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Годишник на Софийския университет „Св. Климент Охридски"
Факултет по математика и информатика

Address for correspondence:

Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5, Blvd J. Bourchier, P.O. Box 48
BG-1164 Sofia, Bulgaria

Fax xx(359 2) 8687 180
Electronic mail:
annuaire@fmi.uni-sofia.bg

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НАУЧНОТО НАСЛЕДСТВО НА ПОАНКАРЕ И СЪВРЕМЕННАТА МАТЕМАТИКА

ЕМИЛ ХОРОЗОВ

The paper discusses the main contributions of Poincare to mathematics. On the basis of his work on authomorphic functions, celestial mechanics and topology we trace his enormous influence on modern mathematics.

В историята на науката има неголям брой гении, които са оставили в наследство за поколенията не само и не толкова блестящите си резултати, а преди всичко насоките в развитието на науката за десетки и стотици години. В математиката това без съмнение са Нютон, Ойлер, Гаус,Галоа, Риман... Спокойно можем да поставим в тази редица и Поанкаре, още повече, че не е лесно да се посочи математик с по-голямо влияние върху съвременната математика от него. По-долу ще се опитам да дам някои аргументи в подкрепа на тази теза, като си давам сметка, че когато се пише и говори по повод сто и петдесет годишнината от рождението му, авторът може и да е пристрастен.

Анри Поанкаре е не само математик, но и физик и философ. (В скоби ще отбележа, че той е ученият с най-много номинации за Нобелова награда по физика в периода 1901-1912 год.) Тук, обаче, ще се спра само върху математическото му творчество, не само поради „липса на място" , колкото поради липса на квалификация. Ясно е, освен това, че в текст като този научните описания ще бъдат опростени и повърхностни, за което се надявам специалистите да проявят разбиране.

Анри Поанкаре

В научно-философското си съчинение „Наука и хипотеза“ Поанкаре казва „Науката е изградена от факти, така както една къща е изградена от камъни, но натрупването на факти представлява толкова наука, колкото купчина камъни представлява къща.“

В този доклад ще се опитам да нахвърлям част от това, което Поанкаре е направил за подреждането и появането на камъните на науката в една постройка, както и за здравия ѝ фундамент. Ще се старая да споменавам факти, дължащи се на Поанкаре, само ако служат на тази цел. Нека специалистите, които останат с впечатление, че тяхната област е пренебрегната, ме извинят.

1. АВТОМОРФНИ ФУНКЦИИ

Първите изследвания на Поанкаре, донесли му световна слава, са изследванията му по фуксови функции (по терминологията на Поанкаре). Днес те са известни под името, дадено им от Ф. Клайн - автоморфни функции. Класът от фуксови функции според Поанкаре му е подсказан от Л. Фукс, който се опитва да изучи обикновени линейни диференциални уравнения с рационални коефициенти. В непълните и неточни работи на Л. Фукс Поанкаре намира средство да „реши всички линейни диференциални уравнения“ . Горните думи означават следното.

Поанкаре дефинира нов клас от трансцедентни функции (чрез решенията на някои класове уравнения от втори ред) и показва, че решенията на всички линейни диференциални уравнения (с алгебрични коефициенти),т.е. уравнения от вида:

$$y^n(z) + a_1(z,w)y^{n-1}(z) + \ldots + a_n(z,w)y(z) = 0, \tag{1}$$

където z, w са свързани с алгебрично съотношение $\Phi(z,w) = 0$, се представят с помощта на функции от този клас.

За да не се създаде погрешно впечатление бързам да кажа, че ролята на тези функции за диференциалните уравнения е скромна. А ролята им за цялата математика е трудно да се преувеличи. На това ще се спра по-късно. Все пак за компенсация ще припомня, че практически всяко занимание на Поанкаре е свързано по някакав начин с диференциалните уравниения. Но областта, в която работи Поанкаре не е „диференциални уравнения“ . Областта на Поанкаре е просто науката – и математика и физика, и философия. Опитите да се разглежда Поанкаре в един пласт може лесно да доведе до профанизиране на творчеството му.

Нека да се върна към автоморфните функции.

С няколко думи основните идеи, от които тръгва Поанкаре, се свеждат до следното. Разглеждаме диференциално уравнение от втори ред в комплексната област

$$y'' + p(z)y' + q(z)y = 0.$$

с рационални коефициенти. Нека $y_1(z)$ и $y_2(z)$ са две линейно независими решения в околност на неособена точка z_0 (т.е., в която коефициентите нямат полюси). Можем да продължаваме тези решения по всеки път в комплексната област. Интересен е случаят, когато пътят обикаля *особена точка* z_1 и се връща в неособената точка z_0. Двойката линейно-независими решения $y_1(z)$ и $y_2(z)$ преминава в друга двойка линейно-независими решения $w_1(z)$ и $w_2(z)$. Последните се изразяват като линейна комбинация на $y_1(z)$ и $y_2(z)$:

$$w_1(z) = \alpha y_1(z) + \beta y_2(z) \tag{2}$$
$$w_2(z) = \gamma y_1(z) + \delta y_2(z) \tag{3}$$

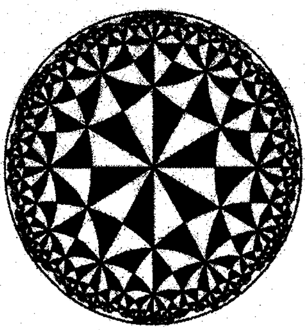
Но оттук следва, че отношението $w = w_1(z)/w_2(z)$ е дробно-линейна функция на отношението $y = y_1(z)/y_2(z)$:

$$w = \frac{\alpha y + \beta}{\gamma y + \delta}$$

С други думи при обикаляне около особената точка z_1 величината y претърпява дробно-линейна трансформация. Всички такива дробно-линейни трансформации образуват група G, която е дискретна подгрупа на $SL(2, \mathbb{C})$. А сега да разгледаме функцията $z(y)$, която е обратна на $y(z)$. Ясно е, че при смяна на y с $w = \Gamma(y)$, където Γ е дробно-линейна трансформация от групата G.

На това място е полезна аналогия с понятието периодична функция върху реалната права. При нея, ако знаем значението на функцията в интервал с дължина един период, функцията се възстановява върху цялата права. А именно с движения с дължина периода или кратни на периода можем да пренасяме графиката на функцията. Същото важи за елиптичните функции, т.е. двойно-периодичните меромерфни функции. Защо казвам тези тривиални неща? Защото класът от функции, открит от Поанкаре, е обобщение именно в тази плоскост. Преди това Фукс е стигнал до това, че някои функции, построени чрез решенията на линейни диференциални уравнения от втори ред, би трябвало да имат свойството, че ако в аргумента им направим дробно-линейна трансформация, принадлежаща на подходяща група, свързана с диференциалното уравнение, функцията не се променя. Но къде са дефинирани, дали са еднозначни, какви свойства имат и др. за Фукс не е ясно.

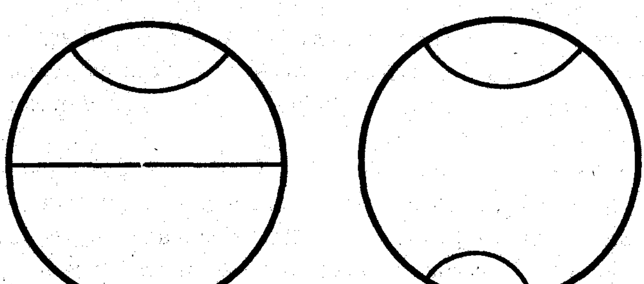
При търсенето на съответния аналог Поанкаре стига до едно забележително откритие – дробно-линейните трансформации, които изпращат единичния кръг в себе си, могат да се интерпретират като *движения в геометрията на Лобачевски*. До това всъщност просто заключение Поанкаре стига чрез работите си по квадратични форми – т.е. по теория на числата! Поанкаре намира и аналог на понятието „интервал с дължина период" . Той получава названието *фундаментална област* (Виж фиг. 1, на която всеки триъгълник е фундаментална област).



Фиг. 1. Единичният кръг, разбит на фундаментални области

Както при пренасянето на интервал върху реалната права можем да получим цялата права без да има застъпване на образите (освен в краищата), така и тук при пренасянето (но сега в геометрията на Лобачевски) на фундаменталната област можем да покрием напълно единичния кръг, но без образите да се застъпват. Тук обаче, за разлика от групите от дискретни движения върху реалната права, дискретните групи не се определят лесно. Нещо повече: това е следващия проблем, с който се сблъсква Поанкаре. Но за него няма да говоря.

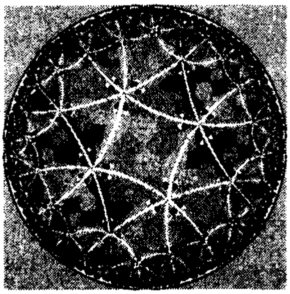
Така се появява едно от откритията на Поанкаре – този път в хиперболичната геометрия на Лобачевски. То носи названието модел на Поанкаре. В него правите са или диаметри или окръжности, перпендикулярни на единичната окръжност (фиг. 2).



Фиг. 2. Модел на Поанкаре

Добре е да припомня, че след Лобачевски и преди Поанкаре хиперболичната геометрия няма кой знае какво развитие. Тя е по-скоро екзотична област. Напротив, с модела на Поанкаре и с изследванията на Поанкаре (и Клайн) по автоморфни функции, тя става централен обект в математиката, какъвто е и сега.

Тук си струва да припомня творчеството на един от най-известните художници на 20 век - Мауритс Ешер. Предполагам, че някои от вас са виждали копия или може би оригинали на неговите рисунки. Но може би не много знаят, че знаменитата му серия - Circle limit (гранична окръжност, виж напр. [4]) е повлияна именно от модела на Поанкаре (фиг.3). Дори думата „повлияна“ е неточна – Ешер е изучавал с детайли модела, включително и с помощта на известния математик Кокстер. Рисунките са правени след внимателни пресмятания на базата на този модел.



Фиг. 3. Circle limite 1

Характерното за творчеството на Поанкаре е, че той не решава конкретно поставени задачи. Поанкаре изследва природата, иска да разбере философията на света. Като правило той няма хитроумни разсъждения. Дори конкретните си резултати той извлича след намирането на движещия механизъм на явлението. Така е и при автоморфните функции. След намирането на връзката с хиперболичната геометрия Поанкаре сравнително лесно получава всичко останало.

Нека резюмираме. За целите на диференциалните уравнения е развита теория, основана на комплексния анализ, теорията на групите, неевклидовата геометрия. Използвани са идеи от теория на числата. Още в началото на заниманията си Поанкаре забелязва, че автоморфните функции са много полезен инструмент в изучаването на римановите повърхнини (удивително е, че Поанкаре по онова време има неособено големи познания в тази област). Например чрез теорията на автоморфните функции се получава едно от доказателствата (това на Поанкаре) на теоремата за униформизацията.

Процесът на това откритие на Поанкаре е описан подробно от самия него в книгата му „Наука и метод“. Там той подчертава особено много ролята на интуицията, на безсъзнателното в научното творчество. Тъй като самонаблюденията му по тези въпроси са извънредно детайлни, той е изнасял и доклади пред Института по обща психология в Париж през 1908 г.,

Днес не можем да си представим математиката без автоморфните функции. Освен в теорията на римановите повърхнини [1] тя е съществена част и важен инструмент на теорията на числата [2]. Теорията на представянията на групи, централна област, пронизваща природо-математическите науки от аритметиката до квантовата химия, е практически невъзможна без нея [3].

На това място сигурно е подходящо да се спомене за една от най-грандиозните програми в математиката - програмата на Р. Ленглендс, свързваща теорията на числата, теорията на представянията и, разбира се, автоморфните функции. Невъзможно е тук да се даде и бегла представа за същността или значението на програмата. Но като страничен аргумент за важността ѝ ще спомена, че филдсовите лауреати – Владимир Дринфелд (за 1990) и Лоран Лафорг (за 2002), са удостоени с високата награда именно за изследвания в тази област.

Надявам се споменатото до тук да дава поне бегла представа за мястото на автоморфните функции в математиката и математическото естествознание.

2. НЕБЕСНА МЕХАНИКА

„Възможно е ключът към разбирането на творчеството на Поанкаре да дават неговите идеи в небесната механика и по-специално - в проблема за трите тела." Може би това мнение на известния историк на математиката Д. Стройк [5] да звучи преувеличено, но то подчертава широко възприетото мнение за фундаменталния характер на идеите на Поанкаре в тази област. Няма съмнение обаче, че работите на Поанкаре в задачата за трите тела са знакови за творчеството му.

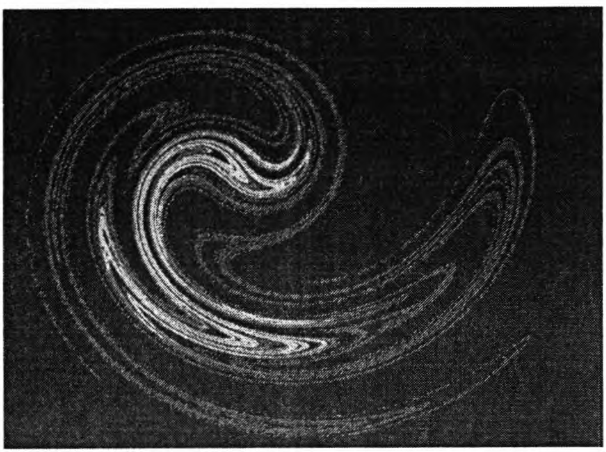
Най-напред ще припомня основната задача. Обектът на изследване е движението на планетите, техните спътници, както естествени като Луната, така и изкуствени. На по-научен език задачата звучи така. Три материални (т.е с ненулева маса) точки, например Слънце, Земя и Юпитер, се движат в пространството под действието на силите на привличане помежду им по закона на Нютон. Иска се да се опише движението. Тази размита постановка не е случайна – задачата всъщност няма строга формулировка. Изследователите сами биха могли да я допълват според интересите и силите си.

За разлика от задачата за двете тела, на която всички движения могат да се опишат детайлно и класифицират, тук намирането на някое движение, например периодично, вече е голям успех. Добре е да се отбележи, че задачата съдържа практически всички трудности присъщи на механични задачи и специално на задачи за много тела.

Като правило специалистите считат за главен резултат на Поанкаре в тази област доказателството за неинтегрумост на задачата за трите тела - например Вайерщрас. Това означава, че всеки опит да се напишат формули, описващи движенията на планетите, е обречен на неуспех. Този резултат е получен в резултат на фин анализ на резултатите на неговите предшественици Линдщедт, Болин и др. (главно астрономи), които получават „решенията" във вид на безкрайни редове. Поанкаре доказва, че редовете са разходящи. Поради това той съсредоточава вниманието си върху проблеми от качествен характер. Откъде обаче да започне? От най-простото, най-естественото, както е типично за Поанкаре. Това са периодичните решения, при които планетите след известно време се връщат точно в първоначаналното си положение. „...*Особената ценност на тези решения се състои в това, че те представляват единствения жалон, по който можем да проникнем в област, считана по-рано за недостъпна.*“ - така Поанкаре мотивира обекта на изследванията си. Той обаче не се задаволява просто да намери големи класове периодични решения - и преди него има намерени такива от Ойлер, Лагранж, Хил. Поанкаре наистина иска да проникне в областта считана по-рано за недостъпна. Новите периодични решения му дават възможност да открие нови явления, далеч надхвърлящи границите на небесната механика. Ще се спра на това, което е просто нова наука.

След изучаването на периодичните решения Поанкаре прави следващата стъпка – да разбере какви решения са свързани с тях по някакъв начин. Най-интересните се оказват решения, при които с течение на времето планетната система все повече се приближава до периодично движеща се, но освен това преди голям интетервал от време е приличала на същата периодична система. Тези решения са наречени от Поанкаре двойно-асимптотични. С тях завършва и тритомният трактат на Поанкаре „Нови методи в небесната механика“ , заемащ над 1000 страници.

При доказването на тяхното съществуване Поанкаре се сблъсква с ново явление. Не мога да се въздържа от цитирането на знаменит пасаж от съчинението на Поанкаре.



Фиг. 4. Хаос

„*Оставаш поразен от сложността на тази фигура, която даже не се опитвам да изобразя. Нищо не е по-подходящо за да ни даде представа за сложността на задачата на трите тела и въобще на задачите на динамиката, в която няма еднозначни интеграли и в която редовете на Болин са разходящи.*“

Специалистите вече са се досетили, че става дума за откриването на хаоса. Оказва се, че повечето движения всъщност не могат **принципиално** да се опишат индивидуално, и съвсем не заради нашето неумение – такава им е природата. Те трябва да се изучават в тяхната съвкупност, а не поотделно. Затворената обвивка на такова решение може да е множество с размерност по-голяма от едно. Днес теорията на хаоса е сред най-активно развиващите се области с много приложения – сред тях например са метеорологията, химическата кинетика и др.

Откриването на хаоса е свързано с един драматичен епизод в историята на науката и в иначе лишения от бурни събития живот на Поанкаре. Ще си позволя да го разкажа, тъй като е свързан с едно от основните съчинения на Поанкаре и с едно от откритията, които дават облика на съвременната математика (фиг. 4).

По повод 60-годишнината на краля на Швеция и Норвегия Оскар II се обявява международен конкурс „*за важно откритие в чистия математически анализ*" – това гласи съобщението в списанието Нейчър от 1885 г. Без да отивам в подробности ще припомня, че Поанкаре е бил обявен тържествено за един от двамата победители. По регламент статиите на победителите трябвало да се публикуват в едно от най-авторитетните математически списания по онова време, а и до сега - **Acta Mathematica**. При подготовката на списанието за печат младият помощник-редактор и по-късно известен математик Едвард Фрагмен открива неясни места в текста на Поанкаре и му ги съобщава. След като поправя съответните текстове обезпокоеният Поанкаре преглежда отново съчинението и открива доста по-сериозни грешки с грамадни последствия. През това време спианието вече е набрано и дори по-лошо – ограничен брой книжки са разпратени на отделни специалисти. Сред тях са членовете на журито – Вайерщрас и Ермит, астрономите - Гилден и Линдщедт, математиците Ковалевска, Ли. Поанкаре съобщава неприятните новини на председателя на журито Митаг-Лефлер. По молба на последния това остава в тайна между тях. И двамата имат сериозни врагове – сред тях са знаменитият математик Кронекер или маститият астроном Гилден, директор на стокхолмската обсерватория. За кратко време, около 2 месеца, Поанкаре практически написва нов труд с различни научни заключения. Погрешните твърдения на Поанкаре са водели до извода, че при планетните движения не се появяват хаотични движения. В новата редакция се появява голямото откритие на Поанкаре – съществуването на хаос в детерминирана система, в случая описвана с обикновени диференциални уравнения.

Поанкаре сам заплаща новото отпечатване на тома. Цената надвишава значително премията, получена от крал Оскар II.

А епизодът наистина остава в пълна тайна повече от сто години, до началото на деветдесетте години на 20 век, когато младата историчка на науката Джун Бароу-Грийн открива първия вариант на тома в Института Митаг-Лефлер, както и съответната кореспонденция.

Пак периодичните решения довеждат Поанкаре до открития, на които е съдено да играят основна роля в развитието на математиката и на теоретичната физика през 20-ти век. Става въпрос за топологията. Аз ще кажа нещо за топологията малко по-нататък, а сега искам да припомня части от разсъжденията на Поанкаре, с които той иска да установи съществуването на периодични решения в задачата за трите тела. Със съображения от механиката той свежда задачата до въпрос за критичните точки на функция върху двумерния тор. Всяка такава функция разбира се, има поне един максимум и един минимум. Следователно, заключава Поанкаре, функцията има и поне две инфлексни точки, което в конкретния случай води до 4 периодични решения. Това сбито разсъждение от два реда съдържа позоваване на топологията на тора и поспециално

1) характеристиката на Ойлер-Поанкаре (за нея ще стане дума по-долу);

2) елементи от теорията на Морс, построена 30 години по-късно от М. Морс,

т.е. до основни въпроси от несъздадената още топология. А заедно с последната геометрична теорема (това е последната статия на Поанкаре) горните въпроси водят до създаването на симплектичната топология, появила се 70 години по-късно след серия резултати и хипотези на Арнолд, както и работите на Громов, Фльор, Хофер и др. С други думи тук срещаме освен небесната механика и серия фрагменти от различни раздели на топологията – алгебрична, диференциална, симплектична. Но както често става с работите на Поанкаре, при него трудно се отделят конкретни области.

Един въпрос , до който достигат класиците Лагранж и Лаплас е въпросът: устойчива ли е Слънчевата система? Това е типичен въпрос от тематиката на задачата за трите тела (тук те са повече). Въпросът трябва да се разбира както в обикновенния, т.е. нематематически език – пита се дали планетите няма да избягат много далече от слънцето, дали няма да се приближават произволно близко до него или помежду си. Няма трудност да се даде и точната математическа формулировка – въпросът е дали решенията остават в някаква компактна област на фазовото пространство -всяко в своя. Знаменитата теорема на Лаплас казва, че Слънчевата система е устойчива, ако се пренебрегнат квадратите на масите. Това не много ясно твърдение означава следното. В задачата за трите тела допълнително се предполага, че масата на едно от телата - Слънцето - е много по-голяма от другите маси (на планетите). Например ако масата на Слънцето е единица, то масите на планетите са хилядни части от единицата. Решенията се записват с безкрайни редове, в които участват масите. Величините, в които масите участват чрез квадратите си просто зачеркваме. Действително те (квадратите на масите) ще бъдат (за Слънчевата система) милиони пъти по-малки от единица. Друг е въпросът, че те са коефициенти пред изрази, в които участвува времето и когато то расте, пренебрегнатите членове евентулно също растат.

Поанкаре се връща към тази теорема в съвършенно друга постановка, произлизаща от Поасон, който доказва, че ако се оставят квадратите, но се пренебрегнат кубовете, Слънчевата система е отново устойчива. Тук обаче смисълът на думата „устойчива“ е доста различен. Сега новото значение е, че планетната система вечно ще се връща близо до сегашното си положение, но планетите биха могли да се отдалечават колкото искаме или да се приближават произволно близко до Слънцето или помежду си. Поанкаре доказва далеч по-силен резултат - заключението на Поасон е вярно без да се пренебрегват кубовете или които и да било степени на масите. За стойността на този резултат ще припомня, че в надгробната реч на Пенлеве той е един от малкото споменати. Припомняйки наградата на шведския крал Пенлеве казва:

„През 1889 г. при съобщението на резултата от състезанието Франция научи с гордост, че златният медал ... е даден на французин, млад учен на 35 год. за блестящото изследване на устойчивостта на слънчевата система и името на Поанкаре стана известно навсякъде.“

Средствата , с които Поанкаре получава този блестящ резултат имат далеч по-голямо значение от самия резултат. Става въпрос за знаменитата теорема на Поанкаре за възвръщането, която има горе-долу същото звучене, но за произволна механична система, а също и за теорията на интегралните инварианти. Това са все общотеоретични резултати, лежащи в основата на ергодичната теория, важни за статистическата физика, за хидромеханиката и т.н.

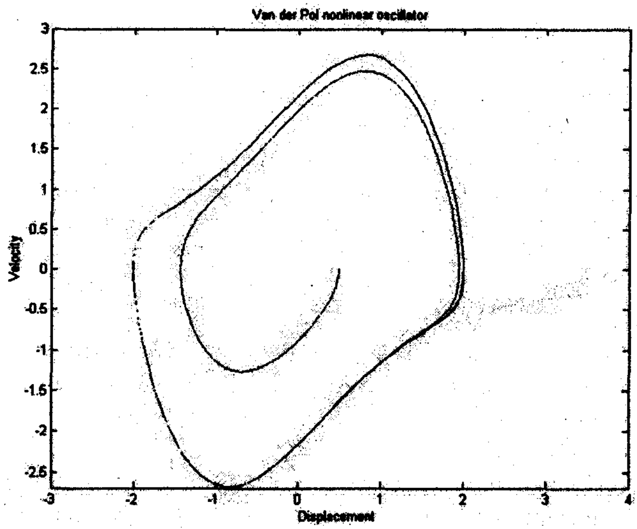
Впрочем въпросът за устойчивост на слънчевата система се оказа доста по-сложен. Въпреки огромното придвижване дължащо се на Поанкаре, въпросът на Лагранж и Лаплас трябваше да чака 70 години за да получи сравнително удовлетворителен отговор. Той е част на така-наречената КАМ-теория - по имената на Колмогоров, Арнолд и Мозер. Конкретният резултат принадлежи на тогава 25-годишния Арнолд и изказан „на пръсти“ гласи: с голяма вероятност слънчевата система е устойчива – приблизително 0.999 .

Тясно свързани с небесната механика са работите на Поанкаре по динамични системи или качествена теория, по-точно създаването й. Именно в контекста на качествената теория са голяма част от изследванията му по небесна механика – например тези по периодични решения и свързаните с тях. Далече преди Поанкаре е било ясно, че повечето диференциални уравнения не могат да се решат в никакъв смисъл. Станало е ясно, че трябва да се изучават решенията без да се решават уравнеията. Но предшествениците му (например Брио и Буке) не са имали убедителни примери. Вероятно защото не са били наясно кое е това, което трябва да се изучава. Поанкаре е успял да намери удивително прости и изключително важни геометрични обекти – фазов портрет на система, съставен от фазовите криви, т.е. кривите зададени от решенията и параметризирани с независимата променлива („времето“). За неспециалистите ще спомена, че това е изучаване на решенията в цялата им съвкупност. И тъкмо защото се въвежда геометрична картина можем да говорим за тяхното взаимно разположение. Една голяма част от тези изследвания сега е част от задължителния материал в обучението по математика не само за математици, но и за представители на други естествени науки, инженери икономисти и др.

Сред детайлите на фазовия портрет най важните са положенията на равновесие и отново периодичните решения. Последните са източник на много дълбоки и много приложни в истинския смисъл на думата изследвания – периодичните движения се срещат на всяка крачка – и в механиката и в радиотехниката и в икономиката. Например работата на радиолампата се описва с уравнението на Ван дер Пол:

$$\ddot{y} + (\mu + y^2)\dot{y} + \omega^2 y = 0$$

Съответната система има фазов портрет посочен на фиг. 5.



Фиг. 5. Трептения, описвани с уравнението на Ван дер Пол

От него се вижда, че едно от решенията е устойчив граничен цикъл – изолирано периодично решение, което „привлича" близките му решения. Един от знаменитите проблеми на Хилберт - шестнадесетият е посветен именно на периодичните решения и по-точно на „граничните цикли на Поанкаре" .

Изследванията в небесната механика имат още много следствия и връзки. Ще засегна и една област, стояща не чак толкова близко до топологията. При изследването на приливите и отливите Поанкаре достига до задачата с тангенциална към част от границата наклонена производна за оператора на Лаплас. Самият той не е успял да постигне успех в нея. Но работата на серия учени във втората половина на 20 век по този проблем стимулираха до известна степен развитието на теорията на псевдодиференциалните оператори, теорията на интегралните оператори на Фурие, деликатни и сложни аспекти на хармоничния анализ. Много от най-видните специалисти по ЧДУ са допринесли в това направление - Бицадзе, Хьормандер, Егоров, а в по-широк план - Стайн и учениците му.

Уместно е да се спомене, че българската математика има достоен принос в тези разработки. В серия работи акад. П. Попиванов и неговата школа установиха важни резултати, от които ще спомена само последния – на Попиванов и Кутев за съществуване и единственост на вискозно решение на тази задача за напълно нелинейно елиптично уравнение.

3. ТОПОЛОГИЯ

„Всички различни пътища, върху които аз последователно се намирах, ме водеха към Analysis situs (т.е. топологията, б.а.)" - пише Поанкаре в своето „Аналитично резюме" . В увода към първата си статия по топология това наблюдение е описано подробно. Поанкаре посочва три примера от своето творчество, които мотивират създаването на топологията. Първият е от алгебричната геометрия - класификацията на комплексни криви и повърхнини. Следващият пример представляват диференциалните уравнения и специално – тези, които описват небесната механика. Накрая Поанкаре посочва проблем от теорията на групите – определяне на крайните или дискретни групи, съдържащи се в общите линейни групи.

Топологията се занимава с геометрични свойства на криви, повърхнини, и т.н., които остават неизменни при деформация на геометричните обекти – при разтягане, огъване, но без късане или лепене. При нея например окръжност и триъгълник са едно и също.

Казват, че Поанкаре е измислил топологията, защото не умеел да рисува – още като ученик при него горните фигури били трудно различими. Аз обаче препоръчвам на тези, които мислят така да погледнат рисунките му свързани с автоморфни функции.

Основната задача на топологията е да класифицира геометричните обекти с точност до хомеоморфизми - това са именно посочените по-горе деформации. Би било несправедливо да кажем, че Поанкаре е започнал от празно място, че не е имал предшественици. Такъв е Ойлер със задачата за Кьонигсбергските мостове или формулата на Ойлер, свързваща броя на стените, ръбовете и върховете на изпъкнали многостени. Такива са Риман и Бети, класифицирали двумерните компактни повърхнини. Въпреки това е трудно да се каже, че тези постижения са представлявали последователна математическа дисциплина. „Поради това преди Поанкаре трябва да говорим за предистория на алгебричната топология" казва един от най-известните математици и историци на науката Ж. Дьодоне. Полагането на основите на стройна наука с нейните основни понятия, факти, задачи и т.н. започва с въвеждането на споменатото по-горе понятие хомеоморфизъм между две многообразия. За Поанкаре последното означава множеството от съвместни решения на система уравнения, зададени с гладки функции (и удовлетворяващи естествени условия за да отговаря множеството от решения на интуитивното понятие, което имаме от крива и повърхнина). Италианският математик Бети е въвел преди това числа, които остават неизменни при хомеоморфизми, т.е. представляват топологични инварианти. Разбира се при Бети (и неговия приятел Риман) тези понятия са „на пръсти" - при тях липсват понятията хомеоморфизъм, многообразие. Поанкаре не само че дава ясна дефиниция, но ги допълва с още числа-инварианти – коефициенти на торзия. И най-важното - след Поанкаре тези числа може да се пресмятат за многообразия с по-голяма размерност. Тези, които познават по-добре предмета, биха казали, че въпросните числа се дефинират от групите от хомологии, отговарящи на многообразието. Поанкаре никъде не употрябава понятието групи от хомологии, макар че при внимателното четене на неговите статии се вижда, че понятието е там - просто не е наречено с тези думи.

Много години след Поанкаре дискусията на тема кой е въвел групите от хомологии продължава – например МакЛейн смята че това е Поанкаре, а Дьодоне – обратното. Според мене не е нужно да приписваме всичко, до което се е докоснал Поанкаре, на неговото име - струва ми се, че той самият би възразил като имам предвид как щедро е раздавал приоритети. Във всеки случай Поанкаре никъде не е използвал груповата структура. Впрочем той е въвел и друго основно понятие на топологията - фундаменталната група, този път с това име. С това той полага основите на друг основен раздел на топологията – хомотопичната топология. Един забележителен проблем свързан с това понятие е знаменият проблем на Поанкаре. Вярно ли е че ако едно многообразие има хомотопичните групи на сферата, то е сфера? За решението на многомерните аналози на този проблем (т.е. за четиримерни, петмерни и т.н. сфери) са дадени две филдсофски премии- на С.Смейл и на М. Фридман. Но истинският проблем на Поанкаре е за тримерната сфера и той все още не е решен. Той е поставен в списъка на най-важните проблеми за новия век. За решението му неотдавна създаденият Институт Клей предоставя награда от 1 милион долара. [1]

В една от първите книги по топология С. Лефшец пише „Може би в нито един дял от математиката Поанкаре не е оставил по-неизгладимо своя отпечатък отколкото в топологията". И това се отнася за математика, който е въвел автоморфните функции, създал е теорията на динамичните системи, преобърнал е възгледите в небесната механика ... Действително, като се започне от въвеждането на основните понятия и методите за пресмятане чрез триангулация, мине се през най-дълбоките свойства на групите от хомологии и се стигне до мястото, където Поанкаре превъзхожда всички известни математици - да свърже различни области от математиката с топологията до такава степен, че да не може да се посочи коя е основната област. Ще си послужа с примери. Посочената по-горе формула на Ойлер изглежда едно забележително комбинаторно равенство. И нищо повече. И макар че опити то да се обобщи не липсват, те са в най-добрия случай красиви, но не отиващи далече забележки. Грандиозното обобщение на това понятие, направено от Поанкаре, което вече носи името характеристика на Ойлер-Поанкаре, сега е централно понятие не само в топологията, но и в диференциалната геометрия, създадена далеч преди Поанкаре, алгебричната геометрия, динамичните системи,

[1] Впрочем има доста сериозни основания да се счита, че скоро тази награда ще бъде дадена.

теорията на числата... Самият Поанкаре открива, че тази величина не зависи от триангулацията, но може да се пресмята така както е у Ойлер. От друга страна хактеристиката на Ойлер-Поанкаре за многообразие е равна на сумата от индексите на особените точки на коя да е система диференциални уравнения върху многообразието (това е знаменитата теорема на Поанкаре-Хопф). Това свързва характеристиката на Ойлер-Поанкаре с качествената теория на диференциалните уравнения. Освен това дава удобен начин за пресмятането ѝ.

Друг пример е теоремата за дуалност на Поанкаре, която твърди, числата на Бети с номера k и $n-k$ съвпадат за компактни многообразия. Нейните обощения и аналози така са се просмукали в съвременната математика, че без тях тя просто ще се върне с един век назад. Като започнем с теоремата на Александър, през въвеждането на по-удобния „двойнствен" обект – кохомологии, и всевъзможните теореми за дуалност в алгебричната геометрия и стигнем до начина на разсъждение чрез дуалности – всичко това е плод на тази естествен резултат.

4. МАТЕМАТИКАТА НА ДВАДЕСЕТИ ВЕК

През 1900 г. гениалният съвременник на Поанкаре Давид Хилберт представя пред II световен математически конгрес списък от проблеми, известен сега като „Проблеми на Хилберт" . Една цел на Хилберт, която личи и от разнобразието на областите, в които са поставени проблемите е да постави ясно въпроса

„...предстои ли на математиката, някога това, което отдавна се случва с другите науки, няма ли тя да се разпадне на отделни частни науки, представителите на които едва се разбират помежду си и поради това, връзките между които стават все по-малко... " Хилберт не отговаря на поставения въпрос, а възкликва емоционално „Аз не вярвам в това и не го искам!"

Страховете на Хилберт не са безпочвени. Разпадането все повече характеризира математиката от онова време. С бурното ѝ развитие през деветнадесети век постепенно се оформят големи и трудни за изучаване области – алгебрична теория на числата, диференциална геометрия, алгебрична геометрия, небесна механика, да не говорим за класическия анализ.

Заедно с това обаче върви и друга тенденция – на обединяване на математиката. С известно опростяване можем да кажем, че това става като идеи, средства и резултати от едни раздели се пренасят в други раздели. Нека да посочим примера, който дължим на гения на Риман – чрез римановата дзетафункция да се изследва разпределението на простите числа. Но в обединяването на науката не по-малка, а може би по-голяма роля играе умението да се намира общ произход на обекти изглеждащи доста различни.

„Ние трябва да съсредоточим своето внимание главно не толкова върху сходствата и различията, колкото върху тези аналогии, които често се скриват в изглеждащите различия" , пише Поанкаре в книгата си „Наука и метод" .

Нека си спомним как правилната интерпретация на трансформациите, запазващи автоморфните функции като пренос, по аналогия (доста неочевидна) на пренос за случая на периодични функции, го довеждат до откритията му в тази област.

Поанкаре е може би първият, който заличава границите между отделните математически дисциплини (без да отричам заслугите на Якоби, Риман, Клайн). Той свободно прехвърля идеи от една област в друга, дотогава считана за съвършенно различна. Преливането отива дотам, че те стават една наука. Например трудно е да определим на кой раздел принадлежи цитираната погоре теорема на Поанкаре-Хопф - на диференциалните уравнения или на алгебричната топология. Или на областта, наречена от С. Ленг „ничия земя“ , т.е. диференцируемите многообразия. Така е с автоморфните функции - те са диференциални уравнения, комплексен анализ, геометрия, теория на числата, теория на групите...

Но преди всичко вероятно трябва да посочим изключителната интуиция на Поанкаре, забелязал обединяващата роля на топологията. А това, че топологията е обединяваща наука, личи от следния околонаучен аргумент. Доста повече от половината математици получили Филдсовска награда (математическата „нобелова“ награда) са използвали съществено топологията в своите работи или просто са допринесли за развитието й.

Счита се, че съвременната математика се характеризира преди всичко с алгебризацията си. Поанкаре не е алгебрист (макар да има фундаментални работи и там). По своя начин на мислене Поанкаре е естествоизпитател, физик, а на математически език - геометър. Но ако проследим пътя на алгебризацията ще се убедим веднага, че една от основните крачки, направени от Еми Ньотер, е дефиницията на групите от хомологии и съществено използване на груповата структура, например на изображения между различните групи (хомоморфизми). Това силно обогатява топологията и тя скоро става мощен инструмент в математиката. Вероятно първите фундаментални приложения са при обосноваването и развитието на алгебричната геометрия, която дотогава е била доста интуитивна наука.

По-нататък идват функциите на много комплексни променливи, диференциалната геометрия. Пълният триумф на топологията настъпва, когато тя става основен инструмент в теорията на числата благодарение на А. Вейл, А. Гротендик, Е. Артин, Дж. Тейт и др. и на нейна база става обединението й с алгебричната геометрия. Покрай това се създават нови математически дисциплини, също имащи до голяма степен обединяващ характер – теорията на сноповете, хомологичната алгебра, К-теорията и др. По този начин участието на Поанкаре в създаването на фундамента на съвременната математика е решаващо - и в определяне на централните направления на изследванията, и в създаването на съвременния универсален език, който обединява класически разделените геометрия и алгебра. Това, от което се е страхувал Хилберт – разпадането на математиката на отделни науки – не стана факт. В интервю с един от най-видните математици на изминалия век – Жан-Пиер Сер, на въпроса за перспективите за обединение на математиката, той отговаря - „Това вече се е случило“ и илюстрира твърдението си така. Стойността на римановата дзета-функция в точката -1 е $-1/12$. Точно толкова е стойността на характеристиката на Ойлер-Поанкаре (орбифолдната) на групата от целочислени матрици с детерминанта 1. Това случайно ли е? Оказва се, че не. Фактът има дълбоко обощение свързващо други групи и други дзета-функции. „Такива въпроси не са теория на числата, нито топология, нито теория на групите: те са просто математика.“ обобщава знаменитият ни съвременник.

Заслугата за това нашата наука да остане единна, като намери универсални средства за разбирането на проблемите ѝ чрез обединяване на тези проблеми, т.е. намирането на по-общи постановки, а оттук и за бурното ѝ развитие, принадлежи на серия блестящи умове. Но днес без преувеличение можем да кажем, че сред тях първи е Поанкаре. Тук си струва един цитат от книгата „Що е математика“ . Нейният автор и достоен наследник на Поанкаре В. И. Арнолд казва:

Поанкаре е създател на математиката на двадесетия век.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: horozov@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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PROPERTIES OF CO-SPECTRA OF JOINT SPECTRA OF STRUCTURES

ALEXANDRA SOSKOVA

Two properties of co-spectrum of joint spectrum of finitely many abstract structures are presented: a minimal pair type theorem and an existence of a quasi-minimal degree for the joint spectrum.

Keywords: enumeration reducibility, enumeration jump, enumeration degrees, forcing
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1. INTRODUCTION

Let $\mathfrak{A}$ be an abstract structure. The degree spectrum $DS(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all enumeration degrees generated by all presentations of $\mathfrak{A}$ on the natural numbers. In [6, 2, 5, 4, 9] several results about degree spectra of structures are obtained.

The co-spectrum of the structure $\mathfrak{A}$ is the set of all lower bounds of the degree spectra of $\mathfrak{A}$. Co-spectra are introduced and studied in [9].

In [10] a generalization of the notions of degree spectra and co-spectra for finitely many structures is presented - the so called joint spectrum and co-spectrum. A normal form of the sets which generates the elements of the co-spectrum of the joint spectrum in terms of some positive recursive Σ^+ formulae, introduced first in [1], is obtained.

Here we shall prove two properties of the co-spectrum of joint spectrum of structures - the Minimal pair type theorem and the existence of a quasi-minimal degree for the joint spectrum.

The proofs use the technique of regular enumerations introduced in [8], combined with partial generic enumerations used in [9].

2. PRELIMINARIES

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$ be a partial structure over the set of all natural numbers $\mathbb{N}$, where each R_i is a subset of $\mathbb{N}^{r_i}$ and "$=$" and "$\neq$" are among $R_1, \dots, R_k$.

An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

If $A \subseteq \mathbb{N}^a$, define

$$f^{-1}(A) = \{\langle x_1 \dots x_a \rangle : (f(x_1), \dots, f(x_a)) \in A\}.$$

Let $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k)$.

For any sets of natural numbers A and B the set A is enumeration reducible to B ($A \leq_e B$) if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$. By $d_e(A)$ we denote the enumeration degree of the set A and by $\mathcal{D}_e$ the set of all enumeration degrees. The set A is total if $A \equiv_e A^+$, where $A^+ = A \oplus (\mathbb{N} \setminus A)$. A degree a is called total if a contains the e-degree of a total set. The jump operation "$'$" denotes here the enumeration jump introduced by Cooper [3].

Given $n+1$ subsets $B_0, \dots, B_n$ of $\mathbb{N}$, $i \leq n$, define the set $\mathcal{P}(B_0, \dots, B_i)$ as follows:

(i) $\mathcal{P}(B_0) = B_0$;

(ii) If $i < n$, then $\mathcal{P}(B_0, \dots, B_{i+1})(\mathcal{P}(B_0, \dots, B_i))' \oplus B_{i+1}$.

3. JOINT SPECTRA OF STRUCTURES

Let $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ be abstract structures on $\mathbb{N}$.

The *joint sSpectrum* of $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set

$$DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)\{\mathbf{a} : \mathbf{a} \in DS(\mathfrak{A}_0), \mathbf{a}' \in DS(\mathfrak{A}_1), \dots, \mathbf{a}^{(n)} \in DS(\mathfrak{A}_n)\}.$$

For every $k \leq n$, *the k-th jump spectrum of* $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ is the set

$$DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)\{\mathbf{a}^{(k)} : \mathbf{a} \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)\}.$$

In [10] we prove that $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$ is closed upwards, i.e. if $\mathbf{a}^{(k)} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, $\mathbf{b}$ is a total e-degree and $\mathbf{a}^{(k)} \leq \mathbf{b}$, then $\mathbf{b} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$.

The k-th co-spectrum of $\mathfrak{A}_0, \dots, \mathfrak{A}_n$, $k \leq n$, is the set of all lower bounds of $DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, i.e.

$$CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)\{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \& (\forall \mathbf{a} \in DS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\}.$$

From [10] we know that the k-th Co-spectrum for $k \leq n$ depends only of the first k structures:

$$CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n) CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_k).$$

In [10] we give a normal form of the sets which generates the elements of the k-th co-spectrum of $DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$, i.e. for every $A \subseteq \mathbb{N}$ the following are equivalent:

(1) $d_e(A) \in CS_k(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$;

(2) For every $f_0, \dots, f_k$ enumerations of $\mathfrak{A}_0, \dots, \mathfrak{A}_k$, respectively,

$$A \leq_e \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k));$$

(3) A is forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$;

(4) A is formally k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$.

In Section 4 we shall recall the definition of the forcing k-definable sets on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$.

The analog of the Minimal pair theorem, which we shall prove in Section 5, is in the following form:

Theorem 3.1. *Let $k \leq n$. There exist enumeration degrees* $\mathbf{f}$ *and* $\mathbf{g}$, *elements of $DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, such that for any enumeration degree* $\mathbf{a}$*:*

$$\mathbf{a} \leq \mathbf{f}^{(k)} \;\&\; \mathbf{a} \leq \mathbf{g}^{(k)} \Longrightarrow \mathbf{a} \in CS_k(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n).$$

The proof uses the technique of the regular enumerations from [8], which we will discuss in Section 6.

Given a set $\mathcal{A}$ of enumeration degrees, denote by $co(\mathcal{A})$ the set of all lower bounds of $\mathcal{A}$. Say that the degree $\mathbf{q}$ is *a quasi-minimal with respect to* $\mathcal{A}$ if the following conditions hold:

(i) $\mathbf{q} \notin co(\mathcal{A})$;

(ii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$;

(iii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

The second property, we are going to prove in Section 7, is the existence of a quasi-minimal degree with respect to $DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

Theorem 3.2. *There exists an enumeration degree* $\mathbf{q}$ *such that:*

(i) $\mathbf{q}' \in DS(\mathfrak{A}_1), \dots, \mathbf{q}^{(n)} \in DS(\mathfrak{A}_n)$, $q \notin CS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$;

(ii) *If* $\mathbf{a}$ *is a total degree and* $\mathbf{a} \geq \mathbf{q}$, *then* $\mathbf{a} \in DS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$;

(iii) *If* $\mathbf{a}$ *is a total degree and* $\mathbf{a} \leq \mathbf{q}$, *then* $\mathbf{a} \in CS(\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

4. FORCING k-DEFINABLE SETS

Suppose that $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ are structures on $\mathbb{N}$. Let $f_0, \dots, f_n$ be enumerations of $\mathfrak{A}_0, \dots, \mathfrak{A}_n$, respectively.

Denote by $\bar{f} = (f_0, \dots, f_n)$ and $\mathcal{P}_k^{\bar{f}}\mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \dots, f_k^{-1}(\mathfrak{A}_k))$ for $k = 0, \dots, n$.

Let $W_0, \dots, W_z, \dots$ be a Goedel enumeration of the r.e. sets and D_v be the finite set having a canonical code v.

For every $i \leq n$, e and x in $\mathbb{N}$ define the relations $\bar{f} \models_i F_e(x)$ and $\bar{f} \models_i \neg F_e(x)$ by induction on i:

(i) $\bar{f} \models_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;

(ii)
$$\begin{aligned} \bar{f} \models_{i+1} F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\ u = \langle 0, e_u, x_u\rangle \;\&\; \bar{f} \models_i F_{e_u}(x_u) \vee \\ u = \langle 1, e_u, x_u\rangle \;\&\; \bar{f} \models_i \neg F_{e_u}(x_u) \vee \\ u = \langle 2, x_u\rangle \;\&\; x_u \in f_{i+1}^{-1}(\mathfrak{A}_{i+1})));\end{aligned}$$

(iii) $\bar{f} \models_i \neg F_e(x) \iff \bar{f} \not\models_i F_e(x)$.

If $A \subseteq \mathbb{N}$ and $k \leq n$, then

$$A \leq_e \mathcal{P}_k^{\bar{f}} \iff (\exists e)(A = \{x : \bar{f} \models_k F_e(x)\}).$$

The forcing conditions, which we shall call *finite parts*, are $n+1$-tuples $\bar{\tau} = (\tau_0, \dots, \tau_n)$ of finite mappings $\tau_0, \dots, \tau_n$ of $\mathbb{N}$ in $\mathbb{N}$. We suppose that an effective coding of the finite parts is fixed, and by the least finite part with a fixed property we mean a finite part with a minimal code.

For every $i \leq n$, e and x in $\mathbb{N}$ and every finite part $\bar{\tau}$ we define the forcing relations $\bar{\tau} \Vdash_i F_e(x)$ and $\bar{\tau} \Vdash_i \neg F_e(x)$ following the definition of relation "$\models_i$".

Definition 4.1. (i) $\bar{\tau} \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; D_v \subseteq \tau_0^{-1}(\mathfrak{A}_0))$;

(ii)
$$\begin{aligned} \bar{\tau} \Vdash_{i+1} F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \;\& \\ (\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \;\&\; \bar{\tau} \Vdash_i F_{e_u}(x_u) \vee \\ u = \langle 1, e_u, x_u\rangle \;\&\; \bar{\tau} \Vdash_i \neg F_{e_u}(x_u) \vee \\ u = \langle 2, x_u\rangle \;\&\; x_u \in \tau_{i+1}^{-1}(\mathfrak{A}_{i+1})));\end{aligned}$$

(iii) $\bar{\tau} \Vdash_i \neg F_e(x) \iff (\forall \bar{\rho} \supseteq \bar{\tau})(\bar{\rho} \nVdash_i F_e(x))$.

Given finite parts $\bar{\delta} = (\delta_0, \ldots, \delta_n)$ and $\bar{\tau} = (\tau_0, \ldots, \tau_n)$, let

$$\bar{\delta} \subseteq \bar{\tau} \iff \delta_0 \subseteq \tau_0, \ldots, \delta_n \subseteq \tau_n.$$

For any $i \leq n, e, x \in \mathbb{N}$ denote $X^i_{\langle e,x\rangle} = \{\bar{\rho} : \bar{\rho} \Vdash_i F_e(x)\}$.
If $\bar{f} = (f_0, \ldots, f_n)$ is an enumeration of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, then

$$\bar{\tau} \subseteq \bar{f} \iff \tau_0 \subseteq f_0, \ldots, \tau_n \subseteq f_n.$$

Definition 4.2. An enumeration $\bar{f}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is *i-generic* if for every $j < i$, $e, x \in \mathbb{N}$

$$(\forall \bar{\tau} \subseteq \bar{f})(\exists \bar{\rho} \in X^j_{\langle e,x\rangle})(\bar{\tau} \subseteq \bar{\rho}) \Longrightarrow (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \in X^j_{\langle e,x\rangle}).$$

From [10] we know that:

(1) If $\bar{f}$ is a k-generic enumeration, then

$$\bar{f} \models_k F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_k F_e(x)).$$

(2) If f is a $(k+1)$-generic enumeration, then

$$\bar{f} \models_k \neg F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_k \neg F_e(x)).$$

Definition 4.3. The set $A \subseteq \mathbb{N}$ is *forcing k-definable* on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ if there exist a finite part $\bar{\delta}$ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k F_e(x)).$$

Proposition 4.1. *Let $\{X^k_r\}_r$, $k = 0, \ldots, n$, be $(n+1)$-sequences of sets of natural numbers. There exists an $(n+1)$-generic enumeration $\bar{f}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ such that for any $k \leq n$ and for all $r \in \mathbb{N}$, if the set X^k_r is not forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, then $X^k_r \not\leq_e \mathcal{P}^f_k$.*

Proof. We shall construct an $(n+1)$-generic enumeration $\bar{f}$ such that for all r and all $k = 0, \ldots, n$, if the set X^k_r is not forcing k-definable, then $X^k_r \not\leq_e \mathcal{P}^{\bar{f}}_k$. Let call the last condition an omitting condition.

The construction of the enumeration $\bar{f}$ will be carried out by steps. On each step j we shall define a finite part $\bar{\delta}^j = (\delta^j_0, \ldots, \delta^j_n)$, so that $\bar{\delta}^j \subseteq \bar{\delta}^{j+1}$, and take $f_i = \cup_j \delta^j_i$ for each $i \leq n$.

On the steps $j = 3q$ we shall ensure that each f_i is a total surjective mapping from $\mathbb{N}$ onto $\mathbb{N}$. On the steps $j = 3q+1$ we shall ensure that $\bar{f}$ is $(n+1)$-generic. On the steps $j = 3q+2$ we shall ensure the omitting condition.

Let $\bar{\delta}^0 = (\emptyset, \ldots, \emptyset)$.

Suppose that $\bar{\delta}^j$ is defined.

Case $j = 3q$. For every i, $0 \le i \le n$, let $x_i = \mu x[x \notin \mathrm{dom}(\delta_i^j)]$ and $y_i = \mu y[y \notin \mathrm{ran}(\delta_i^j)]$. Let $\delta_i^{j+1}(x_i) = y_i$ and $\delta_i^{j+1}(x) \simeq \delta_i^j(x)$ for $x \neq x_i$.

Case $j = 3\langle e, i, x\rangle + 1$, $i \le n$. Check if there exists a finite part $\bar{\rho} \supseteq \bar{\delta}^j$ such that $\bar{\rho} \Vdash_i F_e(x)$. If so, then let $\bar{\delta}^{j+1}$ be the least such ρ. Otherwise, let $\bar{\delta}^{j+1} = \bar{\delta}^j$.

