq 3$ the corresponding matrices (h^α_{ij}) are the following

$$\begin{pmatrix} a_1^\alpha & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & a_1^\alpha & 0 & 0 \\ 0 & \dots & 0 & \dfrac{a_1^\alpha \mp c^\alpha}{2} & a^\alpha_{n-1,n} \\ 0 & \dots & 0 & a^\alpha_{n-1,n} & \dfrac{a_1^\alpha \pm c^\alpha}{2} \end{pmatrix} \tag{$(1.8)_1$}$$

where

$$a_1^\alpha = \frac{h^\alpha}{n-1}; \quad a^\alpha_{n-1,n} \leq \frac{S^\alpha}{2} + \frac{(3-2n)(h^\alpha)^2}{4(n-1)^2},$$

$$c^\alpha = \frac{1}{n-1}\sqrt{(3-2n)(h^\alpha)^2 + 2(n-1)^2[S^\alpha - 2(a^\alpha_{n-1,n})^2]},$$

and

$$\begin{pmatrix} h^\alpha_{11} & h^\alpha_{12} & \dots & h^\alpha_{1,n-1} & h^\alpha_{1n} \\ h^\alpha_{12} & h^\alpha_{22} & \dots & h^\alpha_{2,n-1} & h^\alpha_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ h^\alpha_{1,n-1} & h^\alpha_{2,n-1} & \dots & 0 & 0 \\ h^\alpha_{1n} & h^\alpha_{2n} & \dots & 0 & 0 \end{pmatrix}. \tag{$(1.8)_2$}$$

To find the view (1.7), $(1.8)_1$ and $(1.8)_2$ of the matrices (h^α_{ij}) we apply for them the basic Lemma 2.1 from [1] and obtain that with respect to the suitably chosen orthonormal basis $(*)$ the upper and the lower bounds of the functions

$$h^\alpha_{n-1,n-1}h^\alpha_{nn} - (h^\alpha_{n-1,n})^2, \quad \alpha = 1, 2, \dots p, \tag{1.9}$$

appearing in the expression (1.4) for the sectional curvature $K_M(\sigma)$, namely,

$$\begin{aligned} h^\alpha_{n-1,n-1}h^\alpha_{n,n} - (h^\alpha_{n-1,n})^2 \le \tfrac{1}{2n^2}\big\{&(4-n)(h^\alpha)^2 + n(n-2)S^\alpha \\ &+ 2|h^\alpha|\sqrt{2(n-2)[nS^\alpha - (h^\alpha)^2]}\big\}, \end{aligned} \tag{1.10_1}$$

$$\begin{aligned} &h^\alpha_{n-1,n-1}h^\alpha_{n,n} - (h^\alpha_{n-1,n})^2 \ge \tfrac{1}{2(n-1)}(h^\alpha)^2 - \tfrac{1}{2}S^\alpha, && \text{if } \tfrac{1}{n-1}(h^\alpha)^2 - S^\alpha < 0, \\ &h^\alpha_{n-1,n-1}h^\alpha_{n,n} - (h^\alpha_{n-1,n})^2 \ge 0, && \text{if } \tfrac{1}{n-1}(h^\alpha)^2 - S^\alpha \ge 0 \end{aligned} \tag{1.10_2}$$

are achieved only when (h^α_{ij}) have the forms (1.7), $(1.8)_1$ and $(1.8)_2$, respectively.

We shall formulate some corollaries from this theorem.

Corollary 1.1. *The sectional curvature $K_M(\sigma)$ of M^n at a point x for all 2-planes $\sigma \in T_xM^n$ is non-negative ($K_M(\sigma) \ge 0$) if*

$$K_N(\sigma) \ge \frac{1}{2}S - \frac{n^2}{2(n-1)}H^2 \qquad \textit{when } \frac{n^2}{n-1}H^2 < S, \tag{1.11}$$

or

$$K_N(\sigma) \ge 0 \qquad \textit{when } S \le \frac{n^2}{n-1}H^2. \tag{1.12}$$

Corollary 1.2. *$K_M(\sigma) \ge K_N(\sigma)$ for the plane $\sigma \in T_xM^n$ at a point $x \in M^n$ when*

$$S \le \frac{n^2}{n-1}H^2. \tag{1.13}$$

Corollary 1.3. *$K_M(\sigma) \le 0$ for the plane $\sigma \in T_xM^n$ at a point $x \in M^n$ when*

$$K_N(\sigma) \le -\left(\frac{4-n}{2}H^2 + \frac{n-2}{2n}S + \sqrt{\frac{2(n-2)}{n}H^2(S - nH^2)}\right), \tag{1.14}$$

(1.14) is possible only when $K_N(\sigma)$ is negative as the right side of (1.14) is negative.

Next we will give other estimates of the sectional curvature $K_M(\sigma)$, depending only on the length S of the second fundamental form.

We need the following

Proposition 1.2. *Let M^n be a submanifold in a Riemannian manifold N^{n+p}, then at a point $x \in M^n$ the functions* (1.9) *satisfy*

$$h^\alpha_{n-1,n-1}h^\alpha_{nn} - (h^\alpha_{n-1,n})^2 \leq \frac{1}{2}S^\alpha, \qquad (1.15)_1$$

$$h^\alpha_{n-1,n-1}h^\alpha_{nn} - (h^\alpha_{n-1,n})^2 \geq -\frac{1}{2}S^\alpha. \qquad (1.15)_2$$

The equality in $(1.15)_1$ *holds when the matrices* (h^α_{ij}) *with respect to the basis* (*) *have the view*

$$h^\alpha_{ij} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & h^\alpha_{nn} & 0 \\ 0 & 0 & \dots & 0 & h^\alpha_{nn} \end{pmatrix}, \quad h^\alpha_{nn} = \pm\sqrt{\frac{S^\alpha}{2}}. \qquad (1.16)$$

The equality in $(1.15)_2$ *is valid when* $h^\alpha = 0$ *and* (h^α_{ij}) *are*

$$\begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & c^\alpha & h^\alpha_{n-1,n} \\ 0 & \dots & 0 & h^\alpha_{n-1,n} & -c^\alpha \end{pmatrix} \qquad (1.17)$$

where

$$(h^\alpha_{n-1,n})^2 < \frac{1}{2}S^\alpha, \quad c^\alpha = \pm\frac{1}{2}\sqrt{2[S^\alpha - 2(h^\alpha_{n-1,n})^2]}.$$

The proof of this proposition follows from Lemma 2.2 from [1], applied to the matrices (h^α_{ij}).

From these estimates of the functions (1.9) and the expression (1.4) for the sectional curvature $K_M(\sigma)$ we obtain the following

Theorem 1.3. *The sectional curvature $K_M(\sigma)$ of M^n in a Riemannian manifold N^{n+p} at a point $x \in M^n$ satisfies the following inequalities:*

$$K_M(\sigma) \leq K_N(\sigma) + \frac{1}{2}S, \qquad (1.18)_1$$

$$K_M(\sigma) \leq K_N(\sigma) - \frac{1}{2}S. \qquad (1.18)_2$$

The equalities in $(1.18)_1$ *and* $(1.18)_2$ *are satisfied only when* (h^α_{ij}) *with respect to a suitable basis* (*) *have the forms* (1.16) *and* (1.17), *respectively.*

2. THE EQUALITY CASES IN THE ESTIMATES

Let the ambient space $N^{n+p}(c)$ be a space of constant curvature c, then (1.5), $(1.6)_1$ and $(1.6)_2$ take view, respectively:

$$K_M(\sigma) \leq c + \frac{4-n}{2}H^2 + \frac{n-2}{2n}S + \sqrt{\frac{2(n-2)}{n}H^2(S-nH^2)}, \tag{2.1}$$

$$K_M(\sigma) \geq c + \frac{n^2}{2(n-1)}H^2 - \frac{1}{2}S \quad \text{when} \quad \frac{n^2}{n-1}H^2 - S < 0, \tag{2.2_1}$$

$$K_M(\sigma) \geq c \qquad\qquad \text{when} \quad \frac{n^2}{n-1}H^2 - S \geq 0. \tag{2.2_2}$$

We'll show when the equality in (2.1) holds. From the form (1.7) of the matrices (h^α_{ij}) corresponding to this bound we see that all they are simultaneously diagonalized with respect to the chosen basis $e_1, \dots, e_{n-1} = X, e_n = Y, \dots, e_{n+p}$ $(*)$. Each one of them has exactly $n-2$ eigenvalues equal to the corresponding λ^α_1 and two equal to the corresponding λ^α_n from (1.7) and the vectors X and Y on which the 2-plane σ is spanned are their common eigenvectors corresponding to their 2-multiple eigenvalue λ^α_n. Then, taking in view the fact that every two of the matrices (1.7) are commutative as they can be simultaneously diagonalized, from the Ricci equation

$$R^\alpha_{\beta kl} = h^\alpha_{ks}h^\beta_{sl} - h^\alpha_{ls}h^\beta_{sk} \tag{2.3}$$

where $R^\alpha_{\beta kl}$ is the curvature tensor of the normal bundle $T^\perp_x M^n$, it follows that

$$R^\alpha_{\beta kl} = 0, \tag{2.4}$$

i.e. the normal bundle of M^n is flat. The converse is also true.

Thus we prove the following

Theorem 2.1. *Let M^n be a non-totally geodesic submanifold in a space form $N^{n+p}(c)$. The equality*

$$\max_{\sigma \in T_x M^n} K_M(\sigma) = c + \frac{4-n}{2}H^2 + \frac{n-2}{2n}S + \sqrt{\frac{2(n-2)}{n}H^2(S-nH^2)} \tag{2.5}$$

when σ runs over all 2-plane sections tangent to M^n at a point $x \in M^n$, holds for all points $x \in M^n$, if and only if:

i. the normal bundle of M^n is flat,

ii. each one of the matrices (h^α_{ij}) has exactly $(n-2)$ eigenvalues equal to the corresponding λ^α_1 and two equal to λ^α_n from (1.7) with respect to the basis $(*)$,

iii. the vectors X and Y on which the 2-plane σ is spanned for which $\max K(\sigma)$ *is achieved are their common eigenvectors corresponding to their double eigenvalue λ^α_n.*

With the next theorem two examples of submanifolds satisfying the conditions of the above theorem will be given.

Theorem 2.2. *If the submanifold M^n $(n \geq 4)$ of $N^{n+p}(c)$ satisfies the following conditions:*

i. the normal bundle of M^n is flat,

ii. M^n is a product submanifold of the type $M^n = M^2 \times M^{n-2}$ or $M^n = M^2 \times E^{n-2}$, where M^2, M^{n-2} and E^{n-2} are 2-dimensional umbilical submanifold of $N^{n+p}(c)$, $(n-2)$-dimensional umbilical submanifold of $N^{n+p}(c)$, and $(n-2)$-dimensional Euclidean submanifold of $N^{n+p}(c)$, respectively,

then the equality in (2.1) (or (2.5)) is achieved at a point $x \in M^n$ for a 2-plane σ, which belongs to T_xM^2.

Next, from Theorems 1.3 and 2.1 we obtain the following

Theorem 2.3. *From all n-dimensional submanifolds of $N^{n+p}(c)$ which pass through a point $x \in N^{n+p}(c)$ and have at x the same $S(x)$, only the submanifold M^n which satisfies the following conditions:*

i. the normal bundle of M^n is flat;

ii. each one of the matrices (h^α_{ij}) has exactly $n-2$ eigenvalues equal to zero and two equal to $\lambda^\alpha_n = \pm\sqrt{\dfrac{S^\alpha}{2}}$ with respect to the basis $(*)$,

has the biggest $\max K(\sigma_0)(x) = c + \dfrac{1}{2}S(x)$ achieved for σ_0 spanned by the common eigenvectors X and Y of all (h^α_{ij}), corresponding to their 2-multiple eigenvalue $\lambda^\alpha_n = \pm\sqrt{\dfrac{S^\alpha}{2}}$. The mean curvature of this submanifold is $H(x) = \pm\dfrac{1}{n}\sqrt{2S(x)}$.

The following theorem gives an example of a submanifold satisfying the conditions of Theorem 2.3.

Theorem 2.4. *The product submanifold $M^n = M^2 \times E^{n-2}$ (cylinder) of $N^{n+p}(c)$ with flat normal bundle, where M^2 and E^{n-2} are 2-dimensional umbilical submanifold of $N^{n+p}(c)$ and $(n-2)$-dimensional Euclidean submanifold of $N^{n+p}(c)$, respectively, has at $x \in M^n$ sectional curvature $K(\sigma_0)(x) = c + \dfrac{1}{2}S(x)$ for $\sigma_0 \in T_xM^2$. The mean curvature of M^n is: $H(x) = \dfrac{1}{n}\sqrt{2S(x)}$ or $H(x) = -\dfrac{1}{n}\sqrt{2S(x)}$.*

Let us now see what we can say for the equality case in the lower bound in $(2.2)_1$.

The only thing which can be said for the equality case in $(2.2)_1$ is formulated in the following theorem and follows from Theorems 1.1 and 1.3.

Theorem 2.5. *From all n-dimensional submanifolds of $N^{n+p}(c)$ which pass through a point $x \in N^{n+p}(c)$ and have at x the same $S(x)$, only the minimal submanifold M^n which second fundamental tensors with respect to an orthonormal basis $e_1, \ldots, e_{n-1} = X, e_n = Y, \ldots, e_{n+p}$, have matrices*

$$\begin{pmatrix} 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots & 0 & c^\alpha & h^\alpha_{n-1,n} \\ 0 & \ldots & 0 & h^\alpha_{n-1,n} & -c^\alpha \end{pmatrix}$$

where

$$(h^\alpha_{n-1,n})^2 < \frac{1}{2}S^\alpha, \quad c^\alpha = \pm\frac{1}{2}\sqrt{2[S^\alpha - 2(h^\alpha_{n-1,n})^2]},$$

has the smallest $\min K(\sigma_0)(x) = c - \dfrac{1}{2}S(x)$ *for σ_0 spanned on $X = e_{n-1}$ and $Y = e_n$.*

The mean curvature $H(x)$ of M^n is zero, the sectional curvature of M^n is negative if the ambient space is Euclidean or Hyperbolic.

Example of Theorem 2.1. The hyperellipsoid $M^3 \in E^4$

$$M^3 : x_1^2 + x_2^2 + x_3^2 + mx_4^2 = 1, \quad 0 < m < 1.$$

The principal curvatures of M^3 are:

$$\lambda_1 = \lambda_2 = \frac{1}{\sqrt{1 + (m^2 - m)x_4^2}} = \frac{1}{\sqrt{Q}}; \quad \lambda_3 = \frac{m}{\left(\sqrt{1 + (m^2 - m)x_4^2}\right)^3} = \frac{m}{\left(\sqrt{Q}\right)^3},$$

$$h_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad 0 < \lambda_3 \leq \lambda_1 = \lambda_2.$$

$$3H = h = 2\lambda_1 + \lambda_3 = \frac{2Q + m}{\left(\sqrt{Q}\right)^3}, \quad S = 2\lambda_1^2 + \lambda_3^2 = \frac{2Q^2 + m^2}{Q^3} \tag{2.6}$$

$$\min \lambda_i\lambda_j \leq K_{M^3}(\sigma) \leq \max \lambda_i\lambda_j = \lambda_1\lambda_2 = \lambda_1^2 = \frac{1}{Q} \Rightarrow K_{12} = \frac{1}{Q} = \max_\sigma K_{M^3}(\sigma).$$

On the other hand, according to (2.5) and taking in view (2.6) for the $\max K_{M^3}(\sigma)$ we have:

$$\max K_{M^3}(\sigma) = \frac{1}{2}H^2 + \frac{1}{6}S + \sqrt{\frac{2}{3}H^2(S - 3H^2)} = \frac{1}{18}\left(h^2 + 3S + 2h\sqrt{2(3S - h^2)}\right),$$

which is exactly equal to $\dfrac{1}{Q} = K_{12}$.

3. CHARACTERIZATION OF SOME SUBMANIFOLDS IN N^{N+P}

Theorem 3.1. *A complete simply connected n-dimensional submanifold M^n in a Riemannian manifold N^{n+p} of negative sectional curvature is diffeomorphic to R^n if the second fundamental tensor of M^n satisfies* (1.14).

The proof follows from Corollary 1.3 and the theorem of Hadamard-Cartan.

Corollary 3.1. *If the second fundamental tensor of an n-dimensional complete simply connected submanifold M^n in an $(n+p)$-dimensional Riemannian manifold N^{n+p} of constant negative curvature $(c < 0)$ satisfies*

$$\frac{4-n}{2}H^2 + \frac{n-2}{2n}S + \sqrt{\frac{2(n-2)}{n}H^2(S - nH^2)} \le -c \tag{3.1}$$

then M^n is diffeomorphic to R^n.

Theorem 3.2. *A complete connected n-dimensional submanifold M^n in an $(n+p)$-dimentional Riemannian manifold N^{n+p} of positive curvature bounded below by a constant $c > 0$ is compact with diameter $\le \dfrac{\pi}{\sqrt{c}}$ if its second fundamental form satisfies* (1.13).

Remark. Another proof of this theorem in the case when N^{n+p} is of constant positive curvature is given by M. Okumura [7].

Theorem 3.3. *Let M^n be an n-dimensional non-compact complete connected submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. If at each point $x \in M^n$ for which $\dfrac{n^2}{n-1}H^2 < S$ the inequality $K_N(\sigma) \ge \dfrac{1}{2}S - \dfrac{n^2}{2(n-1)}H^2$ is fulfilled or if at each point x for which $S \le \dfrac{n^2}{n-1}H^2$ the inequality $K_N(\sigma) \ge 0$ holds, then there exists in M^n a compact totally geodesic and totally convex submanifold Q_M without boundary such that M^n is diffeomorphic to the normal bundle of Q_M. In the case when N^{n+p} is of positive curvature which is not bounded bellow by a positive constant then M^n is diffeomorphic to R^n if $S \le \dfrac{n^2}{n-1}H^2$.*

We prove this theorem using Corollary 3.1 and the theorems of Cheeger and Gromoll [5] and Gromoll and Meyer [6].

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier Blvd, 1164 Sofia
BULGARIA
E-mail: hinevast@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 100

ON THE DIVISIBILITY OF ARCS WITH MULTIPLE POINTS[1]

IVAN N. LANDJEV, ASSIA P. ROUSSEVA

In this paper, we generalize a result by Ball, Hill, Landjev and Ward on plane arcs to arcs with multiple points in spaces of arbitrary dimension. This result is further applied to the characterization of some non-Griesmer arcs in the 3-dimensional projective geometry over $\mathbb{F}_4$.

Keywords: Divisible arcs, divisible codes, the polynomial method, the Griesmer bound, Griesmer codes, Griesmer arcs

2000 MSC: main 94B05, secondary 05E30, 94B60, 51C05

1. INTRODUCTION

In a series of papers in the mid-nineties H. N. Ward introduced and investigated the so-called divisibility property of linear codes over finite fields. It turns out that many important classes of codes are divisible. A celebrated result by Ward establishes the divisibility of Griesmer codes of minimum weight divisible by some power of the field order [7].

By the equivalence of the linear codes of full length and the arcs in $\mathrm{PG}(r, q)$, divisibility can be translated into geometric language. This makes it possible to use a geometric technique, the so-called polynomial method [1, 2], in the investigation of divisibility properties for arcs and codes. For instance, the condition on the arc in question being a Griesmer arc can be replaced by a milder condition on the number of points of maximal multiplicity. A result of this type has been obtained

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in [3] for arcs with parameters $(q^2+q+2, q+2)$ in the projective plane $\mathrm{PG}(2, q)$. In what follows, we generalize this result to arcs in projective geometries of arbitrary dimension.

The paper is organized as follows. In Section 2, we give some basic definitions and results on arcs and codes. Section 3 contains the main theorem which establishes the divisibility of non-Griesmer arcs having some additional properties. Section 4 contains a characterization of non-Griesmer arcs in $\mathrm{PG}(3,4)$ with enough maximal points.

2. PRELIMINARIES

Let $\Pi = \mathrm{PG}(r, q)$ be the r-dimensional projective geometry of order q. A multiset of points is a mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$ from the pointset $\mathcal{P}$ of Π into the nonnegative integers. This mapping is extended trivially to the power set of $\mathcal{P}$ by $\mathcal{K}(\mathcal{Q}) = \sum_{x \in \mathcal{Q}} \mathcal{K}(x)$, $\mathcal{Q} \subseteq \mathcal{P}$. The integer $\mathcal{K}(x)$ is called the multiplicity of the point x. Similarly, we define multiplicities of lines, planes, hyperplanes etc. A multiset $\mathcal{K}$ is called a (n, w)-arc if (1) $\mathcal{K}(\mathcal{P}) = n$, (2) $\mathcal{K}(H) \leq w$ for any hyperplane H, (3) $\mathcal{K}(H_0) = w$ for at least one hyperplane H_0. Denote by a_i the number of hyperplanes in Π of multiplicity exactly i and by Λ_i – the number of points of multiplicity i. The sequence $(a_i)_{i \geq 0}$ is called the spectrum of $\mathcal{K}$.

Let $\mathbb{F}_q^n$ be the vector space of all n-tuples over the finite field $\mathbb{F}_q$. Any k-dimensional subspace C of $\mathbb{F}_q^n$ is called a linear code of length n and dimension k. If, in addition, the minimum Hamming distance between different codewords of C is d the code is referred to as an $[n, k, d]_q$-code. It is well-known that with every linear $[n, k, d]_q$-code of full length, i.e. a code in which no coordinate is identically zero, one can associate an $(n, n-d)$-arc in $\mathrm{PG}(k-1, q)$ so that isomorphic codes lead to equivalent arcs and vice versa. This means that linear codes and arcs are in some sense equivalent objects.

A fundamental bound on the parameters of a linear code is the so-called Griesmer bound [4]. It says that if C is an $[n, k, d]_q$-code then

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil . \tag{2.1}$$

A linear code meeting the Griesmer bound is called a Griesmer code. An arc associated with a Griesmer code is called a Griesmer arc.

A divisible linear code is defined as a code whose word weights have a nontrivial common divisor [6]. It has been proved in [7] that every $[n, k, d]_q$-code meeting the Griesmer bound with minimum weight divisible by some power of q is also divisible. Using the equivalence between linear codes and arcs in the projective geometries $\mathrm{PG}(k-1, q)$, we can translate this in geometric language. An (n, w)-arc $\mathcal{K}$ is said to be divisible if there exists an integer $\Delta > 1$ such that $\mathcal{K}(H) \equiv n \pmod{\Delta}$ for any hyperplane H. Ward's divisibility result from [7] can be restated for Griesmer arcs as follows [5].

Theorem 1. *Let $\mathcal{K}$ be a Griesmer (n,w)-arc in* $\mathrm{PG}(k-1,p)$ *with $w \equiv n \pmod{p^e}$, p – a prime, $e \geq 1$. Then $\mathcal{K}(H) \equiv n \pmod{p^e}$ for every hyperplane H in* $\mathrm{PG}(k-1,p)$.

This result can be generalized to arcs and codes over non-prime fields. However the condition on the arc of meeting the Griesmer bound remains essential. Interestingly, some non-Griesmer arcs and codes also exhibit divisibility properties. In their investigation of arcs with parameters $(q^2+q+2, q+2)$ in $\mathrm{PG}(2,q)$ Ball et al. [3] observed that the presence of many double points implies divisibility of the arc. In the next section, we extend this observation to get a divisibility result for non-Griesmer arcs in finite projective geometries of arbitrary dimension.

3. THE MAIN THEOREM

Consider the projective geometry $\mathrm{PG}(r,q)$ and fix a hyperplane H_∞. Clearly, $\mathrm{PG}(r,q)\setminus H_\infty$ can be regarded as the r-dimensional affine geometry $\mathrm{AG}(r,q)$. The finite field $\mathbb{F}_{q^r}$ is an r-dimensional vector space over $\mathbb{F}_q$ and can be identified by the points of $\mathrm{AG}(r,q) = \mathrm{PG}(r,q)\setminus H_\infty$. The line through the points $X, Y \in \mathbb{F}_q^r$ is given parametrically by

$$L = \langle X, Y\rangle = \{tX + (1-t)Y \mid t \in \mathbb{F}_q\} \subset \mathbb{F}_{q^r}.$$

Let $X, Y, X', Y' \in \mathbb{F}_{q^r}$ be four points from $\mathrm{AG}(r,q)$ such that $\langle X, Y\rangle \cap H_\infty = \langle X', Y'\rangle \cap H_\infty$. If $\mathbb{F}_{q^r} = \mathbb{F}_q(\alpha)$, we can write the above four points as

$$\begin{array}{ll} X = x_0 + x_1\alpha + \ldots + x_{r-1}\alpha^{r-1}, & Y = y_0 + y_1\alpha + \ldots + y_{r-1}\alpha^{r-1} \\ X' = x'_0 + x'_1\alpha + \ldots + x'_{r-1}\alpha^{r-1}, & Y' = y'_0 + y'_1\alpha + \ldots + y'_{r-1}\alpha^{r-1} \end{array}$$

where $x_i, y_i, x'_i, y'_i \in \mathbb{F}_q$. In $\mathrm{PG}(r,q)$, the four points can be viewed as

$$(1, x_0, \ldots, x_{r-1}), (1, y_0, \ldots, y_{r-1}), (1, x'_0, \ldots, x'_{r-1}), (1, y'_0, \ldots, y'_{r-1}).$$

The common point of $\langle X, Y\rangle$ and $\langle X', Y'\rangle$, which lies in H_∞, is

$$(0, x_0 - y_0, x_1 - y_1, \ldots, x_{r-1} - y_{r-1}) = t(0, x'_0 - y'_0, x'_1 - y'_1, \ldots, x'_{r-1} - y'_{r-1})$$

where $t \in \mathbb{F}_q^*$. Hence

$$(X - Y) = \sum_{i=0}^{r-1}(x_i - y_i)\alpha^i = t\sum_{i=0}^{r-1}(x'_i - y'_i)\alpha^i = t(X' - Y')$$

and $(X-Y)^{q-1} = t^{q-1}(X'-Y')^{q-1} = (X'-Y')^{q-1}$. Therefore the points on H_∞ can be identified with the $\frac{q^r-1}{q-1}$-st roots of unity in $\mathbb{F}_{q^r}$. Denote by G the subgroup of $\mathbb{F}_{q^r}^*$ that contains the $\frac{q^r-1}{q-1}$-st roots of unity. The element of G identified with the intersecting point of the line L from $\mathrm{AG}(r,q)$ and H_∞ is denoted by ζ_L. The above argument shows that L and L' are parallel if and only if $\zeta_L = \zeta_{L'}$.

The next theorem is an application of the so-called polynomial method in finite geometry.

Theorem 2. *Let $\mathcal{K}$ be a (n,w)-arc in $\mathrm{PG}(r,q)$, $r \geq 2$, $q = p^h$, p – a prime. Let all lines through a point of maximal multiplicity m have the same multiplicity. If $\Lambda_m > (q-1)p^{t-1}$, where $t \leq (r-1)h$, then for every hyperplane H*

$$\mathcal{K}(H) \equiv n \pmod{p^t}.$$

Proof. Denote by s the multiplicity of a line through a maximal point and set, as usual, $v_i = \frac{q^i-1}{q-1}$. Then $n = m + (s-m)v_r$, and the multiplicity of a hyperplane H containing a maximal point is $\mathcal{K}(H) = m + (s-m)v_{r-1}$. Then

$$n - \mathcal{K}(H) = (s-m)(v_r - v_{r-1}) = (s-m)q^{r-1} \equiv 0 \pmod{q^{r-1}}.$$

Now consider a hyperplane which is not incident with points of maximal multiplicity. We can assume with no loss of generality that $0 \in \mathbb{F}_{q^r}$ is not a point of maximal multiplicity (otherwise, we translate the points of the affine geometry to ensure this). Consider the polynomial

$$\begin{aligned} F(x,y) &= \prod_{P\in\mathbb{F}_{q^r}} \left(1-(1-Px)^{q-1}y\right)^{\mathcal{K}(P)} \prod_{\zeta\in G}(1-\zeta x^{q-1}y)^{\mathcal{K}(\zeta)} \\ &= \sum_{i=0}^{n} F_i(x)y^i. \end{aligned}$$

Let $Q \in \mathbb{F}_{q^r}$ be a point of maximal multiplicity and set $x = Q^{-1}$. Note that $Q \neq 0$. When $P \neq Q$ we have

$$(1-PQ^{-1})^{q-1} = (Q-P)^{q-1}Q^{1-q} = \zeta_L Q^{1-q},$$

where $L = \langle P, Q\rangle$. Collecting the factors in the product above, we get

$$F(Q^{-1},y) = \prod_{\zeta\in G}(1-\zeta Q^{-1}y)^{\mathcal{K}(L)-\mathcal{K}(Q)}$$

where L is a line incident with Q and such that $L \cap H_\infty$ is identified with ζ. Further, we have

$$\begin{aligned} F(Q^{-1},y) &= \left(\prod_{\zeta\in G}(1-\zeta Q^{-1}y)\right)^{s-m} \\ &= (1-y^{v_r})^{s-m} \\ &= 1-\binom{s-m}{1}y^{v_r}+\binom{s-m}{2}y^{2v_r}-\dots. \end{aligned}$$

Therefore $F_i(Q^{-1}) = 0$ for $i = 1, \ldots, v_r - 1$. The polynomial $F_i(x)$ is of degree at most $i(q-1)$ and since $\Lambda_m > (q-1)p^{t-1}$, we have $F_i(x) \equiv 0$ for all $i \le p^{t-1}$. On the other hand,

$$\begin{aligned} F(0, y) &= (1-y)^{n-\mathcal{K}(H_\infty)} \\ &= 1 - \binom{n-\mathcal{K}(H_\infty)}{1} y + \binom{n-\mathcal{K}(H_\infty)}{2} y^2 - \ldots. \end{aligned}$$

This implies in particular that

$$\binom{n-\mathcal{K}(H_\infty)}{p^j} \equiv 0 \pmod{p}$$

for $j = 0, \ldots, t-1$. Now by Lucas theorem, $n - \mathcal{K}(H_\infty) \equiv 0 \pmod{p^t}$. □

4. ONE EXAMPLE

As an illustration of Theorem 2, consider the non-Griesmer arcs with parameters $(86, 22)$ in $\mathrm{PG}(3, 4)$. Clearly, every line through a 2-point has multiplicity of 6. Assume $t = 2$ and $\Lambda_2 > (q-1)p^{t-1} = 6$. Recall the classification of the $(22, 6)$-arcs in $\mathrm{PG}(2, 4)$ from [3]. There exist six equivalence classes of such arcs:

(1) arcs with one 2-point and no 0-points;

(2) arcs with two 2-points and one 0-point, which are collinear;

(3) arcs with three 2-points and two 0-points, which are collinear;

(4) arcs with four 2-points and three collinear 0-points, which form a Baer subplane; the 0-points are collinear in the Baer subplane;

(5) arcs with six 2-points and five 0-points; the 2-points form a hyperoval and the 0-points form an external line to the hyperoval;

(6) arcs with seven 2-points and six 0-points, which are represented as a sum of two copies of a hypeoval plus the sum of two external lines to it.

By Theorem 2, the possible multiplicities of hyperplanes are: 2, 6, 10, 14, 18, 22. Planes of multiplicity 2 are impossible by a counting argument since 22- and 18-planes do not have 1-lines; 10-planes are ruled out by the nonexistence of $(10, 3)$-arcs in $\mathrm{PG}(2, 4)$. In order to rule out 6-planes, assume such a plane π exists and consider a projection φ from an arbitrary 0-point in π. The planes through an arbitrary 2-line in π are all 22-planes. Their image under φ is a line of type $(6, 6, 6, 2, 2)$ or $(6, 6, 4, 4, 2)$. In all cases, we get a line in the projection plane of multiplicity larger than 22, which is impossible.

It is easily checked that a 14-plane cannot be the complement of a line and two further points. So, it is the complement of a Baer subplane. If a plane of this size does not exist, the $(86,22)$-arc is a sum of a plane $(22,6)$-arc of type (6) and $AG(3,4)$. Assume there is a 14-plane. Then there is exactly one 14-plane which is easily proved by considering the projection from a 0-point in this plane. But then an easy counting gives $\Lambda_2 = 8, \Lambda_1 = 70, \Lambda_0 = 7$. Such an arc is obtained by taking the 2- and 0-points to form a $PG(3,2)$, where the 0-points are coplanar (all of them are on the 14-plane).

As a matter of fact, all $(86,22)$-arcs with $\Lambda_2 < 7$ are obtained as the sum of a plane $(22,6)$-arc and $AG(3,4)$.

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Ivan N. Landjev
New Bulgarian University
21, Montevideo Str.
1618 Sofia, BULGARIA

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, BULGARIA
E-mail: i.landjev@nbu.bg, ivan@math.bas.bg

Assia P. Rousseva
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier Blvd
1164 Sofia, BULGARIA
E-mail: assia@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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NON-EXISTENCE OF FLAT PARACONTACT METRIC STRUCTURES IN DIMENSION GREATER THAN OR EQUAL TO FIVE [1]

SIMEON ZAMKOVOY, VASSIL TZANOV

An example of a three dimensional flat paracontact metric manifold with respect to Levi-Civita connection is constructed. It is shown that no such manifold exists for odd dimensions greater than or equal to five.

Keywords: paracontact metric manifold, integral submanifold, maximal integral submanifold

2000 MSC: 53C15, 5350, 53C25, 53C26, 53B30

1. INTRODUCTION

The almost paracontact structure on pseudo-Riemannian manifold M of dimension $(2n+1)$ is defined in [7]. An almost paracomplex structure on $M^{(2n+1)} \times \mathbb{R}$ is constructed in [5]. Some properties of an almost paracontact metric manifold and the gauge (conformal) transformations of a paracontact metric manifold, i.e., transformations preserving the paracontact structure, are studied in [8]. Furthermore, in this paper a canonical paracontact connection on a paracontact metric manifold is defined. This connection is the paracontact analogue of the (generalized) Tanaka-Webster connection. It is shown that the torsion of the canonical

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paracontact connection vanishes exactly when the structure is para-Sasakian and the gauge transformation of its scalar curvature is computed.

An example of a paracontact structure of flat canonical connection is the hyperbolic Heisenberg group [3]. The paraconformal tensor gives a necessary and sufficient condition for a $(2n+1)$-dimensional paracontact manifold to be locally paracontact conformal to the hyperbolic Heisenberg group [3].

In this paper, we show that there is no flat, with respect to Levi-Civita connection, paracontact metric structures in dimension greater than or equal to five, whereas in dimension equal to three there is.

2. PRELIMINARIES

A (2n+1)-dimensional smooth manifold $M^{(2n+1)}$ has an *almost paracontact structure* (φ, ξ, η) if it admits a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying the following compatibility conditions

$$\begin{aligned} &(i) \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \\ &(ii) \quad \eta(\xi) = 1 \quad \varphi^2 = id - \eta \otimes \xi, \\ &(iii) \quad \text{the tensor field } \varphi \text{ induces an almost paracomplex structure (see [4])} \\ &\qquad \text{on each fibre on the horizontal distribution } \mathbb{D} = Ker\ \eta. \end{aligned} \tag{2.1}$$

Recall that an almost paracomplex structure on a $2n$-dimensional manifold is a (1,1)-tensor J such that $J^2 = 1$ and the eigensubbundles T^+, T^- corresponding to the eigenvalues $1, -1$ of J respectively, have dimensions equal to n. The Nijenhuis tensor N of J, given by $N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y]$, is the obstruction for the integrability of the eigensubbundles T^+, T^-. If $N = 0$ then the almost paracomplex structure is called paracomplex or integrable.

An immediate consequence of the definition of the almost paracontact structure is that the endomorphism φ has rank $2n$, $\varphi\xi = 0$ and $\eta \circ \varphi = 0$, (see [1, 2] for the almost contact case).

If a manifold $M^{(2n+1)}$ with (φ, ξ, η)-structure admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.2}$$

then we say that $M^{(2n+1)}$ has an almost paracontact metric structure and g is called *compatible* metric. Any compatible metric g of a given almost paracontact structure is necessarily of signature $(n+1, n)$.

Setting $Y = \xi$, we have $\eta(X) = g(X, \xi)$. From here and (2.2) follows

$$g(\varphi X, Y) = -g(X, \varphi Y).$$

The fundamental 2-form

$$F(X,Y) = g(X, \varphi Y) \tag{2.3}$$

is non-degenerate on the horizontal distribution $\mathbb{D}$ and $\eta \wedge F^n \neq 0$.

Definition 2.1. *If* $g(X, \varphi Y) = d\eta(X, Y)$ *(where* $d\eta(X,Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X,Y])$*), then* η *is a paracontact form and the almost paracontact metric manifold* (M, φ, η, g) *is said to be* paracontact metric manifold.

Definition 2.2. *An* r*-dimensional submanifold* N *of* $M^{(2n+1)}$ *is said to be an* integral submanifold *(of the horizontal distribution* $\mathbb{D}$*) if and only if every tangent vector of* N *at every point* p *of* N *belongs to* $\mathbb{D}$.

Definition 2.3. *An integral submanifold of dimension* r *in* $M^{(2n+1)}$ *is said to be a* maximal integral submanifold *if it is not a proper subset of any other integral submanifold of dimension* r.

Similarly to the contact metric case [6], we may obtain the following

Proposition 2.4. *Let* $(M^{2n+1}, \varphi, \eta, g)$ *be a paracontact metric manifold. Then the highest dimension of integral submanifolds of the horizontal distribution* $\mathbb{D}$ *is equal to* n.

The tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ are defined [8] by

$$N^{(1)}(X,Y) = N_\varphi(X,Y) - 2d\eta(X,Y)\xi,$$

$$N^{(2)}(X,Y) = (\pounds_{\varphi X}\eta)Y - (\pounds_{\varphi Y}\eta)X,$$

$$N^{(3)}(X) = (\pounds_\xi \varphi)X,$$

$$N^{(4)}(X) = (\pounds_\xi \eta)X,$$

where $N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y]$.

The tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ are analogs of the tensors denoted in the same way in the almost contact case [1, 2].

They have the following propositions [8].

Proposition 2.5. *For an almost paracontact structure* (φ, ξ, η) *the vanishing of* $N^{(1)}$ *implies the vanishing* $N^{(2)}$, $N^{(3)}$ *and* $N^{(4)}$*;*

For a paracontact structure (φ, ξ, η, g), $N^{(2)}$ *and* $N^{(4)}$ *vanish. Moreover* $N^{(3)}$ *vanishes if and only if* ξ *is a Killing vector field.*

Proposition 2.6. *For an almost paracontact metric structure* (φ, ξ, η, g)*, the covariant derivative* $\nabla\varphi$ *of* φ *with respect to the Levi-Civita connection* ∇ *is given by*

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) = & -dF(X,Y,Z) - dF(X, \varphi Y, \varphi Z) - g(N^{(1)}(Y,Z), \varphi X) \\ & + N^{(2)}(Y,Z)\eta(X) - 2d\eta(\varphi Z, X)\eta(Y) + 2d\eta(\varphi Y, X)\eta(Z). \end{aligned} \tag{2.4}$$

For a paracontact metric structure (φ, ξ, η, g)*, the formula* (2.4) *simplifies to*

$$2g((\nabla_X \varphi)Y, Z) = -g(N^{(1)}(Y, Z), \varphi X) - 2d\eta(\varphi Z, X)\eta(Y) + 2d\eta(\varphi Y, X)\eta(Z) \quad (2.5)$$

Lemma 2.7. *On a paracontact metric manifold,* $h = \frac{1}{2}N^{(3)}$ *is a symmetric operator,*

$$\nabla_X \xi = -\varphi X + \varphi h X, \quad (2.6)$$

h *anti-commutes with* φ *,and* $trh = h\xi = 0$*.*

3. NON-EXISTENCE OF FLAT PARACONTACT METRIC STRUCTURES IN DIMENSION GREATER THAN OR EQUAL TO FIVE

In this section we shall show that every paracontact metric manifold of dimension greater than or equal to five must have some curvature, though not necessarily in the plane sections containing ξ.

Theorem 3.1. *Let* M^{2n+1} *be a manifold of dimension greater than or equal to five. Then* M^{2n+1} *cannot admit a paracontact structure of vanishing curvature.*

Proof. The proof will be by contradiction. We let (φ, ξ, η, g) denote the structure tensors of a paracontact metric structure and assume that g is flat. From [8] we have, for a paracontact metric structure,

$$\frac{1}{2}(R(\xi, X)\xi + \varphi R(\xi, \varphi X)\xi) = \varphi^2 X - h^2 X$$

where $h = \frac{1}{2}\mathcal{L}_\xi \varphi$. Thus if g is flat, $h^2 = \varphi^2$, and hence $h\xi = 0$ and $rank(h) = 2n$. The eigenvectors corresponding to the non-zero eigenvalues of h are orthogonal to ξ and the non-zero eigenvalues are ± 1. Recall that $d\eta(X, Y) = \frac{1}{2}(g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X))$ and that for a paracontact metric structure

$$\nabla_X \xi = -\varphi X + \varphi h X. \quad (3.1)$$

From Lemma 2.7 follows that whenever X is an eigenvector of eigenvalue $+1$, φX is an eigenvector of -1 and vice-versa. Thus the paracontact distribution $\mathbb{D}$ is decomposed into the orthogonal eigenspaces of ± 1 which we denote by $[+1]$ and $[-1]$.

We now show that the distribution $[+1]$ is integrable. If X and Y are vector fields belonging to $[+1]$, equation (3.1) gives $\nabla_X \xi = 0$ and $\nabla_Y \xi = 0$. Thus since M^{2n+1} is flat:

$$0 = R(X, Y)\xi = -\nabla_{[X,Y]}\xi = \varphi[X, Y] - \varphi h[X, Y];$$

but $\eta([X, Y]) = -2d\eta(X, Y) = -2g(X, \varphi Y) = 0$, so that $h[X, Y] = [X, Y]$. Applying the same argument to ξ and $X \in [+1]$ we see that the distribution $[+1] \oplus [\xi]$ spanned by $[+1]$ and ξ is also integrable.

Since $[+1] \oplus [\xi]$ is integrable, we can choose local coordinates $(u^0, u^1, \dots, u^{2n})$ such that $\frac{\partial}{\partial u^0}, \frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n} \in [+1] \oplus [\xi]$. For $i = 1 \dots n$ the vector $\frac{\partial}{\partial u^{n+i}}$ can be uniquely presented as $\frac{\partial}{\partial u^{n+i}} = v^{+1}_{n+i} + v^{\xi}_{n+i} + v^{-1}_{n+i}$ where $v^{+1}_{n+i} \in [+1]$, $v^{\xi}_{n+i} \in [\xi]$, $v^{-1}_{n+i} \in [-1]$, and $v^{-1}_{n+i} \neq \overrightarrow{0}$. Let $v^{+1}_{n+i} + v^{\xi}_{n+i} = -\sum_{j=0}^{n} f_i^j \frac{\partial}{\partial u^j}$. We define local vector fields X_i, $i = 1, \dots, n$ by $X_i = \frac{\partial}{\partial u^{n+i}} + \sum_{j=0}^{n} f_i^j \frac{\partial}{\partial u^j}$, i.e. $X_i = v^{-1}_{n+i}$ so that $X_i \in [-1]$. Note $X_1, \dots, X_n$ are n linearly independent vector fields spanning $[-1]$. Clearly $[\frac{\partial}{\partial u^k}, X_i] \in [+1] \oplus [\xi]$ for $k = 0, \dots, n$ and hence ξ is parallel along $[\frac{\partial}{\partial u^k}, X_i]$. Therefore using (3.1) and the vanishing curvature

$$0 = \nabla_{[\frac{\partial}{\partial u^k}, X_i]} \xi = \nabla_{\frac{\partial}{\partial u^k}} \nabla_{X_i} \xi - \nabla_{X_i} \nabla_{\frac{\partial}{\partial u^k}} \xi = -2 \nabla_{\frac{\partial}{\partial u^k}} \varphi X_i$$

from which we have

$$\nabla_{\varphi X_j} \varphi X_i = 0. \tag{3.2}$$

In particular $\nabla_{\xi} \varphi X_i = 0$. Furthermore, from equation (3.1) we obtain $\nabla_{\varphi X_i} \xi = 0$ and hence $[\varphi X_i, \xi] = 0$.