Case $j = 3\langle e, k, r\rangle + 2$, $k \le n$. Consider the set X_r^k. If X_r^k is forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ then let $\bar{\delta}^{j+1} = \bar{\delta}^j$.

Suppose now that X_r^k is not forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ and let

$$C = \{x : (\exists \bar{\tau} \supseteq \bar{\delta}^j)(\bar{\tau} \Vdash_k F_e(x))\}.$$

Clearly, C is forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$. Hence $C \neq X_r^k$. Then there exists an x such that either $x \in X_r^k$ and $x \notin C$ or $x \in C$ and $x \notin X_r^k$. Take $\bar{\delta}^{j+1} = \bar{\delta}^j$ in the first case.

If the second case holds, then there exists $\bar{\tau} \supseteq \bar{\delta}^j$ such that $\bar{\tau} \Vdash_k F_e(x)$. Let $\bar{\delta}^{j+1}$ be the least such τ.

In all other cases let $\bar{\delta}^{j+1} = \bar{\delta}^j$.

The so received enumeration $\bar{f} = \cup_j \bar{\delta}^j$ is $(n+1)$-generic. Let $i \le n$, $e, x \in \mathbb{N}$ and suppose that for every finite part $\bar{\tau} \subseteq \bar{f}$ there is an extention $\bar{\rho} \Vdash_i F_e(x)$. Consider the step $j = 3\langle e, i, x\rangle + 1$. From the construction we have that $\bar{\delta}^{j+1} \Vdash_i F_e(x)$.

To prove that the enumeration $\bar{f}$ satisfies the omitting condition, let the set X_r^k be not forcing k-definable on $\mathfrak{A}_0, \dots, \mathfrak{A}_n$ and suppose that $X_r^k \le_e \mathcal{P}_k^{\bar{f}}$. Then $X_r^k = \{x : \bar{f} \models_k F_e(x)\}$ for some e. Consider the step $j = 3\langle e, k, r\rangle + 2$. From the construction there is an x such that one of the following two cases holds:

(a) $x \in X_r^k$ and $(\forall \bar{\rho} \supseteq \bar{\delta}^j)(\bar{\rho} \nVdash_k F_e(x))$. So, $\bar{\delta}^j \Vdash_k \neg F_e(x)$.

Since $\bar{f}$ is $(n+1)$-generic, and hence $(k+1)$-generic, $x \in X_r^k \;\&\; \bar{f} \not\models_k F_e(x)$. A contradiction.

(b) $x \notin X_r^k$ & $\bar{\delta}^{j+1} \Vdash_k F_e(x)$. Since $\bar{f}$ is $(k+1)$-generic, $\bar{f} \models_k F_e(x)$. A contradiction. □

5. MINIMAL PAIR THEOREM

First we need a modification of the "type omitting" version of Jump inversion theorem from [8]. In fact, one can see the result from the proof of Theorem 1.7 in [8]. But in this form it is not explicit formulated there. We shall postpone the proof for Section 6, where the technique of regular enumerations will be discussed.

Theorem 5.1. *Let $B_0, \dots, B_n$ be arbitrary sets of natural numbers. Let $\{A_r^k\}_r$, $k = 0, \dots, n$, be $(n+1)$-sequences of subsets of $\mathbb{N}$ such that for every r and for all k, $0 \le k < n$, $A_r^k \not\le_e \mathcal{P}(B_0, \dots, B_k)$. Then there exists a total set F having the following properties:*

(i) *For all $i \le n$, $B_i \le_e F^{(i)}$;*

(ii) *For all r, for all k, $0 \le k < n$, $A_r^k \not\le_e F^{(k)}$.*

Proof of Theorem 3.1. We shall construct total sets F and G such that $d_e(F) \in DS(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$, $d_e(G) \in DS(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$ and for each $k \leq n$ if a total set X, $X \leq_e F^{(k)}$ and $X \leq_e G^{(k)}$, then $d_e(X) \in CS_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$. And take the degree $\mathbf{f} = d_e(F)$ and $\mathbf{g} = d_e(G)$.

First we construct enumerations $\bar{f}$ and $\bar{h}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ such that for any $k \leq n$ if a set $A \subseteq \mathbb{N}$, $A \leq_e \mathcal{P}_k^{\bar{f}}$ and $A \leq_e \mathcal{P}_k^{\bar{h}}$, then A is a forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.

Let $g_0, \ldots, g_n$ be arbitrary enumerations of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. By Theorem 5.1 for $B_0 = g_0^{-1}(\mathfrak{A}_0), \ldots, B_n = g_n^{-1}(\mathfrak{A}_n)$ there exists a total set F such that: $g_0^{-1}(\mathfrak{A}_0) \leq_e F, g_1^{-1}(\mathfrak{A}_1) \leq_e F', \ldots, g_n^{-1}(\mathfrak{A}_n) \leq_e F^{(n)}$. Since $DS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ is closed upwards, then $d_e(F) \in DS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, i.e. $d_e(F) \in DS(\mathfrak{A}_0), d_e(F') \in DS(\mathfrak{A}_1), \ldots, d_e(F^{(n)}) \in DS(\mathfrak{A}_n)$. Hence, there exist $h_0, h_1, \ldots, h_n$ enumerations of $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$, respectively, such that $h_0^{-1}(\mathfrak{A}_0) \equiv_e F, h_1^{-1}(\mathfrak{A}_1) \equiv_e F', \ldots, h_n^{-1}(\mathfrak{A}_n) \equiv_e F^{(n)}$. Notice that for each $k \leq n$, $F^{(k)} \equiv_e \mathcal{P}_k^{\bar{h}}$.

For each k, $0 \leq k \leq n$, let $\{X_r^k\}_r$ be the sequence of all sets enumeration reducible to $\mathcal{P}_k^{\bar{h}}$.

By Proposition 4.1 there is an $(n+1)$-generic enumeration $\bar{f}$ such that for all r and all $k = 0, \ldots, n$ if the set X_r^k is not forcing k-definable then $X_r^k \not\leq_e \mathcal{P}_k^{\bar{f}}$.

Suppose now that the set $A \leq_e \mathcal{P}_k^{\bar{f}}$ and $A \leq \mathcal{P}_k^{\bar{h}}$. Then $A = X_r^k$ for some r. From the omitting condition of $\bar{f}$ it follows that A is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.

Now we apply Theorem 5.1. Let $B_0 = f_0^{-1}(\mathfrak{A}_0), \ldots, B_n f_n^{-1}(\mathfrak{A}_n)$ and $B_{n+1} = N$. For each $k \leq n$ consider the sequence $\{A_r^k\}_r$ of these sets among the sets $\{X_r^k\}_r$, which are not forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. From the choice of the enumeration $\bar{f}$ it follows that each of these sets A_r^k, $A_r^k \not\leq_e \mathcal{P}_k^{\bar{f}}$. Then by Theorem 5.1 there is a total set G such that:

(i) For all $i \leq n$, $f_i^{-1}(\mathfrak{A}_i) \leq_e G^{(i)}$;

(ii) For all r and all $k \leq n$, $A_r^k \not\leq_e G^{(k)}$.

Note that since G is a total set and because of the fact that each spectrum is closed upwards, we have that $d_e(G) \in DS(\mathfrak{A}_0), \ldots, d_e(G^{(n)}) \in DS(\mathfrak{A}_n)$, and hence $d_e(G) \in DS(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

Suppose now that a total set X, $X \leq_e F^{(k)}$ and $X \leq_e G^{(k)}$, $k \leq n$. From $X \leq_e F^{(k)}$ and $F^{(k)} \equiv_e \mathcal{P}_k^{\bar{h}}$ it follows that $X = X_r^k$ for some r. It is clear that $X \leq_e \mathcal{P}_k^{\bar{f}}$. Otherwise, from the choice of G it follows that $X \not\leq_e G^{(k)}$. Hence X is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. By the normal form of the sets, which enumeration degrees are in $CS_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$, we have that $d_e(X) \in CS_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$. □

6. REGULAR ENUMERATIONS

We shall remind the notion of regular enumerations from [8]. Let $B_0, \ldots, B_n$ be non empty subsets of $\mathbb{N}$.

Finite parts are as usual finite mappings of $\mathbb{N}$ into $\mathbb{N}$. The notion of *i-regular finite parts* is defined by induction on $i \leq n$.

The 0-*regular finite parts* are finite parts τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau), \tau(z) \in B_0$.

Let τ be a 0-regular finite part. If $\mathrm{dom}(\tau) = [0, 2q+1]$, then the 0-rank of τ $|\tau|_0 q + 1$ – the number of the odd elements of $\mathrm{dom}(\tau)$. Let B_0^τ be the set of the odd elements of $\mathrm{dom}(\tau)$. If ρ is a 0-regular extention of τ, we shall denote this fact by $\tau \subseteq_0 \rho$. It is clear that if $\tau \subseteq_0 \rho$ and $|\tau|_0 |\rho|_0$, then $\tau = \rho$. Let

$$\tau \Vdash_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\tau((u)_0) \simeq (u)_1)),$$

$$\tau \Vdash_0 \neg F_e(x) \iff \forall(\rho)(\tau \subseteq_0 \rho \Rightarrow \rho \not\Vdash_0 F_e(x)).$$

Suppose that for some $i < n$ we have defined the i-regular finite parts and for every i-regular τ – the i-rank $|\tau|_i$ of τ, the set B_i^τ and the relations $\tau \Vdash_i F_e(x)$ and $\tau \Vdash_i \neg F_e(x)$. Suppose also that if τ and ρ are i-regular, $\tau \subseteq \rho$ (we write $\tau \subseteq_i \rho$) and $|\tau|_i = |\rho|_i$, then $\tau = \rho$.

Denote by $X^i_{\langle e,x\rangle} = \{\rho : \rho \text{ is } i\text{-regular} \;\&\; \rho \Vdash_i F_e(x)\}$.

For any i-regular finite part τ and any set X of i-regular finite parts, denote by $\mu_i(\tau, X) = \mu\rho[\tau \subseteq_i \rho \;\&\; \rho \in X]$ if any, and $\mu_i(\tau, X) = \mu\rho[\tau \subseteq_i \rho]$, otherwise.

Definition 6.1. Let τ be a finite part and $m \geq 0$. The finite part δ is called an *i-regular m omitting extension* of τ if $\delta \supseteq_i \tau$, $\mathrm{dom}(\delta) = [0, q-1]$ and there exist natural numbers $q_0 < \cdots < q_m < q_{m+1} = q$ such that:

(a) $\delta\restriction q_0 = \tau$;

(b) For all $p \leq m$, $\delta\restriction q_{p+1}\mu_i(\delta\restriction(q_p + 1), X^i_{\langle p,q_p\rangle})$.

Denote by K^δ_τ the sequence $q_0, \ldots, q_m$. If δ and ρ are two i-regular m omitting extensions of τ and $\delta \subseteq \rho$, then $\delta = \rho$.

Let $\mathfrak{R}_i$ denote the set of all i-regular finite parts. Given an index j, by S^i_j we shall denote the intersection $\mathfrak{R}_i \cap \Gamma_j(\mathcal{P}(B_0, \ldots, B_i))$, where Γ_j is the j-th enumeration operator.

Let τ be a finite part defined on $[0, q-1]$ and $r \geq 0$. Then τ is $(i+1)$-*regular* with $(i+1)$-*rank* $r+1$ if there exist natural numbers

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 < \cdots < n_r < l_r < b_r < n_{r+1} = q$$

such that $\tau\restriction n_0$ is an i-regular finite part with i-rank equal to 1 and for all j, $0 \leq j \leq r$, the following conditions are satisfied:

(a) $\tau\restriction l_j \simeq \mu_i(\tau\restriction(n_j + 1), S^i_j)$;

(b) $\tau\restriction b_j$ is an i-regular j omitting extension of $\tau\restriction l_j$;

(c) $\tau(b_j) \in B_{i+1}$;

(d) $\tau\lceil n_{j+1}$ is an i-regular extension of $\tau\lceil(b_j+1)$ with i-rank equal to $|\tau\lceil b_j|_i+1$.

Let $B^\tau_{i+1}=\{b_0,\dots,b_r\}$. By K^τ_{i+1} we shall denote the sequence $K^{\tau\lceil b_r}_{\tau\lceil l_r}$.
Let for every $(i+1)$-regular finite part τ

$$\tau \Vdash_{i+1} F_e(x) \iff \exists v(\langle v,x\rangle \in W_e \ \&\ (\forall u\in D_v)((u=\langle e_u,x_u,0\rangle \ \&\ \tau\Vdash_i F_{e_u}(x_u))\vee \\ (u=\langle e_u,x_u,1\rangle \ \&\ \tau\Vdash_i \neg F_{e_u}(x_u)))).$$

$$\tau\Vdash_{i+1}\neg F_e(x) \iff (\forall\rho)(\tau\subseteq_{i+1}\rho\Rightarrow \rho\nVdash_{i+1} F_e(x)).$$

Definition 6.2. Let f be a total mapping of $\mathbb{N}$ in $\mathbb{N}$. Then f is a *regular enumeration* if the following two conditions hold:

(i) For every finite part $\delta\subseteq f$, there exists an n-regular extension τ of δ such that $\tau\subseteq f$.

(ii) If $i\le n$ and $z\in B_i$, then there exists an i-regular extension $\tau\subseteq f$ such that $z\in\tau(B^\tau_i)$.

Let f be a total mapping on $\mathbb{N}$. We define for every $i\le n,e,x$ the relation $f\models_i F_e(x)$ by induction on i:

Definition 6.3.
(i) $f\models_0 F_e(x) \iff \exists v(\langle v,x\rangle\in W_e \ \&\ (\forall u\in D_v)(f((u)_0)\simeq(u)_1))$;
(ii) $f\models_{i+1} F_e(x) \iff \exists v(\langle v,x\rangle\in W_e \ \&\ (\forall u\in D_v)((u=\langle e_u,x_u,0\rangle \ \&$
$f\models_i F_{e_u}(x_u))\vee(u=\langle e_u,x_u,1\rangle \ \&\ f\not\models_i F_{e_u}(x_u))))$.

Set $f\models_i\neg F_e(x)\iff f\not\models_i F_e(x)$.
In [8] it is proven that for every regular enumeration f:

1. $B_0\le_e f$.
2. If $i<n$, then $B_{i+1}\le_e f\oplus\mathcal{P}(B_0,\dots,B_i)'$, and $\mathcal{P}(B_0,\dots,B_i)<_e f^{(i)}$, for $i\le n$.
3. If $A\subset\mathbb{N}$, then
$$A\ \le_e f^{(i)} \iff (\exists e)A=\{x: f\models_i F_e(x)\}.$$
4. For all $i\le n$ (for negation $i<n$),
$$f\models_i(\neg)F_e(x)\iff(\exists\tau\subseteq f)(\tau \text{ is } i\text{-regular} \ \&\ \tau\Vdash_i(\neg)F_e(x)).$$

Notice that if f is a regular enumeration, then $B_i\le_e f^{(i)}$, $i\le n$.

Given a finite mapping τ defined on $[0,q-1]$, by $\tau * z$ we shall denote the extension ρ of τ defined on $[0,q]$ and such that $\rho(q)\simeq z$. We shall use the following Lemma, proved in [8].

Lemma 6.1. [8] *Let $A_0, \dots, A_{n-1}$ be subsets of $\mathbb{N}$ such that $A_i \not\leq_e \mathcal{P}(B_0, \dots, B_i)$. Let τ be an n-regular finite part, defined on $[0, q-1]$. Suppose that $|\tau|_n = r+1$, $y \in \mathbb{N}$, $z_0 \in B_0, \dots, z_n \in B_n$ and $s \leq r+1$. Then there is an n-regular extension ρ of τ such that:*

(i) $|\rho|_n = r+2$;

(ii) $\rho(q) \simeq y$, $z_0 \in \rho(B_0^\rho), \dots, z_n \in \rho(B_n^\rho)$;

(iii) *if $i < n$ and $K_{i+1}^\rho = q_0^i, \dots, q_s^i, \dots, q_{m_i}^i$, then*

- (a) $\rho(q_s^i) \in A_i \Rightarrow \rho \Vdash_i \neg F_s(q_s^i)$;
- (b) $\rho(q_s^i) \notin A_i \Rightarrow \rho \Vdash_i F_s(q_s^i)$.

Now we turn to the proof of Theorem 5.1. Set $B_{n+1} = \mathbb{N}$ and $\mathcal{P}(B_0, \dots, B_{n+1}) = \mathcal{P}(B_0, \dots, B_n)' \oplus B_{n+1}$. By a regular enumeration f we mean a regular one with respect to $B_0, \dots, B_n, B_{n+1}$.

Proof of Theorem 5.1.

Let $\{A_r^k\}_r$, $k \leq n$, be seqences of subsets of $\mathbb{N}$ such that $A_r^k \not\leq_e \mathcal{P}(B_0, \dots, B_k)$.

We shall construct a regular enumeration f such that f "omits" the sets A_r^k for all r, $k \leq n$, i.e. $A_r^k \not\leq_e f^{(k)}$.

The construction of f will be carried out by steps. At each step s we shall construct an $(n+1)$-regular finite part δ_s, so that $|\delta_s|_{n+1} \geq s+1$ and $\delta_s \subseteq_{n+1} \delta_{s+1}$. On the even steps we shall ensure the genericity of f, i.e. conditions (a) and (d) from the definition of i-regular finite part, and on the odd steps we shall ensure the omitting conditions, the conditions (b), (c).

Let $\mathcal{R}_{n+1}$ be the set of all $(n+1)$-regular finite parts and $S_j^{n+1} = \mathcal{R}_{n+1} \cap \Gamma_j(\mathcal{P}(B_0, \dots, B_{n+1}))$. Let $\sigma_0, \dots, \sigma_{n+1}$ be recursive in $\mathcal{P}(B_0, \dots, B_{n+1})$ enumerations of the sets $B_0, \dots, B_{n+1}$, respectively.

Let δ_0 be an arbitrary $(n+1)$-regular finite part with $(n+1)$-rank equal to 1. Suppose that δ_s is defined.

Case $s = 2m$. Check whether there exists a $\rho \in S_m^{n+1}$ such that $\delta_s \subset \rho$. If so, let δ_{s+1} be the least such ρ. Otherwise, let δ_{s+1} be the least $(n+1)$-regular extension of δ_s with $(n+1)$-rank equal to $|\delta_s|_{n+1} + 1$.

Case $s = 2m+1$. Let $|\delta_s|_{n+1} = r+1 \geq s+1$. Let $m\langle p, e\rangle$. We may assume that $e \leq m$ and then $e < r+1$. Let $\sigma_0(m) \simeq z_0, \dots, \sigma_{n+1}(m) \simeq z_{n+1}$. Set $\tau_0 \simeq \mu_n(\delta_s * z_{n+1}, S_{r+1}^n)$. Let $l_{r+1} = \mathrm{lh}(\tau_0)$ and $q_0^n = l_{r+1}$. For $j < e$, let $\tau_{j+1} = \mu_n(\tau_j * 0, X_{\langle j, q_j^n\rangle}^n)$ and $q_{j+1}^n = \mathrm{lh}(\tau_{j+1})$. So, τ_e and q_e^n are defined. Let

$$C = \{x : (\exists \tau \supseteq \tau_e)(\tau \in \mathcal{R}_n \ \& \ \tau(q_e^n) \simeq x \ \& \ \tau \Vdash_n F_e(q_e^n))\}.$$

The set $C \leq_e \mathcal{P}(B_0, \dots, B_{n+1})$ and $A_p^n \not\leq_e \mathcal{P}(B_0, \dots, B_{n+1})$. Then there is an a such that

$$a \in C \ \& \ a \notin A_p^n \vee a \notin C \ \& \ a \in A_p^n. \tag{6.1}$$

Let a_0 be the least a satisfying (6.1).

Next we extend the finite $\tau_e * a_0$ to a finite part τ, so that τ is an n-regular $r+1$ omitting extension of τ_0. Set $b_{r+1} = \mathrm{lh}(\tau)$. Now consider the sets $A_p^0, \ldots, A_p^{n-1}$. By Lemma 6.1 we can construct an n-regular extension ρ of τ such that:

(i) $|\rho|_n = |\tau|_n + 1$;

(ii) $\rho(b_{r+1}) \simeq z_{n+1}$ and $z_0 \in \rho(B_0^\rho), \ldots, z_n \in \rho(B_n^\rho)$;

(iii) if $k < n$ and $K_{k+1}^\rho = q_0^k, \ldots, q_e^k, \ldots, q_{m_k}^k$, then

(a) $\rho(q_e^k) \in A_p^k \Rightarrow \rho \Vdash_k \neg F_e(q_e^k)$;

(b) $\rho(q_e^k) \notin A_p^k \Rightarrow \rho \Vdash_k F_e(q_e^p)$.

Set $\delta_{s+1} = \rho$.

Let $f = \bigcup \delta_s$. From the construction it follows that f is a regular enumeration. For every e, x, $\{\tau : \tau \in \mathcal{R}_{n+1} \;\&\; \tau \Vdash_{n+1} F_e(x)\}$ is e-reducible to $\mathcal{P}(B_0, \ldots, B_{n+1})$. From here, by the even stages of the construction, it follows that for all e, x,

$$f \models_{n+1} (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{n+1} \;\&\; \tau \Vdash_{n+1} (\neg)F_e(x)).$$

Since f is regular, we have that if $k \leq n$, then for all e and x,

$$f \models_k (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_k \;\&\; \tau \Vdash_k (\neg)F_e(x)).$$

Now suppose that for some $k \leq n$ and p, $A_p^k \leq_e f^{(k)}$. Then the set $C_p^k = \{x : f(x) \in A_p^k\}$ is also e-reducible to $f^{(k)}$. Fix an e such that for all x,

$$f(x) \in A_p^k \iff x \in C_p^k \iff f \models_k F_e(x). \tag{6.2}$$

Consider the step $s = 2\langle p, e\rangle + 1$. By the construction, there exists a $q_e^k \in \mathrm{dom}(\delta_{s+1})$ such that

$$(f(q_e^k) \in A_p^k \Rightarrow f \models_k \neg F_e(q_e^k)) \;\&\; (f(q_e^k) \notin A_p^k \Rightarrow f \models_k F_e(q_e^k)).$$

Clearly, $\delta_{s+1}(q_e^k) \simeq f(q_e^k)$. Now assume that $f(q_e^k) \in A_p^k$. Then $\delta_{s+1} \Vdash_k \neg F_e(q_e^k)$. Hence $f \models_k \neg F_e(q_e^k)$, which is impossible. It remains that $f(q_e^k) \notin A_p^k$. In this case $\delta_{s+1} \Vdash_k F_e(q_e^k)$ and hence $f \models_k F_e(q_e^k)$. The last again contradicts (6.2). So $A_p^k \not\leq_e f^{(k)}$. □

7. QUASI-MINIMAL DEGREE

Definition 7.1. Let $B_0 \subseteq \mathbb{N}$. A set F of natural numbers is called *quasi-minimal over* B_0 if the following conditions hold:

(i) $B_0 <_e F$;

(ii) For any total set $A \subseteq \mathbb{N}$, if $A \leq_e F$, then $A \leq_e B_0$.

The following theorem we shall prove in the next section using the technique of partial regular enumerations.

Theorem 7.1. *Let $B_0, \ldots, B_n, n \geq 1$, be arbitrary sets of natural numbers. There exists a set F having the following properties:*

(i) $B_0 <_e F$;

(ii) *For all $1 \leq i \leq n$, $B_i \leq_e F^{(i)}$;*

(iii) *For any total set A, if $A \leq_e F$, then $A \leq_e B_0$.*

In fact, the set F from Theorem 7.1 is a quasi-minimal over B_0.

Let the structures $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ be fixed.

Proof of Theorem 3.2. By [9], there is a quasi-minimal degree $\mathbf{q}_0$ with respect to $DS(\mathfrak{A}_0)$, i.e.:

(i) $\mathbf{q}_0 \notin CS(\mathfrak{A}_0)$;

(ii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}_0$, then $\mathbf{a} \in DS(\mathfrak{A}_0)$;

(iii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}_0$, then $\mathbf{a} \in CS(\mathfrak{A}_0)$.

Let $B_0 \subseteq \mathbb{N}$ such that $d_e(B_0) = \mathbf{q}_0$, and $f_1, \ldots, f_n$ be fixed total enumerations of $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Denote $B_1 = f_1^{-1}(\mathfrak{A}_1), \ldots, B_n = f_n^{-1}(\mathfrak{A}_n)$. By Theorem 7.1, there is a quasi-minimal over B_0 set F such that:

(i) $B_0 <_e F$;

(ii) For all $1 \leq i \leq n$, $f_i^{-1}(\mathfrak{A}_i) \leq_e F^{(i)}$;

(iii) For any total set A, if $A \leq_e F$, then $A \leq_e B_0$.

We will show that $\mathbf{q} = d_e(F)$ is a quasi-minimal with respect to $DS(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$, i.e.:

(i) $\mathbf{q}' \in DS(\mathfrak{A}_1), \ldots, \mathbf{q}^{(n)} \in DS(\mathfrak{A}_n)$, $\mathbf{q} \notin CS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;

(ii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in DS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;

(iii) If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in CS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

In order to prove (i), suppose that $\mathbf{q} \in CS(\mathfrak{A}_0)$. By Theorem 7.1, $\mathbf{q}_0 < \mathbf{q}$ and thus $\mathbf{q}_0 \in CS(\mathfrak{A}_0)$. A contradiction with the fact that $\mathbf{q}_0$ is quasi-minimal with respect to $DS(\mathfrak{A}_0)$. Then $\mathbf{q} \notin CS(\mathfrak{A}_0)$ and hence $\mathbf{q} \notin CS(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

For each i, $1 \leq i \leq n$, the set $F^{(i)}$ is total and $f_i^{-1}(\mathfrak{A}_i) \leq_e F^{(i)}$. Since any degree spectrum is closed upwards, it follows that $d_e(F^{(i)}) \in DS(\mathfrak{A}_i)$, i.e. $\mathbf{q}^{(i)} \in DS(\mathfrak{A}_i)$.

For (ii) consider a total set X such that $X \geq_e F$. Then $d_e(X) \geq \mathbf{q}_0$. From the fact that $\mathbf{q}_0$ is quasi-minimal with respect to $DS(\mathfrak{A}_0)$ it follows that $d_e(X) \in DS(\mathfrak{A}_0)$. Moreover, for each $1 \leq i \leq n$, $X^{(i)} \geq_e F^{(i)} \geq_e f_i^{-1}(\mathfrak{A}_i)$, and $X^{(i)}$ is a total set. Then for each $i \leq n$, $d_e(X^{(i)}) \in DS(\mathfrak{A}_i)$, and hence $d_e(X) \in DS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$.

For (iii) suppose that X is a total set and $X \leq_e F$. Then, from the choice of F, $X \leq_e B_0$. Because $\mathbf{q}_0$ is quasi-minimal with respect to $DS(\mathfrak{A}_0)$, it follows that $d_e(X) \in CS(\mathfrak{A}_0)$. But $CS(\mathfrak{A}_0, \dots, \mathfrak{A}_n) = CS(\mathfrak{A}_0)$ and therefore $d_e(X) \in CS(\mathfrak{A}_0, \dots, \mathfrak{A}_n)$. □

8. PARTIAL REGULAR ENUMERATIONS

Let $B_0 \subseteq \mathbb{N}$.

Definition 8.1. *A partial enumeration f of B_0* is a partial surjective mapping from $\mathbb{N}$ onto $\mathbb{N}$ with the following properties:

(i) For all odd x, if $f(x)$ is defined, then $f(x) \in B_0$;

(ii) For all $y \in B_0$, there is an odd x such that $f(x) \simeq y$.

It is clear that if f is a partial enumeration of B_0, then $B_0 \leq_e f$ since

$$y \in B_0 \iff (\exists n)(f(2n+1) \simeq y).$$

Let $\perp \notin \mathbb{N}$.

Definition 8.2. A *partial finite part* τ is a finite mapping of $\mathbb{N}$ into $\mathbb{N} \cup \{\perp\}$ such that $(\forall x)(x \in \mathrm{dom}(\tau) \ \&\ x \text{ is odd} \Rightarrow (\tau(x) = \perp \vee \tau(x) \in B_0))$.

If τ is a partial finite part and f is a partial enumeration of B_0, say that

$$\tau \subseteq f \iff (\forall x \in \mathrm{dom}(\tau))((\tau(x) = \perp \Rightarrow f(x) \text{ is not defined }) \ \& \\ (\tau(x) \neq \perp \Rightarrow \tau(x) \simeq f(x)).$$

Let $B_0, \dots, B_n$ be fixed sets of natural numbers. Combining the technique of the regular enumerations with the partial (generic) enumerations on the 0-level for B_0, we shall construct a partial regular enumeration f, which will be quasi-minimal over the set B_0 and such that $B_i \leq_e f^{(i)}$ for $i \leq n$.

A 0-*regular partial finite part* is a partial finite part τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau)$, $\tau(z) \in B_0$ or $\tau(z) = \perp$.

Let B_0^τ be the set of all odd elements z of $\mathrm{dom}(\tau)$ such that $\tau(z) \in B_0$. The 0-rank of τ, $|\tau|_0 = q+1$, we call the number of the odd elements of $\mathrm{dom}(\tau)$. If ρ is a 0-regular partial extention of τ, we shall denote this fact again by $\tau \subseteq_0 \rho$. It is clear that if $\tau \subseteq_0 \rho$ and $|\tau|_0 |\rho|_0$, then $\tau = \rho$. Let

$$\tau \Vdash_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(u = \langle s, t\rangle, \ \&\ \tau(s) \simeq t \ \&\ t \neq \perp.)),$$

$$\tau \Vdash_0 \neg F_e(x) \iff \forall(\rho)(\tau \subseteq_0 \rho \Rightarrow \rho \not\Vdash_0 F_e(x)).$$

The definition of $(i+1)$-regular partial finite part τ, the set B^τ_{i+1}, the $(i+1)$-rank of τ and the relations $\tau \Vdash_{i+1} F_e(x)$ and $\tau \Vdash_{i+1} \neg F_e(x)$ are defined in the same way as in Section 6, the only difference is that instead of i-regular finite parts we use i-regular partial finite parts. Notice that again if τ is an i-regular partial finite part, then τ is a j-regular partial finite part for each $j < i$.

Definition 8.3. *A partial regular enumeration* is a partial mapping f from $\mathbb{N}$ onto $\mathbb{N}$ such that the following two conditions hold:

(i) For every partial finite part $\delta \subseteq f$, there exists an n-regular partial extension τ of δ such that $\tau \subseteq f$.

(ii) If $i \leq n$ and $z \in B_i$, then there exists an i-regular partial finite part $\tau \subseteq f$ such that $z \in \tau(B^\tau_i)$.

If f is a partial regular enumeration and $i \leq n$, then for every $\delta \subseteq f$, $\text{dom}(\delta) \subseteq [0, q-1]$, there exists an i-regular partial $\tau \subseteq f$ such that $\delta \subseteq \tau$, and for every $x \in [0, q-1]$ if $f(x)$ is not defined, then $\tau(x) = \bot$. Moreover, there exist i-regular partial finite parts of f of arbitrary large rank.

The relation $f \models_i F_e(x)$ is the same as in Definition 6.3. By induction on i one could check that for any $A \subseteq \mathbb{N}$, $A \leq_e f^{(i)}$ iff there exists e such that for all x,

$$x \in A \iff f \models_i F_e(x).$$

Lemma 8.1. *Suppose that f is a partial regular enumeration. Then:*

(1) *For all $i \leq n$, $f \models_i F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is i-regular & $\tau \Vdash_i F_e(x))$.*

(2) *For all $i < n$, $f \models_i \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is i-regular & $\tau \Vdash_i \neg F_e(x))$.*

The proof follows from the definitions by induction on i as in the total case.

Let $\mathcal{R}_i$ be the set of all i-regular partial finite parts. It is clear that $\mathcal{R}_i \leq_e \mathcal{P}_i$, where $\mathcal{P}_i = \mathcal{P}(B_0, \ldots, B_n)$.

Definition 8.4. A partial enumeration f is *i-generic* if for any $j < i$ and for every enumeration reducible to $\mathcal{P}_j$ set S of j-regular partial finite parts the following condition holds:

$$(\exists \tau \subseteq f)(\tau \in S \vee (\forall \rho \supseteq_j \tau)(\rho \notin S)).$$

Proposition 8.1. *Every partial regular enumeration is $(i+1)$-generic enumeration for every $i < n$.*

Proof. Let S be a set of i-regular partial finite parts such that $S \leq_e \mathcal{P}_i$. Then there exists an e such that $S = \mathcal{R}_i \cap \Gamma_e(\mathcal{P}_i)$. Consider an $(i+1)$-regular partial finite part $\tau \subseteq f$ with $(i+1)$-rank greater than e. From the definition of $(i+1)$-regular partial finite part it follows that there is an i-regular partial finite part $\sigma \subseteq_i \tau$, and hence $\sigma \subseteq f$ such that $\sigma \in S$ or $(\forall \rho \supseteq_i \sigma)(\rho \notin S)$. □

Proposition 8.2. *Suppose that f is a partial regular enumeration. Then:*

(1) *For each $i \leq n$, $B_i \leq_e f^{(i)}$.*

(2) *If $i < n$, then $f \not\leq_e \mathcal{P}_i$.*

Proof. We know that $B_0 \leq_e f$. Let $i < n$. Suppose that for each $j \leq i$, $B_j \leq_e f^{(j)}$. Then $\mathcal{P}_i \leq_e f^{(i)}$.

Since f is partial regular, for every partial finite part δ of f there exists an $(i+1)$-regular partial finite part $\tau \subseteq f$ such that $\delta \subseteq \tau$, where if $f(x)$ is not defined and $x \in \mathrm{dom}(\tau)$, then $\tau(x) = \perp$. For each q denote by $f\lceil_q$ the partial finite part τ with $\mathrm{dom}(\tau) = [0, q-1]$, $\tau \subseteq f$, and for each $x < q$ if $f(x)$ is not defined, then $\tau(x) = \perp$.

Let

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 < \cdots < n_r < l_r < b_r < n_{r+1} < \dots$$

be the numbers satisfying the conditions (a)–(d) from the definition of the $(i+1)$-regular partial finite part τ_r. Clearly, if $B^f_{i+1} = \{b_0, b_1 \dots\}$, then $f(B^f_{i+1}) = B_{i+1}$. We shall show that there exists an effective in $f^{(i+1)}$ procedure which lists $n_0, l_0, b_0, \dots$ in an increasing order.

Using the oracle f', we can generate consecutively the partial finite parts $f\lceil q$ for $q = 1, 2 \dots$. Notice that $f\lceil n_0$ is i-regular and $|f\lceil n_0|_i = 1$, and it is the first element of this sequence which belongs to $\mathcal{R}_i$. Clearly, $n_0 = \mathrm{lh}(f\lceil n_0)$.

Suppose that $n_0, l_0, b_0, \dots, n_r$ have already been listed. Since $f\lceil l_r \simeq \mu_i(f\lceil(n_r + 1), S^i_r)$, we can find effectively in $f^{(i+1)}$ the partial finite part $f\lceil l_r$. Then $l_r = \mathrm{lh}(f\lceil l_r)$. Next $f\lceil b_r$ is an i-regular partial r omitting extension of $f\lceil l_r$. So, there exist natural numbers $l_r = q_0 < \cdots < q_r < q_{r+1} = b_r$. Using the oracle $f^{(i+1)}$, we can find consecutively the numbers $q_0, \dots, q_r, q_{r+1} = b_r$. By definition, $f\lceil n_{r+1}$ is an i-regular partial extension of $f\lceil(b_r + 1)$ having i-rank equal to $|f\lceil b_r|_i + 1$. Using f', we can generate consecutively the partial finite parts $f\lceil(b_r + 1 + q)$, $q = 0, 1, \dots$ Then $f\lceil n_{r+1}$ is the first element of this sequence which belongs to $\mathcal{R}_i$.

Then B^f_{i+1} is effective in $f^{(i+1)}$ and $B_{i+1} \leq_e f^{(i+1)}$.

To prove (2), assume that $f \leq_e \mathcal{P}_i$. Then the set

$$S = \{\tau : \tau \in \mathcal{R}_i \ \& \ (\exists x, y_1 \neq y_2 \in \mathbb{N})(\tau(x) \simeq y_1 \ \& \ f(x) \simeq y_2)\},$$

$S \leq_e \mathcal{P}_i$. Using the fact that f is $(i+1)$-generic, there is an i-regular partial finite part $\tau \subseteq f$ such that either $\tau \in S$ or $(\forall \rho \supseteq_i \tau)(\rho \notin S)$. It is obvious that both of these cases are impossible. A contradiction. □

Lemma 8.2. *Let $i \leq n$ and τ be an i-regular partial finite part with domain $[0, q-1]$.*

(1) *For every $y \in \mathbb{N}, z_0 \in B_0, \dots, z_i \in B_i$, we can find effectively in $\mathcal{P}'_{i-1}$ an i-regular partial extension ρ of τ such that $|\rho|_i = |\tau|_i + 1$ and $\rho(q) \simeq y, z_0 \in \rho(B^\rho_0), \dots, z_i \in \rho(B^\rho_i)$.*

(2) *For every sequence* $\vec{a} = a_0, \ldots, a_m$ *of natural numbers, one can find effectively in* $\mathcal{P}'_i$ *an* i*-regular* m *omitting partial extension* δ *of* τ *such that* $\delta(K^\delta_\tau) = \vec{a}$.

Proof. The proof is as in the total case [8]. By induction on i, (1) and (2) are proven simultaneously. □

Proof of Theorem 7.1. By Proposition 8.2, it is sufficient to show that there exists a partial regular enumeration f which is quasi-minimal over B_0.

We shall construct f as a union of n-regular partial finite parts δ_s such that for all s, $\delta_s \subseteq_n \delta_{s+1}$ and $|\delta_s|_n = s+1$. Suppose that for $i \leq n$ σ_i is a recursively in B_i enumeration of B_i.

Let δ_0 be a 0-regular partial finite part such that $|\delta_0|_n = 1$. Suppose that δ_s is defined. Set $z_0 = \sigma_0(s), \ldots, z_n\sigma_n(s)$. Using Lemma 8.2, we can construct effectively in $\mathcal{P}'_{n-1}$ an n-regular partial finite part $\rho \supseteq_n \delta_s$ such that $|\rho|_n|\delta_s|_n + 1$, $\rho(\mathrm{lh}(\delta_s)) = s$ and $z_0 \in \rho(B^\rho_0), \ldots, z_n \in \rho(B^\rho_n)$. Set $\delta_{s+1} = \rho$.

The obtained enumeration f is surjective on $\mathbb{N}$ and it is a union of n-regular partial finite parts. From the construction is obvious that for every $z \in B_i$ there is an i-regular partial finite part τ of f such that $z \in B^\tau_i$. Hence f is a partial regular enumeration. By Proposition 8.1, f is $(i+1)$-generic for each $i < n$.

Then by Proposition 8.2, for $i \leq n$, $B_i \leq f^{(i)}$. Moreover, f is a partial 1-generic enumeration and hence $B_0 <_e f$.

To prove that f is quasi-minimal over B_0, it is sufficient to show that if ψ is a total function and $\psi \leq_e f$, then $\psi \leq_e B_0$. It is clear that for any total set $A \subseteq \mathbb{N}$ one can construct a total function ψ, $\psi \equiv_e A$. Let ψ be a total function and $\psi = \Gamma_e(f)$. Then

$$(\forall x, y \in \mathbb{N})(f \models_0 F_e(\langle x, y\rangle) \iff \psi(x) \simeq y).$$

Consider the set

$$S_0 = \{\rho : \rho \in \mathcal{R}_0 \ \&\ (\exists x, y_1 \neq y_2 \in \mathbb{N})(\rho \Vdash_0 F_e(\langle x, y_1\rangle) \ \&\ \rho \Vdash_0 F_e(\langle x, y_2\rangle))\}.$$

Since $S_0 \leq_e B_0$, we have that there exists a 0-regular partial finite part $\tau_0 \subseteq f$ such that either $\tau_0 \in S_0$ or $(\forall \rho \supseteq_0 \tau_0)(\rho \notin S_0)$. Assume that $\tau_0 \in S_0$. Then there exist $x, y_1 \neq y_2$ such that $f \models_0 F_e(\langle x, y_1\rangle)$ and $f \models_0 F_e(\langle x, y_2\rangle)$. Then $\psi(x) \simeq y_1$ and $\psi(x) \simeq y_2$, which is impossible. So, $(\forall \rho \supseteq_0 \tau_0)(\rho \notin S_0)$.

Let

$$\begin{aligned} S_1 = \{\rho : \rho \in \mathcal{R}_0 \ \&\ & (\exists \tau \supseteq_0 \tau_0)(\exists \delta_1 \supseteq_0 \tau)(\exists \delta_2 \supseteq_0 \tau) \\ & (\exists x, y_1 \neq y_2 \in \mathbb{N})(\tau \subseteq_0 \rho \ \&\ \delta_1 \Vdash_0 F_e(\langle x, y_1\rangle) \ \&\ \delta_2 \Vdash_0 F_e(\langle x, y_2\rangle) \ \&\ \\ & \mathrm{dom}(\rho) = \mathrm{dom}(\delta_1) \cup \mathrm{dom}(\delta_2) \ \&\ \\ & (\forall x)(x \in \mathrm{dom}(\rho) \setminus \mathrm{dom}(\tau) \Rightarrow \rho(x) \simeq \bot))\}. \end{aligned}$$

We have that $S_1 \leq_e B_0$ and hence there exists a 0-regular partial finite part $\tau_1 \subseteq f$ such that either $\tau_1 \in S_1$ or $(\forall \rho \supseteq_0 \tau_1)(\rho \notin S_1)$.

Assume that $\tau_1 \in S_1$. Then there exists a 0-regular partial finite part τ such that $\tau_0 \subseteq_0 \tau \subseteq_0 \tau_1$ and for some $\delta_1 \supseteq_0 \tau$, $\delta_2 \supseteq_0 \tau$ and $x_0, y_1 \neq y_2 \in \mathbb{N}$ we have

$$\delta_1 \Vdash_0 F_e(\langle x_0, y_1\rangle) \ \&\ \delta_2 \Vdash_0 F_e(\langle x_0, y_2\rangle) \ \&\ \operatorname{dom}(\tau_1) = \operatorname{dom}(\delta_1) \cup \operatorname{dom}(\delta_2) \ \&$$
$$\&\ (\forall x)(x \in \operatorname{dom}(\tau_1) \setminus \operatorname{dom}(\tau) \Rightarrow \tau_1(x) \simeq \bot).$$

Let $\psi(x_0) \simeq y$. Then $f \models_0 F_e(\langle x_0, y\rangle)$. Hence there exists a $\rho \supseteq_0 \tau_1$ such that $\rho \Vdash_0 F_e(\langle x_0, y\rangle)$. Let $y \neq y_1$. Define the partial finite part ρ_0 as follows:

$$\rho_0(x) \simeq \begin{cases} \delta_1(x) & \text{if } x \in \operatorname{dom}(\delta_1), \\ \rho(x) & \text{if } x \in \operatorname{dom}(\rho) \setminus \operatorname{dom}(\delta_1). \end{cases}$$

Then $\tau_0 \subseteq_0 \rho_0$, $\delta_1 \subseteq_0 \rho_0$ and notice that for all $x \in \operatorname{dom}(\rho)$ if $\rho(x) \not\simeq \bot$, then $\rho(x) \simeq \rho_0(x)$. Hence $\rho_0 \Vdash_0 F_e(\langle x_0, y_1\rangle)$ and $\rho_0 \Vdash_0 F_e(\langle x_0, y\rangle)$. So, $\rho_0 \in S_0$. A contradiction.

Thus, $(\forall \rho)(\rho \supseteq_0 \tau_1 \Rightarrow \rho \notin S_1)$.

Let $\tau = \tau_1 \cup \tau_0$. Notice that $\tau \subseteq f$. We shall show that

$$\psi(x) \simeq y \iff (\exists \delta \supseteq_0 \tau)(\delta \Vdash_0 F_e(\langle x, y\rangle)).$$

And hence $\psi \leq_e B_0$.

If $\psi(x) \simeq y$, then $f \models_0 F_e(x)$, and by Lemma 8.1 $(\exists \rho \subseteq f)(\rho \Vdash_0 F_e(x))$ and ρ is 0-regular. Then take $\delta = \tau \cup \rho$.

Assume that $\delta_1 \supseteq_0 \tau$, $\delta_1 \Vdash_0 F_e(\langle x, y_1\rangle)$. Suppose that $\psi(x) \simeq y_2$ and $y_1 \neq y_2$. Then there exists a $\delta_2 \supseteq_0 \tau$ such that $\delta_2 \Vdash_0 F_e(\langle x, y_2\rangle)$. Set

$$\rho(x) \simeq \begin{cases} \tau(x) & \text{if } x \in \operatorname{dom}(\tau), \\ \bot & \text{if } x \in (\operatorname{dom}(\delta_1) \cup \operatorname{dom}(\delta_2)) \setminus \operatorname{dom}(\tau). \end{cases}$$

Clearly, $\rho \supseteq_0 \tau_1$ and $\rho \in S_1$. A contradiction. □

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: asoskova@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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RELATIVELY INTRINSICALLY ARITHMETICAL SETS

STELA NIKOLOVA

Sets that have Σ_n^0 (Π_n^0, arithmetical) associates in every partial enumeration of given countable abstract structure are considered. Some minimal classes of partial enumerations are obtained such that admissibility in every such class yields the respective definability.

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1. INTRODUCTION

Let $\mathfrak{A} = (B; P_1, \ldots, P_m)$ be a total countable relational structure. *Partial enumeration* of $\mathfrak{A}$ is an ordered pair $(f, \mathfrak{B})$, where f is a partial function from the set of all naturals N onto B, $\mathfrak{B}$ is a total structure over N and the mapping $f \upharpoonright Dom(f)$ is a strong homomorphism from $\mathfrak{B} \upharpoonright Dom(f)$ onto $\mathfrak{A}$. An associate of a set $A \subseteq B$ (in the enumeration $(f, \mathfrak{B})$) is a set $W \subseteq N$ such that $W \cap Dom(f) = f^{-1}(A)$, i.e. the pullback $f^{-1}(A)$ is exactly the set W, restricted to $Dom(f)$. Following [2], say that the set A is *relatively intrinsically* Σ_n^0 (Π_n^0, *arithmetical*) if for every partial enumeration $(f, \mathfrak{B})$ it has an associate that is Σ_n^0 (Π_n^0, arithmetical) in $\mathfrak{B}$. This is a typical implicit definition of complexity of a set A over an abstract structure and a natural question that arises here is whether the set A could be described also explicitely. And further, if such an explicite characterization does exist, is it necessary to involve the whole class of partial enumerations in order to obtain it? In other words, does there exist some smaller class of partial enumerations such that the fact that A has an appropriate associate in every enumeration in this class yields the respective explicite characterization of A?

Results that answer the last question can be found in [8], where Σ_n^0-admissible sets are considered (although in another context), and in [4], where a minimal class of enumerations for the Σ_1^0-admissible sets is obtained. Here we further extend the investigations from [5], considering also relatively intrinsically Π_n^0 and arithmetical sets.

2. PRELIMINARIES

Let us fix a relational abstract structure $\mathfrak{A} = (B; P_1, \ldots, P_m)$, where B is at most denumerable and each $P_i, 1 \leq i \leq m$, is a total predicate of k_i arguments on B. The equality relation is not supposed to be among the initial predicates of $\mathfrak{A}$.

Definition 2.1. *Partial enumeration* (of the structure $\mathfrak{A}$) is an ordered pair $(f, \mathfrak{B})$, where f is a partial function from the set of all natural numbers N onto B and $\mathfrak{B} = (N; Q_1, \ldots, Q_m)$ is a total structure in the signature of $\mathfrak{A}$ such that for every $1 \leq i \leq m$ the equivalence

$$Q_i(x_1, \ldots, x_{k_i}) \iff P_i(f(x_1), \ldots, f(x_{k_i}))$$

holds whenever $x_1, \ldots, x_{k_i}$ are in $Dom(f)$.

The set $Dom(f)$ is the *domain* of the enumeration $(f, \mathfrak{B})$. We shall classify the enumerations of $\mathfrak{A}$ with respect to the complexity of their domains.

Let $D(\mathfrak{B})$ be the atomic diagram of $\mathfrak{B}$, more precisely

$$D(\mathfrak{B}) = \{\langle i, x_1, \ldots, x_{k_i}, \varepsilon\rangle | Q_i(x_1, \ldots, x_{k_i}) = \varepsilon, 1 \leq i \leq m\},$$

where $\langle \ldots \rangle$ is some effective coding of all finite sequences over N, which we shall suppose fixed until the end of this work. (We shall identify the boolean constants `true` and `false` with 0 and 1, respectively).

Definition 2.2. The enumeration $(f, \mathfrak{B})$ is Σ_n^0 (Π_n^0) iff the set $Dom(f)$ is Σ_n^0 (Π_n^0) in the diagram $D(\mathfrak{B})$ of $\mathfrak{B}$.

Definition 2.3. The enumeration $(f, \mathfrak{B})$ is *arithmetical* iff the set $Dom(f)$ is arithmetical in the diagram $D(\mathfrak{B})$, i.e. $Dom(f)$ is Σ_n^0 or Π_n^0 in $D(\mathfrak{B})$ for some $n \geq 1$.

Definition 2.4. Let A be a subset of B^k. The set $W \subseteq N^k$ is called an *associate* of A (in the enumeration $(f, \mathfrak{B})$) iff the equivalence

$$(x_1, \ldots, x_k) \in W \iff (f(x_1), \ldots, f(x_k)) \in A$$

holds for all $x_1, \ldots, x_k$ in $Dom(f)$.

Obviously, if f is not total, the set A has many associates.

Definition 2.5. Say that a set $A \subseteq B^k$ is Σ_n^0- (Π_n^0-, *arithmetically*) *admissible in* $(f, \mathfrak{B})$ if A has an associate, which is Σ_n^0 (Π_n^0, arithmetical) in $D(\mathfrak{B})$.

Remark. If we stick to the terminology from [1] and [2], kept also in [6] and [7], we should call the above sets *relatively intrinsically* Σ_n^0 (Π_n^0, *arithmetical*) in $(f, \mathfrak{B})$. We, however, will use the shorter term "admissible", which come from the LACOMBE'S notion of $\forall$-admissibility [3].

Next we introduce Σ_n^0 and Π_n^0, $n \geq 0$, formulas in a recursive fragment of the language $L_{\omega_1,\omega}$ of $\mathfrak{A}$. The definition is by simultaneous induction on n. For that purpose to each formula we assign (at least one) index.

We assume that we have chosen some effective coding κ of all atomic formulas in the first-order language $L_\mathfrak{A}$ of $\mathfrak{A}$, extended with the logical constants T and F (denote it by $L_\mathfrak{A}^+$). Throughout the paper, we shall suppose also that some effective enumeration $W_0, W_1, \ldots$ of all recursively enumerable (r. e.) subsets of N is fixed.

Definition 2.6. (i) Every atomic formula Φ in $L_\mathfrak{A}^+$ is a Σ_0^0 formula with an index $\langle 0, 0, \kappa(\Phi) \rangle$.

Every negated atomic formula $\neg\Phi$ in $L_\mathfrak{A}^+$ is a Π_0^0 formula with an index $\langle 1, 0, \kappa(\Phi) \rangle$.

Every finite conjunction $\Phi_1 \& \ldots \& \Phi_l$ of Σ_0^0 or Π_0^0 formulas with indices $v_1, \ldots, v_l$, respectively, is a Δ_1^0 formula with an index $\langle 2, v_1, \ldots, v_l \rangle$.

(ii) If every $v \in W_e$ is an index of a Δ_{n+1}^0 formula Φ^v, whose free variables are among $X_1, \ldots, X_k$, then

$$\bigvee_{v \in W_e} \Phi^v$$

is a Σ_{n+1}^0 formula with an index $\langle 0, n+1, e \rangle$ (with free variables among $X_1, \ldots, X_k$).

If Φ is $\neg\Psi$, where Ψ is a Σ_{n+1}^0 formula with an index $\langle 0, n+1, e \rangle$, then Ψ is a Π_{n+1}^0 formula with an index $\langle 1, n+1, e \rangle$.

If Φ is $\Psi_1 \& \ldots \& \Psi_l$ and every Ψ_j is a Σ_m^0 or Π_m^0, $0 \leq m \leq n+1$, formula with an index v_j, $1 \leq j \leq l$, then Φ is a Δ_{n+2}^0 formula with an index $\langle 2, v_1, \ldots, v_l \rangle$.

Definition 2.7. A set $A \subseteq B^k$ is Σ_n^0 (Π_n^0) *definable* on $\mathfrak{A}$ iff there exists some Σ_n^0 (Π_n^0) formula Φ with variables among $X_1, \ldots, X_k, Y_1, \ldots, Y_l$ and elements $t_1, \ldots, t_l$ of B such that for every $(s_1, \ldots, s_k) \in B^k$

$$(s_1, \ldots, s_k) \in A \iff \mathfrak{A} \models \Phi(X_1/s_1, \ldots, X_k/s_k, Y_1/t_1, \ldots, Y_l/t_l).$$

Clearly, if a set A is Σ_n^0 (Π_n^0) definable on $\mathfrak{A}$, then A is Σ_n^0 (Π_n^0) -admissible in every enumeration $(f, \mathfrak{B})$ of $\mathfrak{A}$.

In order to save space, from now on we shall consider only subsets of B. All the results can be easily generalized for subsets of B^k for arbitrary $k \geq 1$.

Let $(f, \mathfrak{B})$ be an enumeration of $\mathfrak{A}$. We first introduce a satisfaction relation $(f, \mathfrak{B}) \models_n F_e(x)$. For our purposes, it is suitable to make a slight deviation from the standard satisfaction relation for the Σ^0_n in $D(\mathfrak{B})$ sets (as it is in [8], for example). Let $U(e, x)$ be an universal function for the class of all unary primitive recursive functions. Using the S^m_n-theorem, we obtain a recursive function h such that for every index e

$$W_{h(e)} = \{U(e, x) | x \in N\}.$$

It is well known that a nonempty set $W \subseteq N$ is r. e. iff $W = W_{h(e)}$ for some index e. We shall suppose that the function h is fixed until the end of this work. It will appear in the definitions of the basic notions of forcing and satisfaction relation.

We begin with the definition of the satisfaction relation $\models_n$, which is by induction on n. As customary, D_v will denote the finite set with canonical index v.

Definition 3.1. Set

$$\begin{aligned}
(f, \mathfrak{B}) \models u \quad &\iff \quad \exists i \exists x_1 \ldots \exists x_{k_i} \exists \varepsilon (1 \leq i \leq m \ \&\ u = \langle i, x_1, \ldots, x_{k_i}, \varepsilon \rangle \ \& \\
&\qquad Q_i(x_1, \ldots, x_{k_i}) = \varepsilon), \\
(f, \mathfrak{B}) \models_1 F_e(x) \quad &\iff \quad \exists v (\langle v, x \rangle \in W_{h(e)} \ \&\ \forall u \in D_v (f, \mathfrak{B}) \models u)), \\
(f, \mathfrak{B}) \models_{n+1} F_e(x) \quad &\iff \quad \exists v (\langle v, x \rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y (u = \langle d, y, 0 \rangle \& \\
&\qquad (f, \mathfrak{B}) \models_n F_d(y) \ \vee \ u = \langle d, y, 1 \rangle \ \&\ (f, \mathfrak{B}) \not\models_n F_d(y))).
\end{aligned}$$

Put finally

$$(f, \mathfrak{B}) \models_n \neg F_e(x) \quad \iff \quad (f, \mathfrak{B}) \not\models_n F_e(x).$$

The next fact is a direct consequence of Proposition 3.3 of [8] and our choice of the satisfaction relation $\models_n$.

Proposition 3.1. (i) *If $W \subseteq N$ is Σ^0_n in $D(\mathfrak{B})$, then there exists an index e such that $W = \{x | (f, \mathfrak{B}) \models_n F_e(x)\}$.*

(ii) *If $W \subseteq N$ is Π^0_n in $D(\mathfrak{B})$, then $W = \{x | (f, \mathfrak{B}) \models_n \neg F_e(x)\}$ for some index e.*

Definition 3.2. *Finite part* is an $(m + 2)$–tuple

$$\tau = (f_\tau, H_\tau, q^\tau_1, \ldots, q^\tau_m),$$

where f_τ is a finite function from N into B, $H_\tau \subseteq N$, $Dom(f_\tau) \cap H_\tau = \emptyset$, $Dom(f_\tau) \cup H_\tau = \{0, \ldots, l - 1\}$ for some $l \in N$ and $q^\tau_i, 1 \leq i \leq m$, is a partial predicate of k_i arguments on $\{0, \ldots, l - 1\}$ such that for every $x_1, \ldots, x_{k_i}$ in $Dom(f_\tau)$

$$q^\tau_i(x_1, \ldots, x_{k_i}) \iff P_i(f_\tau(x_1), \ldots, f_\tau(x_{k_i})).$$

The set $Dom(f_\tau) \cup H_\tau$, which is in fact the initial segment $[0, l)$ of N, we shall call *domain* of τ ($Dom(\tau)$); l is the *length* of τ (in symbols $|\tau|$). If $l = 0$, τ is the *empty* finite part. We shall use small Greek letters to denote finite parts.

Below we introduce three types of binary relations between finite parts that model in a different way the notion "extention of a finite part".

Definition 3.3. Let $\tau = (f_\tau, H_\tau, q_1^\tau, \dots, q_m^\tau)$ and $\delta = (f_\delta, H_\delta, q_1^\delta, \dots, q_m^\delta)$ be finite parts. Set

$$\begin{aligned} \tau \subseteq \delta &\iff f_\tau \subseteq f_\delta \ \&\ H_\tau \subseteq H_\delta \ \&\ q_1^\tau \subseteq q_1^\delta \ \&\ \dots \ \&\ q_m^\tau \subseteq q_m^\delta, \\ \tau \leqslant \delta &\iff \tau \subseteq \delta \ \&\ f_\tau = f_\delta, \\ \tau \preccurlyeq \delta &\iff \tau \leqslant \delta \ \&\ H_\tau = H_\delta. \end{aligned}$$

Clearly, these three relations are partial orderings. We shall sometimes write $\tau \supseteq \delta, \tau \geqslant \delta$, etc. for $\delta \subseteq \tau, \delta \leqslant \tau$, etc.

Definition 3.4. The enumeration $(f, \mathfrak{B} = (N; Q_1, \dots, Q_m))$ *extends* τ (in symbols $\tau \subseteq (f, \mathfrak{B})$) iff $f_\tau \subseteq f$, $H_\tau \subseteq N \setminus Dom(f)$ and $q_i^\tau \subseteq Q_i$ for every $i = 1, \dots, m$.

Next we define the forcing relation $\Vdash_n$ again by induction on n. Notice that in the definition of $\Vdash_n \neg \dots$ we use the strongest numerical extention $\geqslant$ instead of the usual $\supseteq$. This type of forcing is called a "starred forcing" in [8].

Definition 3.5.

$$\begin{aligned} \tau \Vdash u &\iff \exists i \exists x_1 \dots \exists x_{k_i} \exists \varepsilon (1 \le i \le m \ \&\ u = \langle i, x_1, \dots, x_{k_i}, \varepsilon \rangle \ \& \\ &\qquad q_i^\tau (x_1, \dots, x_{k_i}) = \varepsilon), \\ \tau \Vdash_1 F_e(x) &\iff \exists v (\langle v, x \rangle \in W_{h(e)} \ \&\ \forall u \in D_v (\tau \Vdash u)), \\ \tau \Vdash_1 \neg F_e(x) &\iff \forall \rho (\rho \geqslant \tau \Rightarrow \rho \nVdash_1 F_e(x)), \\ \tau \Vdash_{n+1} F_e(x) &\iff \exists v (\langle v, x \rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y (u = \langle d, y, 0 \rangle \& \\ &\qquad \tau \Vdash_n F_d(y) \ \vee \ u = \langle d, y, 1 \rangle \ \&\ \tau \Vdash_n \neg F_d(y))), \\ \tau \Vdash_{n+1} \neg F_e(x) &\iff \forall \rho (\rho \geqslant \tau \Rightarrow \rho \nVdash_n F_e(x)). \end{aligned}$$

Lemma 3.1. *Let $n \geq 1$. For every finite part τ :*

(i) $\{(e, x) | \tau \Vdash_n F_e(x)\}$ *is a* Σ_n^0 *set;*

(ii) $\{(e, x) | \tau \Vdash_n \neg F_e(x)\}$ *is a* Π_n^0 *set.*

Proof. Straightforward induction on n. The crucial point here is that we consider numerical extentions $\geqslant$ instead of $\supseteq$ in the definition of the forcing relation. □

In what follows, we shall need the following notion of *restriction* of a finite part τ to δ (for $\tau \supseteq \delta$).

Definition 3.6. Let $\tau \supseteq \delta$. Set

$$\tau | \delta = (f_\delta, H_\delta \cup (Dom(f_\tau) \setminus Dom(f_\delta)), q_1^\tau, \dots, q_m^\tau).$$

It can be easily checked that $\tau|\delta$ is also a finite part and $\tau|\delta \geqslant \delta$.

The next important property of the restrictions will be systematically used in the sequel.

Lemma 3.2. *Let δ be a finite part. For every $n \geq 1$:*

(i) $\forall \tau \supseteq \delta(\tau \Vdash_n F_e(x) \iff \tau|\delta \Vdash_n F_e(x))$;

(ii) $\forall \tau \supseteq \delta(\tau \Vdash_n \neg F_e(x) \iff \tau|\delta \Vdash_n \neg F_e(x))$.

Proof. Induction on n. The validity of (i) for $n = 1$ follows from the obvious equivalence

$$\tau \Vdash u \iff \tau|\delta \Vdash u.$$

Assume that (i) is true for some $n \geq 1$. We shall show first that (ii) is also true for n and then, using this fact and the induction hypothesis, will establish (i) for $n + 1$.

Indeed, take some $\tau \supseteq \delta$ such that $\tau \Vdash_n \neg F_e(x)$. We have to see that $\tau|\delta \Vdash_n \neg F_e(x)$. Assuming that this is not true, we will have that for some $\rho \geqslant \tau|\delta$: $\rho \Vdash_n F_e(x)$. We have $\rho \geqslant \tau|\delta \geqslant \delta$, so by induction hypothesis $\rho|\delta \Vdash_n F_e(x)$. Now consider the tuple

$$\rho_1 = (f_\tau, H_\rho \setminus Dom(f_\tau), q_1^\rho, \dots, q_m^\rho).$$

Let us first check that ρ_1 is a finite part. Obviously, $Dom(f_\tau)$ and $H_\rho \setminus Dom(f_\tau)$ are disjoint. Further, since $f_\delta = f_\rho$, we have that $H_\rho \cap Dom(f_\delta) = \emptyset$ and hence

$$Dom(f_\tau) \cup (H_\rho \setminus Dom(f_\tau)) = (Dom(f_\delta) \cup (Dom(f_\tau) \setminus Dom(f_\delta))) \cup$$

$$(H_\rho \setminus (Dom(f_\tau) \setminus Dom(f_\delta))) = Dom(f_\delta) \cup H_\rho = Dom(f_\rho) \cup H_\rho,$$

which is an initial segment. So,

$$Dom(q_i^{\rho_1}) = Dom(q_i^\rho) \subseteq Dom(f_\rho) \cup H_\rho = Dom(f_{\rho_1}) \cup H_{\rho_1}.$$

Finally, if $x_1, \dots, x_{k_i}$ are in $Dom(f_{\rho_1}) = Dom(f_\tau)$, then

$$!q_i^\tau(x_1, \dots, x_{k_i}) \text{ and } q_i^\tau(x_1, \dots, x_{k_i}) = P_i(f_\tau(x_1), \dots, f_\tau(x_{k_i})).$$

However, $\rho \geqslant \tau|\delta$, hence $q_i^\rho \supseteq q_i^\tau$, so $q_i^{\rho_1}(x_1, \dots, x_{k_i}) = q_i^\rho(x_1, \dots, x_{k_i})$ is defined and is equal to $P_i(f_\tau(x_1), \dots, f_\tau(x_{k_i}))$, which is actually $P_i(f_{\rho_1}(x_1), \dots, f_{\rho_1}(x_{k_i}))$, hence ρ_1 is a finite part indeed.

It can be easily checked that $\rho_1 \geqslant \tau$ and $\rho_1|\delta = \rho|\delta$. As we have seen above, $\rho|\delta \Vdash_n F_e(x)$, hence $\rho_1|\delta \Vdash_n F_e(x)$. From here, using again the induction hypothesis and the fact that $\rho_1|\delta \supseteq \delta$, we get $\rho_1 \Vdash_n F_e(x)$, which contradicts the fact that $\tau \Vdash_n \neg F_e(x)$.

Conversely, suppose that $\tau|\delta \Vdash_n \neg F_e(x)$ and towards contradiction assume that $\tau \nVdash_n \neg F_e(x)$. Therefore there exists $\rho \geqslant \tau$ with $\rho \Vdash_n F_e(x)$. We have $\rho \geqslant \tau \supseteq \delta$, so by induction hypothesis $\rho|\delta \Vdash_n F_e(x)$. Further, $\rho|\delta \geqslant \tau|\delta$, which follows immediately from the fact that $\rho \geqslant \tau \supseteq \delta$. However, $\tau|\delta \Vdash_n \neg F_e(x)$ and we could not have $\rho|\delta \Vdash_n F_e(x)$, which is the desired contradiction.

Let us now check the validity of (i) for $n+1$. Indeed, we have that (i) and (ii) are true for n, so

$$\begin{aligned}
\tau \Vdash_{n+1} F_e(x) \quad &\Longleftrightarrow \quad \exists v(\langle v,x\rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y(u = \langle d,y,0\rangle \& \\
&\qquad \tau \Vdash_n F_d(y) \ \vee \ u = \langle d,y,1\rangle \ \&\ \tau \Vdash_n \neg F_d(y))) \\
&\Longleftrightarrow \quad \exists v(\langle v,x\rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y(u = \langle d,y,0\rangle \& \\
&\qquad \tau|\delta \Vdash_n F_d(y) \ \vee \ u = \langle d,y,1\rangle \ \&\ \tau|\delta \Vdash_n \neg F_d(y))) \\
&\Longleftrightarrow \quad \tau|\delta \Vdash_{n+1} F_e(x). \square
\end{aligned}$$

Using Lemma 3.2, one can easily get the monotonicity of the forcing relation.

Lemma 3.3. (i) $\delta \Vdash_n F_e(x) \ \& \tau \supseteq \delta \ \Rightarrow \tau \Vdash_n F_e(x)$;

(ii) $\delta \Vdash_n \neg F_e(x) \ \& \tau \supseteq \delta \ \Rightarrow \tau \Vdash_n \neg F_e(x)$.

Proof. Let us first see the validity of (ii). Suppose that $\delta \Vdash_n \neg F_e(x)$, $\tau \supseteq \delta$, but $\tau \nVdash_n \neg F_e(x)$. Then for some $\rho \geqslant \tau, \rho \Vdash_n F_e(x)$. Since $\rho \geqslant \tau \supseteq \delta$, applying Lemma 3.2, we get $\rho|\delta \Vdash_n F_e(x)$. This, together with the fact that $\rho|\delta \geqslant \delta$, contradicts the assumption $\delta \Vdash_n \neg F_e(x)$.

Now (i) is by induction on n. Dropping the obvious case $n = 1$, suppose that (i) is true for some n. We have also that (ii) is true for this n, so

$$\begin{aligned}
\delta \Vdash_{n+1} F_e(x) \quad &\Longleftrightarrow \quad \exists v(\langle v,x\rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y(u = \langle d,y,0\rangle \& \\
&\qquad \delta \Vdash_n F_d(y) \vee \ u = \langle d,y,1\rangle \ \&\ \delta \Vdash_n \neg F_d(y))) \\
&\Rightarrow \quad \exists v(\langle v,x\rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y(u = \langle d,y,0\rangle \& \\
&\qquad \tau \Vdash_n F_d(y) \vee \ u = \langle d,y,1\rangle \ \&\ \tau \Vdash_n \neg F_d(y))) \\
&\Longleftrightarrow \quad \tau \Vdash_{n+1} F_e(x). \square
\end{aligned}$$

Let us remind some basic notions from the forcing constructions machinery.

Definition 3.7. (i) Let X be a set of finite parts. The enumeration $(f, \mathfrak{B})$ *meets* X if $\exists \delta(\delta \in X \ \&\ \delta \subseteq (f, \mathfrak{B}))$.

(ii) X is *dense* in $(f, \mathfrak{B})$ if $\forall \delta \subseteq (f, \mathfrak{B}) \exists \tau \supseteq \delta(\tau \in X)$.

(iii) Let $\mathfrak{F}$ be a family of sets of finite parts. The enumeration $(f, \mathfrak{B})$ is $\mathfrak{F}$-*generic* if for every $X \in \mathfrak{F}$ the following condition holds:

if X is dense in $(f, \mathfrak{B})$, then $(f, \mathfrak{B})$ meets X.

Set $X^k_{e,x} = \{\tau | \tau \Vdash_k F_e(x)\}$ and let

$$\mathfrak{F}_n = \bigcup_{e,x \in N, 1 \leq k \leq n} X^k_{e,x}.$$

We have the following Truth Lemma that brings together the forcing and satisfaction relation.

Lemma 3.4. *Let $n \geq 1$. Then*

(i) *If $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic enumeration, then*

$$(f, \mathfrak{B}) \models_n F_e(x) \iff \exists \tau \subseteq (f, \mathfrak{B})(\tau \Vdash_n F_e(x)).$$

(ii) *If $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic enumeration, then*

$$\exists \tau \subseteq (f, \mathfrak{B})(\tau \Vdash_n \neg F_e(x)) \Rightarrow (f, \mathfrak{B}) \models_n \neg F_e(x).$$

(iii) *If $(f, \mathfrak{B})$ is $\mathfrak{F}_n$-generic enumeration, then*

$$(f, \mathfrak{B}) \models_n \neg F_e(x) \Rightarrow \exists \tau \subseteq (f, \mathfrak{B})(\tau \Vdash_n \neg F_e(x)).$$

Proof. Induction on n. It is straightforward that for every enumeration $(f, \mathfrak{B})$

$$(f, \mathfrak{B}) \models_1 F_e(x) \iff \exists \tau \subseteq (f, \mathfrak{B})(\tau \Vdash_1 F_e(x)),$$

hence (i) is true for $n = 1$. Now assume that (i) holds for an arbitrary $n \geq 1$. We shall successively check that (ii) and (iii) also hold for this n and after that – that (i) is true for $n + 1$.

Indeed, let $(f, \mathfrak{B})$ be $\mathfrak{F}_{n-1}$-generic, $\tau \subseteq (f, \mathfrak{B})$ and $\tau \Vdash_n \neg F_e(x)$. Towards a contradiction, assume that $(f, \mathfrak{B}) \models_n F_e(x)$. By induction hypothesis $\exists \delta \subseteq (f, \mathfrak{B})$: $\delta \Vdash_n F_e(x)$. Now denote by $\tau \cup \delta$ the tuple

$$(f_\tau \cup f_\delta, H_\tau \cup H_\delta, q_1^\tau \cup q_1^\delta, \ldots, q_m^\tau \cup q_m^\delta).$$

Since τ and δ have a common extension — the enumeration $(f, \mathfrak{B})$, it can be easily seen that $\tau \cup \delta$ is a finite part, too. We have $\tau \cup \delta \supseteq \tau$, $\tau \cup \delta \supseteq \delta$, and by Lemma 3.3 $\tau \cup \delta \models_n \neg F_e(x)$ and at the same time $\tau \cup \delta \models_n F_e(x)$, which is impossible.

Now let $(f, \mathfrak{B})$ be $\mathfrak{F}_n$-generic and suppose that $(f, \mathfrak{B}) \models_n \neg F_e(x)$. We have to see that there exists $\tau \subseteq (f, \mathfrak{B})$ such that $\tau \Vdash_n \neg F_e(x)$. Indeed, assume that for every finite part $\tau \subseteq (f, \mathfrak{B})$, $\tau \nVdash_n \neg F_e(x)$. This means that

$$\forall \tau \subseteq (f, \mathfrak{B}) \exists \rho \supseteq \tau(\rho \Vdash_n F_e(x)),$$

in other words, $X^n_{e,x}$ is dense in $(f, \mathfrak{B})$. However, $X^n_{e,x}$ is in $\mathfrak{F}_n$ and $(f, \mathfrak{B})$ is $\mathfrak{F}_n$-generic, hence $(f, \mathfrak{B})$ meets $X^n_{e,x}$, i.e. there exists $\tau \subseteq (f, \mathfrak{B})$ such that $\tau \in X^n_{e,x}$, in other words, $\tau \Vdash_n F_e(x)$, according to our choice of $X^n_{e,x}$. Now applying (i) for n (notice that $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic, too), we obtain $(f, \mathfrak{B}) \models_n F_e(x)$ — a contradiction.

It remains to see the validity of (i) for $n + 1$. Again take some $\mathfrak{F}_n$-generic enumeration $(f, \mathfrak{B})$ and suppose that $(f, \mathfrak{B}) \models_{n+1} F_e(x)$. Hence there exists $\langle v, x \rangle \in W_{h(e)}$ such that

$$\forall u \in D_v \exists d \exists y(u = \langle d, y, 0 \rangle \& \ (f, \mathfrak{B}) \models_n F_d(y) \vee u = \langle d, y, 1 \rangle \ \& \ (f, \mathfrak{B}) \models_n \neg F_d(y))).$$

Since (i) and (iii) are true for n, we have that for every $u = \langle d_u, y_u, \varepsilon_u \rangle$ in D_v there is some $\tau_u \subseteq (f, \mathfrak{B})$ such that $\tau_u \Vdash_n (\neg)^{\varepsilon_u} F_{d_u}(y_u)$. Again $\tau = \bigcup\{\tau_u | u \in D_v\}$ is a finite part and by the monotonicity of the forcing relation, $\tau \Vdash_n (\neg)^{\varepsilon_u} F_{d_u}(y_u)$ for every $u \in D_v$, hence $\tau \Vdash_{n+1} F_e(x)$.

The verification of the opposite direction of (i) is very similar – this time use the validity of (i) and (ii) for our n and the monotonicity of the satisfaction relation. Notice that in this direction of (i) it is sufficient to have $\mathcal{F}_{n-1}$-genericity of the enumeration $(f, \mathfrak{B})$ (as it is in the case of the relation $\tau \Vdash_n \neg F_e(x)$, point (ii)). We, however, will not need this refinement for the positive case of the forcing relation. □

4. NORMAL FORMS

Suppose that $\tau = (f_\tau, H_\tau, q_1^\tau, \ldots, q_m^\tau)$ is a finite part, $x \in N$ is the first not in $Dom(\tau)$ (i.e. $x = |\tau|$) and $s \in B$. Then by $\tau * s$ we shall denote the tuple $(g, H_\tau, r_1, \ldots, r_m)$, where g is the function with a graph $G_{f_\tau} \cup \{(x, s)\}$ and for each $1 \leq i \leq m$, r_i is the predicate with a graph

$$G_{q_i} \cup \{(x_1, \ldots, x_{k_i}, \varepsilon) | (x_1, \ldots, x_{k_i}) \in Dom(g) \;\&\; P_i(g(x_1), \ldots, g(x_{k_i})) = \varepsilon\}.$$

Clearly, $\tau * s$ is a finite part, too.

Definition 4.1. (i) A set $A \subseteq B$ has a Σ_n^0 *normal form* if there exist a finite part δ and a natural number e such that for $x = |\delta|$ the equivalence

$$s \in A \iff \exists \rho (\rho \geqslant \delta * s \;\&\; \rho \Vdash_n F_e(x)) \tag{4.1}$$

holds for every $s \in B$.

(ii) A set $A \subseteq B$ has a Π_n^0 *normal form* if there exist a finite part δ and a natural number e such that for $x = |\delta|$ the equivalence

$$s \in A \iff \delta * s \Vdash_n \neg F_e(x))$$

holds for every $s \in B$.

Clearly, if the set A has a Σ_n^0 normal form, then $B \setminus A$ has a Π_n^0 normal form and vice versa.

Now we are in a position to prove a series of auxiliary propositions that make a connection between the implicit notion of admissibility and the explicit notion of normal form. Their proofs make use of generic enumerations and in essence follow the general scheme used in such type of constructions (in particular, the proof of Proposition 4.1 can be found in [8]). We formulate and prove them here not for the results themselves but rather for the precise constructions of the generic enumerations in their proofs. In the next section we shall explain how to refine these constructions, in order to obtain the main results in this work.

Proposition 4.1. *Let $n \geq 1$. If $A \subseteq B$ is Σ_n^0-admissible in every enumeration, then A has a Σ_n^0 normal form.*

Proof. Assume that A does not have a Σ_n^0 normal form. We shall construct an enumeration $(f, \mathfrak{B})$ (a refuting enumeration) such that A does not have a Σ_n^0 associate in it. The construction of $(f, \mathfrak{B})$ will be carried out in steps. Using induction on a, we shall define a sequence

$$\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_a \subseteq \ldots$$

of finite parts such that the set A is not admissible in any enumeration $(f, \mathfrak{B})$ that extends τ_a for every a. We shall make three types of steps. The first type (when $a \equiv 0(mod 3)$) will ensure that s is onto B, the second type is for $\mathfrak{F}_{n-1}$-genericity and the third type of steps will guarantee that A is not admissible in $(f, \mathfrak{B})$.

Let us fix an enumeration $s_0, s_1, \ldots$ of the elements of the basic set B. Set τ_0 to be the empty finite part and suppose that we have built τ_{3a} for some $a \geq 0$. We are going to explain how to define τ_{3a+1}. Let $a = \langle e, x, j\rangle$ and put $k = \min(j+1, n-1)$ (so we always have $1 \leq k < n$ for $n > 1$ and $k = 0$ if $n = 1$). If $k = 0$, set $\tau_{3a+1} = \tau_{3a}$ (since in this case $n = 1$ and no genericity is needed), otherwise ask the question "$\exists \rho(\rho \geqslant \tau_{3a} : \rho \Vdash_k F_e(x)$?". If yes, set $\tau_{3a+1} = \rho$ (take an arbitrary $\rho \geqslant \tau_{3a}$ such that $\rho \Vdash_k F_e(x)$), otherwise set $\tau_{3a+1} = \tau_{3a}$.

In order to define τ_{3a+2}, we will use the fact that the set A does not have a Σ_n^0 normal form. Hence the equivalence (4.1) is not true for $\delta = \tau_{3a+1}$ and $e = a$. This means that for $x = |\tau_{3a+1}|$ there exists $s \in B$ such that one of the following two conditions is true:

i) $s \in A$, but for every $\rho \geqslant \tau_{3a+1} * s$ we have $\rho \nVdash_n F_a(x)$;

ii) $s \notin A$, but there exists $\rho \geqslant \tau_{3a+1} * s$ such that $\rho \Vdash_n F_a(x)$.

In the first case put $\tau_{3a+2} = \tau_{3a+1} * s$, in the second case take an arbitrary $\rho : \rho \geqslant \tau_{3a+1} * s \,\&\rho \Vdash_n F_a(x)$ and set $\tau_{3a+2} = \rho$. Finally, set $\tau_{3a+3} = \tau_{3a+2} * s_a$.

Now define the tuple $(f, \mathfrak{B} = (N; Q_1, \ldots, Q_m))$ as follows:

$$f = \bigcup_a f_{\tau_a},$$

and for every $1 \leq i \leq m$ and $(x_1, \ldots, x_{k_i}) \in N^{k_i}$:

$$Q_i(x_1, \ldots, x_{k_i}) = \begin{cases} q_i^{\tau_a}(x_1, \ldots, x_{k_i}), & \text{if } \exists a \, !q_i^{\tau_a}(x_1, \ldots, x_{k_i}), \\ \text{arbitrary}, & \text{otherwise.} \end{cases}$$

Obviously, $(f,\mathfrak{B}) \supseteq \tau_a$ for every a. Since for every $s \in B$ there exists a such that $s \in Range(\tau_a)$, we have that $Range(f) = B$. The definition of the notion of finite part guarantees that f is a strong homomorphism from $\mathfrak{B}$ onto $\mathfrak{A}$, i. e. $(f, \mathfrak{B} = (N; Q_1, \ldots, Q_m))$ is an enumeration of $\mathfrak{A}$. Let us see now that $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic if $n > 1$. Indeed, take some $X_{e,x}^k$, $1 \leq k < n$, and suppose that $X_{e,x}^k$ is dense in $(f, \mathfrak{B})$, i. e. $\forall \tau \subseteq (f,\mathfrak{B}) \exists \rho \supseteq \tau : \rho \Vdash_k F_e(x)$. Take $a = \langle e, x, k-1\rangle$

and consider the step $3a+1$. From the density of $X^k_{e,x}$ it follows that for $\tau = \tau_{3a}$ there exists $\rho \supseteq \tau_{3a}$ such that $\rho \Vdash_k F_e(x)$. Hence, putting $\rho* = \rho|\tau_{3a}$, we will have, using Lemma 3.2, that $\rho* \geqslant \tau_{3a}$ and $\rho* \Vdash_k F_e(x)$. This means, according to our construction of $\{\tau_a\}_a$, that $\tau_{3a+1} \Vdash_k F_e(x)$ and $\tau_{3a+1} \subseteq (f, \mathfrak{B})$, i. e. $(f, \mathfrak{B})$ meets $X^k_{e,x}$, hence $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic.

Towards a contradiction, assume that A is Σ^0_n-admissible in this $(f, \mathfrak{B})$. Hence A has an associate W, which is Σ^0_n in $\mathfrak{B}$. Therefore, according to Proposition 3.1, $W = \{x | (f, \mathfrak{B}) \models_n F_a(x)\}$ for some a. We have that for every $x \in Dom(f)$

$$(f, \mathfrak{B}) \models_n F_a(x) \iff f(x) \in A. \tag{4.2}$$

Now have a look at step $3a+2$. If the case i) holds at this step, then for some $s \in A$ we will have that $\tau_{3a+2} = \tau_{3a+1} * s$ and $\tau_{3a+2} \Vdash_n \neg F_a(x)$. By definition $\tau_{3a+2}(x) = s$ for $x = |\tau_{3a+1}|$, hence $x \in Dom(f)$ and $f(x) = s \in A$. So according to (4.2), $(f, \mathfrak{B}) \models_n F_a(x)$. On the other hand, $\tau_{3a+2} \Vdash_n \neg F_a(x)$, which, combined with the $\mathfrak{F}_{n-1}$-genericity of $(f, \mathfrak{B})$ and Lemma 3.4, gives us $(f, \mathfrak{B}) \models_n \neg F_a(x)$ — a contradiction.

If it is the case ii) at the step $3a+2$, then we will have $\tau_{3a+2} \Vdash_n F_a(x)$, $\tau_{3a+2}(x) = s$ and $f(x) = s \notin A$. On the other hand, since $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic, according to Lemma 3.4, $(f, \mathfrak{B}) \models_n F_a(x)$, hence by (4.2), $f(x) = s \in A$ — again a contradiction. □

As a consequence we obtain the following proposition.

Proposition 4.2. *Let $n \geq 1$. If $A \subseteq B$ is Π^0_n-admissible in every enumeration, then A has a Π^0_n normal form.*

Proof. Take an arbitrary enumeration $(f, \mathfrak{B})$ of $\mathfrak{A}$. If W is an associate of A in $(f, \mathfrak{B})$, then, clearly, $N \setminus W$ is an associate of $B \setminus A$ in $(f, \mathfrak{B})$. Hence $B \setminus A$ is Σ^0_n-admissible in $(f, \mathfrak{B})$ and, according to the previous proposition, $B \setminus A$ has a Σ^0_n normal form, therefore A has a Π^0_n normal form. □

Proposition 4.3. *Let $n \geq 1, A \subseteq B$ and for every enumeration $(f, \mathfrak{B})$ of $\mathfrak{A}$ the set A is Σ^0_n- or Π^0_n-admissible in $(f, \mathfrak{B})$. Then A has Σ^0_n normal form or Π^0_n normal form (and hence the set A is Σ^0_n-admissible in every $(f, \mathfrak{B})$ or is Π^0_n-admissible in every $(f, \mathfrak{B})$).*

Proof. We shall follow the proof of Proposition 4.1. Assuming that A does not have neither Σ^0_n nor Π^0_n normal form, we will construct an enumeration $(f, \mathfrak{B})$, in which A does not have an appropriate (Σ^0_n or Π^0_n) associate. We shall use four types of steps here in order to define the sequence $\{\tau_a\}_a$. The first three ones will be just as in the proof of Proposition 4.1. At the steps $4a+4$ we do the following. According to our assumption that A does not have a Π^0_n normal form, we have that there exists $s \in B$ such that, putting $x = |\tau_{4a+3}|$, one of the following two cases hold:

i) $s \in A$, but $\tau_{4a+3} * s \nVdash_n \neg F_a(x)$;

ii) $s \notin A$, but $\tau_{4a+3} * s \Vdash_n \neg F_a(x)$.

In the first case we have that for some $\rho \geqslant \tau_{4a+3} * s : \rho \Vdash_n F_a(x)$. Put in this case $\tau_{4a+4} = \rho$. In the second case put $\tau_{4a+4} = \tau_{4a+3} * s$.

Now let $(f, \mathfrak{B})$ be an enumeration that extends τ_a for every $a \in N$. As we have established in the proof of Proposition 4.1, $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic and A is not Σ_n^0-admissible in it. Let us see that A is not Π_n^0-admissible in $(f, \mathfrak{B})$ as well. Assume the contrary and take a Π_n^0 set W that is an associate of A. According to Proposition 3.1, $W = \{x | (f, \mathfrak{B}) \models_n \neg F_a(x)\}$ for some a. Consider the step $4a + 4$. If the case i) holds at this step, then $\tau_{4a+4} \Vdash_n F_a(x)$ for $x = |\tau_{4a+3}|, \tau_{4a+4}(x) = s$ and $s \in A$. Since $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic, using Lemma 3.4 we get $(f, \mathfrak{B}) \models_n F_a(x)$, hence $x \notin W$, whereas $f(x) \in A$ — a contradiction. In the case ii) we put $\tau_{4a+4} = \tau_{4a+3} * s$ with $\tau_{4a+4} \Vdash_n \neg F_a(x)$ and $f(x) = s \notin A$. Now again by Lemma 3.4 we get $(f, \mathfrak{B}) \models_n \neg F_a(x)$, hence $x \in W$, whereas $f(x) = s \notin A$. □

Proposition 4.4. *Let the set $A \subseteq B$ be arithmetically admissible in every enumeration $(f, \mathfrak{B})$. Then there exists $n \geq 1$ such that A has Σ_n^0 or Π_n^0 normal form.*

Proof. Assume the contrary. We generalize the idea used in the proof of Proposition 4.3 in such a way that n is now a parameter of the construction. Again we will make four types of steps. With the first type of steps (of the form $4a + 1$) we shall ensure $\mathfrak{F}_n$-genericity of $(f, \mathfrak{B})$ for every $n \geq 1$; with the second and the third types — that A does not have neither Σ_n^0, nor Π_n^0 associate in $(f, \mathfrak{B})$ for every $n \geq 1$. The fourth type of steps will guarantee that the mapping $f = \bigcup_a f_{\tau_a}$ is onto B.

Let τ_0 be the empty finite part and suppose that we have constructed τ_{4a} for some a. Let $a = \langle e, x, n \rangle$. If there exists $\rho \geqslant \tau_{4a} : \rho \Vdash_{n+1} F_e(x)$, put $\tau_{4a+1} = \rho$, otherwise put $\tau_{4a+1} = \tau_{4a}$. In order to determine τ_{4a+2}, we represent a as $\langle e, n-1 \rangle$ for some e and $n \geq 1$ and use the fact that A does not have a Σ_n^0 normal form. So putting $x = |\tau_{4a+1}|$, we will have that there exists some $s \in B$ such that one of the next two possibilities holds:

i) $s \in A$, but $\forall \rho \geqslant \tau_{4a+1} * s$, $\rho \nVdash_n F_e(x)$;

ii) $s \notin A$, but $\exists \rho \geqslant \tau_{4a+1} * s$, $\rho \Vdash_n F_e(x)$.

Set $\tau_{4a+2} = \tau_{4a+1} * s$ or $\tau_{4a+2} = \rho$ if it is the case i) or ii), respectively.

At the step $4a + 3$ with $a = \langle e, n - 1 \rangle$ for some e and $n \geq 1$ we proceed in a similar way, taking into account this time the fact that A does not have a Π_n^0 normal form and hence for $x = |\tau_{4a+2}|$ there is an $s \in B$ such that:

i) $s \in A$, but $\tau_{4a+2} * s \nVdash_n \neg F_e(x)$;

ii) $s \notin A$, but $\tau_{4a+2} * s \Vdash_n \neg F_e(x)$.

If it is the case i), then for some $\rho \geqslant \tau_{4a+2} * s$ we will have $\rho \Vdash_n F_e(x)$, so put $\tau_{4a+3} = \rho$ in this case. In the second case put $\tau_{4a+3} = \tau_{4a+2} * s$.

At the step $4a + 4$ we put $\tau_{4a+4} = \tau_{4a+3} * s_a$. To complete the proof, proceed just as in the proof of Proposition 4.3. □

5. THE MAIN RESULT

In this section we will introduce a step-wise refinement $\overset{t}{\Vdash}_n$ of the forcing relation $\Vdash_n$ that will allow us to define more precise construction of the generic enumerations $(f, \mathfrak{B})$, built in the proofs of the propositions in the previous section. As a result, we will obtain a refined versions of these propositions that will bring us to our final results.

The definition of $\overset{t}{\Vdash}_n$ will follow the step-wise enumeration of the sets $W_{h(e)}$ by the function $\lambda t.U(e,t)$.

Set for brevity

$$\tau \Vdash_0 D_v \iff \tau \Vdash u \text{ for every } u \in D_v$$

and for $n \geq 1$

$$\tau \Vdash_n D_v \iff \forall u \in D_v \exists d \exists y((u = \langle d, y, 0\rangle \& \tau \Vdash_n F_d(y)) \vee (u = \langle d, y, 1\rangle \ \& \ \tau \Vdash_n \neg F_d(y)))).$$

Let $\lambda x, i.(x)_i$ be a recursive function that returns the i-th component of the sequence with a code x (if it exists). So we have for $n \geq 1$:

$$\tau \Vdash_n F_e(x) \iff \exists v(\langle v, x\rangle \in W_{h(e)} \ \& \ \tau \Vdash_{n-1} D_v) \iff$$
$$\exists t \exists v(U(e,t) = \langle v, x\rangle \ \& \ \tau \Vdash_{n-1} D_v)) \iff \exists t((U(e,t))_1 = x \ \& \ \tau \Vdash_{n-1} D_{(U(e,t))_0}).$$

Definition 5.1. Put

$$\tau \overset{t}{\Vdash}_n F_e(x) \iff \exists t_0(t_0 \leq t \ \& \ (U(e,t_0))_1 = x \ \& \ \tau \Vdash_{n-1} D_{(U(e,t_0))_0}).$$

The first t with $\tau \overset{t}{\Vdash}_n F_e(x)$ may be thought of as the first step at which the validity of $\tau \Vdash_n F_e(x)$ is established.

Here are the main properties of the relation $\overset{t}{\Vdash}_n$ that we will need.

Lemma 5.1. (i) $\tau \Vdash_n F_e(x) \iff \exists t(\tau \overset{t}{\Vdash}_n F_e(x))$;

(ii) $\tau \overset{t}{\Vdash}_n F_e(x) \ \& \ t' > t \Rightarrow \tau \overset{t'}{\Vdash}_n F_e(x)$;

(iii) $\tau \overset{t}{\Vdash}_n F_e(x) \ \& \delta \supseteq \tau \Rightarrow \delta \overset{t}{\Vdash}_n F_e(x)$;

(iv) *The set* $\{(e,x,t) | \tau \overset{t}{\Vdash}_n F_e(x)\}$ *is recursive in* $\emptyset^{(n-1)}$.

Proof. (i) and (ii) are straightforward; the proof of (iii) is by a routine induction on n. In order to establish (iv), notice that according to Lemma 3.1 and the Post's theorem the set $M = \{(e,x,t) | (U(e,t))_1 = x \ \& \ \tau \Vdash_{n-1} D_{(U(e,t))_0}\}$ is recursive in $\emptyset^{(n-1)}$, hence the set $\{(e,x,t) | \exists t_0(t_0 \leq t \ \& \ (e,x,t_0) \in M\}$ is recursive in $\emptyset^{(n-1)}$ as well. □

Let $D(\tau)$ – the *diagram* of τ – be the set $\{\langle i, x_1, \ldots, x_{k_i}, \varepsilon\rangle | q_i^\tau(x_1, \ldots, x_{k_i}) = \varepsilon, 1 \leq i \leq m\}$. Clearly $D(\tau)$ is a finite subset of N.

Definition 5.2. The *code* of τ (in symbols, $\|\tau\|$) is the canonical index of the diagram $D(\tau)$ of τ.

In fact, the code $\|\tau\|$ of τ does not code τ completely since it preserves no information about $Dom(f_\tau)$ and H_τ. We, however, will consider codes $\|\tau\|$ only for finite parts τ such that $\tau \succcurlyeq \tau_0$ for some fixed τ_0. In such case, clearly, $\|\tau\|$ identifies completely τ. If $\|\tau\| \leq \|\delta\|$, we shall say that τ is *less* than δ.

Let $t \in N$. Denote by τ^{+t} the finite part

$$\tau = (f_\tau, H_\tau \cup \{|\tau|, \dots, |\tau| + t - 1\}, q_1^\tau, \dots, q_m^\tau)$$

($\tau^{+0} \stackrel{df}{=} \tau$). Clearly, $\tau^{+t} \geqslant \tau$, $|\tau^{+t}| = |\tau| + t$ and $\|\tau^{+t}\| = \|\tau\|$. The next simple observation will be of use when constructing special generic enumerations.

Lemma 5.2. *Suppose that $\exists \delta \geqslant \tau(\delta \Vdash_n F_e(x))$. Then there exist $t \in N$ and ρ such that $\rho \succcurlyeq \tau^{+t}$ and $\rho \overset{t}{\Vdash}_n F_e(x)$.*

Proof. Let $\delta \Vdash_n F_e(x)$ and $\delta \geqslant \tau$. Then there exists $t_0 : \delta \overset{t_0}{\Vdash}_n F_e(x)$. Put $t = \max(t_0, k)$, where $k = |\delta| - |\tau|$, and consider the finite part $\rho = \delta^{+(t-k)}$. Clearly, $\rho \geqslant \delta \geqslant \tau$ and $|\rho| = |\delta| + t - k = |\tau| + t = |\tau^{+t}|$, hence $\rho \succcurlyeq \tau^{+t}$. We have $\delta \overset{t_0}{\Vdash}_n F_e(x)$, hence $\delta \overset{t}{\Vdash}_n F_e(x)$ and, by monotonicity, $\rho \overset{t}{\Vdash}_n F_e(x)$. □

Now put

$$\mu_0(\tau, n, e, x) \simeq \begin{cases} \min\{t | t > 0 \ \&\ \exists \delta \succcurlyeq \tau^{+t}(\delta \overset{t}{\Vdash}_n F_e(x))\}, & \text{if } \exists \delta \geqslant \tau(\delta \Vdash_n F_e(x)), \\ \neg !, & \text{otherwise}, \end{cases}$$

$$\mu(\tau, n, e, x) \simeq \begin{cases} \min\{\rho | \exists t (t \simeq \mu_0(\tau, n, e, x) \& \rho \succcurlyeq \tau^{+t} \& \rho \overset{t}{\Vdash}_n F_e(x))\}, & \text{if } !\mu_0(\tau, n, e, x), \\ \neg !, & \text{otherwise}. \end{cases}$$

Here by $\min\{\rho | \dots\}$ we mean the least finite part ρ with the respective property.

Using Lemma 5.2, we easily get

$$\exists \delta \geqslant \tau(\delta \Vdash_n F_e(x)) \implies !\mu_0(\tau, n, e, x) \ \&\ !\mu(\tau, n, e, x).$$

Let us notice also that, according to Lemma 5.1 and the fact that there exist finitely many δ: $\delta \succcurlyeq \tau$, both functions μ_0 and μ are computable in $\emptyset^{(n-1)}$.

Proposition 5.1. *Let $n \geq 1$. If $A \subseteq B$ is Σ_n^0-admissible in every Π_n^0 enumeration, then A has a Σ_n^0 normal form.*

Proof. Assume that A does not have a Σ_n^0 normal form. We have to construct a Π_n^0 enumeration $(f, \mathfrak{B})$ in which A is not Σ_n^0-admissible. The construction of the enumeration $(f, \mathfrak{B})$ will follow the scheme, described in the proof of Proposition 4.1. We will, however, be more careful at the positive cases of the steps $k = 3a + 1$ and $k = 3a + 2$, i.e. when we put τ_k to be an arbitrary $\rho \geqslant \tau_{k-1}$ with the respective property. Now we will choose this ρ more precisely. In addition, we will ensure that every τ_k is *total*, i.e. every $q_i^{\tau_k}, 1 \le i \le m$, is a total predicate over $Dom(\tau_k)$. (In fact, the last requirement is not essential for our construction. We will support it only to facilitate the proof that $(f, \mathfrak{B})$ is a Π_n^0 enumeration.)

If $\delta = (f_\delta, H_\delta, q_1^\delta, \dots, q_m^\delta)$ is a finite part, let

$$\delta^+ = (f_\delta, H_\delta, q_1, \dots, q_m),$$

where $q_i \supseteq q_i^\delta$ and $q_i(x_1, \dots, x_{k_i}) = 0$ whenever $(x_1, \dots, x_{k_i}) \in Dom(\delta)^{k_i}$ and $\neg ! q_i^\delta(x_1, \dots, x_{k_i}), 1 \le i \le m$. Clearly $\delta^+ \succcurlyeq \delta$ and δ is total.

Now we are ready to explain how to define τ_k for each step k. Indeed, assume that τ_{3a} is defined. Following the construction, described in the proof of Proposition 4.1, we present a as $a = \langle e, x, j \rangle$. So putting $k(a) = \min(j+1, n-1) = \min((a)_2 + 1, n-1)$, we set

$$\tau_{3a+1} = \begin{cases} \mu(\tau_{3a}, k(a), (a)_0, (a)_1)^+, & \text{if } \exists \rho \geqslant \tau_{3a} (\rho \Vdash_{k(a)} F_{(a)_0}((a)_1)), \\ \tau_{3a}, & \text{otherwise.} \end{cases}$$

In order to explain how to proceed at step $3a+2$, look again at the construction in the proof of Proposition 4.1. At this step we have that at least one $s \in B$ with some special property exists. (Take an arbitrary s with this property, for example take the first one in the enumeration $s_0, s_1, \dots$ of B). Since we will need to cite this s in the future, let us denote it by r_a.

$$\tau_{3a+2} = \begin{cases} \mu((\tau_{3a+1} * r_a)^+, n, a, |\tau_{3a+1}| + 1)^+, & \text{if } \exists \rho \geqslant (\tau_{3a+1} * r_a)^+, \\ & \rho \Vdash_n F_a(|\tau_{3a+1}| + 1) \\ (\tau_{3a+1} * r_a)^+, & \text{otherwise.} \end{cases}$$

Finally, put $\tau_{3a+3} = (\tau_{3a+2} * s_a)^+$.

Now set

$$f = \bigcup_k f_{\tau_k}, \ Q_i = \bigcup_k = q_i^{\tau_k}, \ 1 \le i \le m.$$

The fact that A is not Σ_n^0-admissible in $(f, \mathfrak{B})$ follows immediately from the proof of Proposition 4.1. What we claim here is that $(f, \mathfrak{B})$ is a Π_n^0 enumeration, i.e. that $Dom(f)$ is a Π_n^0 in $D(\mathfrak{B})$, or equivalently, that $N \setminus Dom(f)$ is r. e. in $D(\mathfrak{B})^{(n-1)}$. Below we will see that $N \setminus Dom(f)$ is in fact r. e. in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$, hence $(f, \mathfrak{B})$ is a Π_n^0 enumeration indeed.

Remark. Let us notice that we do not achieve more than we claim for the complexity of $Dom(f)$, since as is well known, for the $\mathfrak{F}_n$-generic enumerations $(f, \mathfrak{B})$, $D(\mathfrak{B})^{(n-1)}$ is Turing equivalent to $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$.

Let $l \in N$. Denote by α_l the finite part $(f | [0, l), (N \setminus Dom(f)) \cap [0, l), q_1, \dots, q_m)$, where each q_i is the predicate Q_i of $\mathfrak{B}$, restricted to the interval $[0, l)$. Clearly, $\alpha_l \subseteq (f, \mathfrak{B})$ and α_l is total. The problem here is that we have at our disposal only the structure $\mathfrak{B}$, not the whole enumeration $(f, \mathfrak{B})$, so we cannot construct α_l. We, however, can determine the finite part

$$\beta_l = (\emptyset; [0, l)); q_1, \dots, q_m).$$

Clearly, β_l is $\alpha_l | \emptyset$, hence, using Lemma 3.2, we get

$$\beta_l \Vdash_n F_e(x) \iff \alpha_l \Vdash_n F_e(x). \tag{5.1}$$

From here for every p, e, x we get

$$\mu_0(\alpha_l, p, e, x) \simeq \mu_0(\beta_l, p, e, x) \ \& \ \mu(\alpha_l, p, e, x) \simeq \mu(\beta_l, p, e, x).$$

Set

$$H_0 = \emptyset, \ H_k = H_{\tau_k} \setminus H_{\tau_{k-1}} \text{ for } k > 0.$$

Clearly, H_k are disjoint and $N \setminus Dom(f) = \bigcup_k H_k$. Hence

$$x \in N \setminus Dom(f) \iff \exists k(x \in H_k).$$

Let us look closely how the sets H_k are constructed. We have $H_{3a} = \emptyset$,

$$H_{3a+1} = \begin{cases} \{|\tau_{3a}|, \dots, |\rho| - 1\}, & \text{if } \rho \simeq \mu(\tau_{3a}, k(a), (a)_0, (a)_1)^+, \\ \emptyset, & \text{if } \neg ! \mu(\tau_{3a}, k(a), (a)_0, (a)_1), \end{cases}$$

$$H_{3a+2} = \begin{cases} \{|\tau_{3a+1}| + 1, \dots, |\rho| - 1\}, & \text{if } \rho \simeq \mu((\tau_{3a+1} * r_a)^+, n, a, |\tau_{3a+1}| + 1)^+, \\ \emptyset, & \text{if } \neg ! \mu((\tau_{3a+1} * r_a)^+, n, a, |\tau_{3a+1}| + 1). \end{cases}$$

Clearly, $x \in N \setminus Dom(f)$ iff $x \in H_{3a+1}$ or $x \in H_{3a+2}$. Consider, for example, the set H_{3a+1} (the case with H_{3a+2} is similar). Suppose also that H_{3a+1} is not empty. We have $\tau_{3a} \subseteq (f, \mathfrak{B})$, $\tau_{3a} = \alpha_l$ for $l = |\tau_{3a}|$, so if we knew the length of τ_{3a}, we could compute $t = \mu_0(\alpha_l, k(a), (a)_0, (a)_1) = \mu_0(\beta_l, k(a), (a)_0, (a)_1)$, using the oracles $\emptyset^{(n-1)}$ and $D(\mathfrak{B})$. Hence (the canonical index of) $H_{3a+1} = \{|\tau_{3a}|, \dots, |\tau_{3a}| + t - 1\}$ would be computable in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$.

The problems here are two. The first one is that we cannot decide recursively in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$ whether $H_{3a+1} \neq \emptyset$. The second problem is that we cannot compute the length $|\tau_a|$, using the oracles $\emptyset^{(n-1)}$ and $D(\mathfrak{B})$. So our idea is to start a recursive in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$ procedure that for every a computes consecutive approximations $l_a^0, l_a^1, \dots$, leading to the "real" length $|\tau_a|$. Using the approximate lengths l_a^s, we will define finite sets H_a^s that will be already recursive in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$. Not all of the sets H_a^s are approximations of our sets H_a, but as we will see, their union $\cup_{a,s} H_a^s$ coincides with $\cup_a H_a$, i. e. with $N \setminus Dom(f)$. The rest of the proof of the proposition consists in precise definitions of these approximations and their properties, and is gathered in the next four lemmas.