Similarly, noting that $[X_i, X_j] \in [+1]$,

$$0 = R(X_i, X_j)\xi = \nabla_{X_i} \nabla_{X_j} \xi - \nabla_{X_j} \nabla_{X_i} \xi - \nabla_{[X_i, X_j]} \xi = -2 \nabla_{X_i} \varphi X_j + 2 \nabla_{X_j} \varphi X_i$$

giving

$$\nabla_{X_i} \varphi X_j = \nabla_{X_j} \varphi X_i, \tag{3.3}$$

or equivalently

$$\varphi[X_i, X_j] = -(\nabla_{X_i} \varphi) X_j + (\nabla_{X_j} \varphi) X_i. \tag{3.4}$$

Using equations (3.1) and (3.2) we obtain

$$0 = R(X_i, \varphi X_j)\xi = -\nabla_{[X_i, \varphi X_j]} \xi = \varphi[X_i, \varphi X_j] - \varphi h [X_i, \varphi X_j]$$

from which

$$g([X_i, \varphi X_j], X_k) = -g(\varphi[X_i, \varphi X_j], \varphi X_k) = g(h[X_i, \varphi X_j], X_k) =$$

$$= g([X_i, \varphi X_j], h X_k) = -g([X_i, \varphi X_j], X_k)$$

and hence

$$g([X_i, \varphi X_j], X_k) = 0. \tag{3.5}$$

Using formula (2.5) and equations (3.2), (3.4) and (3.5) we have

$$2g((\nabla_{X_i} \varphi) X_j, X_k) = -g(N^{(1)}(X_j, X_k), \varphi X_i) = -g([X_j, X_k], \varphi X_i) =$$

$$= -g((\nabla_{X_j} \varphi) X_k, X_i) + g((\nabla_{X_k} \varphi) X_j, X_i).$$

From $F = d\eta$, we obtain $dF = 0$ and hence $\sigma_{i,j,k} g((\nabla_{X_i} \varphi) X_j, X_k) = 0$. Thus our computation yields $g((\nabla_{X_i} \varphi) X_j, X_k) = 0$. Similarly

$$2g((\nabla_{X_i} \varphi) X_j, \varphi X_k) = -g(N^{(1)}(X_j, \varphi X_k), \varphi X_i) =$$

$$= -g([X_j, \varphi X_k], \varphi X_i) - g([\varphi X_j, X_k], \varphi X_i) =$$

$$= -g(\nabla_{X_j}\varphi X_k - \nabla_{\varphi X_k} X_j - \nabla_{X_k}\varphi X_j + \nabla_{\varphi X_j} X_k, \varphi X_i)$$

which vanishes by equations (3.2) and (3.3). Finally

$$2g((\nabla_{X_i}\varphi)X_j, \xi) = -g(N^{(1)}(X_j, \xi), \varphi X_i) + 2d\eta(\varphi X_j, X_i) =$$

$$= -g(\varphi^2[X_j, \xi], \varphi X_i) + 2d\eta(\varphi X_j, X_i) = -4g(X_i, X_j).$$

Thus for any vector fields X and Y in $[-1]$ on a paracontact metric manifold such that ξ is annihilated by the curvature transformation

$$(\nabla_X \varphi)Y = -2g(X, Y)\xi. \tag{3.6}$$

Note that equation (3.4) now gives $[X_i, X_j] = 0$.

Analogously, we obtain

$$2g(\nabla_{\varphi X_i} X_j, X_k) = 2g((\nabla_{\varphi X_i}\varphi)X_j, \varphi X_k) = 0. \tag{3.7}$$

Therefore by equation (3.5), we get

$$g(\nabla_{X_i} X_j, \varphi X_k) = -g(X_j, \nabla_{X_i}\varphi X_k) = -g(X_j, [X_i, \varphi X_k]) = 0.$$

It is trivial that $g(\nabla_{X_i} X_j, \xi) = 0$ and hence we obtain $\nabla_{X_i} X_j \in [-1]$.

Differentiating equation (3.6), we have

$$\nabla_{X_k}\nabla_{X_i}\varphi X_j - (\nabla_{X_k}\varphi)\nabla_{X_i} X_j - \varphi\nabla_{X_k}\nabla_{X_i} X_j =$$

$$= -2X_k(g(X_i, X_j))\xi + 4g(X_j, X_i)\varphi X_k.$$

Taking the inner product with φX_l, having in mind equation (3.6) and $\nabla_{X_i} X_j \in [-1]$, we obtain

$$g(\nabla_{X_k}\nabla_{X_i}\varphi X_j, \varphi X_l) + g(\nabla_{X_k}\nabla_{X_i} X_j, X_l) = -4g(X_j, X_i)g(X_k, X_l) \tag{3.8}$$

Interchanging i and k, $i \neq k$ and subtracting, we have

$$g(X_i, X_j)g(X_k, X_l) - g(X_k, X_j)g(X_i, X_l) = 0$$

by virtue of the flatness and $[X_i, X_j] = 0$.

Setting $i = j$ and $k = l$, we have

$$g(X_i, X_i)g(X_k, X_k) - g(X_i, X_k)g(X_i, X_k) = 0$$

contradicting the linear independence of X_i and X_k. □

Note that in the proof of our theorem, the vanishing of $R(X, Y)\xi$ is enough to obtain the decomposition of the paracontact distribution into ± 1 eigenspaces of the operator $h = \frac{1}{2}\mathcal{L}_\xi \varphi$. Moreover, $R(X,Y)\xi = 0$ for X and Y in $[+1]$ is sufficient for the integrability of $[+1]$. Thus we have the following

Theorem 3.2. *Let M^{2n+1} be a paracontact manifold with paracontact metric structure (φ,ξ,η,g). If the sectional curvatures of all plane sections containing ξ vanish, then the operator h has rank $2n$ and the paracontact distribution is decomposed into the ± 1 eigenspaces of h. Moreover, if $R(X,Y)\xi = 0$ for $X, Y \in [+1]$, M admits a foliation by $n-$dimensional integral submanifolds of the paracontact distribution along which ξ is parallel.*

From Theorem 3.1 and Theorem 3.2 we obtain following

Theorem 3.3. *Let M^{2n+1} be a paracontact metric manifold and suppose that $R(X,Y)\xi = 0$ for all vector fields X and Y. Then locally M^{2n+1} is the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4.*

Proof. We noted in Theorem 3.1 proof that $[X_i, X_j] = 0$ so that the distribution $[-1]$ is also integrable and hence we may take $X_i = \frac{\partial}{\partial u^{n+i}}$. Moreover, locally M^{2n+1} is the product of an integral submanifold M^{n+1} of $[+1] \oplus [\xi]$ and an integral submanifold M^n of $[-1]$. Since $\{\varphi X_i, \xi\}$ is a local basis of tangent vector fields on M^{n+1}, equation (3.2) and $R(X,Y)\xi = 0$ show that M^{n+1} is flat.

Now $\nabla_{\varphi X_i} X_j = 0$ since $g(\nabla_{\varphi X_i} X_j, X_k) = 0$ by equation (3.7). Moreover, $g(\nabla_{\varphi X_i} X_j, \varphi X_k) = 0$ by equation (3.2) and $g(\nabla_{\varphi X_i} X_j, \xi) = 0$ which is trivial. Interchanging i and k in equation (3.8) and subtracting, we have

$$R(X_k, X_i, \varphi X_j, \varphi X_l) + R(X_k, X_i, X_j, X_l) =$$

$$= -4(g(X_i, X_j)g(X_k, X_l) - g(X_k, X_j)g(X_i, X_l)).$$

Using $\nabla_{\varphi X_i} X_j = 0$ and $[\varphi X_i, \varphi X_j] = 0$ we see that $R(X_k, X_i, \varphi X_j, \varphi X_l) =$ $= R(\varphi X_j, \varphi X_l, X_k, X_i) = 0$, and hence

$$R(X_k, X_i, X_j, X_l) = -4(g(X_i, X_j)g(X_k, X_l) - g(X_k, X_j)g(X_i, X_l))$$

completing the proof. □

4. FLAT ASSOCIATED METRICS ON $\mathbb{R}^3_1$

In dimension 3 it is easy to construct flat paracontact structures. For example, consider $\mathbb{R}^3_1$ with coordinates (x^1, x^2, x^3). The 1-form $\eta = \frac{1}{2}(ch(x^3)dx^1 + sh(x^3)dx^2)$ is a paracontact form. In this case $\xi = 2(ch(x^3)\frac{\partial}{\partial x^1} - sh(x^3)\frac{\partial}{\partial x^2})$ and the metric g whose components are $g_{11} = -g_{22} = g_{33} = \frac{1}{4}$ gives flat paracontact metric structure. Following the proof of the Theorem 3.1, we see that $\frac{\partial}{\partial x^3}$ spans the distribution $[-1]$, and $sh(x^3)\frac{\partial}{\partial x^1} + ch(x^3)\frac{\partial}{\partial x^2}$ spans the distribution $[+1]$.

We can now find a flat associated metric on $\mathbb{R}^3_1$ for the standard paracontact form $\eta_0 = \frac{1}{2}(dz - ydx)$. Consider the diffeomorphism $f : \mathbb{R}^3_1 \to \mathbb{R}^3_1$ given by

$$x^1 = zch(x) - ysh(x)$$

$$x^2 = zsh(x) - ych(x)$$

$$x^3 = -x$$

Then $\eta_0 = f^*\eta$, and the pseudo-Riemannian metric $g_0 = f^*g$ is a flat associated metric for the paracontact form η_0.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier Blvd, 1164 Sofia
BULGARIA
E-mail: zamkovoy@fmi.uni-sofia.bg

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ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
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EXACT EFFECTIVE ENUMERATIONS OF TOTAL FUNCTIONAL STRUCTURES[1]

ANGEL V. DITCHEV

In the present paper is considered an introduced by the author notion of exact effective enumeration. In some sense those enumerations are "the least one". It is proved that in case of total structure without predicates there exist exact effective enumerations, even infinitely many such enumerations.

Keywords: Structure, Total structure, Enumeration, Exact effective enumeration

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1. INTRODUCTION

In [1] Lacombe and in [2] Moschovakis have defined different kinds of computability in abstract structure. The first one uses enumerations of the structure and the second one, called search computability, uses only the functions and predicates in the structure. Mosckovakis [3] has proved that both computabilities are equivalent in the case when the equality is among the basic predicates. Soskov [4] has proved that both computabilities coincide in the general case.

Skordev has defined an "effective" version of Lacombe's computability as follows: It is said φ is effective in $\langle \alpha_0, \mathfrak{B}_0 \rangle$ iff φ has a partial recursive "associate". It is said φ is e-admissible iff φ is effective in all effective enumerations of the structure $\mathfrak{A}$. Skordev has stated a conjecture in the case when the structure has at most a denumerable domain, and it admits an effective enumeration. Skordev's conjecture

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is that e-admissibility coincides with search computability. Attempts were made to prove Skordev's conjecture [5, 6, 7, 8]. They were successful for some special cases, but not for the general one. As Manasse, Chisholm, Vencov [9, 10, 11] showed, the above mentioned conjecture wasn't true. Nevertheless, it is interesting to know for what kind of structures Skordev's conjecture is valid. The author puts the question: Which are the structures $\mathfrak{A}$ for which we could find effective enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ such that, for every function φ, φ is effective in $\langle \alpha_0, \mathfrak{A}_0 \rangle$ iff φ is e-admissible. Such kind of enumeration we shall call an *exact effective enumeration.* In his master thesis Stoyan Atanasov showed that there exist exact effective enumerations for the structures with only unary total functions and no predicates. It is natural to expect that this kind of enumerations have to have some minimal(maximal) properties. In this paper we investigate exact enumerations for the structures with only total functions and no predicates. Different partial orders can be taken in the set of enumerations. Here we choose a partial order in the set of enumerations as the one in [12]. We prove that for total structures there exist exact enumerations. Furthermore, there exist infinitely many mutually incomparable exact enumerations. It is shown that above (in the considered partial order) every strongly effective enumeration there exists an exact enumeration.

2. PRELIMINARIES

In this paper we use ω to denote the set of all natural numbers. We shall recall some definitions from [4, 7].

Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; \Sigma_1, \dots, \Sigma_k \rangle$ be a partial structure, where B is an arbitrary most denumerable set, $\theta_1, \dots, \theta_n$ are partial functions of many arguments on B, and $\Sigma_1, \dots, \Sigma_k$ are partial predicates of many arguments on B. The relational type of $\mathfrak{A}$ is the order pair $\langle \langle k_1, \dots, k_n \rangle, \langle l_1, \dots, l_k \rangle \rangle$, where each θ_i is k_i-ary and each Σ_j is l_j-ary. We identify the partial predicates with partial mapping taking values in $\{0, 1\}$, writing 0 for true and 1 for false. We use also $\mathrm{Dom}(f)$ and $\mathrm{Ran}(f)$ to denote the domain and the range of the function f respectively.

An *effective enumeration* of the structure $\mathfrak{A}$ is any ordered pair $\langle \alpha, \mathfrak{B} \rangle$ where $\mathfrak{B} = \langle \omega; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k \rangle$ is a partial structure of the same relational type as $\mathfrak{A}$, and α is a surjective mapping of ω onto B such that the following conditions hold:

a) $\varphi_1, \dots, \varphi_n$ are partial recursive (p.r.) and $\sigma_1, \dots, \sigma_k$ are recursively enumerable (r.e.);

b) $\alpha(\varphi_i(x_1, \dots, x_{k_i})) \cong \theta_i(\alpha(x_1), \dots, \alpha(x_{k_i}))$ for every natural numbers $x_1, \dots, x_{k_i}$, $1 \leq i \leq n$;

c) $\sigma_j(x_1, \dots, x_{l_j}) \cong \Sigma_j(\alpha(x_1), \dots, \alpha(x_{l_j}))$ for all natural numbers $x_1, \dots, x_{l_j}$, $1 \leq j \leq k$.

If θ is a partial function of m variables on B, then we say θ is effective in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ iff $\mathrm{Dom}(\theta) \neq \emptyset$ and there exists such p.r. function f that for all natural numbers $i_1, \dots, i_m$,

$$\theta(\alpha(i_1), \dots, \alpha(i_m)) \cong \alpha(f(i_1, \dots, i_m)).$$

We exclude the trivial case of an empty function because it doesn't depend on any enumeration.

It is said that θ is *e-admissible* in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ iff θ is effective in every effective enumeration of the structure $\mathfrak{A}$.

We say that $\langle \alpha, \mathfrak{B}_0 \rangle$ is an *exact effective enumeration* if it is an effective enumeration, and for every partial function θ, θ is e-admissible iff θ is effective in $\langle \alpha, \mathfrak{B}_0 \rangle$.

It is well known that there are structures which don't have effective enumerations [13].The above definitions don't make good sense in all those cases. We'll consider those definitions only in case when the structure admits an effective enumeration. Actually, when the structure has only total functions and no predicates it admits an effective enumeration.

There isn't an established definition for partial order of the set of enumerations. There are different possibilities to define partial order, depending on different reducibilities in the set of all sets of natural numbers and aims of research. Here we shall take one of the possibilities, connected with m-reducibility.

Definition 1. *It is said that $\langle \alpha_0, \mathfrak{B}_0 \rangle \leq \langle \alpha, \mathfrak{B} \rangle$ iff there exist partial recursive function f such that for all natural numbers n,*

$$\alpha_0(n) \cong \alpha(f(n)).$$

It is said that $\langle \alpha_0, \mathfrak{B}_0 \rangle$, $\langle \alpha, \mathfrak{B} \rangle$ are incomparable iff neither $\langle \alpha_0, \mathfrak{B}_0 \rangle \leq \langle \alpha, \mathfrak{B} \rangle$ nor $\langle \alpha, \mathfrak{B} \rangle \leq \langle \alpha_0, \mathfrak{B}_0 \rangle$.

Let $\mathcal{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i.e. $\mathcal{L}$ consists of n functional symbols $\mathbf{f_1}, \ldots, \mathbf{f_n}$ and k predicate symbols $\mathbf{T_1}, \ldots, \mathbf{T_k}$, where $\mathbf{f_i}$ is k_i-ary and $\mathbf{T_j}$ is l_j-ary. We add a new unary predicate symbol $\mathbf{T_0}$ which will represent the unary total predicate Σ_0, where $\Sigma_0(s) = 0$ for all $s \in B$.

Let us have a denumerable set of variables. We shall use capital letters X, Y, Z and the same letters by indexes to denote variables.

If τ is a term in the language $\mathcal{L}$, then we write $\tau(X_1, \ldots, X_l)$ to denote that all the variables in the term τ are among $X_1, \ldots, X_l$. If $s_1, \ldots, s_l$ are elements of B and $\tau(X_1, \ldots, X_l)$ is a term, then by $\tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_l/s_l)$ we denote the value of the term τ in the structure $\mathfrak{A}$ over the elements $s_1, \ldots, s_l$, if it exists.

We intend to show that all structures with total functions and no predicates have effective exact enumerations. Let from now on $\mathfrak{A} = \langle B; \theta_1, \ldots, \theta_n \rangle$ be an arbitrary structure, where $\theta_1, \ldots, \theta_n$ are total functions and B is a denumerable set. The case when B is a finite set is analogous. As in [7] we shall construct a special kind of enumerations. Later this kind of enumerations is generalized and called normal enumerations [4].

Let $\langle p_1, \ldots, p_n \rangle$be some fixed coding function of all finite sequences of natural numbers.

Define $f_i(p_1, \ldots, p_{k_i}) = \langle i-1, p_1, \ldots, p_{k_i} \rangle$, $i = 1, \ldots, n$ and

$N_0 = \omega \setminus (\mathrm{Ran}(f_1) \cup \cdots \cup \mathrm{Ran}(f_{k_n}))$. It is obvious that N_0 is a recursive set. For every surjective mapping α^0 of N_0 onto B(called basis) we define a partial mapping of ω onto B by the following inductive clauses:

(i) If $p \in N_0$, then $\alpha(p) = \alpha^0(p)$;

(ii) If $p = f_i(q_1, \ldots, q_{k_i})$, $\alpha(q_1) = s_1, \ldots, \alpha(q_{k_i}) = s_{k_i}$ and $\theta_i(s_1, \ldots, s_{k_i}) = t$, then $\alpha(p) = t$.

It is well known that α is well defined and let $\mathfrak{B} = \langle \omega; f_1, \ldots, f_n \rangle$. We shall recall some obvious propositions for such kind $\langle \alpha, \mathfrak{B} \rangle$. The proofs are the same as in the case of normal enumerations [4].

Proposition 1. *For every $1 \leq i \leq n$ and $p_1, \ldots, p_{k_i} \in \omega$,*

$$\alpha(f_i(p_1, \ldots, p_{k_i})) = \theta_i(\alpha(p_1), \ldots, \alpha(p_{k_i})).$$

Corollary 1. *Let $\tau(Y_1, \ldots, Y_m)$ be a term and $p_1, \ldots, p_m \in \omega$. Then*

$$\alpha(\tau_{\mathfrak{B}}(Y_1/p_1, \ldots, Y_m/p_m)) = \tau_{\mathfrak{A}}(\alpha(p_1), \ldots, \alpha(p_m)).$$

Corollary 2. *$\langle \alpha, \mathfrak{B} \rangle$ is an effective enumeration.*

Proposition 2. *There exists an effective way to define for every p of ω elements $q_1, \ldots, q_m \in N_0$ and term $\tau(Y_1, \ldots, Y_m)$ such that*

$$p = \tau_{\mathfrak{B}}(Y_1/p_1, \ldots, Y_m/p_m).$$

A term τ which we define by the above proposition from the natural number p we will denote by τ^p.

We can define the just mentioned enumerations also in the following way. Let $B = \{a_0, a_1, \ldots\}$, where $a_0, a_1, \ldots$ are different. Let $A_0, A_1, \ldots$ be a sequence of disjoint subsets of N_0 such that $\bigcup_{i \in \omega} A_i$. We define $[A_0], [A_1], \ldots$ as follows:

(a) If $p \in A_i$, then $p \in [A_i]$;

(b) $1 \leq i \leq n$ and $p_1 \in [A_{j_1}], \ldots, p_{k_i} \in [A_{j_{k_i}}]$ and $a_q = \theta_i(a_{j_1}, \ldots, a_{j_{k_i}})$, then $f_i(p_1, \ldots, p_{k_i}) \in [A_q]$.

Taking $\alpha^0(A_i) = a_i$ we have the basis and then we have $\alpha^{-1}(a_i) = [A_i]$. From now on if we define some sequence $[A_0], [A_1], \ldots$ of disjoint subsets of N_0 we shall have in mind the above mentioned enumeration.

Corollary 3. *Let $A_0, A_1, \ldots$ be a sequence of disjoint nonempty subsets of N_0. Then $\langle \alpha, \mathfrak{B} \rangle$ is an effective enumeration of the structure $\mathfrak{A}$.*

3. THE MAIN RESULTS

Theorem 1. *Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n \rangle$ be a structure, where $\theta_1, \dots, \theta_n$ are total functions. Then there exists an exact effective enumeration $\langle \alpha, \mathfrak{B} \rangle$.*

Proof. First we shall recall that in [7] it is shown that all e-admissible functions in the structure $\mathfrak{A}$, which are defined at least in one point, are exactly all search computable functions, which in this case are all superpositions of the functions $\theta_1, \dots, \theta_n$, projecting and constant functions of many variables.

We will build an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ of the structure $\mathfrak{A}$ building a sequence $A_0, A_1, \dots$ by steps. In each step s we will define a sequence $A_{0,s}, A_{1,s}, \dots$ of N_0 such that:

(i) $A_{i,s}$ is a finite subset of N_0, $i, s \in \omega$;

(ii) $A_{i,s} \subseteq A_{i,s+1}$, $i, s \in \omega$.

At the end we will take $A_i = \cup_{s=0}^{+\infty} A_{i,s}$ and $\alpha([A_i]) = a_i$, $i \in \omega$. With the even steps we shall ensure that there is no subset of some Cartesian product of B different of that Cartesian product of B which is a domain of some e-admissible function. With the odd steps we shall ensure that the only e-admissible functions are all superpositions of the functions $\theta_1, \dots, \theta_n$.

Let $\varphi_0^{(k)}, \varphi_1^{(k)}, \dots$ be the standard enumeration of all partial recursive functions on k variables, $W_0^{(k)}, W_1^{(k)}, \dots$ be the standard enumeration of all recursively enumerable subsets of ω^k and $B = \{a_0, a_1, \dots\}$, where $a_0, a_1, \dots$ are different.

Step $s = 0$. Set $A_{i,s} = \emptyset$, $i \in \omega$.

Step $s = 2\langle e, k \rangle + 1$. We check if there exist different elements $p_1, \dots, p_k$, $p'_1, \dots, p'_k$ of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ such that:

(i) $\varphi_e^{(k)}(p_1, \dots, p_k)$ and $\varphi_e^{(k)}(p'_1, \dots, p'_k)$ are defined;

(ii) $\varphi_e^{(k)}(p_1, \dots, p_k) = p =$
$\tau_{\mathfrak{B}}^p(X_1/p_1, \dots, X_k/p_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,
$\varphi_e^{(k)}(p'_1, \dots, p'_k) = q =$
$\tau_{\mathfrak{B}}^q(X'_1/p'_1, \dots, X'_k/p'_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,
where $q_1, \dots, q_l \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k, p'_1, \dots, p'_k\})$ and $r_1, \in A_{j_1,s-1}, \dots, r_m \in A_{j_m,s-1}$;

(iii) There exist $a_{i_1}, \dots, a_{i_k}, a_{n_1}, \dots, a_{n_l} \in B$ such that
$\tau_{\mathfrak{A}}^p(X_1/a_{i_1}, \dots, X_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) \neq$
$\tau_{\mathfrak{A}}^q(X'_1/a_{i_1}, \dots, X'_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$.

It it is so we set $A_{i_1,s} = A_{i_1,s-1} \cup \{p_1, p'_1\}, \dots, A_{i_k,s} = A_{i_k,s-1} \cup \{p_k, p'_k\}$, $A_{n_1,s} = A_{n_1,s-1} \cup \{q_1\}, \dots, A_{n_l,s} = A_{n_l,s-1} \cup \{q_l\}$ and $A_{i,s} = A_{i,s-1}$ for all $i \notin \{i_1, \dots, i_k, n_1, \dots, n_l\}$. Otherwise set $A_{i,s} = A_{i,s-1}$ for all $i \in \omega$.

Step $s = 2\langle e, j \rangle + 2$. Let p be the least element of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ and $p \in W_e^{(1)}$, if such elements p exist. Set $A_{j,s} = A_{j,s-1} \cup \{p\}$ and $A_{i,s} = A_{i,s-1}$ for all $i \neq j$, if such elements exist. Otherwise set $A_{i,s} = A_{i,s-1}$ for all $i \in \omega$.

We fix $A_i = \cup_{s=0}^{+\infty} A_{i,s}$, $\alpha([A_i]) = a_i$, $i \in \omega$ and construction is completed.

Lemma 1. *For every natural number s, $A_{0,s-1} \cup A_{1,s-1} \cup \ldots$ is finite.*

Proof. For every step s we add only finitely many numbers to

$$A_{0,s-1} \cup A_{1,s-1} \cup \ldots .$$ □

Lemma 2. *Let e be such that $W_e^{(1)}$ is infinite. Then for every $j \in \omega$ there exists $p \in W_e^{(1)}$, such that $p \in A_j$.*

Proof. Let j be an arbitrary element of ω. Then on step $s = 2\langle e, j\rangle + 2$ we find $p \in W_e^{(1)}$ such that $p \in N_0$. Then we set $p \in A_{j,s} \subseteq A_j$. □

Corollary 4. *For every natural i, A_i is infinite and immune and $[A_i]$ is not recursively enumerable.*

Proof. Indeed, for every infinite r.e. subset $W_e^{(1)}$ of N_0 and every element $a_i \in B$ there exists an element $p \in W_e^{(1)}$ such that $p \in A_i$. Therefore, A_i is infinite and $W_e^{(1)} \cap (N_0 \setminus A_i) = W_e^{(1)} \cap (\cup_{j \neq i} A_j) \neq \emptyset$, i.e. A_i is immune and not recursively enumerable. □

Analogously one can prove the following

Corollary 5. *For every nonempty subset L of ω, $L \neq \omega$, $\cup_{i \in L} A_i$ is infinite and immune and $\cup_{i \in L}[A_i]$ is not recursively enumerable.*

Corollary 6. *For every natural $m \geq 1$ and every nonempty subset L of ω^m such that $L \neq \omega^m$ the set*

$$M = \cup\{(p_1, \ldots, p_m) | \exists j_1 \ldots \exists j_m [(j_1, \ldots, j_m) \in L \& p_1 \in A_{j_1} \& \ldots \& p_m \in A_{j_m}]\}$$

is not recursively enumerable.

Proof. First we claim: there exist coordinate i, $1 \leq i \leq m$, and $j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_m$ such that the set $L' = \{j | (j_1, \ldots, j_{i-1}, j, j_{i+1}, \ldots, j_m) \in L\}$, which is an i-th projection of L for the fixed $j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_m$, is nonempty and $L' \neq \omega$. For the sake of simplicity let $m = 2$. Since $L \neq \emptyset$, there exists (j_1', j_2') such that $(j_1', j_2') \in L$. Analogously, $L \neq \omega^2$ and there exists (i_1, i_2) such that $(i_1, i_2) \notin L$. If $(j_1', i_2) \in L$, then fix $i = 1$ and $j_2 = i_2$ and the claim is true; otherwise $(j_1', i_2) \notin L$, $(j_1', j_2') \in L$, fix $i = 2$ and $j_1 = j_1'$. Thus the claim is true again.

Let us assume that M is r.e. Then for some fixed i and $j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_m$ the set

$L' = \{j | (j_1, \ldots, j_{i-1}, j, j_{i+1}, \ldots, j_m) \in L\} \neq \emptyset$ and $L' \neq \omega$. Let

$A' = \{a_k | (a_{j_1}, \ldots, a_{j_{i-1}}, a_k, a_{j_{i+1}}, \ldots, a_{j_m}) \in A\}$. Then
$M' = \{p | \exists j[p \in A_j \& a_j \in A']\} = \cup_{j \in L'} A_j$. According to the previous corollary, M' is not r.e. On the other hand, if we fix
$p_1 \in A_{j_1}, \ldots, p_{i-1} \in A_{j_{i-1}}, p_{i+1} \in A_{j_{i+1}}, p_m \in A_{j_m}$, then
$M' = \{p | \exists j[p \in A_j \& (j_1, \ldots, j_{i-1}, j, j_{i+1}, \ldots, j_m) \in L]\} =$
$\{p | \exists p_{j_1} \ldots \exists p_{j_{i-1}} \exists p_{j_{i+1}} \ldots \exists p_{j_m} [(p_{j_1}, \ldots, p_{j_{i-1}}, p, p_{j_{i+1}}, \ldots, p_{j_m}) \in M]\}$ is r.e. The obtained contradiction shows that the assumption is wrong. □

It is easy to check the following

Corollary 7. *For every natural $m \geq 1$ and every nonempty subset L of ω^m, such that $L \neq \omega^m$*

$$\cup\{(p_1, \ldots, p_m) | \exists j_1 \ldots \exists j_m [(j_1, \ldots, j_m) \in L \& p_1 \in [A_{j_1}] \& \ldots \& p_m \in [A_{j_m}]]\}$$

is not recursively enumerable.

Corollary 8. *For every natural $m \geq 1$ and every nonempty subset A of B^m such that $A \neq B^m$*

$$\cup\{(p_1, \ldots, p_m) | \exists j_1 \ldots \exists j_m [(a_{j_1}, \ldots, a_{j_m}) \in A \& p_1 \in A_{j_1} \ldots \& p_m \in A_{j_m}]\}$$

is not recursively enumerable.

Corollary 9. *For every function θ such that $\mathrm{Dom}(\theta) \subseteq B^m$ and θ is effective in the enumeration $\langle \alpha, \mathfrak{B} \rangle$, the equality $\mathrm{Dom}(\theta) = B^m$ holds.*

Proof. It is an immediate corollary of the previous one. □

Lemma 3. $N_0 \subseteq \mathrm{Dom}(\alpha)$.

Proof. Let us assume that $N_0 \setminus \mathrm{Dom}(\alpha) \neq \emptyset$ and p_0 is the least element of $N_0 \setminus \mathrm{Dom}(\alpha)$. Then there exists a step $s = 2\langle e, j \rangle + 2$ such that p_0 is the least element of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \ldots)$ and $W_e^{(1)} = \omega$. At that step s we have to put p_0 in some $A_j \subseteq \mathrm{Dom}(\alpha)$. The contradiction obtained shows that $N_0 \subseteq \mathrm{Dom}(\alpha)$. □

Now it is obvious that

Corollary 10. $\mathrm{Dom}(\alpha) = \omega$.

Let θ be effective in $\langle \alpha, \mathfrak{B} \rangle$ and $\mathrm{Dom}(\theta) \subseteq B^k$ for some natural $k \geq 1$. Then $\mathrm{Dom}(\theta) = B^k$ and there exists p.r. function f such that for all natural numbers $i_1, \ldots, i_k$,

$$\theta(\alpha(i_1), \ldots, \alpha(i_k)) \cong \alpha(f(i_1, \ldots, i_k)).$$

Therefore f is a total function and $f = \varphi_e^{(k)}$ for some natural e. Let us consider step $s = 2\langle e, k\rangle + 1$.

First assume there are different elements $p_1, \dots, p_k, p'_1, \dots, p'_k$ of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ satisfying the conditions (i)–(iii) at step $s = 2\langle e, k\rangle + 1$ and fix such elements $p_1, \dots, p_k, p'_1, \dots, p'_k$. Then according to Corollary 1, $\alpha(f(p_1, \dots, p_k)) = \alpha(\varphi_e^{(k)}(p_1, \dots, p_k)) =$
$\alpha(\tau^p_{\mathfrak{B}}(X_1/p_1, \dots, X_k/p_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)) =$
$\tau^p_{\mathfrak{A}}(X_1/\alpha(p_1), \dots, X_k/\alpha(p_k), Y_1/\alpha(q_1), \dots, Y_l/\alpha(q_l), Z_1/\alpha(r_1), \dots, Z_m/\alpha(r_m))$
$= \tau^p_{\mathfrak{A}}(X_1/a_{i_1}, \dots, X_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) \neq$
$\tau^q_{\mathfrak{A}}(X'_1/a_{i_1}, \dots, X'_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) =$
$\tau^q_{\mathfrak{A}}(X'_1/\alpha(p'_1), \dots, X'_k/\alpha(p'_k), Y_1/\alpha(q_1), \dots, Y_l/\alpha(q_l), Z_1/\alpha(r_1), \dots, Z_m/\alpha(r_m))$
$= \alpha(\tau^q_{\mathfrak{B}}(X'_1/p'_1, \dots, X'_k/p'_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)) =$
$\alpha(\varphi_e^{(k)}(p'_1, \dots, p'_k)) = \alpha(f(p'_1, \dots, p'_k))$.

On the other hand, $\alpha(f(p_1, \dots, p_k)) = \theta(\alpha(p_1), \dots, \alpha(p_k)) = \theta(a_{i_1}, \dots, a_{i_k}) = \theta(\alpha(p'_1), \dots, \alpha(p'_k)) = \alpha(f(p'_1, \dots, p'_k))$. That contradiction shows this case isn't possible. Therefore, there aren't different elements $p_1, \dots, p_k, p'_1, \dots, p'_k$ of $N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ satisfying the conditions (i) – (iii)

Let us fix different $p_1, \dots, p_k \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$. Then
$f(p_1, \dots, p_k) = \varphi_e^{(k)}(p_1, \dots, p_k) = p =$
$\tau^p_{\mathfrak{B}}(X_1/p_1, \dots, X_k/p_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$, where $q_1, \dots, q_l \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k\})$, $r_1, \in A_{j_1,s-1}, \dots, r_m \in A_{j_m,s-1}$. Furthermore, for every different $p'_1, \dots, p'_k \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k\})$, $f(p'_1, \dots, p'_k) = \varphi_e^{(k)}(p'_1, \dots, p'_k) = q =$
$\tau^q_{\mathfrak{B}}(X'_1/p'_1, \dots, X'_k/p'_k, Y'_1/q'_1, \dots, Y'_{l'}/q'_{l'}, Z_1/r_1, \dots, Z_m/r_m)$, where
$q'_1, \dots, q'_{l'} \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k, p'_1, \dots, p'_k\})$,
$r_1, \in A_{j_1,s-1}, \dots, r_m \in A_{j_m,s-1}$ and for every
$a_{i_1}, \dots, a_{i_k}, a_{n_1}, \dots, a_{n_l}, a_{n'_1}, \dots, a_{n'_{l'}} \in B$
$\tau^p_{\mathfrak{A}}(X_1/a_{i_1}, \dots, X_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) =$
$\tau^q_{\mathfrak{A}}(X'_1/a_{i_1}, \dots, X'_k/a_{i_k}, Y'_1/a_{n'_1}, \dots, Y'_{l'}/a_{n'_{l'}}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$.

Therefore, $\theta(\alpha(p'_1), \dots, \alpha(p'_k)) = \alpha(f(p'_1, \dots, p'_k)) =$
$\alpha(\tau^q_{\mathfrak{B}}(X'_1/p'_1, \dots, X'_k/p'_k, Y'_1/q'_1, \dots, Y'_{l'}/q'_{l'}, Z_1/r_1, \dots, Z_m/r_m)) =$
$\tau^q_{\mathfrak{A}}(X'_1/\alpha(p'_1), \dots, X'_k/\alpha(p'_k), Y'_1/\alpha(q'_1), \dots, Y'_{l'}/\alpha(q'_{l'}), Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) =$
$\tau^p_{\mathfrak{A}}(X_1/\alpha(p'_1), \dots, X_k/\alpha(p'_k), Y_1/\alpha(q_1), \dots, Y_l/\alpha(q_l), Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$.

Let $q_1 \in A_{n_1}, \dots, q_l \in A_{n_l}$ and θ' be the function
$\theta'(b_1, \dots, b_k) = \tau^p_{\mathfrak{A}}(X_1/b_1, \dots, X_k/b_k, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$ for fixed $a_{n_1}, \dots, a_{n_l}, a_{j_1}, \dots, a_{j_m}$. We'll prove that $\theta = \theta'$. Let $(b_1, \dots, b_k)$ be an arbitrary k-tuple of B^k. Take $p'_1, \dots, p'_k \in N_0 \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k\})$ such that $\alpha(p'_1) = b_1, \dots, \alpha(p'_k) = b_k$. It is possible because every element of B has infinitely many numbers. Then $\theta(\alpha(p'_1), \dots, \alpha(p'_k)) = \tau^p_{\mathfrak{A}}(X_1/\alpha(p'_1), \dots, X_k/\alpha(p'_k),$ $Y_1/\alpha(q_1), \dots, Y_l/\alpha(q_l), Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) = \theta(b_1, \dots, b_k)$ and θ' is a superposition of the function $\theta_1, \dots, \theta_{n},$ projecting and constant functions of many variables. □

Theorem 2. *Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n \rangle$ be a structure, where $\theta_1, \dots, \theta_n$ are total functions. If $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is an effective enumeration in $\mathfrak{A}$, then there exists an exact effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ such that $\langle \alpha_0, \mathfrak{B}_0 \rangle \leq \langle \alpha, \mathfrak{B} \rangle$.*

Proof. We shall give only the construction of the effective enumeration. The proof that it has the required properties is analogous to the previous one.

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an effective enumeration in $\mathfrak{A}$, N_0 the same as in the proof of the previous Theorem and $N_0 = N_0' \cup N_0''$, where N_0', N_0'' are infinite recursive sets. Take recursive $f(i) = p_i''$, where $N_0'' = \{p_0'', p_1'', \dots\}$, $p_0'' < p_1'', < \dots$. As in the proof of the previous theorem we will build an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ of the structure $\mathfrak{A}$ building a sequence $A_0, A_1, \dots$ by steps. In each step s we will define a sequence $A_{0,s}, A_{1,s}, \dots$ of N_0 such that:

(i) $A_{i,s} \cap N_0'$ is a finite subset of N_0', $i, s \in \omega$;

(ii) $A_{i,s} \subseteq A_{i,s+1}$, $i, s \in \omega$.

At the end we will take $A_i = \cup_{s=0}^{+\infty} A_{i,s}$ and $\alpha([A_i]) = a_i$, $i \in \omega$.

Let $\varphi_0^{(k)}, \varphi_1^{(k)}, \dots$ be the standard enumeration of all partial recursive functions of k variables, $W_0^{(k)}, W_1^{(k)}, \dots$ be the standard enumeration of all recursively enumerable subsets of ω^k and $B = \{a_0, a_1, \dots\}$ where $a_0, a_1, \dots$ are different.

Step $s = 0$. Set $A_{i,s} = \{p | p \in N_0'' \& \exists q [f(q) = p \& \alpha_0(q) = a_i]\}$, $i \in \omega$.

Step $s = 2\langle e, k \rangle + 1$. We check if there exist different elements $p_1, \dots, p_k$, $p_1', \dots, p_k'$ of $N_0' \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ such that:

(i) $\varphi_e^{(k)}(p_1, \dots, p_k)$ and $\varphi_e^{(k)}(p_1', \dots, p_k')$ are defined;

(ii) $\varphi_e^{(k)}(p_1, \dots, p_k) = p =$
$\tau_{\mathfrak{B}}^p(X_1/p_1, \dots, X_k/p_k, Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,
$\varphi_e^{(k)}(p_1', \dots, p_k') = q =$
$\tau_{\mathfrak{B}}^q(X_1'/p_1', \dots, X_k'/p_k', Y_1/q_1, \dots, Y_l/q_l, Z_1/r_1, \dots, Z_m/r_m)$,
where $q_1, \dots, q_l \in N_0' \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots \cup \{p_1, \dots, p_k, p_1', \dots, p_k'\})$ and $r_1, \in A_{j_1,s-1}, \dots, r_m \in A_{j_m,s-1}$;

(iii) There exist $a_{i_1}, \dots, a_{i_k}, a_{n_1}, \dots, a_{n_l} \in B$ such that
$\tau_{\mathfrak{A}}^p(X_1/a_{i_1}, \dots, X_k/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m}) \neq$
$\tau_{\mathfrak{A}}^q(X_1'/a_{i_1}, \dots, X_k'/a_{i_k}, Y_1/a_{n_1}, \dots, Y_l/a_{n_l}, Z_1/a_{j_1}, \dots, Z_m/a_{j_m})$.

If it is so we set $A_{i_1,s} = A_{i_1,s-1} \cup \{p_1, p_1'\}, \dots, A_{i_k,s} = A_{i_k,s-1} \cup \{p_k, p_k'\}$, $A_{n_1,s} = A_{n_1,s-1} \cup \{q_1\}, \dots, A_{n_l,s} = A_{n_l,s-1} \cup \{q_l\}$ and $A_{i,s} = A_{i,s-1}$ for all $i \notin \{i_1, \dots, i_k, n_1, \dots, n_l\}$. Otherwise set $A_{i,s} = A_{i,s-1}$ for all $i \in \omega$.

Step $s = 2\langle e, j \rangle + 2$. Let p be the least element of $N_0' \setminus (A_{0,s-1} \cup A_{1,s-1} \cup \dots)$ and $p \in W_e^{(1)}$, if such elements p exist. Set $A_{j,s} = A_{j,s-1} \cup \{p\}$ and $A_{i,s} = A_{i,s-1}$ for all $i \neq j$, if such elements exist. Otherwise set $A_{i,s} = A_{i,s-1}$ for all $i \in \omega$.

We fix $A_i = \cup_{s=0}^{+\infty} A_{i,s}$, $\alpha([A_i]) = a_i$, $i \in \omega$ and construction is completed. □

Theorem 3. *There exist infinitely many mutually incomparable exact effective enumerations.*

Proof. We will build effective enumerations $\langle \alpha_j, \mathfrak{B}_j \rangle$, $j \in \omega$ of the structure

$\mathfrak{A}$ building a sequence $A_0^j, A_1^j, \ldots$, $j \in \omega$, by steps. In each step s we will define a sequence $A_{0,s}^j, A_{1,s}^j, \ldots$, of subsets of N_0, $j \in \omega$, such that:

(i) $A_{i,s}^j$ is a finite subset of N_0, $i, j, s \in \omega$;

(ii) $A_{i,s}^j \subseteq A_{i,s+1}^j$, $i, j, s \in \omega$.

With the steps of the kind $3k+1$ we shall ensure for every j that there isn't a subset of some Cartesian product of B different of that Cartesian product of B which is a domain of some e-admissible function for the enumeration $\langle \alpha_j, \mathfrak{B}_j \rangle$. With the steps of the kind $3k+2$ we shall ensure that the only e-admissible functions for the enumeration $\langle \alpha_j, \mathfrak{B}_j \rangle$ are all superpositions of the functions $\theta_1, \ldots, \theta_n$. With the steps of the kind $3k+3$ we shall ensure that $\langle \alpha_j, \mathfrak{B}_j \rangle \not\leq \langle \alpha_k, \mathfrak{B}_k \rangle$ for $j \neq k$, $j, k \in \omega$.

As above, $\varphi_0^{(k)}, \varphi_1^{(k)}, \ldots$ is the standard enumeration of all partial recursive functions on k variables, $W_0^{(k)}, W_1^{(k)}, \ldots$ is the standard enumeration of all recursively enumerable subsets of ω^k and $B = \{a_0, a_1, \ldots\}$, where $a_0, a_1, \ldots$ are different.

Step $s = 0$. Set $A_{i,s}^j = \emptyset$, $i, j \in \omega$.

Step $s = 3\langle e, k, j \rangle + 1$. We check if there exist different elements $p_1, \ldots, p_k, p_1', \ldots, p_k'$ of $N_0 \setminus (A_{0,s-1}^j \cup A_{1,s-1}^j \cup \ldots)$ such that:

(i) $\varphi_e^{(k)}(p_1, \ldots, p_k)$ and $\varphi_e^{(k)}(p_1', \ldots, p_k')$ are defined;

(ii) $\varphi_e^{(k)}(p_1, \ldots, p_k) = p =$
$\tau_{\mathfrak{B}}^p(X_1/p_1, \ldots, X_k/p_k, Y_1/q_1, \ldots, Y_l/q_l, Z_1/r_1, \ldots, Z_m/r_m)$,

$\varphi_e^{(k)}(p_1', \ldots, p_k') = q =$
$\tau_{\mathfrak{B}}^q(X_1'/p_1', \ldots, X_k'/p_k', Y_1/q_1, \ldots, Y_l/q_l, Z_1/r_1, \ldots, Z_m/r_m)$,
where $q_1, \ldots, q_l \in N_0 \setminus (A_{0,s-1}^j \cup A_{1,s-1}^j \cup \cdots \cup \{p_1, \ldots, p_k, p_1', \ldots, p_k'\})$ and $r_1, \in A_{j_1,s-1}^j, \ldots, r_m \in A_{j_m,s-1}^j$;

(iii) There exist $a_{i_1}, \ldots, a_{i_k}, a_{n_1}, \ldots, a_{n_l} \in B$ such that
$\tau_{\mathfrak{A}}^p(X_1/a_{i_1}, \ldots, X_k/a_{i_k}, Y_1/a_{n_1}, \ldots, Y_l/a_{n_l}, Z_1/a_{j_1}, \ldots, Z_m/a_{j_m}) \neq$
$\tau_{\mathfrak{A}}^q(X_1'/a_{i_1}, \ldots, X_k'/a_{i_k}, Y_1/a_{n_1}, \ldots, Y_l/a_{n_l}, Z_1/a_{j_1}, \ldots, Z_m/a_{j_m})$.

If it is so we set $A_{i_1,s}^j = A_{i_1,s-1}^j \cup \{p_1, p_1'\}, \ldots, A_{i_k,s}^j = A_{i_k,s-1}^j \cup \{p_k, p_k'\}$, $A_{n_1,s}^j = A_{n_1,s-1}^j \cup \{q_1\}, \ldots, A_{n_l,s}^j = A_{n_l,s-1}^j \cup \{q_l\}$ and $A_{i,s}^j = A_{i,s-1}^j$ for all $i \notin \{i_1, \ldots, i_k, n_1, \ldots, n_l\}$. Otherwise set $A_{i,s}^j = A_{i,s-1}^j$ for all $i \in \omega$.

Step $s = 3\langle e, k, j \rangle + 2$. Let p be the least element of $N_0 \setminus (A_{0,s-1}^j \cup A_{1,s-1}^j \cup \ldots)$ and $p \in W_e^{(1)}$, if such elements p exist. Set $A_{k,s}^j = A_{k,s-1}^j \cup \{p\}$ and $A_{i,s}^j = A_{i,s-1}^j$ for all $i \neq k$, if such elements exist. Otherwise set $A_{i,s}^j = A_{i,s-1}^j$ for all $i \in \omega$.

Step $s = 3\langle e, k, j \rangle + 3$.

Let first $k \neq j$, $\varphi_e^{(1)}$ be a total function. Let p be the least element of $N_0 \setminus (A_{0,s-1}^k \cup A_{1,s-1}^k \cup \ldots)$,

$\varphi_e^{(1)}(p) = q = \tau_{\mathfrak{B}}^q(X/p, Y_1/q_1, \ldots, Y_l/q_l, Z_1/r_1, \ldots, Z_m/r_m)$, where $q_1, \ldots, q_l \in N_0 \setminus (A_{0,s-1}^j \cup A_{1,s-1}^j \cup \cdots \cup \{p\})$ and $r_1 \in A_{j1,s-1}^j, \ldots, r_m \in A_{j_m,s-1}^j$.

Fix $A_{j,s}^j = A_{j,s-1}^j \cup \{p, q_1, \ldots, q_l\}$ and $A_{i,s}^j = A_{i,s-1}^j$, for $i \neq j$, $i \in \omega$.

We check if $\tau^q_{\mathfrak{A}}(X/a_j, Y_1/a_j, \ldots, Y_l/a_j, Z_1/a_{j_1}, \ldots, Z_m/a_{j_m}) = a_j$. If so, fix $A^k_{k,s} = A^k_{k,s-1} \cup \{p\}$ and $A^k_{i,s} = A^k_{i,s-1}$ for $i \neq k$, $i \in \omega$; otherwise fix $A^k_{k-1,s} = A^k_{k-1,s-1} \cup \{p\}$ and $A^k_{i,s} = A^k_{i,s-1}$ for $i \neq k-1$, $i \in \omega$.

Fix $A^{i'}_{i,s} = A^{i'}_{i,s-1}$ for $i, i' \in \omega$, $i' \notin \{j, k\}$.

In case either $k = j$ or $\varphi^{(1)}_e$ is not a total function, fix $A^{i'}_{i,s} = A^{i'}_{i,s-1}$ for $i, i' \in \omega$, $i' \notin \{j, k\}$.

At the end we fix $A^j_i = \cup^{+\infty}_{s=0} A^j_{i,s}$, $\alpha^{-1}_j([A^j_i]) = a_i$, $i, j \in \omega$, and construction is completed.

The proof that $\langle \alpha_j, \mathfrak{B}_j \rangle$, $j \in \omega$, is an exact effective enumeration is analogous to the previous ones. We'll concentrate on the proof that $\langle \alpha_j, \mathfrak{B}_j \rangle$ and $\langle \alpha_k, \mathfrak{B}_k \rangle$ are incomparable.

Let us assume that $\langle \alpha_k, \mathfrak{B}_k \rangle \leq \langle \alpha_j, \mathfrak{B}_j \rangle$. Then there exists a total recursive function f such that for all natural p, $\alpha_k(p) = \alpha(f(p))$. Let $f = \varphi^{(1)}_e$, consider the step $s = 3\langle e, k, j \rangle + 3$ and let p be the element belonging to $N_0 \setminus (A^k_{0,s-1} \cup A^k_{1,s-1} \cup \ldots)$ chosen on that step.

Then $\varphi^{(1)}_e(p) = q = \tau^q_{\mathfrak{B}}(X/p, Y_1/q_1, \ldots, Y_l/q_l, Z_1/r_1, \ldots, Z_m/r_m)$, where $q_1, \ldots, q_l \in N_0 \setminus (A^j_{0,s-1} \cup A^j_{1,s-1} \cup \cdots \cup \{p\})$, $r_1 \in A^j_{j_1,s-1}, \ldots, r_m \in A^j_{j_m,s-1}$ and $A^j_{j,s} = A^j_{j,s-1} \cup \{p, q_1, \ldots, q_l\}$.

We have to consider two cases. We'll consider only the first one:
$\tau^q_{\mathfrak{A}}(X/a_j, Y_1/a_j, \ldots, Y_l/a_j, Z_1/a_{j_1}, \ldots, Z_m/a_{j_m}) = a_j$.
Then $A^k_{k,s} = A^k_{k,s-1} \cup \{p\}$, $\alpha_k(p) = a_k \neq a_j =$
$\alpha_j(\tau^q_{\mathfrak{A}}(X/a_j, Y_1/a_j, \ldots, Y_l/a_j, Z_1/a_{j_1}, \ldots, Z_m/a_{j_m})) =$
$\tau^q_{\mathfrak{A}}(X/\alpha_j(p), Y_1/\alpha_j(q_1), \ldots, Y_l/\alpha_j(q_l), Z_1/\alpha_j(r_1), \ldots, Z_m/\alpha_j(r_m)) =$
$\alpha_j(\tau^q_{\mathfrak{B}}(X/p, Y_1/q_1, \ldots, Y_l/q_l, Z_1/r_1, \ldots, Z_m/r_m)) = \alpha_j(q) = \alpha_j(\varphi^{(1)}_e(p)) =$
$\alpha_j(f(p))$.
The contradiction obtained shows the assumption is not true. □

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier Blvd, 1164 Sofia
BULGARIA
E-mail: ditchev@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 100

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FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 100

MODAL OPERATORS FOR RATIONAL GRADING[1]

TINKO TINCHEV, MITKO YANCHEV

A generalization of the majority operator based on a rational degree of grading is introduced. In the natural semantics of the language, Kripke frames such that any world can see finitely many worlds, the set of all valid formulae is a non-normal modal logic, RGML. Decidability of RGML and its completeness with respect to the class of all finite tree-like Kripke frames are the main results of the paper.

Keywords: Graded modal logic, majority logic

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1. INTRODUCTION

The language of modal logic is widely used to formalize notions like knowledge, possibility and necessity. The modal logics for grading formalize our ability to express assertions about quantity, number or a part of the whole and there are well known results in the integer grading. In this paper we examine grading with rational values. We start with a brief review of two distinctive modal logics for grading.

1.1. GRADED MODAL LOGIC GML

The Graded modal logic formalizes the reasonings about a finite number of objects, i.e. it is connected with integer "grading" of the number of objects.

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Graded modal logic (GML) was introduced for the first time by Kit Fine [2]. Some important results were obtained by Fattorosi-Barnaba and Cerrato [3] and by Caro [4]. We introduce briefly the language of GML, its interpretation and the main results known.

The language of GML contains a countable set of propositional variables $P = \{p_1, p_2, \ldots\}$, the Boolean connectives $\neg$ and $\vee$, and a countable set of modal operators $\Diamond_n,\ n \in \mathbb{N}$.

A formula α of GML has the following syntactic form:

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n \alpha,$$

where $p \in P$ and $n \in \mathbb{N}$.

The "integer" modal operators $\Diamond_n$ extend in a natural way the language of normal modal logics, in which $\Diamond\alpha$ says that "α is true in at least one accessible world". The meaning of $\Diamond_n\alpha$ is that "α is true in (strictly) more than n accessible worlds".

The GML formulae are interpreted in the usual Kripke structures. Let $\mathfrak{M} = \langle U, R, V\rangle$ be a Kripke model, where U is a non-empty set of worlds, $R \subseteq U^2$ is an accessibility relation in U and $V : P \to 2^U$ is a valuation function. The propositional variables and the Boolean connectives are evaluated as usual. The evaluation of modal operators is defined in the following way:

$$\mathfrak{M}, x \models \Diamond_n\alpha \Leftrightarrow |\{y|xRy \text{ and } \mathfrak{M}, y \models \alpha\}| > n.$$

A formula α is *valid in a model* $\mathfrak{M}$ iff α is true in any world of the model. α is *valid* iff it is valid in all models (based on a certain class of frames[2]). We will also use the following notation throughout this paper: $R(x) := \{y|xRy\}$ and for an arbitrary formula α (of the corresponding language) $R_\alpha(x) := \{y|\ xRy \text{ and } y \models \alpha\}$. So for the definition above we get the alternative notation:

$$\mathfrak{M}, x \models \Diamond_n\alpha \Leftrightarrow |R_\alpha(x)| > n.$$

GML is shown to be sound and complete with respect to the class of all frames.

GML is shown also to be decidable as it has the finite model property. It is proven that the decidability problem for GML is $PSPACE$-complete.

1.2. MAJORITY LOGIC MJL

The Majority logic was introduced by Eric Pacuit and Samer Salame [5] with the aim to model the concept of majority, i.e. to formalize the reasoning how far a given number of objects is a majority (part) of the whole. The concept of majority plays an important role in different social situations – from taking a decision of a group of friends how to spend the evening, to determining the result of a given

[2] Unless otherwise stated we assume the models, based on the class of all frames.

vote. MJL axiomatizes that concept. Now we give in this section a brief overview of basic ideas, formulations and results from [5].

As an example of the type of reasoning, captured in MJL, a variant of the muddy children puzzle is considered. Suppose that there are $n > 1$ children who have been playing outside and $k > 1$ of them have mud on their forehead. (At that we assume that the children are perfect reasoners, honest, and cannot see the mud on their forehead.) After a while an adult comes and announces: “strictly more than half of you have mud on your forehead”. The man then proceeds to ask the children to say if they have mud on their forehead. It is not too hard to see that the $(k - \lfloor \frac{n}{2} \rfloor)^{th}$ time the children are asked, the dirty ones will correctly respond.

The language of MJL extends GML with a new unary modal operator W, where $W\alpha$ has the meaning “α is true in more than or equal to half of the accessible worlds” (Weak majority). Hence the dual $M\alpha$ means α is true in more than half of the accessible worlds (strict Majority). It is shown that the operator W cannot be defined from the standard modal operators ($\Box$ and $\Diamond$), the same is true for the operator $\Diamond_n$. Furthermore, the modal operator M cannot be expressed by the operators of GML. Hence as in GML, in MJL more expressive power of the language is achieved with the new modal operators.

The intuitive semantics of W and M, described above, makes sense only when the half of a finite set “is measured”. The key problem is what is the majority (i.e. $\geq \frac{1}{2}$ or at least 50%) of an infinite set. As a decision the so called *majority systems*, which generalize the concept of the ultrafilters, are introduced. Having in mind the majority systems, the valid formulae are axiomatized and soundness and completeness are proven.

1. Syntax and semantics of MJL. A *formula* α of MJL has the syntactic form:

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n \alpha \mid W\alpha,$$

where $p \in P$, $P = \{p_1, p_2, \dots\}$ is a countable set of propositional variables, and $n \in \mathbb{N}$, $M\alpha := \neg W \neg \alpha$.

MJL formulae are interpreted in the usual Kripke models [1]. If for any accessible world x the set $R(x)$ is a finite set then the natural semantics is the following:

$$\mathfrak{M}, x \models W\alpha \Leftrightarrow |R_\alpha(x)| \geq \frac{1}{2}|R(x)|.$$

But in the common case the set $R(x)$ can be infinite (and that is the case in proving the completeness, for example). The solution of the problem is found by giving to any set $R(x)$ the opportunity to determine which of its subsets are majority ones. That is achieved by defining for each $R(x)$ a family of subsets, called *a majority system*, members of which satisfy properties in accordance with our intuition of majority subset. For the finite sets this definition completely agrees with the well-known properties of the majority subsets. For the infinite sets it was proven that these properties also hold. The connection between the majority systems and ultrafilters, is also proven: namely every non-principal ultrafilter is a majority system; the reverse is not true. Next, the *majority models* are defined by adding to the definition of a standard Kripke model a *majority function*, comparing to any set of accessible worlds $R(x)$, $x \in U$, its majority system.

2. The main results, stated and proven in [5], are: *Soundness theorem*, saying that MJL is sound with respect to the class of all majority models, and *canonical model theorem*, proving completeness of MJL by means of canonical majority model.

It is pointed out that the main question remains open — the decidability of MJL — with the expectation MJL to possess the finite model property, already proven for GML.

Finally, a possible application of logics for grading is noted and in particular of MJL, in the so called social software, for example in the voting systems.

Now we shall proceed to presenting the modal logic, suggested by us, which introduces modal operators for *rational grading* and which thus develops modal grading, moving the things forward as in comparison with GML, so with MJL. At that we shall follow the semantic approach — we define the language of the new logic and we give the appropriate semantics without axiomatizing the system. Also, we shall consider finite sets of admissible worlds only, i.e. we want the set $R(x)$ to be finite for any $x \in U$. The main results we shall present are: the finite model property with respect to the class of tree-like models, and the decidability of the new logic. The basic idea of the proofs originates in [6] and uses a variant of a theorem from [7].

2. MODAL LOGIC FOR RATIONAL GRADING

2.1. SYNTAX

We define *a modal language* $\mathcal{L}_M$, containing a countable set of propositional variables $P = \{p_1, p_2, \dots\}$, the Boolean connectives $\neg$, $\wedge$ and $\vee$, and the modal operator $M^{p,q}$, where p and q are relatively prime integers and $1 \leq p < q$.

Formulae in $\mathcal{L}_M$ are defined inductively:

1. The elements of P are formulae;

2. If φ and ψ are formulae, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $M^{p,q}\varphi$ are also formulae.

In what follows a *formula* will mean a formula in $\mathcal{L}_M$.

We define, in addition, in the standard manner the rest of the usual Boolean connectives $\rightarrow$ and $\leftrightarrow$, and the dual to $M^{p,q}$ modal operator $W^{p,q} := \neg M^{p,q} \neg$. We also denote: $\top := \varphi \rightarrow \varphi$, $\bot := \neg\top$, where φ is an arbitrary formula.

A frame for $\mathcal{L}_M$ is a tuple $\langle U, R\rangle$, where U is a non-empty set, called *universe*, consisting of *points* or *(possible) worlds*; the elements of U we denote: x, y, dots, eventually with indices. $R \subseteq U^2$ is *an accessibility relation* in U, and we want for any $x \in U$ the set $R(x) := \{y|(x,y) \in R\}$ to be finite. For $(x,y) \in R$ we use the notation $R(x,y)$, also xRy and we say that y is an $(R\text{-})$*successor* of x or $(R\text{-})$*accessible* from x.

A model for $\mathcal{L}_M$ is a triplet $\langle U, R, V\rangle$, where $\langle U, R\rangle$ is a frame for $\mathcal{L}_M$ and $V : P \to 2^U$ is *a valuation* of the variables. We call the model $\langle U, R, V\rangle$ *a model based on the frame* $\langle U, R\rangle$. We denote models by $\mathfrak{M}, \mathfrak{N}$.

Truth: The valuation V from a model $\mathfrak{M}$ is inductively extended for an arbitrary formula; so we obtain a valuation of all the formulae in the model. For $x \in V(\varphi)$, $x \in U$, we say that φ *is true in a world* x and we denote also $\mathfrak{M}, x \models \varphi$ or simply $x \models \varphi$, when that makes no confusion.

The set of successors of the world x in which the formula φ is true, i.e. the set $\{y|(x,y) \in R \text{ and } y \models \varphi\}$, we denote by $R_\varphi(x)$.

For the Boolean connectives the inductive definition is standard:

$$V(\neg\varphi) := U \backslash V(\varphi)$$

$$V(\varphi \wedge \psi) := V(\varphi) \cap V(\psi)$$

$$V(\varphi \vee \psi) := V(\varphi) \cup V(\psi)$$

For the modal operator we define:

$$V(M^{p,q}\varphi) := \{x : |R_\varphi(x)| > \frac{p}{q}|R(x)|\}$$

or, with the equivalent notation,

$$\mathfrak{M}, x \models M^{p,q}\varphi \Leftrightarrow |R_\varphi(x)| > \frac{p}{q}|R(x)|,$$

i.e. $M^{p,q}\varphi$ is true in a world x if φ is true in a (strict) greater than $\frac{p}{q}$ part of the successors of x. About the dual modal operator we obtain

$$\mathfrak{M}, x \models W^{p,q}\varphi \Leftrightarrow |R_{\neg\varphi}(x)| \leq \frac{p}{q}|R(x)| \quad \Big(\Leftrightarrow |R_\varphi(x)| \geq \frac{q-p}{q}|R(x)|\Big),$$

i.e. $W^{p,q}\varphi$ is true in a world x if φ is refused in not greater than $\frac{p}{q}$ part of the successors of x.

A formula φ is *valid in a model* $\mathfrak{M} = \langle U, R, V\rangle$, notation $\mathfrak{M} \vDash \varphi$, if φ is true in any world of the model, i.e.

$$\mathfrak{M} \vDash \varphi \Leftrightarrow (\forall x \in U)(\mathfrak{M}, x \vDash \varphi).$$

A formula φ is *valid in a frame* $\mathfrak{F} = \langle U, R\rangle$, denoted by $\mathfrak{F} \vDash \varphi$, if φ is valid in any model on the frame. A formula φ is *valid* (or *tautology*), if φ is valid in any frame (from a given class), notation $\vDash \varphi$.

A formula φ is *satisfiable in a model* if it is true in some world of the model. A formula φ is *satisfiable in a frame* if it is valid in some model, based on the frame. A formula φ *satisfiable* if it is satisfiable in some model.

Definition 1. The logic *RGML is the set of all formulae, valid in the class of all frames.*

2.3. MODAL ELEMENTARY CONJUNCTIONS

Definition 2. *We define by induction* modal depth (md) of a formula*:*

$md(p) = 0$

$md(\neg\varphi) = md(\varphi)$

$md(\varphi \bigtriangleup \psi) = max(md(\varphi), md(\psi)), \bigtriangleup \rightleftharpoons \wedge, \vee$

$md(M^{p,q}\varphi) = md(\varphi) + 1,$ *hence*:

$md(\varphi\Box\psi) = max(md(\varphi), md(\psi)), \Box \rightleftharpoons \rightarrow, \leftrightarrow$

$md(W^{p,q}\varphi) = md(\varphi) + 1$

Using induction on the complexity of formula, it is easy to prove the following properties of the formulae:

Fact 1. $md(\varphi) = 0$ *iff* φ *is a Boolean formula.*

Fact 2. $md(\varphi) \geq md(\chi)$*, for any subformula* χ *of* φ*.*

Definition 3. Modal elementary conjunction *is a formula of the kind:*

$$\theta = p_1^{\varepsilon_1} \wedge \cdots \wedge p_n^{\varepsilon_n} \wedge M^{p,q}\varphi_1 \wedge \cdots \wedge M^{p,q}\varphi_m \wedge W^{p,q}\psi_1 \wedge \cdots \wedge W^{p,q}\psi_l, \qquad (2.1)$$

where $\varepsilon_i, i = 1, \ldots, n$*, are 0 or 1 and for any formula* φ *we use the notation* $\varphi^1 := \varphi$*,* $\varphi^0 := \neg\varphi$*.*

Proposition 1. *There exists an algorithm* $\mathcal{N}$*, acting on an arbitrary formula* φ*, which terminates in a finite number of steps and the result* $\mathcal{N}(\varphi)$ *is a (finite) disjunction of modal elementary conjunctions (i.e.* $\mathcal{N}$ *transforms an arbitrary formula in a modal equivalent of DNF) and the following holds:*

1. For arbitrary model $\mathfrak{M}$ *and world* x *from it*

$$\mathfrak{M}, x \models \varphi \leftrightarrow \mathcal{N}(\varphi),$$

2. $md(\varphi) \geq md(\mathcal{N}(\varphi))$*.*

The role of $\mathcal{N}$ can play any algorithm, transforming a Boolean formula in DNF, just treating the subformulae with a modal operator on the outer level as variables.

For 2. it is sufficient to note that the transformation in modal DNF cannot increase the modal depth, while a modal operator can be reduced in case of exclusion in disjunction, if a subformula occurs both in positive and negative form.

From Proposition 1 and Fact 2 follows:

Fact 3. *Any modal elementary conjunction from $\mathcal{N}(\varphi)$ has depth, not exceeding the depth of φ, i.e. $md(\theta) \leq md(\varphi)$, for any θ – a modal elementary conjunction from $\mathcal{N}(\varphi)$.*

From Proposition 1 it also follows that the question for the satisfiability of a formula can be reduced to the one for the satisfiability of a finite number of modal elementary conjunctions, each of which with modal depth, not exceeding a constant — the modal depth of the formula itself.

Let us consider one such modal elementary conjunction θ and its satisfiability.

Case 1. The Boolean part of θ, we denote it by $B_\theta := p_1^{\varepsilon_1} \wedge \ldots \wedge p_n^{\varepsilon_n}$, is not satisfiable (when for some $i \neq j \leq n$ it holds $p_i = p_j$ and $\varepsilon_i \neq \varepsilon_j$).

Then θ is not satisfiable.

Case 2. B_θ is satisfiable.

Then we examine the rest (modal) part of θ — the satisfiability of the whole θ depends on it:

$$M^{p,q}\varphi_1 \wedge \ldots \wedge M^{p,q}\varphi_m \wedge W^{p,q}\psi_1 \wedge \ldots \wedge W^{p,q}\psi_l,$$

$m \geq 0,\ l \geq 0$.

Case 2.1. $m = 0$.

Then the modal part is true in any one-world model. Really, let $\mathfrak{M}$ be a model with a single world x. From the valuation of $W^{p,q}$ for any formula ψ holds:

$$\mathfrak{M}, x \models W^{p,q}\psi \Leftrightarrow |R_\psi(x)| \geq \frac{q-p}{q}|R(x)|.$$

As x has no successors, $R_\psi(x) = R(x) = \varnothing$ and the inequality on the right is fulfilled as the equality $0 = 0$. Hence $\mathfrak{M}, x \models W^{p,q}\psi$.

Therefore, as every modal conjunct, so the whole modal part of θ are true in every one-world model, and particularly in every one-world model for B_θ, and such a model exists. Hence θ is satisfiable.

Case 2.2. $m > 0$, i.e. at least one conjunct of the form $M^{p,q}\varphi$ participates in θ.

Let some model $\mathfrak{M}$ and a world x from the model fulfil $\mathfrak{M}, x \models \theta$, then $\mathfrak{M}, x \models M^{p,q}\varphi$. If we assume that $R(x) = \varnothing$, from the evaluation of $M^{p,q}$ we obtain $0 = |R_\varphi(x)| > \frac{p}{q}|R(x)| = 0$ — a contradiction.

Hence, if θ is true in the world x, then x has $a > 0$ successors.

We will formulate and prove a proposition connected with the satisfiability of θ in Case 2.2. First we will give a definition.

Let us consider a model $\mathfrak{M}$ and a world x_0 from the model, having $a > 0$ successors (i.e. $|R(x_0)| = a > 0$), and examine the truth of θ in x_0.

We consider all the conjunctions of the form:

$$\varphi_1^{\varepsilon_1} \wedge \ldots \wedge \varphi_m^{\varepsilon_m} \wedge \psi_1^{\varepsilon_{m+1}} \wedge \ldots \wedge \psi_l^{\varepsilon_{m+l}}, \tag{2.2}$$

where ε_j, $j = 1, \ldots, m+l$, are 0 or 1 and let $\tau_1, \ldots, \tau_t$ are all of them, which are satisfiable, $0 \leq t \leq 2^{m+l}$, i.e.

$$\begin{aligned} \tau_1 &= \varphi_1^{\varepsilon_{11}} \wedge \ldots \wedge \varphi_m^{\varepsilon_{1m}} \wedge \psi_1^{\varepsilon_{1m+1}} \wedge \ldots \wedge \psi_l^{\varepsilon_{1m+l}} \\ &\vdots \\ \tau_t &= \varphi_1^{\varepsilon_{t1}} \wedge \ldots \wedge \varphi_m^{\varepsilon_{tm}} \wedge \psi_1^{\varepsilon_{tm+1}} \wedge \ldots \wedge \psi_l^{\varepsilon_{tm+l}} \end{aligned}$$

where ε_{ij} is short for $\varepsilon_{i,j}$ and ε_{ij}, $i = 1, \ldots, t$, $j = 1, \ldots, m+l$, are 0 or 1.

We denote $T := \{\tau_1, \ldots, \tau_t\}$.

Let $\mathfrak{M}' = \langle U', R', V' \rangle$ be an arbitrary model. For all formulae in θ, φ_j, $j = 1, \ldots, m$, ψ_{j-m}, $j = m+1, \ldots, m+l$, consider the corresponding sets $V'(\varphi_j)$, and $V'(\psi_{j-m})$. For $x' \in U'$, we define

$$\varepsilon'_j = \begin{cases} 1, & x' \in V'(\varphi_j) \\ 0, & x' \notin V'(\varphi_j), \ j = 1, \ldots, m, \end{cases}$$

$$\varepsilon'_j = \begin{cases} 1, & x' \in V'(\psi_{j-m}) \\ 0, & x' \notin V'(\psi_{j-m}), \ j = m+1, \ldots, m+l \end{cases}$$

We consider $\tau' = \varphi_1^{\varepsilon'_1} \wedge \ldots \wedge \varphi_m^{\varepsilon'_m} \wedge \psi_1^{\varepsilon'_{m+1}} \wedge \ldots \wedge \psi_l^{\varepsilon'_{m+l}}$. Then $x' \models \tau'$ and hence $\tau' \in T \neq \varnothing$ and $t > 0$, i.e. $1 \leq t \leq 2^{m+l}$.

Suppose τ_i is satisfiable in a_i worlds from $R(x_0)$, i.e. $|R_{\tau_i}(x_0)| = a_i$, $0 \leq a_i \leq a$, $i = 1, \ldots, t$. We form the following system of $m+l$ linear inequalities, in which we consider $a_1, \ldots, a_t$ as unknowns:

$$(\sigma_\theta) \left| \begin{array}{ll} \sum_{i=1}^{t} \varepsilon_{ij} a_i > \frac{p}{q}(a_1 + \cdots + a_t), & j = 1, \ldots, m \\ \\ \sum_{i=1}^{t} \varepsilon_{ij} a_i \geq \frac{q-p}{q}(a_1 + \cdots + a_t), & j = m+1, \ldots, m+l \end{array} \right. \tag{2.3}$$

Definition 4. *We call the above system of linear inequalities* corresponding to the modal elementary conjunction θ.

We also define a system of linear inequalities, corresponding to a modal elementary conjunction θ, consisting just of a (satisfiable!) Boolean part, i.e. we define a system of linear inequalities, corresponding to a satisfiable Boolean formula B, in the following way:

$$(\sigma_B) \mid a \geq a$$

Note 1. In the condition of Case 2.2 ($m > 0$), if (σ_θ) has a solution, then this solution is not zero, as (σ_θ) contains at least one strict inequality.

Note 2. In Case 2.1 ($m = 0$) we can also consider a system (σ_θ), corresponding to θ, but that system has always the zero solution as a system of non-strict homogeneous linear inequalities. This exactly corresponds to the fact that the modal part of θ is always satisfiable, but we already know that. That is why the interesting case is when (σ_θ) has (only) non-zero solution.

Now we formulate a proposition related to the satisfiability of θ in Case 2.2.

Proposition 2. *For any modal elementary conjunction θ with a satisfiable Boolean part and a modal part which is either empty or has at least one conjunct with modal operator $M^{p,q}$, the following three statements are equivalent:*

(i) θ is satisfiable;

(ii) θ is satisfiable in the root of a finite tree-like model;

(iii) the system of linear inequalities (σ_θ), corresponding to θ, has non-negative (non-zero) integer solution.

Proof. We use induction on the modal depth of θ.

1. For $md(\theta) = 0$, θ is a Boolean formula. Then θ is satisfiable iff it is satisfiable in a single-world model. As it is satisfiable under the terms of the proposition, it is also satisfiable in a single-world (tree-like) model. (σ_θ) has (a trivial) solution 1. So (i), (ii) and (iii) are fulfilled, and hence are equivalent.

2. (ih) Let, for any modal elementary conjunction θ with $md(\theta) \leq n$, (i), (ii) and (iii) be equivalent.

3. Consider θ: $md(\theta) = n + 1$.

3.1. (ii) $\Rightarrow$ (i) is always (trivially) fulfilled;

3.2. (i) $\Rightarrow$ (iii)

Let θ be satisfiable. Then there exist a model $\mathfrak{M}$ and a world x_0 from the model in which θ is true and let the number of successors of x_0, $|R(x_0)|$, is $a^0 > 0$.

Let $\tau_1, \dots, \tau_t$ be all the conjunctions from T (as defined above), and let τ_i be true in a_i^0 in number worlds from $R(x_0)$, i.e. $|R_{\tau_i}(x_0)| = a_i^0$, $0 \leq a_i^0 \leq a^0$, $i = 1, \dots, t$.

For any $x \in R(x_0)$ there exists $\tau \in T$: $x \models \tau$, i.e. $x \in R_\tau(x_0)$, (τ can be constructed as in the proof of $T \neq \varnothing$), i.e. $x \in R_{\tau_i}(x_0)$ for some i, $1 \leq i \leq t$. Hence $R(x_0) \subseteq \bigcup_{i=1}^t R_{\tau_i}(x_0)$. The reverse inclusion obviously holds, so $R(x_0) = \bigcup_{i=1}^t R_{\tau_i}(x_0)$. Then

$$a^0 = |R(x_0)| = |\bigcup_{i=1}^{t} R_{\tau_i}(x_0)| \leq \sum_{i=1}^{t} |R_{\tau_i}(x_0)| = a_1^0 + \dots + a_t^0$$

Let $x' \in R_{\tau_i}(x_0)$. As for $i \neq j$, τ_j differs from τ_i at least in one conjunct, $x' \notin R_{\tau_j}(x_0)$. Hence $R_{\tau_i}(x_0) \cap R_{\tau_j}(x_0) = \varnothing$, for all $i \neq j$, $i, j \in \{1, \dots, t\}$, and $|\bigcup_{i=1}^{t} R_{\tau_i}(x_0)| = \sum_{i=1}^{t} |R_{\tau_i}(x_0)|$. Hence

$$a^0 = a_1^0 + \cdots + a_t^0 > 0$$

Now, as θ is true in x_0, any of $M^{p,q}\varphi_j$, $j = 1, \dots, m$, and any of $W^{p,q}\psi_{j-m}$, $j = m+1, \dots, m+l$, is also true in x_0. Now consider any of the sets $R_{\varphi_j}(x_0)$, $j = 1, \dots, m$. From the truth definition for $M^{p,q}$ follows:

$$|R_{\varphi_j}(x_0)| > \frac{p}{q} a^0 = \frac{p}{q}(a_1^0 + \cdots + a_t^0) \tag{2.4}$$

But $R_{\varphi_j}(x_0)$ is a union of sets $R_{\tau_i}(x_0)$ for those τ_i in which φ_j is in positive form. As these sets are pairwise disjoint and taking into account how the coefficients ε_{ij} are defined, we get that the following holds:

$$|R_{\varphi_j}(x_0)| = \sum_{i=1}^{t} \varepsilon_{ij} |R_{\tau_i}(x_0)| = \sum_{i=1}^{t} \varepsilon_{ij} a_i^0 \tag{2.5}$$

From (2.4) and (2.5) we obtain that $|R_{\tau_i}(x_0)| = a_i^0$, $i = 1, \dots, t$, satisfy the inequalities from the system (σ_θ), corresponding to θ, for $j = 1, \dots, m$.

In the same way, considering any of the sets $R_{\psi_{j-m}}(x_0)$, $j = m+1, \dots, m+l$, from the truth definition for $W^{p,q}$, we get

$$\sum_{i=1}^{t} \varepsilon_{ij} a_i^0 = \sum_{i=1}^{t} \varepsilon_{ij} |R_{\tau_i}(x_0)| = |R_{\psi_{j-m}}(x_0)| \geq \frac{q-p}{q} a^0 = \frac{q-p}{q}(a_1^0 + \cdots + a_t^0) \tag{2.6}$$

From (2.6) we obtain that $|R_{\tau_i}(x_0)| = a_i^0$, $i = 1, \dots, t$, satisfy also the inequalities from (σ_θ) for $j = m+1, \dots, m+l$.

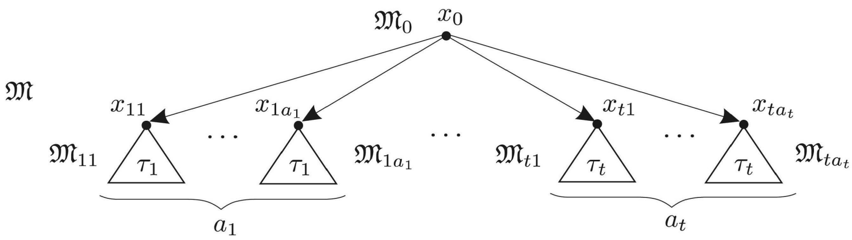
It follows that $(|R_{\tau_1}(x_0)|, \dots, |R_{\tau_t}(x_0)|)$ is a solution of (σ_θ), moreover non-negative (non-zero) integer one, i.e. (iii) holds.

3.3. (iii) $\Rightarrow$ (ii)

Let (σ_θ) be the system, corresponding to θ (i.e. it is formed in the way, described above), and let it have at least one non-negative (non-zero) integer solution, i.e. there exist numbers $a_1^0, \dots, a_t^0$, $a_i^0 \in \mathbb{N}$, $i = 1, \dots, t$, $a^0 := \sum_{i=1}^{t} a_i^0 > 0$, and $(a_1^0, \dots, a_t^0)$ is a solution of (σ_θ).

So, there exist just t different conjunctions of form (2) — we denote them by $\tau_1, \dots, \tau_t$ — which are got from θ and which are satisfiable. As $md(\tau_i) \leq md(\theta) - 1 = n$, by (ih) the proposition holds for τ_i, $i = 1, \dots, t$. Hence any τ_i is satisfiable in the root of a finite tree-like model. For any τ_i we consider a_i^0 finite tree-like models $\mathfrak{M}_{ik} = \langle U_{ik}, R_{ik}, V_{ik} \rangle$, $k = 1, \dots, a_i^0$, with roots, denoted by $x_{i1}, \dots, x_{ia_i^0}$ respectively, in each of which τ_i is true. In addition, we want the universes of the models $\mathfrak{M}_{ik}$, $i = 1, \dots, t$, $k = 1, \dots, a_i^0$, to be pairwise disjoint (we can obtain these models by some procedure of "copying" or "colouring").

Let the Boolean part of θ, B_θ, be true in a single-world model $\mathfrak{M}_0 = \langle U_0, R_0, V_0 \rangle$, where $U_0 = \{x_0\}$, $R_0 = \varnothing$ and V_0 is an evaluation in $\mathfrak{M}_0$ for which $x_0 \models B_\theta$; such a model exists as B_θ is satisfiable. We define a model $\mathfrak{M} = \langle U, R, V \rangle$ as a natural union of the above models in the following way:



$$U = \{x_0\} \cup \bigcup_{i=1}^{t} \bigcup_{k=1}^{a_i^0} U_{ik},$$

$$R = \{(x_0, x_{ik}),\ 1 \le i \le t,\ 1 \le k \le a_i^0\} \cup \bigcup_{i=1}^{t} \bigcup_{k=1}^{a_i^0} R_{ik},$$

$$V = V_0 \cup \bigcup_{i=1}^{t} \bigcup_{k=1}^{a_i^0} V_{ik},$$

and the definition of V is extended to the set of all formulae.

From the definition of $\mathfrak{M}$ follow:

Fact 4. *$\mathfrak{M}$ is a finite tree-like model with a root — the world x_0, which has $a^0 = a_1^0 + \cdots + a_t^0 > 0$ successors.*

Fact 5. *For any Boolean formula A: $x_0 \in V(A) \iff x_0 \in V_0(A)$, i.e.*

$$\mathfrak{M}, x_0 \models A \Leftrightarrow \mathfrak{M}_0, x_0 \models A$$

So for the modal part B_θ of θ holds:

$$\mathfrak{M}, x_0 \models B_\theta \tag{2.7}$$

Fact 6. *For any world $x \in U$, $x \neq x_0$, there exist $1 \le i \le t$, $1 \le k \le a_i^0$ such that $x \in U_{ik}$, and for any formula φ: $x \in V(\varphi) \iff x \in V_{ik}(\varphi)$, i.e.*

$$\mathfrak{M}, x \models \varphi \Leftrightarrow \mathfrak{M}_{ik}, x \models \varphi$$

so that

$$\mathfrak{M}, x_{ik} \models \tau_i,\ i = 1, \ldots, t,\ k = 1, \ldots, a_i^0. \tag{2.8}$$

Now we examine the truth of φ_j from θ, $j = 1, \ldots, m$. Any φ_j is true just in those worlds, in which are true these τ_i, $i = 1, \ldots, t$, in which φ_j takes part in positive form, i.e. $\varepsilon_{ij} = 1$. For the number of R-successors of x_0, in which φ_j is true, we have:

$$|R_{\varphi_j}(x_0)| = \sum_{i=1}^{t} \varepsilon_{ij} |R_{\tau_i}(x_0)| = \sum_{i=1}^{t} \varepsilon_{ij} a_i^0 \overset{\text{from } (\sigma_\theta)}{>} \frac{p}{q} \sum_{i=1}^{t} a_i^0 = \frac{p}{q} |R(x_0)|. \tag{2.9}$$

From (2.9), using the truth definition for $M^{p,q}$, we get:

$$\mathfrak{M}, x_0 \models M^{p,q} \varphi_j, \quad j = 1, \ldots, m. \tag{2.10}$$

In the same way, for the number of R-successors of x_0, in which ψ_{j-m}, $j = m+1, \ldots, m+l$, are true, holds:

$$|R_{\psi_{j-m}}(x_0)| = \sum_{i=1}^{t} \varepsilon_{ij} |R_{\tau_i}(x_0)|$$

$$= \sum_{i=1}^{t} \varepsilon_{ij} a_i^0 \overset{\text{from } (\sigma_\theta)}{\geq} \frac{q-p}{q} \sum_{i=1}^{t} a_i^0 = \frac{q-p}{q} |R(x_0)|. \tag{2.11}$$

From (2.11), using the truth definition for $W^{p,q}$, we get:

$$\mathfrak{M}, x_0 \models W^{p,q} \psi_{j-m}, \quad j = m+1, \ldots, m+l. \tag{2.12}$$

From (2.7), (2.10) and (2.12) we get:

$$\mathfrak{M}, x_0 \models \theta,$$

which, taking into account Fact 4, means that θ is satisfiable in the root of a finite tree-like model, i.e. (ii) holds. □

The equivalence of (i) and (ii) from the Proposition 2 gives, as an immediate corollary, the following proposition:

Proposition 3. *The logic RGML coincides with the set of formulae, valid in the finite trees.*

Later on we will use Proposition 2 to prove that the logic RGML is decidable. But first we state briefly some elements from the theory of systems of linear inequalities and we prove a proposition connected with them.

3. SYSTEMS OF LINEAR INEQUALITIES

3.1. A METHOD OF SOLVING OF SYSTEMS OF LINEAR INEQUALITIES BY CONSECUTIVE REDUCTION OF THE NUMBER OF UNKNOWNS

Here we present a modified version of the method stated in [7]. Consider a system of linear inequalities (σ):

$$(\sigma)\left|\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n + a_1 \geqslant 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + a_m \geqslant 0 \end{array}\right. \tag{3.1}$$

where the sign $\geqslant$ stands for $\geq$ or $>$.

We associate with (σ) a system (σ'), called *attendant on* (σ), which has just one unknown less than (σ) and for definiteness let it be the one with the greatest index — x_n. Let

$$b_1x_1 + \cdots + b_nx_n + b \geqslant 0 \tag{3.2}$$

be an arbitrary inequality from (σ). The following possibilities for b_n exist:

1) $b_n = 0$ — in that case we do not change the inequality (3.2);

2) $b_n > 0$ — in that case we divide (3.2) by b_n and take all the members except x_n from the right side; we get the inequality

$$x_n \geqslant c_1x_1 + \cdots + c_{n-1}x_{n-1} + c \tag{3.3}$$

3) $b_n < 0$ — in that case we take the member with x_n on the right side and divide (3.2) by $-b_n$, so we get:

$$d_1x_1 + \cdots + d_{n-1}x_{n-1} + d \geqslant x_n. \tag{3.4}$$

Applying this procedure to every inequality from (σ), we get (with possible change in the order of the inequalities) the system (σ^*), equivalent to (σ) and having the form

$$\left|\begin{array}{l} P_1 \geqslant x_n \\ \vdots \\ P_p \geqslant x_n \end{array}\right.$$

$$(\sigma^*)\left|\begin{array}{l} x_n \geqslant Q_1 \\ \vdots \\ x_n \geqslant Q_q \end{array}\right. \tag{3.5}$$

$$\left| \begin{array}{l} R_1 \geqslant 0 \\ \vdots \\ R_r \geqslant 0 \end{array} \right.$$

The first block of (σ^*) includes the inequalities from (σ), falling in case 3), the second — in case 2 and the third — in case 1, and obviously P_α, $1 \leq \alpha \leq p$, Q_β, $1 \leq \beta \leq q$ and R_γ, $1 \leq \gamma \leq r$, are linear functions of $x_1, \ldots, x_{n-1}$, not containing x_n.

(σ^*) can be written shortly:

$$\left| \begin{array}{ll} P_\alpha \geqslant x_n \geqslant Q_\beta, & \alpha = \overline{1,p},\ \beta = \overline{1,q} \\ R_\gamma \geqslant 0, & \gamma = \overline{1,r} \end{array} \right.$$

We consider the system (σ'):

$$(\sigma') \left| \begin{array}{ll} P_\alpha \geqslant Q_\beta, & \alpha = \overline{1,p},\ \beta = \overline{1,q} \\ R_\gamma \geqslant 0, & \gamma = \overline{1,r} \end{array} \right.$$

Definition 5. *The system (σ') considered as derived from system (σ^*) is called* attendant on the system (σ).

Obviously (σ') has $n-1$ unknowns $x_1, \ldots, x_{n-1}$.

Note. If there is no inequality in (σ), falling in case 1, then the second group of inequalities in (σ') is missing. If there is no inequality in (σ), falling in case 2 or if there is no inequality in (σ), falling in case 3, then the first group of inequalities in (σ') is missing.

The following theorem about the systems above holds:

Theorem 1. *For any solution $(x_1, \ldots, x_n)$ of (σ), $(x_1, \ldots, x_{n-1})$ is a solution of (σ'). Conversely, for any solution $(x_1, \ldots, x_{n-1})$ of (σ') there is a number x_n such that $(x_1, \ldots, x_{n-1}, x_n)$ is a solution of (σ), i.e. any solution of the attendant system can be extended to a solution of the initial one.*

Proof. The proof follows the steps of the one stated in [7], with just one additional check to ensure that everything goes well in both cases – in a non-strict as well as in a strict inequality. □

The next proposition is an immediate corollary of Theorem 1.

Proposition 4. *The system (σ) has a solution iff the system (σ') has a solution.*

Definition 6. Admissible vector *for the first k unknowns of (σ) is the vector of numbers $(x_1^0, \ldots, x_k^0)$, if it can be extended to a solution of (σ), i.e. if there exist* *numbers* $x_{k+1}^0, \ldots, x_n^0$ *such that* $(x_1^0, \ldots, x_k^0, x_{k+1}^0, \ldots, x_n^0)$ *is a solution of* (σ).

Now we denote with $\mathcal{C}'$ the algorithm which, for a given system of linear inequalities, constructs (in a finite number of steps) its attendant system, excluding the unknown with the greatest index. So $\mathcal{C}'((\sigma)) = (\sigma')$, and if (σ) has unknowns $x_1, \ldots, x_n$, then (σ') has unknowns $x_1, \ldots, x_{n-1}$. We can use again $\mathcal{C}'$ to act on (σ'). After $n-1$ usages of $\mathcal{C}'$ we get $\mathcal{C}'^{n-1}((\sigma)) = (\sigma^{n-1})$, where (σ^{n-1}) is a system of linear inequalities with just one unknown x_1.

Using Proposition 4 $n-1$ times we get that (σ) is compatible iff (σ^{n-1}) is compatible. In case that (σ^{n-1}) is compatible, we can easily find a solution x_1^0 of (σ^{n-1}), which is also an admissible vector for (σ^{n-2}). Substituting x_1^0 for x_1 in (σ^{n-2}) and solving (σ^{n-2}) regarding x_2, we get x_2^0. (x_1^0, x_2^0) is a solution of (σ^{n-2}) and an admissible vector for (σ^{n-3}). Continuing in that reverse way for $n-1$ steps we get a solution $(x_1^0, \ldots, x_n^0)$ of (σ).

3.2. ALGORITHM FOR SYSTEMS OF LINEAR HOMOGENEOUS INEQUALITIES WITH RATIONAL COEFFICIENTS

Proposition 5. *There exists an algorithm, acting on systems of linear homogeneous inequalities with rational coefficients, which terminates for any such a system (σ) in a finite number of steps, giving a result* yes *if (σ) has a non-negative non-zero integer solution, and* no *in the opposite case.*

Let, for definiteness, (σ) has n unknowns $x_1, \ldots, x_n$. We expand the system with the inequalities

$$\begin{aligned} &x_1 \geq 0 \\ &\vdots \\ &x_n \geq 0 \\ &x_1 + \cdots + x_n > 0 \end{aligned}$$

Note. If at least one of the inequalities in (σ) is strict, the last one of the upper inequalities is redundant and we do not add it.