We define by simultaneous recursion the functions l_k^s and t_s^k as follows:

$$l_{-1}^s = 0,\ l_k^0 = k \text{ for every } k \in N, s \in N,$$

$$t_k^0 = 1 \text{ for every } k \in N,$$

and for every k, s in N

$$l_k^{s+1} = \begin{cases} l_{k-1}^{s+1} + 1, & \text{if } l_{k-1}^s \neq l_{k-1}^{s+1}, \\ l_k^s + t_k^s, & \text{if } l_{k-1}^s = l_{k-1}^{s+1} \ \&\ t_k^s > 0 \ \&\ k = 2a \ \& \\ & \exists \rho \succcurlyeq \beta_{l_k^s}^{+t_k^s} : \rho \Vdash_{k(a)}^{t_k^s} F_{(a)_0}((a)_1), \\ l_k^s + t_k^s, & \text{if } l_{k-1}^s = l_{k-1}^{s+1} \ \&\ t_k^s > 0 \ \&\ k = 2a+1 \ \& \\ & \exists \rho \succcurlyeq \beta_{l_k^s}^{+t_k^s} : \rho \Vdash_n^{t_k^s} F_a(l_k^s), \\ l_k^s, & \text{in the remained cases,} \end{cases}$$

$$t_k^{s+1} = \begin{cases} 1, & \text{if } l_{k-1}^s \neq l_{k-1}^{s+1}, \\ 0, & \text{if } l_{k-1}^s = l_{k-1}^{s+1} \ \&\ t_k^s > 0 \ \&\ k = 2a \ \& \\ & \exists \rho \succcurlyeq \beta_{l_k^s}^{+t_k^s} : \rho \Vdash_{k(a)}^{t_k^s} F_{(a)_0}((a)_1), \\ 0, & \text{if } l_{k-1}^s = l_{k-1}^{s+1} \ \&\ t_k^s > 0 \ \&\ k = 2a+1 \ \& \\ & \exists \rho \succcurlyeq \beta_{l_k^s}^{+t_k^s} : \rho \Vdash_n^{t_k^s} F_a(l_k^s), \\ t_k^s + 1, & \text{in the remained cases.} \end{cases}$$

Our first lemma establishes some basic properties of l_k^s and t_k^s.

Lemma 5.3. *For every $k \in N$ and $s \in N$*

(i) $l_k^s \leq l_k^{s+1}$;

(ii) $l_k^s \leq k + s$;

(iii) $l_k^s < l_k^{s+1} \Rightarrow l_k^{s+1} = k + s + 1$;

(iv) $l_k^s + t_k^s \leq k + s + 1$;

(v) $t_k^s > 0 \Rightarrow l_k^s + t_k^s = k + s + 1$.

Proof. Induction on k. The case $k = 0$ is by a straightforward induction on s. Assume now that for some $k > 0$

$$\forall s(l_{k-1}^s \leq l_{k-1}^{s+1} \ \&\ l_{k-1}^s \leq k - 1 + s \ \&\ (l_{k-1}^s < l_{k-1}^{s+1} \Rightarrow l_{k-1}^{s+1} = k + s), \tag{5.2}$$
$$l_{k-1}^s + t_{k-1}^s \leq k + s \ \&\ (t_{k-1}^s > 0 \Rightarrow l_{k-1}^s + t_{k-1}^s = k + s)).$$

In order to establish (i) – (v) for k, we shall proceed by induction on s. For $s = 0$ the only points that are not obvious are (i) and (iii). We consider the cases in the definition of l_k^1. If $l_{k-1}^0 \neq l_{k-1}^1$, then $l_k^1 = l_{k-1}^1 + 1 = (k-1)+1+1 = k+1$, according to (5.2). Hence $l_k^1 > l_k^0 = k$, i. e. (i) and (iii) hold. If $l_{k-1}^0 = l_{k-1}^1$, but $l_k^1 \neq l_k^0$ (i. e. it is the second or the third case of the definition of l_k^1), then $l_k^1 = l_k^0 + t_k^0 = l_k^0 + 1 = k+1$, hence again (i) and (iii) are true. They are evidently true if $l_k^1 = l_k^0$.

Now suppose that the conditions (i) — (v) hold for some s. We are going to check them for $s+1$. If $l_k^s < l_k^{s+1}$, then by induction hypothesis for s, $l_k^{s+1} = k+s+1$, i. e. (ii) is true for $s+1$. We check similarly conditions (iv) and (v), using the fact that (iv) and (v) hold for s. To see that (i) and (iii) also hold for $s+1$, we consider separately the cases in the definition of l_k^{s+2}. If $l_{k-1}^{s+1} \neq l_{k-1}^{s+2}$, then $l_k^{s+2} = l_{k-1}^{s+2} + 1$, which is by (5.2) exactly $(k-1)+(s+2)+1 = k+s+2$, so we checked (i) and (iii) for $s+1$. The next two cases of the definition of l_k^{s+2} are treated similarly to the case $s = 0$. The last case of the definition is again obvious. □

Clearly, the functions $\lambda s.l_k^s$ have finitely many different values (since they depend on the switches in the values of $\lambda s.l_m^s$ for $m < k$). This fact, combined with the previous lemma, means that for every k there exists least S_k such that

$$l_k^0 \leq l_k^1 \leq \cdots \leq l_k^{S_k} = l_k^{S_k+1} \ldots \tag{5.3}$$

The following additional properties follow directly or by an easy induction from the definition of l_k^s, Lemma 5.3 and the choice of S_k.

Lemma 5.4. (i) $0 \leq S_0 \leq S_1 \leq \ldots$;

(ii) $S_k = 0 \Rightarrow L_k = k$;

(iii) $S_k > 0 \Rightarrow l_{k+1}^{S_k} = l_k^{S_k} + 1$;

(iv) $S_k > 0 \Rightarrow l_{k+1}^{S_k} = l_{k+1}^{S_{k+1}-1}$;

(v) $l_k^{s-1} < l_k^s \;\&\; s < S_k \Rightarrow \exists m < k \exists s'(l_m^{s'-1} < l_m^{s'})$;

(vi) $l_k^{s-1} < l_k^s \Rightarrow \forall m > k(l_m^{s-1} + 1 = l_m^s)$;

(vii) $k \leq m \Rightarrow l_k^s \leq l_m^s$.

Put $L_k = l_k^{S_k}$.

Our next lemma makes connection between (the lengths of) the finite parts τ_k and the function l_k^s (in fact, it clarifies the definition of this function).

Lemma 5.5. *For every $a \in N$ we have that $|\tau_{3a+1}| = L_{2a}$, $|\tau_{3a+2}| = L_{2a+1}$ and $|\tau_{3a+3}| = L_{2a+1} + 1$.*

Proof. We have by definition that $|\tau_{3a+3}| = |\tau_{3a+2}| + 1$, hence we have to check the first two equalities. We shall proceed by induction on a. Let $a = 0$. We shall see in turn that $|\tau_1| = L_0$ and $|\tau_2| = L_1$ ($|\tau_0| = 0$ by definition).

Case 1. $\exists \rho \geqslant \tau_0(\rho \Vdash_{k(0)} F_{(0)_0}((0)_1))$. In this case $\tau_1 = \mu(\tau_0, k(0), (0)_0, (0)_1)^+$ and the length $|\tau_1|$ of τ_1 is $T = \mu_0(\tau_0, k(0), (0)_0, (0)_1)$. Then clearly, $l_0^s = 0, t_0^s = s+1$

for $s < T$, and $l_0^T = l_0^{T-1} + t_0^{T-1} = T$, $t_0^T = 0$. Therefore $l_0^s = T, t_0^s = 0$ for $s > T$, so $L_0 = T$, i. e. $|\tau_1| = L_0$.

Case 2. $\neg\exists\rho \geqslant \tau_0(\rho \Vdash_{k(0)} F_{(0)_0}((0)_1))$. Then by definition $\tau_1 = \tau_0$, the length $|\tau_1|$ is 0, and in this case $L_0 = 0$, too.

Let us now see that $|\tau_2| = L_1$. If $|\tau_1| = 0$, then clearly $|(\tau_1 * r_0)^+| = 1$, hence $(\tau_1 * r_0)^+ = \alpha_1$ and τ_2 is an appropriate extention of α_1.

Case 1. $\exists\rho \geqslant (\tau_1 * r_0)^+(\rho \Vdash_n F_0(|\tau_1| + 1))$. Taking into account (5.1) and the fact that $|\tau_1| = 0$, we can rewrite equivalently this condition as $\exists\rho \geqslant \beta_1(\rho \Vdash_n F_0(1))$, which is closer to the definition of l_1^s and t_1^s. Clearly, $|\tau_2| = T + 1$, where $T = \mu_0((\tau_1 * r_0)^+, n, 0, 1)$, and since in this case $l_0^s = 0$ for every s, using the appropriate definitions, we notice that $l_1^s = 1, t_1^s = s + 1$ for every $s < T$ and $l_1^T = l_1^{T-1} + t_1^{T-1} = T + 1$, $t_1^T = 0$. Hence $l_1^s = T + 1, t_1^s = 0$ for $s > T$, so $L_1 = T + 1$, which means that $|\tau_2| = L_1$.

Case 2. $\neg\exists\rho \geqslant (\tau_1 * r_0)^+(\rho \Vdash_n F_0(|\tau_1| + 1))$, or equivalently $\neg\exists\rho \geqslant \beta_1(\rho \Vdash_n F_0(1))$. It can be easily checked that in this case $l_1^s = 1$ for every s, hence $L_1 = 1$, which is exactly the length of τ_2.

Suppose now that $|\tau_1| = L_0 > 0$. We have $L_0 = l_0^{S_0}$, hence $l_0^{S_0 - 1} < l_0^{S_0}$ and $l_1^{S_0} = l_0^{S_0} + 1 = |\tau_1| + 1$. So $|(\tau_1 * r_0)^+| = l_1^{S_0}$ and $(\tau_1 * r_0)^+$ is in fact $\beta_{l_1^{S_0}}$. Now using this fact and having in mind the respective definitions, proceeding as in the case $|\tau_1| = 0$, we see that $|\tau_2| = L_1$.

Now suppose that for some a the lemma is true. We have to check that $|\tau_{3a+4}| = L_{2a+2}$ and $|\tau_{3a+5}| = L_{2a+3}$. Indeed, by induction hypothesis, $|\tau_{3a+3}| = L_{2a+1} + 1 = l_{2a+1}^{S_{2a+1}} + 1$. We consider separately the cases $S_{2a+1} = 0$ (hence $L_{2a+1} = 2a + 1$) and $S_{2a+1} > 0$, and obtain that $l_{2a+2}^{S_{2a+1}} = l_{2a+1}^{S_{2a+1}} + 1 = L_{2a+1} + 1$, which is exactly $|\tau_{3a+3}|$. Hence $\alpha_{l_{2a+2}^{S_{2a+1}}}$ is in fact τ_{3a+3}. Now the condition $\exists\rho \geqslant \tau_{3a+3}(\rho \Vdash_{k(a+1)} F_{(a+1)_0}((a+1)_1))$, which is used in the definition of τ_{3a+4}, is equivalent to $\exists\rho \geqslant \beta_{l_{2a+2}^{S_{2a+1}}}(\rho \Vdash_{k(a+1)} F_{(a+1)_0}((a+1)_1))$. If $S_{2a+2} = S_{2a+1}$, then such ρ does not exist, hence $\tau_{3a+4} = \tau_{3a+3}$, $L_{2a+2} = L_{2a+1} + 1$, therefore $|\tau_{3a+4}| = L_{2a+2}$. If $S_{2a+2} > S_{2a+1}$, then $L_{2a+2} = l_{2a+2}^{S_{2a+2}} = l_{2a+2}^{S_{2a+1}} + t_{2a+2}^{S_{2a+1}}$. However $t_{2a+2}^{S_{2a+1}}$ is in fact $\mu_0(\tau_{3a+3}, k(a+1), (a+1)_0, (a+1)_1)$ (the verification is as in the case $a = 0$). So $L_{2a+2} = |\tau_{3a+3}| + \mu_0(\tau_{3a+3}, k(a+1), (a+1)_0, (a+1)_1)$, which is, according to our construction, the length of τ_{3a+4}. In order to prove the equality $L_{2a+3} = |\tau_{3a+5}|$ we proceed in a similar way. □

Set

$$H_k^s = \begin{cases} \{l_k^{s-1}, \ldots, l_k^s - 1\}, & \text{if } l_k^{s-1} < l_k^s \ \& \ \forall m < k(l_m^{s-1} = \cdots = l_m^{s-1+k-m}), \\ \emptyset, & \text{else.} \end{cases}$$

Let us notice that (for $k > 0$) if $H_k^s \neq \emptyset$, then $l_{k-1}^{s-1} = l_{k-1}^s$, hence the change of the value of l_k^{s-1} is due to the existence of an appropriate $\rho \geqslant \beta_{l_k^{s-1}}$.

Since the function $\lambda k, s.l_k^s$ is recursive in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$, the function $H(k,s) =$ the canonical index of H_k^s is recursive in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$, too. Hence the set $H =$

$\cup_{k,s} H_k^s$ is r. e. in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$. We shall see below that $N \setminus Dom(f)$ coincides with H, hence $N \setminus Dom(f)$ is r. e. in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$, which will bring us to the end of the proof of the proposition.

Lemma 5.6. $N \setminus Dom(f) = \cup_{k,s} H_k^s$.

Proof. For the first inclusion, let us see that $H_{3a+1} = H_{2a}^{S_{2a}}$ and $H_{3a+2} = H_{2a+1}^{S_{2a+1}}$. We may suppose that H_{3a+1} and H_{3a+2} are nonempty. We shall consider separately the cases $a = 0$ and $a > 0$. If $a = 0$, then by Lemma 5.5, $|\tau_1| = L_0 = l_0^{S_0}$ and since $H_1 \neq \emptyset$, $l_0^{S_0} > 0$. Therefore $l_0^{S_0 - 1} = \cdots = l_0^0 = 0$, hence

$$H_0^{S_0} = \{l_0^{S_0-1}, \ldots, l_0^{S_0} - 1\} = \{0, \ldots |\tau_1| - 1\} = H_1.$$

If $a > 0$, again by Lemma 5.5, $|\tau_{3a}| = L_{2a-1} + 1 = l_{2a-1}^{S_{2a-1}} + 1$. Now using Lemma 5.4 we get $l_{2a}^{S_{2a-1}} = l_{2a-1}^{S_{2a-1}} + 1 = |\tau_{3a}|$ and $l_{2a}^{S_{2a-1}} = l_{2a}^{S_{2a}-1}$. Hence $l_{2a}^{S_{2a}-1} = |\tau_{3a}|$, while $l_{2a}^{S_{2a}} = |\tau_{3a+1}|$. So $H_{3a+1} = \{l_{2a}^{S_{2a}-1}, \ldots, l_{2a}^{S_{2a}} - 1\}$. If $m < 2a$, then $l_m^s = l_m^{S_m}$ for every $s \geq S_m$, in particular, for every $s \geq S_{2a}$, hence $H_{2a}^{S_{2a}} = \{l_{2a}^{S_{2a}-1}, \ldots, l_{2a}^{S_{2a}} - 1\}$. The verification of the equality $H_{3a+2} = H_{2a+1}^{S_{2a+1}}$ is similar.

Conversely, take some $H_k^s \neq \emptyset$. Clearly $s \leq S_k$ (otherwise $l_k^{s-1} = l_k^s$ and $H_k^s = \emptyset$). If $s = S_k$, as we saw above, $H_k^s = H_{3a+1}$ or H_{3a+2}, depending on whether $k = 2a$ or $k = 2a+1$. Suppose now that $s < S_k$. We cannot claim anymore that H_k^s coincides with some H_a. We shall see, however, that $H_k^s \subseteq H_m^{S_m}$ for some $m < k$, hence if $x \in H_k^s$, then $x \in N \setminus Dom(f)$ again. Indeed, since $H_k^s \neq \emptyset$, we have $l_k^{s-1} < l_k^s$ and $l_k^s < l_k^{S_k}$. Hence, by Lemma 5.4, there exist $m < k$ and s' such that $l_m^{s'-1} = l_m^{s'}$. We may suppose that m is the minimal one with this property. Using the definition of H_k^s, we get that $s' \leq s - 1$ and $s' - 1 \geq s - 1 + k - m$. From the last equality, $s' + m \geq s + k$ hence $l_m^{s'} = s' + m \geq s + k = l_k^s$. On the other hand, $s' \leq s - 1$, i. e. $s' - 1 < s - 1$, so by Lemma 5.4, $l_m^{s'-1} \leq l_k^{s-1}$. From here, $\{l_m^{s'-1}, \ldots, l_m^{s'} - 1\} \leq_k^{s-1}, \ldots, l_k^s - 1\}$. Using the minimality of m, one can easily get that s' is in fact S_m, hence $\{l_m^{s'-1}, \ldots, l_m^{s'} - 1\} = H_m^{S_m}$. □

We can apply this idea of refined refuting enumerations to the constructions used in the proofs of Proposition 4.2, Proposition 4.3 and Proposition 4.4. Thus we obtain that the following is true:

Proposition 5.2. (i) *Let $n \geq 1$. If $A \subseteq B$ is Π_n^0-admissible in every Π_n^0 enumeration of $\mathfrak{A}$, then A has a Π_n^0 normal form.*

(ii) *Let $n \geq 1$. If the set $A \subseteq B$ is Σ_n^0- or Π_n^0-admissible in every Π_n^0 enumeration of $\mathfrak{A}$, then A has Σ_n^0 normal form or Π_n^0 normal form.*

(iii) *Let the set $A \subseteq B$ be arithmetically admissible in every arithmetical enumeration of $\mathfrak{A}$. Then there exists $n \geq 1$ such that A has Σ_n^0 or Π_n^0 normal form.*

In order to formulate our final results, we use a syntactical characterizations of the sets that have Σ_n^0 (Π_n^0) normal form, obtained in [8], Lemma 6.3, namely, that a set has a Σ_n^0 (Π_n^0) normal form iff it is Σ_n^0 (Π_n^0) definable on $\mathfrak{A}$. This statement, combined with the above two propositions, brings us to our final result.

Theorem 5.1. *Let $n \geq 1$. Then the following is true:*

(i) *If $A \subseteq B$ is Σ_n^0-admissible in every Π_n^0 enumeration, then A is Σ_n^0 definable on $\mathfrak{A}$.*

(ii) *If $A \subseteq B$ is Π_n^0-admissible in every Π_n^0 enumeration, then A is Π_n^0 definable on $\mathfrak{A}$.*

(iii) *If $A \subseteq B$ is Σ_n^0-admissible or Π_n^0-admissible in every Π_n^0 enumeration, then A is Σ_n^0 definable or Π_n^0 definable on $\mathfrak{A}$.*

(iv) *If $A \subseteq B$ is arithmetically admissible in every arithmetical enumeration, then there exists $n \geq 1$ such that A is Σ_n^0 or Π_n^0 definable on $\mathfrak{A}$.*

Let us notice that the class of all Π_n^0 enumerations in points (i) – (iii) of the above theorem cannot be reduced anymore. Indeed, let us take a set A, which is definable by means of existential Σ_n^0 formula (cf. [8)], but is not Σ_n^0 definable on $\mathfrak{A}$ (it can be easily seen that such A does exist, if the structure $\mathfrak{A}$ is interesting enough). Clearly, A has a Σ_n^0 associate in every Σ_n^0 enumeration. Hence A is Σ_n^0 admissible in every class of enumerations, that is included in the class of all Σ_n^0 enumerations, and at the same time A is not Σ_n^0 definable on $\mathfrak{A}$.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: stenik@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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AFFINE APPLICATIONS IN OPERATIVE SPACES [1]

JORDAN ZASHEV

We prove a new normal form theorem for a special kind of expressible mappings in operative spaces with iteration. As a consequence, this provides a large class of models for the type-free implicative linear logic and a natural connection between operative spaces and the systems of algebraic recursion theory based on the linear or affine application which were studied previously.

Keywords: algebraic recursion theory, combinatory logic, linear logic

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INTRODUCTION

Operative spaces are a special class of partially ordered algebras developed in [2] for the purposes of the axiomatization of recursion theory. Recently, they were shown ([8]) to give rise to a large class of combinatory algebras. There is, on the other hand, an alternative to combinatory algebras - a kind of algebras called 'type-free models of the linear logic' below, since they can be regarded as models of a type-free version of the proof calculus for a Hilbert-styled system of a suitable fragment of linear logic. The last algebras have a natural connection with recursion theory and some other advantages which make reasonable the question whether we can model them in a way similar to that in which combinatory algebras were modeled in [8]. In the present paper we discuss this question and indicate a large class of operative spaces (including the iterative ones in the sense of [2]) and combinatory spaces in the sense of [4] in which the type-free models of the linear

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logic can be modeled. Some of these results were briefly mentioned in the last section of [8]; in the present paper we give them a detailed exposition. As in the paper [8], the last results will follow from a suitable normal form theorem for some kind of expressible mappings in operative spaces with iteration. The normal form theorem of the present paper can be regarded as a refinement of a normal form theorem of Georgieva [1].

The type-free models of the implicative linear logic and other partially ordered algebras based on linear (otherwise called BCI-) application were studied previously by the present author ([5, 6, 7]) for the same purposes of axiomatization of recursion theory; but until recently no natural connection was known with other systems of the algebraic recursion theory like operative spaces or combinatory spaces. The normal form theorem of the present paper changes this situation since it enables one to define a natural affine (or otherwise BCK-) application in every operative space with iteration. This opportunity is used in Section 3 below, where we show how to model one of the most important application-based systems of the algebraic recursion theory in such spaces, indicating in this way that the last system comprises the majority of the kinds of recursivenesses dealt with in the theory of operative spaces.

1. BASIC DEFINITIONS

An *operative space*, according to [2], up to some notational modifications, is a partially ordered algebra $\mathcal{F}$ with two binary operations called multiplication and pairing and three constants I,T,F, considered as 0-ary operations, which satisfy the conditions (OS1) and (OS2) below. (Note that the condition that $\mathcal{F}$ is a partially ordered algebra includes the requirement that all operations of $\mathcal{F}$ are increasing on each argument.) We use the following notations for the operations in question: the multiplication is denoted by juxtaposition and the result of applying pairing to the arguments $\varphi, \psi \in \mathcal{F}$ will be denoted by $[\varphi, \psi]$. The conditions defining an operative space $\mathcal{F}$ are the following ones:

(OS1) $\mathcal{F}$ is a monoid with unit I with respect to the multiplication;

(OS2) the identities $\chi[\varphi,\psi] = [\chi\varphi,\chi\psi]$, $[\varphi,\psi]T = \varphi$, and $[\varphi,\psi]F = \psi$ are satisfied for all elements φ, ψ, χ of $\mathcal{F}$.

The operations of multiplication and pairing and the constants I,T,F are called *basic* operations of operative spaces or, more briefly, basic OS-operations. We shall denote by m^+ the standard representation $F^mT = F...FT$ of a natural number m in arbitrary operative space $\mathcal{F}$. We employ also the shorthand notation

$$[\varphi_0, \varphi_1, ..., \varphi_{m-1}] = [\varphi_0, [\varphi_1, ...[\varphi_{m-2}, \varphi_{m-1}]...]];$$

so we have the identities

$$[\varphi_0, \varphi_1, ..., \varphi_m]i^+ = \varphi_i$$

for $i \leq m-1$ and

$$[\varphi_0, \varphi_1, \ldots, \varphi_m]F^m = \varphi_m$$

in every operative space.

For a unary operation $\mathbf{I} : \mathcal{F} \to \mathcal{F}$ in an operative space $\mathcal{F}$ we shall say that it is an *iteration* iff it satisfies the inequality

$$[I, \mathbf{I}(\varphi)]\varphi \leq \mathbf{I}(\varphi)$$

for all $\varphi \in \mathcal{F}$, and for all $\alpha, \xi \in \mathcal{F}$ the inequality $[\alpha, \xi]\varphi \leq \xi$ implies $\alpha\mathbf{I}(\varphi) \leq \xi$. Therefore the iteration $\mathbf{I}$ in $\mathcal{F}$, if it exists, is uniquely determined by the fact that for every $\varphi \in \mathcal{F}$ the element $\mathbf{I}(\varphi)$ is bound to be the least solution of the inequality $[I, \xi]\varphi \leq \xi$ with respect to ξ; in particular, it satisfies the corresponding equality

$$[I, \mathbf{I}(\varphi)]\varphi = \mathbf{I}(\varphi).$$

When the iteration $\mathbf{I}$ exists in the space $\mathcal{F}$, we shall say that $\mathcal{F}$ is an operative space with iteration; this notion is equivalent to the notion of G-space in [4] and to that of an operative space satisfying the axiom $(\mathcal{LL})$ of Ivanov [2]. Hence every *iterative* operative space in the sense of [2] is an operative space with iteration, but the reverse is not necessarily true. Every operative space $\mathcal{F}$ in which the least upper bound $supB$ exists for every well ordered part $B \subseteq \mathcal{F}$ and commutes with the left multiplication: $\varphi supB = sup\{\varphi x \mid x \in B\}$, is iterative and therefore has iteration. Another important and most commonly apearing class of iterative operative spaces is that of *continuous* ones, i.e. those in which the least upper bounds of countable increasing sequences exist and commute with all basic OS-operations.

In every operative space $\mathcal{F}$ with iteration the element $O = \mathbf{I}(F)$ is the least element of $\mathcal{F}$, and it satisfies also the equality $\alpha O = O$ for all $\alpha \in \mathcal{F}$. This follows from the equality $[\alpha, \xi]F = \xi$ by the definition of iteration. The last definition implies also that the iteration is an increasing operation.

Let $\mathcal{F}$ be arbitrary operative space and $B \subseteq \mathcal{F}$. We shall say for a mapping $f : \mathcal{F}^n \to \mathcal{F}$ that it is OS-expressible in B iff f can be defined by an explicit expression constructed by means of the basic operations of operative spaces and the elements of B as constants. Similarly, when the space $\mathcal{F}$ has an iteration, the mapping f will be called OSI-expressible in B iff it can be defined by similar expression, which may contain also the operation of iteration $\mathbf{I}$. Instead of OS- or OSI-expressible in B mappings of zero arguments we shall speak also of OS- or OSI-expressible in B elements of $\mathcal{F}$, respectively. We shall also drop 'in B' when B is clear from the context or arbitrary.

2. NORMAL FORM OF SINGULAR MAPPINGS

An OSI-expressible (in B) mapping $f : \mathcal{F}^n \to \mathcal{F}$ in an operative space $\mathcal{F}$ with iteration will be called $(B\text{-})$*singular* iff in the expression defining f all the variables for the arguments of f occur exactly once. More precisely, the B-singular mappings of n arguments are those which belong to the least class $\mathcal{O}$ of operations in $\mathcal{F}$ satisfying the following conditions:

0) $\mathcal{O}$ contains the identity operation in $\mathcal{F}$ of one argument;

1) $\mathcal{O}$ contains the basic constants I, T, F and the elements of B, considered as operations of zero arguments;

2) for all two operations f and g in $\mathcal{O}$ of n and m arguments, respectively, the operation h of $n+m$ arguments defined by

$$h(\xi_0, ..., \xi_{n-1}, \eta_0, ..., \eta_{m-1}) = f(\xi_0, ..., \xi_{n-1})g(\eta_0, ..., \eta_{m-1})$$

is also in $\mathcal{O}$;

3) for all two operations f and g in $\mathcal{O}$ of n and m arguments, respectively, the operation h of $n+m$ arguments defined by

$$h(\xi_0, ..., \xi_{n-1}, \eta_0, ..., \eta_{m-1}) = [f(\xi_0, ..., \xi_{n-1}), g(\eta_0, ..., \eta_{m-1})]$$

is also in $\mathcal{O}$;

4) for all operations f in $\mathcal{O}$ of n arguments the operation h of n arguments defined by

$$h(\xi_0, ..., \xi_{n-1}) = \mathbf{I}(f(\xi_0, ..., \xi_{n-1}))$$

is also in $\mathcal{O}$;

5) for all operations f in $\mathcal{O}$ of n arguments and every bijection

$$p : \{0, ..., n-1\} \to \{0, ..., n-1\}$$

the operation f_p of n arguments defined by

$$f_p(\xi_0, ..., \xi_{n-1}) = f(\xi_{p(0)}, ..., \xi_{p(n-1)})$$

is also in $\mathcal{O}$.

The main purpose of the present section is to prove the following normal form theorem for singular mappings.

Theorem 1. *For every B-singular mapping $f:\mathcal{F}^n \to \mathcal{F}$ in an operative space $\mathcal{F}$ with iteration there is an element $\varphi \in \mathcal{F}$, OS-expressible in $B \cup \{O\}$, such that*

$$f(\xi_0, ..., \xi_{n-1}) = \mathbf{I}([I, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}]\varphi)T$$

for all $\xi_0, ..., \xi_{n-1} \in \mathcal{F}$.

For that purpose we shall employ the technique of homogeneous systems introduced by Skordev [3], [4]. Let $\mathcal{F}$ be an arbitrary operative space. Following Skordev, we shall call a mapping $f : \mathcal{F}^n \to \mathcal{F}$ *left homogeneous* iff it satisfies the equality

$$f(\varphi\vartheta_0, ..., \varphi\vartheta_{n-1}) = \varphi f(\vartheta_0, ..., \vartheta_{n-1})$$

for all $\varphi, \vartheta_0, ..., \vartheta_{n-1} \in \mathcal{F}$. Left homogeneous mappings f are easily seen to be increasing on each argument since they satisfy the equalities

$$\begin{aligned} f(\vartheta_0, ..., \vartheta_{n-1}) &= f([\vartheta_0, ..., \vartheta_{n-1}]0^+, ..., [\vartheta_0, ..., \vartheta_{n-1}]F^n) \\ &= [\vartheta_0, ..., \vartheta_{n-1}]f(0^+, ..., F^{n-1}). \end{aligned}$$

By a homogeneous system we shall mean a system of inequalities of the form

$$\Phi_i(\alpha, x_0, ..., x_{n-1}) \leq x_i \tag{2.1}$$

where i ranges over natural numbers less than n, $\Phi_i : \mathcal{F}^{n+1} \to \mathcal{F}$ are left homogeneous mappings of $n+1$ arguments, $\alpha \in \mathcal{F}$ is a parameter, and $x_0, ..., x_{n-1}$ are unknowns. The following fundamental result for such systems, up to nonessential modifications, belongs to Skordev [3], [4].

Theorem 2. *Suppose the space $\mathcal{F}$ has an iteration. Then the elements μ_i defined by $\mu_i = \alpha\mathbf{I}(\varphi)i^+$, where $i = 0, ..., n-1$,*

$$\varphi = [\varphi_0, ..., \varphi_{n-1}, O],$$

and $\varphi_i = \Phi_i(T, 1^+, ..., n^+)$, form the least solution of the system 2.1 in $\mathcal{F}$ with respect to x_0,...,x_{n-1}, respectively, for all $\alpha \in \mathcal{F}$.

Proof. $(\mu_0, ..., \mu_{n-1})$ is a solution of (2.1) since

$$\begin{aligned} \Phi_i(\alpha, \mu_0, ..., \mu_{n-1}) &= \Phi_i(\alpha, \alpha\mathbf{I}(\varphi)0^+, ..., \alpha\mathbf{I}(\varphi)(n-1)^+) \\ &= \alpha\Phi_i(I, \mathbf{I}(\varphi)0^+, ..., \mathbf{I}(\varphi)(n-1)^+) \\ &= \alpha[I, \mathbf{I}(\varphi)]\Phi_i(T, 1^+, ..., n^+) = \alpha[I, \mathbf{I}(\varphi)]\varphi_i \\ &= \alpha[I, \mathbf{I}(\varphi)]\varphi i^+ = \alpha\mathbf{I}(\varphi)i^+ = \mu_i; \end{aligned}$$

and for an arbitrary solution $(\xi_0, ..., \xi_{n-1})$ of (2.1) with respect to $x_0, ..., x_{n-1}$, respectively, define $\xi = [\xi_0, ..., \xi_{n-1}, O]$; then for arbitrary $\alpha \in \mathcal{F}$ we have

$$\begin{aligned} [\alpha, \xi]\varphi &= [\alpha, \xi][\varphi_0, ..., \varphi_{n-1}, O] = [[\alpha, \xi]\varphi_0, ..., [\alpha, \xi]\varphi_{n-1}, O] \\ &= [[\alpha, \xi]\Phi_0(T, 1^+, ..., n^+), ..., [\alpha, \xi]\Phi_{n-1}(T, 1^+, ..., n^+), O] \\ &= [\Phi_0(\alpha, \xi 0^+, ..., \xi(n-1)^+), ..., \Phi_{n-1}(\alpha, \xi 0^+, ..., \xi(n-1)^+), O] \\ &= [\Phi_0(\alpha, \xi_0, ..., \xi_{n-1}), ..., \Phi_{n-1}(\alpha, \xi_0, ..., \xi_{n-1}), O] \\ &\leq [\xi_0, ..., \xi_{n-1}, O] = \xi, \end{aligned}$$

whence by the definition of iteration $\alpha\mathbf{I}(\varphi) \leq \xi$ and

$$\mu_i = \alpha\mathbf{I}(\varphi)i^+ \leq \xi i^+ = \xi_i$$

for all $i \leq n-1$.□

Note that the last theorem implies that the least solutions of homogeneous systems in operative spaces with iteration are left homogeneous in the following sense: If $(\mu_0, ..., \mu_{n-1})$ is the least solution of a homogeneous system of the form (2.1), then $(\beta\mu_0, ..., \beta\mu_{n-1})$ is the least solution of the system

$$\Phi_i(\beta\alpha, x_0, ..., x_{n-1}) \leq x_i$$

for all $\beta \in \mathcal{F}$.

Henceforth in this section we shall suppose that $\mathcal{F}$ is an operative space with iteration.

Next we shall specify some kind of homogeneous systems, which we shall call canonical. Namely, let $\xi = (\xi_0, ..., \xi_{n-1}) \in \mathcal{F}^n$ and $B \subseteq \mathcal{F}$; then by a $(B;\xi)$-*canonical* system of inequalities we shall mean a homogeneous system of the form

$$\Gamma_i(I, x_0, x_1\xi_0, ..., x_n\xi_{n-1}, x_{n+1}, ..., x_{n+m}) \leq x_i, \tag{2.2}$$

where $i \leq n+m$, $x_0, ..., x_{n+m}$ are unknowns and $\Gamma_0, ..., \Gamma_{n+m} : \mathcal{F}^{n+m+2} \to \mathcal{F}$ are OS-expressible in B left homogeneous mappings of $n+m+2$ arguments. A mapping $f : \mathcal{F}^n \to \mathcal{F}$ will be called *canonically definable* in B iff there is a $(B;\xi)$-canonical system of the form (2.2) such that the first member (corresponding to x_0) of the least solution of (2.2) equals $f(\xi)$ for all $\xi \in \mathcal{F}^n$. The systems of the form (2.2) being homogeneous, the Theorem 2 applies to them, whence we obtain the following

Corollary 1. *Every $(B;\xi)$–canonical system (2.2) has a least solution whose components μ_i ($i \leq n-1$) have the form*

$$\mu_i = \mathbf{I}([I, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}]\gamma)i^+$$

for a suitable element $\gamma \in \mathcal{F}$ which is OS-expressible in $B \cup \{O\}$.

Proof. By Theorem 2 the least solution of (2.2) is given by

$$\mu_i = \mathbf{I}(\varphi)i^+,$$

where $i \leq n+m$, $\varphi = [\varphi_0, ..., \varphi_{n+m}, O]$, and

$$\varphi_i = \Gamma_i(T, 1^+, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}, (n+2)^+, ..., (n+m+1)^+).$$

Thence

$$\varphi_i = [I, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}]\gamma_i,$$

where

$$\gamma_i = \Gamma_i(T^2, T1^+, 1^+, ..., (n-1)^+, F^n, T(n+2)^+, ..., T(n+m+1)^+).$$

Then defining $\gamma = [\gamma_0, ..., \gamma_{n+m}, O]$, we obtain

$$\varphi = [I, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}]\gamma$$

and the required representation of μ_i. □

Corollary 2. *Every canonically definable in $B \subseteq \mathcal{F}$ mapping $f : \mathcal{F}^n \to \mathcal{F}$ of n arguments is representable of the form*

$$f(\xi_0, ..., \xi_{n-1}) = \mathbf{I}([I, 2^+\xi_0, ..., (n+1)^+\xi_{n-1}]\gamma)T$$

for a certain element $\gamma \in \mathcal{F}$ which is OS*-expressible in $B \cup \{O\}$.*□

The last corollary shows that to establish Theorem 1 it is enough to prove that all singular mappings are canonically definable. For that purpose it will be convenient to introduce some more terminology about homogeneous systems. Consider two homogeneous systems

$$\Phi_i(I, x_0, ..., x_{n-1}) \leq x_i \tag{2.3}$$

and

$$\Psi_j(I, y_0, ..., y_{m-1}) \leq y_j, \tag{2.4}$$

where $i \leq n-1$ and $j \leq m-1$ and the variables x_i,y_j are supposed pairwise different. Then the *product* of the systems (2.3) and (2.4) is defined as the homogeneous system of $n+m$ inequalities, consisting of the inequalities of (2.3) and the m inequalities

$$\Psi_j(x_0, y_0, ..., y_{m-1}) \leq y_j. \tag{2.5}$$

Similarly, the homogeneous system of $n+m+1$ inequalities, consisting of the inequalities of (2.3) and (2.4) and the inequality $[x_0, y_0] \leq z$, where z is a new variable, will be called *pairing* of the systems (2.3) and (2.4); and the homogeneous system of n inequalities

$$\Phi_i([I, x_0], x_0, ..., x_{n-1}) \leq x_i \tag{2.6}$$

will be called *iteration* of the system (2.3).

Lemma 1. *Suppose $(\mu_0, ..., \mu_{n-1})$ and $(\nu_0, ..., \nu_{m-1})$ are the least solutions in $\mathcal{F}$ of the systems* (2.3) *and* (2.4) *with respect to $x_0, ..., x_{n-1}$ and $y_0, ..., y_{m-1}$, respectively. Then:*

(a) *the $(n+m)$-tuple $(\mu_0, ..., \mu_{n-1}, \mu_0\nu_0, ..., \mu_0\nu_{m-1})$ is the least solution in $\mathcal{F}$ of the product of the systems* (2.3) *and* (2.4) *with respect to $x_0, ..., x_{n-1}, y_0, ..., y_{m-1}$, respectively;*

(b) *the $(n+m+1)$-tuple $(\mu_0, ..., \mu_{n-1}, \nu_0, ..., \nu_{m-1}, [\mu_0, \nu_0])$ is the least solution in $\mathcal{F}$ of the pairing of the systems* (2.3) *and* (2.4) *with respect to $x_0, ..., x_{n-1}$, $y_0, ..., y_{m-1}, z$, respectively; and*

(c) *the n-tuple $(\mathbf{I}(\mu_0), [I, \mathbf{I}(\mu_0)]\mu_1, ..., [I, \mathbf{I}(\mu_0)]\mu_{n-1})$ is the least solution in $\mathcal{F}$ of the iteration of the system* (2.3) *with respect to $x_0, ..., x_{n-1}$, respectively.*

Proof. The n-tuple $(\mu_0, ..., \mu_{n-1})$ satisfies the inequalities of (2.3), and by the left homogeneity of Ψ_j the $(m+1)$-tuple

$$(\mu_0, \mu_0\nu_0, ..., \mu_0\nu_{m-1})$$

satisfies those of (2.5). Hence the $(n+m)$-tuple

$$(\mu_0, ..., \mu_{n-1}, \mu_0\nu_0, ..., \mu_0\nu_{m-1})$$

is a solution of the product of the systems (2.3) and (2.4). Consider an arbitrary solution $(\xi_0, ..., \xi_{n-1}, \eta_0, ..., \eta_{m-1})$ of the last product with respect to $x_0, ..., x_{n-1}$, $y_0, ..., y_{m-1}$, respectively. Since $(\mu_0, ..., \mu_{n-1})$ is the least solution of (2.3), we have $\mu_i \leq \xi_i$ for all natural numbers i less than n. On the other hand, by Theorem 2 it follows that $(\mu_0\nu_0, ..., \mu_0\nu_{m-1})$ is the least solution of the system

$$\Psi_j(\mu_0, y_0, ..., y_{m-1}) \leq y_j$$

with respect to $y_0, ..., y_{m-1}$, respectively; and from the inequality $\mu_0 \leq \xi_0$ we can conclude that $(\eta_0, ..., \eta_{m-1})$ satisfies the last system. Therefore $\mu_0\nu_j \leq \eta_j$ for all $j \leq m-1$, which proves (a). The proof of (b) is similar, but simpler and straightforward, and we leave it to the reader. Finally, to prove (c), denote shortly by λ the iteration $\mathbf{I}(\mu_0)$; then using the equality $[I,\lambda]\mu_0 = \lambda$ and the left homogeneity of Φ_i, we see that the n-tuple $(\lambda, [I,\lambda]\mu_1, ..., [I,\lambda]\mu_{n-1})$ is a solution of (2.6):

$$\Phi_i([I,\lambda], \lambda, [I,\lambda]\mu_1, ..., [I,\lambda]\mu_{n-1}) = [I,\lambda]\Phi_i(I, \mu_0, ..., \mu_{n-1}) \leq [I,\lambda]\mu_i.$$

Let $(\xi_0, ..., \xi_{n-1})$ be an arbitrary solution of (2.6) with respect to $x_0, ..., x_{n-1}$, respectively. Then it is a solution of the system

$$\Phi_i([I,\xi_0], x_0, ..., x_{n-1}) \leq x_i$$

with respect to $x_0, ..., x_{n-1}$, respectively. But Theorem 2 implies that the n-tuple

$$([I,\xi_0]\mu_0, ..., [I,\xi_0]\mu_{n-1})$$

is the least solution of the last system. Therefore $[I,\xi_0]\mu_i \leq \xi_i$ for all $i \leq n-1$, whence $[I,\xi_0]\mu_0 \leq \xi_0$ and $\lambda = \mathbf{I}(\mu_0) \leq \xi_0$, and finally $[I,\lambda]\mu_i \leq \xi_i$. □

Proof of Theorem 1. According to Corollary 2 it suffices to show that all B-singular operations in $\mathcal{F}$ of n arguments are canonically definable in B; and this follows from the fact that the class $\mathcal{O}$ of all canonically definable in B operations in $\mathcal{F}$ satisfies the conditions 0) - 5) in the definition of B-singular mapping above. Indeed, the identity mapping of one argument $e(\xi) = \xi$ is canonically definable by the system of two inequalities $x_1\xi \leq x_0$ and $I \leq x_1$. If b is a basic constant or element of B, then b as an operation in $\mathcal{F}$ of zero arguments is canonically definable by the system of one inequality $Ib \leq x_0$. Thus $\mathcal{O}$ satisfies conditions 0) and 1). To see that it satisfies conditions 2) - 4), it is enough to note, respectively, that the product, the pairing and the iteration of canonical systems are also canonical systems, and to apply Lemma 1. Finally, the class $\mathcal{O}$ satisfies 5), because a canonical definition by a system of the form (2.2) of an operation f in $\mathcal{F}$ of n arguments can be regarded as a canonical definition of the operation f_p after a suitable permutation of the variables $x_1, ..., x_n$. □

3. AFFINE APPLICATIONS IN OPERATIVE SPACES

The normal form Theorem 1 proved in the previous section enables us to define in arbitrary operative space $\mathcal{F}$ with iteration a binary operation which has the properties of affine application. Namely, for all φ, $\psi \in \mathcal{F}$ define

$$(\varphi \cdot \psi) = \mathbf{I}([I, 2^{+}\psi]\varphi)T. \tag{3.1}$$

We shall adopt the shorthand notation $(\varphi_0 \cdot \varphi_1 \cdot ... \cdot \varphi_{n-1} \cdot \varphi_n)$ for the iterated application $((...(\varphi_0 \cdot \varphi_1) \cdot ... \cdot \varphi_{n-1}) \cdot \varphi_n)$ and we shall also omit the external parentheses in such expressions. In the case $n = 0$ the last notation should be interpreted as φ_0. We have the following

Corollary 3. *Let $\mathcal{F}$ be an operative space with iteration and let f be a B-singular operation in $\mathcal{F}$ of n arguments. Then there is an element $\varphi \in \mathcal{F}$, OS-expressible in $B \cup \{O\}$, which represents f in the sense that for all $\xi_0,...,\xi_{n-1} \in \mathcal{F}$ we have*

$$f(\xi_0, ..., \xi_{n-1}) = \varphi \cdot \xi_0 \cdot ... \cdot \xi_{n-1}.$$

Proof. Induction on n. The case $n = 0$ is trivial; suppose $n \geq 1$ and assume the induction hypothesis for $n - 1$. By Theorem 1 we have an element $\varphi_0 \in \mathcal{F}$ OS-expressible in $B \cup \{O\}$ such that for all $\xi_0, ..., \xi_{n-1} \in \mathcal{F}$

$$f(\xi_0, ..., \xi_{n-1}) = \mathbf{I}([I, 2^{+}\xi_{n-1}, 3^{+}\xi_0, ..., (n+1)^{+}\xi_{n-2}]\varphi_0)T.$$

Thence for the B-singular mapping $f_1 : \mathcal{F}^{n-1} \to \mathcal{F}$ defined by

$$f_1(\xi_0, ..., \xi_{n-2}) = [T, F, T3^{+}\xi_0, ..., T(n+1)^{+}\xi_{n-2}]\varphi_0$$

we have

$$f(\xi_0, ..., \xi_{n-1}) = \mathbf{I}([I, 2^{+}\xi_{n-1}]f_1(\xi_0, ..., \xi_{n-2}))T = f_1(\xi_0, ..., \xi_{n-2}) \cdot \xi_{n-1},$$

and by the induction hypothesis applied to f_1 we obtain the required representation of f.□

Corollary 4. *In an arbitrary operative space $\mathcal{F}$ with iteration, the binary operation defined by* (3.1) *is an affine application in $\mathcal{F}$ in the sense that there are three elements A, C, $K \in \mathcal{F}$, OS-expressible in $\{O\}$, such that for all $\varphi, \psi, \chi \in \mathcal{F}$ we have*

$$A \cdot \varphi \cdot \psi \cdot \chi = \varphi \cdot (\psi \cdot \chi), \tag{3.2}$$

$$C \cdot \varphi \cdot \psi = \psi \cdot \varphi, \tag{3.3}$$

$$K \cdot \varphi \cdot \psi = \varphi. \tag{3.4}$$

Moreover, there are two elements $C_*, D_* \in \mathcal{F}$, OS-expressible in $\{O\}$, which satisfy for all φ, ψ , $\chi \in \mathcal{F}$ the equalities

$$C_* \cdot \varphi \cdot (D_* \cdot \psi \cdot \chi) = \varphi \cdot \psi \cdot \chi \tag{3.5}$$

and

$$D_* \cdot \psi \cdot \chi = [T\psi, F\chi]. \qquad (3.6)$$

Proof. The existence of A, C and D_* satisfying (3.2), (3.3) and (3.6), respectively, follows immediately from Corollary 3 since the right-hand sides of the last three equalities are the values of suitable $\emptyset$-singular mappings. The same holds also for K and (3.4), since we can replace φ with $[\varphi, \psi]T$. The right-hand side of (3.5) is also the value of a singular mapping for the arguments φ, ψ, χ, whence by Theorem 1 we have

$$\varphi \cdot \psi \cdot \chi = \mathbf{I}([I, 2^+\varphi, 3^+\psi, 4^+\chi]C_0)T = \mathbf{I}([I, 2^+\varphi, [3^+, 4^+][T\psi, F\chi]]C_0)T$$

for a certain OS-expressible in $\{O\}$ element $C_0 \in \mathcal{F}$, and we can define C_* by the help of Corollary 3 as an element of $\mathcal{F}$ satisfying

$$C_* \cdot \varphi \cdot \vartheta = \mathbf{I}([I, 2^+\varphi, [3^+, 4^+]\vartheta]C_0)T$$

for all φ, $\vartheta \in \mathcal{F}$. □

The partially ordered algebras having one binary operation called application, five constants A,C,K,C_* and D_*, satisfying (3.2)-(3.5), and a least element O for which $D_* \cdot O \cdot O = O$ were studied before by the present author under the name of CLCA (cartesian linear combinatory algebras; [6], [7]). However, a more appropriate (for the traditions of algebraic recursion theory) terminology would be, for instance, 'applicative spaces' instead of CLCA, and we shall follow this terminology below. The applicative spaces were shown to provide a simple abstract algebraic treatment of graph models of lambda calculus, which can comprise all the basic recursive algebra of sets of natural numbers under appropriate conditions (consisting in a suitable strengthening of the supposition of existence of least solutions of all inequalities of the form $\varphi \cdot \xi \leq \xi$ with respect to ξ) and which have a good variety of models inspired besides the graph models also by continuous functionals, Scott domains and others. Now the last corollary shows that applicative spaces admit a still greater variety of models, every operative space with iteration in which the equality $[O, O] = O$ holds providing such a model.

Properly speaking, we obtained a functor $\Phi : \mathbf{OSI} \to \mathbf{AS}$ from the category **OSI** of operative spaces with iteration satisfying the last equality to the category **AS** of applicative spaces (morphisms of **OSI** are the mappings preserving the basic OS-operations and the iteration, and the morphisms of **AS** are the mappings preserving the application and the basic constants A,C,K,C_*,D_* and O), which sends every object $\mathcal{F}$ of **OSI** to the applicative space $\Phi(\mathcal{F})$ described by (the proof of) Corollary 4 (in particular, $\Phi(\mathcal{F})$ has the same set of elements and the same partial order as $\mathcal{F}$) and every **OSI**-morphism $f : \mathcal{F} \to \mathcal{F}'$ to the same mapping f. This functor Φ is obviously faithful, but it is a problematical question whether Φ is full, which amounts to the question whether the original basic operations of an operative space $\mathcal{F} \in \mathbf{OSI}$ are explicitly expressible via the basic constants and operations of the applicative space $\Phi(\mathcal{F})$; and the question of equivalence of the categories **OSI** and **AS** is still more problematical.

A nice feature of the functor Φ is that it preserves the storage operations in the following sense. As it was shown in [7], the unary operation ∇ in an applicative space $\mathcal{A}$, which assigns to each element φ of $\mathcal{A}$ the least solution $\nabla(\varphi)$ of the inequality $D_*{\cdot}\varphi{\cdot}\xi \leq \xi$ with respect to ξ, has the properties of the storage operations arising in semantics of linear logic, namely there are elements $I_*, M_*, Q_*, K_*, W_* \in \mathcal{A}$, such that the equalities

$$
\begin{aligned}
I_* \cdot \nabla(\varphi) &= \varphi, && (3.7)\\
M_* \cdot \nabla(\varphi) \cdot \nabla(\psi) &= \nabla(\varphi \cdot \psi), && (3.8)\\
Q_* \cdot \nabla(\varphi) &= \nabla^2(\varphi) = \nabla(\nabla(\varphi)), && (3.9)\\
K_* \cdot \psi \cdot \nabla(\varphi) &= \psi, && (3.10)\\
W_* \cdot \psi \cdot \nabla(\varphi) &= \psi \cdot \varphi \cdot \varphi && (3.11)
\end{aligned}
$$

hold for all φ, $\psi \in \mathcal{A}$, provided the space $\mathcal{A}$ satisfies the conditions mentioned above and specified in [6] and [7]. The algebra of proofs for the Hilbert-type axiomatization of the fragment of linear logic restricted to the linear implication and the exponential connective 'of course' can be regarded, by the well known formulae-as-types correspondence, as a typed version of the algebras with two operations – (linear) application and storage ∇ – and seven constants A, C, I_*, M_*, Q_*, K_*, W_* satisfying (3.2), (3.3) and (3.7)-(3.11). Hence we use the term *type-free models of linear logic* for the last algebras. Now the natural storage operation ∇ defined above in the applicative space $\Phi(\mathcal{F})$ assigned to an operative space $\mathcal{F} \in$ **OSI** coinsides with the natural storage operation in the last space, which is called *translation* (in [2]) and is defined as the least solution of the inequality $[T\varphi, F\xi] \leq \xi$ with respect to ξ. This follows immediately from the equality (3.6) of Corollary 4 and is the reason to say that Φ preserves the storage operations.