We denote the expanded system by (σ^+). Obviously the system (σ) has a non-negative non-zero solution iff (σ^+) has a solution.

Let $\mathcal{D}$ be an algorithm, acting on systems of linear homogeneous inequalities with just one unknown, which terminates in a finite number of steps with a result *yes* if the system is compatible, and *no* in the opposite case. It is easy to see that such an algorithm exists and it can be easily constructed.

Then we put $\mathcal{C}((\sigma)) = \mathcal{D}(\mathcal{C}'^{n-1}(\sigma^+))$ and state that $\mathcal{C}$ is the algorithm we ask for.

Proof of Proposition 5.

1. $\mathcal{C}((\sigma))$ is well defined and terminates in a finite number of steps.

Really, (σ^+) is a system of linear (homogeneous) inequalities with n unknowns, so $\mathcal{C}'((\sigma^+))$ is defined and, after being applied $n-1$ times, $\mathcal{C}'$ transforms (in a finite number of steps) (σ^+) into a system (σ^{+n-1}) with just one unknown (from subsection). Then the algorithm $\mathcal{D}(\mathcal{C}'^{n-1}(\sigma^+))$ is also defined and in a finite number of steps gives a result *yes* or *no*.

2. $\mathcal{C}((\sigma)) = no \Rightarrow (\sigma)$ has no non-negative non-zero integer solution.

Let $\mathcal{D}(\mathcal{C}'^{n-1}(\sigma^+)) = \mathcal{D}((\sigma^{+n-1})) = no$. Then (σ^{+n-1}) is incompatible and by Proposition 4 (σ^+) is also incompatible, i.e. has no solution. Hence (σ) has no non-negative non-zero (integer) solution.

3. $\mathcal{C}((\sigma)) = yes \Rightarrow (\sigma)$ has non-negative non-zero integer solution.

Let $\mathcal{D}(\mathcal{C}'^{n-1}(\sigma^+)) = \mathcal{D}((\sigma^{+n-1})) = yes$. Then (σ^{+n-1}) has a solution and by Proposition 4 (σ^+) is also has a solution.

Now consider the system (σ^{+n-1}). It contains the inequality

$$x_1 \geq 0 \tag{3.6}$$

and (eventually) other inequalities of that kind and of the following kinds:

$$x_1 > 0 \tag{3.7}$$

$$-x_1 \geq 0 \tag{3.8}$$

$$-x_1 > 0 \tag{3.9}$$

As (σ^{+n-1}) has a solution, it has no inequalities of the kind (3.9) and also it has no together inequalities of kinds (3.7) and (3.8), i.e. it contains, except the inequality (3.6), (eventually) inequalities of kind (3.7) or inequalities of kind (3.8). Hence the set of solutions of (σ^{+n-1}) is either the point $x_1^0 = 0$ or a positive half-line with the beginning at the point 0 (eventually not including the point 0 itself). In the second case we can chose $x_1^0 = 1$ (or arbitrary rational number). Thus obtained x_1^0 is an admissible vector for (σ^{+n-2}). Substituting it for x_1 in (σ^{+n-2}) we can get x_2^0, by Theorem 1. Besides, as all the coefficients in (σ^+), and finally consecutively obtained from it attendant systems, are rational and x_1^0 is also rational, we can get x_2^0 also to be rational. (x_1^0, x_2^0) is an admissible vector for (σ^{+n-3}) and, continuing in that way, we get (in $n-1$ steps) $X^0 = (x_1^0, \ldots, x_n^0)$ — a (non-negative, non-zero) rational solution of (σ^+). Hence X^0 is a non-negative non-zero rational solution of (σ).

Let k be the lowest common denominator of the integers $x_1^0, \ldots, x_n^0$. Then $kX^0 = (kx_1^0, \ldots, kx_n^0)$ is also a solution of (σ), moreover kX^0 is a non-negative non-zero integer solution of (σ). □

4. DECIDABILITY OF THE LOGIC RGML

Theorem 2. *There exists an algorithm $\mathcal{A}$, acting on formulae, which, applied on an arbitrary formula φ, terminates in a finite number of steps with result* yes *or* no *such that*

$$\mathcal{A}(\varphi) = yes \quad \textit{iff } \varphi \textit{ is satisfiable.} \tag{4.1}$$

Proof. We construct $\mathcal{A}$ by induction on the modal depth of the formulae, namely we build a sequence of algorithms $\mathcal{A}_0, \mathcal{A}_1, \ldots$ such that the algorithm $\mathcal{A}_n$ acts only on formulae with modal depth not greater than n and for these formulae it carries out the equivalence (4.1).

1. $\mathcal{A}_0$ is an algorithm, acting on the Boolean formulae.

2. (ih) Let $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ be defined and let the assertion of the theorem hold for them.

3. We define $\mathcal{A}_{n+1}$ in the following way:

3.1. Let φ be such that $md(\varphi) \leq n$. We put $\mathcal{A}_{n+1}(\varphi) := \mathcal{A}_n(\varphi)$.

3.2. Let φ be such that $md(\varphi) = n+1$. We present φ in the form $\bigvee_{i=1}^{k} \theta_i$, where θ_i, $i = 1, \ldots, k$, are modal elementary conjunctions. By Proposition 1 there exists an algorithm $\mathcal{N}$, transforming φ in this form in a finite number of steps, and $md(\theta_i) \leq n+1$, $i = 1, \ldots, k$.

We define the algorithm $\mathcal{B}_{n+1}$, acting on modal elementary conjunctions with $md \leq n+1$, which, for θ — a modal elementary conjunction with $md(\theta) \leq n+1$, gives the result *yes*, if θ is satisfiable, or *no* in the opposite case.

3.2.a. $md(\theta) \leq n$, $\mathcal{B}_{n+1}(\theta) := \mathcal{A}_n(\theta)$.

3.2.b. $md(\theta) = n+1$. Let B_θ is the Boolean part of θ. Then:

b.1. If $\mathcal{A}_0(B_\theta) = no$, i.e. if B_θ is not satisfiable, then θ is not satisfiable also, and we put $\mathcal{B}_{n+1}(\theta) = no$.

b.2. Let $\mathcal{A}_0(B_\theta) = yes$, i.e. B_θ is satisfiable, and the modal part of θ contains only conjuncts with $W^{p,q}$ on the outer level. Then the whole θ is satisfiable (in a single-world model), so we put $\mathcal{B}_{n+1}(\theta) = yes$.

b.3. Let $\mathcal{A}_0(B_\theta) = yes$, i.e. B_θ is satisfiable, and the modal part of θ contains at least one conjunct with $M^{p,q}$ on the outer level. Then we consider all the conjunctions of the form (2.2) for θ. They are with modal depth not greater than n and the algorithm $\mathcal{A}_n$ acts on them. We implement $\mathcal{A}_n$ on any of them consecutively and pick out which of them are satisfiable: let these be $\tau_1, \ldots, \tau_t$, $1 \leq t \leq 2^{m+l}$.

Then we consider the system (σ_θ), attendant on θ and having the form (2.3). By Proposition 2 the formula θ is satisfiable iff (σ_θ) has non-negative (non-zero) integer solution. Let $\mathcal{C}$ be the algorithm from Proposition 5 (which for any system of the above kind tells if the system has non-negative non-zero integer solution or not). In this case we put $\mathcal{B}_{n+1}(\theta) := \mathcal{C}((\sigma_\theta))$.

Now we proceed with $\mathcal{A}_{n+1}$ in case 3.2. Having the algorithm $\mathcal{B}_{n+1}$ just defined, we define $\mathcal{A}_{n+1}$ as an implementation of the algorithm $\mathcal{N}$, followed the implementations of $\mathcal{B}_{n+1}$ on each of the modal elementary conjunctions θ_i, $i = 1, \ldots, k$, of φ. If for some i, $1 \leq i \leq k$, $\mathcal{B}_{n+1}(\theta_i) = yes$, we put $\mathcal{A}_{n+1}(\varphi) = yes$; in the opposite case, i.e. if all the results are *no*, we put $\mathcal{A}_{n+1}(\varphi) = no$.

Thus the inductive definition of $\mathcal{A}_n$ for any natural number n is finished. Let $\mathcal{M}$ be an algorithm, calculating the modal depth of the formulae, i.e. $\mathcal{M}$ acts on an arbitrary formula and gives (in a finite number of steps) as a result a natural number, so that $\mathcal{M}(\varphi) = n$ iff $md(\varphi) = n$. It is easy to see that, as the formulae are of finite length and the modal depth is inductively defined, such an algorithm exists.

Now we define the algorithm $\mathcal{A}$ for an arbitrary formula φ in the following way: first we implement the algorithm $\mathcal{M}$ on φ, next we implement the algorithm $\mathcal{A}_{\mathcal{M}(\varphi)}$ on φ, and then we put

$$\mathcal{A}(\varphi) := \mathcal{A}_{\mathcal{M}(\varphi)}(\varphi)$$

It is clear from the above definitions that $\mathcal{A}$ always terminates in a finite number of steps and satisfies the equivalence (4.1). □

As a corollary of Theorem 2 we obtain the main theorem of this paper:

Theorem 3. *The logic RGML is decidable.*

Proof. Let us use the notation $\overline{yes} := no$ and $\overline{no} := yes$. We define the algorithm $\mathcal{R}$, acting on formulae, in the following way: $\mathcal{R}(\varphi) := \overline{\mathcal{A}(\neg\varphi)}$. We state that $\mathcal{R}$ is a decision method for RGML and the formula φ belongs to RGML iff $\mathcal{R}(\varphi) = yes$.

Indeed, for an arbitrary formula φ the algorithm $\mathcal{R}$ terminates in a finite number of steps and $\mathcal{R}(\varphi) = yes$ iff $\mathcal{A}(\neg\varphi) = no$, i.e. just when $\neg\varphi$ is not satisfiable. But that holds iff φ is valid or — what is the same — φ belongs to RGML. □

5. SOME EXTENSIONS OF THE LANGUAGE

We extend the language $\mathcal{L}_M$ with additional "rational" modal operators.

1. We consider n fractions $\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}$ and the corresponding to them modal operators M^{p_k,q_k}, $1 \leq p_k < q_k$, p_k and q_k are relatively prime integers, $k = 1, \ldots, n$, with the interpretation in a model

$$\mathfrak{M}, x \models M^{p_k,q_k}\varphi \Leftrightarrow |R_\varphi(x)| > \frac{p_k}{q_k}|R(x)|, \quad k = 1, \ldots, n.$$

We consider also the dual modal operators W^{p_k,q_k}, respectively with the interpretation

$$\mathfrak{M}, x \models W^{p_k,q_k}\varphi \Leftrightarrow |R_{\neg\varphi}(x)| \leq \frac{p_k}{q_k}|R(x)|, \quad k = 1, \ldots, n.$$

The logic, consisting of all valid formulae, we denote by RGML^n.

The common form of a modal elementary conjunction is:

$$\theta = B \wedge \bigwedge_{j=1}^{m_1} M^{p_1,q_1}\varphi_j \wedge \ldots \wedge \bigwedge_{j=m_{n-1}+1}^{m_n} M^{p_n,q_n}\varphi_j \wedge$$

$$\bigwedge_{j=1}^{l_1} W^{p_1,q_1}\psi_j \wedge \ldots \wedge \bigwedge_{j=l_{n-1}+1}^{l_n} W^{p_n,q_n}\psi_j .$$

For the system of linear inequality, attendant on θ, we get:

$$(\sigma_\theta)\quad \left| \begin{array}{l} \sum_{i=1}^{t} \varepsilon_{ij} a_i > \frac{p_k}{q_k} \sum_{i=1}^{t} a_i, \\ \quad k = 1, \ldots, n,\ j = 1, \ldots, m_1 + \cdots + m_n \\ \\ \sum_{i=1}^{t} \varepsilon_{ij} a_i \geq \frac{q_k - p_k}{q_k} \sum_{i=1}^{t} a_i, \\ \quad k = 1, \ldots, n,\ j = m_1 + \cdots + m_n + 1, \ldots, m_1 + \cdots + m_n + l_1 + \cdots + l_n \end{array} \right.$$

Besides, the tree-like model which we build in Proposition 2, is still finite and all the propositions from the case with just one modality hold.

2. If we consider all fractions $\frac{p}{q}$ with relatively prime integers p, q with $1 \leq p < q$, we obtain the logic RGML$^\omega$.

As in the language under consideration there are only finite formulae, any modal elementary conjunction θ is finite and therefore contains only finitely many modal operators. Hence we can consider the attendant on it (finite) system (σ_θ). The tree-like frame and model are finite again, and all the propositions from the case with a single modality, including the decidability, hold.

The following proposition holds for the above defined logics:

Theorem 4. *The logic RGML^n (RGML$^\omega$) is decidable. It coincides with the set of formulae, valid in the finite trees.*

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier Blvd, 1164 Sofia
BULGARIA
E-mail: tinko@fmi.uni-sofia.bg, yanchev@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 100

MODEL REPRESENTATIONS OF THE LIE–GEIZENBERG ALGEBRA OF THREE LINEAR NON-SELFADJOINT OPERATORS

VICTORIYA A. KUZNETSOVA

This work is dedicated to the study of Lie algebra of linear non-selfadjoint operators $\{A_1, A_2, A_3\}$ given by the relations $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, besides, we assume that none of the operators A_1, A_2, A_3 is dissipative. For Lie algebra $\{A_1, A_2, A_3\}$ such that $\{A_1, A_2, A_3\}$ given by the relations $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, take place, and when one of the operators is dissipative, the functional models were constructed earlier.

In Paragraph 1 it is shown that the open system corresponding to this Lie algebra $\{A_1, A_2, A_3\}$, $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, should be considered on the Lie – Geizenberg group $H(3)$. Paragraph 2 is dedicated to the construction of triangular model for this Lie algebra, A_1, A_3 in which are bounded, and A_2 is an unbounded operator. Note that even in the dissipative case such dissipative models haven't been constructed. Using the models from Paragraph 2, in the following Paragraph 3 functional models for the Lie algebra $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, of the special form and acting in the L. de Branges Hilbert space of whole functions are listed. In Paragraph 4 the special class of Lie algebras $[A_1, A_2] = iA_3$; $[A_1, A_3] = 0$; $[A_2, A_3] = 0$, having the reasonable model representations in L. de Branges spaces on Riemann surfaces is displayed.

Keywords: Functional models, L. de Branges transform, Lie algebra
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1. LIE–GEIZENBERG GROUP

I. Following the works [4, 6] for the study of Lie algebra of linear non-selfadjoint operators $\{A_1, A_2, A_3\}$ given by the commutation relations $[A_1, A_2] = iA_3$; $[A_1, A_3]$

$= 0$; $[A_2, A_3] = 0$, we ought to find such Lie group G, the Lie algebra $\{\partial_1, \partial_2, \partial_3\}$ of which is such that $[\partial_1, \partial_2] = \partial_3$, $[\partial_1, \partial_3] = 0$; $[\partial_2, \partial_3] = 0$. Let x, y, $z \in \mathbb{R}$. Consider the Lie – Geizenberg group $G = H(3)$ formed by the elements $g = g(x, y, z)$, the multiplication law in G is given by [8, 9]

$$g\left(x_1, y_1, z_1\right) \circ g\left(x_2, y_2, z_2\right) \stackrel{\text{def}}{=} g\left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2\right). \tag{1.1}$$

Hence it follows that every subgroup

$$G_1 = \{g(x, 0, 0) \in G\}; \quad G_2 = \{g(0, y, 0) \in G\}; \quad G_3 = \{g(0, 0, z) \in G\}; \tag{1.2}$$

is equivalent to the additive group of real numbers $\mathbb{R}$.

It is easy to prove that the group G is isomorphic to the following group of matrices of the third order

$$B_g = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

This immediately follows from the equality

$$B_{g_2} \cdot B_{g_1} = \begin{bmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_2 & z_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x_1 + x_2 & z_1 + z_2 + x_1 y_2 \\ 0 & 1 & y_1 + y_2 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= B_{g_1 \circ g_2}.$$

Consider a complex-valued function $f(g)$ on the group G, which means that we have a function $f(x, y, z)$ on $\mathbb{R}^3$. Define one-parameter subgroup in G corresponding to G_1, G_2, G_3 (1.2),

$$g_1(t) = (t, 0, 0) \in G_1; \quad g_2(t) = (0, t, 0) \in G_2; \quad g_3(t) = (0, 0, t) \in G_3. \tag{1.3}$$

Find the vector fields corresponding to these subgroups

$$F_t^1 = f\left(g_1(t) \circ g(x, y, z)\right) = f(x + t, y, z + ty).$$

Therefore the derivative by t at the identity $e = (0, 0, 0)$ of group G of this function

$$\left.\frac{d}{dt} F_t^1\right|_{t=0} = \partial_1 f$$

where $\partial_1 = \dfrac{\partial}{\partial x} + y \dfrac{\partial}{\partial z}$. Since

$$F_t^2 = f\left(g_2(t) \circ g(x, y, z)\right) = f(x, y + t, z),$$

it is obvious that

$$\left.\frac{d}{dt} F_t^2\right|_{t=0} = \partial_2 f,$$

besides,

$$\partial_2 = \frac{\partial}{\partial y}.$$

Finally, taking into account

$$F_t^3 = f\left(g_3(t) \circ g(x, y, z)\right) = f\left(x, y, z_1 + t\right)$$

we obtain

$$\left.\frac{d}{dt} F_t^3\right|_{t=0} = \partial_3 f,$$

where $\partial_3 = \dfrac{\partial}{\partial z}$. Thus the Lie algebra of vector fields $h(3)$ corresponding to $G = H(3)$ is generated by the differential operators of first order

$$\partial_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}; \quad \partial_2 = \frac{\partial}{\partial y}; \quad \partial_3 = \frac{\partial}{\partial z}. \tag{1.4}$$

Obviously, for this Lie algebra $h(3)$ the commutation relations

$$[\partial_2, \partial_1] = \partial_3; \quad [\partial_1, \partial_3] = 0; \quad [\partial_2, \partial_3] = 0 \tag{1.5}$$

take place. It is well-known [8, 9] that the simply connected Lie group $G = H(3)$ "uniquely" corresponds to this Lie algebra of differential operators (1.4).

II. Consider in a Hilbert space H the Lie algebra of linear operators $\{A_1, A_2, A_3\}$ satisfying the relations

$$[A_1, A_2] = iA_3; \quad [A_1, A_3] = 0; \quad [A_2, A_3] = 0. \tag{1.6}$$

Note that the operators A_1, A_2, A_3 cannot be bounded simultaneously. Otherwise, (1.6) yields

$$[A_1^n, A_2] = inA_1^{n-1}A_3$$

and thus $2\left\|A_1^n\right\| \cdot \left\|A_2\right\| \geq n\left\|A_3\right\|\left\|A_1^{n-1}\right\|$ $(\forall n \in \mathbb{Z}_+)$. In connection with this it is sensible to rewrite the relations (1.6) in terms of resolvents,

$$R_3(w)\left[R_1(\lambda)R_2(z) - R_2(z)R_1(\lambda)\right] = iR_1^2(\lambda)R_2^2(z)R_3(w)w + iR_1^2(\lambda)R_2^2(z);$$

$$[R_1(\lambda), R_3(w)] = 0; \quad [R_2(z), R_3(w)] = 0 \tag{1.7}$$

where $R_1(\lambda) = (A_1 - \lambda I)^{-1}$; $R_2(z) = (A_2 - zI)^{-1}$; $R_3(w) = (A_3 - wI)^{-1}$; and λ, z, w are regularity points of the operators A_1, A_2, A_3, respectively.

III. For the given Lie algebra $\{A_1, A_2, A_3\}$ (1.6) of non-selfadjoint operators construct the colligation of Lie algebra [4, 5, 6].

Definition 1.1. *A family*

$$\Delta = \left(\{A_1, A_2, A_3\}; H; \varphi; E; \{\sigma_k\}_1^3; \left\{\gamma_{k,s}^-\right\}_1^3; \left\{\gamma_{k,s}^+\right\}_1^3 \right) \tag{1.8}$$

is said to be the colligation of Lie algebra if

$$\begin{aligned} &1) \quad [A_1, A_2] = iA_3; \quad [A_1, A_3] = 0; \quad [A_2, A_3] = 0; \\ &2) \quad 2\mathrm{Im}\,\langle A_k h, h\rangle = \langle \sigma_k \varphi h, \varphi h\rangle; \quad \forall h \in \vartheta(A_k); \\ &3) \quad \sigma_k \varphi A_s - \sigma_s \varphi A_k = \gamma_{k,s}^+ \varphi; \quad \gamma_{k,s}^+ = -\gamma_{s,k}^+; \\ &4) \quad \gamma_{k,s}^- = \gamma_{k,s}^+ + i(\sigma_s \varphi\varphi^* \sigma_k - \sigma_k \varphi\varphi^* \sigma_s); \end{aligned} \tag{1.9}$$

for all k and s $(1 \le k,\ s \le 3)$.

Relations (3.6) (§1.3) imply

$$\gamma_{1,3}^\pm = \left(\gamma_{1,3}^\pm\right)^*; \quad \gamma_{2,3}^\pm = \left(\gamma_{2,3}^\pm\right)^*; \quad \gamma_{1,2}^\pm - \left(\gamma_{1,2}^\pm\right)^* = i\sigma_3. \tag{1.10}$$

Consider the differential operators

$$\partial_1 = \frac{\partial}{\partial x}; \quad \partial_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}; \quad \partial_3 = \frac{\partial}{\partial z}; \tag{1.11}$$

coinciding with operators (1.4) after the substitution $x \to y$, $y \to x$. It is obvious that the commutation relations (1.5) now are written in the following way:

$$[\partial_1, \partial_2] = \partial_3; \quad [\partial_1, \partial_3] = 0; \quad [\partial_2, \partial_3] = 0. \tag{1.12}$$

Equations of the open system (3.13), (3.14) (§1.3) are given by

$$\begin{cases} i\partial_k h(x,y,z) + A_k h(x,y,z) = \varphi^* \sigma_k u(x,y,z); \\ h(0) = h_0 \quad (1 \le k \le 3) \quad (x,y,z) \in G; \\ v(x,y,z) = u(x,y,z) - i\varphi h(x,y,z). \end{cases} \tag{1.13}$$

It is easy to show that $u(x,y,z)$ is the solution of the equation system

$$\left\{\sigma_k i\partial_s - \sigma_s i\partial_k + \gamma_{k,s}^-\right\} u(x,y,z) = 0 \quad (1 \le k, s \le 3), \tag{1.14}$$

and the function $v(x,y,z)$ also satisfies the equation system

$$\left\{\sigma_k i\partial_s - \sigma_s i\partial_k + \gamma_{k,s}^+\right\} v(x,y,z) = 0 \quad (1 \le k, s \le 3). \tag{1.15}$$

If σ_1 is invertible, then relations eliminating the overdetermination of equation system (1,14) are given by

$$\begin{aligned} &1. \quad \left[\sigma_1^{-1}\sigma_2, \sigma_1^{-1}\sigma_3\right] = 0; \\ &2. \quad \left[\sigma_1^{-1}\sigma_2, \sigma_1^{-1}\gamma_{1,3}^-\right] - \left[\sigma_1^{-1}\sigma_3, \sigma_1^{-1}\gamma_{1,2}^-\right] = i\sigma_1^{-1}\sigma_3\sigma_1^{-1}\sigma_3; \\ &3. \quad \left[\sigma_1^{-1}\gamma_{1,2}^-, \sigma_1^{-1}\gamma_{1,3}^-\right] = i\sigma_1^{-1}\sigma_3\sigma_1^{-1}\gamma_{1,3}^-. \end{aligned} \tag{1.16}$$

Moreover,

$$\gamma_{2,3}^{-} = \sigma_2 \sigma_1^{-1} \gamma_{1,3}^{-} - \sigma_3 \sigma_1^{-1} \gamma_{1,2}^{-}. \tag{1.17}$$

Similar relations also take place for the family $\left\{ \gamma_{k,s}^{+} \right\}_1^3$.

So, we assume that the operators $\gamma_{1,2}^{-}$, $\gamma_{1,3}^{-}$, for which (1.10) takes place, are specified and the operator $\gamma_{2,3}^{-}$ is specified by formula (1.17). Note that the self-adjointness of $\gamma_{2,3}^{-}$ automatically follows from 2. (1.16) and corresponding relations (1.10) for $\gamma_{1,3}^{-}$ and $\gamma_{1,2}^{-}$.

2. TRIANGULAR MODEL

I. Consider the colligation Δ (1.8) corresponding to the Lie algebra of linear operators $\{A_1, A_2, A_3\}$ given by the commutation relations 1) (1.9) assuming that $\dim E = r < \infty$ and $\sigma_1 = J$ is an involution in E. Let the characteristic function $S_1(\lambda) = I - i\varphi (A_1 - \lambda I)^{-1} \varphi^* J$ be given by

$$S_1(\lambda) = \int_0^{\overleftarrow{l}} \exp \frac{iJdF_t}{\lambda}$$

where F_x is a non-decreasing function on $[0, l]$ such that $\mathrm{tr} F_x = x$. Besides, we assume that measure dF_x is absolutely continuous, $dF_x = a_x dx$ ($\mathrm{tr} a_x = 1$). Define the Hilbert space $L_{r,l}^2 (F_x)$ [1, 3]. Specify in this space the operator system

$$\left(\overset{\circ}{A}_1 f \right)_x = i \int_x^l f_t a_t J dt;$$

$$\left(\overset{\circ}{A}_3 f \right)_x = f_x J \gamma_{x,3} + i \int_x^l f_t a_t \sigma_3 dt;$$

$$\left(\overset{\circ}{A}_2 f \right)_x = f_x' b_x + f_x J \gamma_{x,2} + i \int_x^l f_t a_t \sigma_2 dt; \tag{2.1}$$

where b_x, $\gamma_{x,3}$, $\gamma_{x,2}$ are some operator-functions in E specified on $[0, l]$ and σ_2, σ_3 are selfadjoint operators in E. The domain of definition $\mathcal{D}(A_2)$ is formed by the linear span of smooth functions in $L_{r,l}^2 (F_x)$ such that A_1, A_3 are bounded and A_2 is unbounded non-selfadjoint operator. Find the necessary and sufficient conditions on a_x, b_x, $\gamma_{x,3}$, $\gamma_{x,2}$, σ_2, σ_3 for this operator system (2.1) to form the Lie algebra,

$$\left[\overset{\circ}{A}_1, \overset{\circ}{A}_3 \right] = 0; \quad \left[\overset{\circ}{A}_2, \overset{\circ}{A}_3 \right] = 0; \quad \left[\overset{\circ}{A}_1, \overset{\circ}{A}_2 \right] = i \overset{\circ}{A}_3 . \tag{2.2}$$

It is easy to see [4] that the commutativity of operators $\left[\overset{\circ}{A}_1, \overset{\circ}{A}_3\right] = 0$ signifies that the operator-function $\gamma_{x,3}$ satisfies the relations

$$\begin{cases} \gamma'_{x,3} = i\left(Ja_x\sigma_3 - \sigma_3 a_x J\right); \quad \gamma_{0,3} = \gamma^+_{1,3}; \\ Ja_x\gamma_{x,3} = \gamma_{x,3}a_x J. \end{cases} \tag{2.3}$$

Hence it follows [4] that

$$\overset{\circ}{A}_1 - \overset{\circ}{A}{}^*_1 = i\,\overset{\circ}{\varphi}{}^*\, J\,\overset{\circ}{\varphi}, \quad \overset{\circ}{A}_3 - \overset{\circ}{A}{}^*_3 = i\,\overset{\circ}{\varphi}{}^*\,\sigma_3\,\overset{\circ}{\varphi} \tag{2.4}$$

and, moreover,

$$J\,\overset{\circ}{\varphi}\overset{\circ}{A}_3 - \sigma_3\,\overset{\circ}{\varphi}\overset{\circ}{A}_1 = \gamma^+_{1,3}\,\overset{\circ}{\varphi};$$

$$\gamma^-_{1,3} = \gamma^+_{1,3} + i\left(\sigma_3\,\overset{\circ}{\varphi}\overset{\circ}{\varphi}{}^*\, J - J\,\overset{\circ}{\varphi}\overset{\circ}{\varphi}{}^*\,\sigma_3\right) \tag{2.5}$$

where $\gamma^-_{1,3} = \gamma_{x,3}\big|_{x=l}$ and the operator $\overset{\circ}{\varphi}$ from $L^2_{r,l}(F_x)$ into E is given by

$$\left(\overset{\circ}{\varphi}\, f\right)_x \overset{\text{def}}{=} \int_0^l f_t dF_t. \tag{2.6}$$

Note that (2.4), (2.5) coincide, respectively, with the conditions of colligation 1), 3) 4) (1.9).

II. Find the conditions on a_x, b_x, $\gamma_{x,3}$, $\gamma_{x,2}$ for the relation

$$\left[\overset{\circ}{A}_1, \overset{\circ}{A}_2\right] = i\,\overset{\circ}{A}_3 \tag{2.7}$$

to hold. It is easy to see that

$$\left(\overset{\circ}{A}_1\overset{\circ}{A}_2\, f\right)_x = i\int_x^l f'_t b_t a_t dtJ + i\int_x^l f_t J\gamma_{t,2}a_t dtJ - \int_x^l dt\int_t^l ds f_s a_s\sigma_2 a_t J =$$

$$= -if_x b_x a_x J - i\int_x^l f_t\left(b_t a_t\right)' dtJ + i\int_x^l f_t J\gamma_{t,2}a_t dtJ - \int_x^l dt\int_t^l ds f_s a_s\sigma_2 a_t J,$$

in view of the fact that $f_l = 0$. Similarly,

$$\left(\overset{\circ}{A}_2\overset{\circ}{A}_1\, f\right)_x = -if_x a_x Jb_x + i\int_x^l f_t a_t dt\gamma_{x,2} - \int_x^l dt\int_t^l ds f_s a_s Ja_t\sigma_2.$$

Consider the vector-function Φ_x in $L^2_{r,l}(F_x)$,

$$\Phi_x \overset{\text{def}}{=} \left\{ \left[\overset{\circ}{A}_1, \overset{\circ}{A}_2 \right] - i\, \overset{\circ}{A}_3 \right\} f_x = -if_x \left[b_x a_x J - a_x J b_x + J\gamma_{x,3} \right] - i \int\limits_x^l f_t \left(b_t a_t \right)' dtJ +$$

$$+i \int\limits_x^l f_t J\gamma_{t,2} a_t dtJ - i \int\limits_x^l f_t a_t dt \gamma_{x,2} - i^2 \int\limits_x^l f_t a_t dt \sigma_3 - \int\limits_x^l dt \int\limits_t^l ds f_s a_s \left(\sigma_2 a_t J - J a_t \sigma_2 \right).$$

Suppose

$$b_x a_x J - a_x J b_x + J\gamma_{x,3} = 0 \tag{2.8}$$

and let $\gamma_{x,2}$ be differentiable, then it is easy to see that the derivative of function Φ_x is

$$\Phi'_x = if_x \left(b_x a_x \right)' J - if_x J\gamma_{x,2} a_x J + if_x a_x \gamma_{x,2} + if_x a_x \sigma_3 -$$

$$-i \int\limits_x^l f_t a_t dt \gamma'_{x,2} + \int\limits_x^l f_t a_t dt \left(\sigma_2 a_x J - J a_x \sigma_2 \right).$$

Hence it follows that $\Phi'_x = 0$ if

$$\begin{cases} \left(b_x a_x \right)' J - J\gamma_{x,2} a_x J + a_x \gamma_{x,2} + i a_x \sigma_3 = 0; \\ i\gamma'_{x,2} = \sigma_2 a_x J - J a_x \sigma_2. \end{cases} \tag{2.9}$$

Thus, $\Phi'_x = 0$, and since $\Phi_l = 0$, then $\Phi_x \equiv 0$.

Lemma 2.1. *Suppose that* (2.8), (2.9) *take place, then the operator system* $\left\{ \overset{\circ}{A}_1, \overset{\circ}{A}_2, \overset{\circ}{A}_3 \right\}$ (2.1) *satisfies the commutation relation* (2.7).

III. Prove that condition 3) (1.9) is true for $\overset{\circ}{A}_1$, $\overset{\circ}{A}_2$ (2.1). To do this, calculate

$$\left(J\, \overset{\circ}{\varphi} \overset{\circ}{A}_2 - \sigma_2\, \overset{\circ}{\varphi} \overset{\circ}{A}_1 \right) f_x = \int\limits_0^l \left(f'_x b_x + f_x J\gamma_{x,2} + \int\limits_x^l f_t a_t \sigma_2 dt \right) a_x dx J -$$

$$- \int\limits_0^l i \int\limits_x^l f_t a_t dt J a_x dx \sigma_2 =$$

$$= \int\limits_0^l f_x \left\{ J\gamma_{x,2} a_x J - \left(b_x a_x \right)' J + i a_x \int\limits_0^x \left(\sigma_2 a_t J - J a_t \sigma_2 \right) dt \right\} dx.$$

The second equality in (2.9) implies

$$\gamma_{x,2} = \gamma_{1,2}^{+} - i\sigma_3 + i \int_0^x (J a_t \sigma_2 - \sigma_2 a_t J)\, dt. \tag{2.11}$$

Here we use the equality

$$\gamma_{1,2}^{+} - \left(\gamma_{1,2}^{+}\right)^{*} = i\sigma_3 \tag{2.12}$$

taking place in virtue of (1.10) §3.1. Thus

$$\left(J \overset{\circ}{\varphi} \overset{\circ}{A}_2 - \sigma_2 \overset{\circ}{\varphi} \overset{\circ}{A}_1\right) f_x =$$

$$= \int_0^l f_x \left\{J\gamma_{x,2} a_x J - (b_x a_x)' J + a_x \gamma_{1,2}^{+} - i a_x \sigma_3 - a_x \gamma_{x,2}\right\} dx =$$

$$= \int_x^l f_x a_x dx \gamma_{1,2}^{+} = \gamma_{1,2}^{+} \left(\overset{\circ}{\varphi} f\right)_x$$

in virtue of the first condition in (2.9) and definition (2.6) of the operator $\overset{\circ}{\varphi}$.

Lemma 2.2. *Let the family* $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, J, \sigma_2, \sigma_3\}$ *be such that* (2.8), (2.9) *are true and, moreover,* $\gamma_{x,2}$, *solution of the second equation in* (2.9) *satisfies the initial condition* $\gamma_{0,2} = \left(\gamma_{1,2}^{+}\right)^{*}$, *besides,* $\gamma_{1,2}^{+} - \left(\gamma_{1,2}^{+}\right)^{*} = i\sigma_3$ (2.12). *Then the colligation relation* 3) (1.9)

$$J \overset{\circ}{\varphi} \overset{\circ}{A}_2 - \sigma_2 \overset{\circ}{\varphi} \overset{\circ}{\varphi} \overset{\circ}{A}_1 = \gamma_{1,2}^{+} \overset{\circ}{\varphi} \tag{2.13}$$

is true.

IV. Study when the colligation relation 2) (1.9) takes place for the operator $\overset{\circ}{A}_2$ (2.1). Calculate the expression

$$2\mathrm{Im}\left\langle \overset{\circ}{A}_2 f, f\right\rangle = \frac{1}{i}\int_0^l \left[f_x' b_x + f_x J\gamma_{x,2} + i\int_x^l f_t a_t dt \sigma_2\right] a_x f_x^* dx -$$

$$-\frac{1}{i}\int_0^l dx f_x a_x \left[b_x^* (f_x^*)' + \gamma_{x,2}^* J f_x^* - i\int_x^l \sigma_2 a_t f_t^* dt\right] =$$

$$= \frac{1}{i}\int_0^l \left[f_x' b_x a_x f_x^* - f_x a_x b_x^* (f_x^*)' + f_x J\gamma_{x,2} a_x f_x^* - f_x a_x \gamma_{x,2}^* J f_x^*\right] dx +$$

$$+\int_0^l \left\{ \int_x^l f_t a_t \sigma_2 dt a_x f_x^* + f_x a_x \int_x^l \sigma_2 a_t f_t^* dt \right\} dx.$$

Obviously, the second integral after the transfer of the order of integration is

$$\int_0^l f_x a_x dx \sigma_2 \int_0^l a_t f_t^* dt = \left\langle \sigma_2 \overset{\circ}{\varphi} f, \overset{\circ}{\varphi} f \right\rangle$$

in virtue of the definition of operator $\overset{\circ}{\varphi}$ (2.6). So, for the colligation relation 2) (1.9) to hold for $\overset{\circ}{A}_2$, one has to ascertain when the first integral vanishes.

The integrand of this integral equals

$$\Psi_x \overset{\text{def}}{=} f_x' b_x a_x f_x^* - f_x a_x b_x^* (f_x^*)' + f_x J \gamma_{x,2} a_x f_x^* - f_x a_x (\gamma_{x,2} + i\sigma_3) J f_x$$

in virtue of $\gamma_{x,2}^* - \gamma_{x,2} = i\sigma_3$. This easily follows from (2.11). Thus,

$$\Psi_x = f_x' b_x a_x f_x^* - f_x a_x b_x^* (f_x^*)' + f_x (a_x b_x)' f_x^*,$$

we took into account the first equality in (2.9).

Let the condition

$$a_x b_x^* = b_x a_x \tag{2.14}$$

hold, then $\Psi_x = (f b_x a_x f_x^*)'$ and thus

$$\int_0^l \Psi_t dt = 0$$

since $f_0 = f_l = 0$ for $f_x \in \mathcal{D}(A_2)$.

Lemma 2.3. *Suppose that for the family* $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, J, \sigma_2, \sigma_3\}$ (2.8), (2.9) *are true and* $\gamma_{x,2}$ *as the solution of the second equation in* (2.9) *is such that* $\gamma_{0,2} = \gamma_{1,2}^+$ *and* (2.12) *takes place. Then, if* (2.14) *holds* $\forall f_x \in \mathcal{D}(A_2)$, *the colligation relation*

$$2\text{Im}\left\langle \overset{\circ}{A}_2 f, f \right\rangle = \left\langle \sigma_2 \overset{\circ}{\varphi} f, \overset{\circ}{\varphi} f \right\rangle \tag{2.15}$$

is true.

V. Study the interchangeability (2.2) of operators $\overset{\circ}{A}_2$, $\overset{\circ}{A}_3$ (2.1). It is easy to see that

$$\overset{\circ}{A}_2 \overset{\circ}{A}_3 f_x = \left(f_x J \gamma_{x,3} + i \int_x^l f_t a_t dt \sigma_3 \right)' b_x + \left(f_x J \gamma_{x,3} + i \int_x^l f_t a_t dt \sigma_3 \right) J \gamma_{x,2} +$$

$$+i\int_x^l\left(f_tJ\gamma_{t,3}+i\int_t^l f_sa_sds\sigma_3\right)a_t\sigma_2dt=f_xJ\gamma_{x,3}b_x+f_xJ\gamma'_{x,3}b_x-if_xa_x\sigma_3b_x+$$

$$+f_xJ\gamma_{x,3}J\gamma_{x,2}+i\int_x^l f_ta_tdt\sigma_3J\gamma_{x,2}+i\int_x^l f_tJ\gamma_{t,3}a_t\sigma_2dt-\int_x^l dt\int_t^l dsa_s\sigma_3a_t\sigma_2.$$

Similarly,

$$\overset{\circ}{A}_3\overset{\circ}{A}_2\ f_x=\left(f'_xb_x+f_xJ\gamma_{x,2}+i\int_x^l f_ta_tdt\sigma_2\right)J\gamma_{x,3}+$$

$$+i\int_x^l\left(f'_tb_t+f_tJ\gamma_{t,2}+i\int_t^l f_sa_sds\sigma_2\right)a_t\sigma_3dt=f'_xb_xJ\gamma_{x,3}+f_xJ\gamma_{x,2}J\gamma_{x,3}+$$

$$+i\int_x^l f_ta_tdt\sigma_2J\gamma_{x,3}-i\int_x^l f_t\left(b_ta_t\right)'\sigma_3dt+i\int_x^l f_tJ\gamma_{t,2}a_t\sigma_3dt-\int_x^l dt\int_t^l dsa_s\sigma_2a_t\sigma_3.$$

Thus function G_x from $L^2_{r,l}\left(F_x\right)$ is

$$G_x\overset{\text{def}}{=}\left[\overset{\circ}{A}_2,\overset{\circ}{A}_3\right]f_x=f'_x\left[J\gamma_{x,3}b_x-b_xJ\gamma_{x,3}\right]+$$

$$+f_x\left\{J\gamma'_{x,3}b_x-ia_x\sigma_3b_x+J\gamma_{x,2}J\gamma_{x,3}+ib_xa_x\sigma_3\right\}+i\int_x^l f_ta_tdt\left[\sigma_3J\gamma_{x,2}-\sigma_2J\gamma_{x,3}\right]+$$

$$+i\int_x^l f_t\left[J\gamma_{t,3}a_t\sigma_2-J\gamma_{t,2}a_t\sigma_3\right]dt-\int_x^l dt\int_t^l dsa_s\left(\sigma_3a_t\sigma_2-\sigma_2a_t\sigma_3\right).$$

Suppose that the equalities

$$\begin{cases} J\gamma_{x,3}b_x=b_xJ\gamma_{x,3};\\ J\gamma'_{x,3}b_x+ib_xa_x\sigma_3-ia_x\sigma_3b_x+J\gamma_{x,3}J\gamma_{x,2}-J\gamma_{x,2}J\gamma_{x,3}\end{cases}$$

hold. Then, taking into account smoothness of $\gamma_{x,2}$ and $\gamma_{x,3}$, we obtain

$$G'_x=-if_x\left\{a_x\sigma_3\gamma_{x,2}-a_x\sigma_2J\gamma_{x,3}+J\gamma_{x,3}a_x\sigma_2-J\gamma_{x,2}a_x\sigma_3\right\}+$$

$$+\int_x^l f_ta_tdt\left\{i\left[\sigma_3J\gamma_{x,2}-\sigma_2J\gamma_{x,3}\right]'+\sigma_3a_x\sigma_2-\sigma_2a_x\sigma_3\right\}.$$

Requirement $G'_x = 0$ leads to the equalities

$$\begin{cases} a_x\sigma_3 J\gamma_{x,2} - J\gamma_{x,2}a_x\sigma_3 + J\gamma_{x,3}a_x\sigma_2 - a_x\sigma_2 J\gamma_{x,3} = 0; \\ \sigma_3 J\gamma'_{x,2} - \sigma_2 J\gamma'_{x,3} = i\left(\sigma_3 a_x\sigma_2 - \sigma_2 a_x\sigma_3\right). \end{cases} \tag{2.17}$$

Since $G_l = 0$, hence it follows that $G_x \equiv 0$. As a result, we obtain the statement.

Lemma 2.4. *If relations* (2.16), (2.17) *hold for the family* $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, J, \sigma_2\sigma_3\}$, *then the operators* $\overset{\circ}{A}_2$ *and* $\overset{\circ}{A}_3$ *commute,*

$$\left[\overset{\circ}{A}_2, \overset{\circ}{A}_3\right] = 0. \tag{2.18}$$

Observation 2.1. *Last equality in* (2.17) *is the obvious corollary of equations for* $\gamma_{x,2}$ *(2.9) and* $\gamma_{x,3}$ (2.3) *since*

$$\sigma_3 Ji\left(Ja_x\sigma_2 - \sigma_2 a_x J\right) - \sigma_2 Ji\left(Ja_x\sigma_3 - \sigma_3 a_x J\right) = i\left(\sigma_3 a_x\sigma_2 - \sigma_2 a_x\sigma_3\right)$$

in virtue of 1. (1.6). *Note that this fact is completely coordinated with* (1.17).

VI. Summarizing considerations of previous clauses, we obtain the following

Theorem 2.1. *Suppose operators* $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, \sigma_2, \sigma_3\}$ *in* E *are such that:*

$$\begin{aligned} &1) \quad \gamma_{x,3} \text{ satisfies relations (2.3);} \\ &2) \quad \gamma_{x,3} = Ja_xJb_x - Jb_xa_xJ; \\ &3) \quad (b_xa_x)' = J\gamma_{x,2}a_x - a_x\gamma_{x,2}J - ia_x\sigma_3 J; \\ &4) \quad \gamma'_{x,2} = i\left(Ja_x\sigma_2 - \sigma_2 a_x J\right); \quad \gamma_{0,2} = \left(\gamma^{+}_{1,2}\right)^*; \end{aligned} \tag{2.19}$$

and $\gamma_{1,2} - \gamma^*_{1,2} = i\sigma_3$. *Moreover,*

$$\begin{aligned} &5) \quad J\gamma_{x,3}b_x = b_xJ\gamma_{x,3}; \\ &6) \quad J\gamma'_{x,3}b_x = \left[J\gamma_{x,2}, J\gamma_{x,3}\right] + i\left[a_x\sigma_3, b_x\right]; \\ &7) \quad \left[a_x\sigma_3, J\gamma_{x,2}\right] - \left[a_x\sigma_2, J\gamma_{x,3}\right] = 0 \end{aligned} \tag{2.20}$$

take place. Then the family

$$\overset{\circ}{\Delta} = \left(\left\{\overset{\circ}{A}_1, \overset{\circ}{A}_2, \overset{\circ}{A}_3\right\}; L^2_{r,l}\left(F_x\right); \overset{\circ}{\varphi}; E; \{\sigma_k\}_1^3; \left\{\gamma^-_{k,s}\right\}_1^3; \left\{\gamma^+_{k,s}\right\}_1^3\right) \tag{2.21}$$

is the colligation of Lie algebra (1.8)–(1.9) *where* $\overset{\circ}{A}_1$, $\overset{\circ}{A}_2$, $\overset{\circ}{A}_3$ *are given by* (2.1) *and* $\overset{\circ}{\varphi}$, *respectively, by* (2.6), *besides,* $\gamma^-_{1,k} = \gamma_{x,k}|_{x=l}$ $(k = 2, 3)$, *the operators* $\gamma^{\pm}_{k,s}$ *when* $s \neq 1$ *are given by formula* (1.17) *and* $\sigma_1 = J$ *is an involution.*

Now use the theorem on unitary equivalence [1, 2].

Theorem 2.2. *Let Δ, simple colligation of Lie algebra* (1.8), (1.9), *be given by* (1.16), (1.17). *If the spectrum of operator A_1 is concentrated at zero and the characteristic function $S_1(\lambda) = I - i\varphi (A_1 - \lambda I)^{-1} \varphi^* J$ is given by*

$$S_1(\lambda) = \int_0^{\overleftarrow{l}} \exp \frac{iJdF_t}{\lambda},$$

besides, dF_x is absolutely continuous, $dF_x = a_x dx$, and a_x is such that for the family $\{a_x, b_x, \gamma_{x,2}, \gamma_{x,3}, J, \sigma_2, \sigma_3\}$ (2.19), (2.20) *take place, then the colligation Δ is unitarily equivalent to the simple part of colligation $\overset{\circ}{\Delta}$* (2.21).

3. FUNCTIONAL MODEL OF LIE ALGEBRA

I. Consider the triangular model (2.1) of Lie algebra of linear operators $\left\{ \overset{\circ}{A}_1, \overset{\circ}{A}_2, \overset{\circ}{A}_3 \right\}$ (2.2) assuming that $\dim E = 2$ and $J = J_N$ is given by

$$J_N = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}. \tag{3.0}$$

Under the action of the L. de Branges transform [3, 7], the operator $\overset{\circ}{A}_1$ (2.1) turns into the shift operator in $\mathcal{B}(A, B)$ since

$$\mathcal{B}_L \left(\overset{\circ}{A}_1 f_t \right) = \frac{1}{\pi} \int_0^l \left\{ i \int_t^l f_s dF_s J \right\} dF_t L_t^* (\bar{z}) = \frac{1}{\pi} \int_0^l f_t dF_t \left\{ \frac{L_t^* (\bar{z}) - L_t^*(0)}{z} \right\}^*$$

and thus operator $\overset{\circ}{A}_1$ after the transform $\mathcal{B}_L$ turns into $\tilde{A}_1$,

$$\tilde{A}_1 = \frac{F(z) - F(0)}{z}, \tag{3.1}$$

where $F(z) \overset{\text{def}}{=} \mathcal{B}_L (f_t)$. To calculate $\mathcal{B}_L \left(\overset{\circ}{A}_3 f_t \right)$ and $\mathcal{B}_L \left(\overset{\circ}{A}_2 f_t \right)$, note that

$$L_t(z) = \left(I - z \overset{\circ}{A}{}_1^* \right)^{-1} \tilde{\varphi}^*(1, 0). \tag{3.2}$$

Since

$$\mathcal{B}_L \left(\overset{\circ}{A}_k f_t \right) = \left\langle \overset{\circ}{A}_k f_t, L_t (\bar{z}) \right\rangle = \left\langle f_t, \overset{\circ}{A}{}_k^* L_t (\bar{z}) \right\rangle$$

($k = 2$, 3), then using (3.2) we ought to find the expressions

$$\overset{\circ}{A}{}_3^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0); \quad \overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0). \tag{3.3}$$

Commutativity of $\left[\overset{\circ}{A}_1, \overset{\circ}{A}_3\right]$, the colligation relation $J\tilde{\varphi} \overset{\circ}{A}_3 = \sigma_3\tilde{\varphi} \overset{\circ}{A}_1 + \gamma_{1,3}^+\tilde{\varphi}$, and the self-adjointness of $\gamma_{1,3}^+ = \left(\gamma_{1,3}^+\right)^*$ (1.10) yields

$$\overset{\circ}{A}{}_3^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* \tilde{\varphi}^*\sigma_3 J + \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\gamma_{1,3}^+ J =$$

$$= \frac{\left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} - I}{z} \tilde{\varphi}^*\sigma_3 J + \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\gamma_{1,3}^+ J.$$

Thus, expression (3.3) for the operator $\overset{\circ}{A}_3$ is given by

$$\overset{\circ}{A}{}_3^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) = \frac{1}{z}\left\{\left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^*\right\}\sigma_3 J(1,0) +$$

$$+ \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\gamma_{1,3}^+ J(1,0). \tag{3.4}$$

Expand $\sigma_3 J(1,0)$ and $\gamma_{1,3}^+ J(1,0)$ in terms of the basis $\{(1,0),(0,1)\}$ in E^2,

$$\sigma_3 J(1,0) = \bar{\alpha}_3(1,0) + \bar{\beta}_3(0,1);$$

$$\gamma_{1,3}^+ J(1,0) = \bar{\mu}_3(1,0) + \bar{\vartheta}_3(0,1); \tag{3.5}$$

where

$$\bar{\alpha}_3 = (1,0)\sigma_3 J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\beta}_3 = (1,0)\sigma_3 J \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\bar{\mu}_3 = (1,0)\gamma_{1,3}^+ J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\vartheta}_3 = (1,0)\gamma_{1,3}^+ J \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.6}$$

As a result, we obtain that expression (3.4) can be written in the following form:

$$\overset{\circ}{A}{}_3^* \left(I - z \overset{\circ}{A}{}_1^*\right) \tilde{\varphi}^*(1,0) = \bar{\alpha}_3 \frac{1}{z}\left\{\left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^*\right\}(1,0) +$$

$$+ \bar{\beta}_3 \frac{1}{z}\left\{\left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^*\right\}(0,1) + \bar{\mu}_3 \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) +$$

$$+ \bar{\vartheta}_3 \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(0,1). \tag{3.7}$$

Along with the integral equation

$$L_x(z) + iz \int_0^x L_t(z) dF_t J = (1, 0) \tag{3.8}$$

for $L_x(z)$, consider the integral equation

$$N_x(z) + iz \int_0^x N_t(z) dF_t J = (0, 1) \tag{3.9}$$

for the row vector $N_x(z)$ [3, 7].

Thus expression (3.7) can be written as

$$\overset{\circ}{A}{}_3^* \, L_t(\bar{z}) = \bar{\alpha} \frac{L_t(\bar{z}) - L_t(0)}{\bar{z}} + \bar{\beta}_3 \frac{N_t(\bar{z}) - N_t(0)}{\bar{z}} + \bar{\mu}_3 L_t(\bar{z}) + \bar{\vartheta}_3 N_t(\bar{z}) \,. \tag{3.10}$$

Construct the L. de Branges space $\mathcal{B}(C, D)$ [3, 7] by the row vector $N_x(z) = [C_x(z), D_x(z)]$ and specify the L. de Branges space $\mathcal{B}_L$ from $L^2_{2,l}(F_x)$ onto $\mathcal{B}(C, D)$ using the formula

$$G(z) \overset{\text{def}}{=} \mathcal{B}_N(f_t) = \frac{1}{\pi} \int_0^l f_t dF_t N_t^*(\bar{z}) \,. \tag{3.11}$$

A function $G(z) \in \mathcal{B}(C, D)$ is said to be **dual** to $F(z) \in \mathcal{B}(A, B)$ if

$$F(z) = \mathcal{B}_L(f_t) \,, \quad G(z) = \mathcal{B}_N(f_t) \,. \tag{3.12}$$

Using these notations and (3.10), we obtain that the operator $\overset{\circ}{A}_3$ after the L. de Branges transform equals

$$\tilde{A}_3 F(z) = \frac{\alpha_3 F(z) + \beta_3 G(z) - \alpha_3 F(0) - \beta_3 G(0)}{\bar{z}} + \mu_3 F(z) + \vartheta_3 G(z) \tag{3.13}$$

where the complex numbers α_3, β_3, μ_3, ϑ_3 are given by (3.6) and functions $F(z)$ and $G(z)$, respectively, equal (3.12).

Observation 3.1. *Generally speaking, function $G(z)$ (3.12) does not belong to the space $\mathcal{B}(A, B)$ but, nevertheless, there exist such numbers α_3, β_3, μ_3, ϑ_3 (3.6) from $\mathbb{C}$ that the expressions*

$$\mu_3 F(z) + \vartheta_3 G(z); \qquad \frac{\alpha_3 F(z) + \beta_3 G(z) - \alpha_3 F(0) - \beta_3 G(0)}{\bar{z}}$$

belong to the space $\mathcal{B}(A, B)$. Besides, numbers α_3, β_3, μ_3, ϑ_3 do not depend on $F(z)\mathcal{B}(A, B)$.

To obtain the formula similar to (3.13) for $\tilde{A}_2$, it is necessary, in virtue of (3.3), to calculate the expression $\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0)$.

The commutation relation $\left[\overset{\circ}{A}_1, \overset{\circ}{A}_2\right] = i \overset{\circ}{A}_3$ implies

$$\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} - \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_2^* = iz \overset{\circ}{A}{}_3^*,$$

therefore

$$\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} = \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_2^* - iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_3^*$$

in virtue of $\left[\overset{\circ}{A}_3, \overset{\circ}{A}_1\right] = 0$. Taking into account the colligation relation $J\tilde{\varphi} \overset{\circ}{A}_2 = \sigma\tilde{\varphi} \overset{\circ}{A}_2 = \sigma_2\tilde{\varphi} \overset{\circ}{A}_1 + \gamma_{1,2}^{+}\tilde{\varphi}$, $J\tilde{\varphi} \overset{\circ}{A}_3 = \sigma_3\tilde{\varphi} \overset{\circ}{A}_1 + \gamma_{1,3}^{+}\tilde{\varphi}$ 3) from (1.9), we obtain

$$\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* \tilde{\varphi}^* \sigma_2 J + \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi} \left(\gamma_{1,2}^{+}\right)^* J -$$

$$-iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_1^* \tilde{\varphi}^* \sigma_3 J - iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \tilde{\varphi}^* \gamma_{1,3}^{+} J.$$

Use an obvious equality

$$z \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* = \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} - I,$$

then

$$\overset{\circ}{A}{}_2^* \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \frac{1}{z} \left\{ \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^* \right\} \sigma_2 J +$$

$$+ \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* \left(\gamma_{1,2}^{+}\right)^* J - iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_1^* \tilde{\varphi}^* \sigma_3 J -$$

$$-iz^2 \left(I - z \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_1^* \tilde{\varphi}^* \gamma_{1,3}^{+} J + iz \left(I - z \overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* \gamma_{1,3}^{+} J. \tag{3.14}$$

Similar to (3.5), expand the vectors $\sigma_2 J(1,0)$ and $\left(\gamma_{1,2}^{+}\right)^* J(1,0)$ in terms of the basis $\{(1,0), (0,1)\}$ in E^2,

$$\sigma_2 J(1,0) = \bar{\alpha}_2(1,0) + \bar{\beta}_2(0,1);$$

$$\left(\gamma_{1,2}^{+}\right)^* J(1,0) = \bar{\mu}_2(1,0) + \bar{\vartheta}_2(0,1); \tag{3.15}$$

where

$$\bar{\alpha}_2 = (1,0)\sigma_2 J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\beta}_2 = (1,0)\sigma_2 J \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\bar{\mu}_2 = (1,0)\left(\gamma_{1,2}^{+}\right)^* J \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \bar{\vartheta}_2 = (1,0)\left(\gamma_{1,2}^{+}\right)^* J \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.16}$$

Then we obtain that expression (3.14) equals

$$\overset{\circ}{A}{}_2^* \left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) = \bar{\alpha}_2 \frac{1}{z}\left\{\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^*\right\}(1,0) +$$

$$+\bar{\beta}_2 \frac{1}{z}\left\{\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* - \tilde{\varphi}^*\right\}(0,1) + \bar{\mu}_2\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) +$$

$$+\bar{\vartheta}_2\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(0,1) - iz\bar{\alpha}_3 \frac{d}{dz}\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) -$$

$$-iz\bar{\beta}_3 \frac{d}{dz}\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) - iz^2\bar{\mu}_3 \frac{d}{dz}\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^{(}1,0) -$$

$$-iz^2\bar{\vartheta}_3 \frac{d}{dz}\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) + iz\bar{\mu}_3\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0) +$$

$$+iz\bar{\vartheta}_3\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*(1,0). \tag{3.17}$$

Using the definition of $F(z)$ and $G(z)$ (3.12), we obtain that the operator $\overset{\circ}{A}_2$ after the L. de Branges transform turns into the operator $\tilde{A}_2$,

$$\tilde{A}_2 F(z) = \frac{\bar{\alpha}_2 F(z) + \beta_2 G(z) - \alpha_2 F(0) - \beta_2 G(0)}{\bar{z}} + \mu_2 F(z) + \vartheta_2 G(z) -$$

$$-iz\frac{d}{dz}\left\{\alpha_3 F(z) + \beta_3 G(z)\right\} - iz^2 \frac{d}{dz}\left\{\mu_3 F(z) + \vartheta_3 G(z)\right\} + iz\left\{\mu_3 F(z) + \vartheta_3 G(z)\right\}, \tag{3.18}$$

which in elementary way follows from (3.17).

Observation 3.2. *The dual function $G(z)$ to $F(z)$ does not necessarily belong to the space $\mathcal{B}(A,B)$ but, nevertheless, there always exist such constants α_2, α_3, β_2, β_3, μ_2, μ_3, ϑ_2, ϑ_3 from $\mathbb{C}$ (not depending on $F(z)$) that the expressions*

$$\frac{\alpha_2 F(z) + \beta_2 G(z) - \alpha_2 F(0) - \beta_2 G(0)}{\bar{z}}; \quad F(z)\left(\mu_2 + iz\mu_3\right) + G(z)\left(\vartheta_2 + iz\vartheta_3\right);$$

$$z\frac{d}{dz}\left\{\alpha_3 F(z) + \beta_3 G(z)\right\}; \quad z^2\frac{d}{dz}\left\{\mu_3 F(z) + \vartheta_3 G(z)\right\}$$

already belong to $\mathcal{B}(A, B)$.

Define the operator $\tilde{\varphi}$ from $\mathcal{B}(A, B)$ into E^2 by the formula

$$\tilde{\varphi}F(z)\, \langle F(z), e_1(z)\rangle\, (1,0) + \langle F(z), e_2(z)\rangle\, (0,1) \tag{3.19}$$

where

$$e_1(z) = \frac{B_l^*(z)}{z}; \quad e_2(z) = 1 - A_l^*(z)z. \tag{3.20}$$

Theorem 3.1. *Let* Δ *be the simple colligation of Lie algebra* (1.8), (1.9), *spectrum of the operator* A_1 *be concentrated at zero and the characteristic function* $S_1(\lambda) = I - i\varphi(A_1 - \lambda I)^{-1}\varphi^* J$ *be given by*

$$S_1(\lambda) = \overset{\curvearrowleft}{\int_0^l} \exp \frac{iJdF_t}{\lambda}.$$

Besides, measure dF_x *is absolutely continuous,* $dF_x = a_x dx$, $a_x \geq 0$, a_x *is matrix-function in* E^2, *and* J *is given by* (3.0). *And, moreover, let the selfadjoint operators* σ_2, σ_3, $\gamma_{1,3}^+$ *be given in* E^2, *the operator* $\gamma_{1,2}^+$ *be such that* $\gamma_{1,2}^+ - \left(\gamma_{1,2}^+\right)^* = i\sigma_3$, *and* (1.16), (1.7) *take place. Then the colligation* Δ (1.8) *is unitarily equivalent to the functional model*

$$\tilde{\Delta} = \left(\left\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3\right\}; \mathcal{B}(A, B); \tilde{\varphi}; \{J, \sigma_2, \sigma_3\}; \left\{\gamma_{k,s}^+\right\}_1^3; \left\{\gamma_{k,s}^-\right\}_1^3\right) \tag{3.21}$$

where the operators $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ *are given by* (3.1), (3.13), (3.18) *respectively; operator* $\tilde{\varphi}$ *equals* (3.19)*; the numbers* $\{\alpha_k, \beta_k, \mu_k, \vartheta_k\}_2^3$ *are given by the formulas* (3.6), (3.15)*; and, finally,* $\{e_k(z)\}_1^2$ *are given by* (3.20).

4. FUNCTIONAL MODELS ON RIEMANN SURFACE

I. Let $\dim E = r < \infty$, and $\sigma_1 = J$ be an involution, then the relation [4, 5, 6]

$$J\left(\sigma_2 + z\left(\gamma_{1,2}^+\right)^*\right) J\left(\sigma_3 + z\gamma_{1,3}^+\right) = J\left(\sigma_3 + z\gamma_{1,3}^+\right) J\left(\sigma_2 + z\gamma_{1,2}^+\right) \tag{4.1}$$

is true $\forall z \in \mathbb{C}$. We used the fact that $\gamma_{1,2}^+ = \left(\gamma_{1,2}^+\right)^* + i\sigma_3$ in virtue of (1.16) §3.1. Suppose that $\dim E = r = 2n$ is even and the matrix-function in E specified on $[0, l]$ equals

$$a_x = I_n \otimes \hat{a}_x \tag{4.2}$$

where I_n is the unit operator in E^n, $\hat{a}_x$ is the non-negative (2×2) matrix-function such that $\operatorname{tr}\hat{a}_x = n^{-1}$. Knowing $dF_x = a_x dx$, define the Hilbert space $L^2_{2n,l}(F_x)$ formed by the vector-functions $f(x) = (f_1(x), \dots, f_n(x))$ such that

$$\int_0^l f_k(x)\hat{a}_x f_k^*(x)dx < \infty$$

$\forall k$ $(1 \le k \le n)$, besides, $f_k(x)$ is a row vector from E^2 $(x \in [0, l])$.

Let the operators σ_1 $(= J)$, σ_2, σ_3 and $\gamma^+_{1,3}$, $\gamma^-_{1,2}$ be given by

$$\sigma_1 = J = I_n \otimes J_N; \quad \sigma_2 = \tilde{\sigma}_2 \otimes J_N; \quad \sigma_3 = \tilde{\sigma}_3 \otimes J_N;$$
$$\gamma^+_{1,3} = \tilde{\gamma}_3 \otimes J_N; \quad \gamma^+_{1,2} = \tilde{\gamma}_2 \otimes J_N \tag{4.3}$$

where $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\gamma}_3$ are selfadjoint operators in E^n, and $\tilde{\gamma}_2$ is such that

$$\tilde{\gamma}_2 - \tilde{\gamma}_2^* = i\tilde{\sigma}_3. \tag{4.4}$$

Then the conditions (1.10) §1 hold. Equality (4.1) in terms of $\{\tilde{\sigma}_k, \tilde{\gamma}_k\}_1^3$ is written in the following way:

$$(\tilde{\sigma}_2 + z\tilde{\gamma}_2^*)(\tilde{\sigma}_3 + z\tilde{\gamma}_3) = (\tilde{\sigma}_3 + z\tilde{\gamma}_3)(\tilde{\sigma}_2 + z\tilde{\gamma}_2). \tag{4.5}$$

The L. de Branges transform $\mathcal{B}_L$ [3, 7] of a vector-function $f(x)$ from $L^2_{2n,l}(F_x)$ associates each of its components $f_k(x) \in L^2_{2,l}(\hat{a}_x dx)$ (here $dF_x = a_x dx$ and a_x is given by (4.2)) with the function

$$F_k(x) \stackrel{\text{def}}{=} \mathcal{B}_L(f_k) = \frac{1}{\pi}\int_0^l f_k(x)\hat{a}_x L_x^*(\bar{z})\,dx \tag{4.6}$$

from the L. de Branges $\mathcal{B}(A, B)$, besides, $L_x(z)$ is the solution of the integral equation (3.8) by the measure $\hat{a}_x dx$. As a result, we obtain the Hilbert space $\mathcal{B}^n(A, B) = E^n \otimes \mathcal{B}(A, B)$ formed by the vector-functions $F(z) = (F_1(z), \dots, F_n(z))$,

$$\mathcal{B}^n(A, B) = \{F(z) = (F_1(z), \dots, F_n(z)) : F_k(z) \in \mathcal{B}(A, B)\,(1 \le k \le n)\}. \tag{4.7}$$

Scalar product in $\mathcal{B}^n(A, B)$ is given by

$$\langle F(z), G(z)\rangle_{\mathcal{B}^n(A,B)} = \sum_{k=1}^n \langle F_k(z), G_k(z)\rangle_{\mathcal{B}(A,B)}.$$

Taking into account the form of the matrix-function a_x (4.2) and the operator σ_1 (4.3), it is easy to show that the L. de Branges transform (4.6) translates the triangular model $\overset{\circ}{A}_1$ (2.1) in the shift operator

$$\left(\tilde{A}_1 F\right)(z) = \frac{1}{z}(F(z) - F(0)), \tag{4.8}$$

$\forall F(z) \in \mathcal{B}^n(A,B)$. To obtain the model representation for $\overset{\circ}{A}_3$ in the space $\mathcal{B}^n(A,B)$, use that

$$\overset{\circ}{A}{}_3^* \left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_3^*\,\tilde{\varphi}^* =$$

$$= \frac{1}{z}\left\{\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\sigma_3 J - \tilde{\varphi}^*\sigma_3 J\right\} + \left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* \left(\gamma_{1,3}^+\right)^* J$$

in virtue of (2.5), §3.2, $\left[\overset{\circ}{A}_1, \overset{\circ}{A}_3\right] = 0$ (2.2), §2 and selfadjointness of $\gamma_{1,3}^+$.

The form of the operators J, σ_3, $\gamma_{1,3}^+$ (4.3) yields

$$\sigma_3 J = \tilde{\sigma}_3 \otimes I_2; \quad \gamma_{1,3}^+ J = \tilde{\gamma}_3 \otimes I_2. \tag{4.9}$$

Taking into account that $L_x(z) = (I - zA_1^*)^{-1}\,\tilde{\varphi}^*(1,0)$, we obtain that the operator $\overset{\circ}{A}_3$ (2.1) after the L. de Branges transform $\mathcal{B}_L$ (4.6) is given by

$$\left(\tilde{A}_3 F\right)(z) = \frac{1}{z}(F(z) - F(0))\sigma_3 + F(z)\tilde{\gamma}_3. \tag{4.10}$$

Thus

$$\tilde{A}_3 F(z) = \frac{1}{z}\left\{F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) - F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right)|_0\right\} \tag{4.11}$$

where, as always, $F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right)|_0 = F(0)\tilde{\sigma}_3$.

To find the representation for $\overset{\circ}{A}_2$ (2.1) in $\mathcal{B}^n(A,B)$ similar to (4.8), (4.11), note that $\overset{\circ}{A}{}_2^*\overset{\circ}{A}{}_1^* - \overset{\circ}{A}{}_1^*\overset{\circ}{A}{}_2^* = i\,\overset{\circ}{A}{}_3^*$ (in virtue of (2.2), §2), therefore

$$\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_2^* - \overset{\circ}{A}{}_2^*\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} = iz\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_3^*\,. \tag{4.12}$$

Taking into account (2.5) and (2.13), §2, we obtain

$$\overset{\circ}{A}{}_2^*\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^* = \left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_2^*\,\tilde{\varphi}^* - iz\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{A}{}_3^*\,\tilde{\varphi}^* =$$

$$= \frac{1}{z}\left\{\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\sigma_2 J - \tilde{\varphi}^*\sigma_2 J\right\} +$$

$$-iz\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \tilde{\varphi}^*\left(\gamma_{1,2}^+\right)^* J -$$

$$-iz\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1}\left\{\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^*\,\tilde{\varphi}^*\sigma_3 J + \left(I - z\,\overset{\circ}{A}{}_1^*\right)\tilde{\varphi}^*\gamma_{1,3}^+ J\right\}.$$

In connection with $\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} = z\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1}\overset{\circ}{A}{}_1^* - I$, we have

$$\overset{\circ}{A}{}_2^*\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^* = \frac{1}{z}\left\{\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*\sigma_2 J - \tilde{\varphi}^*\sigma_2 J\right\} +$$

$$+\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*\left(\gamma_{1,2}^+\right)^* J - iz\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-2}\overset{\circ}{A}{}_1^*\,\tilde{\varphi}^*\sigma_3 J -$$

$$-iz^2\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-2}\overset{\circ}{A}{}_1^*\,\tilde{\varphi}^*\gamma_{1,3}^+ J - iz\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1}\tilde{\varphi}^*\gamma_{1,3}^+ J.$$

Since

$$\sigma_2 J = \tilde{\sigma}_2 \otimes I_2; \quad \gamma_{1,2}^+ J = \tilde{\gamma}_2 \otimes I_2, \tag{4.13}$$

then using (4.9) and $\dfrac{d}{dz}\left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-1} = \left(I - z\,\overset{\circ}{A}{}_1^*\right)^{-2}\overset{\circ}{A}{}_1^*$, we obtain that the operator $\overset{\circ}{A}_2$ (2.1) after the L. de Branges transform (4.6) in the space $\mathcal{B}^n(A,B)$ is given by

$$\left(\tilde{A}_2 F\right)(z) = \frac{1}{z}\left\{F(z)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right) - F(z)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right)\big|_0\right\} + iz\frac{d}{dz}F(z)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right), \tag{4.14}$$

besides, $F(z)\left(\tilde{\sigma}_2 + z\tilde{\gamma}_2\right)\big|_0 = F(0)\tilde{\sigma}_2$.

Now define the colligation of Lie algebra (1.8), (1.9)

$$\tilde{\Delta} = \left(\left\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3\right\}; \mathcal{B}^n(A,B); \tilde{\varphi}; E; \{\sigma_k\}; \left\{\gamma_{k,s}^-\right\}_1^3; \left\{\gamma_{k,s}^+\right\}_1^3\right) \tag{4.15}$$

assuming that the operators $\left\{\sigma_k, \gamma_{1,k}^+\right\}_1^3$ are given by (4.3), the operator $\gamma_{2,3}^+$ is given by formula (1.17), and $\left\{\gamma_{k,s}^-\right\}_1^3$ are found by the formulas 4) (1.9) where $\tilde{\varphi}$ on every component acts in a standard way (3.19), (3.20).

Theorem 4.1. *Suppose that the simple colligation Δ of Lie algebra* (1.8), (1.9) *is given, besides,* $\dim E = 2n$, *and the operators* $\left\{\sigma_k, \gamma_{1,k}^+\right\}_1^3$ *in E are given by* (4.3) *and condition* (4.4) *is true. And let the spectrum of operator A_1 lie at zero, and the characteristic function $S_1(\lambda)$ of operator A_1 be given by*

$$S_1(\lambda) = \int_0^{\overleftarrow{l}} \exp\frac{iJdF_t}{\lambda},$$

and be such that the measure dF_x is absolutely continuous, $dF_x = a_x dx$ and a_x equals (4.1). *Then the colligation Δ is unitarily equivalent to the simple part of* *functional model* $\tilde{\Delta}$ (4.15) *where the operators* $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ *are given by* (4.8), (4.11), (4.14) *respectively.*

II. Consider the linear operator bundle

$$\tilde{\sigma}_3 + z\tilde{\gamma}_3$$

which is a selfadjoint operator when $z \in \mathbb{R}$. Denote by $h(z, w)$ eigenvectors of the given bundle,

$$h(P)\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3\right) = wh(P), \tag{4.17}$$

where $P = (z, w)$ belongs to the algebraic curve $\mathbb{Q}$,

$$\mathbb{Q} = \left\{P = (z, w) \in \mathbb{C}^2 : \mathbb{Q}(z, w) = 0\right\}, \tag{4.18}$$

specified by the polynomial

$$\mathbb{Q}(z, w) \stackrel{\text{def}}{=} \det\left(\tilde{\sigma}_3 + z\tilde{\gamma}_3 - wI_n\right). \tag{4.19}$$

Suppose that the curve $\mathbb{Q}$ is nonsingular [4], then $z = z(P)$ and $w = w(P)$ are correspondingly 'l-valued' and 'n-valued' functions on $\mathbb{Q}$ ($l = \text{rank}\tilde{\gamma}_3$). Norm the rational function $h(P)$ (4.17) using the condition $h_n(P) = 1$ where $h_n(P)$ is the 'nth' component of vector $h(P)$. It is easy to show [4] that the quantity of poles (subject to multiplicity) of vector-function $h(P)$ equals $N = g+n-1$ where g is type of the Riemann surface $\mathbb{Q}$ (4.18). Isolate on $\mathbb{Q}$ (4.18) analogues of the semi-planes $\mathbb{C}_\pm$ and real axis $\mathbb{R}$,

$$\mathbb{Q}_\pm = \{P = (z, w) \in \mathbb{Q} : \pm\text{Im}z(P) > 0\}; \quad \mathbb{Q}^0 = \partial\mathbb{Q}_\pm. \tag{4.20}$$

Roots $w^k(z)$ of the polynomial $\mathbb{Q}$, $\left(z, w^k(z)\right) = 0$, (4.19) are different when $z \in \mathbb{R}$ in virtue of non-singularity of the curve $\mathbb{Q}$ (4.18) (excluding the points of branching). Therefore the eigenvectors $h\left(P_k\right)$ (4.17) corresponding to $P_k = \left(z, w^k(z)\right) \in \mathbb{Q}$ (4.18) are orthogonal. Therefore we can expand every vector-function $F(z) \in \mathcal{B}^n(A, B)$ in terms of the orthogonal basis $\{h\left(P_k\right)\}_1^n$,

$$F(z) = \sum_{k=1}^{n} g\left(P_k\right) \left\|h\left(P_k\right)\right\|_E^{-2} h\left(P_k\right), \tag{4.21}$$

where $g\left(P_k\right) = \left\langle F(z), h\left(P_k\right)\right\rangle_E$ $(1 \le k \le n)$. It is easy to see that $w^k(z)$, $h\left(P_k\right)$ and $g\left(P_k\right)$ represent branches of the 'n-valued' algebraic functions $w(P)$, $h(P)$ and $g(P)$, respectively. In view of this, we can rewrite the last equality in the following form:

$$F(P) = F(z(P)) = g(P) \cdot \|h(P)\|_E^{-2} h(P). \tag{4.22}$$

Since the basis $h(P)$ in E^n is fixed, the function $F(P)$ is defined by the scalar component $g(P)$. Note that $g(P)$ is meromorphic on $\mathbb{Q}$ (4.18) and its poles can lie only at the poles of $h(P)$ (4.17), besides, their aggregate multiplicity does not exceed $N = g + n - 1$.

Construct the L. de Branges space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$ corresponding to the Riemann surface $\mathbb{Q}$ (4.18). Operator $\tilde{A}_1$ (4.8) in the space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$ is given by

$$\left(\hat{A}_1 g\right)(P) = \frac{g(P) - \psi(P, P_0)\, g(P_0)}{z(P) - z(P_0)} \tag{4.23}$$

where

$$\psi(P, P_0) = \langle h(P_0), h(P) \rangle_{E^n} \cdot \|h(P)\|_{E^n}^{-2}, \tag{4.24}$$

besides, $P_0 = (0, w) \in \mathbb{Q}$. Similarly, operator $\tilde{A}_3$ (4.11) in the space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$ is given by the formula

$$\left(\hat{A}_3 g\right)(P) = \frac{w(P) g(P) - w(P_0)\, \psi(P, P_0)\, g(P_0)}{z(P) - z(P_0)}, \tag{4.25}$$

besides, $\psi(P, P_0)$ is given by (4.24).

Now consider the operator $\tilde{A}_2$ (4.14). Let $\{h(P_k)\}_1^n$ be the orthogonal basis of eigenvectors (4.17),

$$h(P_k)(\tilde{\sigma}_3 + z\tilde{\gamma}_3) = w^k(z) h(P_k) \tag{4.26}$$

where $P_k = \left(z, w^k(z)\right) \in \mathbb{Q}$ (4.18) and $z \in \mathbb{R}$. Then (4.5) implies

$$w^k(z) h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2) = h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2^*)(\tilde{\sigma}_3 + z\tilde{\gamma}_3).$$

Taking into account (4.4), we can rewrite this equality in the following form:

$$\begin{aligned} w^k(z) h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2) = \\ = h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2)(\tilde{\sigma}_3 + z\tilde{\gamma}_3) - izh(P_k)\tilde{\sigma}_3(\tilde{\sigma}_3 + z\tilde{\gamma}_3) = \\ = h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2)(\tilde{\sigma}_3 + z\tilde{\gamma}_3) + \\ + iz^2 w^k(z) h(P_k)\tilde{\gamma}_3(\tilde{\sigma}_3 + z\tilde{\gamma}_3) - iz\left(w^k(z)\right)^2 h(P_k). \end{aligned} \tag{4.27}$$

To simplify the last summand in this sum, differentiate equality (4.26) by z,

$$h(P_k)\tilde{\gamma}_3 + h'(P_k)(\tilde{\sigma}_3 + z\tilde{\gamma}_3) = \left(w^k(z)\right)' h(P_k) + w^k(z) h'(P_k) \tag{4.28}$$

where prime signifies the derivative by z. Expand vector $h'(P_k)$ in terms of the basis $\{h(P_s)\}_1^n$:

$$h'(P_k) = \sum_{s=1}^{n} a(P_k, P_s) \|h(P_s)\|_E^{-2} \cdot h(P_s) \tag{4.29}$$

where

$$a(P_k, P_s) = \langle h'(P_k), h(P_s) \rangle_E. \tag{4.30}$$

Then (4.28) implies

$$h(P_k)\tilde{\gamma}_3 = \left(w^k(z)\right)' h(P_k) + \sum_{s=1}^{n} a(P_k, P_s)\left(w^k(z) - w^s(z)\right) \|h(P_s)\|_E^{-2} \cdot h(P_s).$$

Now realize the expansion of vector $h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2)$ from (4.27) in terms of the basis $\{h(P_s)\}_1^n$:

$$h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2) = \sum_{s=1}^{n} b(P_k, P_s) \|h(P_s)\|_E^{-2} \cdot h(P_s) \tag{4.31}$$

where

$$a(P_k, P_s) = \langle h'(P_k), h(P_s)\rangle . \tag{4.30}$$

Then (4.28) yields

$$h(P_k)\tilde{\gamma}_3 = \left(w^k(z)\right)' h(P_k) + \sum_{s=1}^{n} a(P_k, P_s)\left(w^k(z) - w^s(z)\right) \|h(P_s)\|_E^{-2} \cdot h(P_s) .$$

Now realize expansion of the vector $h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2)$ from (4.27) in terms of the basis $\{h(P_s)\}_1^n$:

$$h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2) = \sum_{s=1}^{n} b(P_k, P_s) \|h(P_s)\|_E^{-2} \cdot h(P_s) \tag{4.31}$$

where

$$b(P_k, P_s) = \langle h'(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2), h(P_s)\rangle_E . \tag{4.32}$$

Then equality (4.27) has the form

$$\begin{gathered}\sum_{s=1}^{n} b(P_k, P_s)\left(w^k(z) - w^s(z)\right) \|h(P_s)\|_E^{-2} \cdot h(P_s) = -iz\left(w^k(z)\right)^2 h(P_k) + \\ +iz\left(w^k(z)\right)' w^k(z) h(P_k) + \\ +iz^2 \sum_{s=1}^{n} a(P_k, P_s)\left(w^k(z) - w^s(z)\right) w^s(z) \|h(P_s)\|_E^{-2} h(P_s) .\end{gathered}$$

Linear independence of $\{h(P_s)\}_1^n$ yields

$$\begin{cases} b(P_k, P_s) = iza(P_k, P_s) w^s(z) \quad (s \neq k); \\ w^k(z) = z\left(w^k(z)\right)' \quad (s = k). \end{cases} \tag{4.33}$$

Using (4.27), it is easy to show that $b(P_k, P_k) = 0$.

Thus knowing the function $a(P_k, P_s)$ (4.30) defined by the vector-functions $h(P_k)$ (4.25), we can construct $b(P_k, P_s)$ and find expansion of the vector $h(P_k) \times \times (\tilde{\sigma}_2 + z\tilde{\gamma}_2)$:

$$h(P_k)(\tilde{\sigma}_2 + z\tilde{\gamma}_2) = iz \sum_{s=1}^{n} a(P_k, P_s) \cdot \|h(P_s)\|_E^{-2} \cdot h(P_s) . \tag{4.34}$$

This implies that action of the bundle $\tilde{\sigma}_2 + z\tilde{\gamma}_2$ on $F(z)$ (4.21) in terms of the components $g(P_k)$ appears as follows:

$$g(P_k) \longrightarrow izw^k(z) \sum_{s=1}^{n} g(P_s)\, a(P_k, P_s) \cdot \|h(P_s)\|_E^{-2} \cdot h(P_s). \tag{4.35}$$

Now consider the second summand in (4.14), use (4.21), then

$$iz\frac{d}{dz}F(z)(\tilde{\sigma}_3 + z\tilde{\gamma}_3) = iz\frac{d}{dz}\left\{\sum_{k=1}^{n} g(P_k)\, \|h(P_k)\|_E^{-2}\, w^k(z) h(P_k)\right\} =$$

$$= iz\sum_{k=1}^{n}\left(g(P_k)\, w^k(z)\right) \|h(P_k)\|_E^{-2} \cdot$$

$$\cdot h(P_k) - 2iz\sum_{k=1}^{n} g(P_k)\, w^k(z) \cdot \|h(P_k)\|_E^{-3} \cdot \|h(P_k)\|_E^{1}\, h(P_k) +$$

$$+iz\sum_{k=1}^{n} g(P_k)\, w^k(z) \cdot \|h(P_k)\|_E^{-2} \cdot \sum_{s=1}^{n} a(P_k, P_s) \cdot \|h(P_s)\|_E^{-2} \cdot h(P_s).$$

Thus action of the expression $\frac{d}{dz}F(z)(\tilde{\sigma}_3 + z\tilde{\gamma}_3)$ in terms of the scalar component $g(P_k)$ can be written as

$$g(P_k) \longrightarrow iz\left(w^k(z) g(P_k)\right)' - 2izw^k(z) g(P_k)\, \|h(P_k)\|_E^{-1} \cdot \|h(P_k)\|_E^{1} +$$

$$+iz\sum_{s=1}^{n} g(P_s)\, w^s(z) a(P_s, P_k) \cdot \|h(P_s)\|_E^{-2}. \tag{4.36}$$

To rewrite the formulas (4.35), (4.36) in a compact form, consider the kernel

$$a(P', P) = \left\langle \frac{d}{dz} h(P'), h(P) \right\rangle_E \tag{4.37}$$

coinciding with (4.30) as $P' = P_k$, $P = P_s$. Define action of this kernel on the function $g(P)$ in the following way:

$$(a * g)(P) \stackrel{\text{def}}{=} \sum_{P'} g(P')\, a(P', P) \cdot \|h(P')\|_E^{-2} \tag{4.38}$$

where P' varies over all the values (branches) of the function $g(P')$.

Now taking into account (4.35) and (4.36), we can write form of the operator $\tilde{A}_2$, which, in view of (4.14), is given by

$$\left(\tilde{A}_2 g\right)(P) = \frac{iz(P)w(P)(a*g)(P) - iz(P_0)\, w(P_0)\, \psi(P, P_0)\, (a*g)(P_0)}{z(P) - z(P_0)} +$$

$$+iz(P)\frac{d}{dz}(w(P)g(P)) - 2iz(P)w(P)b(P)g(P) + iz(P)(a*g)(P) \qquad (4.39)$$

where

$$b(P) = \|h(P)\|_E^{-1} \cdot \frac{d}{dz}\|h(P)\|. \qquad (4.40)$$

Construct colligation of the Lie algebra (1.8), (1.9)

$$\tilde{\Delta} = \left(\left\{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3\right\}; \mathcal{B}_{\mathbb{Q}}(A, B, h); \tilde{\varphi}, E; \{\sigma_k\}_1^3, \left\{\gamma_{k,s}^-\right\}_1^3, \left\{\gamma_{k,s}^+\right\}_1^3\right) \qquad (4.41)$$

where the operators $\left\{\sigma_k, \gamma_{1,k}^+\right\}_1^3$ are given by (4.3), $\gamma_{2,3}^+$ is defined by formula (1.17), and the operators $\left\{\gamma_{k,s}^-\right\}_1^3$ are defined from 4) (1.9), $\tilde{\varphi}$ is given by

$$\tilde{\varphi}g(P) = \sum_{k=1}^{2} \langle g(P), e_k(z(P))\rangle_{\mathcal{B}_{\mathbb{Q}}(A,B,h)} \cdot e_k, \qquad (4.42)$$

e_k are given by

$$\begin{aligned} e_1(z) &= \frac{1-\alpha z}{z} B^*(\bar{z}); \quad & e_2(z) &= \frac{1-\alpha z}{z}(1 - A^*(\bar{z})); \\ e_1 &= (1, 0); & e_2 &= (0, 1). \end{aligned} \qquad (4.43)$$

Theorem 4.2. *Suppose that for the colligation Δ of Lie algebra* (1.8), (1.9) *requirements of Theorem* 4.1 *hold and let curve $\mathbb{Q}$* (4.18) *be non-singular, besides, $zw' = w(z)$. Then colligation Δ* (1.8), (1.9) *is unitarily equivalent to the simple part of colligation $\tilde{\Delta}$* (4.41) *where operators $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ are given by* (4.23), (4.25) *and* (4.39), *respectively.*

In this work for a Lie algebra of linear non-selfadjoint operators $\{A_1, A_2, A_3\}$ ($[A_1, A_2] = iA_3$, $[A_1, A_3] = 0$, $[A_2, A_3] = 0$) are obtained the following results.

1) The triangular model (2.1) for this Lie algebra in the space $L^2_{r,l}(F_x)$ is constructed.

2) In §3 using the triangular model from §2, the functional model (Theorem 3.1) for the studied in this chapter Lie algebra $\{A_1, A_2, A_3\}$ is stated.

3) For special classes of Lie algebra $\{A_1, A_2, A_3\}$, the functional model on Riemann surface in special L. de Branges spaces (Theorem 4.1 and Theorem 4.2) is constructed.

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V. N. Karazin Kharkov National University
4, Svobody sq.
61077 Kharkov
UKRAINE
E-mail: victoriya979@mail.ru

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 100

MODEL REPRESENTATIONS OF THE LIE ALGEBRA $[A_2, A_1] = iA_1$ OF LINEAR NON-SELFADJOINT OPERATORS

VICTORIYA A. KUZNETSOVA

Construction of functional models of Lie algebra $\{A_1, A_2\}$ ($[A_2, A_1] = iA_1$), one of which is dissipative, was realized earlier. The question of construction of model realizations for the given Lie algebra not containing dissipative operator remained open. This work is dedicated to the construction of model representation of the Lie algebra $\{A_1, A_2\}$ of linear non-selfadjoint operators not containing a dissipative operator which is generated by the commutation relation $[A_2, A_1] = iA_1$. In Paragraph 1 the preliminary information is stated, the definitions of colligation of Lie algebra and corresponding open system on Lie group of affine transformations of the line $M(1)$ are given. Paragraph 2 is dedicated to the construction of triangular model for the Lie algebra $[A_2, A_1] = iA_1$ in the case of finite dimension of the general space of non-hermicity of operator system $\{A_1, A_2\}$. In Paragraph 3 functional model of the Lie algebra $[A_2, A_1] = iA_1$ is presented, it is realized in L. de Branges spaces of whole functions. In the last paragraph of this paper, functional model of the Lie algebra $[A_2, A_1] = iA_1$ on Riemann surface is constructed.