The type-free models of linear logic have various instances closely connected with recursion theory; that is why they were introduced and used even before the discovery of linear logic for the purposes of axiomatization of recursion (e.g. in [5]). From purely formal point of view, they provide a substitute for the (models of the) combinatory logic, which is easier to deal with, being based on the binary operation of linear application. The last operation is more natural and easier to model, and has the advantage of being free of the gross algebraic complexity of the basic laws for the traditional application operation of combinatory logic, replacing them with some kind of generalized associative and commutative laws. The usual combinatory logic can be easily modeled in the type-free linear logic by defining the application operation Ω as follows: $\Omega(\varphi, \psi) = \varphi \cdot \nabla(\psi)$.

Now the results of [7] imply that for every object $\mathcal{F}$ of **OSI** the applicative space $\Phi(\mathcal{F})$ forms a model of the type-free linear logic with respect to the storage operation defined as the least solution of the inequality $D_* \cdot \varphi \cdot \xi \leq \xi$, provided the space $\Phi(\mathcal{F})$ satisfies the conditions specified in [7], which is always the case for continuous operative spaces $\mathcal{F}$. However, one can naturally construct models of the type-free linear logic in a larger class of operative spaces (not necessarily continuous), namely the iterative ones, as well as in all combinatory spaces in the sense of [4] satisfying some weak suppositions of iterativity, by a direct use of translation and other suitable storage operations. This we shall discuss in the next section.

4. STRONG STORAGE OPERATIONS IN OPERATIVE SPACES

Let $\mathcal{F}$ be an operative space with iteration and let \$ be a unary operation in $\mathcal{F}$. We shall say that \$ is a *strong storage* operation in $\mathcal{F}$ iff there are three elements $D, P, Q \in \mathcal{F}$ such that the equalities

$$\begin{aligned} \$(\varphi)n^+ &= n^+\varphi, && (4.1)\\ \$(\varphi\psi) &= \$(\varphi)\$(\psi), && (4.2)\\ \$([\varphi,\psi]) &= [\$(\varphi),\$(\psi)]D, && (4.3)\\ \$(\$(\varphi)) &= Q\$(\varphi)P, && (4.4)\\ \$(\mathbf{I}(\varphi)) &= \mathbf{I}(D\$(\varphi)) && (4.5)\end{aligned}$$

hold for all φ, $\psi \in \mathcal{F}$ and all natural numbers n. A natural example of a strong storage operation provides the operation of translation in iterative operative spaces, which are defined ([2]) as operative spaces with iteration satisfying the following additional axiom

($\mathcal{L}$) There is a unary operation $\varphi \mapsto \langle\varphi\rangle$ in $\mathcal{F}$ called translation, such that the inequality $[T\varphi, F\langle\varphi\rangle] \leq \langle\varphi\rangle$, and the implication

$$(\alpha F \leq \psi\alpha \,\&\, [\alpha T\varphi, \psi\tau] \leq \tau) \Rightarrow \alpha\langle\varphi\rangle \leq \tau$$

hold for all φ, α, ψ, $\tau \in \mathcal{F}$.

Proposition 1. *In every iterative operative space the operation of translation is a strong storage operation such that the corresponding constants D,P and Q are explicitly expressible by means of the basic operations, iteration and translation.*

Proof. This is proved in [2]. Namely, the equality (4.1) for the translation operation is Proposition 5.6 in the quoted book; the equalities (4.2), (4.3) and (4.4) are Propositions 6.21, 6.36 and 6.40, respectively; and the equality (4.5) is Proposition 6.37 in view of the expressions $\mathbf{I}(\varphi) = [\varphi]F = [\varphi]\varphi$ and $[\varphi] = (I, \mathbf{I}(\varphi))$ of the operations of iteration in the sense of the present paper and that in the sense of [2] with each other, which is easy to check directly and which also occurs, for instance, in [8], pp. 1739-1740. □

Generally speaking, there are many other fixed point definable strong storage operations in every iterative operative space, but the translation is one of the simplest of them.

Another important example of strong storage operations is provided by the notion of combinatory space of Skordev [4]. Consider a combinatory space $\mathcal{S} = \langle \mathcal{F}, I, \mathcal{C}, \Pi, L, R, \Sigma, T, F\rangle$; we shall use the notations and the terminology concerning combinatory spaces from [4], and we shall suppose that $T, F \in \mathcal{C}$, which does not make an essential difference with the original definition in [4]. We shall call the space $\mathcal{S}$ *weakly iterative* (compare with the notion of *iterative* combinatory space from [4]) iff for all $\varphi,\ \chi \in \mathcal{F}$ the least solution $[\varphi, \chi] \in \mathcal{F}$ of the inequality $(\chi \to \xi\varphi, I) \leq \xi$ with respect to ξ exists, and for all $\alpha \in \mathcal{F}$ the element $\alpha[\varphi, \chi]$ is the least solution of $(\chi \to \xi\varphi, \alpha) \leq \xi$ with respect to ξ in $\mathcal{F}$.

Proposition 2. *Suppose in the combinatory space $\mathcal{S}$ the equality $(L, R) = I$ holds. Then there are elements G, T_+, F_+, D, P, $Q \in \mathcal{F}$ elementary in $\emptyset$ such that: the poset $\mathcal{F}$ forms an operative space $\mathcal{S}_+$ with respect to the same unit I and a multiplication operation as in $\mathcal{S}$, the operation $[-,-]$ defined by*

$$[\varphi, \psi] = (L \to \varphi R, \psi R)G,$$

and the elements T_+ and F_+ as the basic OS-constants T and F, respectively; and the operation \$ defined by $\$(\varphi) = (L, \varphi R)$ satisfies (4.1)- (4.4) *in $\mathcal{S}_+$. Moreover, if the space $\mathcal{S}$ is weakly iterative, then the operative space $\mathcal{S}_+$ has an iteration and* (4.5) *also holds, i.e. \$ is a strong storage operation in $\mathcal{S}_+$.*

Proof. The proof makes use of the technique for combinatory spaces developed in [4] and [2]. More specifically, it is proved in [2] that the partially ordered monoid $\mathcal{F}$ is an operative space $\mathcal{S}_*$ (called the companion operative space of $\mathcal{S}$) with respect to the pairing operation $\varphi, \psi \mapsto (L \to \varphi R, \psi R)$ and the elements $T' = (T, I)$ and $F' = (F, I)$ as the basic constants T and F, respectively; and the elements $C = (LR \to T'\$(R), F'\$(R))$, $P = ((L, LR), RR)$ and $Q = (LL, (RL, R))$ satisfy the equalities (4.2)-(4.4) in $\mathcal{S}_*$, replacing D by C (see Propositions 27.13, 10.12, 10.13 and 10.16 in [2]). Then define $T_+ = PT'T'$, $F_+ = PF'$ and $G = (LL \to T'R, F'(RL, R))$, and note that Proposition 27.8 in [2] combined with the supposition $(L, R) = I$ implies the following fact:

(*) *For all φ, $\psi \in \mathcal{F}$ such that $\varphi(c, I) \leq \psi(c, I)$ for all $c \in \mathcal{C}$ we have $\varphi \leq \psi$.*

Using this fact and the equalities

$$GF_+(c, I) = GP(F, (c, I)) = G((F, c), I) = F'(RL, R)((F, c), I) = F'(c, I)$$

we obtain $GF_+ = F'$, and similarly,

$$GT_+ = GPT'T' = GP(T, (T, I)) = G((T, T), I) = T'R((T, T), I) = T'.$$

Hence

$$[\varphi, \psi]F_+ = (L \to \varphi R, \psi R)GF_+ = (L \to \varphi R, \psi R)F' = \psi,$$

and similarly, $[\varphi,\psi]T_+ = \varphi$, which shows that $\mathcal{S}_+$ is an operative space, the equality $\chi[\varphi,\psi] = [\chi\varphi,\chi\psi]$ being obvious. Using again (*) and the equalities

$$\$(\varphi)(c,I) = (L,\varphi R)(c,I) = (c,\varphi) = (c,I)\varphi, \tag{4.6}$$

which hold for all $\varphi \in \mathcal{F}$ and $c \in \mathcal{C}$, we can prove the equality $\$(\varphi)F_+ = F_+\$(\varphi)$ as follows:

$$\begin{aligned} \$(\varphi)F_+(c,I) &= \$(\varphi)PF'(c,I) = \$(\varphi)((L,LR),RR)(F,(c,I)) \\ &= (L,\varphi R)((F,c),I) = ((F,c),\varphi) = ((F,c),I)\varphi \\ &= F_+(c,I)\varphi = F_+\$(\varphi)(c,I). \end{aligned}$$

Similarly, $\$(\varphi)PT' = PT'\$(\varphi)$, whence

$$\begin{aligned} \$(\varphi)T_+ &= \$(\varphi)PT'T' = PT'\$(\varphi)T' = PT'(L,\varphi R)(T,I) \\ &= PT'(T,\varphi) = PT'(T,I)\varphi = T_+\varphi. \end{aligned}$$

The equalities $\$(\varphi)F_+ = F_+\$(\varphi)$ and $\$(\varphi)T_+ = T_+\varphi$ imply (4.1) for the operative space $\mathcal{S}_+$; and (4.2) and (4.4) are the same as in the companion space $\mathcal{S}_*$. The equality (4.3) follows from the same one in $\mathcal{S}_*$:

$$\begin{aligned} \$([\varphi,\psi]) &= \$((L \to \varphi R,\psi R))\$(G) = (L \to \$(\varphi)R, \$(\psi)R)C\$(G) \\ &= [\$(\varphi),\$(\psi)](L \to T_+R, F_+R)C\$(G) = [\$(\varphi),\$(\psi)]D, \end{aligned}$$

where

$$D = (L \to T_+R, F_+R)C\$(G) = (LR \to T_+\$(R), F_+\$(R))\$(G).$$

Now suppose the combinatory space $\mathcal{S}$ is weakly iterative. To prove that the operative space $\mathcal{S}_+$ has iteration, it is enough to show that the companion space $\mathcal{S}_*$ has an iteration, as it follows easily from the definition of the pairing operation in $\mathcal{S}_+$. We shall see that the iteration $\mathbf{I}_*$ in $\mathcal{S}_*$ can be defined by $\mathbf{I}_*(\varphi) = R[(L \to F'R, T'R)\varphi R, L]T'$, i.e. that for all φ, $\alpha \in \mathcal{F}$ the element $\alpha\mathbf{I}_*(\varphi)$ is the least solution of

$$(L \to \alpha R, \xi R)\varphi \le \xi \tag{4.7}$$

with respect to ξ in $\mathcal{F}$. Indeed, for all $\vartheta \in \mathcal{F}$ we have $(L \to [\vartheta R, L]\vartheta R, I) = [\vartheta R, L]$, whence

$$[\vartheta R, L]T' = [\vartheta R, L]\vartheta.$$

Using the last equality and writing shortly E for $(L \to F'R, T'R)$, we can check that $\alpha\mathbf{I}_*(\varphi)$ satisfies (4.7) as follows:

$$\begin{aligned} (L \to \alpha R,\, \alpha\mathbf{I}_*(\varphi)R)\varphi &= (L \to \alpha R,\, \alpha R[E\varphi R,\, L]T'R)\varphi \\ &= (L \to \alpha R[E\varphi R,\, L]T'R,\, \alpha R)(L \to F'R, T'R)\varphi \\ &= \alpha R(L \to [E\varphi R,\, L]T'R,\, I)E\varphi \\ &= \alpha R(L \to [E\varphi R,\, L](L \to F'R, T'R)\varphi R,\, I)E\varphi \\ &= \alpha R[E\varphi R,\, L]E\varphi = \alpha R[E\varphi R,\, L]T' = \alpha\mathbf{I}_*(\varphi). \end{aligned}$$

Assuming that ξ is an arbitrary solution of (4.7) in $\mathcal{F}$, we have as well

$$\begin{aligned}(L \to (L \to \xi R, \alpha R)E\varphi R,\ \alpha R)\\ = (L \to (L \to \alpha R, \xi R)\varphi R,\ \alpha R) \le (L \to \xi R, \alpha R),\end{aligned}$$

whence by the weak iterativity of $\mathcal{S}$ we obtain

$$\alpha R[E\varphi R, L] \le (L \to \xi R, \alpha R)$$

and

$$\alpha \mathbf{I}_*(\varphi) = R[E\varphi R, L]T' \le (L \to \xi R, \alpha R)T' = \xi,$$

completing the proof that $\mathbf{I}_*$ is an iteration in $\mathcal{S}_*$. Hence the operation $\mathbf{I}_+$ defined by $\mathbf{I}_+(\varphi) = \mathbf{I}_*(G\varphi)$ is an iteration in $\mathcal{S}_+$. To establish the equality (4.5) for $\mathcal{S}_+$, we shall do this first for $\mathcal{S}_*$, namely we shall show that

$$\$(\mathbf{I}_*(\varphi)) = \mathbf{I}_*(C\$(\varphi))$$

for all $\varphi \in \mathcal{F}$. Indeed, using the equalities (4.3) (for $\mathcal{S}_*$) and $\$(I) = (L, R) = I$, we have

$$\begin{aligned}(L \to R, \$(\mathbf{I}_*(\varphi))R)C\$(\varphi) &= (L \to \$(I)R, \$(\mathbf{I}_*(\varphi))R)C\$(\varphi)\\ &= \$((L \to R, \mathbf{I}_*(\varphi)R)\varphi) = \$(\mathbf{I}_*(\varphi)),\end{aligned}$$

which shows that $\mathbf{I}_*(C\$(\varphi)) \le \$(\mathbf{I}_*(\varphi))$. To prove the reverse inequality, take an arbitrary $c \in C$ and note, by the help of the definition of the constant C and the equalities (4.6), that

$$C(c, I) = (LR \to T'\$(R),\ F'\$(R))(c, I) = (L \to T'(c, I)R,\ F'(c, I)R).$$

Then

$$\begin{aligned}&(L \to (c, I)R,\ \mathbf{I}_*(C\$(\varphi))(c, I)R)\varphi\\ &= (L \to R, \mathbf{I}_*(C\$(\varphi))R)(L \to T'(c, I)R,\ F'(c, I)R)\varphi\\ &= (L \to R, \mathbf{I}_*(C\$(\varphi))R)C(c, I)\varphi\\ &= (L \to R, \mathbf{I}_*(C\$(\varphi))R)C\$(\varphi)(c, I) = \mathbf{I}_*(C\$(\varphi))(c, I),\end{aligned}$$

whence, $\mathbf{I}_*$ being iteration in $\mathcal{F}_*$, we obtain

$$(c, I)\mathbf{I}_*(\varphi) \le \mathbf{I}_*(C\$(\varphi))(c, I),$$

which by (4.6) and (*) implies $\$(\mathbf{I}_*(\varphi)) \le \mathbf{I}_*(C\$(\varphi))$ and completes the proof of the equality $\$(\mathbf{I}_*(\varphi)) = \mathbf{I}_*(C\$(\varphi))$. The last one implies

$$\$(\mathbf{I}_+(\varphi)) = \$(\mathbf{I}_*(G\varphi)) = \mathbf{I}_*(C\$(G\varphi)) = \mathbf{I}_*(C\$(G)\$(\varphi)).$$

On the other hand

$$(L \to \varphi R, \psi R) = [\varphi, \psi](L \to T_+R, F_+R)$$

whence

$$\mathbf{I}_*(\varphi) = \mathbf{I}_+((L \to T_+R, F_+R)\varphi)$$

for all $\varphi \in \mathcal{F}$, and

$$\$(\mathbf{I}_+(\varphi)) = \mathbf{I}_+((L \to T_+R, F_+R)C\$(G)\$(\varphi)) = \mathbf{I}_+(D\$(\varphi)).\square$$

Remark. The supposition $(L, R) = I$ in the last proposition was made for the sake of simplicity – we could avoid it at the expense of some complications of the exposition, especially in Section 2. On the other hand, this supposition is natural, but no special reasons are discussed in the books [3] and [4] for its abandonment; it seems it was abandoned just for the reason of its not being necessary for the exposition presented in those books. The paper [9] however, combined with Proposition 2 above, indicates a better exposition which can be simplified by adding the supposition in question to the axioms. Also, the examples of combinatory spaces occurring in the quoted books do not give reasons to consider the abandonment of $(L, R) = I$ as essential for the scope of the theory: all of them have more or less obvious variants in which the last equality is true.

Now, returning to the general case of operative spaces, we have the following

Corollary 5. *Suppose $\mathcal{F}$ is an operative space with iteration, and \$ is a strong storage operation in $\mathcal{F}$ with corresponding constants D, P, $Q \in \mathcal{F}$ satisfying* (4.1)-(4.5). *Then the poset $\mathcal{F}$ forms a model of the type-free linear logic with respect to the application operation defined by* (3.1), *the operation \$ as the storage ∇, and certain constants A, C, I_*, M_*, Q_*, K_* and W_* which are OS-expressible in $\{D, P, Q, O\}$.*

Proof. The existence of A and C satisfying (3.2) and (3.3) is established in Corollary 4. By Corollary 3 there is $I_0 \in \mathcal{F}$ OS-expressible in $\{O\}$ and such that $\varphi = I_0 \cdot \varphi$ for all $\varphi \in \mathcal{F}$, whence by (4.1)

$$\varphi = \mathbf{I}([I, 2^+\varphi]I_0)T = \mathbf{I}([I, \$(\varphi)2^+]I_0)T,$$

and taking I_* to represent the unary singular operation f_0 defined by

$$f_0(\xi) = \mathbf{I}([I, \xi 2^+]I_0)T$$

in the sense of Corollary 3 we obtain (3.7). By (4.2),(4.5) and (4.3) we have as well

$$\nabla(\varphi \cdot \psi) = \$(\mathbf{I}([I, 2^+\psi]\varphi)T) = \mathbf{I}(D[\$(I), \$(2^+)\$(\psi)]D\$(\varphi))\$(T)),$$

whence the element M_* representing the binary singular operation f_1 defined by

$$f_1(\xi, \eta) = \mathbf{I}(D[\$(I), \$(2^+)\eta]D\xi)\$(T))$$

in the sense of Corollary 3 satisfies (3.8). Similarly, the equality (4.4) shows that the element Q_* representing in the sense of Corollary 3 the unary operation f_2 defined by $f_2(\xi) = Q\xi P$ satisfies (3.9); and to satisfy (3.10) we can obviously take the constant K from Corollary 4 for K_*. Finally, by Theorem 1 we have an element $W_0 \in \mathcal{F}$ OS-expressible in $\{O\}$ such that $\mathbf{I}([I, 2^+\xi, 3^+\eta, 4^+\zeta]W_0)T = \xi \cdot \eta \cdot \zeta$ for all $\xi,\eta,\zeta \in \mathcal{F}$. Then

$$\psi \cdot \varphi \cdot \varphi = \mathbf{I}([I, 2^+\psi, 3^+\varphi, 4^+\varphi]W_0)T = \mathbf{I}([I, 2^+\psi, \$(\varphi)[3^+, 4^+]]W_0)T,$$

and taking W_* to represent in the sense of Corollary 3 the binary singular operation f_3 defined by

$$f_3(\xi, \eta) = \mathbf{I}([I, 2^+\xi, \eta[3^+, 4^+]]W_0)T$$

we obtain (3.11). □

Properly speaking, the last corollary yields a faithful functor Ψ from the category of operative spaces with iteration and strong storage operation to the category of type-free models of linear logic; but, as with the functor Φ, the questions of whether Ψ is full and of existence of equivalence of the last two categories are open.

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Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bontchev str., bl. 8, 1113 Sofia
BULGARIA
E-mail: zashev@math.bas.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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BALANCED VERTEX SETS IN GRAPHS

NIKOLAY KHADZHIIVANOV, NEDYALKO NENOV

Let $v_1, \ldots, v_r$ be a β-sequence (Definition 1.2) in an n-vertex graph G and $v_{r+1}, \ldots, v_n$ be the other vertices of G. In this paper we prove that if $v_1, \ldots, v_r$ is balansed, that is

$$\frac{1}{r}(d(v_1) + \ldots + d(v_r)) = \frac{1}{n}(d(v_1) + \ldots + d(v_n)),$$

and if the number of edges of G is big enough, then G is regular.

Keywords: saturated sequence, balanced sequence, generalized r-partite graph, generalized Turan's graph

2000 MSC: 05C35

1. INTRODUCTION

$e(G) = |E(G)|$ – the number of edges of G;
$G[M]$ – the subgraph of G, induced by M, where $M \subset V(G)$;
$\Gamma_G(M)$ –the set of all vertices of G adjacent to any vertex of M;
$d_G(v) = |\Gamma_G(v)|$ – the degree of a vertex v in G;
K_n and $\overline{K}_n$ – the complete and discrete n-vertex graphs, respectively.

Let r be an integer. A graph G is called r-partite with partition classes $V_i, i = 1, \ldots, r$ if $V(G) = V_1 \cup \ldots \cup V_r$, $V_i \cap V_j = \varnothing$ for $i \neq j$ and the sets V_i are independent sets in G. If every two vertices from different partition classes are adjacent, then G is called complete r-partite graph. Let G be an n-vertex r-partite graph with partition classes V_i and $p_i = |V_i|, i = 1, \ldots, r$. Obviously, $d_G(v) \leq n - p_i$, for any $v \in V_i, i = 1, \ldots, r$ and $d_G(v) = n - p_i$ if and only if G is a complete r-partite graph. The symbol $K(p_1, \ldots, p_r)$ denotes the complete r-partite graph with partition classes $V_1, \ldots, V_r$ such that $|V_i| = p_i, i = 1, \ldots, r$. If $p_1, \ldots, p_r$ are as equal as possible (in the sense that $|p_i - p_j| \leq 1$ for all pairs $\{i, j\}$), then if $p_1 + \ldots + p_r = n$, $K(p_1, \ldots, p_r)$ is denoted by $T_r(n)$ and is called r-partite n-vertex Turan's graph. Clearly

$$e(K(p_1, \ldots, p_r)) = \sum \{p_i p_j \mid 1 \leq i < j \leq r\}.$$

Thus, if $p_i - p_j \geq 2$, then

$$e(K(p_1 - 1, p_2 + 1, p_3, \ldots, p_r)) - e(K(p_1, p_2, \ldots, p_r)) = p_1 - p_2 - 1 > 0$$

This observation implies the following elementary proposition, we make shall use of later:

Lemma 1.1. *Let n and r be positive integers. Then the inequality*

$$e(K(p_1, \ldots, p_r)) \leq e(T_r(n))$$

holds for each r-tuple $(p_1, \ldots, p_r)$ of nonnegative integers p_i such that $p_1 + \ldots + p_n = n$. The equality occurs only when $K(p_1, \ldots, p_r) = T_r(n)$.

Let $V_1, \ldots, V_{r-1}$ be partition classes of $T_{r-1}(n), 2 \leq r \leq n$. Then $T_{r-1}(n)$ is r-partite graph with partition classes $V_1, \ldots, V_{r-1}, \{\varnothing\}$. Since $2 \leq r \leq n$, $T_{r-1}(n) \neq T_r(n)$. Thus, from Lemma 1.1 it follows that

$$e(T_{r-1}(n)) < e(T_r(n)) \tag{1.1}$$

Let $V(G) = \{v_1, \ldots, v_n\}$. We call the graph G regular, if

$$d_G(v_1) = d_G(v_2) = \ldots = d_G(v_n).$$

A simple calculation shows that

$$e(T_r(n)) = \frac{(n^2 - \nu^2)(r-1)}{2r} + \binom{\nu}{2}, \tag{1.2}$$

where $n = kr + \nu$, $0 \leq \nu \leq r - 1$. □

Definition 1.1 Let G be a graph and $v_1, \ldots, v_r \in V(G)$ be a vertex sequence such that

$$v_i \in \Gamma_G(v_1, \ldots, v_{i-1}),\ 2 \leq i \leq r.$$

Define $V_1 = V(G) \backslash \Gamma_G(v_1)$, $V_2 = \Gamma_G(v_1) \backslash \Gamma_G(v_2)$, $V_3 = \Gamma_G(v_1, v_2) \backslash \Gamma_G(v_3), \ldots$, $V_{r-1} = \Gamma_G(v_1, \ldots, v_{r-2}) \backslash \Gamma_G(v_{r-1})$, $V_r = \Gamma_G(v_1, \ldots, v_{r-1})$.

Definition 1.2 The sequence of vertices $v_1, \ldots, v_r$ in a graph G is called β-sequence, if the following conditions are satisfied: v_1 is a vertex of maximal degree in G, and for $i \geq 2$, $v_i \in \Gamma_G(v_1, \ldots, v_{i-1})$ and

$$d_G(v_i) = \max \{d_G(v) | v \in \Gamma_G(v_1, \ldots, v_{i-1})\}.$$

Definition 1.3 Let G be an n-vertex graph and $v_1, \ldots, v_r \in V(G)$. Then the sequence $v_1, \ldots, v_r$ is called saturated, if

$$\frac{1}{r}(d_G(v_1) + \ldots + d_G(v_r)) > \frac{2e(G)}{n}.$$

This sequence is called balanced, if

$$\frac{1}{r}(d_G(v_1) + \ldots + d_G(v_r)) = \frac{2e(G)}{n}.$$

Obviously, if G is regular, then any vertex sequence in G is balanced. Let $V(G) = \{v_1, \ldots, v_n\}$. Then

$$d(v) \geq \frac{2e(G)}{n} = \frac{1}{n}(d_G(v_1) + \ldots + d_G(v_n))$$

for any vertex of maximal degree in G. Thus, if $d(v) = \frac{2e(G)}{n}$ for some vertex of maximal degree in G, then G is regular.

Let r and n be positive integers,$2 \leq r \leq n$. Define

$$f(n,r) = \begin{cases} \frac{n^2(r-1)}{2r} - \frac{n}{2r} & \text{if } n \equiv 0 (mod\, r); \\ \frac{n^2(r-1)}{2r} - \frac{\nu n}{2r(r-1)} & \text{if } n \equiv \nu (mod\, r), 1 \leq \nu \leq r-1. \end{cases}$$

It straightforward to show that

$$f(n,r) > \frac{(r-2)n^2}{2(r-1)}, \; r \geq 2$$

Since $\frac{(r-2)n^2}{2(r-1)} > f(n, r-1)$, we have

$$f(n, r-1) < f(n,r), \; 2 \leq r \leq n \tag{1.3}$$

Our main result is the following theorem:

Theorem 1.1 (The Main Theorem). *Let G be an n-vertex graph and r be a positive integer, $2 \leq r \leq n$, such that $e(G) > f(n,r)$. Let for some s, $1 \leq s \leq r$, there exists a balanced β-sequence $v_1, \ldots, v_s \in V(G)$. Then G is regular.*

Example 1.1. Consider the graph G shown in Fig.1. The β-sequence $\{v_1, v_3\}$ is balanced, because

$$\frac{1}{2}(d_G(v_1) + d_G(v_2)) = \frac{2e(G)}{8} = \frac{5}{2}.$$

Obviously, G is not regular.

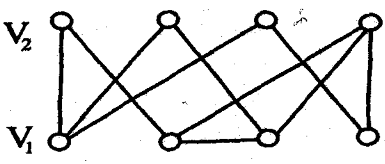

Fig. 1.

2. GENERALIZED r-PARTITE GRAPHS

Definition 2.1. ([2]) An n-vertex graph G is called generalized r-partite with partition classes V_i, $i = 1, \ldots, r$, if $V(G) = V_1 \cup \ldots \cup V_r$, $V_i \cap V_j = \varnothing$, $i \neq j$ and $d_G(v) \leq n - p_i$ for any $v \in V_i$, $i = 1, \ldots, r$, where $p_i = |V_i|$. If $d_G(v) = n - p_i$ for any $v \in V_i$, $i = 1, \ldots, r$, then G is called generalized complete r-partite graph with partition classes $V_1, \ldots, V_r$. We call G generalized Turan's r-partite graph if G is a generalized complete r-partite graph with partition classes $V_1, \ldots, V_r$ and $|p_i - p_j| \leq 1$ for all pairs $\{i, j\}$.

Proposition 2.1. *Let r and n be natural numbers, $1 \leq r \leq n$. Let G be an n-vertex graph, such that*

$$d(v) \leq \frac{(r-1)n}{r}, \ \forall v \in V(G).$$

Then G is generalized r-partite graph.

Proof. Let

$$V(G) = V_1 \cup \ldots \cup V_r, \ V_i \cap V_j = \varnothing, \ i \neq j$$

and $\lfloor \frac{n}{2} \rfloor \leq |V_i| \leq \lceil \frac{n}{2} \rceil$, $i = 1, \ldots, r$.

From $d(v) \leq \frac{(r-1)n}{r} = n - \frac{n}{r}$ it follows that $d(v) \leq n - \lceil \frac{n}{r} \rceil$, $\forall v \in V(G)$. Thus $d(v) \leq n - |V_i|$, $\forall v \in V_i$, $i = 1, \ldots, r$, and G is generalized r-partite graph with partition classes $V_1, \ldots, V_r$. □

Observe that, if $n \equiv 0 (mod\, r)$ and $d(v) = \frac{(r-1)n}{r}$, $\forall v \in V(G)$, then G is generalized r-partite Turan's graph.

We shall make use of the following result:

Theorem 2.1. ([2]) *Let G be a generalized r-partite graph with partition classes $V_1, \ldots, V_r$, where $|V_i| = p_i$, $i = 1, \ldots, r$. Then*

$$e(G) \leq e(K(p_1, \ldots, p_r)).$$

The equality holds if and only if G is generalized complete r-partite graph with partition classes $V_1, \ldots, V_r$.

Theorem 2.2. ([2]) *Let G be a generalized r-partite graph and $|V(G)| = n$. Then*

$$e(G) \leq e(T_r(n))$$

and equality occurs if and only if G is generalized r-partite Turan's graph.

Example 2.1. Consider the graph $K_3 + C_5 = K_8 - C_5$. Obviously, $e(K_3 + C_5) = 23 < e(T_4(8)) = 24$. This graph is not generalized 4-partite graph. Assume the opposite, i.e. that $K_3 + C_5$ is generalized 4-partite graph with partition classes V_1, V_2, V_3, V_4. Let $V(K_3)) = \{v_1, v_2, v_3\}$. If $v_i \in V_j$, then from $d(v_i) = 7 \leq 8 - |V_j|$ it follows that $|V_j| = 1$, i.e. $V_j = \{v_i\}$. Thus, we may assume that $V_i = \{v_i\}, i = 1, 2, 3$. Hence, $V_4 = V(C_5)$. Let $v \in V(C_5)$. Then $d(v) = 5 > 8 - |V_4| = 3$, which is a contradiction.

3. β-SEQUENCES AND GENERALIZED r-PARTITE GRAPHS

We shall use the following:

Theorem 3.1. ([2]) *Let $v_1, \ldots, v_r$ be a β-sequence in an n-vertex graph G, which is not contained in an $(r+1)$-clique. If V_i is the i-th stratum of the stratification induced by this sequence and $p_i = |V_i|$ (see Definition 1.1), then*

(a) *G is generalized r-partite graph with partition classes $V_1, \ldots, V_r$;*

(b) *$e(G) \leq e(K(p_1, \ldots, p_r))$, and the equality occurs if and only if G is a generalized complete r-partite graph with partition classes $V_1, \ldots, V_r$;*

(c) *$e(G) \leq e(T_r(n))$ and we have $e(G) = e(T_r(n))$ only when G is a generalized r-partite Turan's graph.*

The proof of the theorem 3.1, given in [2], actually establishes the following stronger statement:

Theorem 3.2. ([2]) *Let $v_1, \ldots, v_r$ be a β-sequence in an n-vertex graph G such that*

$$d_G(v_r) \leq n - |\Gamma_G(v_1, \ldots, v_{r-1})|$$

Then the statements (a), (b) *and* (c) *of the Theorem* 3.1 *hold.*

Denote by $\psi(G)$ the smallest integer r for which there exist a β-sequence $v_1, \ldots, v_r$, $r \geq 2$, in n-vertex graph G, such that

$$d_G(v_r) \leq n - |\Gamma_G(v_1, \ldots, v_{r-1})|.$$

Theorem 3.3. *Let G be an n-vertex graph and $e(G) \geq e(T_r(n))$. Then $\psi(G) \geq r$ and $\psi(G) = r$ only when G is a generalized r-partite Turan's graph.*

Proof. Let $\psi(G) = s$. By Theorem 3.2, $e(G) \leq e(T_s(n))$. Thus $e(T_r(n)) \leq e(T_s(n))$. From (1.1) it follows that $s \geq r$. If $s = r$, then $e(G) = e(T_r(n))$. According Theorem 3.2, G is a generalized r-partite Turan's graph. □

The following lemma generalizes the Proposition 2.1.

Lemma 3.1. ([3]) *Let G be a graph and $v_1, \ldots, v_r$ be a β-sequence in G such that*

$$d(v_1) + \ldots + d(v_k) \leq \frac{k(r-1)n}{r}, \text{ for some } 1 \leq k \leq r. \quad (3.1)$$

Then G is a generalized r-partite graph. If inequality (3.1) is strict, then G is not generalized r-partite Turan's graph.

Denote the smallest integer r for which there exists a a β-sequence $v_1, \ldots, v_r$ in n-vertex graph G, such that

$$d_G(v_1) + \ldots + d_G(v_r) \leq (r-1)n \quad (3.2)$$

by $\xi(G)$.

Theorem 3.4. *Let G be an n-vertex graph and $e(G) \geq e(T_r(n))$. Then $\xi(G) \geq r$ and $\xi(G) = r$ only when G is generalized r-partite Turan's graph.*

Proof. Let $\xi(G) = s$ and let $v_1, \ldots, v_s$ be a β-sequence in G, such that

$$d_G(v_1) + \ldots + d_G(v_s) \leq (s-1)n$$

By Lemma 3.1 ($r = k = s$), the graph G is generalized r-partite. According to Theorem 2.2 $e(G) \leq e(T_s(n))$. Thus, the inequality $e(G) \geq e(T_r(n))$ implies $e(T_s(n)) \geq e(T_r(n))$. By (1.1) we have $s \geq r$.

Let $s = r$. Then $e(G) = e(T_r(n))$ and from the Theorem 2.2 it follows that G is a generalized r-partite Turan's graph. □

4. SATURATED AND BALANCED β-SEQUENCES

The following results were proved by us:

Theorem 4.1. ([3]) *Let G be an n-vertex graph and $v_1, \ldots, v_r$ be a β-sequence in G, which is not balanced and not saturated. Then G is generalized r-partite graph, which is not a generalized r-partite Turan's graph. Thus $e(G) < e(T_r(n))$.*

Theorem 4.2. ([3]) *Let G be an n-vertex graph and let $v_1, \ldots, v_r$ be a β-sequence in G, $r \geq 2$, which is not balanced and not saturated. Then*

$$d(v_1) + \ldots + d(v_{r-1}) < \frac{(r-1)^2}{r} n.$$

In this section we improve Theorem 4.2.

Theorem 4.3. *Let G be an n-vertex graph and $v_1, \ldots, v_r$ $r \geq 2$ be a β-sequence in G, which is not saturated but $v_1, \ldots, v_{r-1}$ is saturated. Then*

$$d(v_1) + \ldots + d(v_{r-1}) \leq \frac{(r-1)^2}{r} n. \quad (4.1)$$

If there is equality in (4.1), *then:*
(a) $v_1, \ldots, v_r$ *is balanced;*
(b) $n \equiv 0 (mod\, r)$ *and G is a generalized (noncomplete) r-partite graph with partition classes* $V_1', \ldots, V_r'$, *such that* $|V_i'| = \frac{n}{r}$, $i = 1, \ldots, r$ *and*

$$d(v) = \frac{r-1}{r} n, \; \forall v \in \bigcup_{i=1}^{r-1} V_i'$$

$$d(v) = \frac{2e(G)r}{n} - \frac{(r-1)^2 n}{r}, \; \forall v \in V_r';$$

(c) $\dfrac{(r-1)^2 n^2}{r^2} + \dfrac{r-1}{2r} n \le e(G) \le \dfrac{(r-1)n^2}{2r} - \dfrac{n}{2r}$.

Proof. Since $(r-2)n < \frac{(r-1)^2 n}{r}$, in case $d(v_1) + \ldots + d(v_{r-1}) \le (r-2)n$ the inequality (4.1) holds. Therefore, we shall assume that

$$d(v_1) + \ldots + d(v_{r-1}) > (r-2)n. \tag{4.2}$$

Let V_i be the i-stratum of the stratification, induced by sequence $v_1, \ldots, v_r$. Obviously, $v_i \in V_i$, $i = 1, \ldots, r$ and

$$V(G) = V_1 \cup \ldots \cup V_r, \; V_i \cap V_j = \varnothing, \; i \neq j. \tag{4.3}$$

Since $V_i \subset V(G) \backslash \Gamma(v_i)$, $i = 1, \ldots, r-1$, we have

$$|V_i| \le n - d(v_i), \; i = 1, \ldots, r-1. \tag{4.4}$$

It follows from (4.3), (4.4) and (4.2) that

$$|V_r| = n - \sum_{i=1}^{r-1} |V_i| \ge \sum_{i=1}^{r-1} d(v_i) - (r-2)n > 0.$$

Thus $V_r \neq \varnothing$. Let V_r' be a subset of V_r such that

$$|V_r'| = \sum_{i=1}^{r-1} d(v_i) - (r-2)n. \tag{4.5}$$

Define $W = V(G) \backslash V_r'$. By (4.5) we have

$$|W| = \sum_{i=1}^{r-1} (n - d(v_i)). \tag{4.6}$$

Since $V_i \subset W$, $i = 1, \ldots, r-1$, from (4.3), (4.4) and (4.6) it follows that there exist disjoint sets V_i', $i = 1, \ldots, r-1$, such that $V_i \subseteq V_i' \subset W$ and $|V_i'| = n - d(v_i)$.

Since $V_i \subseteq V_i'$, we have $v_i \in V_i'$, $i = 1, \ldots, r-1$. From (4.6) it follows that $W = \bigcup_{i=1}^{r-1} V_i'$. Hence,

$$V(G) = V_1' \cup \ldots \cup V_r', \; V_i' \cap V_j' = \varnothing, \; i \neq j. \tag{4.7}$$

Observe that

$$V_i' \backslash V_i \subset V_r = \Gamma(v_1, \ldots, v_{r-1}) \subset \Gamma(v_1, \ldots, v_{i-1})$$

and $V_i \subset \Gamma(v_1, \ldots, v_{i-1})$. Thus $V_i' \subset \Gamma(v_1, \ldots, v_{i-1})$, $i = 1, \ldots, r-1$ and $d(v) \leq d(v_i)$, $\forall v \in V_i'$, $i = 1, \ldots, r-1$. From the inclusion $V_r' \subset V_r$ it follows that $d(v) \leq d(v_r)$, $\forall v \in V_r'$. So, we have

$$d(v) \leq d(v_i), \; \forall v \in V_i', \; i = 1, \ldots, r. \tag{4.8}$$

By (4.7), we have

$$2e(G) = \sum_{v \in V(G)} d(v) = \sum_{v \in V_1'} d(v) + \ldots + \sum_{v \in V_r'} d(v).$$

Let $d(v_i) = d_i$, $i = 1, \ldots, r$. From $|V_i'| = n - d_i$, $i = 1, \ldots, r-1$, (4.8) and (4.5) it follows that

$$2e(G) \leq \sum_{i=1}^{r-1} d_i(n - d_i) + \Big(\sum_{i=1}^{r-1} d_i - (r-2)n \Big) d_r. \tag{4.9}$$

The equality in (4.9) occurs if and only if

$$d(v) = d_i, \; \forall v \in V_i', \; i = 1, \ldots, r.$$

Let $\sigma = d_1 + \ldots + d_{r-1}$. We have $\dfrac{\sigma + d_r}{r} \leq \dfrac{2e(G)}{n}$ because the sequence $v_1, \ldots, v_r$ is not saturated. Thus,

$$d_r \leq \frac{2re(G)}{n} - \sigma. \tag{4.10}$$

By the Caushy-Schwarz inequality $\left(\sum x_i y_i\right)^2 \leq \sum x_i^2 \sum y_i^2$, applied to $x_i = d_i$, $y_i = 1$, we have

$$\sum_{i=1}^{r-1} d_i^2 \geq \frac{\sigma^2}{r-1}. \tag{4.11}$$

and the equality holds if and only if $d_1 = \ldots = d_{r-1}$. We obtain by (4.10) and (4.11)

$$2e(G) \leq n\sigma - \frac{\sigma^2}{r-1} + (\sigma - (r-2)n)\Big(\frac{2re(G)}{n} - \sigma\Big).$$

This inequality is equivalent to

$$\frac{2e(G)}{n}\left((r-1)^2n - r\sigma\right) \leq \frac{\sigma}{r-1}\left((r-1)^2n - r\sigma\right). \tag{4.12}$$

The equality in (4.12) occurs simultaneously with the equalities in (4.9), (4.10) and (4.11), i.e. when

$$d(v) = d_i = d_1,\ \forall v \in V_i',\ i = 1, \ldots, r-1 \text{ and} \tag{4.13}$$

$$d(v) = d_r = \frac{2re(G)}{n} - \sigma,\ \forall v \in V_r'.$$

Since $v_1, \ldots, v_{r-1}$ is saturated, we have

$$\frac{\sigma}{r-1} > \frac{2e(G)}{n}.$$

Thus, (4.12) is equivalent to the inequality $\sigma \leq \dfrac{(r-1)^2n}{r}$. The inequality (4.1) is proved.

It remains to examine the case of the equality in (4.1). Assume, that

$$\sigma = \frac{(r-1)^2n}{r}. \tag{4.14}$$

Then $n \equiv 0(mod\ r)$ and the equality holds in (4.12), i.e. (4.13) is realized. From (4.14) and (4.13) it follows that

$$d(v) = d_1 = \ldots = d_{r-1} = \frac{(r-1)n}{r},\ \forall v \in V_i',\ i = 1, \ldots, r-1 \tag{4.15}$$

and

$$d(v) = d_r = \frac{2re(G)}{n} - \frac{(r-1)^2}{r}n,\ \forall v \in V_r'. \tag{4.16}$$

By (4.15) and (4.16) it follows that

$$\frac{d_1 + \ldots + d_r}{r} = \frac{2e(G)}{n},$$

i.e. $v_1, \ldots, v_r$ is balansed. Since $v_1, \ldots, v_{r-1}$ is saturated, we have

$$\frac{d_1 + \ldots + d_{r-1}}{r-1} > \frac{2e(G)}{n} = \frac{d_1 + \ldots + d_r}{r},$$

Hence $d_r < d_1 = \dfrac{r-1}{r}n$.Thus

$$d(v) = d_r < \frac{r-1}{r}n,\ v \in V_r'. \tag{4.17}$$

Since $|V_i'| = n - d_i$, $i = 1, \dots, r-1$ and $|V_r'| = \sum_{i=1}^{r-1} d_i - (r-2)n$, we obtain by (4.15)

$$|V_i'| = \frac{n}{r},\ i = 1, \dots, r$$

Thus, from (4.15) and (4.17) it follows that G generalized (noncomplete) r-partite graph with equal partite classes $V_1', \dots, V_r'$.

So, (a) and (b) are proved. It remains to prove (c). The number $\dfrac{(r-1)n}{r}$ is integer, because $n \equiv 0 (mod\ r)$ and consequently from (4.17) it follows that

$$d_r \leq \frac{(r-1)n}{r} - 1.$$

Since $v_1, \dots, v_r$ is balanced, by this inequality and (4.15) we have

$$\frac{2e(G)}{n} = \frac{d_1 + \dots + d_r}{r} \leq \frac{\frac{(r-1)^2 n}{r} + \frac{(r-1)n}{r} - 1}{r} = \frac{(r-1)n - 1}{r}.$$

Thus, $e(G) \leq \dfrac{(r-1)}{2r} n^2 - \dfrac{n}{2r}$.

Since $v_r \in \Gamma_G(v_1, \dots, v_{r-1})$, $d(v_r) \geq r - 1$. From this inequality and (4.16) we conclude that

$$e(G) \geq \frac{(r-1)^2}{2r^2} n^2 + \frac{r-1}{2r} n.$$

The proof of (c) is over and Theorem 4.3 is proved. □

Corollary 4.1. *Let G be an n-vertex graph and r be integer, $1 \leq r \leq n$. Let $e(G) \geq e(T_r(n))$ and for some s, $1 \leq s \leq r$ there exists a balanced β-sequence $v_1, \dots, v_s \in V(G)$. Then G is regular.*

Proof. We prove this corollary by induction on s. The base $s = 1$ is clear, since $d(v_1) = \dfrac{2e(G)}{n}$ implies that G is regular.

Let $s \geq 2$. Since $\dfrac{d(v_1) + \dots + d(v_s)}{s} = \dfrac{2e(G)}{n}$, from $d(v_1) \geq d(v_2) \geq \dots \geq d(v_s)$ it follows that

$$\frac{d(v_1) + \dots + d(v_{s-1})}{s-1} \geq \frac{2e(G)}{n},$$

i.e. $v_1, \dots, v_{s-1}$ is balanced or saturated. We prove that $v_1, \dots, v_{s-1}$ is balanced. Assume the opposite.

Since $v_1, \dots, v_s$ is not saturated, by Theorem 4.3

$$d(v_1) + \dots + d(v_{s-1}) \leq \frac{(s-1)^2 n}{s}. \tag{4.18}$$

By Lemma 3.1, G is a generalized s-partite graph. From Theorem 2.2 it follows $e(G) \leq e(T_s(n))$.

Thus, we have $e(T_r(n)) \leq e(G) \leq e(T_s(n))$. Since $s \leq r$, (1.1) implies that $s = r$ and $e(G) = e(T_s(n))$. According to Lemma 3.1, there is equality in (4.18). Thus, Theorem 4.3 implies that $n \equiv 0(mod\ s)$ and $e(G) \leq \frac{(s-1)n^2}{2s} - \frac{n}{2s}$. This contradicts the equality $e(G) = e(T_s(n)) = \frac{(s-1)n^2}{2s}$.

So, $v_1, \ldots, v_{s-1}$ is balanced. By inductive hypothesis, G is regular and the proof of Corollary 4.1 is over. □

5. PROOF OF THE MAIN THEOREM

We prove that G is regular by induction on s. The base $s = 1$ is clear, since $d(v_1) = \frac{2e(G)}{n}$ implies that G is regular.

Let $s \geq 2$. From $d(v_1) \geq \ldots \geq d(v_s)$ it follows that

$$\frac{d(v_1) + \ldots + d(v_{s-1})}{s-1} \geq \frac{2e(G)}{n}.$$

Hence, $v_1, \ldots, v_{s-1}$ is balanced or saturated. We prove that $v_1, \ldots, v_{s-1}$ is balanced. Assume the opposite. Then

$$\frac{d(v_1) + \ldots + d(v_{s-1})}{s-1} > \frac{2e(G)}{n}. \tag{5.1}$$

By Theorem 4.3, the inequality (4.18) holds. If there is equality in (4.18), then according to Theorem 4.3, $n \equiv 0(mod\ s)$ and $e(G) \leq \frac{(s-1)n^2}{2s} - \frac{n}{2s} = f(n, s)$. But $f(n, s) \leq f(n, r)$, because $s \leq r$ (see (1.3)). Therefore, $e(G) \leq f(n, r)$ which is a contradiction. Assume that (4.18) is strict.

Case 1. $n \equiv 0(mod\ s)$. Since (4.18) is strict, it follows that

$$d(v_1) + \ldots + d(v_{s-1}) \leq \frac{(s-1)^2 n}{s} - 1. \tag{5.2}$$

From (5.1) and (5.2) it follows that

$$e(G) < \frac{(s-1)n^2}{2s} - \frac{n}{2(s-1)} < f(n, s).$$

By $s \leq r$ and (1.3), $f(n, s) \leq f(n, r)$. Hence $e(G) < f(n, r)$, which is a contradiction.

Case 2. $n \equiv \nu(mod\ s)$, $1 \leq \nu \leq s-1$. Since (4.18) is strict, we have

$$d(v_1) + \ldots + d(v_{s-1}) \leq \lfloor \frac{(s-1)^2 n}{s} \rfloor = \frac{(n-\nu)(s-1)^2}{s} + \nu(s-2). \tag{5.3}$$

From (5.1) and (5.3) it follows

$$e(G) \leq f(n,s) \leq f(n,r),$$

which is a contradiction.

The Main Theorem is proved.

Remark. If $n \equiv 0(mod\ r)$, then $f(n,r) < e(T_r(n)) = \frac{n^2(r-1)}{2r}$. Therefore, in this case the Corollary 4.1 follows from Main Theorem. Let $n \equiv \nu(mod\ r)$, $1 \leq \nu \leq r-1$. From (1.2) it follows that

$$e(T_r(n)) = \frac{n^2(r-1)}{2r} - \frac{\nu(r-\nu)}{2r}. \tag{5.4}$$

The equality (5.4) implies, that if

$$\frac{\nu(r-\nu)}{2r} < \frac{\nu n}{2r(r-1)},$$

i.e. $n > (n-\nu)(r-1)$, then $f(n,r) < e(T_r(n))$. Hence, if $n > (r-\nu)(r-1)$, Corollary 4.1 follows from the Main Theorem.

6. α-SEQUENCES IN GRAPHS

Let G be a graph and $v_1, \ldots, v_r \in V(G)$. Define $\Gamma_0 = V(G)$ and $\Gamma_i = \Gamma_G(v_1, \ldots, v_i)$, $i = 1, \ldots, r-1$. In our articles [4] and [5] we introduced the following concept:

Definition 6.1. The sequence $v_1, \ldots, v_r \in V(G)$ is called α-sequences if $v_i \in \Gamma_{i-1}$ and v_i has maximal degree in the graph $G[\Gamma_{i-1}]$, $i = 1, \ldots, r$.

α-sequences appears later in [7-10] under the name "degree-greedy algorithm" and in [11] under the name "s-stable algorithm".

The following result was proved by us:

Theorem 6.1. ([2]) *Let $v_1, \ldots, v_r$ be a α-sequence in an n-vertex graph G, which is not contained in an $(r+1)$-qlique. If V_i is the i-th stratum of the stratification induced by this sequence and $p_i = |V_i|$, $i = 1, \ldots, r$ (see Definition 1.1), then*

(a) *G is generalized r-partite graph with partition classes $V_1, \ldots, V_r$ and*

$$e(G) \leq e(K(p_1, \ldots, p_r)); \tag{6.1}$$

(b) *There is equality in* (6.1) *only when $G = K(p_1, \ldots, p_r)$.*

The proof of Theorem 6.1, given in [2], actually establishes the following statement:

Theorem 6.2. *Let $v_1, \ldots, v_r$ be an α-sequence in an n-vertex graph G such that*

$$d(v) \le n - |\Gamma_{r-1}|, \ \forall v \in \Gamma_{r-1}. \tag{6.2}$$

If V_i is the i-th stratum of the stratification induced by this sequence and $p_i = |V_i|$, $i = 1, \ldots, r$, then

(a) *G is generalized r-partite graph with partition classes $V_1, \ldots, V_r$ and inequality* (6.1) *holds;*

(b) *There is equality in* (6.1) *only when G is generalized complete r-partite graph with partition classes $V_1, \ldots, V_r$.*

Denote by $\varphi(G)$ the smallest integer r for which there exists an α-sequence $v_1, \ldots, v_r \in V(G)$, such that (6.2) holds.