Keywords: Functional models, L. de Branges transform, Lie algebra
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1. LIE GROUP OF AFFINE TRANSFORMATIONS OF LINE AND COLLIGATION OF LIE ALGEBRA

I. To study a Lie algebra of linear non-selfadjoint operators specified by the commutation relation $[A_2, A_1] = iA_1$, one has [4] to find such Lie group G, vector $\{\partial_1, \partial_2\}$ Lie algebra of which is such that

$$[\partial_2, \partial_1] = \partial_1.$$

Let $\mathbb{R}$ be the real line. Define $G = M(1)$ [7, 8] the group of transformations of $\mathbb{R}$ preserving the orientation. Associate with each $\xi \in \mathbb{R}$ number $\eta = y\xi + x$ ($y > 0$, $x \in \mathbb{R}$). Denote a group element by $g = g(x, y)$. If $\eta = y_1\xi + x_1$ and $\zeta = y_2\eta + x_2$ then

$$\zeta = y_1 y_2 \xi + x_1 y_2 + x_2.$$

Therefore the group operation on G is given by

$$g(x_2, y_2) \circ g(x_1, y_1) = g(x_1 y_2 + x_2, y_2 y_1). \tag{1.1}$$

Hence it follows that the elements $g(x, 1)$ form the subgroup in G, isomorphic to the additive group of real numbers $\mathbb{R}$.

$$g(x_2, 1) \circ g(x_1, 1) = g(x_1 + x_2, 1).$$

And the elements $g(0, y)$ form the subgroup in G equivalent to the multiplicative group of positive numbers in $\mathbb{R}_+$.

$$g(0, y_2) \circ g(0, y_1) = g(0, y_2 y_1).$$

The group G is isomorphic to the group of matrices of the second order given by

$$B_g = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}.$$

This fact immediately follows from the equality

$$B_{g_2} \cdot B_{g_1} = \begin{bmatrix} y_2 & x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 & x_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} y_2 y_1 & y_2 x_1 + x_2 \\ 0 & 1 \end{bmatrix} = B_{g_1 \circ g_2}.$$

Specify two subgroups in G,

$$G_x^1 = \{g(x, 1) \in G\}; \quad G_y^2 = \{g(y, 0) \in G\}; \tag{1.2}$$

as is stated above, they are isomorphic to $\mathbb{R}$ and $\mathbb{R}_+$, respectively. To specify a function $f(g)$ on the group $G = M(1)$, $f\colon G \to \mathbb{C}$, signifies that we define complex-valued function $f(x, y)$ in the upper half-plane $\mathbb{R} \times \mathbb{R}_+$. Calculate vector fields corresponding to the one-parametric semigroups (1.2) [8]. Let $g_t = (t, 1) \in G_x^1$ in (1.2). Then

$$F_t = f(g_t \circ g(x, y)) = f(g(ty + x, y)) = f(ty + x, y).$$

Therefore the derivative by t at unit, $e = e(0, 1) \in G$, of the given function equals

$$\left.\frac{d}{dt} F_t\right|_{t=0} = \partial_1 f$$

where $\partial_1 = y\dfrac{\partial}{\partial x}$. Similarly, consider the functions

$$\tilde{F}_t = f(g_t \circ g(x, y)) = f(x, ty)$$

where $\tilde{g}_t = (0, t) \in G_x^2$ in (1.2). Then

$$\frac{d}{dt}\tilde{F}_t\bigg|_{t_0=0} = \partial_2 f$$

where $\partial_2 = y\dfrac{\partial}{\partial y}$. Thus we construct the Lie algebra of vector fields $m(1)$ of the group $M(1)$ specified by the differential operators of the first order

$$\partial_1 = y\frac{\partial}{\partial x}, \quad \partial_2 = y\frac{\partial}{\partial y}. \tag{1.3}$$

It is easy to see that the Lie algebra $\{\partial_2, \partial_1\}$ is specified by the commutation relation

$$[\partial_2, \partial_1] = \partial_1. \tag{1.4}$$

It is well-known that the simply connected Lie group $M(1)$ is "uniquely" restored by the Lie algebra $m(1)$ of differential operators (1.3) [7, 8].

II. Consider in a Hilbert space H the Lie algebra of linear operators $\{A_1, A_2\}$ satisfying the relation

$$[A_2, A_1] = iA_1. \tag{1.5}$$

Note that A_1 and A_2 cannot be bounded simultaneously, since otherwise (1.5) implies

$$[A_2, A_1^n] = inA_1^n$$

which results in the inequality $2\|A_2\| \geq n$ $(\forall n \in \mathbb{Z}_t)$.

It seems natural to write relation (1.5) in the "integral form" similarly to the Weyl identity in Quantum Mechanics [4]. Let $Z_t(t_k) = \exp(it_k A_k)$ $k = 1, 2$. (1.5) implies

$$Z_1(t_1) A_2 = (A_2 + t_1 A_1) Z_1(t_1). \tag{1.6}$$

Indeed, it is easy to see that $f'(t_1) = iA_1 f(t_1)$ and $f(0) = 0$ where $f(t_1) = Z_1(t_1) A_2 - (A_2 + t_1 A_1) Z_1(t_1)$. Therefore it is obvious that

$$Z_1(t_1) Z_2(t_2) = \exp\{it_2(A_2 + t_1 A_1)\} Z_1(t_1). \tag{1.7}$$

III. Construct the colligation of Lie algebra for the given Lie algebra (1.5) of linear non-selfadjoint operators.

Definition 1.1. *Family*

$$\Delta = \Big(\{A_1, A_2\}; H; \varphi; E; \{\sigma_k\}_1^2; \{\gamma^-, \gamma^+\}\Big), \tag{1.8}$$

where φ: $H \to E$, σ_k, $\gamma^{\pm}$: $E \to E$ ($\sigma_k^* = \sigma_k$, $k = 1, 2$), *is said to be the* colligation *of the Lie algebra* (1.5), *if*

$$\begin{aligned} &1) \quad [A_2, A_1] = iA_1; \\ &2) \quad 2\mathrm{Im}\,\langle A_k h, h\rangle = \langle \sigma_k \varphi h, \varphi h\rangle\,; \quad \forall h \in \vartheta\,(A_k)\,; \\ &3) \quad \sigma_1 \varphi A_2 - \sigma_2 \varphi A_1 = \gamma^+ \varphi; \\ &4) \quad \gamma^- = \gamma^+ + i\,(\sigma_2 \varphi\varphi^* \sigma_1 - \sigma_1 \varphi\varphi^* \sigma_2)\,. \end{aligned} \tag{1.9}$$

It is obvious that $\gamma^{\pm}$ are non-selfadjoint operators [4] and

$$\gamma^{\pm} - \left(\gamma^{\pm}\right)^* = -i\sigma_1. \tag{1.10}$$

Equations of the open system [2, 3, 4, 5] are given by

$$\begin{cases} i\partial_k h(x, y) + A_k h(x, y) = \varphi^* \sigma_k u(x, y) \quad (k = 1, 2) \\ h(e) = h_0 \quad (k = 1, 2); \quad (x, y) \in G; \\ v(x, y) = u(x, y) - i\varphi h(x, y). \end{cases} \tag{1.11}$$

Besides, ∂_k in (1.11) are equal to (1.11). It is not hard to show [2, 4, 5] that

$$\left\{\sigma_1 i\partial_2 - \sigma_2 i\partial_1 + \gamma^-\right\} u(x, y) = 0;$$

$$\left\{\sigma_1 i\partial_2 - \sigma_2 i\partial_1 + \gamma_s^+\right\} v(x, y) = 0.$$

2. TRIANGULAR MODEL OF LIE ALGEBRA

I. Consider the colligation Δ (1.8) corresponding to the Lie algebra of linear operators $\{A_1, A_2\}$ assuming that (1.9), (1.10) take place, besides, $\dim E = r < \infty$, operator $\sigma_1 = J$ is involution, and let $\sigma_2 = \sigma$. Define the Hilbert space $L^2_{r,l}\,(F_x)$ [1, 3] assuming that the measure dF_x is absolutely continuous, $dF_x = a_x dx$, $a_x \leq 0$, $\mathrm{tr} a_x \equiv 1$. Specify in this space the operator system

$$\left(\overset{\circ}{A}_1 f\right)_x = i \int\limits_x^l f_t a_t J dt;$$

$$\left(\overset{\circ}{A}_2 f\right)_x = f'_x b_x + f_x J\gamma_x + i \int\limits_x^l f_t a_t dt\sigma \tag{2.1}$$

$\left(f_x \in L^2_{r,l}\,(F_x)\right)$ where b_x, γ_x are some operator-functions in E specified on $[0, l]$. Linear span of continuously differentiable functions from $L^2_{2,l}\,(F_x)$ such that $f'_x b_x \in L^2_{2,l}\,(F_x)$ and $f_0 = f_l = 0$ is the domain $\mathcal{D}\,(A_2)$. Note that the structure of A_1 (2.2) coincides with the triangular model [1, 3] when the spectrum $\sigma(A_1) = 0$. Find the necessary and sufficient conditions on a_x, b_x, γ_x, J, σ for this operator system (2.1) to form the Lie algebra,

$$\left[\overset{\circ}{A}_2, \overset{\circ}{A}_1\right] = i\,\overset{\circ}{A}_1\ . \tag{2.2}$$

It is easy to see that

$$\overset{\circ}{A}_2\overset{\circ}{A}_1\ f_x = -if_x a_x J b_x + i\int_x^l f_t a_t dt \gamma_x - \int_x^l \left(\int_t^l f_s a_s J ds\right) a_t \sigma dt.$$

Similarly,

$$\overset{\circ}{A}_1\overset{\circ}{A}_2\ f_x = i\int_x^l f_t' b_t a_t J dt + \int_x^l f_t J \gamma_t a_t J dt - \int_x^l \left(\int_t^l f_s a_s \sigma ds\right) a_t J dt =$$

$$= -if_x b_x a_x J - i\int_x^l f_t \left(b_t a_t\right)' J dt + i\int_x^l f_t J \gamma_t a_t J dt - \int_x^l \left(\int_t^l f_s a_s \sigma ds\right) a_t J dt$$

by virtue of $f_l = 0$. Suppose that

$$a_x J b_x = b_x a_x J. \tag{2.3}$$

Then

$$\Psi_x \overset{\text{def}}{=} \left(\left[\overset{\circ}{A}_2, \overset{\circ}{A}_1\right] - i\,\overset{\circ}{A}_1\right) f_x = i\int_x^l f_t a_t dt \gamma_x - \int_x^l \left(\int_t^l f_s a_s J ds\right) a_t \sigma dt -$$

$$-i\int_x^l f_t \left\{J\gamma_t a_t J - \left(b_t a_t\right)' J\right\} dt + \int_x^l \left(\int_x^l f_s a_s \sigma ds\right) a_t J dt + \int_x^l f_t a_t J dt.$$

Supposing that γ_x is continuously differentiable operator-function, calculate derivative of the function Ψ_x:

$$\Psi_x' = -if_x a_x \gamma_x + i\int_x^l f_t a_t dt \gamma_x' + \int_x^l f_t a_t dt J a_x \sigma +$$

$$+if_x \left\{J\gamma_x a_x J - \left(b_x a_x\right)' J\right\} - \int_x^l f_t a_t dt \sigma a_x J - f_x a_x J.$$

Hence it follows that $\Psi'_x = 0$, if

$$\begin{cases} i\gamma'_x = \sigma a_x J - J a_x \sigma; \\ a_x \gamma_x J = J\gamma_x a_x - (b_x a_x)' + i a_x. \end{cases} \tag{2.4}$$

Thus $\Psi_x \equiv 0$ since $\Psi_l = 0$.

Lemma 2.1. *Suppose that there exists a family* $\{a_x, \gamma_x, b_x, J, \sigma\}$ *such that* (2.3) *and* (2.4) *take place. Then the operator system* $\left\{\mathring{A}_1, \mathring{A}_2\right\}$ (2.1) *satisfies the commutation relation* (2.2).

II. In order to include the operator system $\left\{\mathring{A}_1, \mathring{A}_2\right\}$ (2.1) in the colligation Δ (1.8), it is necessary to verify that the colligation relations (1.9) are true. It is easy [1, 3] to show that $\mathring{A}_1 - \mathring{A}_1^* = i\,\mathring{\varphi}^*\,J\,\mathring{\varphi}$ where the operator $\mathring{\varphi}$: $L^2_{2,l}(F_x) \to E$ is given by

$$\mathring{\varphi}\, f_x = \int_0^l f_t dt. \tag{2.5}$$

Calculate $2\mathrm{Im}\left\langle \mathring{A}_2\, f, f\right\rangle$ where $f \in \mathcal{D}\left(\mathring{A}_2\right)$. Then

$$2\mathrm{Im}\left\langle \mathring{A}_2\, f, f\right\rangle = \frac{1}{i}\int_0^l \left(f'_x b_x + f_x J\gamma_x + i\int_x^l f_t a_t \sigma\right) dt a_x f_x^* dx -$$

$$-\frac{1}{i}\int_0^l dx f_x a_x \left(b_x^* \left(f_x^*\right)' + \gamma_x^* J f_x - i\int_x^l \sigma a_t f_t^* dt\right) =$$

$$= \frac{1}{i}\int_0^l \left(f'_x b_x a_x f_x^* - f_x a_x b_x^* \left(f_x^*\right)' + f_x \left\{J\gamma_x a_x - a_x \gamma_x^* J\right\} f_x^*\right) dx +$$

$$+\int_0^l \left(\int_x^l f_t a_t \sigma dt a_x f_x^* + f_x a_x \int_x^l \sigma a_t f_t^*\right) dt.$$

It is easy to see that the second integral after the change of order of integration equals

$$\int_0^l f_t a_t dt \sigma \int_0^l a_t f_t^* dt = \langle \sigma\varphi f, \varphi f \rangle_E .$$

Therefore in order to the colligation relation 2) for $\overset{\circ}{A}_2$ (2.1) take place, it is necessary to ascertain under which conditions the first integral vanishes. The integrand of this integral equals

$$\Phi_x \overset{\text{def}}{=} f_x' b_x a_x f_x^* - f_x a_x b_x^* \left(f_x^*\right)' + f_x \left\{J\gamma_x a_x - a_x \gamma_x^* J\right\} f_x^* =$$

$$= f_x' a_x J b_x J f_x^* - f_x a_x b_x^* \left(f_x^*\right)' + f_x \left\{a_x \gamma_x J + (b_x a_x)' - i a_x - a_x \gamma_x^* J\right\} f_x^*$$

in virtue of (2.3) and the second equation in (2.4). It is obvious that the solution γ_x of equation (2.4) is given by

$$\gamma_x = \gamma_0 + i \int_0^x \left(J a_t \sigma - \sigma a_t J\right) dt. \tag{2.6}$$

Choose the initial condition $\gamma_0 = \left(\gamma^+\right)^*$. Since the second summand in (2.6) is a selfadjoint operator, then taking into account $\gamma^+ - \left(\gamma^+\right)^* = -iJ$ (1.10) we obtain

$$\gamma_x - \gamma_x^* = \gamma_0 - \gamma_0^* = \left(\gamma^+\right)^* - \gamma^+ = iJ. \tag{2.7}$$

So $\gamma_x^* = \gamma_x - iJ$. Substituting this expression in the formula for Φ_x, we obtain

$$\Phi_x = f_x' a_x J b_x J f_x^* - f_x a_x b_x^* \left(f_x^*\right)' + f_x \left\{a_x \gamma_x J + (b_x a_x)' - i a_x - ax\left(\gamma_x - iJ\right) J\right\} f_x =$$

$$= f_x' a_x J b_x J f_x^* + f_x a_x \left(-b_x^*\right) f_x^{*'} + f_x \left(a_x J b_x J\right)' f_x^*$$

in virtue of (2.3). Let

$$b_x^* = -J b_x J. \tag{2.8}$$

Then $\Phi_x = \left\{f_x a_x J b_x J f_x^*\right\}'$, and hence

$$\int_0^l \Phi_x dx = 0$$

since $f_0 = f_l$ as $f \in \mathcal{D}\left(\overset{\circ}{A}_2\right)$.

Lemma 2.2. *Let the family $\{a_x, \gamma_x, b_x, J, \sigma\}$ be such that the relations* (2.3), (2.4) *are true and, moreover, γ_x, the solution of the first equation in* (2.4), *satisfies* *the initial condition* $\gamma_0 = (\gamma^+)^*$, *besides,* $\gamma^+ - (\gamma^+)^* = -iJ$ (1.9). *Then, if* (2.8) *takes place,* $\forall f \in \mathcal{D}\left(\overset{\circ}{A}_2\right)$ *the colligation relation*

$$2\mathrm{Im}\left\langle \overset{\circ}{A}_2 f, f\right\rangle = \left\langle = \sigma \overset{\circ}{\varphi} f, \overset{\circ}{\varphi} f\right\rangle$$

where f *is given by* (2.5).

Verify that the colligation condition 3) (1.9) also is true. Really, find the function Ψ_x,

$$\Psi_x \overset{\mathrm{def}}{=} \left(J \overset{\circ}{\varphi}\overset{\circ}{A}_2 - \sigma \overset{\circ}{\varphi}\overset{\circ}{A}_1 - \gamma^+ \overset{\circ}{\varphi}\right) f_x = \int\limits_0^l \left(f_x' b_x + f_x J\gamma_x + i\int\limits_x^l f_t a_t dt\sigma\right) a_x dx J -$$

$$- \int\limits_0^l i \int\limits_x^l f_t a_t J dt a_x \sigma - \int\limits_0^l f_x a_x dx \gamma^+.$$

Integrating by parts and changing the order of integration, we obtain

$$\Psi_x = \int\limits_0^l dx \left\{ -f_x \left(b_x a_x\right)' J + f_x J\gamma_x a_x J + f_x a_x \left[i\int\limits_0^x \left(\sigma a_t J - J a_t \sigma\right) dt\right] - f_x a_x \gamma^+ \right\}.$$

Now taking into account (2.6) and the second equality in (2.4), we have

$$\Psi_x = \int\limits_0^l \left\{ f_x a_x \gamma_x - f_x J\gamma_x a_x J - i a_x J + f_x J \gamma_x a_x J + f_x a_x \left(\gamma_0 - \gamma_x\right) - f_x a_x \gamma^+ \right\} dx =$$

$$= \int\limits_0^l f_x a_x dx \left(\gamma_0 - \gamma^+ - iJ\right) = 0$$

in virtue of $\gamma_0 = (\gamma^+)^*$ and condition (1.10). So $\Psi_x \equiv 0$ and relation 3) (1.9) is proved. If one takes into account (2.5), then (2.6) yields

$$\gamma_l = \gamma_0 + i \int\limits_0^l \left(J a_t \sigma - \sigma a_t J\right) dt = \gamma_0 + i\left(J \overset{\circ}{\varphi}\overset{\circ}{\varphi}{}^* \sigma - \sigma \overset{\circ}{\varphi}\overset{\circ}{\varphi}{}^* J\right),$$

therefore

$$\gamma_l^* = \gamma^+ + i\left(J \overset{\circ}{\varphi}\overset{\circ}{\varphi}{}^* \sigma - \sigma \overset{\circ}{\varphi}\overset{\circ}{\varphi}{}^* J\right).$$

And we obtain the colligation relation 4) (1.9) where $\gamma_l^* = \gamma^-$.

Theorem 2.1. *Suppose that an operator family* $\{a_x, \gamma_x, b_x, J, \sigma\}$ *is such that*

$$\begin{aligned} &1) \quad a_x J b_x = b_x a_x J; \\ &2) \quad b_x^* = -J b_x J; \\ &3) \quad i\gamma_x^- = \sigma a_x J - J a_x \sigma; \quad \gamma_0 = (\gamma^+)^*; \\ &4) \quad (b_x a_x)' = J\gamma_x a_x - a_x \gamma_x J + i a_x; \end{aligned} \tag{2.9}$$

besides, $\gamma^+ - (\gamma^+)^* = -iJ$. *Then the set*

$$\overset{\circ}{\Delta} = \left(\left\{ \overset{\circ}{A}_1, \overset{\circ}{A}_2 \right\}; L^2_{2,l}(F_x); \overset{\circ}{\varphi}; E; \{J, \sigma_k\}; \{\gamma^-, \gamma^+\} \right) \tag{2.10}$$

is the colligation of Lie algebra (1.8)–(1.9) *where* $\overset{\circ}{A}_1$, $\overset{\circ}{A}_2$ *are given by* (2.1)*, the operator* $\overset{\circ}{\varphi}$ *equals* (2.5) *and* $\gamma^- = \gamma_l^*$.

Now use the Theorem on unitary equivalence [1, 3, 4].

Theorem 2.2. *Let* Δ *be a simple colligation* (1.8), (1.9). *If the spectrum of operator* A_1 *is concentrated at zero and the characteristic function* $S_1(\lambda) = I - i\varphi(A_1 - \lambda I)^{-1}\varphi^* J$ *is given by*

$$S_1(\lambda) = \overset{\leftarrow}{\int_0^l} \exp \frac{iJdF_t}{\lambda},$$

besides, $dF_x = a_x dx$ *and* a_x *is such that for the family* $\{a_x, \gamma_x, b_x, J, \sigma\}$ ($\sigma_1 = J$ *is involution and* $\sigma = \sigma^*$) *the equation system* 1) – 4) (2.9) *is solvable. Then the colligation* Δ *is unitary equivalent to the simple part of colligation* $\overset{\circ}{\Delta}$ (2.10).

Observation 2.1. 1), 2) (2.9) *imply*

$$a_x b_x^* + b_x a_x = 0, \tag{2.11}$$

$\forall x \in [0, l]$.

3. FUNCTIONAL MODEL IN L. DE BRANGES SPACE

This section is concerned with the construction of functional model of the studied in this paper Lie algebra in L. de Branges space [3]. Consider the triangular model of the colligation of Lie algebra (2.10) assuming that $r = 2$ and J is given by

$J = j_N$ (2.1). Under the action of L. de Branges transformation [3], the operator $\overset{\circ}{A}_1$ (2.1) changes into the shift operator since

$$\mathcal{B}_L\left(\overset{\circ}{A}_1 f_t\right) = \frac{1}{\pi}\int_0^l \left\{ i\int_t^l f_s dF_s J\right\} dF_t L_t^*(\bar{z}) =$$

$$= \frac{1}{\pi}\int_0^l f_t dF_t \left\{ -\frac{L_t^*(\bar{z}) - L_t^*(0)}{z}\right\}$$

and thus

$$\mathcal{B}_L\left(\overset{\circ}{A}_1 f_1\right) = \frac{F(z) - F(0)}{z} \tag{3.1}$$

where $F(z) \in \mathcal{B}_L(f_t)$. In order to find $\mathcal{B}_L\left(\overset{\circ}{A}_1 f_t\right)$, first of all note that

$$L_t(z) = \left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^*(1,0). \tag{3.2}$$

Since

$$\mathcal{B}_L\left(\overset{\circ}{A}_2 f_t\right) = \left\langle \overset{\circ}{A}_2 f_t, L_t(\bar{z})\right\rangle = \left\langle f_t, \overset{\circ}{A}{}_2^* L_t(\bar{z})\right\rangle,$$

then, taking into account (3.2), we have to calculate the expression

$$\overset{\circ}{A}{}_2^*\left(I - z\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^*(1,0). \tag{3.3}$$

(2.2) implies

$$\overset{\circ}{A}_2\left(I - z\overset{\circ}{A}_1\right) - \left(I - z\overset{\circ}{A}_1\right)^{-1} = -iz\overset{\circ}{A}_1,$$

therefore

$$\left(I - z\overset{\circ}{A}_1\right)^{-1}\overset{\circ}{A}_2 - \overset{\circ}{A}_2\left(I - z\overset{\circ}{A}_1\right)^{-1} = -iz\overset{\circ}{A}_1\left(I - z\overset{\circ}{A}_1\right)^{-2}.$$

Thus

$$\overset{\circ}{A}{}_2^*\left(I - \bar{z}\overset{\circ}{A}{}_1^*\right)^{-1} - \left(I - z\overset{\circ}{A}{}_1^*\right)\overset{\circ}{A}{}_2^* = -i\bar{z}\overset{\circ}{A}{}_1^*\left(I - \bar{z}\overset{\circ}{A}{}_1^*\right)^{-2}. \tag{3.4}$$

Using (3.4), we obtain

$$\overset{\circ}{A}{}_2^*\left(I - \bar{z}\overset{\circ}{A}{}_1^*\right)^{-1}\overset{\circ}{\varphi}{}^* = \left(I - \bar{z}\overset{\circ}{A}{}_1^*\right)^{-1}\overset{\circ}{A}{}_2^*\overset{\circ}{\varphi}{}^* + i\bar{z}\overset{\circ}{A}{}_1^*\left(I - \bar{z}\overset{\circ}{A}{}_1^*\right)^{-2}\overset{\circ}{\varphi}{}^*.$$

The colligation relation $J \overset{\circ}{\varphi} \overset{\circ}{A}_2 = \sigma \overset{\circ}{\varphi} \overset{\circ}{A}_1 + \gamma^+ \overset{\circ}{\varphi}$ yields

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* = \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* \overset{\circ}{\varphi}{}^* \sigma J + \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* \left(\gamma^+\right)^* J +$$

$$+ i\bar{z}\, \overset{\circ}{A}{}_1^* \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{\varphi}{}^* .$$

Now taking into account $\left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* = \frac{1}{z}\left\{\left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} - I\right\}$, we finally obtain

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* = \frac{1}{z}\left\{\left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* \sigma J - \overset{\circ}{\varphi}{}^* \sigma J\right\} +$$

$$+ \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* \left(\gamma^+\right)^* J + i\bar{z}\, \overset{\circ}{A}{}_1^* \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{\varphi}{}^*$$

Thus expression (3.3) has the form

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi} (1,0) = \frac{1}{z}\left\{\left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* - \overset{\circ}{\varphi}{}^*\right\} \sigma J(1,0) +$$

$$+ \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* \left(\gamma^+\right)^* J(1,0) + i\bar{z}\, \overset{\circ}{A}{}_1^* \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{\varphi}{}^* (1,0). \tag{3.5}$$

Expand the vectors $\sigma J(1,0)$ and $\left(\gamma^+\right)^* J(1,0)$ by the basis $(1,0)$ and $(0,1)$ in E^2.

$$\sigma J(1,0) = \bar{\alpha}(1,0) + \bar{\beta}(0,1);$$

$$\left(\gamma^+\right)^* J(1,0) = \bar{\mu}(1,0) + \bar{\nu}(0,1) \tag{3.6}$$

where

$$\bar{\alpha} = (1,0)\sigma J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\beta} = (1,0)\sigma J \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{\mu} = (1,0)\left(\gamma^+\right)^* J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\nu} = (1,0)\left(\gamma^+\right)^* J \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.7}$$

As a result, we obtain that expression (3.5) is written in the following form:

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* (1,0) = \bar{\alpha}\frac{1}{z}\left\{\left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* - \overset{\circ}{\varphi}{}^*\right\}(1,0) +$$

$$+\bar{\beta}\frac{1}{z}\left\{\left(I-\bar{z}\overset{\circ}{A}{}_1^*\right)^{-1}\overset{\circ}{\varphi}{}^*-\overset{\circ}{\varphi}{}^*\right\}(0,1)+\bar{\mu}\left(I-\bar{z}\overset{\circ}{A}{}_1^*\right)^{-1}\overset{\circ}{\varphi}{}^*(1,0)+$$

$$+\bar{\nu}\left(I-\bar{z}\overset{\circ}{A}{}_1^*\right)^{-1}\overset{\circ}{\varphi}{}^*(0,1)+i\bar{z}\overset{\circ}{A}{}_1^*\left(I-\bar{z}\overset{\circ}{A}{}_1^*\right)^{-2}\overset{\circ}{\varphi}{}^*(1,0). \tag{3.8}$$

Along with the integral equation for $L_x(z)$

$$L_x(z)+iz\int_0^x L_t(z)dF_tJ=(1,0), \tag{3.9}$$

consider [3] the integral equation for $N_x(z)$

$$N_x(z)+iz\int_0^x N_t(z)dF_tJ=(0,1). \tag{3.10}$$

So we can rewrite expression (3.8) as

$$\overset{\circ}{A}{}_2^*L_t(\bar{z})=\bar{\alpha}\frac{L_t(\bar{z})-L_t(0)}{\bar{z}}+\bar{\beta}\frac{N_t(\bar{z})-N_t(0)}{\bar{z}}+\bar{\mu}L_t(\bar{z})+\bar{\nu}N_t(\bar{z})-i\bar{z}\frac{d}{dz}L_t(\bar{z}). \tag{3.11}$$

By the vector-row $N_x(z)=[C_x(z);D_x(z)]$, similar to [3], construct the L. de Branges space $\mathcal{B}(C,D)$ and specify the L. de Branges transform from $L^2_{2,l}(F_x)$ on $\mathcal{B}(C,D)$ by the formula

$$G(z)\overset{\text{def}}{=}\mathcal{B}_N(f_t)=\frac{1}{\pi}\int_0^l f_tdF_tN_t^*(\bar{z}). \tag{3.12}$$

A function $G(z)\in\mathcal{B}(C,D)$ is said to be the dual to $F(z)\in\mathcal{B}(A,B)$ if

$$F(z)=\mathcal{B}_L(f_t),\quad G(z)=\mathcal{B}_N(f_t). \tag{3.13}$$

Using the notation (3.11) and (3.13), we obtain

$$\mathcal{B}_L\left(\overset{\circ}{A}_2f_t\right)=\alpha\frac{F(z)-F(0)}{z}+\beta\frac{G(z)-G(0)}{z}+\mu F(z)+\nu G(z)-iz\frac{d}{dz}F(z). \tag{3.14}$$

Thus the Lie algebra (2.2) of linear operators $\left\{\overset{\circ}{A}_1,\overset{\circ}{A}_2\right\}$ (2.1) after the L. de Branges transform $\mathcal{B}_L$ changes into the following operator system

$$\tilde{A}_1F(z)=\frac{F(z)-F(0)}{z};$$

$$\tilde{A}_2 F(z) = \frac{\alpha F(z) + \beta G(z) - \alpha F(0) - \beta G(0)}{z} +$$

$$+ \mu F(z) + \nu G(z) - iz \frac{d}{dz} F(z) \tag{3.15}$$

where the numbers α, β, μ, ν are given by (3.7) and the functions $F(z)$ and $G(z)$ are equal to(3.13).

Observation 3.1. *The dual function $G(z)$ (3.13) does not necessarily belong to the space $\mathcal{B}(A, B)$, nevertheless, under such selection of α, β, μ, ν (3.7), the expressions*

$$\mu F(z) + \nu G(z); \quad \frac{\alpha F(z) + \beta G(z) - \alpha F(0) - \beta G(0)}{z}$$

already belong to $\mathcal{B}(A, B)$. Note that the numbers α, β, μ, ν do not depend on $F(z)$.

Specify now the operator $\tilde{\varphi}$ from $\mathcal{B}(A, B)$ into E^2 using the formula

$$\tilde{\varphi} F(z) = \langle F(z), e_1(z) \rangle (1, 0) + \langle F(z), e_2(z) \rangle (0, 1) \tag{3.16}$$

where

$$\hat{e}_1(z) = \frac{B_l^*(\bar{z})}{z}; \quad \hat{e}_2(z) = \frac{1 - A_l^*(\bar{z})}{z}. \tag{3.17}$$

Theorem 3.1. *Let Δ be the simple colligation of Lie algebra (1.8), (1.9), besides, the spectrum of operator A is concentrated at zero and the characteristic function $S_1(\lambda) = I - i\varphi (A_1 - \lambda I)^{-1} \varphi^* J$ is given by*

$$S_1(\lambda) = \int_0^l \exp\left\{\frac{iJdF_t}{\lambda}\right\},$$

besides, the measure dF_x is absolutely continuous, $dF_x = a_x dx$, $a_x \geq 0$, a_x is a matrix-function in E^2 and J is given by (2.1) [3]. And, moreover, a selfadjoint operator σ and operators $\gamma^\pm$ are given in E^2 such that $\gamma^\pm - (\gamma^\pm)^* = iJ$. Then the colligation Δ (1.8) is unitary equivalent to the functional model

$$\tilde{\Delta} = \left(\left\{\tilde{A}_1, \tilde{A}_2\right\}; \mathcal{B}(A, B); \tilde{\varphi}, E^2, \{J, \sigma\}; \{\gamma^+, \gamma^-\}\right) \tag{3.18}$$

where $\tilde{A}_1$, $\tilde{A}_2$ are given by (3.15); the operator $\tilde{\varphi}$ equals (3.16); the numbers α, β, μ, $\nu \in \mathbb{C}$ are given by the formulas (3.7); $G(z)$ is the dual function of $F(z)$, and, finally, $\{e_k(z)\}_1^2$ are given by (3.17).

4. FUNCTIONAL MODELS ON RIEMANN SURFACE

I. Let $r = 2n$ be even, and let a_x be given by

$$a_x = I_n \times \hat{a}_x \tag{4.1}$$

where I_x is the unit operator in E^n and $\hat{a}_x$ is such non-negative matrix (2×2) that $\mathrm{tr}\hat{a}_x = n^{-1}$. It is obvious that the Hilbert space $L^2_{2n,l}(F_x)$ [1, 3] is formed by the vector-functions $f(x) = (f_1(x), \ldots, f_n(x))$ such that

$$\int_0^l f_k(x)\hat{a}_x f_k^*(x)dx < \infty$$

for all k $(1 \geq k \geq n)$ where $f_k(x) \in E^2$ for every $x \in [0, l]$.

Suppose that the operators $\sigma_1 = J$, $\sigma_1 = \sigma$, $\gamma^{\pm}$ are given by

$$\sigma_1 = J = I_n \otimes J_N; \quad \sigma = \tilde{\sigma} \otimes J_N; \quad \gamma^{\pm} = \tilde{\gamma} \otimes J_N \tag{4.2}$$

where $\tilde{\sigma}$ is a selfadjoint operator in E^n and $\tilde{\gamma}$ is such operator in E^n that

$$\tilde{\gamma} - (\tilde{\gamma})^* = -iI_n. \tag{4.3}$$

Realize the L. de Branges transform $\mathcal{B}_L$ [3] of each component $f_k(x) \in L^2_{2,l}(\hat{a}_x dx)$ of the vector-function $f(x)$ from $L^2_{2n,l}(F_x)$ assuming that a_x is given by (4.1),

$$F_k(x) \stackrel{\text{def}}{=} \mathcal{B}_L(f_k) = \frac{1}{\pi}\int_0^l f_k(x)\hat{a}_x L_x^*(\bar{z})\,dx \tag{4.4}$$

where $L_x(z)$ is the solution of the integral equation (3.9) by the measure $\hat{a}_x dx$.

As a result, we obtain the Hilbert space $\mathcal{B}^n(A, B) = E^n \otimes \mathcal{B}(A, B)$ which is formed by the vector-functions $F(z) = (F_1(z), \ldots, F_n(z))$,

$$\mathcal{B}^n(A, B) = \{F(z) = (F_1(z), \ldots, F_n(z)) : F_k(z) \in \mathcal{B}(A, B)\,(1 \leq k \leq n)\}, \tag{4.5}$$

besides, the scalar product in $\mathcal{B}^n(A, B)$ is given by

$$\langle F(z), G(z)\rangle_{\mathcal{B}^n(A,B)} = \sum_{k=1}^{n} \langle F_k(z), G_k(z)\rangle_{\mathcal{B}(A,B)}\,. \tag{4.6}$$

Taking into account the form of a_x (4.1) and J (4.2), we obtain that the L. de Branges transform $\mathcal{B}_L$ [3] translates the triangular model $\overset{\circ}{A}_1$ (2.1) into the shift operator

$$\left(\tilde{A}_1 F\right) = \frac{1}{z}(F(z) - F(0)); \quad \forall F(z) \in \mathcal{B}^N(A, B). \tag{4.7}$$

To obtain the model representation $\overset{\circ}{A}_2$ (2.1), use the formula

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* = \frac{1}{z}\left\{\left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^*\, \sigma J\right\} +$$

$$+ \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* \left(\gamma^+\right)^* J + i\bar{z}\, \overset{\circ}{A}{}_1^* \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{\varphi}{}^* \tag{4.8}$$

and the fact that $L_x^*(\bar{z}) = \left(I - \bar{z}\, \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^*\,(1,0)$. Taking into account the concrete form of the operators J, σ, γ^+ (4.2), we obtain

$$\sigma J = \tilde{\sigma} \otimes I_2, \quad \left(\gamma^+\right)^* J = \tilde{\gamma}^* \otimes I_2. \tag{4.9}$$

Therefore, after the L. de Branges transform (2.24), the operator $\overset{\circ}{A}_2$ (2.1) is given by

$$\left(\tilde{A}_2 F\right)(z) = \frac{1}{z}(F(z) - F(0))\tilde{\sigma} + F(z)\tilde{\gamma} - iz\frac{d}{dz}F(z). \tag{4.10}$$

Thus

$$\tilde{A}_2 F(z) = \frac{1}{z}\left\{F(z)\left(\tilde{\sigma} + z\tilde{\gamma}\right) - F(z)\left.\left(\tilde{\sigma} + z\tilde{\gamma}\right)\right|_0\right\} + iz\frac{d}{dz}F(z) \tag{4.11}$$

where $F(z)\left.\left(\tilde{\sigma} + z\tilde{\gamma}\right)\right|_0 = F(0)\sigma$.

Now define the colligation of Lie algebra

$$\tilde{\Delta} = \left(\left\{\tilde{A}_1, \tilde{A}_2\right\}; \mathcal{B}^n(A,B); \tilde{\varphi}, E^{2n}; J, \sigma, \gamma^+, \gamma^-\right), \tag{4.12}$$

besides, J, σ, γ^+ are given by (4.2), $\gamma^- = \gamma^+$, and the operator $\tilde{\varphi}$ on each component of $F_k(z)$ acts in a standard way [1, 3].

Theorem 4.1. *Let Δ (1.8) be the simple colligation of Lie algebra such that $\dim E = 2n$, $\sigma_1 = J$, $\sigma_1 = \sigma$, γ^+ are given by (4.2), spectrum of the operator A_1 lies at zero, and the characteristic function of operator A_1 is such that the measure dF_x in multiplicative representation of $S_1(\lambda)$ (see Theorem 3.1) is absolutely continuous, $dF_x = a_x dx$ and a_x equals (4.1). Then the colligation Δ is unitarily equivalent to the simple part of the functional model $\tilde{\Delta}$ (4.12) where the operators $\tilde{A}_1$ and $\tilde{A}_2$ are given by the formulas (4.7) and (4.11), respectively.*

II. Consider the linear bundle

$$\tilde{\sigma} + z\tilde{\gamma} = \sigma + z\tilde{\gamma}_R - \frac{i}{2}zI_n \tag{4.13}$$

in view of (4.3) where $\tilde{\gamma}_R = \tilde{\gamma}_R^* = \dfrac{1}{2}\left(\tilde{\gamma} + \tilde{\gamma}^*\right)$.

Denote by $h(z,w)$ the eigenvectors of selfadjoint (when $z \in \mathbb{R}$) bundle $\tilde{\sigma} + z\tilde{\gamma}_R$,

$$h(P)\left(\tilde{\sigma} + z\tilde{\gamma}_R\right) = wh(P) \tag{4.14}$$

where $P = (z,w)$ belongs to the algebraic curve

$$\mathbb{Q} = \left\{P = (z,w) \in \mathbb{C}^2 : \mathbb{Q}(z,w) = 0\right\} \tag{4.15}$$

specified by the polynomial

$$\mathbb{Q}(z,w) = \det\left(\tilde{\sigma} + z\tilde{\gamma}_R - wT_n\right). \tag{4.16}$$

Suppose that the curve $\mathbb{Q}$ (4.15) is nonsingular [4, 9], then $z = z(P)$ and $w = w(P)$ are "l-valued" and, respectively, "n-valued" functions on $\mathbb{Q}$ ($l = \operatorname{rank}\tilde{\gamma}_R$). We normalize the rational function $h(P)$ (4.14) using the condition $h_n(P) = 1$ where $h_n(P)$ is the nth component of the vector $h(P)$.

It is easy to see [4] that the quantity of poles, taking into account the multiplicity, of vector-function $h(P)$, equals $N = g + n - 1$ where g is the type of Riemann surface $\mathbb{Q}$ (4.15). Specify on $\mathbb{Q}$ analogues of halfplanes $\mathbb{C}_\pm$ and $\mathbb{R}$,

$$\mathbb{Q}_\pm = \{P = (z,w) \in \mathbb{Q} : \pm\operatorname{Im} z(P) > 0\}; \quad \mathbb{Q}^0 = \partial\mathbb{Q}_\pm. \tag{4.17}$$

Expand every function $F(z) \in \mathcal{B}^n(A,B)$ by the basis $h\left(P_k\right)$ ($z \in \mathbb{R}$),

$$F(z) = \sum_{k=1}^{n} g\left(P_k\right) \left\|h\left(P_k\right)\right\|_{E^n}^{-2} h\left(P_k\right)$$

where $P_k = \left(z, w^k(z)\right) \in \mathbb{Q}$ and $w^k(z)$ are different roots of the polynomial $\mathbb{Q}(z,w) = 0$ (4.16); $g\left(P_k\right) = \left\langle F(z), h\left(P_k\right)\right\rangle_{E^n}$ ($1 \le k \le n$). It is obvious that $w^k(P)$, along with $h\left(P_k\right)$, $g\left(P_k\right)$, represents branches of "n-valued" algebraic functions $w(P)$, $h(P)$, $g(P)$. Therefore the last equality signifies that

$$F(P) = F(z(P)) = g(P)\|h(P)\|_{E^n}^{-2} h(P). \tag{4.18}$$

And since the basis $h(P)$ in E^n is constant, the vector-function $F(P)$ is defined by the scalar component $g(P)$. The function $g(P)$ is a meromorphic function on $\mathbb{Q}$ (4.15), the poles of which may lay only in the poles of $h(P)$ (4.14), and their joint multiplicity could not exceed $N = g + n - 1$.

Define [3] the L. de Branges space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$ on the Riemann surface $\mathbb{Q}$ (4.15). It is easy to see that the operator $\tilde{A}_1$ (4.7) in L. de Branges space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$ acts in the following way:

$$\left(\tilde{A}_1 g\right)(P) = \frac{g(P) - \psi\left(P, P_0\right) g\left(P_0\right)}{z(P) - z\left(P_0\right)} \tag{4.19}$$

where $\psi\left(P, P_0\right)$ is given by

$$\psi\left(P, P_0\right) = \left\langle h\left(P_0\right), h(P)\right\rangle_{E^n} \|h(P)\|_{E^n}^{-2}, \tag{4.20}$$

besides, $P_0 = (0, w) \in \mathbb{Q}$.

Now study how the operator $\tilde{A}_2$ (4.11) acts in the space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$. (4.11), (4.13) imply

$$\left(\tilde{A}_2 F\right)(P) = \frac{1}{z(P) - z(P_0)} \left\{ g(P) \left[w(P) + \frac{i}{2} z(P) \right] \cdot \|h(P)\|_E^{-2} h(P) - \right.$$

$$\left. - g(P_0) \left[w(P_0) + \frac{i}{2} z(P_0) \right] \cdot \|h(P_0)\|_{E^n}^{-2} h(P_0) \right\} -$$

$$-iz(P) \left\{ \frac{d}{dz} g(P) \cdot \|h(P)\|_{E^n}^{-2} h(P) - 2g(P) \cdot \|h(P)\|_{E^n}^{-3} \frac{d}{dz} \|h(P)\|_{E^n} h(P) + \right.$$

$$\left. + g(P) \cdot \|h(P)\|_{E^n}^{-2} \frac{d}{dz} h(P) \right\}.$$

Therefore we arrive at the following structure of the operator $\tilde{A}_2$ in L. de Branges space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$:

$$\left(\tilde{A}_2 g\right)(P) = \frac{1}{z(P) - z(P_0)} \left\{ g(P) \left[w(P) + \frac{i}{2} z(P) \right] - \right.$$

$$\left. - \psi(P, P_0)\, g(P_0) \left[w(P_0) + \frac{i}{2} z(P_0) \right] \right\} - iz(P) \frac{d}{dz} g(P) - iz(P) b(P) g(P) \quad (4.21)$$

where the function $b(P)$ equals

$$b(P) = \left\langle \frac{d}{dz} h(P), h(P) \right\rangle_{E^n} \|h(P)\|_{E^n}^{-4} - 2\|h(P)\|_{E^n}^{-3} \frac{d}{dz} \|h(P)\|_{E^n}. \quad (4.22)$$

Now construct the colligation of Lie algebra

$$\tilde{\Delta} = \left(\left\{ \tilde{A}_1, \tilde{A}_2 \right\}; \mathcal{B}_{\mathbb{Q}}(A, B, h); \tilde{\varphi}, E^{2n}; J, \sigma, \gamma^+, \gamma^- \right) \quad (4.23)$$

where the operators $\tilde{A}_1$ and $\tilde{A}_2$ are given by (4.19), (4.21); the functions $\psi(P, P_0)$ and $b(P)$ are given by the formulas (4.20) and (4.22); the operators J, σ, γ^+ are represented by (4.2), $\gamma^- = \gamma^+$; and the operator $\tilde{\varphi}$ acts on the function $g(P)$ in the following way:

$$\tilde{\varphi} g(P) = \sum_{k=1}^{2} \langle g(P), e_k(z(P)) \rangle_{\mathcal{B}_{\mathbb{Q}}(A,B,h)} \cdot e_k,$$

besides, $e_k(z)$ are given by

$$e_1(z) = \frac{1 - \alpha z}{z} B^*(\bar{z}); \quad e_2(z) = \frac{1 - \alpha z}{z} (1 - A^*(\bar{z})); \quad e_1 = (1, 0); \quad e_2 = (0, 1). \quad (4.24)$$

Theorem 4.2. *Let there be given such simple colligation* Δ (1.8) *of Lie algebra that* $\dim E = 2n$*,* $\sigma_1 = J$*,* $\sigma_1 = \sigma$*,* γ^+ *is given by* (4.2)*, spectrum of* A_1 *is concentrated at zero, and the characteristic function of operator* A_1 *is such that the measure* dF_x *in the multiplicative representation of* $S_1(\lambda)$ (*see Theorem* 3.1) *is absolutely continuous,* $dF_x = a_x dx$*, besides,* a_x *equals* (4.1)*. Then the colligation* Δ (1.8) *is unitarily equivalent to the simple part of functional model* $\tilde{\Delta}$ (4.23)*.*

III. Consider the following example. Let $\dim E = 6$, the operators $\tilde{\sigma}$ and $\tilde{\gamma}$ in E^3 be equal

$$\tilde{\sigma} = \begin{bmatrix} -\frac{1}{k} & 0 & 0 \\ 0 & 1 & b \\ 0 & b & \frac{1}{k} - 2 \end{bmatrix}; \quad \tilde{\gamma} = \begin{bmatrix} -\frac{i}{2} & 0 & a \\ 0 & -\frac{i}{2} & 0 \\ a & & -\frac{i}{2} \end{bmatrix}; \tag{4.25}$$

where $a > 0$; $k \in (0, 1)$; $b = \sqrt{2\left(\frac{1}{k} - 1\right)}$. In this case the curve $\mathbb{Q}$ is given by the polynomial

$$k^2 a^3 z^2 (1 - w) = (1 + w)\left(1 - k^2 w^2\right). \tag{4.26}$$

Assuming that $\xi = ka\lambda(1 - w)$, we obtain the Legendre algebraic curve

$$\xi^2 = \left(1 - w^2\right)\left(1 - k^2 w^2\right). \tag{4.27}$$

The two-sheeted Riemann surface (4.27) has the genus $g = 1$ and is formed by the "crosswise" gluing of two w planes along the cut $\left(-\infty, -\frac{1}{k}\right] \cup [-1,] \cup \left(\frac{1}{k}, \infty\right)$. The imaginary part

$$ka\mathrm{Im}z = \mathrm{Im}\sqrt{\frac{1 + w}{1 - w}\left(1 - k^2 w^2\right)}$$

changes its sign on the cuts, therefore $\mathbb{Q}^+$ and $\mathbb{Q}^-$ (4.17) are sheets of the Riemann surface (4.15) and $\mathbb{Q}^0 = \partial\mathbb{Q}^\pm$ coincides with the mentioned cuts. On surface (4.27) there exists the Abelian differential of genus one [9],

$$\bar{\omega} = \frac{dw}{\sqrt{\left(1 - w^2\right)\left(1 - k^2 w^2\right)}}. \tag{4.28}$$

Using the elliptic integral

$$u(P) = \int_{P_1}^{P} \omega \quad (P = (\lambda, w) \in \mathbb{Q}), \tag{4.29}$$

specify the conform map [9] between (4.27) and the rectangle

$$\Gamma = \{u \in \mathbb{C} : \mathrm{Re}u \in [-2k, 2k]; \mathrm{Im}u \in [-k', k]\} \tag{4.30}$$

where $P_1 = (0, 1)$ and the numbers $4k$ and $2ik'$ are the periods of the closed differential ω (4.28). Inversion of the elliptic integral (4.29) results in the uniformization of curve (4.27) in terms of the elliptic Jacobi functions [9]. Therefore for (4.26) we obtain

$$z(u) = \frac{\mathrm{sn}'u}{ka(1 - \mathrm{sn}u)}; \quad w(u) = \mathrm{sn}u. \tag{4.31}$$

The eigenvectors $h(P) = h(u)$ of linear bundle $h(P)\left(\tilde{\sigma} + z\tilde{\gamma}_R\right) = wh(P)$ are given by

$$h(P) = \left[\frac{kaz}{1+kw}, \frac{b}{w-1}, 1\right]; \quad h(u) = \left[\frac{\mathrm{sn}'u}{(1-\mathrm{sn}u)(1+k\mathrm{sn}u)}, \frac{b}{\mathrm{sn}u - 1}, 1\right]. \tag{4.32}$$

It is easy to show that the function $\psi\left(P, P_0\right)$ (4.20) equals 1, $\psi\left(P, P_0\right) = 1$. The function $b(P)$ (4.22) is given by

$$b(P) = \|h(P)\|_{E^n}^{-4} \left\{\frac{b^2}{(w-1)^2} - \frac{k^2a^2z}{(1+kw)^2}\right\}. \tag{4.33}$$

Thus in this case the functional model of Lie algebra is

$$\left(\tilde{A}_1 g\right)(P) = \frac{g(P) - g\left(P_0\right)}{z(P) - z\left(P_0\right)};$$

$$\left(\tilde{A}_2 g\right)(P) = \frac{g(P)\left[w(P) + \frac{i}{2}z(P)\right] - g\left(P_0\right)\left[w\left(P_0\right) + \frac{i}{2}z\left(P_0\right)\right]}{z(P) - z\left(P_0\right)} - \tag{4.34}$$

$$-iz(P)\frac{d}{dz}g(P) - iz(P)b(P)g(P).$$

Thus,

1) for the Lie algebra $\{A_1, A_2\}$ ($[A_2, A_1] = iA_1$), the triangular model (2.1) in the space of functions $L^2_{r,l}\left(F_x\right)$ is constructed (see Theorem 2.2);

2) functional model (3.15) for the studied in this paper Lie algebra $\{A_1, A_2\}$ in spaces of entire L. de Branges functions is determined (Theorem 3.1);

3) model realization of the given Lie algebra on Riemann surface is presented (Theorem 4.1 and Theorem 4.2).