Theorem 6.3. *Let G be an n-vertex graph, such that $e(G) \ge e(T_r(n))$, $1 \le r \le n$. Then $\varphi(G) \ge r$ and $\varphi(G) = r$ only when G is generalized r-partite Turan's graph.*

Proof. Let $\varphi(G) = s$ and $v_1, \ldots, v_s$ be α-sequence in G, such that $d(v) \le n - |\Gamma_{s-1}|$, $\forall v \in \Gamma_{s-1}$. By Theorem 6.2 and Theorem 2.2, we have $e(T_r(n)) \le e(T_s(n))$. From (1.1) it follows $s \ge r$. If $s = r$, then $e(G) = e(T_r(n))$. According to Theorem 2.2(c), G is generalized r-partite Turan's graph. This completes the proof of Theorem 6.3. □

Let $v_1, \ldots, v_r$ be α-sequence in graph G, and $G_{i-1} = G[\Gamma_{i-1}]$, $i = 1, \ldots, r$, where Γ_i, $i = 1, \ldots, r-1$ are defined above. Define

$$d'_1 = d_G(v_1),\ d'_2 = d_{G_1}(v_2), \ldots, d'_r = d_{G_{r-1}}(v_r).$$

Theorem 6.4. *Let G be an n-vertex graph and $v_1, \ldots, v_r$ be α-sequence in G, such that for some s, $1 \le s \le r$,*

$$d'_1 + \ldots + d'_s \le \frac{n}{r}\left(\binom{r}{2} - \binom{r-s}{2}\right). \tag{6.3}$$

Then G is generalized r-partite graph.

Proof. We prove Theorem 6.4 by induction on s. The induction base is $s = 1$. From (6.3) it follows that $d'_1 \le \dfrac{(r-1)n}{r}$. Since $d_1 = d_G(v_1)$ and v_1 has maximal degree in G, we have $d(v) \le \dfrac{(r-1)n}{r}$, $\forall v \in V(G)$. By Proposition 1.1, G is generalized r-partite graph.

Let $s \ge 2$ and suppose, that assertion is true for $s - 1$.

Case 1. $d'_2 + \ldots + d'_s \le \dfrac{d'_1}{r-1}\left(\binom{r-1}{2} - \binom{r-s}{2}\right)$.

Obviously $v_2, \ldots, v_r$ be α-sequence in $G_1 = G[\Gamma_G(v_1)]$. By inductive hypothesis, we may assume that G_1 is generalized $(r-1)$-partite graph with partition classes $W_2, \ldots, W_r$. Thus, G is generalized r-partite graph with partition classes $W_1 = V(G)\backslash\Gamma_G(v_1), W_2, \ldots, W_r$.

Case 2. $d'_2 + \ldots + d'_s > \frac{d'_1}{r-1}\left(\binom{r-1}{2} - \binom{r-s}{2}\right)$.

From (6.3) it follows that

$$d'_1 + \frac{d'_1}{r-1}\left(\binom{r-1}{2} - \binom{r-s}{2}\right) < \frac{n}{r}\left(\binom{r}{2} - \binom{r-s}{2}\right).$$

Hence

$$d'_1 \leq \frac{n}{r}A, \text{ where } A = \frac{\binom{r}{2} - \binom{r-s}{2}}{1 + \frac{1}{r-1}\left(\binom{r-1}{2} - \binom{r-s}{2}\right)}. \tag{6.4}$$

Note that $A = r-1$. Thus, by (6.4), we have $d'_1 \leq \frac{n}{r}(r-1)$. Hence $d(v) \leq \frac{n(r-1)}{r}$, $\forall v \in V(G)$. By Proposition 2.1, G is generalized r-partite graph. □

Theorem 6.5. *Let G be an n-vertex graph and $v_1, \ldots, v_k$ be α-sequence in G, such that*

$$d'_1 + \ldots + d'_k \leq \frac{ke(G)}{n}.$$

Then G is generalized k-partite graph.

Proof. If $k = 1$, then $d'_1 \leq \frac{e(G)}{n}$. Since $e(G) \leq \frac{d'_1 n}{2}$, it follows that $d'_1 = 0$. Thus, $E(G) = \varnothing$ and G is 1-partite graph.

Let $k \geq 2$. Then

$$d'_2 + \ldots + d'_k \leq \frac{ke(G)}{n} - d'_1.$$

From this inequality and $e(G) \leq \frac{nd'_1}{2}$, it follows that

$$d'_2 + \ldots + d'_k \leq \frac{(k-2)d'_1}{2} = \frac{d'_1}{k-1}\binom{k-1}{2}.$$

Since $v_2, \ldots, v_k$ is an α-sequence in $G_1 = G[\Gamma_G(v_1)]$, by this inequality and Theorem 6.4 (with $r = s = k-1$), it follows that the graph G_1 is generalized $(k-1)$-partite graph. Let $W_2, \ldots, W_k$ be partition classes of G_1. Then G is generalized r-partite graph with partition classes $W_1 = V(G)\backslash\Gamma_G(v_1), W_2, \ldots, W_k$.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: nenov@fmi.uni-sofia.bg
hadji@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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PARTITIONED GRAPHS AND DOMINATION RELATED PARAMETERS

VLADIMIR D. SAMODIVKIN

Let G be a graph of order $n \geq 2$ and $n_1, n_2, .., n_k$ be integers such that $1 \leq n_1 \leq n_2 \leq .. \leq n_k$ and $n_1 + n_2 + .. + n_k = n$. Let for $i = 1, .., k$: $\mathcal{A}_i \subseteq \mathcal{K}_{n_i}$ where $\mathcal{K}_m$ is the set of all pairwise non-isomorphic graphs of order m, $m = 1, 2, ...$ In this paper we study when for a domination related parameter μ (such as domination number, independent domination number and acyclic domination number) is fulfilled $\mu(G) = \mu(\cup_{i=1}^{k} < V_i, G >)$ for all vertex partitions $\{V_1, V_2, .., V_k\}$, $k \geq 2$, of a vertex set of G such that $< V_i, G >$ is isomorphic to some a member of $\mathcal{A}_i$, $i = 1, 2, .., k$. In the process several results for acyclic domination vertex critical graphs are presented. Results for independence number of double vertex graphs are obtained.

Keywords: domination number, acyclic domination number, independent domination number, independence number, double vertex graph

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1. NOTATION AND DEFINITIONS

For a graph theory terminology not presented here, we follow Haynes, et al.[8]. All our graphs are finite and undirected with no loops or multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $< S, G >$. We denote by K_n and $\overline{K}_n$ complete graph on n vertices and its complement. If $n \geq 3$ then C_n is a connected 2 - regular graph of order n. P_m is a tree of order m and diameter $m-1$, $m \geq 1$. By $\mathcal{K}_s$ we denote the set of all pairwise non-isomorphic graphs of order s, $s \geq 1$. A subset of vertices A in a graph G is said to be *acyclic* if $< A, G >$

contains no cycles. A subset of vertices I in a graph G is said to be *independent* if $< I, G >$ contains no edges. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set in G. A *dominating set* in a graph G is a set of vertices D such that every vertex of G is either in D or is adjacent to an element of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets of G. The *independent domination number* $i(G)$ (*acyclic domination number* $\gamma_a(G)$) of a graph G is the minimum cardinality of an independent dominating (acyclic dominating) set of G.

Throughout this paper, let a property $\mathcal{P}$ of graphs be given and $\mu(G)$ be a numeral invariant of a graph G defined in a such a way that it is the minimum or maximum number of vertices of a set $S \subseteq V(G)$ which has the property $\mathcal{P}$. A set with property $\mathcal{P}$ and with $\mu(G)$ vertices is called a μ - *set* of G. A vertex v of a graph G is μ - critical if $\mu(G-v) \neq \mu(G)$. The graph G is μ - critical if all its vertices are μ - critical. Much has been written about the effects on a parameter(such connectedness, chromatic number, domination number) when a graph is modified by deleting a vertex. μ - critical graphs for $\mu = \gamma, i$ was investigated by Brigham et al.[4] and Ao and MacGillivray (see [9, ch. 16]) respectively. Further properties on these graphs can be found in [6], [7], [8, ch.5], [9, ch. 16], [10].

In this work, by a partition of a graph G into k parts, $k \geq 2$, we mean a family $A = \{G_1, G_2, .., G_k\}$ of pairwise disjoint induced subgraphs of G, with $\cup_{i=1}^{k} V(G_i) = V(G)$ and $1 \leq |V(G_1)| \leq |V(G_2)| \leq .. \leq |V(G_k)|$. We denote by $G[A]$ the graph $\cup_{i=1}^{k} G_i$.

Let G be a graph of order $n \geq 2$ and $n_1, n_2, .., n_k$ be integers such that $1 \leq n_1 \leq n_2 \leq .. \leq n_k$ and $n_1 + n_2 + .. + n_k = n$. Let $\mathcal{A}_i \subseteq \mathcal{K}_{n_i}$, $i = 1, .., k$. We say that a partition $A = \{G_1, G_2, .., G_k\}$ of G is of type $[\mathcal{A}_1, \mathcal{A}_2, , .., \mathcal{A}_k]$ if G_i is isomorphic to some a member of $\mathcal{A}_i$, $i = 1, .., k$. The set of all partitions of a graph G which are of type $[\mathcal{A}_1, \mathcal{A}_2, , .., \mathcal{A}_k]$ will be denoted by $\mathcal{F}_G$ $(\mathcal{A}_1, \mathcal{A}_2, , .., \mathcal{A}_k)$.

For a graph invariant μ and a family $\{\mathcal{A}_1, \mathcal{A}_2, , .., \mathcal{A}_k\}$, where $\mathcal{A}_i \subseteq \mathcal{K}_{n_i}$, $i = 1, .., k$ and $1 \leq n_1 \leq n_2 \leq .. \leq n_k$ it is important to characterize/study the graphs G with $\mu(G) = \mu(G[A])$ for all $A \in \mathcal{F}_G$ $(\mathcal{A}_1, \mathcal{A}_2, , .., \mathcal{A}_k)$.

We proceed as follows. In Section 2, we deals with critical vertices in a graph with respect to the acyclic domination number and give a necessary and sufficient condition for a graph to be γ_a- critical. In Section 3 we study when $\mu(G) = \mu(G[A])$ for all $A \in \mathcal{F}_G$ $(\mathcal{A}_1, \mathcal{A}_2, , .., \mathcal{A}_k)$ for some families $\{\mathcal{A}_1, \mathcal{A}_2, , .., \mathcal{A}_k\}$.

2. ACYCLIC DOMINATION NUMBER

The concept of acyclic domination was introduced by Hedetniemi et al.[11]. In this section some properties of critical vertices with respect to γ_a will be given.

Theorem 2.1. *Let G be a graph of order $n \geq 2$ and $u, v \in V(G)$.*

(i) *Let $\gamma_a(G-v) < \gamma_a(G)$.*

(i.1) [15] *If* $uv \in E(G)$ *then* u *belongs to no* γ_a *- set of* $G - v$;

(i.2) *If* M *is a* γ_a *- set of* $G - v$ *then* $M \cup \{v\}$ *is a* γ_a *- set of* G;

(i.3) [15] $\gamma_a(G - v) = \gamma_a(G) - 1$;

(ii) *Let* $\gamma_a(G - v) > \gamma_a(G)$. *Then* v *belongs to every* γ_a *- set of* G;

(iii) *If* $\gamma_a(G - v) < \gamma_a(G) < \gamma_a(G - u)$ *then* $uv \notin E(G)$;

(iv) *If* v *belongs to no* γ_a *- set then* $\gamma_a(G - v) = \gamma_a(G)$.

Proof. (i) For reason of completeness, we shall give here the proofs of (i.1) and (i.3).

(i.1): Let $uv \in E(G)$ and M be a γ_a - set of $G - v$. If $u \in M$ then M will be an acyclic dominating set of G with $|M| < \gamma_a(G)$ - a contradiction.

(i.2)and (i.3): If M is a γ_a - set of $G - v$ then (i.1) implies that $M_1 = M \cup \{v\}$ is an acyclic dominating set of G with $|M_1| = \gamma_a(G - v) + 1 \leq \gamma_a(G)$. Hence M_1 is a γ_a - set of G and $\gamma_a(G - v) = \gamma_a(G) - 1$.

(ii) If $M$is a γ_a - set of G and $v \notin M$ then M is an acyclic dominating set of $G - v$. But then $\gamma_a(G) = |M| \geq \gamma_a(G - v) > \gamma_a(G)$ and the result follows.

(iii) Let $\gamma_a(G - v) < \gamma_a(G)$ and M be a γ_a - set of $G - v$. Then by (i.2), $M \cup \{v\}$ is a γ_a-set of G. Let $\gamma_a(G - u) > \gamma_a(G)$. Now (ii) implies that $u \in M$ and by (i.1) - $uv \notin E(G)$.

(iv) By (ii), $\gamma_a(G - v) \leq \gamma_a(G)$. Assume $\gamma_a(G - v) < \gamma_a(G)$. It follows from (i.2) that $M \cup \{v\}$ is a γ_a – set of G, where M is a γ_a – set of $G - v$ – a contradiction. □

Theorem 2.2. Let G be a graph of order at least two. Then

(i) [3, 10] G *is* γ *- critical if and only if* $\gamma(G - v) = \gamma(G) - 1$ *for all* $v \in V(G)$;

(ii) (Ao and MacGillivray (see the bibliography in [9, ch.16])) G *is* i *- critical if and only if* $i(G - v) = i(G) - 1$ *for all* $v \in V(G)$.

Analogously result is valid and for γ_a - critical graphs.

Theorem 2.3. *Let* G *be a graph of order* $n \geq 2$. *Then* G *is a* γ_a *- critical graph if and only if* $\gamma_a(G - v) = \gamma_a(G) - 1$ *for all* $v \in V(G)$.

Proof. Necessity is obvious.

Sufficiency: Let G be a γ_a - critical graph. Clearly for every isolated vertex $v \in V(G)$, $\gamma_a(G - v) = \gamma_a(G) - 1$. Hence if G is isomorphic to $\overline{K}_n$ then $\gamma_a(G - v) = \gamma_a(G) - 1$ for all $v \in V(G)$. So, let G have a component of order at least two, say Q. Because of Theorem 2.1 (iii), either for all $v \in V(Q)$, $\gamma_a(Q - v) > \gamma_a(Q)$ or for all $v \in V(Q)$, $\gamma_a(Q - v) < \gamma_a(Q)$. Suppose, for all $v \in V(Q)$, $\gamma_a(Q - v) > \gamma_a(Q)$. It follows by Theorem 2.1 (ii) that $V(Q)$ is the unique acyclic dominating set of Q. Since $V(Q)$ is an acyclic set then Q is a tree which implies $\gamma_a(Q) = \gamma(Q) = |V(Q)|$

- a contradiction with the well known Ore's theorem [12] that for every connected graph H of order at least two, $\gamma(H) \leq |V(H)|/2$. □

Theorem 2.4. *Let G_1 and G_2 be two connected graphs both of order at least two with $V(G_1) \cap V(G_2) = \{x\}$. If $\gamma_a(G_1 - x) < \gamma_a(G_1)$ and $\gamma_a(G_2 - x) < \gamma_a(G_2)$ then $\gamma_a(G) = \gamma_a(G_1) + \gamma_a(G_2) - 1$ and $\gamma_a(G - x) = \gamma_a(G) - 1$.*

Proof. It follows from Theorem 2.1 (i.2) that there exist a γ_a - set U_1 of G_1 and a γ_a - set U_2 of G_2 such that $x \in U_1 \cap U_2$. Hence $U_1 \cup U_2$ is an acyclic dominating set of G of cardinality $\gamma_a(G_1) + \gamma_a(G_2) - 1$. So we prove $\gamma_a(G) \leq \gamma_a(G_1) + \gamma_a(G_2) - 1$.

Let M be a γ_a - set of G and $M_i = M \cap V(G_i)$, $i = 1, 2$. There exist three possibilities:

(*) $x \notin M$ and M_i is an acyclic dominating set of G_i, $i = 1, 2$;

(**) $x \notin M$ and there are i, j such that $\{i, j\} = \{1, 2\}$, M_i is an acyclic dominating set of G_i and M_j is an acyclic dominating set of $G_j - x$;

(***) $x \in M$.

If (*) holds, then $\gamma_a(G) = |M| = |M_1| + |M_2| \geq \gamma_a(G_1) + \gamma_a(G_2)$ - a contradiction. If (**) holds, then $\gamma_a(G) = |M| = |M_1| + |M_2| \geq \gamma_a(G_i) + \gamma_a(G_j - x) = \gamma_a(G_1) + \gamma_a(G_2) - 1$. If (***) holds then $\gamma_a(G) = |M| = |M_1| + |M_2| - 1 \geq \gamma_a(G_1) + \gamma_a(G_2) - 1$.

Thus we have $\gamma_a(G) = \gamma_a(G_1) + \gamma_a(G_2) - 1$.

Clearly $\gamma_a(G - x) = \gamma_a(G_1 - x) + \gamma_a(G_2 - x)$ and by Theorem 2.1 (i.3) it follows $\gamma_a(G - x) = \gamma_a(G_1) + \gamma_a(G_2) - 2$. Hence $\gamma_a(G - x) = \gamma_a(G) - 1$.□

Corollary 2.5. *Let G be a connected graph with blocks $G_1, G_2, .., G_n$. If the all $G_1, G_2, .., G_n$ are γ_a - critical then $\gamma_a(G) = \Sigma_{i=1}^{n} \gamma_a(G_i) - n + 1$.*

Proof. We proceed by induction on the number of blocks n. The statement is immediate if $n = 1$. Let the blocks of G be $G_1, G_2, .., G_n, G_{n+1}$ and without loss of generality let G_{n+1} contain only one cut-vertex of G. Hence Theorem 2.4 implies that $\gamma_a(G) = \gamma_a(G_{n+1}) + \gamma_a(Q) - 1$ where $Q = < \cup_{i=1}^{n} V(G_i), G >$. The result now follows from the inductive hypothesis. □

It is not possible to characterize γ - critical graphs in terms of forbidden graphs as it is shown in [3]. We shall prove a similar result for γ_a - critical graphs. We need the following example which is analogous to the one used in the proof of Theorem 6 in [3].

Example 2.6. Let G be a graph. If $\gamma_a(G) \geq 3$ then let $T = G$, otherwise $T = G \cup K_1 \cup K_1$. Let $V(T) = \{v_1, v_2, .., v_n\}$. Define the graph H as follows: $V(H) = \cup_{i=1}^{n} \{v_i, u_i, w_i\}$ and $E(H) = E(G) \cup \{v_i u_j, u_i w_j, w_i v_j \mid 1 \leq i, j \leq n, j \neq i\}$. It is straightforward to verify that no two vertices dominate H. Hence $\gamma_a(H) \geq 3$.

But by the definition of H, for each $i = 1, 2, .., n$, $\{u_i, v_i, w_i\}$ is a dominating and independent set (hence and an acyclic set) of H. So, $\gamma_a(H) \leq 3$. Thus $\gamma_a(H) = 3$. Clearly $\{u_i, v_i\}$ is a γ_a - set of $H - w_i$, $\{u_i, w_i\}$ is a γ_a - set of $H - v_i$ and $\{w_i, v_i\}$ is a γ_a - set of $H - u_i$. Therefore H is a γ_a - critical graph and G is its own induced subgraph.

From the above example we immediately have:

Theorem 2.7. *There does not exist a forbidden subgraph characterization of the class of* γ_a *- critical graphs.*

3. PARTITIONED GRAPHS

We begin with the family $\{\mathcal{A}_1 = \mathcal{K}_1, \mathcal{A}_2 = \mathcal{K}_{n-1}\}$ and $\mu \in \{\gamma, \gamma_a, i\}$.
From Theorem 2.2 and Theorem 2.3 we immediately have:

Theorem 3.1. *Let G be a graph of order $n \geq 2$ and $\mu \in \{\gamma, \gamma_a, i\}$. Then $\mu(G) = \mu(G[A])$ for all $A \in \mathcal{F}_G(\mathcal{K}_1, \mathcal{K}_{n-1})$ if and only if G is a μ - critical graph.*

Now, let us consider the family $\{\mathcal{K}_1, \mathcal{K}_1, \mathcal{K}_{n-2}\}$, $n \geq 3$ and $\mu \in \{\gamma, \gamma_a, i\}$.

Theorem 3.2. *Let G be a graph of order $n \geq 3$ and $\mu \in \{\gamma, \gamma_a, i\}$. Then $\mu(G) = \mu(G[A])$ for all $A \in \mathcal{F}_G(\mathcal{K}_1, \mathcal{K}_1, \mathcal{K}_{n-2})$ if and only if $G = \overline{K}_n$.*

Proof. Clearly if $G = \overline{K}_n$ then $\mu(G) = \mu(G[A])$ for all $A \in \mathcal{F}_G(\mathcal{K}_1, \mathcal{K}_1, \mathcal{K}_{n-2})$. So, let we have $\mu(G) = \mu(G[A])$ for all $A \in \mathcal{F}_G(\mathcal{K}_1, \mathcal{K}_1, \mathcal{K}_{n-2})$ and suppose $G \neq \overline{K}_n$. Note that if H is a graph of order at least two and $u \in V(H)$ then $\mu(H - u) \geq \mu(H) - 1$, which follows from [3, 5], [9, ch.16] and Theorem 2.1.(i) for $\mu = \gamma$, $\mu = i$ and $\mu = \gamma_a$ respectively. Choose $x, y \in V(G)$ to be adjacent and let $A = \{\{x\}, \{y\}, V(G) - \{x, y\}\}$. If $\mu(G - x) \geq \mu(G)$ then $\mu(G - \{x, y\}) \geq \mu(G - x) - 1 \geq \mu(G) - 1$ which implies $\mu(G[A]) \geq 1 + 1 + \mu(G) - 1 > \mu(G)$. Hence $\mu(G - x) = \mu(G) - 1$ and therefore if M is a μ - set of $G - x$ then M does not dominate x in G. Hence y belongs to no μ - set of $G - x$. But if a vertex u of a graph H belongs to no μ - set of H then $\mu(H) = \mu(H - u)$, which follows from [5, 13], [14] and Theorem 2.1 (iv) for $\mu = \gamma$, $\mu = i$ and $\mu = \gamma_a$ respectively. Therefore $\mu(G[A]) = 1 + 1 + \mu(G - \{x, y\}) = 2 + \mu(G - x) = 1 + \mu(G)$, which is a contradiction. □

The next family is $\{\{P_2\}, \mathcal{K}_{n-2}\}$, $n \geq 4$ and again $\mu \in \{\gamma, \gamma_a, i\}$.

Theorem 3.3. *Let G be a μ - critical graph of order $n \geq 4$ and size at least 1, where $\mu \in \{\gamma, \gamma_a, i\}$. Then $\mu(G) = \mu(G[A])$ for all $A \in \mathcal{F}_G(\{P_2\}, \mathcal{K}_{n-2})$.*

Proof. As we have seen, $\mu(G - x) = \mu(G) - 1$ for all $x \in V(G)$. By the proof of Theorem 3.2, if $yx \in E(G)$ then y belongs to no μ - set of $G - x$ which implies $\mu(G - \{x,y\}) = \mu(G - x)$. Hence if $xy \in E(G)$ and $A= \{\{x,y\}, V(G - \{x,y\})\}$ then $\mu(G[A]) = 1 + \mu(G - \{x,y\}) = 1 + \mu(G - x) = \mu(G)$. □

Let G be a graph of order $n \geq 2$. The *double vertex graph* $U_2(G)$ of G is the graph whose vertex set consists of all 2-subsets of $V(G)$ such that two distinct vertex $\{x,y\}$ and $\{u,v\}$ are adjacent if and only if $|\{x,y\} \cap \{u,v\}| = 1$ and if $x = u$, they y and v are adjacent in G. The concept of double vertex graphs was introduced by Alavi et al. [1]. For this class of graphs, there are many results about regularity, eulerian, hamiltonian, and bipartite properties of these graphs. For a survey of double vertex graphs see [2]. Here we deal with the independence number of double vertex graphs.

Theorem 3.4. *Let G be a graph and $V(G) = \{v_1, v_2, .., v_n\}$, $n \geq 3$. Then $\beta_0(U_2(G)) \leq \Sigma_{k=1}^{n-1}\beta_0(< \{v_{k+1}, v_{k+2}, .., v_n\}, G >)$.*

Proof. Let for each $k \in \{1, 2, .., n-1\}$, $V_k = \{v_{k+1}, v_{k+2}, .., v_n\}$, $W_k = \{\{v_k, v_j\} | k < j \leq n\}$, $H_k =< V_k, G >$ and $Q_k =< W_k, U_2(G) >$. Certainly $\{Q_{n-1}, Q_{n-2}, .., Q_1\}$ is a partition of $U_2(G)$. For all $k \in \{1, 2, .., n-1\}$ define the map $\pi_k : W_k \to V_k$ by $\pi_k(\{v_k, v_j\}) = v_j$, where $j = k+1, .., n$. Clearly π_k is a bijection and if $k < j \leq n$, $k < s \leq n$, $j \neq s$ then $\{v_k, v_j\}\{v_k, v_s\} \in E(Q_k)$ if and only if $\pi_k(\{v_k, v_j\})\pi_k(\{v_k, v_s\}) = v_j v_s \in E(H_k)$ which follows by the definition of the double vertex graph. Then the graphs Q_k and H_k are isomorphic, $k = 1, 2, .., n-1$. Combining this with the well known fact that if T is a graph and $e \in E(T)$ then $\beta_0(T - e) \geq \beta_0(T)$ [8], we obtain $\beta_0(U_2(G)) \leq \beta_0(\cup_{k=1}^{n-1} Q_k) = \Sigma_{k=1}^{n-1}\beta_0(Q_k) = \Sigma_{k=1}^{n-1}\beta_0(H_k)$. □

Corollary 3.5 *If G is hamiltonian graph of order n then $\beta_0(U_2(G)) \leq \lfloor n^2/4 \rfloor$.*

Proof. Let $v_1, v_2, .., v_n, v_1$ be a hamiltonian cylle in G. Since $H_k =< \{v_{k+1}, v_{k+2}, .., v_n\}, G >$ has a spanning subgraph isomorphic to P_{n-k} then Theorem 3.4 implies $\beta(U_2(G)) \leq \Sigma_{k=1}^{n-1}\beta_0(H_k) \leq \Sigma_{k=1}^{n-1}\beta_0(P_{n-k})$. Clearly $\beta_0(P_s) = \lceil s/2 \rceil$ for all positive integers s. Hence $\beta_0(U_2(G)) \leq \Sigma_{k=1}^{n-1}\lceil (n-k)/2 \rceil$. It is easy to see that $\Sigma_{k=1}^{n-1}\lceil (n-k)/2 \rceil = \lfloor n^2/4 \rfloor$. □

In the next theorem we will find $\beta_0(U_2(C_n))$.

Theorem 3.6. $\beta_0(U_2(C_n)) = \lfloor n^2/4 \rfloor$.

Proof. By the definition of double vertex graph it immediately follows that the set $M = \{\{v_i, v_{i+1+2r}\} \in V(U_2(C_n)) \mid 1 \leq i \leq n-1, 0 \leq r \leq (n-i-1)/2\}$ (r is an integer) is independent. Hence $\beta_0(U_2(C_n)) \geq |M| = \Sigma_{i=1}^{n-1}\lceil (n-i)/2 \rceil = \lfloor n^2/4 \rfloor$. The result now follows because of Corollary 3.5. □

Theorem 3.7. *$\beta_0(U_2(C_n)[A]) = \beta_0(U_2(C_n))$ for all $A \in \mathcal{F}_{U_2(C_n)}(\{P_1\}, \{P_2\}, ..., \{P_{n-1}\})$.*

Proof. Let $V(C_n) = \{v_1, v_2, .., v_n\}$, $E(C_n) = \{v_1v_2, v_2v_3, .., v_{n-1}v_n, v_nv_1\}$ and for $k = 1, 2, .., n-1$: $Q_k =< \{\{v_k, v_j\} | k < j \leq n\}, U_2(C_n) >$. By the proof of Theorem 3.4 we have that $A = \{Q_{n-1}, Q_{n-2}, .., Q_1\}$ is a partition of $U_2(C_n)$ and for $k = 1, 2, .., n-1$, the graph Q_k is isomorphic to $H_k =< \{v_{k+1}, v_{k+2}, .., v_n\}, C_n >$. But obviously H_k is isomorphic to P_{n-k}. Thus we obtain $A \in \mathcal{F}_{U_2(C_n)}(\{P_1\}, \{P_2\}, .., \{P_{n-1}\})$. Now, choose an arbitrary $B \in \mathcal{F}_{U_2(C_n)}(\{P_1\}, \{P_2\}, .., \{P_{n-1}\})$. Hence $\beta_0(U_2(C_n)[B]) = \Sigma_{m=1}^{n-1}\beta_0(P_m) = \Sigma_{k=1}^{n-1}\beta_0(P_{n-k}) = \Sigma_{k=1}^{n-1}\lceil (n-k)/2 \rceil = \lfloor n^2/4 \rfloor = \beta_0(U_2(C_n))$. □

4. OPEN QUESTIONS

We close with a list of open problems and questions.

1. Which graphs are γ - critical and γ_a - critical (or one but not the other).
2. Characterize/study those graphs achieving equality in Theorem 3.4.
3. Characterize/study the all graphs G with $\mu(G) = \mu(G[A])$ for all $A \in \mathcal{F}_G(\{P_s\}, \mathcal{K}_{n-s})$, $s \geq 2$ where $\mu \in \{\gamma, \gamma_a, i, ..\}$.

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Department of Mathematics
University of Architecture, Civil Engineering and Geodesy
1, Christo Smirnenski blvd.
1046 Sofia
BULGARIA
E-mail: vlsam_fte@uacg.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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(2, 3)-GENERATION OF THE GROUPS $PSL_5(q)$

KEROPE TCHAKERIAN

We prove that the group $PSL_5(q)$ is (2, 3)-generated for any q.

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1. INTRODUCTION

A group G is said to be (2, 3)-generated if $G = \langle x, y\rangle$ for some elements x and y of orders 2 and 3, respectively. This generation property has been proved for a number of series of finite simple groups. Concerning the projective special linear groups $PSL_n(q)$, (2, 3)-generation is known in the cases $n = 2$, $q \neq 9$ ([4]), $n = 3$, $q \neq 4$ (see [1]), $n = 4$, $q \neq 2$ ([7], [8]; for even q also proved independently and later in [5]), $n \geq 5$, q odd, $q \neq 9$ ([2], [3]), and $n \geq 13$, any q ([6]). The present paper is another contribution to the problem. We prove the following

Theorem. *The group $PSL_5(q)$ is* (2, 3)*-generated for any* q.

We note that our approach is quite different from that in [2].

2. PROOF OF THE THEOREM

Let $G = SL_5(q)$ and $\overline{G} = G/Z(G) = PSL_5(q)$, where $q = p^m$ and p is a prime. Set $Q = (q^5 - 1)/(q - 1)$ and $d = (5, q - 1) = (5, Q)$.

We first look for elements x and y of G of respective orders 2 and 3 such that the element $z = xy$ has order Q. Choose x in the form

$$x = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \lambda & \mu & 1 & 0 \\ 0 & \nu & \xi & 0 & 1 \end{pmatrix} \quad (x \in G,\ |x| = 2 \text{ for any } \lambda, \mu, \nu, \xi \in \mathrm{GF}(q))$$

and

$$y = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (y \in G,\ |y| = 3).$$

Then

$$z = xy = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 \\ \mu & 0 & \lambda & -1 & \lambda + \mu - 1 \\ \xi & 0 & \nu & 1 & \nu + \xi \end{pmatrix}.$$

The characteristic polynomial of z is

$$f_z(t) = t^5 + (2 - \nu - \xi)t^4 + (2 - \lambda - \mu - \nu - 2\xi)t^3 + (\nu - \mu)t^2 + (\lambda + \mu + \nu + \xi - 1)t - 1.$$

Let ω be an element of order Q in the group $\mathrm{GF}(q^5)^*$ and

$$f(t) = (t - \omega)(t - \omega^q)(t - \omega^{q^2})(t - \omega^{q^3})(t - \omega^{q^4}) = \\ t^5 - \alpha t^4 + \beta t^3 - \gamma t^2 + \delta t - 1.$$

Then $f(t) \in \mathrm{GF}(q)[t]$ and the roots of $f(t)$ are pairwise distinct (in fact, the polynomial $f(t)$ is irreducible over $\mathrm{GF}(q)$). Now choose λ, μ, ν, ξ so that

$$2 - \nu - \xi = -\alpha,\ 2 - \lambda - \mu - \nu - 2\xi = \beta,\ \nu - \mu = -\gamma,\ \lambda + \mu + \nu + \xi - 1 = \delta,$$

i.e.

$$\lambda = -2\alpha - \beta - \gamma - 2,\ \mu = \alpha + \beta + \gamma + \delta + 1,\ \nu = \alpha + \beta + \delta + 1,\ \xi = -\beta - \delta + 1.$$

This implies $f_z(t) = f(t)$. Then, in $\mathrm{GL}_5(q^5)$, z is conjugate to $\mathrm{diag}(\omega, \omega^q, \omega^{q^2}, \omega^{q^3}, \omega^{q^4})$ and hence z is an element of G of order Q.

Now, in $\overline{G}$, $\overline{x}$, $\overline{y}$ and $\overline{z} = \overline{x}\,\overline{y}$ are elements of orders 2, 3 and Q/d, respectively, and $\overline{H} = \langle \overline{x}, \overline{y} \rangle$ is a subgroup of order divisible by $6Q/d$. We claim that $\overline{H} = \overline{G}$. To prove this, we make use of the subgroup structure of $\overline{G}$.

The irreducible subgroups of $\mathrm{PSL}_5(q)$ are classified in [9] and [10]. This readily implies that if $\overline{M}$ is a maximal subgroup of $\overline{G}$ then one of the following holds.

1) $|\overline{M}| = q^{10}(q-1)(q^2-1)(q^3-1)(q^4-1)/d$.

2) $|\overline{M}| = q^{10}(q-1)(q^2-1)^2(q^3-1)/d$.

3) $|\overline{M}| = 120(q-1)^4/d$ if $q \geq 5$.

4) $\overline{M} \cong Z_{Q/d} \,.\, Z_5$.

5) $\overline{M} \cong \mathrm{PSL}_5(q_0) \,.\, Z_{(d,r)}$ if $q = q_0^r$ and r is a prime.

6) $\overline{M} \cong \mathrm{PSU}_5(q_0)$ if $q = q_0^2$.

7) $\overline{M} \cong \mathrm{PSO}_5(q)$ if q is odd.

8) $\overline{M} \cong E_{5^2} \,.\, \mathrm{SL}_2(5)$ if $q = p \equiv 1 \pmod 5$.

9) $\overline{M} \cong \mathrm{PSU}_4(2)$ if $q = p \equiv 1 \pmod 3$.

10) $\overline{M} \cong \mathrm{PSL}_2(11)$ if $q = p > 3$, $p \equiv 1, 3, 4, 5, 9 \pmod{11}$.

11) $\overline{M} \cong M_{11}$ if $q = 3$.

It can be easily checked (directly or using Zsigmondy's well-known theorem) that the only maximal subgroup of $\overline{G}$ whose order is a multiple of Q/d is that in 4), of order $5Q/d$. This implies that no proper subgroup of $\overline{G}$ has order divisible by $6Q/d$. Hence $\overline{H} = \overline{G}$ and $\overline{G} = \langle \overline{x}, \overline{y} \rangle$ is a (2, 3)-generated group.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: kerope@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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ON THE COMPUTATION OF WEIGHT DISTRIBUTION OF THE COSETS OF CYCLIC CODES

E. VELIKOVA, T. BAICHEVA

Using the algebraic structure of cyclic codes an efficient method for the determination of the weight distribution of the cosets of cyclic codes is presented. As an illustration of the method weight distributions of the coset leaders of all ternary cyclic codes of lengths up to 14 are calculated.

Keywords: cyclic codes, covering radius, coset weight distribution

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1. INTRODUCTION

Cyclic codes form an important subclass of linear codes. These codes are attractive by two reasons: first, encoding and syndrome computation can be implemented easily by employing shift registers with feedback connections and second, because they have well known algebraic structure, it is possible to find various methods for decoding them. To be able to evaluate the performance of a cyclic code for some application we have to know the exact values of all its basic characteristics among them covering radius, coset leaders and coset weight distributions.

Using an exhaustive search covering radii of some binary and ternary cyclic codes are determined in [1], [2], [3], [4], [5], [6], [7]. In this work we suggest a method for efficient calculation of the complete coset weight distributions of cyclic codes.

2. COSETS OF CYCLIC CODES

Let C be a cyclic $[n,k]$ code over the finite field of q elements $F_q = GF(q)$ and let the generator polynomial of C be $g(x)$ with the degree $deg(g(x)) = n-k$. By V we will denote the n-dimensional vector space over F_q. Then the map $\sigma : V \to V$ will be the cyclic shift of the words of V

$$\sigma(a_0, a_1, a_2, \ldots, a_{n-1}) = (a_{n-1}, a_0, a_1, \ldots, a_{n-2}).$$

Theorem 2.1. *Let C be a cyclic $[n,k]$ code with the generator polynomial $g(x) = x^{n-k} + g_{n-k-1}x^{n-k-1} + \ldots + g_1 x + g_0$ and let $a = (a_0, a_1, \ldots, a_{n-k-1}, 0, \ldots, 0)$ be a vector from the space V. Then the following two cosets coincide:*

$$\sigma(a) + C = r + C,$$

where $r = (0, a_0, a_1, \ldots, a_{n-k-2}, 0, \ldots, 0) - a_{n-k-1}(g_0, g_1, \ldots, g_{n-k-1}, 0, \ldots, 0)$.

Proof. Let us consider the standard correspondence between a vector from V and a polynomial from the ring of the polynomials $F_q[x]$

$$v = (v_0, v_1, \ldots, v_{n-1}) \to v(x) = v_0 + v_1 x + \ldots + v_{n-1}x^{n-1}.$$

If C is a cyclic code with the generator polynomial $g(x)$ of degree $m = n-k$, then it is well known that

$$b \in a + C \Leftrightarrow g(x) | (b(x) - a(x)).$$

Let $a = (a_0, a_1, \ldots, a_{n-k-1}, 0, \ldots, 0)$ be a vector of V. Then

$$b = \sigma(a) = (0, a_0, a_1, \ldots, a_{n-k-1}, 0, \ldots, 0)$$

and its corresponding polynomial is

$$b(x) = \sigma(a)(x) = xa(x) = a_0 x + a_1 x^2 + \ldots + a_{n-k-1}x^{n-k}.$$

The remainder of the division of $b(x)$ by $g(x) = x^{n-k} + g_{n-k-1}x^{n-k-1} + \ldots + g_1 x + g_0$ is

$$r(x) = b(x) - a_{n-k-1}g(x) =$$
$$= a_0 x + a_1 x^2 + \ldots + a_{n-k-2}x^{n-k-1} - a_{n-k-1}(g_{n-k-1}x^{n-k-1} + \ldots + g_1 x + g_0)$$

and its corresponding vector is

$$r = (0, a_0, a_1, \ldots, a_{n-k-2}, 0, \ldots, 0) - a_{n-k-1}(g_0, g_1, \ldots, g_{n-k-1}, 0, \ldots, 0). \square$$

From the well known fact that if two vectors a and b belong to one and the same coset of the code C then their corresponding polynomials have the same remainders by division by $g(x)$ we can conclude that we will get one representative from each coset if we take all vectors of the type

$$a = (a_0, a_1, \ldots, a_{n-k-1}, 0, \ldots, 0).$$

Let the parity check matrix of the code C be in the form $H = [I_{n-k}|B]$. If $a = (a_0, a_1, \ldots, a_{n-k-1}, 0, \ldots, 0)$ is a vector from V then its syndrome is $s(a) = Ha^t = (a_0, a_1, \ldots, a_{n-k-1})^t$. According to Theorem 2.1 we have $\sigma(a) + C = r + C$ and therefore

$$s(\sigma(a)) = (0, a_0, a_1, \ldots, a_{n-k-2}) - a_{n-k-1}(g_0, g_1, \ldots, g_{n-k-1}).$$

Therefore from the syndrome of a word of V we are able to compute the syndromes of all its cyclic shifts.

3. ACTING OF THE CYCLIC GROUP ON THE COSETS OF A CYCLIC CODE

Let $G = <\sigma>$ be a cyclic group generated by σ. The group has n elements.

Lemma 3.1. *Let C be a cyclic $[n, k]$ code and $a \in V$. Let $B = \{\sigma(z)|z \in a + C\}$. Then B is a coset for the code C and $B = \sigma(a) + C$.*

Proof. $\sigma(a + c_1) - \sigma(a + c_2) = \sigma(a) + \sigma(c_1) - \sigma(a) - \sigma(c_2) = \sigma(c_1 - c_2) = = \sigma(c_3) \in C$. □

It follows from this lemma that we can consider the action of G over the set of all cosets of the code C in the following way $\sigma(a + C) = \sigma(a) + C$. By this action the set of all cosets is partitioned to non intersecting orbits $O(a + C) = \{\sigma^t(a) + C|t = 0, \ldots n - 1\}$ and the length of each orbit (i.e. the number of the different cosets) is a divisor of n. All cosets belonging to one and the same orbit have one and the same weight distribution. We can obtain one representative from each coset of one orbit by taking the vectors $a = (a_0, a_1, \ldots, a_{n-k-1}, 0, \ldots, 0)$; $\phi(a) = (0, a_0, a_1, \ldots, a_{n-k-2}, 0, \ldots, 0) - a_{n-k-1}(g_0, g_1, \ldots, g_{n-k-1}, 0, \ldots, 0)$ and $\phi^2(a), \ldots, \phi^{n-1}(a)$. If the last k coordinates of the vectors a and b are zeroes then they belong to the cosets from one and the same orbit iff there exists s such that $b = \phi^s(a)$.

4. COSET LEADERS WEIGHT DISTRIBUTIONS OF TERNARY CYCLIC CODES WITH $N \leq 14$

Using the results from the previous sections we have calculated the coset leaders weight distributions of some ternary cyclic codes of small lengths. For the calculations we have used the definition of the covering radius of a code as the weight of the coset leader of greatest weight. For a code in a standard form a vector of each coset can be found by generating all the vectors of the form $(a, \underbrace{0, \ldots, 0}_{k})$, $a \in GF(3^{n-k})$.

Then the number of steps required to find $R(C)$ by an exhaustive search is proportional to $n3^n$. If we check only one coset from each orbit we can considerably reduce the required time. More precisely, if we have s different orbits the number of steps will be $n3^{k+s}$ and this number is less then $n3^n$ because $s < n - k$. The time complexity can be additionally decreased if we take into the consideration the fact that all vectors of weight less than or equal to $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ are unique coset leaders. So, we have to check only vectors of greater than t weights.

The classification from [6] was used as source for all ternary cyclic codes of lengths up to 14. The results (a list of the nonequivalent ternary cyclic codes of length up to 14, the roots of the generator polynomials and the coset leaders weight distributions) are presented in the Table below.

Table 1. Coset leaders weight distributions of ternary cyclic codes of length ≤ 14

No	n	k	d	Roots	Coset leaders weight distribution
1.	4	3	2	2	$\alpha_1 = 2$
2.	4	2	2	1	$\alpha_1 = 4, \alpha_2 = 4$
3.	4	1	4	0,1	$\alpha_1 = 8, \alpha_2 = 18$
4.	8	7	2	4	$\alpha_1 = 2$
5.	8	6	2	1	$\alpha_1 = 8$
6.	8	6	2	2	$\alpha_1 = 4, \alpha_2 = 4$
7.	8	5	3	0,1	$\alpha_1 = 16, \alpha_2 = 10$
8.	8	5	2	0,2	$\alpha_1 = 8, \alpha_2 = 18$
9.	8	4	4	1,2	$\alpha_1 = 16, \alpha_2 = 60, \alpha_3 = 4$
10.	8	4	2	1,5	$\alpha_1 = 8, \alpha_2 = 24, \alpha_3 = 32, \alpha_4 = 16$
11.	8	3	5	0,1,2	$\alpha_1 = 16, \alpha_2 = 112, \alpha_3 = 108, \alpha_4 = 6$
12.	8	3	4	0,1,5	$\alpha_1 = 16, \alpha_2 = 82, \alpha_3 = 96, \alpha_4 = 48$
13.	8	2	6	0,1,2,4	$\alpha_1 = 16, \alpha_2 = 112, \alpha_3 = 368, \alpha_4 = 216, \alpha_5 = 16$
14.	8	2	4	0,1,4,5	$\alpha_1 = 16, \alpha_2 = 100, \alpha_3 = 288, \alpha_4 = 324$
15.	8	1	8	0,1,2,5	$\alpha_1 = 16, \alpha_2 = 112, \alpha_3 = 448, \alpha_4 = 1050, \alpha_5 = 560$
16.	10	9	2	5	$\alpha_1 = 2$
17.	10	8	2	0,5	$\alpha_1 = 4, \alpha_2 = 4$
18.	10	6	2	1	$\alpha_1 = 10, \alpha_2 = 40, \alpha_3 = 30$
19.	10	5	4	0,1	$\alpha_1 = 20, \alpha_2 = 132, \alpha_3 = 90$
20.	10	5	2	0,2	$\alpha_1 = 10, \alpha_2 = 40, \alpha_3 = 80, \alpha_4 = 80, \alpha_5 = 32$
21.	10	4	4	0,1,5	$\alpha_1 = 20, \alpha_2 = 132, \alpha_3 = 240, \alpha_4 = 240,$ $\alpha_5 = 96$
22.	10	2	5	1,2	$\alpha_1 = 20, \alpha_2 = 180, \alpha_3 = 860, \alpha_4 = 2200, \alpha_5 = 2400,$ $\alpha_6 = 900$
23.	10	1	10	0,1,2	$\alpha_1 = 20, \alpha_2 = 180, \alpha_3 = 960, \alpha_4 = 3360, \alpha_5 = 7812,$ $\alpha_6 = 7350$
24.	11	6	5	1	$\alpha_1 = 22, \alpha_2 = 220$
25.	11	5	6	0,1	$\alpha_1 = 22, \alpha_2 = 220, \alpha_3 = 440, \alpha_4 = 44, \alpha_5 = 2$
26.	11	1	11	1,2	$\alpha_1 = 22, \alpha_2 = 220, \alpha_3 = 1320, \alpha_4 = 5280,$ $\alpha_5 = 14784, \alpha_6 = 25872, \alpha_7 = 11550$

No	n	k	d	Roots	Coset leaders weight distribution
27.	13	10	3	1	$\alpha_1 = 26$
28.	13	9	3	0,1	$\alpha_1 = 26, \alpha_2 = 52, \alpha_3 = 2$
29.	13	7	5	1,4	$\alpha_1 = 41, \alpha_2 = 362, \alpha_3 = 324$
30.	13	7	4	1,2	$\alpha_1 = 41, \alpha_2 = 302, \alpha_3 = 384$
31.	13	6	6	0,1,4	$\alpha_1 = 29, \alpha_2 = 348, \alpha_3 = 1274, \alpha_4 = 32, \alpha_5 = 3$
32.	13	6	6	0,1,2	$\alpha_1 = 29, \alpha_2 = 352, \alpha_3 = 1432, \alpha_4 = 373$
33.	13	4	7	1,2,4	$\alpha_1 = 26, \alpha_2 = 312, \alpha_3 = 2288, \alpha_4 = 8788,$ $\alpha_5 = 8060, \alpha_6 = 208$
34.	13	3	9	0,1,2,4	$\alpha_1 = 26, \alpha_2 = 312, \alpha_3 = 2288, \alpha_4 = 11440,$ $\alpha_5 = 30342, \alpha_6 = 14352, \alpha_7 = 288$
35.	13	1	13	1,2,4,7	$\alpha_1 = 26, \alpha_2 = 312, \alpha_3 = 2288, \alpha_4 = 11440,$ $\alpha_5 = 41184, \alpha_6 = 109824, \alpha_7 = 204204,$ $\alpha_8 = 162162$
36.	14	13	2	7	$\alpha_1 = 2$
37.	14	12	2	0,7	$\alpha_1 = 4, \alpha_2 = 4$
38.	14	8	2	1	$\alpha_1 = 14, \alpha_2 = 84, \alpha_3 = 280, \alpha_4 = 350$
39.	14	7	4	0,1	$\alpha_1 = 30, \alpha_2 = 300, \alpha_3 = 1015, \alpha_4 = 841$
40.	14	7	2	0,2	$\alpha_1 = 14, \alpha_2 = 84, \alpha_3 = 280, \alpha_4 = 560,$ $\alpha_5 = 672, \alpha_6 = 448, \alpha_7 = 128$
41.	14	6	4	0,1,7	$\alpha_1 = 44, \alpha_2 = 343, \alpha_3 = 1102, \alpha_4 = 1930,$ $\alpha_5 = 1935, \alpha_6 = 1003, \alpha_7 = 202$
42.	14	2	7	1,2	$\alpha_1 = 28, \alpha_2 = 364, \alpha_3 = 2912, \alpha_4 = 15596,$ $\alpha_5 = 56840, \alpha_6 = 137200, \alpha_7 = 196000,$ $\alpha_8 = 122500$
43.	14	1	14	0,1,2	$\alpha_1 = 28, \alpha_2 = 364, \alpha_3 = 2912, \alpha_4 = 16016,$ $\alpha_5 = 64064, \alpha_6 = 192192, \alpha_7 = 435864,$ $\alpha_8 = 630630, \alpha_9 = 252252$

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Evgenia Velikova
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., BG-1164 Sofia
BULGARIA
E-mail: velikova@fmi.uni-sofia.bg

Tsonka Baicheva
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
P.O. Box 323, Veliko Turnovo 5000
BULGARIA
E-mail: tsonka@moi.math.bas.bg

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THE WEIGHT DISTRIBUTION OF THE COSET LEADERS OF TERNARY CYCLIC CODES WITH GENERATING POLYNOMIAL OF SMALL DEGREE

E. VELIKOVA

Using the algebraic structure of cyclic codes, it is proved that the cyclic codes with one and the same generating polynomial have equal weight distribution of cosets' leaders. As an illustration,the weight distribution of the leaders of the cosets of all ternary cyclic codes with generating polynomial of degree less than 6 is presented.

Keywords: cyclic codes, covering radius, coset weight distribution

2000 MSC: 94B15

1. INTRODUCTION

Let C be a cyclic code of length n over the finite field $F_q = GF(q)$. Let us consider the standard correspondence between a vector from $n-$dimensional vector space ${F_q}^n$ and a polynomial from the ring of the polynomials $F_q[x]$

$$v = (v_0, v_1, \dots, v_{n-1}) \to v(x) = a_0 + a_1 x + \dots + a_{n-1}x^{n-1}.$$

A generator polynomial $g(x)$ of code C is a nonzero polynomial of the smallest degree of code and $c \in C$ if and only if $g(x)|c(x)$. If C is a cyclic $[n, k]$ code with the generator polynomial $g(x)$, then the degree of $g(x)$ is $m = n-k$ and the number of cosets $a + C$ of code C is equal to q^m.

Leader of a coset $a + C$ is the vector with the smallest Hamming weight in that coset and by $wt(a + C)$ we denote the weight of the coset leader $a + C$, i.e.

$wt(a+C) = min\{wt(x)|x \in a+C\}$. The covering radius of the code is the weight of the leader with maximum weight. The covering radii of some binary and ternary cyclic codes are determined in [1], [2], [3], [4], [5], [6], [7].

Some applications of codes require the knowledge of not only the covering radius but also of spectrum of leaders of all cosets of a code. Let us denote by ω_e the number of cosets $a+C$ for which $wt(a+C) = e$. It is clear that $\omega_0 = 1$; $\omega_0 + \omega_1 + ... + \omega_n = q^{n-k}$ and $\omega_t = 0$, for every $t > n-k$. The spectrum of the of cosets leaders of the code C is $\omega(C) = (\omega_0, \omega_1, ..., \omega_{n-k})$. In [8] a method for computation of weight distribution of spectrum of coset leaders of an cyclic code is presented.

In all the known tables the cyclic codes are grouped by the code length and by the roots of the generating polynomials. It is proved in this paper that there is a connection between spectrum of coset leaders for cyclic codes over a finite field $GF(q)$ with equal generating polynomial and non equal lengths. As an illustration,the weight distribution of the leaders of the cosets of all ternary cyclic codes with generating polynomial with degree less than 6 is presented.