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V. N. Karazin Kharkov National University
4, Svobody sq.
61077 Kharkov
UKRAINE
E-mail: victoriya979@mail.ru

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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SYMMETRIES OF THE HERGLOTZ VARIATIONAL PRINCIPLE IN THE CASE OF ONE INDEPENDENT VARIABLE[1]

BOGDANA A. GEORGIEVA

This paper provides a method for calculating the symmetry groups of the functional defined by the generalized variational principle of Herglotz in the case of one independent variable. A variational description is found for several named ordinary differential equations. Variational symmetry groups are calculated for a Liouville's equation and a Lane-Emden equation.

Keywords: variational symmeties, Herglotz variational principle, invariant functional, Herglotz

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1. INTRODUCTION

It is well known that a variational description of a differential equation or a system of such equations is very desirable both from mathematical and from physical point of view. The classical variational principle, although far-reaching and very powerful, can not describe many important differential equations. In 1930 Gustav Herglotz proposed a *generalized variational principle* with one independent variable, which generalizes the classical variational principle by defining the functional, whose extrema are sought, by a certain differential equation, see Herglotz [7] and Guenther et al. [5]. Herglotz variational principle contains the classical variational principle as a special case. His original idea was published in 1979 in

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his collected works, Herglotz [6] and [8]. This variational principle can describe not only all physical processes with one independent variable which the classical variational principle can, but also many others for which the classical variational principle is not applicable. For example, it can give a variational description of nonconservative processes even when the Lagrangian is not dependent on time, something which can not be done with the classical variational principle. It is also related to contact transformations.

The generalized variational principle of Herglotz defines the functional z, whose extrema are sought, by the differential equation

$$\frac{dz}{dt} = L\Big(t, x(t), \frac{dx(t)}{dt}, z\Big) \tag{1.1}$$

where t is the independent variable, and $x(t) \equiv (x_1(t), \ldots, x_n(t))$ stands for the argument functions. In order for the equation (1.1) to define a functional $z = z[x]$ of $x(t)$ equation (1.1) must be solved with the same fixed initial condition $z(0)$ for all argument functions $x(t)$, and the solution $z(t)$ must be evaluated at the same fixed final time $t = T$ for all argument functions $x(t)$.

The equations whose solutions produce the extrema of this functional are

$$\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k} = 0 , \qquad k = 1, \ldots, n , \tag{1.2}$$

where $\dot{x}_k$ denotes dx_k/dt. Herglotz called them the *generalized Euler-Lagrange equations*. See Guenther et al. [5] for a derivation of this system. The solutions of these equations, when written in terms of the dependent variables x_k and the associated momenta $p_k = \partial L/\partial \dot{x}_k$, determine a family of *contact transformations*. See Guenther et al. [5].

For equations which can be obtained from Herglotz variational principle as (1.2) one can systematically derive conserved quantities, as shown in Georgieva et al. [2], by applying the first Noether-type theorem formulated and proven in the same paper. For convenience of the reader we state this result here. Consider the one-parameter group of transformations

$$\bar{t} = \phi(t, x, \varepsilon), \tag{1.3}$$

$$\bar{x}_k = \psi_k(t, x, \varepsilon), \qquad k = 1, ..., n$$

where ε is the parameter, $\phi(t, x, 0) = t$, and $\psi_k(t, x, 0) = x_k$, with infinitesimal generator

$$\mathbf{v} = \tau(t, x)\frac{\partial}{\partial t} + \xi_k(t, x)\frac{\partial}{\partial x_k}$$

where

$$\tau(t, x) = \frac{d\phi}{d\varepsilon}\Big|_{\varepsilon=0} \text{and } \ \xi_k(t, x) = \frac{d\psi_k}{d\varepsilon}\Big|_{\varepsilon=0} . \tag{1.4}$$

Throughout this paper we assume that the summation convention on repeated indices holds and that $\cdot$ denotes differentiation with respect to t.

Theorem 1.1 (First Noether-type theorem for the generalized variational principle). *If the functional* $z = z[x(t)]$ *defined by the differential equation* $\dot{z} = L(t, x, \dot{x}, z)$ *is invariant under the one-parameter group of transformations* (1.3) *then the quantity*

$$\exp\left(-\int_0^t \frac{\partial L}{\partial z}\, d\theta\right)\left(\left(L - \dot{x}_k \frac{\partial L}{\partial \dot{x}_k}\right)\tau + \frac{\partial L}{\partial \dot{x}_k}\xi_k\right) \tag{1.5}$$

is conserved along the solutions of the generalized Euler-Lagrange equations (1.2).

The present paper shows how a group of transformations can be found under which the functional of a given Herglotz variational principle is invariant. The importance of this problem is that once such a group of symmetries is found, conserved quantities for the corresponding system of generalized Euler-Lagrange equations can be written down directly, applying the first Noether-type theorem. The symmetry group generators are obtained from a system of first order partial differential equations as shown in section 3. In section 2 a variational description is found for several named ordinary differential equations. Several examples of calculating variational symmetry groups, and from them the corresponding first integrals, are given in section 4.

The interested reader can find a generalization of the Herglotz variational principle to one with several independent variables in Georgieva et al. [3]. There a theorem of Noether-type is formulated and proven, for the case of finite-dimensional symmetry groups of the functional, and applications are given. That paper also contains a proposition characterizing the variational symmetry groups of differential equations describing physical fields.

Historically, the question of calculating the symmetries of a given Lagrangian functional was answered by W. Killing [9] in 1892 in the context of describing the motions of a n-dimensional manifold of fundamental form given by

$$L = \frac{1}{2} g_{kl} \dot{x}^k \dot{x}^l$$

(see Eisenhart [1] and Logan [10]). In the case of a classical variational functional, some authors refer to the system of partial differential equations for the unknown symmetry group generators as the *generalized Killing equations.* For the derivation of these equations in the case of the classical variational principle, see Logan [10].

2. VARIATIONAL DESCRIPTION VIA HERGLOTZ VARIATIONAL PRINCIPLE

In this section we use the generalized variational principle of Herglotz to give a variational description of several named ordinary differential equations.

First we show that the class of ordinary differential equations

$$\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} + h(x) = 0 \tag{2.1}$$

for the function $x = x(t)$ can be given a variational description via the Herglotz variational principle, by letting L in the defining equation (1.1) be

$$L = \frac{1}{2}\dot{x}^2 - \big(2f(x)\dot{x} + g(t)\big)z - U(x) \tag{2.2}$$

where $U(x)$ is any solution of the ODE

$$\frac{dU(x)}{dx} + 2f(x)U(x) = h(x).$$

Indeed,

$$\begin{aligned}
&\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} = \\
&= -\frac{dU}{dx} - 2\dot{x}\frac{df}{dx}z - \frac{d}{dt}\Big(\dot{x} - 2fz\Big) - (2f\dot{x} + g)(\dot{x} - 2fz) \\
&= -\frac{dU}{dx} - \ddot{x} + 2f\Big(\frac{1}{2}\dot{x}^2 - 2f\dot{x}z - gz - U\Big) - 2f\dot{x}^2 + 4f^2\dot{x}z - \dot{x}g + 2fgz \\
&= -\ddot{x} - f\dot{x}^2 - g\dot{x} - 2fU - \frac{dU}{dx} = -(\ddot{x} + f\dot{x}^2 + g\dot{x} + h).
\end{aligned}$$

Equation (2.1) contains several well known named equations as special cases:
a. When $h(x) = kx$, with $k = constant$, $f(x) = 0$ and $g(t) = a = constant$, (2.1) is the equation of the damped harmonic oscillator

$$\ddot{x} + a\dot{x} + kx = 0. \tag{2.3}$$

The corresponding Lagrangian is

$$L = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - az. \tag{2.4}$$

b. In the special case when $h(x) = kx$, $k = constant$ and $f(x) = 0$, equation (2.1) becomes the Lienard's equation

$$\ddot{x} + g(t)\dot{x} + kx = 0. \tag{2.5}$$

The corresponding Lagrangian is

$$L = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - g(t)z. \tag{2.6}$$

c. In the case when $h(x) = x^n$, $f(x) = 0$ and $g(t) = 2/t$, equation (2.1) becomes the Lane-Emden equation

$$\ddot{x} + \frac{2}{t}\dot{x} + x^n = 0\ , \quad n \neq -1. \tag{2.7}$$

In that case the Lagrangian is

$$L = \frac{1}{2}\dot{x}^2 - \frac{x^{n+1}}{n+1} - \frac{2}{t}z. \tag{2.8}$$

d. As a final example consider the special case when $h(x) = 0$. Then equation (2.1) is the Liouville's equation

$$\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} = 0 \ . \tag{2.9}$$

The Lagrangian for it is

$$L = \frac{1}{2}\dot{x}^2 - \big(2f(x)\dot{x} + g(t)\big)z. \tag{2.10}$$

3. THE FUNDAMENTAL INVARIANCE IDENTITY AND THE GENERALIZED KILLING EQUATIONS

Let the functional z be defined by the ordinary differential equation (1.1). Consider the one-parameter group of transformations (1.3) with the coefficients of its generator given in (1.4).

Lemma 3.1. *The following identity holds*

$$\frac{d}{d\varepsilon}\frac{d\bar{x}_k}{d\bar{t}}\Big|_{\varepsilon=0} = \frac{d\xi_k}{dt} - \dot{x}_k\frac{d\tau}{dt} \ , \tag{3.1}$$

provided $\bar{x}_k$ *and* $\bar{t}$ *are defined by the transformation*

$$\bar{t} = t + \tau(t,x)\,\varepsilon$$

$$\bar{x}_k = x_k + \xi_k(t,x)\,\varepsilon \ , \tag{3.2}$$

corresponding to (1.3) *and* (1.4).

Proof. We have

$$\frac{d\bar{x}_k}{dt} \equiv \frac{\partial \bar{x}_k}{\partial t} + \frac{\partial \bar{x}_k}{\partial x_h}\dot{x}_h = \frac{d\bar{x}_k}{d\bar{t}}\frac{d\bar{t}}{dt} \equiv \frac{d\bar{x}_k}{d\bar{t}}\Big(\frac{\partial \bar{t}}{\partial t} + \frac{\partial \bar{t}}{\partial x_h}\dot{x}_h\Big). \tag{3.3}$$

Setting $\varepsilon = 0$ yields

$$\frac{d\bar{x}_k}{d\bar{t}}\Big|_{\varepsilon=0} = \delta^k_h\dot{x}_h = \dot{x}_k. \tag{3.4}$$

Differentiate (3.3) with respect to ε and expand both sides

$$\frac{d}{d\varepsilon}\frac{\partial \bar{x}_k}{\partial t} + \dot{x}_h\frac{d}{d\varepsilon}\frac{\partial \bar{x}_k}{\partial x_h} = \frac{d\bar{x}_k}{d\bar{t}}\Big(\frac{d}{d\varepsilon}\frac{\partial \bar{t}}{\partial t} + \dot{x}_h\frac{d}{d\varepsilon}\frac{\partial \bar{t}}{\partial x_h}\Big) + \Big(\frac{\partial \bar{t}}{\partial t} + \frac{\partial \bar{t}}{\partial x_h}\dot{x}_h\Big)\frac{d}{d\varepsilon}\frac{d\bar{x}_k}{d\bar{t}}. \tag{3.5}$$

We set $\varepsilon = 0$ in this equation, substitute in it (3.4) and use the following relations:

$$\frac{d}{d\varepsilon}\frac{\partial \bar{x}_k}{\partial t}\Big|_{\varepsilon=0} = \frac{\partial \xi_k}{\partial t}\,, \quad \frac{d}{d\varepsilon}\frac{\partial \bar{x}_k}{\partial x_h}\Big|_{\varepsilon=0} = \frac{\partial \xi_k}{\partial x_h}\,, \quad \frac{d}{d\varepsilon}\frac{\partial \bar{t}}{\partial t}\Big|_{\varepsilon=0} = \frac{\partial \tau}{\partial t}\,, \quad \frac{d}{d\varepsilon}\frac{\partial \bar{t}}{\partial x_h}\Big|_{\varepsilon=0} = \frac{\partial \tau}{\partial x_h}$$

$$\frac{\partial \bar{t}}{\partial t}\Big|_{\varepsilon=0} = 1\,, \qquad \frac{\partial \bar{t}}{\partial x_h}\Big|_{\varepsilon=0} = 0.$$

Then equation (3.5) becomes

$$\frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial x_h}\dot{x}_h = \dot{x}_k\Big(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x_h}\dot{x}_h\Big) + \frac{d}{d\varepsilon}\frac{d\bar{x}_k}{d\bar{t}}\Big|_{\varepsilon=0}.$$

Observe that the total derivatives of ξ_k and τ, with respect to t, appear in the last equation. Solving for the last term in it yields the statement of the lemma. □

The following theorem gives a method for finding symmetry groups of the Herglotz functional.

Theorem 3.2. *The coefficients $\tau(t,x)$ and $\xi_k(t,x)$ of the infinitesimal generator of a one-parameter group of transformations which preserve the value of the functional $z = z[x(t)]$, defined by the differential equation (1.1), are solutions of the system of partial differential equations obtained from the identity*

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\Big(\frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial x_j}\dot{x}_j - \dot{x}_k\frac{\partial \tau}{\partial t} - \dot{x}_k\dot{x}_j\frac{\partial \tau}{\partial x_j}\Big) + L\Big(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x_j}\dot{x}_j\Big) = 0 \quad (3.6)$$

by equating to zero the coefficients of the powers of z and $\dot{x}_k$ and of products of such powers.

In analogy with the classical case, we call this identity the *fundamental invariance identity* and the resulting partial differential equations for the coefficients of the infinitesimal generator of the variational symmetry group the *generalized Killing equations.*

Proof. Apply the transformation (3.2)

$$\bar{t} = t + \tau(t,x)\,\varepsilon$$

$$\bar{x}_k = x_k + \xi_k(t,x)\,\varepsilon\,,$$

corresponding to (1.3) and (1.4), to the differential equation (1.1). The result is the defining equation

$$\frac{d\bar{z}}{d\bar{t}} = L\Big(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}, \bar{z}\Big)$$

for the transformed functional $\bar{z} = \bar{z}[\bar{x}(\bar{t})]$. Since $d\bar{z}/d\bar{t} = (d\bar{z}/dt)\,(dt/d\bar{t})$, we have

$$\frac{d\bar{z}}{dt} = L\Big(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}(\bar{t})}{d\bar{t}}, \bar{z}\Big)\frac{d\bar{t}}{dt}. \quad (3.7)$$

Differentiate (3.7) with respect to ε and set $\varepsilon = 0$ to obtain

$$\frac{d}{d\varepsilon}\Big(\frac{d\bar{z}}{dt}\Big)\Big|_{\varepsilon=0} = \frac{d}{dt}\Big(\frac{d\bar{z}}{d\varepsilon}\Big)\Big|_{\varepsilon=0} = \frac{dL}{d\varepsilon}\Big|_{\varepsilon=0}\frac{d\bar{t}}{dt}\Big|_{\varepsilon=0} + L\frac{d}{d\varepsilon}\Big(\frac{d\bar{t}}{dt}\Big)\Big|_{\varepsilon=0}. \tag{3.8}$$

From $\phi(t, x, 0) = t$ it follows that

$$\frac{d\bar{t}}{dt}\Big|_{\varepsilon=0} = 1.$$

Similarly, we have

$$\frac{d}{d\varepsilon}\Big(\frac{d\bar{t}}{dt}\Big)\Big|_{\varepsilon=0} = \frac{d}{dt}\Big(\frac{d}{d\varepsilon}\phi(t, x, \varepsilon)\Big)\Big|_{\varepsilon=0} = \frac{d}{dt}\tau(t, x).$$

Denote by $\zeta = \zeta(t)$ the total variation of the functional $z = z[x]$ produced by the transformation (3.2), i.e.

$$\zeta(t) = \frac{d}{d\varepsilon}z[x;\varepsilon]\Big|_{\varepsilon=0}.$$

Thus, equation (3.8) becomes

$$\frac{d\zeta}{dt} = \frac{dL}{d\varepsilon}\Big|_{\varepsilon=0} + L\frac{d\tau}{dt}.$$

Expanding the derivative $dL/d\varepsilon$ and setting $\varepsilon = 0$, we obtain

$$\frac{d\zeta}{dt} = \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\frac{d}{d\varepsilon}\Big(\frac{d\bar{x}_k}{d\bar{t}}\Big)\Big|_{\varepsilon=0} + \frac{\partial L}{\partial z}\zeta + L\frac{d\tau}{dt}. \tag{3.9}$$

We now use the assertion of lemma 3.1 and insert expression (3.1) in equation (3.9) to obtain the linear differential equation

$$\frac{d\zeta}{dt} = \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\Big(\frac{d\xi_k}{dt} - \dot{x}_k\frac{d\tau}{dt}\Big) + L\frac{d\tau}{dt} + \frac{\partial L}{\partial z}\zeta \tag{3.10}$$

for the variation ζ of the functional z. For clarity we denote by $A(s)$ the expression

$$A(s) = \frac{\partial L}{\partial s}\tau + \frac{\partial L}{\partial x_k}\xi_k + \frac{\partial L}{\partial \dot{x}_k}\Big(\frac{d\xi_k}{ds} - \dot{x}_k\frac{d\tau}{ds}\Big) + L\frac{d\tau}{ds}.$$

The solution $\zeta(t)$ of (3.10) is given by

$$\exp\Big(-\int_0^t \frac{\partial L}{\partial z}\,d\theta\Big)\zeta - \zeta(0) = \int_0^t \exp\Big(-\int_0^s \frac{\partial L}{\partial z}\,d\theta\Big)A(s)\,ds.$$

Notice that $\zeta(0) = 0$. Indeed, as explained earlier, in order to have a well-defined functional z of $x(t)$, we must evaluate the solution $z(t)$ of the equation (1.1) with the same fixed initial condition $z(0)$, independently of the function $x(t)$. Then $z(0)$ is independent of ε. Hence, the variation of z evaluated at $t = 0$ is zero.

Since by hypothesis the one-parameter group of transformations (1.3) leaves the functional $z = z[x(t)]$ stationary, we have $\zeta(t) = 0$. Thus, it follows that

$$\int_0^t \exp\left(-\int_0^s \frac{\partial L}{\partial z}\, d\theta\right) A(s)\; ds = 0. \tag{3.11}$$

Taking in consideration the fact that equation (3.11) is valid for all values of t, and that the exponent is always positive, we obtain the identity $A(t) = 0$ which, after writing the total derivatives explicitly, becomes (3.6).

Equation (3.6) is an identity in (t, x_k) for arbitrary directional arguments $\dot{x}_k$. Therefore, we can regard this identity as a set of partial differential equations in the unknowns τ and ξ_k . Due to the arbitrariness of $\dot{x}_k$ and the fact that z depends on $\dot{x}_k$, we can further reduce (3.6) to obtain a system of first order partial differential equations in τ and ξ_k, by equating to zero the coefficients of the powers of $\dot{x}_k$, the powers of z, as well as the coefficients of products of such powers. The solution of this system, if it exists, determines a group of transformations under which the functional defined by equation (1.1) is invariant. □

4. APPLICATIONS

In this section we calculate variational symmetries of several ordinary differential equations and use the first Noether-type theorem 1.1 to find the corresponding conserved quantities.

We start with the equation for the damped harmonic oscillator $\ddot{x}+a\dot{x}+kx = 0$, where a and k are constants. The Lagrangian is $L = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - az$. The fundamental invariance identity (3.6) of theorem 3.2 assumes the form

$$-kx\xi + \dot{x}\Big(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\dot{x} - \dot{x}\frac{\partial \tau}{\partial t} - \dot{x}^2\frac{\partial \tau}{\partial x}\Big) + \Big(\frac{1}{2}\dot{x}^2 - \frac{k}{2}x^2 - az\Big)\Big(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial x}\dot{x}\Big) = 0.$$

The system of partial differential equations for τ and ξ is obtained by equating to zero the coefficients in front of z, $\dot{x}z$, and powers of $\dot{x}$. The solution of this system is $\tau = constant$ and $\xi = 0$. Without loss of generality, we take $\tau = 1$. Applying the first Noether-type theorem 1.1 we obtain the corresponding conserved quantity

$$Q = -e^{at}\Big(\frac{1}{2}\big(\dot{x}^2 + kx^2\big) + az\Big).$$

It will be nice to express the conserved quantity in terms of x and $\dot{x}$ only, and for this we need to express z in terms of x and $\dot{x}$. The reader can check that $z = \frac{1}{2}x\dot{x}$ satisfies the defining equation for z, namely $\dot{z} = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - az$, with x being a solution of the damped harmonic oscillator. Thus, the conserved quantity is

$$e^{at}\Big(\dot{x}^2 + ax\dot{x} + kx^2\Big) = constant. \tag{4.1}$$

This method produces no non-trivial variational symmetries of the Lienard's equation (2.5), except in the case when $g(t) = constant$, which is the case of the damped harmonic oscillator presented above.

As another application we calculate a variational symmetry group of the Liouville's equation $\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} = 0$ with a specific choice of the coefficient functions, namely

$$f(x) = \frac{h}{kx + a} \qquad g(t) = \frac{c}{2kt + b} \tag{4.2}$$

where a, b, c, h, and k are arbitrary constants (if $k = 0$ then a and b must be non-zero). As noted in section 1, this equation can be given a variational description via the Herglotz variational principle if the functional z is defined by the differential equation

$$\dot{z} = \frac{1}{2}\dot{x}^2 - (2f(x)\dot{x} + g(t))z.$$

The fundamental invariance identity (3.6) takes the form

$$-\frac{dg}{dt}z\tau - 2\frac{df}{dx}\dot{x}z\xi + (\dot{x} - 2f(x)z)\Big(\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\dot{x} - \dot{x}\frac{\partial\tau}{\partial t} - \dot{x}^2\frac{\partial\tau}{\partial x}\Big) +$$
$$+ \Big(\frac{1}{2}\dot{x}^2 - 2f(x)\dot{x}z - g(t)z\Big)\Big(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x}\dot{x}\Big) = 0.$$

With the specific choices (4.2) for $f(x)$ and $g(t)$ the system of partial differential equations obtained from this identity after equating to zero the proper coefficients has the solutions $\xi = kx + a$ and $\tau = 2kt + b$. Thus, the variational symmetry of the Liouville's equation produced by this method is

$$\bar{x} = x + (kx + a)\varepsilon\ , \quad \bar{t} = t + (2kt + b)\varepsilon. \tag{4.3}$$

The corresponding conserved quantity of the Liouville's equation is obtained through an application of theorem 1.1, and is

$$Q = \Big(\frac{kx(t) + a}{kx(0) + a}\Big)^{2h/k}\Big(\frac{2kt + b}{b}\Big)^{c/2k}\Big(\dot{x}(kx + a) - (2kt + b)\frac{\dot{x}^2}{2} - (2h + c)z\Big). \tag{4.4}$$

As a last application, we calculate a variational symmetry for the equation

$$\ddot{x} + \frac{2}{t}\dot{x} + \frac{1}{x^3} = 0\ .$$

In this case the functional z is defined by the equation

$$\dot{z} = \frac{1}{2}\dot{x}^2 + \frac{1}{2x^2} - \frac{2}{t}z\ ,$$

and the fundamental invariance identity (3.6) assumes the form

$$2\frac{1}{t^2}z\tau - \frac{1}{x^3}\xi + \dot{x}\Big(\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\dot{x} - \dot{x}\frac{\partial\tau}{\partial t} - \dot{x}^2\frac{\partial\tau}{\partial x}\Big)$$

$$+\Big(\frac{1}{2}\dot{x}^2 + \frac{1}{2x^2} - \frac{2}{t}z\Big)\Big(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x}\dot{x}\Big) = 0.$$

The system of PDE's for the coefficients $\tau(t,x)$ and $\xi(t,x)$ of the infinitesimal generator of the variational symmetry group has the solution $\tau = 2kt$ $\xi = kx$, where k is an arbitrary constant. The corresponding conserved quantity is

$$Q = -k\frac{t^2}{e^2}\bigg(\Big(\dot{x}^2 - \frac{1}{x^2}\Big)t - x\dot{x} + 4z\bigg). \tag{4.5}$$

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Department of Analytical Mechanics
Faculty of Mathematics and Informatcs
Sofia University "St. Kliment Ohridski"
1164 Sofia, BULGARIA
E-mail: georgieva@fmi.uni-sofia.bg

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COMBINATORY SPACES VERSUS OPERATIVE SPACES WITH STORAGE OPERATION

JORDAN ZASHEV

Some versions of the notion of operative space with storage operation were used previously [10, 11] for uniform treatment of both theories of operative and combinatory spaces [2, 7]. In this paper we show that the scope of this notion is essentially greater than that considered in [10]. Formally, we describe a general categorical model of operative spaces with storage operation and specify some particular cases of this model, which cannot be directly treated by operative and combinatory spaces. On the other hand, these examples arise naturally in an attempt to comprise in the sense of algebraic recursion theory some important kinds of nondeterministic computing notions like that of quantum (and more generally reversible) one, which were not treated before in the last theory.

Keywords: Algebraic recursion theory, combinatory space, operative space, coherence space

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1. INTRODUCTION

Combinatory spaces of Skordev [6, 7] were the first system of algebraic recursion theory, a branch of abstract recursion theory based on a specific algebraic treatment of least fixed points. Later on, the same approach was employed for other algebraic systems by L. Ivanov and the present author, in a search for the best such system for which the approach in question works. Among these systems, the operative spaces of Ivanov [2] are the most remarkable. They are natural and simple objects with a huge variety of models, being in a sense very near to practically arising ideas and universes for computation. There is, however, a practically important operation which can not be directly treated by operative spaces; that is the operation of encoding a pair of data objects into one such object. This operation can be viewed as a computable retraction of the set X of data objects on its cartesian square $X \times X$, and in this sense the notion of operative space does not directly handle the cartesian product $\times$. That is why the notions of computability like that of Moschovakis are difficult to comprise immediately by operative spaces. The notion of combinatory space, however, can handle in this sense the cartesian product, while, on the other hand, it does not work well (or does not work at all) with more general kinds of products, and in particular with the tensor product of Hilbert spaces which has to be used, instead, for the corresponding treatment of quantum computing. In the present paper we show how the notion of combinatory space can be modified (or rather generalized) in order to avoid this difficulty. For that purpose we use the categorical language, and thus indicate also that there is a large variety of natural models for the notion of storage operation in an operative space besides those given by the operation of translation in an iterative one, as well as by other inductively definable operations of similar kind.

In this way we propose a revision of the notion of combinatory space, replacing it by another one called regular OSS below. The last notion almost coincides with the notion of operative space with strong storage operation from [11] and is a special case of that of intensional combinatory space from [10]. It avoids some basic algebraic disadvantages of the notion of combinatory space like using constants for data objects, and the projection objects L and R, which obstructs the treatment of reversible computing notions, retaining in the same time its capacity to express various possible forms of (tensor) product operations.

2. OPERATIVE SPACES AND STORAGE OPERATIONS

We shall remind the basic notions of operative space and storage operation in such spaces. Detailed information about these notions and their role in algebraic recursion theory is available in the book [2] and the papers [9, 10]. By *protoring* we shall mean a set $\mathcal{R}$ with two binary operations and three constants I, T, F such that:

1) $\mathcal{R}$ is a monoid with unit I w.r.t. one of the operations called multiplication and denoted by juxtaposition;

2) the other operation denoted by $[-,-]$ satisfies the identities $\chi[\varphi,\psi] = [\chi\varphi,\chi\psi]$, $[\varphi,\psi]T=\varphi$, and $[\varphi,\psi]F=\psi$ for all elements φ, ψ, χ of $\mathcal{R}$.

A *storage operation* in a protoring $\mathcal{R}$ is a quadruple $(\$, D, P, Q)$ consisting of a unary operation $\$$ and three constants D, P, Q in $\mathcal{R}$ such that the following equalities hold for all $\varphi,\psi \in \mathcal{R}$:

$$\$(\varphi\psi) = \$(\varphi)\$(\psi); \tag{2.1}$$

$$\$([\varphi, \psi]) = [\$(\varphi), \$(\psi)]D; \tag{2.2}$$

$$\$(\$(\varphi)) = Q\$(\varphi)P; \tag{2.3}$$

$$[T\$(I), F\$(I)]D = D\$(I). \tag{2.4}$$

Below we shall often use the shorthand $\hat{\varphi}$ for $\$(\varphi)$. A storage operation in a protoring $\mathcal{R}$ will be called *regular*, iff it satisfies the equalities

$$\$(\varphi)T = T\varphi, \quad \$(\varphi)F = F\$(\varphi) \tag{2.5}$$

for all $\varphi \in \mathcal{R}$.

An *operative space* (shortly OS) is a partially ordered protoring, i.e. protoring with a partial order $\leq$ in the set of its elements, such that the basic binary operations of this algebra are increasing w.r.t. $\leq$ in both arguments. Similarly, by *operative space with storage* (OSS) we mean operative space in which a storage operation $(\$, D, P, Q)$ with increasing first component $\$$ is given, and when the last operation is regular we shall say that the OSS in question is *regular*.

An OSS $\mathcal{F}$ is called *iterative*, iff for every $\varphi \in \mathcal{F}$ the inequality $[I, \xi]\varphi \leq \xi$ has least solution $\mathbf{I}(\varphi)$ w.r.t. ξ in $\mathcal{F}$, such that for every $\alpha \in \mathcal{F}$ the element $\alpha\mathbf{I}(\varphi)$ is the least solution of $[\alpha, \xi]\varphi \leq \xi$, and the equality

$$\$(\mathbf{I}(\varphi)) = \mathbf{I}(D\$(\varphi))$$

holds for every $\varphi \in \mathcal{F}$. The iterative regular OSS are special case of iterative intensional combinatory spaces from [10]. Hence the results of [10] imply that the basic theorems of algebraic recursion theory (inductive completeness and the normal form theorem) hold in all iterative regular OSS $\mathcal{F}$ in which the following condition is fulfilled:

(S7) For all elements α,β,φ of $\mathcal{F}$, s.t. the inequalities $\vartheta \leq [I, \alpha\$(\vartheta)]\beta$ and $\vartheta \leq [I, \vartheta]\varphi$ imply $[I, \vartheta]\varphi \leq [I, \alpha\$([I, \vartheta]\varphi)]\beta$ for all $\vartheta \in \mathcal{F}$, we have the inequality $\mathbf{I}(\varphi) \leq [I, \alpha\$(\mathbf{I}(\varphi))]\beta$.

The last condition is rather weak, and is fulfilled in all cases in which the existence of the operation $\mathbf{I}$ can be established by the usual methods. (The normal form theorem holds even without (S7)).

Iterative operative spaces of Ivanov [2] provide a general and natural model for iterative regular OSS, the operation $\$$ being interpreted as translation. Weakly iterative combinatory spaces ([11]) in which $(L, R) = I$ are also a special case of iterative regular OSS.

3. CATEGORICAL MODELS OF REGULAR OSS

As mentioned in [9], protorings can be described categorically as follows. Consider a category $\mathcal{C}$, an object X of $\mathcal{C}$ for which the coproduct $X+X$ exists, and a retraction $r : X \to X+X$ with section $s : X+X \to X$ in $\mathcal{C}$. Then we have a protoring $\mathbf{R}(\mathcal{C}, X, r)$ whose elements are the $\mathcal{C}$-morphisms $\varphi : X \to X$ with the composition in $\mathcal{C}$ as multiplication, the identity 1_X of X in $\mathcal{C}$ as unit, and the second binary operation and the constants T, F defined by $[\varphi, \psi] = [\varphi, \psi]_+ \circ r$, $T = s \circ I_0$ and $F = s \circ I_1$ respectively, where I_0 and I_1 are the canonical injections $X \to X+X$ of the last sum and $[\varphi, \psi]_+ : X+X \to X$ is the unique arrow in $\mathcal{C}$ such that $[\varphi, \psi]_+ \circ I_0 = \varphi$ and $[\varphi, \psi]_+ \circ I_1 = \psi$. Conversely, every protoring $\mathcal{R}$ can be regarded as one-object category with the multiplication in $\mathcal{R}$ as composition law and the unit I as the identity arrow; the Karoubi envelope $\mathcal{K}$ of the last category consists of all elements $\varepsilon \in \mathcal{R}$ such that $\varepsilon^2 = \varepsilon$ as objects, and all elements $\varphi \in \mathcal{R}$ such that $\eta\varphi\varepsilon = \varphi$ as arrows $\varphi : \varepsilon \to \eta$. The category $\mathcal{K}$ has binary coproducts $\varepsilon + \eta = [T\varepsilon, F\eta]$ with canonical injections $T\varepsilon : \varepsilon \to [T\varepsilon, F\eta]$ and $F\eta : \eta \to [T\varepsilon, F\eta]$. In particular, we have a retraction $[T, F] : I \to I + I$ in the last category with section $[T, F]$; and the protoring $\mathbf{R}(\mathcal{K}, I, [T, F])$ coincides with the original one $\mathcal{R}$. Thus all protorings are of the form $\mathbf{R}(\mathcal{C}, X, r)$; and working with categories enriched over the category of posets we get a similar description of operative spaces.

Below we shall extend this observation to obtain a large class of models for regular OSS. Let $\mathcal{C}$ be a category with binary coproducts. We denote by $I_0 = I_0(X_0, X_1)$ and $I_1 = I_1(X_0, X_1)$ the canonical injections $I_i : X_i \to X_0 + X_1$ of the coproduct $X_0 + X_1$ in $\mathcal{C}$; they are natural in $X_0, X_1 \in \mathcal{C}$. We shall use to omit the arguments X_0, X_1 in I_0 and I_1, as well as in all natural transformations occurring below; this can not create confusion since these arguments can be obviously restored in every expression involving such transformations in order to make this expression meaningful. Similarly, we shall write 1_X for the identity arrow of an object $X \in \mathcal{C}$, often omitting the subscript X. For every two arrows $f_i : X_i \to X$ in $\mathcal{C}$ we denote by $[f_0, f_1]_+$ the unique arrow $f : X_0 + X_1 \to X$ for which $f \circ I_i = f_i$ for both $i = 0, 1$. Suppose there are a bi-endofunctor $\otimes$ and two, natural in $X, Y, Z \in \mathcal{C}$, transformations

$$\underline{a}_\otimes : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \tag{3.1}$$

and

$$\bar{d}_\otimes : X \otimes (Y + Z) \to X \otimes Y + X \otimes Z$$

in $\mathcal{C}$ such that

i) $\underline{a}_\otimes$ is a retraction with section $\bar{a}_\otimes : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ also natural in X, Y, Z;

ii) the arrow

$$\underline{d}_\otimes = [1 \otimes I_0, 1 \otimes I_1]_+ : X \otimes Y + X \otimes Z \to X \otimes (Y + Z)$$

is a retraction with section $\bar{d}_\otimes$, i.e. $\underline{d}_\otimes \circ \bar{d}_\otimes = 1$.

Then given an object X of $\mathcal{C}$ and two retractions $r_+ : X \to X + X$ and $r_\otimes : X \to X \otimes X$ with sections s_+ and $s_\otimes$ respectively, we have the following:

Proposition 1. *There is a storage operation* $(\$, D, P, Q)$ *in the protoring* $\mathbf{R}(\mathcal{C}, X, r_+)$ *defined by*

$$\begin{aligned}
\$(\varphi) &= s_\otimes \circ (1 \otimes \varphi) \circ r_\otimes \\
D &= s_+ \circ (s_\otimes + s_\otimes) \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
P &= s_\otimes \circ (s_\otimes \otimes 1) \circ \bar{a}_\otimes \circ (1 \otimes r_\otimes) \circ r_\otimes \\
Q &= s_\otimes \circ (1 \otimes s_\otimes) \circ \underline{a}_\otimes \circ (r_\otimes \otimes 1) \circ r_\otimes .
\end{aligned}$$

Proof. A direct calculation. Here are the details for the identities (2.2)–(2.4), the identity (2.1) being obvious. For (2.2):

$$\begin{aligned}
\$([\varphi, \psi]) &= s_\otimes \circ (1 \otimes [\varphi, \psi]_+ \circ r_+) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes [I, I]_+ \circ (\varphi + \psi) \circ r_+) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes [I, I]_+) \circ (1 \otimes (\varphi + \psi)) \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes [I, I]_+) \circ \underline{d}_\otimes \circ (1 \otimes \varphi + 1 \otimes \psi) \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes [I, I]_+) \circ [1 \otimes I_0, 1 \otimes I_1]_+ \\
&\quad \circ (r_\otimes \circ \hat{\varphi} \circ s_\otimes + r_\otimes \circ \hat{\psi} \circ s_\otimes) \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_\otimes \circ [1 \otimes I, 1 \otimes I]_+ \circ (r_\otimes \circ \hat{\varphi} + r_\otimes \circ \hat{\psi}) \circ (s_\otimes + s_\otimes) \\
&\quad \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_\otimes \circ [(1 \otimes 1) \circ r_\otimes \circ \hat{\varphi}, (1 \otimes 1) \circ r_\otimes \circ \hat{\psi}]_+ \circ r_+ \circ D \\
&= [s_\otimes \circ r_\otimes \circ \hat{\varphi}, s_\otimes \circ r_\otimes \circ \hat{\psi}] \circ D \\
&= [\hat{\varphi}, \hat{\psi}] \circ D;
\end{aligned}$$

for (2.3):

$$\begin{aligned}
\$^2(\varphi) &= s_\otimes \circ (1 \otimes s_\otimes \circ (1 \otimes \varphi) \circ r_\otimes) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes s_\otimes) \circ (1 \otimes (1 \otimes \varphi)) \circ (1 \otimes r_\otimes) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes s_\otimes) \circ \underline{a}_\otimes \circ ((1 \otimes 1) \otimes \varphi)) \circ \bar{a}_\otimes \circ (1 \otimes r_\otimes) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes s_\otimes) \circ \underline{a}_\otimes \circ ((r_\otimes \circ s_\otimes) \otimes \varphi)) \circ \bar{a}_\otimes \circ (1 \otimes r_\otimes) \circ r_\otimes \\
&= s_\otimes \circ (1 \otimes s_\otimes) \circ \underline{a}_\otimes \circ (r_\otimes \otimes 1) \circ (1 \otimes \varphi) \circ (s_\otimes \otimes 1) \\
&\quad \circ \bar{a}_\otimes \circ (1 \otimes r_\otimes) \circ r_\otimes \\
&= Q \circ s_\otimes \circ (1 \otimes \varphi) \circ r_\otimes \circ P \\
&= Q\$(\varphi)P;
\end{aligned}$$

for (2.4):

$$\begin{aligned}
[T\$(I), F\$(I)]D &= [s_+ \circ I_0 \circ s_\otimes \circ r_\otimes, s_+ \circ I_1 \circ s_\otimes \circ r_\otimes]_+ \circ r_+ \circ D \\
&= [s_+ \circ I_0 \circ s_\otimes \circ r_\otimes, s_+ \circ I_1 \circ s_\otimes \circ r_\otimes]_+ \circ (s_\otimes + s_\otimes) \\
&\quad \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= [s_+ \circ I_0 \circ s_\otimes, s_+ \circ I_1 \circ s_\otimes]_+ \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= s_+ \circ [I_0, I_1]_+ \circ (s_\otimes + s_\otimes) \circ \bar{d}_\otimes \circ (1 \otimes r_+) \circ r_\otimes \\
&= D = D \circ s_\otimes \circ r_\otimes = D\$(I). \qquad \square
\end{aligned}$$

Let $\mathcal{C}$ be a category with a bi-endofunctor $\otimes : \mathcal{C}^2 \to \mathcal{C}$ and associativity transformation (3.1) which is a natural isomorphism with inverse $\bar{a}_\otimes$, and with a unit w.r.t. $\otimes$, i.e. an object $\mathbf{1}$ of $\mathcal{C}$ such that the isomorphisms $X \otimes \mathbf{1} \cong X \cong \mathbf{1} \otimes X$ hold naturally in $X \in \mathcal{C}$. We shall call such categories *premonoidal* (the usual definition of monoidal category requiring some coherence conditions to be fulfilled, which are not supposed for a premonoidal one), and we shall denote by $\bar{e}_\otimes$ and $\underline{e}_\otimes$ the canonical isomorphism $X \to \mathbf{1} \otimes X$ and its inverse respectively, omitting the argument X as usual. All categories occurring below, for which the opposite is not especially stated, will be supposed to have ω-coproducts, i.e. all coproducts $\sum\limits_{n \in A} X_n$ of families of objects X_n indexed by a set A of natural numbers. The canonical injections $X_i \to \sum\limits_{n \in A} X_n$ of such coproducts will be denoted by I_n; they are natural in $(X_n) \in \mathcal{C}^A$ and as usual we shall omit to write the arguments of the natural transformations I_n. A premonoidal category $\mathcal{C}$ with such coproducts will be called *preclosed*, iff the canonical natural transformations

$$\delta : \sum_i (X \otimes Y_i) \to X \otimes \sum_i Y_i, \qquad \delta' : \sum_i (Y_i \otimes X) \to (\sum_i Y_i) \otimes X \tag{3.2}$$

determined by the condition that $\delta \circ I_i = 1 \otimes I_i$ and $\delta' \circ I_i = I_i \otimes 1$ for all i, respectively, are isomorphisms. We shall say that an object X of a preclosed category $\mathcal{C}$ is *strictly reflexive*, iff it satisfies the isomorphisms $X \otimes X \cong X \cong X + X \cong \mathbf{1} + X$, and X will be called *reflexive* iff $X \otimes X \cong X$ and both $X + X$ and $\mathbf{1} + X$ are retracts of X in $\mathcal{C}$.

It is very easy to construct reflexive objects in this sense in preclosed categories. In fact, every object of such category can be extended to a strictly reflexive one in the following sense:

Proposition 2. *In every preclosed category $\mathcal{C}$ there is an endofunctor R and a natural in X transformation $X \to R(X)$ such that the object $R(X)$ is strictly reflexive for every object $X \in \mathcal{C}$.*

Proof. The isomorphisms (3.2) imply the isomorphism

$$\sum_{i=0}^{\infty} X_i \otimes \sum_{j=0}^{\infty} Y_j \cong \sum_{n=0}^{\infty} \sum_{i=0}^{n} (X_i \otimes Y_{n-i})$$

for all sequences X_i and Y_j of objects. On the other hand, for every object X the coproduct $X_\omega = \mathbf{1} + X + \mathbf{1} + X + \cdots$ satisfies the isomorphisms

$$X_\omega \cong X_\omega + X_\omega \cong \mathbf{1} + X_\omega,$$

whence for the progression

$$R(X) = \mathbf{1} + X_\omega + X_\omega \otimes X_\omega + X_\omega \otimes X_\omega \otimes X_\omega + \cdots$$

we have

$$\begin{aligned} R(X) + R(X) &\cong \mathbf{1} + \mathbf{1} + X_\omega + X_\omega + X_\omega \otimes X_\omega + X_\omega \otimes X_\omega + \cdots \\ &\cong \mathbf{1} + X_\omega + X_\omega \otimes (X_\omega + X_\omega) + \cdots \cong R(X) \end{aligned}$$

and

$$\begin{aligned} R(X) \otimes R(X) &\cong (\mathbf{1} + X_\omega + X_\omega \otimes X_\omega + \cdots) \otimes (\mathbf{1} + X_\omega + X_\omega \otimes X_\omega + \cdots) \\ &\cong \mathbf{1} + X_\omega + X_\omega \\ &\quad + (X_\omega \otimes X_\omega + X_\omega \otimes X_\omega + X_\omega \otimes X_\omega) + \cdots \cong R(X). \end{aligned}$$ □

Note that the morphism $X \to R(X)$ in the last Proposition is a canonical injection of certain coproduct, and hence a monomorphism (even section of a retraction) in very general suppositions for the category $\mathcal{C}$ and the object X. (For instance, it suffices to assume that the category $\mathcal{C}$ has terminal object $\mathbf{t}$ and a morphism $\mathbf{t} \to X$.)

Theorem 1. *Every reflexive object X in a preclosed category $\mathcal{C}$ canonically determines a protoring $\mathcal{F}(\mathcal{C}, X)$ with regular storage operation.*

Proof. Consider the protoring $\mathcal{R} = \mathbf{R}(\mathcal{C}, X, r_+)$,where $r_+ : X \to X + X$ is the retraction given with X as reflexive object, together with a section s_+ of it. Similarly, let $r_\otimes$ and $s_\otimes$ be the isomorphism $X \to X \otimes X$ and its inverse, respectively, and let $r_1 : X \to \mathbf{1} + X$ be the retraction with a section s_1, which are given with X. We have a natural in $Y \in \mathcal{C}$ transformation

$$\vartheta(Y) = ((s_1 \circ I_0) \otimes 1_Y) \circ \bar{e}_\otimes : Y \to X \otimes Y.$$

Indeed, for every $\mathcal{C}$-arrow $\varphi : Y \to Z$ we have

$$\begin{aligned}(1_X \otimes \varphi) \circ \vartheta(Y) &= (1_X \otimes \varphi) \circ ((s_1 \circ I_0) \otimes 1_Y) \circ \bar{e}_\otimes \\ &= ((s_1 \circ I_0) \otimes 1_Z) \circ (1_1 \otimes \varphi) \circ \bar{e}_\otimes \\ &= ((s_1 \circ I_0) \otimes 1_Z) \circ \bar{e}_\otimes \circ \varphi = \vartheta(Z) \circ \varphi.\end{aligned}$$

Denote by $\underline{d}^\otimes$ and $\bar{d}^\otimes$ the canonical natural in $Y, Z, W \in \mathcal{C}$ isomorphism

$$\underline{d}^\otimes = [I_0 \otimes 1, I_1 \otimes 1]_+ : Y \otimes W + Z \otimes W \to (Y + Z) \otimes W$$

and its inverse respectively, and define an arrow

$$G = s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ \bar{d}^\otimes \circ (r_1 \otimes 1) \circ r_\otimes : X \to X$$

as the obvious composition of the string

$$X \to X \otimes X \to (\mathbf{1} + X) \otimes X \to \mathbf{1} \otimes X + X \otimes X \to X + X \to X.$$

Similarly, define the arrows $T', F' : X \to X$ by

$$T' = s_\otimes \circ \vartheta(X), \qquad F' = s_\otimes \circ (\zeta \otimes 1) \circ r_\otimes,$$

where $\zeta = s_1 \circ I_1 : X \to X$. The arrows G, T' and F' are elements of the protoring $\mathcal{R}$ satisfying in it the equalities $GT' = T$ and $GF' = F$. Indeed, we have

$$\begin{aligned}GT' &= s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ \bar{d}^\otimes \circ (r_1 \otimes 1) \circ r_\otimes \circ s_\otimes \circ \vartheta(X) \\ &= s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ \bar{d}^\otimes \circ (r_1 \otimes 1) \circ ((s_1 \circ I_0) \otimes 1) \circ \bar{e}_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ \bar{d}^\otimes \circ (I_0 \otimes 1) \circ \bar{e}_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ \bar{d}^\otimes \circ \underline{d}^\otimes \circ I_0 \circ \bar{e}_\otimes = s_+ \circ (\underline{e}_\otimes + r_\otimes) \circ I_0 \circ \bar{e}_\otimes \\ &= s_+ \circ I_0 \circ \underline{e}_\otimes \circ \bar{e}_\otimes = s_+ \circ I_0 = T\end{aligned}$$

and

$$\begin{aligned}GF' &= s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ \bar{d}^\otimes \circ (r_1 \otimes 1) \circ r_\otimes \circ s_\otimes \circ (s_1 \otimes 1) \circ (I_1 \otimes 1) \circ r_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ \bar{d}^\otimes \circ (I_1 \otimes 1) \circ r_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ \bar{d}^\otimes \circ \underline{d}^\otimes \circ I_1 \circ r_\otimes \\ &= s_+ \circ (\underline{e}_\otimes + s_\otimes) \circ I_1 \circ r_\otimes \\ &= s_+ \circ I_1 \circ s_\otimes \circ r_\otimes = s_+ \circ I_1 = F.\end{aligned}$$

Then we have a protoring $\mathcal{F}$ with the same set of elements, multiplication operation and unit as $\mathcal{R}$; the second binary operation defined by

$$[\varphi, \psi]' =_{def} [\varphi, \psi]G,$$

where $[-,-]$ is the corresponding operation of $\mathcal{R}$; and the elements T' and F' as the basic constants T and F, respectively. Indeed,

$$[\varphi,\psi]'T' = [\varphi,\psi]GT' = [\varphi,\psi]T = \varphi,$$

and similarly $[\varphi,\psi]'F' = \psi$ for all $\varphi,\psi \in \mathcal{F}$. By Proposition 1, there is a storage operation $(\$, D, P, Q)$ in $\mathcal{R}$. Defining $D' = [T', F']D\$(G)$, we have a storage operation $(\$, D', P, Q)$ in $\mathcal{F}$, because

$$\$([\varphi,\psi]') = \$([\varphi,\psi]G) = [\$(\varphi),\$(\psi)]D\$(G) = [\$(\varphi),\$(\psi)]'[T',F']D\$(G)$$

for all $\varphi,\psi \in \mathcal{F}$, and

$$\begin{aligned} [T'\$(I), F'\$(I)]'D' &= [T'\$(I), F'\$(I)]'[T',F']D\$(G) \\ &= [T'\$(I), F'\$(I)]D\$(G) \\ &= [T',F'][T\$(I), F\$(I)]D\$(G) \\ &= [T',F']D\$(I)\$(G) = D'\$(I). \end{aligned}$$

The storage operation $(\$, D', P, Q)$ is regular since

$$\hat{\varphi}T' = s_\otimes \circ (1\otimes\varphi) \circ r_\otimes \circ s_\otimes \circ \vartheta = s_\otimes \circ (1\otimes\varphi) \circ \vartheta = s_\otimes \circ \vartheta \circ \varphi = T'\varphi$$

and

$$\begin{aligned} \hat{\varphi}F' &= s_\otimes \circ (1\otimes\varphi) \circ r_\otimes \circ s_\otimes \circ (\zeta\otimes 1) \circ r_\otimes = s_\otimes \circ (1\otimes\varphi) \circ (\zeta\otimes 1) \circ r_\otimes \\ &= s_\otimes \circ (\zeta\otimes 1) \circ (1\otimes\varphi) \circ r_\otimes = s_\otimes \circ (\zeta\otimes 1) \circ r_\otimes \circ s_\otimes \circ (1\otimes\varphi) \circ r_\otimes \\ &= F'\hat{\varphi} \end{aligned}$$

for all $\varphi \in \mathcal{F}$. □

Definition. *We say that a preclosed category is* partially ordered, *iff a partial order is given in every* hom-set *such that the composition,* $\otimes$ *and the* ω*-coproduct functors are increasing (in all arguments together) w.r.t. this partial order. For a partially ordered preclosed category we say that it is* continuous, *iff in every* hom-*set there is least element* o *such that* $\varphi \circ o = o$ *for every suitable arrow* φ *and* $\psi \otimes o = o$ *for every arrow* ψ*, and in every* hom-*set the suprema of increasing sequences exist and are preserved in each argument by the composition,* $\otimes$ *and the coproducts. If in each* hom-*set of a partially ordered preclosed category every chain* C *has least upper bound* $\sup C$ *such that the equalities* $\varphi \circ \sup C = \sup(\varphi \circ C)$ *and* $\psi \otimes \sup C = \sup(\psi \otimes C)$ *hold for all suitable arrows* φ *and* ψ*, we shall say that this category is* semicontinuous. *Finally, if the last condition holds for the composition (not necessarily for* $\otimes$*) and the element* $\mathbf{1}$ *is a* $\otimes$*-generator for the category in question (in the following sense: for every pair of arrows* $f, g : X \otimes Y \to Z$ *such* *that* $f \circ (x \otimes y) = g \circ (x \otimes y)$ *for all arrows* $x : \mathbf{1} \to X$ *and* $y : \mathbf{1} \to Y$, *we have the equality* $f = g$, *then we shall say that this partially ordered preclosed category is* quasisemicontinuous.

Theorem 2. *Let* $\mathcal{C}$ *be a partially ordered preclosed category. Then for every reflexive object* X *of* $\mathcal{C}$ *the canonically generated protoring* $\mathcal{F}(\mathcal{C}, X)$ *is an iterative regular OSS satisfying* (S7) *provided someone of the following three conditions holds:*

i) $\mathcal{C}$ *is continuous*;

ii) $\mathcal{C}$ *is semicontinuous and the morphism* $r_\otimes$ *satisfies the equality*

$$(\sup C) \circ r_\otimes = \sup(C \circ r_\otimes)$$

for every chain C *in every suitable* hom-*set of* $\mathcal{C}$;

iii) $\mathcal{C}$ *is quasisemicontinuous.*

Proof. The set of elements of $\mathcal{F}(\mathcal{C}, X)$ is the hom-set $\mathcal{C}(X, X)$, whence it is partially ordered by the partial order of $\mathcal{C}$, and $\mathcal{F} = \mathcal{F}(\mathcal{C}, X)$ obviously forms a regular OSS w.r.t. this partial order. If $\mathcal{C}$ is continuous, then there is least element $o \in \mathcal{F}$ such that $\varphi o = o$ for all $\varphi \in \mathcal{F}$. Moreover all increasing sequences have suprema in $\mathcal{F}$ which is preserved in each argument by the multiplication and the operations $[-,-]$ and \$. Therefore for every $\varphi \in \mathcal{F}$ the element $\mathbf{I}(\varphi) = \sup \varphi_n$, where φ_n is the sequence defined by $\varphi_0 = o$ and $\varphi_{n+1} = [I, \varphi_n]\varphi$, is the least solution of the inequality $[I, \xi]\varphi \leq \xi$ w.r.t. ξ in $\mathcal{F}$. For arbitrary $\alpha \in \mathcal{F}$ the sequence $\psi_n = \alpha\varphi_n$ satisfies $\psi_0 = o$ and $\psi_{n+1} = [\alpha, \psi_n]\varphi$ for all n, whence the least solution of $[\alpha, \xi]\varphi \leq \xi$ is $\sup \psi_n = \alpha\mathbf{I}(\varphi)$. Moreover the sequence $\chi_n = \hat{\varphi}_n$ satisfies the equalities $\chi_0 = o$ and $\chi_{n+1} = [I, \chi_n]D\hat{\varphi}$, and therefore

$$\mathbf{I}(D\hat{\varphi}) = \sup \hat{\varphi}_n = \$(\sup \varphi_n) = \$(\mathbf{I}(\varphi)).$$

Suppose $\mathcal{C}$ satisfies condition ii). Then for all $\varphi, \alpha \in \mathcal{F}$ we have a transfinite increasing sequence $\psi_i(\alpha, \varphi) \in \mathcal{F}$ uniquely determined by the condition

$$\psi_i(\alpha, \varphi) = \sup_{j<i} [\alpha, \psi_j(\alpha, \varphi)]\varphi \tag{3.3}$$

and $\psi_i(\alpha, \varphi) \leq [\alpha, \psi_i(\alpha, \varphi)]\varphi$ for all $i < k$ where k is a cardinal number greater than the power of $\mathcal{F}$. Then the element $\psi_j(\alpha, \varphi)$ is the least solution of $[\alpha, \xi]\varphi \leq \xi$, where $j < k$ is any ordinal number among those for which $\psi_j(\alpha, \varphi) = \psi_{j+1}(\alpha, \varphi)$. In particular, for $\alpha = I$ denote this least solution by $\mathbf{I}(\varphi)$. Then the supposition of semicontunuity of $\mathcal{C}$ implies $\psi_i(\alpha, \varphi) = \alpha\psi_i(I, \varphi)$ for all $i < k$, which shows that $\alpha\mathbf{I}(\varphi)$ is the least solution of $[\alpha, \xi]\varphi \leq \xi$. Similarly, applying \$ to (3.3) and using the supposition for $r_\otimes$ we obtain , $\$(\psi_i(I, \varphi)) = \psi_i(I, D\hat{\varphi})$ whence $\$(\mathbf{I}(\varphi)) = \mathbf{I}(D\hat{\varphi})$.

Now, when $\mathcal{C}$ is quasisemicontinuous, we see in the same way that $\alpha\mathbf{I}(\varphi)$ is the least solution of $[\alpha, \xi]\varphi \leq \xi$. Hence, also, the inequality

$$\mathbf{I}(D\hat{\varphi}) \leq \$(\mathbf{I}(\varphi)),$$

because $[I, \$(\mathbf{I}(\varphi))]D\hat{\varphi} = \$([I, \mathbf{I}(\varphi)]\varphi) = \$(\mathbf{I}(\varphi))$. For arbitrary $\mathcal{C}$-arrow $x : \mathbf{1} \to X$ the arrow

$$\vartheta_Y(x) = (x \otimes 1_Y) \circ \bar{e}_{\otimes} : Y \to X \otimes Y$$

is a natural in $Y \in \mathcal{C}$ transformation such that $\hat{\varphi}x^* = x^*\varphi$ for all $\varphi \in \mathcal{F}$ where $x^* : X \to X$ is the arrow $x^* = s_{\otimes} \circ \vartheta_X(x)$. Then for all $\varphi, \psi \in \mathcal{F}$ the following will be true: if $\varphi x^* = \psi x^*$ for every $x : \mathbf{1} \to X$, then $\varphi = \psi =$ Indeed, for arbitrary arrow $y : \mathbf{1} \to X$ we have

$$x^*y = s_{\otimes} \circ (x \otimes 1) \circ \bar{e}_{\otimes} \circ y = s_{\otimes} \circ (x \otimes 1) \circ (1 \otimes y) \circ \bar{e}_{\otimes} = s_{\otimes} \circ (x \otimes y) \circ \bar{e}_{\otimes};$$

therefore $\varphi x^* = \psi x^*$ implies $\varphi \circ s_{\otimes} \circ (x \otimes y) \circ \bar{e}_{\otimes} = \psi \circ s_{\otimes} \circ (x \otimes y) \circ \bar{e}_{\otimes}$, whence

$$\varphi \circ s_{\otimes} \circ (x \otimes y) = \psi \circ s_{\otimes} \circ (x \otimes y).$$

By the supposition that $\mathbf{1}$ is a $\otimes$-generator this shows that when $\varphi x^* = \psi x^*$ holds for all x we have $\varphi \circ s_{\otimes} = \psi \circ s_{\otimes}$, and therefore $\varphi = \psi$ since $I = s_{\otimes} \circ r_{\otimes}$.

Denote by $[-,-]_0$, T_0, F_0 and D_0 the operation $[-,-]$ and the constants T, F and D in the protoring $\mathbf{R}(\mathcal{C}, X, r_+)$, respectively (see Proposition 1). Then we have

$$\begin{aligned} D_0 x^* &= s_+ \circ (s_{\otimes} + s_{\otimes}) \circ \bar{d}_{\otimes} \circ (1 \otimes r_+) \circ r_{\otimes} \circ s_{\otimes} \circ (x \otimes 1) \circ \bar{e}_{\otimes} \\ &= s_+ \circ (s_{\otimes} + s_{\otimes}) \circ \bar{d}_{\otimes} \circ (x \otimes 1) \circ (1 \otimes r_+) \circ \bar{e}_{\otimes} \\ &= s_+ \circ (s_{\otimes} + s_{\otimes}) \circ (x \otimes 1 + x \otimes 1) \circ \bar{d}_{\otimes} \circ \bar{e}_{\otimes} \circ r_+ \\ &= s_+ \circ (s_{\otimes} \circ (x \otimes 1) + s_{\otimes} \circ (x \otimes 1)) \circ \bar{d}_{\otimes} \circ \bar{e}_{\otimes} \circ r_+. \end{aligned}$$

But

$$\begin{aligned} \underline{d}_{\otimes} \circ (\bar{e}_{\otimes} + \bar{e}_{\otimes}) &= [1 \otimes I_0, 1 \otimes I_1]_+ \circ (\bar{e}_{\otimes} + \bar{e}_{\otimes}) \\ &= [(1 \otimes I_0) \circ \bar{e}_{\otimes}, (1 \otimes I_1) \circ \bar{e}_{\otimes}]_+ \\ &= [\bar{e}_{\otimes} \circ I_0, \bar{e}_{\otimes} \circ I_1]_+ = \bar{e}_{\otimes} \circ [I_0, I_1]_+ = \bar{e}_{\otimes}, \end{aligned}$$

whence $\bar{d}_{\otimes} \circ \bar{e}_{\otimes} = \bar{e}_{\otimes} + \bar{e}_{\otimes}$. Therefore

$$\begin{aligned} D_0 x^* &= s_+ \circ (s_{\otimes} \circ (x \otimes 1) + s_{\otimes} \circ (x \otimes 1)) \circ (\bar{e}_{\otimes} + \bar{e}_{\otimes}) \circ r_+ \\ &= s_+ \circ (x^* + x^*) \circ r_+ = s_+ \circ [I_0 \circ x^*, I_1 \circ x^*]_+ \circ r_+ \\ &= [s_+ \circ I_0 \circ x^*, s_+ \circ I_1 \circ x^*]_+ \circ r_+ = [T_0 x^*, F_0 x^*]_0. \end{aligned}$$

Then in the protoring $\mathcal{F}$ we have:

$$\begin{aligned} Dx^* &= [T,F]_0 D_0 \$(G) x^* = [T,F]_0 D_0 x^* G = [T,F]_0 [T_0 x^*, F_0 x^*]_0 G \\ &= [Tx^*, Fx^*]_0 G = [Tx^*, Fx^*]. \end{aligned}$$

Hence for all $\varphi \in \mathcal{F}$ and all arrows $x : \mathbf{1} \to X$ we have the following equalities in the protoring $\mathcal{F}$:

$$\begin{aligned} [x^*, \mathbf{I}(D\hat{\varphi})x^*]\varphi &= [I, \mathbf{I}(D\hat{\varphi})][Tx^*, Fx^*]\varphi = [I, \mathbf{I}(D\hat{\varphi})]Dx^*\varphi \\ &= [I, \mathbf{I}(D\hat{\varphi})]D\hat{\varphi}x^* = \mathbf{I}(D\hat{\varphi})x^*, \end{aligned}$$

which implies $\$(\mathbf{I}(\varphi))x^* = x^*\mathbf{I}(\varphi) \leq \mathbf{I}(D\hat{\varphi})x^*$ since $x^*\mathbf{I}(\varphi)$ is the least solution of $[x^*, \xi]\varphi \leq \xi$. Then the inequality $\mathbf{I}(D\hat{\varphi}) \leq \$(\mathbf{I}(\varphi))$ shows that

$$\$(\mathbf{I}(\varphi))x^* = \mathbf{I}(D\hat{\varphi})x^*$$

for all $x : \mathbf{1} \to X$. This, as we have already seen, implies $\$(\mathbf{I}(\varphi)) = \mathbf{I}(D\hat{\varphi})$.

The condition (S7) follows from Proposition 3 in [10] in all of the cases i)–iii). □

4. COHERENCE SPACES

Coherence spaces ([1], sometimes called Girard domains) are well known objects used for semantical treatment of linear logic and other systems of typed lambda calculus. We shall note here that they form a continuous preclosed category w.r.t. so called linear maps, thus giving by Theorem 2 various models for iterative regular OSS. It is essential to stress the intuitive interpretation of these models in terms of some kind of data processing which preserves information, indicating in this way their naturalness, and hence importance for abstract recursion theory.

In detail, we define *coherence spaces* as pairs $(X, \sim)$ consisting of a set X and a binary reflexive and symmetric relation $\sim$ in X. (We shall write also X for $(X, \sim)$ and $\sim_X$ for $\sim$.) A *linear* map $f : X \to Y$ of coherence spaces $X = (X, \sim_X)$ and $(Y, \sim_Y)$ is a multivalued mapping $f : X \to 2^Y$ such that:

1) f is coherently injective in the sense that $y \in f(x) \cap f(x')$ and $x \sim_X x'$ imply $x = x'$ for all $x, x' \in X$ and $y \in Y$;

2) f preserves $\sim$ in the sense that $x \sim_X x'$, $y \in f(x)$ and $y' \in f(x')$ imply $y \sim_Y y'$ for all $x, x' \in X$ and $y, y' \in Y$.

The following Theorem is a (special case of a) well known result.

Theorem 3. **LCoh** *is a continuous preclosed category.*

Proof. We remind only the definitions of the components of the structure of continuous preclosed category in **LCoh**, omitting the straightforward details. The category **LCoh** has coproducts for all families $X_i \in \mathbf{LCoh}$ of objects $X_i = (X_i, \sim_i)$ defined by $\sum_i X_i = (X, \sim)$ where $X =_{def} \bigcup_i(\{i\} \times X_i)$ and

$$(i, x) \sim (j, y) \Leftrightarrow_{def} i = j \,\&\, x \sim_i y.$$

The canonical injections $I_i : X_i \to X$ of these coproducts are $I_i(x) = \{(i, x)\}$. The tensor product of two coherence spaces $X = (X, \sim_X)$ and $Y = (Y, \sim_Y)$ is defined by $X \otimes Y = (X \times Y, \sim)$ where

$$(x, y) \sim (x', y') \Leftrightarrow_{def} x \sim_X x' \,\&\, y \sim_Y y';$$

and the tensor product of two arrows $f : X \to X'$ and $g : Y \to Y'$ in **LCoh** is $(f \otimes g)(x, y) =_{def} f(x) \times g(y)$. The associativity maps $\underline{a}_\otimes : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ and its inverse are $\underline{a}_\otimes((x, y), z) =_{def} \{(x, (y, z))\}$ and $\bar{a}_\otimes(x, (y, z) =_{def} \{((x, y), z)\}$, respectively. The object **1** is defined as the one-element coherence space, and the natural isomorphisms $\underline{e}_\otimes$ and $\bar{e}_\otimes$ are obvious. The distributivity map

$$\underline{d}_\otimes : \sum_i (Z \otimes X_i) \to Z \otimes \sum_i X_i$$

is the unique arrow for which $\underline{d}_\otimes \circ I_i = 1_Z \otimes I_i$ i.e. $\underline{d}_\otimes(i, (z, x)) = \{(z, (i, x))\}$; and its inverse is defined by $\bar{d}_\otimes(z, (i, x)) = \{(i, (z, x))\}$. **LCoh** is a symmetric premonoidal category in the sense that the isomorphism $X \otimes Y \cong Y \otimes X$ holds naturally in $X, Y \in \mathbf{LCoh}$. The partial order for parallel arrows $f, g : X \to Y$ in $LCoh$ is defined by

$$f \leq g \Leftrightarrow_{def} \forall x \in X (f(x) \subseteq g(x)).$$

The least element $o : X \to Y$ in a hom-set is the empty multivalued map, i.e. $o(x) = \emptyset$ for all $x \in X$. The least upper bound of an increasing sequence $f_n : X \to Y$ is given by $(\sup f_n)(x) = \bigcup_{n=0}^{\infty} f_n(x)$. □

Thus every reflexive coherence space X canonically gives rise, according to Theorems 1 and 2, to an iterative regular OSS $\mathcal{F} = \mathcal{F}(\mathbf{LCoh}, X)$ whose elements are the linear mappings $\varphi : X \to X$. Intuitively, the elements of X can be regarded as data units considered not as quite separate entities, but rather in a context of some 'internal' information which connects them with a set of other such units, the connection relation thus arising being represented by $\sim$. Accordingly, the elements of $\mathcal{F}$ are regarded as a mathematical idealization describing a kind of nondeterministic processing of those units which takes care not to annihilate internal information by identifying (transforming to identical) units which are connected with $\sim$, and, on the other hand, preserves the 'external' information of connectedness of these units with each other. This is the intuitive interpretation meant above.

On the other hand, no reasonable way is seen to treat these models or, more precisely, the notion of computability associated with the OSS $\mathcal{F}(\mathbf{LCoh}, X)$, by means of combinatory spaces. The abstract notion of combinatory space requires the data objects $x, y, \dots$ to be presented as elements of the abstract structure in question, as well as a pairing operation $x, y \mapsto (x, y)$ and projections L and R restoring the components of pairs in the sense of the equalities $L(x, y) = x$ and $R(x, y) = y$. This is incompatible with the idea of nondistinctness of the elements of a coherence space; formally, the interpretation of L and R is obstructed by the fact that the set-theoretical projections $X \otimes Y \to X$ and $X \otimes Y \to Y$ of coherence spaces are generally not coherently injective.

5. MODELS OF OSS INVOLVING THE IDEA OF IMPLEMENTATION

Implementation of computations, which practically means physical simulation (theoretically or philosophically other nonphysical realizations may be possible), is an important issue, being fundamental for the modern computer development. It is even hard to separate the theoretical notion of computability from the idea of implementation, as it is seen, for instance, in the notion of Turing machine; and one of the aims of abstract theory of computation is to find a mathematical idealization which characterizes this notion in its pure form, independently of concrete realizations. In the present section we shall indicate how the idea of implementation suggests some natural and mathematically interesting models of iterative regular OSS.

The physical simulation of a computational process requires the data object to be encoded into the state of a physical system (which may generally depend on this object), the time evolution of which is used for modeling the process in question. The keeping of the information conveyed by such object x into the physical system has some energy cost $e(x)$ measured by the energy which will be dissipated into the environments if the information in question is erased; and the cost $e(x)$ is to be supposed proportional to the quantity of information conveyed by x ([5]). This cost has to remain unchanged during the process of computation, which is a basic requirement of the so called reversible computing. Thus the computational process in question may be characterized by a partial function f defined exactly for those data objects x for which the process terminates and satisfying in this case the equality $e(f(x)) = e(x)$. This suggests to consider the category $\mathbf{E}(M)$ defined below. Note that formally it is not essential that $e(x)$ is just this energy cost; one may conceive of any physical invariant of the state encoding the data into the physical system in question. It suffices to suppose that the values of e can be operated algebraically by some operation called addition and having the usual properties, except the commutativity law.

Now let M be a monoid (not necessarily commutative) with basic operations written additively, and denote by $\mathbf{E}(M)$ the category with objects the functions

$e : X \to M$ and arrows with source $e : X \to M$ and target $e' : X' \to M$ the partial mappings $f : X \to X'$ such that $e'(f(x)) = e(x)$ whenever $f(x)$ is defined.

Theorem 4. *The category* $\mathbf{E}(M)$ *can be naturally provided with a structure of continuous preclosed one.*

Proof. The details being quite straightforward, we shall indicate only the required structure. The category $\mathbf{E}(M)$ has coproducts for all families of objects $e_i : X_i \to M$ defined as follows. Consider the coproduct $X = \bigcup_i \{i\} \times X_i$ of the family X_i in **Set** with canonical injections $I_i(x) = (i, x)$ for all $x \in X_i$ and all i. Then the unique mapping $e : \bigcup_i \{i\} \times X_i \to M$ such that $e(i, x) = e_i(x)$ for all $x \in X_i$ and all i is an object of $\mathbf{E}(M)$ which is coproduct of the family e_i with canonical injections I_i (the same as in **Set**). For arbitrary family of $\mathbf{E}(M)$-arrows $f_i : e_i \to e'$ with target the object $e' : X' \to M$ of $\mathbf{E}(M)$ the unique $\mathbf{E}(M)$-arrow $f : e \to e'$ such that $f \circ I_i = f_i$ for all i is the partial mapping $f : X \to X'$ for which $f(I_i(x))$ is defined and equals $f_i(x)$ when $f_i(x)$ is defined and $f(I_i(x))$ is undefined otherwise.

For every two objects $e_0 : X_0 \to M$ and $e_1 : X_1 \to M$ of $\mathbf{E}(M)$ define an object $e_0 \otimes e_1 : X_0 \times X_1 \to M$ by

$$(e_0 \otimes e_1)(x_0, x_1) = e_0(x_0) + e_1(x_1).$$

Given a pair of morphisms $f_0 : e_0 \to e'_0$ and $f_1 : e_1 \to e'_1$ in $\mathbf{E}(M)$ with targets $e'_i : X'_i \to M$, $i = 0, 1$, define the morphism $f_0 \otimes f_1 : e_0 \otimes e_1 \to e'_0 \otimes e'_1$ as the partial mapping $f_0 \otimes f_1 : X_0 \times X_1 \to X'_0 \times X'_1$ such that $(f_0 \otimes f_1)(x_0, x_1)$ is defined for a pair $(x_0, x_1) \in X_0 \times X_1$ iff both $f_0(x_0)$ and $f_1(x_1)$ are, and in the last case

$$(f_0 \otimes f_1)(x_0, x_1) = (f_0(x_0), f_1(x_1)).$$

This defines a bi-endofunctor in $\mathbf{E}(M)$ which is naturally associative, the associativity isomorphisms $\underline{a}_{\otimes}$ being the same as in **Set**. The object $\mathbf{1}$ is defined as the function $\mathbf{1} : \{0\} \to M$ which sends the element 0 into the neutral element of the monoid M. For arbitrary object $e : X \to M$ of $\mathbf{E}(M)$ the projections $\underline{e}^{\otimes} : X \times \{0\} \to X$ and $\underline{e}_{\otimes} : \{0\} \times X \to X$ define isomorphisms $e \otimes \mathbf{1} \to e$ and $\mathbf{1} \otimes e \to e$ in $\mathbf{E}(M)$ which are natural in e. Given a sequence of objects $e : X \to M$, $e_i : X_i \to M$ $(i \in N)$ of $\mathbf{E}(M)$, the distributivity isomorphisms

$$\bar{d}_{\otimes} : e \otimes (e_0 + e_1 + \cdots) \to e \otimes e_0 + e \otimes e_1 + \cdots$$

and

$$\bar{d}^{\otimes} : (e_0 + e_1 + \cdots) \otimes e \to e_0 \otimes e + e_1 \otimes e + \cdots$$

are defined by $\bar{d}_{\otimes}(x, (i, y)) = (i, (x, y))$ and $\bar{d}^{\otimes}((i, y), x) = (i, (y, x))$ respectively for all $x \in X, y \in X_i$ and $i \in N$. If the monoid M is commutative, then the category $\mathbf{E}(M)$ is a symmetric premonoidal one. The partial order in a hom-set of $\mathbf{E}(M)$ is the relation of extension of partial functions, the least element is the function with empty domain, and the suprema of increasing sequences of morphisms is the union of the corresponding partial functions. □

A reflexive object $e : X \to M$ of $\mathbf{E}(M)$ has necessarily (as a consequence of the isomorphism $e \otimes e \cong e$) to have a binary operation $x, y \mapsto \langle x, y\rangle$ in X which maps $X \times X$ bijectively on X and satisfies the equality

$$e(\langle x, y\rangle) = e(x) + e(y),$$

in accordance with the requirement that the energy cost $e(x)$ is proportional to the quantity of information contained in x. The elements of the iterative regular OSS $\mathcal{F} = \mathcal{F}(\mathbf{E}(M), e)$ arising canonically from e are the partial functions $f : X \to X$ preserving the energy cost in the sense that $e(f(x)) = e(x)$ whenever $f(x)$ is defined. As in the case with coherence spaces in the previous section, the treatment of the OSS $\mathcal{F}$ by means of combinatory spaces is obstructed by the fact that the projections $L(\langle x, y\rangle) = x$ and $R(\langle x, y\rangle) = y$ do not generally preserve the energy cost.

Another idea is to describe the physical simulation of a computational process by a mathematical idealization involving the time evolution operator of the physical system through which the process is simulated. This operator can be conceived, in accordance with the requirement of reversibility, as an isomorphism in a certain category. For instance, in the case of quantum computation, the physical system in question is mathematically a Hilbert space, mostly finite dimensional; and we shall use for the mentioned category that one which has finite dimensional Hilbert spaces as objects and isometrical linear operators as morphisms. In the case of 'classical' computation we use the category of finite sets instead (or rather finite sets of special kind – the sets of subsets of finite sets – the usual registers being representable as sets of units which can have two possible states).

To detail this idea consider a premonoidal category $\mathcal{K}$ with tensor product bi-endomorphism $\otimes_K$ and unit-object $\mathbf{1}_K$, not necessarily having ω-coproducts. Let $\mathbf{P}(\mathcal{K})$ be the category with objects the pairs (B, K) consisting of a set B and a mapping K which assigns to every element $b \in B$ an object $K(b)$ of $\mathcal{K}$, and with arrows $(B, K) \to (B', K')$ the pairs (f, φ) of two partial functions with the same domain $D_f \subseteq B$ such that the values of f are in B' and for every $b \in D_f$ the value $\varphi(b)$ is an isomorphism $\varphi(b) : K(b) \to K'(f(b))$. The composition of two arrows $(f, \varphi) : (B, K) \to (B', K')$ and $(g, \psi) : (B', K') \to (B'', K'')$ in $\mathbf{P}(\mathcal{K})$ is

$$(g, \psi) \circ (f, \varphi) =_{def} (g \circ f, \psi(f)\varphi),$$

where $g \circ f : B \to B''$ is the composition of partial functions and

$$(\psi(f)\varphi)(b) = \psi(f(b)) \circ \varphi(b) : K(b) \to K''(g(f(b)))$$

for all $b \in B$ such that $(g \circ f)(b)$ is defined. The identity $1_{(B,K)}$ of an object (B, K) is the pair $(1_B, \iota_K)$ of the identity map 1_B of B and the mapping assigning to each $b \in B$ the identity $\iota_K(b) = 1_{K(b)}$ of $K(b)$ in $\mathcal{K}$.

Theorem 5. *The category* $\mathbf{P}(\mathcal{K})$ *can be naturally provided with a structure of continuous preclosed one.*

Proof. The proof is straightforward and similar to that of Theorem 4. As before, we shall indicate the components of the required structure. Let $(B_n, K_n) \in \mathbf{P}(\mathcal{K})$ be a countable family of objects. The coproduct of this family in $\mathbf{P}(\mathcal{K})$ is defined as the pair $(B, K) \in \mathbf{P}(\mathcal{K})$ where $B = \sum_{n=0}^{\infty} B_n$ is the coproduct in **Set** and $K(i_n(b)) = K_n(b)$ for all $b \in B_n$ and all n, and $i_n : B_n \to B$ are the canonical injections of the coproduct in **Set** so that B is the disjoint union of the sets $i_n(B_n)$. The canonical injections $I_n : (B_n, K_n) \to (B, K)$ of the coproduct in question are the pairs $I_n = (i_n, \iota_n)$ where $\iota_n(b) : K_n(b) \to K(i_n(b))$ is the identity arrow for all n and all $b \in B_n$. Given a family

$$(f_n, \varphi_n) : (B_n, K_n) \to (B', K')$$

of arrows in $\mathbf{P}(\mathcal{K})$, the unique arrow $(f, \varphi) : (B, K) \to (B', K')$ such that

$$(f, \varphi) \circ I_n = (f_n, \varphi_n)$$

for all n is the pair of partial functions such that $f(i_n(b))$ and $\varphi(i_n(b))$ are defied whenever $f_n(b)$ is, and in this case $f(i_n(b)) = f_n(b)$ and

$$\varphi(i_n(b)) = \varphi_n(b) : K(i_n(b)) = K_n(b) \to K'(f_n(b))$$

for all $b \in B_n$ and all n.

The tensor product $(B_0, K_0) \otimes (B_1, K_1)$ of two objects of $\mathbf{P}(\mathcal{K})$ is defined as the pair $(B_0 \times B_1, K_0 \otimes K_1)$ where $(K_0 \otimes K_1)(b_0, b_1) = K_0(b_0) \otimes_K K_1(b_1)$ for every pair $(b_0, b_1) \in B_0 \times B_1$. The tensor product $(f_0, \varphi_0) \otimes (f_1, \varphi_1)$ of two morphisms

$$(f_j, \varphi_j) : (B_j, K_j) \to (B'_j, K'_j), \quad j = 0, 1$$

is the pair (f, φ) where f and φ are the partial mappings defined for those $(b_0, b_1) \in B_0 \times B_1$ for which both $f_j(b_j)$ are defined with values the pair

$$f(b_0, b_1) = (f_0(b_0), f_1(b_1)) \in B'_0 \times B'_1$$

and the isomorphism

$$\varphi(b_0, b_1) = \varphi_0(b_0) \otimes_K \varphi_1(b_1) : K_0(b_0) \otimes_K K_1(b_1) \to K'_0(f_0(b_0)) \otimes_K K'_1(f_1(b_1))$$

respectively. The associativity isomorphisms $\bar{a}_\otimes : X_0 \otimes (X_1 \otimes X_2) \to (X_0 \otimes X_1) \otimes X_2)$, where $X_j = (B_j, K_j) \in \mathbf{P}(\mathcal{K})$ for all $j = 0, 1, 2$ are $\bar{a}_\otimes =_{def} (\bar{a}_\times, \bar{\alpha}_\otimes)$ where $\bar{a}_\times$ is the usual associativity isomorphism for the cartesian product $\times$ in **Set**, and

$$\bar{\alpha}_\otimes(b_0, (b_1, b_2)) : K_0(b_0) \otimes_K (K_1(b_1) \otimes_K K_2(b_2)) \to (K_0(b_0) \otimes_K K_1(b_1)) \otimes_K K_2(b_2)$$

is the corresponding associativity isomorphism for $\otimes_K$ in $\mathcal{K}$ for all $b_j \in B_j$, $j = 0, 1, 2$. The object $\mathbf{1}$ of $\mathbf{P}(\mathcal{K})$ is defined as $(\{0\}, u)$ where $u(0) = \mathbf{1}_K$. The natural isomorphism $\underline{e}_\otimes : \mathbf{1} \otimes (B, K) \to (B, K)$ is $\underline{e}_\otimes = (p, \underline{\varepsilon})$ where $p : \{0\} \times B \to B$ is the projection and $\underline{\varepsilon}(0, b) = \underline{e}_K(K(b)) : \mathbf{1}_K \otimes_K K(b) \to K(b)$ is the natural isomorphism given in $\mathcal{K}$; and similarly is defined the other natural in the object (B, K) isomorphism $\underline{e}^\otimes : (B, K) \otimes \mathbf{1} \to (B, K)$. The distributivity isomorphisms

$$\underline{d}_\otimes : \sum_{n=0}^{\infty}(X \otimes Y_n) \to X \otimes \sum_{n=0}^{\infty} Y_n$$

and

$$\underline{d}^\otimes : \sum_{n=0}^{\infty}(Y_n \otimes X) \to \sum_{n=0}^{\infty} Y_n \otimes X$$

are the unique morphisms $\underline{d}_\otimes$ and $\underline{d}^\otimes$ such that $\underline{d}_\otimes \circ I_n = 1 \otimes I_n$ and $\underline{d}^\otimes \circ I_n = I_n \otimes 1$ respectively for all n. The morphism $\underline{d}_\otimes$ has the form $(\underline{d}_\times, \iota_\otimes)$ where $\underline{d}_\times$ is the corresponding distributivity isomorphism in **Set** and $\iota_\otimes$ at every argument is the identity map of certain object of $\mathcal{K}$; hence $\underline{d}_\otimes$ is invertible, and similarly for $\underline{d}^\otimes$. Note that the category $\mathbf{P}(\mathcal{K})$ is symmetric w.r.t. $\otimes$ if $\mathcal{K}$ is such w.r.t. $\otimes_K$. The partial order in a hom-set of $\mathbf{P}(\mathcal{K})$ is defined as the relation of extension of functions, i.e. $(f, \varphi) \leq (g, \psi)$ iff g is extension of f and ψ – of φ. The least element is the pair of partial functions with empty domain, and the suprema of increasing sequences are the pairwise unions of the corresponding sequences of partial functions. □

As in the case with the category $\mathbf{E}(M)$, for every reflexive object $X = (B, K)$ of $\mathbf{P}(\mathcal{K})$ there is a binary operation $b, c \mapsto \langle b, c\rangle$ in B, obtained from the isomorphism $X \otimes X \cong X$, which maps $B \times B$ bijectively on B and satisfies the isomorphism $K(\langle b, c\rangle) \cong K(b) \otimes K(c)$ for all $b, c \in B$. Hence the projections $p_0(\langle b, c\rangle) = b$ and $p_1(\langle b, c\rangle) = c$ cannot be reasonably expected to satisfy

$$K(p_0(b)) \cong K(b) \cong K(p_1(b)).$$

This, as in the case with $\mathbf{E}(M)$, obstructs the treatment of (the notion of computability definable by) the OSS $\mathcal{F} = \mathcal{F}(\mathbf{P}(\mathcal{K}), X)$ through a combinatory space.

6. FINAL REMARKS

The notion of regular OSS in its previous version from [11] was a subject of some polemic since it does not comprise formally that of combinatory space in full generality. Instead of regular OSS one can use intensional combinatory spaces, of which the combinatory ones are special case ([10]). But the former spaces are somewhat complicated notion, and this complication seems not to be justified by the gain in generality it provides. The polemic started with a remark formulated in [11], which expressed this view. The last remark was objected in [8], but in some misleading way[1]. The real situation can be described as follows. The regular OSS do not comprise all combinatory spaces up to isomorphism in the usual algebraic sense, in which the basic operations are required to be preserved exactly. This is obvious since the equality $(L, R) = I$ is to be preserved by such isomorphisms of combinatory spaces, and it is not clear how we can treat the combinatory spaces in which this equality is violated as regular OSS. On the other hand, from the view point of recursion theory it is more natural to consider another kind of morphisms of iterative combinatory spaces and other similar objects of algebraic recursion theory, namely those which preserve all operations, including the inductively definable ones only up to explicit expressibility. These are the morphisms preserving the notion of computability, expressed by the given space, hence it is natural to call them recursive morphisms. So a natural question is whether the notion of iterative regular OSS can comprise that of iterative combinatory space up to recursive isomorphism. Generally, the question is open, but the works of Ivanov [3, 4] strongly suggest that the answer is positive. What is shown in [8] is that if we consider another kind of morphisms of combinatory spaces of hybrid nature, namely those which preserve one of the basic operations (multiplication) exactly, and the other ones only up to expressibility, then there is an example of iterative combinatory space (expressing a degenerate version of Moschovakis computability), which is not isomorphic in the hybrid sense to one in which $(L, R) = I$ holds, thus retaining the difficulty to be treated as regular OSS up to such isomorphism.

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[1]The author states in [8] that the remark in question fails under a formal interpretation he claims to have described there. This is not very meaningful since every nontautological unformal statement fails under suitable interpretation, as it follows from the completeness of predicate calculus and the well known observation that everything is formalizable in this calculus. Formally however, his claim is incorrect due to insufficient formalization.

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Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia
BULGARIA
E-mail: j_zashev@math.bas.bg