2. COSETS OF CYCLIC CODES WITH EQUAL GENERATING POLYNOMIAL

Let C be a cyclic $[n,k]$ code over the finite field with q elements F_q. The generator polynomial $g(x)$ of C is of degree $deg(g(x)) = n-k$, $g(x)|(x^n - 1)$ and $h(x) = \frac{x^n-1}{g(x)}$ is a parity check polynomial of code C.

Let n_0 be the smallest integer such that $g(x)|(x^{n_0} - 1)$ and C_0 is the cyclic code with length n_0 and generator polynomial $g(x)$. From $gcd(x^n - 1, x^{n_0} - 1) = x^{gcd(n,n_0)} - 1$ we obtain that $n_0|n$. If $n = s.n_0$ then the parity check polynomial of code C is

$$h(x) = \frac{x^n - 1}{g(x)} = \frac{x^{n_0.s} - 1}{x^{n_0} - 1}.h_0$$

and the dual of the code C is s times repeated the dual of code C_0.

Theorem 2.1. *Let C be a cyclic $[n,k]$ code with the generator polynomial $g(x)$ and let n_0 is the small integer such that $g(x)|(x^{n_0} - 1)$. If the $C_0 =< g(x) >$ is the cyclic code with length n_0 and the generator polynomial $g(x)$ then the spectra of cosets leaders for codes C and C_0 are equal $\omega(C) = \omega(C_0)$.*

Proof. Let $a \in {F_q}^{n_0}$ and $\breve{a}$ be the extended vector $\breve{a} = (a, 0, ..., 0)$ from F_q^n. Let a correspondence $\varphi : \{a + C_0 | a \in {F_q}^{n_0}\} \to \{a + C | a \in {F_q}^n\}$ between the cosets of code C_0 and C be defined as $\varphi(a + C_0) = \breve{a} + C$. Then it is clear that $b \in a + C_0 \Leftrightarrow \breve{b} \in \breve{a} + C$. Hence the correspondence φ is a bijection as the number of cosets of codes C and C_0 are equal.

For $z = (z_0, ..., z_{n-1}) \in {F_q}^n$, let us consider the vector $z^{(n_0)} = (y_0, ..., y_{n_0-1}) \in {F_q}^{n_0}$, where $y_i = z_i + z_{i+n_0} + ... + z_{i+(s-1)n_0}$ for all $i \in \{0, ..., n_0 - 1\}$. It is clear that if $y_i \neq 0$ then $wt(z_i) + wt(z_{i+n_0}) + ... + wt(z_{i+(s-1)n_0}) \geq 1$. Hence $wt(z^{(n_0)}) \leq wt(z)$. The polynomial $z^{(n_0)}(x)$ is the remainder of the division of $z(x)$ by $x^{n_0} - 1$. Therefore $\check{z}^{(n_0)} \in z + C$. If $a \in {F_q}^{n_0}$ is the leader of the coset $a + C_0$ then $wt(\varphi(a + C_0)) \leq wt(a)$. Let z be the leader of $\varphi(a + C_0)$ then $z^{(n_0)} \in a + C_0$ hence $wt(z) \geq wt(a + C_0)$. Therefore $wt(a + C_0) = wt(\varphi(a + C_0))$. □

From that theorem we can conclude that if C_1 and C_2 are two cyclic codes with different lengths but with one and the same generator polynomial $g(x)$ then $\omega(C_1) = \omega(C_2)$.

3. COSET LEADERS WEIGHT DISTRIBUTIONS OF SOME TERNARY CYCLIC CODES

As an illustration of the previous section we calculate the coset leaders weight distributions of some ternary cyclic codes with generator polynomial of degree≤ 5. For the calculations we have used mostly the definition of the spectrum of the coset leaders and the following methods:

Method 1. If the linear $[n, k]$ code C over F_q has a parity check matrix H and $a \in {F_q}^n, a \notin C$ then $wt(a + C)$ is the least integer e such that the syndrome $S(a) = Ha^t$ can be represented as a linear combination of the e from the columns of matrix H. So, we can calculate the coset leader's spectrum if for any nonzero syndrome S calculate the minimal number of columns of H that linear generate S.

Method 2. In [8] is considered the action of the cyclic group $G_n =< \sigma >$ (σ is a cyclic shift of coordinates) with n elements on the cosets of one cyclic $[n, k]$ code as $\sigma(a + C) = \sigma(a) + C$. This action splits the cosets in disjoint orbits and from [8] it is clear how to obtain one representative from each coset.Thus we calculate the weight of the coset leader only for one coset from each orbit.

Let C be a cyclic code with generator polynomial $g(x)$ and the minimum length of cyclic code with generator polynomial $g(x)$ be n_0. In the following tables are presented basic parameters of some cyclic codes . In the tables the polynomials are represented by their coefficients, namely $g(x) = g_0 + g_1x + ... + g_mx^m$ is given as a string $g_0g_1...g_m$. As the reciprocals polynomials generate equivalent codes, the table contains only one from any couple of such polynomials.

3.1. SPECTRUM OF COSET LEADERS FOR IRREDUCIBLE POLYNOMIALS

Let $g(x)$ be an irreducible polynomial over F_q of degree m. Then $g(x)|(x^{q^m-1} - 1)$ and if n_0 is the smallest integer such that $g(x)|x^{n_0} - 1$, then $n_0|(q^m - 1)$. Let α be a root of $g(x)$ and C_0 be the cyclic $[n_0, n_0 - m]$ code, with generator polynomial $g(x)$, then a parity check matrix of code C_0 is the following $H = (1, \alpha, \alpha^2, ..., \alpha^{n_0-1})$. A polynomial $g(x)$ for which is hold $n_0 = q^m - 1$ is called primitive polynomial and every parity check matrix for the code with length $q^m - 1$ consists of every nonzero vector column from F_q^m . Hence for that code spectrum of coset's leaders is $(1, q^m - 1, 0, ..., 0)$. If C_1 and C_2 are $[n, k]$ cyclic codes generated with irreducible polynomials of degree m the codes C_1 and C_2 are equivalent.

The table from [9] was used as a source for all irreducible polynomials over F_3.

TABLE 1. Coset leaders weight distributions of irreducible ternary cyclic codes with generator polynomial of degree ≤ 5

N	deg	polynomial	n	k	d	R	Spectrum
1	1	21	n	$n-1$	2	1	$(1,2)$
2	1	11	$2s$	$2s-1$	2	1	$(1,2)$
3	2	101	$4s$	$4s-2$	2	2	$(1,4,4)$
4	2	211	$8s$	$8s-2$	2	1	$(1,8,0)$
5	3	2201; 2111	$13s$	$13s-3$	3 or 2	1	$(1,26,0,0)$
6	3	1201; 1211	$26s$	$26s-3$	2	1	$(1,26,0,0)$
7	4	11111	$5s$	$5s-4$	5 or 2	3	$(1,10,40,30,0)$
8	4	12121	$10s$	$10s-4$	2	3	$(1,10,40,30,0)$
9	4	20201	$16s$	$16s-4$	2	2	$(1,16,64,0,0)$
10	4	12011	$20s$	$20s-4$	2	2	$(1,20,60,0,0)$
11	4	10111; 12101	$40s$	$40s-4$	2	2	$(1,40,40,0,0)$
12	4	20021;22001; 22111;21121	$80s$	$80s-4$	2	1	$(1,80,0,0,0)$
13	5	221201	$11s$	$11s-5$	5 or 2	2	$(1,22,220,0,0,0)$
14	5	122201	$22s$	$22s-5$	2	2	$(1,22,220,0,0,0)$
15	5	220001;211001;210101; 201101;221101;211201; 210011;221011;212111; 212021;211121	$121s$	$121s-5$	3 or 2	1	$(1,242,0,0,0,0)$
16	5	120001;112001;110101; 102101;122101;112201; 120011;111011;121111; 112111;122021	$242s$	$242s-5$	2	1	$(1,242,0,0,0,0)$

3.2. SPECTRA FOR REDUCIBLE POLYNOMIALS WITHOUT MULTIPLE ROOTS

If $g(x)$ is a reducible polynomial over F_q and it does not have multiple roots then for the minimum integer n_0 for which $g(x)|(x^{n_0}-1)$ is hold $gcd(q,n_0)=1$. If α is a primitive $n-$th root of unity in some field F_{q^t} then all zeros of $g(x)$ will be $\alpha^{i_1},...,\alpha^{i_m}$. It is known that if C_1 and C_2 are cyclic $[n_0,n_0-m]$ codes and the sets of roots of the codes C_1 and C_2 are, respectively, $\alpha^{i_1},...,\alpha^{i_m}$ and $\alpha^{j_1},...,\alpha^{j_m}$ and there exists a integer v, such that $gcd(n_0,v)=1$ and $j_s=v,i_s$ for $s\in\{1,...,m\}$ then the codes C_1 and C_2 are equivalent. In that table we omit all equivalent codes, obtained by the upper procedure.

TABLE 2. Coset leaders weight distributions of ternary cyclic codes without multiple roots and generator polynomial of degree ≤ 5

No	deg	polynomial	n	k	d	R	Spectrum
1	2	201	$2s$	$2s-2$	2	2	$(1,4,4)$
2	3	1111; 2121	$4s$	$4s-3$	4 or 2	2	$(1,8,18,0)$
3	3	1101; 2021	$8s$	$8s-3$	3 or 2	2	$(1,16,10,0)$
4	4	20001	$4s$	$4s-4$	2	4	$(1,8,24,32,16)$
5	4	10001	$8s$	$8s-4$	2	4	$(1,8,24,32,16)$
6	4	12111	$8s$	$8s-4$	4 or 2	3	$(1,16,60,4,0)$
7	4	21011	$8s$	$8s-4$	4 or 2	3	$(1,16,60,4,0)$
8	4	10221; 11001	$13s$	$13s-4$	3 or 2	3	$(1,26,52,2,0)$
9	4	10211; 10021	$26s$	$26s-4$	2	3	$(1,26,52,2,0)$
10	4	21211; 20221; 22221; 22101	$26s$	$26s-4$	3 or 2	2	$(1,52,28,0,0)$
11	5	200001	$5s$	$5s-5$	2	5	$(1,10,40,80,80,32)$
12	5	111201; 201121	$8s$	$8s-5$	5 or 2	4	$(1,16,112,108,6,0)$
13	5	210021; 110011	$8s$	$8s-5$	4 or 2	4	$(1,16,82,96,48,0)$
14	5	100001	$10s$	$10s-5$	2	5	$(1,10,40,80,80,32)$
15	5	122221; 221211	$10s$	$10s-5$	4 or 2	3	$(1,20,132,90,0,0)$
16	5	121221; 222211	$16s$	$16s-5$	3 or 2	2	$(1,32,210,0,0,0)$
17	5	222201; 121201	$20s$	$20s-5$	4 or 2	2	$(1,40,202,0,0,0)$
18	5	112101;121011; 211101;210111	$26s$	$26s-5$	3 or 2	3	$(1,52,184,6,0,0)$
19	5	212001;221121; 111221;111001	$40s$	$40s-5$	3 or 2	2	$(1,80,162,0,0,0)$
20	5	222001;210211; 121001;122011	$52s$	$52s-5$	3 or 2	2	$(1,104,138,0,0,0)$
21	5	120111; 102021; 101001; 110211; 201011; 212011; 210221; 202001	$80s$	$80s-5$	3 or 2	2	$(1,160,82,0,0,0)$
22	5	111101; 212101	$104s$	$104s-5$	3 or 2	2	$(1,208,34,0,0,0)$
23	5	121021; 222011	$104s$	$104s-5$	3 or 2	2	$(1,208,34,0,0,0)$
24	5	102001; 201001	$104s$	$104s-5$	3 or 2	2	$(1,208,34,0,0,0)$
26	5	121211; 211111	$104s$	$104s-5$	3 or 2	2	$(1,208,34,0,0,0)$

3.3. SPECTRUM CODES GENERATED BY POLYNOMIALS WITH MULTIPLE ROOTS

If $x^{qn}-1=(x^n-1)^q$ over the field F_q and $g(x)$ has multiple roots then $g(x)|(x^n-1)$ where $q|n$. Very few is known about such a codes so in the following table may contain equivalent codes.

TABLE 3. Coset leaders weight distributions of ternary cyclic codes with multiple roots and generator polynomial of degree ≤ 5

N	deg	polynomial	n	k	d	R	Spectrum
1	2	111	$3s$	$3s-2$	3 or 2	2	$(1,6,2)$
2	2	121	$6s$	$6s-2$	2	2	$(1,6,2)$
3	3	2001	$3s$	$3s-3$	2	3	$(1,6,12,8)$
4	3	1001	$6s$	$6s-3$	2	3	$(1,6,12,8)$
5	3	2211	$6s$	$6s-3$	4 or 2	2	$(1,12,14,0)$
6	3	1221	$6s$	$6s-3$	3 or 2	2	$(1,12,14,0)$
7	4	10101	$6s$	$6s-4$	3 or 2	4	$(1,12,40,24,2)$
8	4	22011	$6s$	$6s-4$	4 or 2	3	$(1,12,44,24,0)$
9	4	21021	$6s$	$6s-4$	4 or 2	3	$(1,12,44,24,0)$
10	4	12021	$9s$	$9s-4$	3 or 2	3	$(1,18,38,24,0)$
11	4	10201	$12s$	$12s-4$	2	4	$(1,12,40,24,4)$
12	4	11211	$12s$	$12s-4$	3 or 2	2	$(1,24,56,0,0)$
13	4	12221	$12s$	$12s-4$	3 or 2	2	$(1,24,56,0,0)$
14	4	11011	$18s$	$18s-4$	2	3	$(1,18,38,24,0)$
15	4	20121	$24s$	$24s-4$	3 or 2	2	$(1,48,32,0,0)$
16	4	22201	$24s$	$24s-4$	3 or 2	2	$(1,48,32,0,0)$
17	4	11221	$24s$	$24s-4$	2	2	$(1,24,56,0,0)$
18	5	212121	$6s$	$6s-5$	6 or 2	4	$(1,12,60,140,30,0)$
19	5	111111	$6s$	$6s-5$	6 or 2	4	$(1,12,60,140,30,0)$
20	5	222111	$9s$	$9s-5$	3 or 2	4	$(1,18,114,108,2,0)$
21	5	120021	$12s$	$12s-5$	3 or 2	4	$(1,24,74,96,48,0)$
22	5	220011	$12s$	$12s-5$	3 or 2	4	$(1,24,74,96,48,0)$
23	5	112211	$12s$	$12s-5$	3 or 2	4	$(1,24,134,72,12,0)$
24	5	211221	$12s$	$12s-5$	3 or 2	4	$(1,24,134,72,12,0)$
25	5	101101	$12s$	$12s-5$	4 or 2	3	$(1,24,146,72,0,0)$
26	5	202101	$12s$	$12s-5$	4 or 2	3	$(1,24,146,72,0,0)$
27	5	121121	$18s$	$18s-5$	2	4	$(1,18,114,108,2,0)$
28	5	102201	$18s$	$18s-5$	3 or 2	3	$(1,36,134,72,0,0)$
29	5	201201	$18s$	$18s-5$	3 or 2	3	$(1,36,134,72,0,0)$
30	5	122211	$24s$	$24s-5$	3 or 2	3	$(1,48,122,72,0,0)$
31	5	211211	$24s$	$24s-5$	3 or 2	3	$(1,48,122,72,0,0)$
32	5	221001	$24s$	$24s-5$	3 or 2	3	$(1,48,182,12,0,0)$
33	5	100221	$24s$	$24s-5$	3 or 2	3	$(1,48,182,12,0,0)$
34	5	202011	$24s$	$24s-5$	3 or 2	2	$(1,48,194,0,0,0)$
35	5	120101	$24s$	$24s-5$	3 or 2	2	$(1,48,194,0,0,0)$
36	5	211011	$39s$	$39s-5$	3 or 2	3	$(1,78,158,6,0,0)$
37	5	201021	$39s$	$39s-5$	3 or 2	3	$(1,78,158,6,0,0)$

N	deg	polynomial	n	k	d	R	Spectrum
38	5	112021	$78s$	$78s-5$	2	3	$(1,78,158,6,0,0)$
39	5	110201	$78s$	$78s-5$	2	3	$(1,78,158,6,0,0)$
40	5	200021	$78s$	$78s-5$	3 or 2	2	$(1,156,86,0,0,0)$
41	5	222101	$78s$	$78s-5$	3 or 2	2	$(1,156,86,0,0,0)$
42	5	100011	$78s$	$78s-5$	3 or 2	2	$(1,156,86,0,0,0)$
43	5	101121	$78s$	$78s-5$	3 or 2	2	$(1,156,86,0,0,0)$

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: velikova@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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CONNECTION BETWEEN THE LOWER P-FRAME CONDITION AND EXISTENCE OF RECONSTRUCTION FORMULAS IN A BANACH SPACE AND ITS DUAL

DIANA T. STOEVA

In the present paper it is proved that under an additional assumption (which is automatically satisfied in case $p = 2$) validity of the lower p-frame condition for a sequence $\{g_i\} \subset X^*$ implies that for f in a subset of X there exists a representation $f = \sum g_i(f)f_i$, where $\{f_i\} \subset X$ satisfies the upper q-frame condition, $\frac{1}{q} + \frac{1}{p} = 1$. An example showing that the above representation is not necessarily valid for all f in X (neither reconstruction formula of type $g = \sum g(f_i)g_i$ for all $g \in X^*$) is given. It is shown that when $\mathcal{D}(U)$ is dense in X, $g \in X^*$ can be represented as $g = \sum g(f_i)g_i$ if and only if $\sum g(f_i)g_i$ converges.

Keywords: p-frames, lower bound, reconstructions, Banach spaces, dual spaces

2000 MSC: 42C15, 40A05

1. INTRODUCTION

It is well known that if a sequence $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$ is a frame for a Hilbert space $\mathcal{H}$, i.e. there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{i=1}^{\infty} |< f, g_i >|^2 \leq B||f||^2, \quad \forall f \in \mathcal{H},$$

then every $f \in \mathcal{H}$ can be represented by a dual frame $\{f_i\}_{i=1}^{\infty} \subset \mathcal{H}$:

$$f = \sum_{i=1}^{\infty} < f, f_i > g_i = \sum_{i=1}^{\infty} < f, g_i > f_i. \tag{1.1}$$

Sequences, which satisfy the lower frame condition, but may fail the upper one, are used in some applications (for example, in irregular sampling). For this reason the existence of reconstruction formulas like (1.1) when only the lower frame condition is assumed has become a topic of investigation. The first study in this direction may be found in [3]. There an operator is associated to a family $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$ and under some assumptions on that operator it is proved that $\{g_i\}_{i=1}^{\infty}$ satisfies the lower frame condition and there exists a Bessel sequence $\{f_i\}_{i=1}^{\infty} \subset \mathcal{H}$ (i.e. sequence satisfying the upper frame condition) such that $f = \sum_{i=1}^{\infty} < f, f_i > g_i$, $\forall f \in \mathcal{H}$. Later, aim of investigation has been to get reconstruction formulas when the lower frame condition is assumed to be valid. In [2] it is proved that if $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$ satisfies the lower frame condition, then there exists a Bessel sequence $\{f_i\}_{i=1}^{\infty} \subset \mathcal{H}$ such that

$$f = \sum_{i=1}^{\infty} < f, g_i > f_i, \ \forall f \in \mathcal{D}(U), \tag{1.2}$$

where

$$\mathcal{D}(U) = \{f \in X \mid \sum_{i=1}^{\infty} | < f, g_i > |^2 < \infty\}, \tag{1.3}$$

$$U : \mathcal{D}(U) \subseteq \mathcal{H} \to \ell^2, \ \ Uf := \{< f, g_i >\}_{i=1}^{\infty}. \tag{1.4}$$

Recently, frames in Hilbert spaces have been generalized to p-frames in Banach spaces [1]. A sequence $\{g_i\}_{i=1}^{\infty} \subset X^*$ is called p-frame for X $(1 < p < \infty)$ if there *exist constants* $A, B > 0$ *such that*

$$A\|f\|_X \leq \left(\sum_{i=1}^{\infty} |g_i(f)|^p\right)^{1/p} \leq B\|f\|_X, \ \forall f \in X.$$

$\{g_i\}_{i=1}^{\infty}$ is called a p-Bessel sequence for X if it satisfies the upper p-frame inequality for all $f \in X$. In [4], p-frames $\{g_i\}_{i=1}^{\infty} \subset X^*$ in general Banach spaces are considered and necessary and sufficient condition for existence of reconstruction formulas like

$$f = \sum_{i=1}^{\infty} g_i(f) f_i, \ \forall f \in X, \tag{1.5}$$

$$g = \sum_{i=1}^{\infty} g(f_i) g_i, \ \forall g \in X^* \tag{1.6}$$

via a dual q-frame is found, namely the condition "the range of the operator U is complemented in ℓ^p", where $U : X \to \ell^p$, $Uf = \{g_i(f)\}_{i=1}^{\infty}$. In the present paper we are interested in reconstruction formulas when only the lower p-frame condition is assumed. In Section 3, generalization of (1.2) to the case when a family $\{g_i\}_{i=1}^{\infty} \subset X^*$ is assumed to satisfy the lower p-frame condition is investigated; for this case a necessary and sufficient condition for validity of formula like (1.2) via a q-Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is given. Section 4 concerns the question whether the lower p-frame condition implies existence of reconstruction formulas not only in $\mathcal{D}(U)$, but in the whole spaces X and X^* (like (1.5) and (1.6)). In Section 5 we investigate the lower p-frame inequality in Banach spaces in case the corresponding operator U is assumed to be densely defined. This is motivated by the work by Christensen and Li, who investigated the lower frame inequality in Hilbert spaces in case the operator U, given by (1.3) and (1.4), is densely defined, with the aim of obtaining reconstruction formulas in the weak sense. Our aim in this section is to obtain representation in X^* with convergence in norm-sense. In Proposition 5.1 a necessary and sufficient condition for an element $g \in X^*$ to be represented via a formula like (1.6) is given.

2. NOTATIONS AND BASIC FACTS

Throughout the paper $(\mathcal{H}, <\cdot,\cdot>)$ denotes a separable Hilbert space; X denotes a separable Banach space and X^* denotes its dual space; p and q are assumed to satisfy $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$; the canonical basis of ℓ^p $(1 < p < \infty)$ is the basis consisting of the elements (1,0,0,0,...), (0,1,0,0,...), (0,0,1,0,...),...

A sequence $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition when there exists a constant $A > 0$ such that

$$A\|f\|_X \leq \left(\sum_{i=1}^{\infty} |g_i(f)|^p\right)^{1/p}, \ \forall f \in X. \tag{2.1}$$

To such a sequence the (possibly unbounded) linear operator

$$U : \mathcal{D}(U) \subseteq X \to \ell^p, \ \ Uf := \{g_i(f)\}_{i=1}^{\infty}, \tag{2.2}$$

where $\mathcal{D}(U) = \{f \in X \mid \sum_{i=1}^{\infty} |g_i(f)|^p < \infty\}$, is associated. $\mathcal{R}(U)$ denotes the range of U.

Recall that a linear operator $U : \mathcal{D}(U) \subseteq X \to Y$, whose domain is a linear subset of a Banach space X and whose range lies in a Banach space Y, is closed if the conditions $\{x_j\} \subset \mathcal{D}(U), x_j \to x$ in X and $Ux_j \to y$ in Y when $j \to \infty$ imply $x \in \mathcal{D}(U)$ and $Ux = y$ or, equivalently, if the graph of U is closed in the product space $X \times Y$ [5, p. 57].

The following known results are needed:

Lemma 2.1 [7, p.156]. *Let E, F be linear normed spaces and $U : E \to F$ be a linear operator. Then, for $A > 0$, the inequality $\|Uf\|_F \geq A\|f\|_E$ holds for all $f \in \mathcal{D}(U)$ if and only if U has a bounded inverse $U^{-1} : \mathcal{R}(U) \to E$ for which $\|U^{-1}\| \leq \frac{1}{A}$.*

Lemma 2.2 [4]. *$\{g_i\}_{i=1}^{\infty} \subset X^*$ is a p-Bessel sequence for X with bound B if and only if*

$$T : \{d_i\}_{i=1}^{\infty} \to \sum_{i=1}^{\infty} d_i g_i$$

is a well defined (hence bounded) operator from ℓ^q into X^* and $\|T\| \leq B$.

Lemma 2.3 [6, 8]. *For every $1 \leq r < p < \infty$, ℓ^r is a linear subset of ℓ^p and $\|\{c_i\}_{i=1}^{\infty}\|_{\ell^p} \leq \|\{c_i\}_{i=1}^{\infty}\|_{\ell^r}$ for all $\{c_i\}_{i=1}^{\infty} \in \ell^r$. Furthermore, no space of the family ℓ^p, $1 \leq p < \infty$, is isomorphic to a subspace of another member of this family.*

Corollary 2.1. *For every $1 \leq r < p < \infty$, the space ℓ^r, considered as a subset of ℓ^p, is not closed in ℓ^p.*

3. CONSEQUENCES OF THE LOWER P-FRAME CONDITION IN THE GENERAL CASE

We begin with a consequence of the lower p-frame condition concerning the associated operator U, which is a generalization of a result concerning the lower frame condition in Hilbert spaces [2]:

Lemma 3.1. *Suppose that $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition* (2.1). *Then the operator U given by* (2.2) *is an injective closed operator with closed range. Furthermore, the inverse $U^{-1} : \mathcal{R}(U) \to \mathcal{D}(U)$ is bounded and $\|U^{-1}\| \leq \frac{1}{A}$.*

Proof. To prove that U is closed, consider a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathcal{D}(U)$ for which

$$x_j \to x \text{ in } X \text{ and } Ux_j \to \{c_i\}_{i=1}^{\infty} \text{ in } \ell^p \text{ when } j \to \infty.$$

Since all g_i are continuous functionals and since convergence in ℓ^p implies convergence by coordinates, the assumptions imply that for all i,

$$g_i(x_j) \to g_i(x) \text{ as } j \to \infty$$

and

$$g_i(x_j) \to c_i \text{ as } j \to \infty.$$

Thus $\{g_i(x)\}_{i=1}^{\infty} = \{c_i\}_{i=1}^{\infty}$, i.e. $x \in \mathcal{D}(U)$ and $Ux = \{c_i\}_{i=1}^{\infty}$, and hence U is closed.

To prove that U has closed range, consider again a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathcal{D}(U)$, and assume that $Ux_j \to y$ as $j \to \infty$. Thus $\{Ux_j\}_{j=1}^{\infty}$ is a Cauchy sequence, which implies by (2.1) that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence. Thus $x_j \to x$ for a certain element x of the Banach space X. Since U is closed, one can now conclude that $x \in \mathcal{D}(U)$ and $y = Ux$, i.e. y belongs to the range of U.

The rest follows by Lemma 2.1 □.

Note that when $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition and $\mathcal{D}(U) = X$, then $\{g_i\}_{i=1}^{\infty}$ is a p-frame for X. Indeed, in this case Lemma 3.1 implies the existence of a bounded inverse U^{-1} from the closed subspace $\mathcal{R}(U)$ of ℓ^p onto X, which by the Inverse Mapping Theorem implies boundedness of U, i.e. validity of the upper p-frame condition. Similarly, if $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition and $\mathcal{D}(U)$ is a closed subspace of X, then $\{g_i|_{\mathcal{D}(U)}\}_{i=1}^{\infty}$ is a p-frame for the Banach space $\mathcal{D}(U)$. Reconstruction formulas when both the lower and the upper p-frame conditions are satisfied have been studied in [4]. In this paper we are mostly interested in cases when $\{g_i\}_{i=1}^{\infty} \subset X$ satisfies the lower p-frame condition and $\mathcal{D}(U) \subsetneqq X$ is not closed in X (i.e. $\{g_i\}_{i=1}^{\infty} \subset X^*$ fails to be a p-Bessel sequence for X or for $\mathcal{D}(U)$). For examples of this kind see 4.1, 5.1 and 3.1.

When (2.1) is satisfied, the above lemma assures that the operator U given by (2.2) has a bounded inverse $U^{-1} : \mathcal{R}(U) \to \mathcal{D}(U)$. The next theorem shows that the existence of a bounded extension of U^{-1} on ℓ^p is a necessary and sufficient condition for existence of representations of the elements in $\mathcal{D}(U)$ via a q-Bessel sequence:

Theorem 3.1. *Suppose that $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition* (2.1). *Then the following are equivalent:*

(i) *there exists a q-Bessel sequence $\{f_i\}_{i=1}^{\infty} \subset X (\subseteq X^{**})$ for X^* such that*

$$f = \sum_{i=1}^{\infty} g_i(f) f_i, \quad \forall f \in \mathcal{D}(U); \tag{3.1}$$

(ii) *the operator $U^{-1} : \mathcal{R}(U) \to X$ can be extended to a linear bounded operator on ℓ^p.*

Proof. Assume (i). By Lemma 2.2, the operator $V : \{c_i\}_{i=1}^{\infty} \to \sum_{i=1}^{\infty} c_i f_i$ is a well defined linear bounded operator from ℓ^p into X. For every $f \in \mathcal{D}(U)$ we have

$$V(Uf) = \sum_{i=1}^{\infty} g_i(f) f_i = f = U^{-1} U f$$

and hence V is an extension of U^{-1}.

Assume now (ii). Let $\{e_i\}_{i=1}^{\infty}$ be the canonical basis for ℓ^p and let $f_i := V e_i$ for all i. Then, by construction, for all $f \in \mathcal{D}(U)$ we have

$$f = VUf = \sum_{i=1}^{\infty} g_i(f) f_i.$$

Now let $g \in X^*$. Considering the functional $gV \in (\ell^p)^*$, the natural isometrical isomorphism between $(\ell^p)^*$ and ℓ^q implies that the sequence $\{g(f_i)\}_{i=1}^\infty = \{gV(e_i)\}_{i=1}^\infty$ belongs to ℓ^q and

$$\left(\sum_{i=1}^{\infty}|g(f_i)|^q\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{\infty}|gV(e_i)|^q\right)^{\frac{1}{q}} = \|gV\|_{(\ell^p)^*} \leq \|V\| \cdot \|g\|_{X^*}, \ \forall g \in X^*.$$

Hence $\{f_i\}_{i=1}^\infty$, considered as a family in X^{**}, is a q-Bessel sequence for X^* □.

Note that when $\{g_i\}_{i=1}^\infty \subset X^*$ is a p-frame for X, then the above conditions (i) and (ii) are equivalent to the condition

(iii) *$\mathcal{R}(U)$ is complemented in ℓ^p*

(see [4]). When only the lower p-frame condition is assumed, (iii) implies (ii). Indeed, if P is a bounded projection from ℓ^p onto $\mathcal{R}(U)$, then, clearly, $U^{-1}P$ is a linear bounded extension of U^{-1} on ℓ^p. In special cases the inverse implication is also true. For example, if $p = 2$ and $\{g_i\}_{i=1}^\infty \subset X^*$ satisfies the lower 2-frame condition, then $\mathcal{R}(U)$ is closed in the Hilbert space ℓ^2 and hence (iii) and (ii) are satisfied; thus (i) is always valid in this case. Example 3.1 and Example 5.1 are examples of cases, when $\{g_i\}_{i=1}^\infty \subset X^*$ satisfies the lower p-frame condition for X, $\mathcal{D}(U) \subsetneqq X$ is not closed in X and (i), (ii) and (iii) are satisfied. It is still an open question whether there exists an example of a family, which satisfies the lower p-frame condition, $\mathcal{D}(U) \subsetneqq X$ is not closed in X, (i) and (ii) are satisfied, but (iii) fails.

Example 3.1. Let $1 < p < s < \infty$. Consider the Banach space $X = \ell^s$. Let $\{e_i\}_{i=1}^\infty$ be the canonical basis for ℓ^s and let $\{E_i\}_{i=1}^\infty \subset (\ell^s)^*$ be the associated coefficient functionals. By Lemma 2.3, the set $\mathcal{D}(U) = \{\{d_i\}_{i=1}^\infty \in \ell^s : \{E_j(\{d_i\}_{i=1}^\infty)\}_{j=1}^\infty \in \ell^p\}$ is actually $\ell^p \subsetneqq X$ and for all $\{d_i\}_{i=1}^\infty \in \mathcal{D}(U)$ we have $\|U(\{d_i\}_{i=1}^\infty)\|_{\ell^p} = \|\{d_i\}_{i=1}^\infty\|_{\ell^p} \geq \|\{d_i\}_{i=1}^\infty\|_{\ell^s}$; for the elements $\{d_i\}_{i=1}^\infty \in \ell^s \setminus \ell^p$ the lower p-frame inequality is clearly satisfied. By Corollary 2.1, $\mathcal{D}(U)$ is not closed in X. The range of the operator U is $\mathcal{R}(U) = \ell^p$ and thus (iii), and hence (ii) and (i) are valid.

4. A COUNTEREXAMPLE

As it was shown in the previous section, under the additional assumption on complementability of $\mathcal{R}(U)$ in ℓ^p or the existence of a bounded extension of U^{-1} on ℓ^p, the lower p-frame condition implies the existence of reconstruction formulas in $\mathcal{D}(U)$. This section concerns the question whether the same assumptions imply existence of reconstruction formulas in the whole space X or in the whole X^*. Example 4.1 below answers negative; it shows a case when there are no reconstruction formulas neither in the whole space X^* via the sequence $\{g_i\}_{i=1}^\infty \subset X^*$, satisfying the lower p-frame condition, nor in the whole space X via a dual family $\{f_i\}_{i=1}^\infty \subset X$

(a q-Bessel sequence satisfying (3.1)). The example concerns a case when X is a Hilbert space and $p = 2$ (in this case the assumption "$\mathcal{R}(U)$-complemented in ℓ^2" is automatically satisfied) and thus it shows that the answer is negative even for this special most considered case. Note that in a recent paper [2], concerning the lower frame condition in Hilbert spaces, it has been shown that the representation in (1.2) is not necessarily valid for all $f \in \mathcal{H}$; the counterexample given in [2] and the one given in the present paper are obtained independently; the counterexample given in [2] is more complicated than the one below.

Example 4.1. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space $(\mathcal{H}, <\cdot, \cdot>)$ and consider the family $\{g_i\}_{i=2}^{\infty} := \{i(e_1 + e_i)\}_{i \geq 2} \subset \mathcal{H}$. The family $\{g_i\}_{i=2}^{\infty}$ has the following properties:

(i) $\{g_i\}_{i=2}^{\infty}$ satisfies the lower frame inequality, but it is not a frame for $\mathcal{H}$;

(ii) e_1 can not be written as $\sum_{i=2}^{\infty} c_i g_i$ for any numbers $\{c_i\}_{i=2}^{\infty}$;

(iii) if $\{f_i\}_{i=2}^{\infty}$ is a Bessel sequence, satisfying (1.2), e_1 can not be written as $\sum_{i=2}^{\infty} c_i f_i$ for any numbers $\{c_i\}_{i=2}^{\infty}$.

Proof. (i) Let $x \in H$ be arbitrary fixed. If $< x, e_1 >= 0$, then

$$\begin{aligned} \sum_{i=2}^{\infty} |< x, g_i >|^2 &= \sum_{i=2}^{\infty} i^2 |< x, e_i >|^2 \geq \sum_{i=2}^{\infty} |< x, e_i >|^2 \\ &= \sum_{i=1}^{\infty} |< x, e_i >|^2 = \|x\|^2. \end{aligned}$$

Let now $< x, e_1 > \neq 0$. Since $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$,

$$\sum_{i=2}^{\infty} |< x, e_i >|^2 < \infty$$

and hence $< x, e_i > \to 0$ when $i \to \infty$. Therefore

$$\sum_{i=2}^{\infty} |< x, e_i > + < x, e_1 >|^2 = \infty, \tag{4.1}$$

because otherwise $< x, e_i >$ would converge to $(- < x, e_1 >) \neq 0$, which is a contradiction. Now (4.1) implies that

$$\sum_{i=2}^{\infty} |< x, g_i >|^2 = \sum_{i=2}^{\infty} i^2 |< x, e_i > + < x, e_1 >|^2 = \infty$$

and hence the inequality

$$\sum_{i=2}^{\infty} |< x, g_i >|^2 \geq \|x\|^2$$

is satisfied.

The fact, that $\{g_i\}_{i=2}^{\infty}$ does not satisfy the upper frame inequality follows from the equalities

$$\sum_{i=2}^{\infty} | < e_k, g_i > |^2 = k^2 = k^2 \|e_k\|, \quad \forall k \geq 2.$$

(ii) If there exist constants $c_2, c_3, c_4, ...$ such that $e_1 = \sum_{i=2}^{\infty} c_i i(e_1 + e_i)$, then the orthogonality $< e_k, e_1 >= 0$, $\forall k \geq 2$, implies that all c_i are zero, which is a contradiction.

(iii) Let now $\{f_i\}_{i=2}^{\infty}$ be a Bessel sequence, satisfying (1.2). For every $k \geq 2$, e_k belongs to $\mathcal{D}(U)$ and thus, by (1.2),

$$e_k = \sum_{i=2}^{\infty} < e_k, i(e_1 + e_i) > f_i = k f_k.$$

If we assume that $e_1 = \sum_{i=2}^{\infty} c_i f_i$ for some numbers $\{c_i\}_{i=2}^{\infty}$, this would imply that $e_1 = \sum_{i=2}^{\infty} \frac{c_i}{i} e_i$, which is a contradiction. □

5. THE LOWER P-FRAME CONDITION IN A SPECIAL CASE

Let $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition. In this section we are interested in representation of elements in the dual space X^*. In the previous section we have seen an example of a case when $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition and $\{f_i\}_{i=1}^{\infty} \subset X$ is a q-Bessel sequence satisfying (3.1), but not all g in X^* can be represented as $g = \sum_{i=1}^{\infty} g(f_i) g_i$. Here the elements $g \in X^*$ which allow such representations are investigated. We consider the special case when the given sequence $\{g_i\}_{i=1}^{\infty}$ satisfies one more assumption, namely that the domain of the associated operator U, defined by (2.2), is a dense subset of X. The following result holds true:

Theorem 5.1. *Let $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfy the lower p-frame condition, $\mathcal{D}(U)$ be dense in X and $\{f_i\}_{i=1}^{\infty} \subset X$ be a q-Bessel sequence satisfying* (3.1). *Then an element $g \in X^*$ can be represented as*

$$g = \sum_{i=1}^{\infty} g(f_i) g_i$$

if and only if

$$\text{the sequence } \left\{ \sum_{i=1}^{n} g(f_i) g_i \right\}_{n=1}^{\infty} \text{ is convergent.}$$

Proof. Fix an arbitrary $g \in X^*$.

It is only needed to prove that if $\{\sum_{i=1}^{n} g(f_i)g_i\}_{n=1}^{\infty}$ is convergent, then it converges to g. Suppose that $\sum_{i=1}^{\infty} g(f_i)g_i = \lim_{n\to\infty} \sum_{i=1}^{n} g(f_i)g_i$ exists. Denote the canonical basis of ℓ^p by $\{e_i\}_{i=1}^{\infty}$ and the canonical basis of ℓ^q by $\{z_i\}_{i=1}^{\infty}$ $(\frac{1}{p}+\frac{1}{q}=1)$. Let $V : \ell^p \to X$ be the linear bounded extension of U^{-1} defined in the proof of Theorem 3.1; then $f_i = V(e_i)$, $\forall i$. By the isometrical isomorphism of $(\ell^p)^*$ and ℓ^q, $\{gV(e_i)\}_{i=1}^{\infty} = \{g(f_i)\}_{i=1}^{\infty} \in \ell^q$ can be identified with $V^*(g) = gV \in (\ell^p)^*$ and thus

$$\sum_{i=1}^{n} g(f_i)z_i \underset{n\to\infty}{\longrightarrow} \sum_{i=1}^{\infty} g(f_i)z_i = V^*g. \tag{5.1}$$

Under the assumptions of the theorem we can consider the adjoint operator

$$U^* : \mathcal{D}(U^*) \to X^*,$$

where

$$\mathcal{D}(U^*) = \{G \in (\ell^p)^* | \text{ the functional } G \circ U \text{ is continuous on } \mathcal{D}(U)\}.$$

By definition, U^*G is the unique extension of GU to a continuous functional on X (the continuous extension is unique, because $\mathcal{D}(U)$ is assumed to be dense in X). It is not difficult to see that U^* is a densely defined closed operator. Every z_i belongs to $\mathcal{D}(U^*)$ (considered as a subset of ℓ^q) and $U^*z_i = g_i$, because $(g_i - U^*z_i)(f) = g_i(f) - g_i(f) = 0$ for all f in $\mathcal{D}(U)$, which is dense in X. Then for every $n \in \mathbb{N}$, the finite sum $\sum_{i=1}^{n} g(f_i)z_i$ belongs to $\mathcal{D}(U^*)$ and

$$U^*\left(\sum_{i=1}^{n} g(f_i)z_i\right) = \sum_{i=1}^{n} g(f_i)U^*z_i = \sum_{i=1}^{n} g(f_i)g_i \;\to\; \sum_{i=1}^{\infty} g(f_i)g_i. \tag{5.2}$$

Now (5.1), (5.2) and the closeness of U^* imply that V^*g belongs to $\mathcal{D}(U^*)$ and

$$U^*V^*g = \sum_{i=1}^{\infty} g(f_i)g_i.$$

Since $U^*V^*(g)(f) - g(f) = gVU(f) - g(f) = 0$ for all f in $\mathcal{D}(U)$, which is dense in X, one can conclude that $U^*V^*(g) = g$. Therefore $g = \sum_{i=1}^{\infty} g(f_i)g_i$. □

As a consequence of Theorem 5.1, for the Hilbert frame case we get:

Corollary 5.1. *Let $\mathcal{H}$ be a Hilbert space and assume that $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$ satisfies the lower frame condition with $\mathcal{D}(U)$ dense in $\mathcal{H}$. Let $h \in \mathcal{H}$ and $\{f_i\}_{i=1}^{\infty} \subset \mathcal{H}$ be a Bessel sequence satisfying* (1.2). *Then*

$$h = \sum_{i=1}^{\infty} < h, f_i > g_i$$

if and only if

$$\text{the sequence } \left\{ \sum_{i=1}^{n} < h, f_i > g_i \right\}_{n=1}^{\infty} \text{ is convergent.}$$

Below an example of a sequence satisfying the assumptions of Theorem 5.1 is given.

Example 5.1. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$ and let $\{g_i\}_{i=1}^{\infty} := \{ie_i\}_{i=1}^{\infty}$. Since

$$\sum_{i=1}^{\infty} | < h, g_i > |^2 = \sum_{i=1}^{\infty} i^2 | < h, g_i > |^2 \geq \sum_{i=1}^{\infty} | < h, e_i > |^2 = \|h\|_{\mathcal{H}}^2, \ \forall h \in \mathcal{H},$$

$\{g_i\}_{i=1}^{\infty}$ satisfies the lower frame condition. Clearly,

$$\mathcal{D}(U) = \{c = \sum_{i=1}^{\infty} c_i e_i \in H \ : \ \sum_{i=1}^{\infty} |ic_i|^2 < \infty\}.$$

Since $span\{e_i\} \subseteq \mathcal{D}(U)$, but $\overline{span}\{e_i\} = \mathcal{H} \not\subseteq \mathcal{D}(U)$ (for example $\sum_{i=1}^{\infty} \frac{1}{i} e_i \in \mathcal{H} \setminus \mathcal{D}(U)$), $\mathcal{D}(U)$ is dense, but not closed in $\mathcal{H}$. For every $g \in H^* = H$, the sequence $\left\{ \sum_{i=1}^{n} < g, \frac{1}{i} e_i > g_i \right\}_{n=1}^{\infty}$ converges to g and $\{\frac{1}{i} e_i\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}$.

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Department of Mathematics
University of Architecture, Civil Engineering and Geodesy
1, Christo Smirnenski blvd.
1046 Sofia
BULGARIA
E-mail: stoeva_fte@uacg.bg

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AN EXAMPLE OF ROTATIONAL HYPERSURFACE IN $\mathbb{R}^{n+1}$ WITH INDUCED IP METRIC FROM $\mathbb{R}^{n+1}$

YULIAN TSANKOV

We find a rotated hypersurface M^n whose induced metric from $\mathbb{R}^{n+1}$ is isometric to metric of IP manifolds and therefore the hypersurface is conformally flat. In the case of 4-dimensional hypersurface with IP metric we have presented explicitly a skew-symmetric curvature operator and have proved directly that its eigenvalues are pointwise. We find the mean curvature of the hypersurface.

Keywords: IP manifolds, curvature operator, rotated hypersurfaces

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Let ∇ be the Levi-Civita connection of a Riemannian manifold (M^m, g). Let x, y and z be tangent vector fields on M^m. Then the associated curvature tensor $R(x, y, z)$ is defined by

$$R(x, y, z) = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z.$$

The value of $R(x, y, z)$ at a point p of M depends only of values of x, y and z at p.

The skew-symmetric curvature operator $K_{x,y}$ is defined by

$$K_{x,y}(u) = R(x, y, u)$$

for any orthonormal pair (x, y) of tangent vectors at any point p in M and $u \in T_pM$. It is easy to see that the curvature operator $K_{x,y}$ does not depend on the orientated orthonormal basis which is chosen for the orientated 2-plane $E^2 = span\{x, y\}$ [1]. A unit vector u is an eigenvector to $K_{x,y}$ with the corresponding eigenvalues c iff

$$K_{x,y}(u) = cu,$$

where, generally, c is a function of the point p and the plane E^2, $c = c(p; E^2)$. G. Stanilov first has stated a problem for the investigation of Riemannian manifolds of pointwise constant eigenvalues of $K_{x,y}$ [5].

In [4] Ivanov and Petrova have given a local classification of four dimension manifolds, where the skew-symmetric curvature operator $K_{x,y}$ has pointwise eigenvalues:

Theorem 1. *Let (M, g) be a four dimensional Riemannian manifold such that the eigenvalues of the skew-symmetric curvature operator are pointwise constants at any point p of the manifold M. Then (M, g) is locally (almost everywhere) isometric to one of the following spaces:*

a) *real space form;*

b) *a warped product $B \times_F N$, where B is an open interval on the real line, N is a 3-dimensional space form of the constant sectional curvature K, and F is a smooth function on B given by $F(u) = \sqrt{Ku^2 + Cu + D}$, K, C, D being constants such that $C^2 - 4KD \neq 0$.*

We say that (M, g) is IP if the eigenvalues of $K_{x,y}$ depend only on the point p in M and do not depend on the plane $E^2 = span\{x, y\}$.

This result is generalized for n-dimensional manifolds for $n \geq 4$ and $n \neq 7$ by Gilkey, Leahy and Sadofsky in [3], [2].

Later, we are going to give an example of a rotational hypersurface in $\mathbb{R}^{n+1}$, whose induced metric from $\mathbb{R}^{n+1}$ is isometric of IP n-dimensional manifolds.

Every rotated hypersurface M^n in $\mathbb{R}^{n+1}$ can be represented locally by

$$\begin{cases} x^1 &= f(u^1)\sin(u^2)\sin(u^3)\dots\sin(u^n), \\ x^2 &= f(u^1)\sin(u^2)\sin(u^3)\dots\cos(u^n), \\ \vdots & \\ x^{n-1} &= f(u^1)\sin(u^2)\cos(u^3), \\ x^n &= f(u^1)\cos(u^2), \\ x^{n+1} &= h(u^1), \end{cases} \tag{1}$$

$$u^i \in J_i, \ J_i \subset \mathbb{R}^1, \quad i = 1, \dots, n.$$

We write

$$x = (x^1, x^2, \dots, x^{n-1}, x^n, x^{n+1}),$$
$$x^i = x^i(u^1, u^2, \dots, u^n), \quad i = 1, \dots, n+1.$$

Suppose that

$$h(u^1) = \varepsilon \int_{u_o^1}^{u^1} \sqrt{1 - \left(\frac{df(v)}{dv}\right)^2} dv, \quad \varepsilon = \pm 1, \tag{2}$$

or

$$\left(\frac{df(u^1)}{du^1}\right)^2 + \left(\frac{dh(u^1)}{du^1}\right)^2 = 1.$$

This means that u^1 is a natural parameter of the curve

$$c \begin{cases} x^1 & = & f(u^1), \\ x^2 & = & 0, \\ \vdots & & \\ x^n & = & 0, \\ x^{n+1} & = & h(u^1). \end{cases} \tag{3}$$

Let

$$f(u^1) = \sqrt{q(u^1)}, \ \ q(u^1) > 0, \ u^1 \in J_1. \tag{4}$$

Then we can evaluate $h(u^1)$ from (2).

Below, we will consider a rotational surface generated from the rotation of a curve that satisfies conditions (2) and (4).

Using that $g_{ij} = \dfrac{\partial x}{\partial u^i}\dfrac{\partial x}{\partial u^j}$, we can evaluate directly the components of the metric tensor g of surface M, induced from the inner product of $\mathbb{R}^{n+1}$. The matrix of g is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q(u^1) & 0 & \cdots & 0 \\ 0 & 0 & q(u^1)\sin^2(u^2) & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & q(u^1)\sin^2(u^2)\ldots\sin^2(u^{n-1})\sin^2(u^n) \end{pmatrix}. \tag{5}$$

Therefore, the metric of the rotational surface given by (1) coincides with the metric of a warped product $B^1 \times_q S^{n-1}$, where B is an open interval in $\mathbb{R}$ and S^{n-1} is the $(n-1)$-dimensional sphere. The radius of the sphere S^{n-1} is 1.

If we set $q(u^1) = (u^1)^2 + C(u^1) + D$, $C^2 - 4D < 0$, the metric of (1) will coincide with the metric of IP manifolds which are not with constant sectional curvature. *This kind of rotational surfaces we will call rotational IP hypersurfaces.*

We can check directly that for rotational IP hypersurfaces in $\mathbb{R}^5$, generated by the rotation of the curve (3) when $q(u^1) = (u^1)^2 + Cu^1 + D$, it holds that the skew-symmetric curvature operator has pointwise eigenvalues. For this purpose we are going to use a local parametrization of 4-dimensional hypersurface. Explicitly, the parametrization of this surface is

$$\begin{cases} x^1 & = & ((u^1)^2 + Cu^1 + D)\sin(u^2)\sin(u^3)\sin(u^4), \\ x^2 & = & ((u^1)^2 + Cu^1 + D)\sin(u^2)\sin(u^3)\cos(u^4), \\ x^3 & = & ((u^1)^2 + Cu^1 + D)\sin(u^2)\cos(u^3), \\ x^4 & = & ((u^1)^2 + Cu^1 + D)\cos(u^2), \\ x^5 & = & \frac{1}{2}\sqrt{4D - C^2}\ln(C + 2(u^1 + \sqrt{(u^1)^2 + Cu^1 + D})). \end{cases} \tag{6}$$

Its metric tensor g is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q(u^1) & 0 & 0 \\ 0 & 0 & q(u^1)\sin^2(u^2) & 0 \\ 0 & 0 & 0 & q(u^1)\sin^2(u^2)\sin^2(u^3) \end{pmatrix},$$

where $q(u^1) = D + Cu^1 + (u^1)^2$. The inverse matrix g^{-1} of g is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{D+Cu^1+(u^1)^2} & 0 & 0 \\ 0 & 0 & \frac{csc^2(u^2)}{D+Cu^1+(u^1)^2} & 0 \\ 0 & 0 & 0 & \frac{csc^2(u^2)csc^2(u^3)}{D+Cu^1+(u^1)^2} \end{pmatrix}.$$

When we have the metric tensor of a given manifold, we can calculate the Christofel symbols Γ^k_{ij} and the components of the curvature tensor R^l_{ijk} on the following way:

$$\Gamma^k_{ij} = \frac{1}{2} g^{hk} \left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right),$$

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial u^i} - \frac{\partial \Gamma^l_{ik}}{\partial u^j} + \Gamma^s_{jk}\Gamma^l_{si} - \Gamma^s_{ki}\Gamma^l_{sj}.$$

After some algebra we find that the sectional curvature $k_{i,j}$, $i,j = 1,\dots,4$, of two-dimensional planes given from the base vectors $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^3}, \frac{\partial}{\partial u^4}$ is

$$\begin{aligned}
k_{1,2} &= \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\
k_{1,3} &= \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\
k_{1,4} &= \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\
k_{2,3} &= -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\
k_{2,4} &= -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}, \\
k_{3,4} &= -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}.
\end{aligned}$$

Using the components of the curvature tensor, we can find the matrix of the skew-symmetric curvature operator K_{e_i,e_j}, $i \neq j$, $i,j = 1,\dots,4$, where $e_i = \dfrac{\frac{\partial}{\partial u^i}}{\sqrt{g(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^i})}}$, $i = 1,\dots,4$, relative to orthonormal base e_i. For example, the matrix of K_{e_1,e_2} is

$$K_{e_1,e_2} = \begin{pmatrix} 0 & -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2} & 0 & 0 \\ \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (7)$$

The eigenvalues λ_i, $i = 1, \dots, 4$, of this operator are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}, \quad \lambda_4 = -\frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}.$$

In a similar way, the matrix of the skew-symmetric curvature operator K_{e_1,e_3} is

$$K_{e_1,e_3} = \begin{pmatrix} 0 & 0 & -\frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (8)$$

The eigenvalues of this operator are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}, \quad \lambda_4 = -\frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}.$$

In general, let us consider two orthonormal vectors $a, b \in T_pM$, i. e.

$$a = a^i e_i, \quad b = b^i e_i,$$

$$g(a,a) = 1, \quad g(b,b) = 1, \quad g(a,b) = 0. \qquad (9)$$

Then the matrix of the skew-symmetric curvature operator $K_{u,v}$ with respect to the orthonormal base e_i is

$$K_{u,v} = k \begin{pmatrix} 0 & a^2b^1 - a^1b^2 & a^3b^1 - a^1b^3 & a^4b^1 - a^1b^4 \\ -a^2b^1 + a^1b^2 & 0 & -a^3b^2 + a^2b^3 & -a^4b^2 + a^2b^4 \\ -a^3b^1 + a^1b^3 & a^3b^2 - a^2b^3 & 0 & -a^4b^3 + a^3b^4 \\ -a^4b^1 + a^1b^4 & a^4b^2 - a^2b^4 & a^4b^3 - a^3b^4 & 0 \end{pmatrix}, \qquad (10)$$

where $k = \dfrac{C^2-4D}{4(D+Cu^1+(u^1)^2)^2}$.

The eigenvalues of the curvature operator $K_{u,v}$ are

$$\begin{aligned} &\lambda_1 = 0, \quad \lambda_2 = 0, \\ &\lambda_3 = \frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}\sqrt{A}, \quad \lambda_4 = -\frac{i(C^2-4D)}{4(D+Cu^1+(u^1)^2)^2}\sqrt{A}, \end{aligned}$$

where

$$\begin{aligned} A = \; & (a^4)^2(b^1)^2 + (a^1)^2(b^2)^2 + (a^4)^2(b^2)^2 + (a^1)^2(b^3)^2 \\ & + (a^4)^2(b^3)^2 - 2a^1a^4b^1b^4 + (a^1)^2(b^4)^2 - a^3b^3(a^1b^1 + a^4b^4) \\ & - 2a^2b^2(a^1b^1 + a^3b^3 + a^4b^4) + (a^3)^2((b^1)^2 + (b^2)^2 + (b^4)^2) \\ & + (a^2)^2((b^1)^2 + (b^3)^2 + (b^4)^2). \end{aligned}$$

Using (9), we obtain that $A = 1$. Therefore, the eigenvalues of the operator $K_{u,v}$ do not depend on the two-dimensional plane determined by the vectors u, v.

We are going to generalize the derived results in the following

Theorem 2. *The rotational 4-dimensional hypersurface given by* (6) *has pointwise eigenvalues.*

We can prove directly also the similar results in dimensions 3, 5, 6, 7.

Let us point out that the two-dimensional IP hypersurface in $\mathbb{R}^3$

$$\begin{cases} x^1 &= ((u^1)^2 + Cu^1 + D)\sin(u^2), \\ x^2 &= ((u^1)^2 + Cu^1 + D)\cos(u^2), \\ x^3 &= \frac{1}{2}\sqrt{4D - C^2}\ln(C + 2(u^1 + \sqrt{(u^1)^2 + Cu^1 + D})) \end{cases}$$

has a vanishing mean curvature, i. e. this is a minimal surface in $\mathbb{R}^3$.

But we prove directly that when we have k-dimensional IP rotational surfaces for $n = 3, 4, 5, 6, 7$, they are not minimal. More exactly, the mean curvature H of n-dimensional IP rotational surfaces is:

$$H = (n-2)\frac{\sqrt{-C^2 + 4D}}{2((u^1)^2 + Cu^1 + D)}.$$

We remark that the IP rotational hypersurfaces are conformally flat as rotational hypersurfaces. We can also see this from (5) or, using the local parametrization of the IP rotational surfaces, we can directly evaluate the components of the Weyl tensor W_{ijkl} and find that $W_{ijkl} = 0$.

We have used computer for some of the evaluations above.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: ucankov@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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ON INFINITESIMAL RIGIDITY OF HYPERSURFACES [1] IN EUCLIDEAN SPACE

IVANKA IVANOVA-KARATOPRAKLIEVA

The infinitesimal rigidity of hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 3$, is considered. In section 1 we remind some definitions in the theory of the infinitesimal bendings (inf. b.). In section 2 we discuss the results in the papers [5 - 9]. In section 3 we consider our main result in the paper [10] and we give some geometric interpretations of the investigations in [10]. Finally we consider an example.

Keywords: hypersurfaces, Euclidian space, infinitesimal rigidity

2000 MSC: 53C24

1. INTRODUCTION

The theory of the bendings of the surfaces is one of the most important sections of the classical differential geometry. The first definitions of the notion bending of the surfaces are in some 19th century works and concern only 2-dimensional surfaces in $\mathbb{R}^3$ (3-dimensional euclidean space). In these works the difference between bending and infinitesimal bending was not made. First Darboux in the end of the 19th century pointed out the difference between these two notions. The first results of the infinitesimal bendings (inf. b.) of the surfaces in $\mathbb{R}^3$ belong to Cauchy (1813) - for a closed convex polyhedron and to Liebmann (1901, 1919) - for an analytic convex surface. During the 20th century too many results on the inf. b. of the

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surfaces in $\mathbb{R}^{n+1}$ have been obtained (see [1 – 4]). In this paper we shall discuss the inf. b. of the hypersurfaces in $\mathbb{R}^{n+1}, n \geq 3$. First we shall give some definitions.

Let $S : r = r(u^1, ..., u^n)$ be a smooth hypersurface in $\mathbb{R}^{n+1}$ and let

$$S_t : r_t = r + 2tU + o(t), \quad t \longrightarrow 0\,,$$

be an infinitesimal deformation of S. Let c be an arbitrary smooth curve on S and let c_t be its corresponding curve on S_t. We denote the lengths of c and c_t with l and l_t correspondingly. The infinitesimal deformation S_t is called an inf. b. of S if the equality

$$l_t - l = o(t), \quad t \longrightarrow 0, \tag{1.1}$$

is true, i.e. the vector field U of inf. b. satisfies the equation

$$dr\, dU = 0. \tag{1.2}$$

The field U of inf. b. is called trivial if it has the form

$$U = \Omega r + \omega, \tag{1.3}$$

where Ω is a constant skew-symmetric matrix and ω is a constant vector. If the equation (1.2) has, under some conditions, only a trivial solution, i.e. the vector field U is of the form (1.3), then the hypersurface S is called infinitesimally rigid under these conditions.

2. THE RESULTS IN THE PAPERS [5]-[9]

There are 6 known papers ([5] - [10]) on inf. rigidity of hypersurfaces in $R^{n+1}, n \geq 3$, in the literature. The first results concerning inf. rigidity of hypersurfaces belong to Sen'kin [5]. He investigated inf.b. of general convex hypersurfaces, i.e. the convex hypersurfaces for which smoothness was not assumed. The theory of inf. b. of such surfaces in R^3 was developed by A. D. Alexandrov (1936). Sen'kin used this theory and the results of A. V. Pogorelov (1959) for inf. b. of general convex surfaces in $\mathbb{R}^3$ and proved ([5]) the following

Theorem 2.1 (Sen'kın, 1972). *A closed convex hypersurface S in $\mathbb{R}^{n+1}, n \geq 3$, which does not contain flat n-dimensional domains is infinitesimally rigid. If S contains n-dimensional domains it is inf. rigid outside of these domains.*

Theorem 2.2 (Sen'kin, 1975). *A closed convex hypersurface S in $\mathbb{R}^{n+1}, n \geq 3$, which does not contain flat n-dimensional domains is inf. rigid in neighborhood of each point, which does not lie in a flat $(n-1)$-dimensional and $(n-2)$-dimensional domain. If S contains flat n-dimensional domains, it is inf. rigid in neighborhood of the indicated points outside of the flat n-dimensional domains.*

In 1975 Goldstein and Ryan, using the theory of the conformal vector fields, proved ([6]) that the sphere S^n in $\mathbb{R}^{n+1}, n \geq 3$, is inf. rigid and in 1980 Nannicini proved ([7]) that a C^∞ smooth compact strictly convex hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, is inf. rigid. It is obvious that these two results are contained in Theorem 2.1 of Sen'kin.

In [7] Nannicini proved the following

Theorem 2.3 (Nannicini, 1980). *Let $\widetilde{S}$ be a $(n-1)$-dimensional C^∞ smooth compact strictly convex surface which lie on a hyperplane $\mathbb{R}^n \subset \mathbb{R}^{n+1}, n \geq 3$. Let $\mu = \mathbb{R}^{n-1}$ be a subspace of $\mathbb{R}^n$ and $\widetilde{S} \cap \mu = \varnothing$. The rotation hypersurface $S = \widetilde{S} \times S^1$ in $\mathbb{R}^{n+1}$ obtained by rotation of $\widetilde{S}$ around μ is inf. rigid.*

In [8] Markov proved the following

Theorem 2.4 (Markov, 1980). *Let S be a C^3 smooth hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, with type number $\tau \geq 3$, i.e. S has at least 3 nonzero principal curvatures at each point. Then each neighborhood on S of every point of S is inf. rigid.*

This result of Markov is infinitesimal analog of the well known classical result of Beez (1876) and Killing (1885) for isometric rigidity of hypersurface S in $\mathbb{R}^{n+1}, n \geq 3$. The surface in theorem 4 can be compact or noncompact.

In [9] Dajczer and Rodrigues prove the following

Theorem 2.5 (Daiczer, Rodrigues, 1990). *Every smooth compact hypersurphace in $\mathbb{R}^{n+1}, n \geq 3$, which does not contain flat n-dimensional domains is infinitesimally rigid.*

This result is an infinitesimal version of a very beautiful result of Sacksteder (1960) for isometric rigidity.

3. THE MAIN RESULT IN THE PAPER [10] AND SOME GEOMETRIC INTERPRETATIONS

In the paper [10] we obtain sufficient conditions for inf. rigidity of a class of hypersurfaces with boundary in $\mathbb{R}^{n+1}, n \geq 3$, which are projected one-to-one orthogonally on a region G in a hyperplane. Such a hypersurface S is represented by:

$$S : x^{n+1} = f(x), x = (x^1, ..., x^n) \in G \tag{3.1}$$

We assume that G is a bounded finitely connected region with piecewise smooth boundary $\partial G = \Gamma$ and the function $f(x)$ and the field $U(x)$ $(\xi^1(x), \ldots, \xi^n(x), \zeta(x))$ of the inf. b. of S belong to the class $C^3(\overline{G})$. We assume that the inequalities

$$f_{\beta\beta\beta} > 0, f_{\beta\beta\beta}f_{\alpha\alpha\beta} - f^2_{\alpha\beta\beta} > 0 \ \text{(respectively } f_{\beta\beta\beta} < 0, f_{\beta\beta\beta}f_{\alpha\alpha\beta} - f^2_{\alpha\beta\beta} > 0) \tag{3.2}$$

are fulfilled on a set $\widetilde{G}_{\alpha\beta}$ everywhere dense in $\overline{G} = G \cup \partial G$, $\beta = \alpha + 1$, $\alpha = 1, 3, ..., n-3, n-1$ for n even and $\alpha = 1, 3, ..., n-2, n-1$ for n odd. Here we denote with $f_\beta, f_{\beta\alpha}, f_{\beta\beta\beta}, ...$ the partial derivatives $f_{x^\beta}, f_{x^\beta x^\alpha}, f_{x^\beta x^\beta x^\beta}, ...$. Further our presentation will be for n even - when n is odd the things are analogous.

We shall give a geometric interpretation of the inequalities (3.2). Let $P(a^1, a^2, ..., a^n)$ be an arbitrary point of $\overline{G}$. We consider, for fixed $\alpha \in \{1, 3, ..., n-3, n-1\}$ and $\beta = \alpha + 1$, the 2-dimensional surface $S^{\alpha\beta} = S \cap \mathbb{R}^3_{\alpha\beta}$, where $\mathbb{R}^3_{\alpha\beta}$ is the 3-dimensional plane which contains P and is parallel to the coordinate 3-dimentional plane $O_{e_\alpha e_\beta e_{n+1}}$. The surface $S^{\alpha\beta}$ has the representation

$$S^{\alpha\beta} : x^{n+1} = f(a^1, ..., a^{\alpha-1}, x^\alpha, x^\beta, a^{\alpha+2}, ..., a^n), \ (x^\alpha, x^\beta) \in \overline{G}^P_{\alpha\beta} = \overline{G} \cap \mathbb{R}^3_{\alpha\beta} \quad (3.3)$$

with respect to the coordinate system $O'_{e_\alpha e_\beta e_{n+1}}$, $O'(a^1, ..., a^{\alpha-1}, 0, 0, a^{\alpha+2}, ..., a^n, 0)$. The following statement is valid

Proposition 3.1. *Let S be of the class* (3.1), (3.2). *Then the surfaces*

$$S^{\alpha\beta}_\beta : x^{n+1} = f_\beta(a^1, ..., a^{\alpha-1}, x^\alpha, x^\beta, a^{\alpha+2}, ..., a^n), \alpha \in \{1, 3, ..., n-3, n-1\}, \beta = \alpha+1, \quad (3.4)$$

have Gaussian curvature $K > 0$ on $\pi^{-1}(\widetilde{G}^P_{\alpha\beta}) = \pi^{-1}(\widetilde{G} \cap \mathbb{R}^3_{\alpha\beta})$ [2] *and they are convex (correspondingly locally convex) if $G^P_{\alpha\beta} = G \cap \mathbb{R}^3_{\alpha\beta}$ is convex (correspondingly nonconvex).*

Proof. From (3.2) for the sign of the Gaussian curvature K of $S^{\alpha\beta}_\beta$ on $\pi^{-1}(\widetilde{G}^P_{\alpha\beta})$ we have

$$\operatorname{sgn} K = \operatorname{sgn}(f_{\beta\beta\beta} f_{\beta\alpha\alpha} - f^2_{\beta\alpha\beta}) > 0. \quad (3.5)$$

Let $G^P_{\alpha\beta} = G \cap \mathbb{R}^3_{\alpha\beta}$ be convex. For fixed $\alpha \in \{1, 3, ..., n-3, n-1\}$ and $\beta = \alpha + 1$ we consider the quadratic form

$$C(\xi_\alpha, \xi_\beta) = f_{\beta\alpha\alpha}\xi^2_\alpha + 2 f_{\beta\alpha\beta}\xi_\beta\xi_\alpha + f_{\beta\beta\beta}\xi^2_\beta \quad (3.6)$$

of the function $f_\beta(a^1, ..., a^{\alpha-1}, x^\alpha, x^\beta, a^{\alpha+2}, ..., a^n)$. From the inequalities (3.2) it follows that quadratic form (3.6) is nonnegative (nonpositive) in $\overline{G}^P_{\alpha\beta}$. Hence the function $f_\beta(a^1, ..., a^{\alpha-1}, x^\alpha, x^\beta, a^{\alpha+2}, ..., a^n)$ is convex. Therefore the surface $S^{\alpha\beta}_\beta$ is convex.

Let $G^P_{\alpha\beta} = G \cap \mathbb{R}^3_{\alpha\beta}$ be nonconvex, where $\alpha \in \{1, 3, ..., n-3, n-1\}$ is fixed, $\beta = \alpha + 1$. For every point of $G^P_{\alpha\beta}$ we take a convex neighborhood and repeating the above reasonings we obtain that the surface $S^{\alpha\beta}_\beta$ is locally convex.

Let us consider the curve $c^{\alpha\beta}_\beta = S^{\alpha\beta}_\beta \cap \mathbb{R}^2_\beta$, where $\mathbb{R}^2_\beta$ is 2-dimensional plane across an arbitrary point of $\widetilde{G}^P_{\alpha\beta}$ and is parallel to $O_{e_\beta e_{n+1}}$. Let $\nu_{c^{\alpha\beta}_\beta}$ be the normal

[2] We denote with π the orthogonal projection $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n = O_{e_1, ..., e_n}$.

curvature of $c_\beta^{\alpha\beta}$ relative to the unit normal vector $\bar{l}$ to $S_\beta^{\alpha\beta}$ for which $(\bar{l}, e_{n+1})_e < \frac{\pi}{2}$. We have

$$\operatorname{sgn}(f_{\beta\beta\beta}) = \operatorname{sgn}(\nu_{c_\beta^{\alpha\beta}}) \tag{3.7}$$

on $\pi^{-1}(\widetilde{G}^P_{\alpha\beta} \bigcap \mathbb{R}^2_\beta)$. Then from (3.2), (3.5) and (3.7)we obtain

Corollary 3.1. *If S is of the class* (3.1), (3.2_1) *(correspondingly* (3.1), (3.2_2), *then the surfaces $S_\beta^{\alpha\beta}, \alpha = 1, 2, \ldots, n-3, n-1$, $\beta = \alpha + 1$ are convex below (correspondingly above).*

Let $\widetilde{n} = (\widetilde{n}^1, \ldots, \widetilde{n}^n)$ be the unit vector of the exterior normal to $\partial G = \Gamma$. Then $\widetilde{n}^\gamma = \cos\theta^\gamma$, where $\theta^\gamma = (e_\gamma, \widetilde{n})_e$ and e_γ is the unit vector of the axis $Ox^\gamma, \gamma = 1, ..., n$. We decompose ([10]) the smooth parts of Γ (for fixed $\alpha \in \{1, 3, ..., n-3, n-1\}$ and $\beta = \alpha + 1$) in nonintersecting subsets $\Gamma_i^{\alpha\beta}, i = 1, 2, 3, 4$, as follows:

1) on $\Gamma_1^{\alpha\beta}$: $H^{\alpha\beta}\widetilde{n}^\beta \geq 0$, $f_{\beta\beta}\widetilde{n}^\beta > 0$ (respectively $H^{\alpha\beta}\widetilde{n}^\beta \leq 0, f_{\beta\beta}\widetilde{n}^\beta < 0$);

2) on $\Gamma_2^{\alpha\beta}$: $\begin{cases} a)\, H^{\alpha\beta}\widetilde{n}^\beta < 0, \quad f_{\beta\beta}\widetilde{n}^\beta \leq 0 \\ \text{(respectively } H^{\alpha\beta}\widetilde{n}^\beta > 0, \quad f_{\beta\beta}\widetilde{n}^\beta \geq 0) \text{ or} \\ b)\, \widetilde{n}^\beta = 0, \quad f_{\beta\beta} \neq 0 \quad \text{or} \quad \widetilde{n}^\beta = 0, \quad f_{\beta\beta} = 0, f_{\alpha\beta}\widetilde{n}^\alpha > 0 \\ \text{(respectively } \widetilde{n}^\beta = 0, \quad f_{\beta\beta} \neq 0 \text{ or } \widetilde{n}^\beta = 0, \quad f_{\beta\beta} = 0, f_{\alpha\beta}\widetilde{n}^\alpha < 0); \end{cases}$

3) on $\Gamma_3^{\alpha\beta}$: $H^{\alpha\beta}\widetilde{n}^\beta < 0$, $f_{\beta\beta}\widetilde{n}^\beta > 0$ (respectively $H^{\alpha\beta}\widetilde{n}^\beta > 0, f_{\beta\beta}\widetilde{n}^\beta < 0$);

4) on $\Gamma_4^{\alpha\beta}$: $\begin{cases} a)\, \widetilde{n}^\beta \neq 0, H^{\alpha\beta}\widetilde{n}^\beta \geq 0 \quad f_{\beta\beta}\widetilde{n}^\beta \leq 0 \\ \text{(respectively } \widetilde{n}^\beta \neq 0, H^{\alpha\beta}\widetilde{n}^\beta \leq 0 \quad f_{\beta\beta}\widetilde{n}^\beta \geq 0) \text{ or} \\ b)\, \widetilde{n}^\beta = 0, \quad f_{\beta\beta} = 0, f_{\alpha\beta}\widetilde{n}^\alpha \leq 0 \\ \text{(respectively } \widetilde{n}^\beta = 0, \quad f_{\beta\beta} = 0, f_{\alpha\beta}\widetilde{n}^\alpha \geq 0); \end{cases}$

where $H^{\alpha\beta} = f_{\beta\beta}(\widetilde{n}^\alpha)^2 - 2f_{\alpha\beta}\widetilde{n}^\alpha\widetilde{n}^\beta + f_{\alpha\alpha}(\widetilde{n}^\beta)^2$.

The decomposition of $\Gamma = \partial G$ induces corresponding decomposition of the boundary $\partial S = \pi^{-1}(\partial G)$ in nonintersecting parts ${}^S\Gamma_i^{\alpha\beta} = \pi^{-1}(\Gamma_i^{\alpha\beta}), i = 1, 2, 3, 4$, which depends only on the geometric properties of ∂S. Indeed, a) $\widetilde{n}^\beta = \cos\theta^\beta, \theta^\beta = (e_\beta, \widetilde{n})_e, \widetilde{n}^\alpha = \cos\theta^\alpha, \theta^\alpha = (e_\alpha, \widetilde{n})_e$; b) $\operatorname{sgn} H^{\alpha\beta} = \operatorname{sgn} \nu_{\widehat{l}}^{\alpha\beta}$, where $\nu_{\widehat{l}}^{\alpha\beta}$ is the normal curvature to the curve $c^{\alpha\beta} = \partial S \cap \mathbb{R}^3_{\alpha\beta}$ relative to unit normal vector $\widehat{l}$ to the surface S for which $(\widehat{l}, e_{n+1})_e < \frac{\pi}{2}$ ($c^{\alpha\beta}$ has equations $x^i = c^i = const$, $i = 1, ..., n+1$, $i \neq \alpha, \beta$, $x^\alpha = x^\alpha(s)$, $x^\beta = x^\beta(s)$, $\dot{c}^{\alpha\beta}(0, ..., 0, \dot{x}^\alpha(s), \dot{x}^\beta(s), , 0, ..., 0)$ and $\dot{x}^\alpha = \sigma\widetilde{n}^\beta$, $\dot{x}^\beta = -\sigma\widetilde{n}^\alpha$, $\sigma \neq 0$, since $\widetilde{n} \perp \underline{\dot{c}}^{\alpha\beta}$, where $\underline{c}^{\alpha\beta} = \pi(c^{\alpha\beta})$); c)$f_{\beta\beta}$ is the normal curvature of the x^β-line on S relative to $\widehat{l}$ and $f_{\alpha\beta}$ is the polar form of the second fundamental form relative to $\widehat{l}$ for x^α-line and x^β-line of S in the points of ∂S.

Let $L \subset S$ be a surface of dimension k, $1 \leq k \leq n-1$. Every inf. b. S_t of S with a field U which satisfies the condition

$$Ue_{n+1}|_L = 0 \quad \text{(respectively } \ Ue_{n+1}|_L = const) \tag{3.8}$$

is called inf. b. with sliding (respectively generalized sliding) along L with respect to $\gamma = Ox^1...x^n$. Let L_1 be the orthogonal projection of L on γ i.e. $L_1 = \pi(L)$.

We shall call x^β - inf. b. along L with respect to γ every inf. b. S_t of S for which the field U of inf. b. satisfy the condition

$$(Ue_{n+1})_\beta|_{L_1} = 0. \tag{3.9}$$

We denote

$$^S\Gamma_{13} = \bigcup_{\substack{\alpha=1,3,\ldots,n-3,n-1\\ \beta=\alpha+1}} ({}^S\Gamma_1^{\alpha\beta} \cup {}^S\Gamma_3^{\alpha\beta}),$$

$$^S\Gamma_4 = \bigcap_{\substack{\alpha=1,3,\ldots,n-3,n-1,\\ \beta=\alpha+1}} {}^S\Gamma_4^{\alpha\beta},$$

$$^S\Gamma_1 = \bigcup_{\substack{\alpha=1,3,\ldots,n-3,n-1,\\ \beta=\alpha+1}} {}^S\Gamma_1^{\alpha\beta}.$$

We proved in [10] the following

Theorem 3.1. *The hypersurface* (3.1), (3.2) *is rigid under inf. b. with sliding (or generalized sliding) along* $^S\Gamma_{13}$ *with respect to the hyperplane* γ *and* x^β *- inf. b. along* $^S\Gamma_2^{\alpha\beta} \cup {}^S\Gamma_3^{\alpha\beta}, \alpha = 1, 3, ..., n-3, n-1,\ \beta = \alpha+1$, *with respect to* γ.

Remark. There are ([10]) $m \leq \frac{n}{2}$ conditions on the field U of the inf. b. at every point $P \in \partial S, P \notin {}^S\Gamma_4$, since there are $\frac{n}{2}$ decompositions of the boundary ∂S. Certainly we assume that these conditions are consistent.

We denote with $^S\Gamma_{21}^{\alpha\beta}$ for $\alpha \in \{1, 3, \ldots, n-3, n-1\}$ and $\beta = \alpha + 1$ this part of $^S\Gamma_2^{\alpha\beta}$, whose orthogonal project on the hyperplane $\gamma = 0e_1 \ldots e_n$ is composed of $(n-1)$-dimensional planes parallel to the coordinate vectors e_β, $\beta = 2, 4, \ldots, n$ or it is composed of $(n-1)$-dimensional ruled surfaces, whose generatrixes are parallel to l_β, $\beta = 2, 4, \ldots, n$. Let

$$^S\Gamma_{21} = \bigcup_{\substack{\alpha=1,3,\ldots,n-3,n-1,\\ \beta=\alpha+1}} {}^S\Gamma_{21}^{\alpha\beta}.$$

Then we have

Corollary 3.2. *The hypersurface* (3.1), (3.2), *which has a boundary* $\partial S = {}^S\Gamma_1 \cup {}^S\Gamma_{21} \cup {}^S\Gamma_4$ *is rigid under inf.b. with sliding (or generalized sliding) along* $\partial S \backslash {}^S\Gamma_4$ *with respect to the hyperplane* γ.

Corollary 3.3. *The hypersurface* (3.1), (3.2), *which has a boundary* $\partial S = {}^S\Gamma_1 \cup {}^S\Gamma_{21} \cup {}^S\Gamma_4$ *is rigid if the part* $^S\Gamma_1 \cup {}^S\Gamma_{21}$ *of* ∂S *is fixed.*

4. AN EXAMPLE

The hypersurface

$$S : x^{n+1} = \sum_{\alpha=1,3,\dots,n-3,n-1} [(x^{\alpha+1})^3 + x^{\alpha+1}(x^{\alpha})^2] + (x^1)^2,\ x = (x^1, \dots, x^n) \in G$$

is of the class (3.1), (3.2) since

$$f_{\beta\beta\beta} = 6, f_{\beta\beta\beta} f_{\beta\alpha\alpha} - f^2_{\beta\beta\alpha} = 12,\ \alpha = 1, 3, \dots, n-3, n-1,\ \beta = \alpha + 1.$$

It is not from Beez-Killing's class. Its type number τ for example at the point $O(0, ..., 0)$ is 1.

Let $n = 4$ and G be a 4-dimensional cube, i.e.

$$S : x^5 = (x^2)^3 + (x^4)^3 + x^2(x^1)^2 + x^4(x^3)^2 + (x^1)^2, \tag{4.1}$$

$$G = \{(x^1, ..., x^4) \in \mathbb{R}^4 : -1 \le x^i \le 1,\ i = 1, ..., 4\}.$$

We have $U(\xi^1(x), \dots, \xi^4(x), \zeta(x)),\ x = (x^1, \dots, x^4)$, and:

(a) $f_{22} = 6x^2, f_{12} = 2x^1, f_{44} = 6x^4, f_{34} = 2x^3$;

(b) $H^{12} = 6x^2(\widetilde{n}^1)^2 - 4x^1\widetilde{n}^1\widetilde{n}^2 + (2x^2 + 2)(\widetilde{n}^2)^2$,
$H^{34} = 6x^4(\widetilde{n}^3)^2 - 4x^3\widetilde{n}^3\widetilde{n}^4 + 2x^4(\widetilde{n}^4)^2$;

(c) $\partial G = \sum_{i=1}^{4} \partial G_i^{\pm}$,
$\partial G_1^{\pm} : x^1 = \pm 1, -1 \le x^2, x^3, x^4 \le 1$,
$\partial G_2^{\pm} : x^2 = \pm 1, -1 \le x^1, x^3, x^4 \le 1$,
$\partial G_3^{\pm} : x^3 = \pm 1, -1 \le x^1, x^2, x^4 \le 1$,
$\partial G_4^{\pm} : x^4 = \pm 1, -1 \le x^1, x^2, x^3 \le 1$;

(d) $\widetilde{n}|_{\partial G_1^{\pm}}(\pm 1, 0, 0, 0)$, $\widetilde{n}|_{\partial G_2^{\pm}}(0, \pm 1, 0, 0)$,
$\widetilde{n}|_{\partial G_3^{\pm}}(0, 0, \pm 1, 0)$, $\widetilde{n}|_{\partial G_4^{\pm}}(0, 0, 0, \pm 1)$.

From 1) - 4) and (a) - (d) we find

$$\partial G_1^{\pm} \subset \Gamma_2^{12},\ \ \partial G_1^{\pm} \backslash \theta_1^{\pm} \subset \Gamma_2^{34},\ \ \theta_1^{\pm} \subset \Gamma_4^{34},\ \ \theta_1^{\pm} : \begin{cases} x^1 = \pm 1, \\ x^4 = 0, \\ -1 \le x^2,\ x^3 \le 1; \end{cases} \tag{4.2}$$

$$\partial G_2^{\pm} \subset \Gamma_1^{12},\ \ \partial G_2^{\pm} \backslash \theta_2^{\pm} \subset \Gamma_2^{34},\ \ \theta_2^{\pm} \subset \Gamma_4^{34},\ \ \theta_2^{\pm} : \begin{cases} x^2 = \pm 1, \\ x^4 = 0, \\ -1 \le x^1,\ x^3 \le 1; \end{cases} \tag{4.3}$$

$$\partial G_3^{\pm} \backslash \theta_3^{\pm} \subset \Gamma_2^{12},\ \ \theta_3^{\pm} \subset \Gamma_4^{12},\ \ \theta_3^{\pm} : \begin{cases} x^3 = \pm 1, \\ x^2 = 0, \\ -1 \le x^1,\ x^4 \le 1 \end{cases},\ \ \partial G_3^{\pm} \subset \Gamma_2^{34}; \tag{4.4}$$

$$\partial G_4^{\pm} \backslash \theta_4^{\pm} \subset \Gamma_2^{12},\ \ \theta_4^{\pm} \subset \Gamma_4^{12},\ \ \theta_4^{\pm} : \begin{cases} x^4 = \pm 1, \\ x^2 = 0, \\ -1 \le x^1,\ x^3 \le 1 \end{cases},\ \ \partial G_4^{\pm} \subset \Gamma_1^{34}; \tag{4.5}$$

We note that:

a_1) if $\zeta|_{\partial G_1^\pm} = const$ then $\zeta_2 = \zeta_4 = 0$ since $\partial G_1^\pm || Ox^2, Ox^4$;

b_1) if $\zeta|_{\partial G_2^\pm} = const$ then $\zeta_4 = 0$ since $\partial G_2^\pm || Ox^4$;

c_1) if $\zeta|_{\partial G_3^\pm} = const$ then $\zeta_2 = \zeta_4 = 0$ since $\partial G_3^\pm || Ox^2, Ox^4$;

d_1) if $\zeta|_{\partial G_4^\pm} = const$ then $\zeta_2 = 0$ since $\partial G_4^\pm || Ox^2$;

From Theorem 3.1, (4.2) - (4.5) and a_1) - d_1) we obtain

Proposition 4.1. *The hypersurface S (4.1) in $\mathbb{R}^5$ is rigid under inf. b. with sliding (or generalized sliding)along its boundary ∂S with respect to the hyperplane $\gamma = Ox^1x^2x^3x^4$.*

Proposition 4.2. *The hypersurface S (4.1) in $\mathbb{R}^5$ is rigid if its boundary ∂S is fixed.*

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: karatopraklieva@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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A DOLBEAULT ISOMORPHISM FOR COMPLETE INTERSECTIONS IN INFINITE-DIMENSIONAL PROJECTIVE SPACE

BORIS KOTZEV

We consider a complex submanifold X of finite codimension in an infinite-dimensional complex projective space $\boldsymbol{P}$ and a holomorphic vector bundle E over X. Given a covering $\mathcal{U}$ of X with Zariski open sets, we define a subcomplex $\mathcal{C}(X, E)$ of the Čech complex corresponding to the vector bundle E and the covering $\mathcal{U}$. For a special class of coverings $\mathcal{U}$, we prove that the complex $\mathcal{C}(X, E)$ is acyclic when X is a complete intersection and $\boldsymbol{P}$ admits smooth partitions of unity.

Keywords: infinite-dimensional complex manifolds, projective manifolds, vanishing theorems

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1. INTRODUCTION

In finite dimensions the Čech cohomology groups and the Dolbeault cohomology groups of a vector bundle over a complex manifold are the same, by the Dolbeault isomorphism. When we try to extend the Dolbeault isomorphism to complex manifolds modeled on infinite-dimensional complex Banach spaces, we encounter a serious obstacle: the existence of Banach spaces for which the Dolbeault lemma about the local solvability of the $\overline{\partial}$-equation is no longer true (see [7]). In this paper we offer a way to overcome this obstacle for a projective space $\boldsymbol{P}(V)$ where V is an arbitrary Banach space. Given a holomorphic vector bundle $E \to \boldsymbol{P}(V)$ and a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $\boldsymbol{P}(V)$ with Zariski open sets, we define a subcomplex

$\mathcal{C}(\boldsymbol{P}(V), E)$ of the Čech complex corresponding to E and $\mathcal{U}$. We show in Theorem 5.1 that if $\dim \boldsymbol{P}(V) = \infty$ and $\boldsymbol{P}(V)$ admits smooth partitions of unity, then the cohomology groups $H^q(\mathcal{C}(\boldsymbol{P}(V), E))$, $q \geq 0$, of $\mathcal{C}(\boldsymbol{P}(V), E)$ are isomorphic to the Dolbeault cohomology groups $H^{0,q}(\boldsymbol{P}(V), E)$, $q \geq 0$, of E. Since the groups $H^{0,q}(\boldsymbol{P}(V), E)$ vanish for $q \geq 1$ ([4, Theorem 7.3]), we obtain a vanishing theorem for the higher cohomology groups of the complex $\mathcal{C}(\boldsymbol{P}(V), E)$.

The definition of the complex $\mathcal{C}(\boldsymbol{P}(V), E)$ carries over without modifications to submanifolds of finite codimension in $\boldsymbol{P}(V)$ - given a holomorphic vector bundle E over a submanifold X of finite codimension in $\boldsymbol{P}(V)$ and a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X with Zariski open sets - we define a subcomplex $\mathcal{C}(X, E)$ of the Čech complex corresponding to E and $\mathcal{U}$. We show in Section 6 that if X is a complete intersection (e.g. hypersurface) in $\boldsymbol{P}(V)$ and $\mathcal{U}$ is a suitable covering of X, then the complex $\mathcal{C}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, has an acyclic resolution of finite length. This allows us to prove the vanishing Theorem 6.5: If X is a complete intersection in $\boldsymbol{P}(V)$, $\dim \boldsymbol{P}(V) = \infty$ and $\boldsymbol{P}(V)$ admits smooth partitions of unity, then the higher cohomology groups of the complex $\mathcal{C}(X, E)$ vanish. This vanishing theorem is used in [3] to prove that $H^{0,1}(X, \mathcal{O}_X(n)) = 0$, $n \in \mathbb{Z}$, when X is a complete intersection in an infinite-dimensional complex projective space $\boldsymbol{P}(V)$ which admits smooth partitions of unity.

Let us describe briefly the contents of the paper.

In the book [8] J.-P. Ramis has extended Chow's lemma to all projective spaces modeled on complex Banach spaces. He has proved that if X is a closed analytic set of finite codimension in $\boldsymbol{P}(V)$ for which there exists a fixed number N such that for any $x \in X$ there is a neighbourhood U of x in which $X \cap U$ is the set of common zeros of N holomorphic functions on U, then X is an algebraic set of finite codimension in $\boldsymbol{P}(V)$ [8, Théorème III.2.3.1]. Hence every submanifold of finite codimension n in $\boldsymbol{P}(V)$ is the set of common zeros of a finite number of homogeneous polynomials on V. Since almost all proofs in this paper rest on the algebraic nature of the submanifolds of finite codimension in $\boldsymbol{P}(V)$, Sections 2 and 3 are devoted to the study of infinite-dimensional affine and projective algebraic sets. The results presented in them are well known in finite dimensions but since there was not a suitable reference at hand, it was necessary to give detailed proofs. Our approach is heavily influenced by the book [8] which contains a similar treatment of infinite-dimensional analytic sets.

In Section 4 we consider a finite holomorphic covering $\pi : Y \to Z$ between complex manifolds along with a holomorphic line bundle $L \to Z$ and show that in certain circumstances differential forms on Y with values in $\pi^* L$ can be represented in terms of differential forms on X with values in L. A special case of this representation is used immediately in the proof of Proposition 4.6 which plays important role in Section 6. The general case of Propositions 4.2 and 4.3 is used in [3].

In Section 5 we define the complex $\mathcal{C}(X, E)$ and prove that it is acyclic when $X = \boldsymbol{P}(V)$ admits smooth partitions of unity, and E is a finite rank holomorphic vector bundle over $\boldsymbol{P}(V)$.

In Section 6 we prove the main result of the paper by making use of the Koszul complex in order to construct an acyclic resolution of $\mathcal{C}(X, E)$ when X is a complete intersection in $\boldsymbol{P}(V)$.

This paper is based on the author's Ph.D. thesis (Purdue University, 2001).

2. AFFINE ALGEBRAIC SETS IN BANACH SPACES

Let V be a complex Banach space. A subset $X \subset V$ is an *analytic set of finite codimension* in V, if for any $x \in X$ there exist a neighbourhood U and a finite number of holomorphic functions $\varphi_1, \dots, \varphi_s \in \mathcal{O}(U)$ such that $X \cap U = Z(\varphi_1, \dots, \varphi_s)$. For any open set $U \subset V$ we denote by $\mathcal{I}(X)(U)$ the set of all holomorphic functions on U that vanish on $X \cap U$. The correspondence $U \mapsto \mathcal{I}(X)(U)$ defines a subsheaf $\mathcal{I}(X)$ of $\mathcal{O}_V$. The sheaf $\mathcal{I}(X)$ is an ideal in $\mathcal{O}_V$, which is called the *ideal sheaf* of X. For any $x \in X$ the stalk $\mathcal{I}_x(X)$ of $\mathcal{I}(X)$ at x consists of all holomorphic germs at x that vanish on X in some neighbourhood of x. We say that the point $x \in X$ is *regular*, if there exist a neighbourhood U of x and a finite number of holomorphic functions $\psi_1, \dots, \psi_n \in \mathcal{O}(U)$ such that $X \cap U = Z(\psi_1, \dots, \psi_n)$ and the differentials $\mathrm{d}\psi_1, \dots, \mathrm{d}\psi_n$ are linearly independent at x. By the implicit function theorem the germs $\psi_{1x}, \dots, \psi_{nx}$ generate the ideal $\mathcal{I}_x(X)$, and the tangent space $T_xX = \{\xi \in V : (\mathrm{d}\psi)_x(\xi) = 0 \text{ for all } \psi \in \mathcal{I}_x(X)\}$ to X at x has codimension n in V. The subset X_{reg}, consisting of all regular points of X, is open in X and it is known that X_{reg} is dense in X (see [8]). An analytic set X of finite codimension in V is a *submanifold* of finite codimension in V, if every point of X is regular.

A function $F : V \to \mathbb{C}$ is a homogeneous polynomial of degree d on V if there is a bounded multilinear map $M : V^d \to \mathbb{C}$ such that $F(v) = M(v, \dots, v)$ for any $v \in V$. The vector space of all homogeneous polynomials on V of degree d will be denoted by $\mathbb{C}[V]_d$.

Let X be an analytic set of finite codimension in V, $x_0 \in X$ and $f \in \mathcal{O}_x(V)$, $f \neq 0$. Then there exist a natural number d and unique homogeneous polynomials $F_i \in \mathbb{C}[V]_i$, $i \geq d$, such that $F_d \neq 0$ and $f(x) = \sum_{i \geq d} F_i(x - x_0)$ for all x in some neighbourhood of x_0. The homogenous polynomial F_d is called the *leading term* of the holomorphic germ $f \in \mathcal{O}_x(V)$. The set of common zeros of the leading terms of all holomorphic germs $f \in \mathcal{I}_x(X)$, $f \neq 0$, is called the *tangent cone* C_xX of x at X.

Remark 1. If x is a regular point of X then $C_xX = T_xX$. To see this, we may assume without loss of generality that $x = 0$, $V = V' \times \mathbb{C}^n$ and U and B are neighbourhoods of 0 in V' and $\mathbb{C}^n$ respectively such that

$$X \cap U \times B = \{ (v', Z_1, \dots, Z_n) \in V' \times \mathbb{C}^n : Z_i = \varphi_i(v'),\ i = 1, \dots, n \}$$

where $\varphi_i : U \to \mathbb{C}$, $i = 1, \dots, n$, are holomorphic functions. We may even further assume that all differentials $\mathrm{d}\varphi_i$, $i = 1, \dots n$, vanish at 0, so that $T_xX = V'$. For a given $f \in \mathcal{O}_{V,x}$, let $g \in \mathcal{O}_{V',x}$ be given by $g(v') = f(v', \varphi_1(v'), \dots, \varphi_n(v'))$ for all v' in some neighbourhood of 0 in V'. Let $g = \sum_{i\geq 0} G_{ix}$ with $G_i \in \mathbb{C}[V']_i$. Suppose $f \neq 0$ and let $F_d \in \mathbb{C}[V]_d$ be the leading term of f. Then $G_d = F_d|_{V'}$ because all functions φ_i, $i = 1, \dots n$, vanish of order > 1 at 0. In particular if $f \in \mathcal{I}_x(X)$ and $f \neq 0$ then $F_d|_{V'} = 0$ which yields $C_x X = T_x X$.

A function $f : V \to \mathbb{C}$ is a polynomial on V of degree d if $f = \sum_{k=1}^{d} f_d$, where each f_k, $k = 1, \dots, d$, is a homogeneous polynomial of degree k and $f_d \neq 0$. The ring of all polynomials on V will be denoted by $\mathbb{C}[V]$. Since $\mathbb{C}[V] = \bigoplus_d \mathbb{C}[V]_d$, the ring of all polynomials on V is a graded ring. It is known that $\mathbb{C}[V]$ is a factorial domain (see [8]). In particular the ring $\mathbb{C}[V]$ is integrally closed. For any $f \in \mathbb{C}[V]$ and any $v_0 \in V$ the function $g : V \to \mathbb{C}$ given by $g(v) = f(v + v_0)$, $v \in V$, is also a polynomial on V. Thus we may also speak about polynomials on the *affine* Banach space $A(V)$ associated with V.

A subset X of V is an *algebraic set of finite codimension* in V, if there is a finite number of polynomials $f_1, \dots, f_r \in \mathbb{C}[V]$ such that $X = Z(f_1, \dots, f_r)$. Every algebraic set of finite codimension in V is a closed analytic set of finite codimension in V. The ideal consisting of all polynomials on V that vanish on X is denoted by $I(X)$. The factor-ring $\mathbb{C}[V]/I(X)$ is denoted by $\mathbb{C}[X]$ and is called the *coordinate ring* of X. An algebraic set X of finite codimension in V is said to be *irreducible* if the coordinate ring of X is an integral domain.

Let W be a finite-dimensional complex vector space of dimension n, and let $Z_1, \dots, Z_n$ be a basis of the dual space W^*. It is clear that the ring of all polynomials on $V \times W$ is isomorphic to the polynomial ring $\mathbb{C}[V][Z_1, \dots, Z_n]$. Let X be an algebraic subset of finite codimension in $V \times W$ and let $p : X \to V$ be the restriction of the projection $\pi : V \times W \to V$ to X. We will say that p is a *finite projection* if the homomorphism $p^* : \mathbb{C}[V] \to \mathbb{C}[X]$, given by $\mathbb{C}[V] \hookrightarrow \mathbb{C}[V \times W] \to \mathbb{C}[X]$, is finite, i.e. $\mathbb{C}[X]$ is a finitely-generated $\mathbb{C}[V]$-module. It is easy to see that $p : X \to V$ is a finite projection if and only if for any $1 \leq j \leq n$ there is a monic polynomial $F_j(T) = T^{k_j} + \sum_{i=1}^{k_j} a_i(v) T^{k_j - i} \in \mathbb{C}[V][T]$ such that $F_j(Z_j) \in I(X)$. All fibers $p^{-1}(v)$, $v \in V$, of a finite projection $p : X \to V$ are finite sets. Moreover, for any $v_0 \in V$, there exists a neighbourhood $U \ni v_0$ and a compact set $K \subset W$ such that $p^{-1}(U) \subset U \times K$. This follows from the well known estimate

$$|\alpha| \leq \max(1, \sum_{i=1}^{k} |a_i(v)|) \tag{2.1}$$

for any root α of a given polynomial $F(v,T) = T^k + \sum_{i=1}^{k} a_i(v) T^{k-i} \in \mathbb{C}[V][T]$. Indeed, for any $v_0 \in V$ there is a neighbourhood O such that $|a_i(v) - a_i(v_0)| < 1$ for any $v \in O$ and any $i = 1, \dots, k$. If $v \in O$ and α is a root of $F(v,T)$ then $|\alpha| < k + \sum_{i=1}^{k} |a_i(v_0)|$ by estimate (2.1). Hence there is a neighbourhood U of v_0 such that all functions $Z_1|_X, \dots, Z_n|_X$ are bounded on $p^{-1}(U)$ which is equivalent to $p^{-1}(U)$ being contained in $U \times K$ for some compact set $K \subset W$. This shows that for any compact set $K' \subset V$ the preimage $p^{-1}(K')$ is compact, i.e. $p : X \to V$ is a *proper* map. In particular p is a closed map: if B is a closed subset of X, then $p(B)$ is a closed subset of V. Hence if $v_0 \in V$ and $N \subset X$ is a neighbourhood of the fiber $p^{-1}(v_0)$, then there is a neighbourhood $U \ni v_0$ such that $p^{-1}(U) \subset N$.

Proposition 2.1. Let X be an algebraic set of finite codimension in $V \times W$ for which $p : X \to V$ is a finite projection. Then the image $p(X)$ of p is an algebraic set of finite codimension in V and $I(p(X)) = I(X) \cap \mathbb{C}[V]$. In particular, a finite projection $p : X \to V$ is surjective if and only if $I(X) \cap \mathbb{C}[V] = (0)$.

Proof. In the proof we assume that the complex space W is one-dimensional because, as soon as the claim is known to be true for such spaces, the general case follows immediately by induction on the complex dimension of W. Let $Z \in W^*$ be a basis of W^*. Since $\mathbb{C}[X]$ is a finite extension of $\mathbb{C}[V]$, there are polynomials $g_1, \dots, g_r \in \mathbb{C}[V][Z]$ such that $X = Z(g_1, \dots, g_r)$ and the leading coefficient of at least one of them is 1. Now we use the existence of a resultant system of several polynomials in a single variable (see [9]):

Let $f_1, \dots, f_r$ be r polynomials of given degrees in a single variable with indeterminate coefficients. Then there exists a system $D_1, \dots, D_h$ of integral polynomials in these coefficients with the property that if these coefficients are assigned values from the field K the conditions $D_1 = 0, \dots, D_h = 0$ are necessary and sufficient in order that either the equations $f_1 = 0, \dots, f_r = 0$ have a solution in a suitable extension field, or that the formal leading coefficients of all polynomials $f_1, \dots, f_r$ vanish.

Let $D_1, \dots, D_h$ be a resultant system of $g_1, \dots, g_r$. Let $d_1, \dots d_h \in \mathbb{C}[V]$ be the system obtained from $D_1, \dots, D_h$ after substituting the coefficients of $g_1, \dots, g_r$ in $D_1, \dots, D_h$. Then the image $p(X)$ coincides with the set of common zeros of the polynomials $d_1, \dots, d_h$. □

We observe that if $\emptyset \neq X \subset V \times W$ is the set of common zeros of r polynomials in $\mathbb{C}[V \times W]$ and the projection $p : X \to V$ is finite then $\dim W \leq r$. Indeed, if $\dim W > r$ the dimension of every non-empty fiber of p would be positive.

Now we are going to prove the normalisation lemma for algebraic subsets of finite codimension in V. Let W be a closed complex vector subspace of V. We will say that a complex vector subspace $V' \subset V$ is complementary to W, or that V' and W are complementary, if V' is closed and the natural linear map $V' \times W \to V$ is an isomorphism.

Proposition 2.2. (The Normalisation Lemma) Let X be a non-empty algebraic set of finite codimension in V, and let (V_0', W_0), $\dim W_1 < \infty$, be a pair of complementary complex vector subspaces of V such that the projection $p_0 : X \to V_0'$ is finite. Then there is a pair (V', W), $\dim W < \infty$, of complementary complex vector subspaces of V such that $W \supset W_0$, $V' \subset V_0'$, and the projection $p : X \to V'$ is finite and surjective.

Proof. Let $X = Z(f_1, \dots f_r)$ with $f_1, \dots, f_r \in \mathbb{C}[V]$. Denote by S be the set of all pairs (V', W) of complementary complex vector subspaces of V such that $\dim W < \infty$, $W \supset W_0$, $V' \subset V_0$, and the projection $p : X \to V'$ is finite. It is clear that $\dim W \leq r$ for any $(V', W) \in S$. Let $(V', W) \in S$ be a pair for which $\dim W$ is maximal. Suppose that $p(X) \neq V'$. Then there is a polynomial $f \in I(X) \cap \mathbb{C}[V']$ with a leading term $f_d \neq 0$. Choose a vector $v' \in V'$ and a bounded linear functional T on V' such that $f_d(v') = 1$ and $T(v') = 1$. Then $V' \cong U' \times \{\mathbb{C}v\}$, where $U' = \operatorname{Ker} T$, and $f = T^d + a_1 T^{d-1} + \cdots + a_d$ with $a_1, \dots, a_d \in \mathbb{C}[U']$. Now the projection $p(X) \to U'$ along $\{\mathbb{C}v'\}$ is finite, which implies that the projection $X \to U'$ along $\{\mathbb{C}v'\} + W$ is also finite. Hence $(U', \{\mathbb{C}v'\} + W) \in S$, which contradicts the assumption that W has maximal dimension. Thus $p(X) = V'$ and the pair (V', W) has the required properties. □

Definition. Let X be an algebraic set of finite codimension in a complex Banach space V. We will say that the pair (V', W) of complex vector subspaces of V is an *admissible factorisation* for X, if $\dim W < \infty$, V' is complementary to W, and the projection map $p : X \to V'$ is finite and surjective.

The Normalisation Lemma shows that admissible factorisations exist for any non-empty algebraic set X of finite codimension in V.

Let X be an irreducible algebraic set of finite codimension in $V \times W$ such that (V, W) is an admissible factorisation for X. Then the homomorphism $p^* : \mathbb{C}[V] \to \mathbb{C}[X]$ is injective and the field of fractions L of $\mathbb{C}[X]$ is a finite extension of the field of fractions K of $\mathbb{C}[V]$. We note that for any $z \in \mathbb{C}[X]$ the coefficients of the minimal polynomial F of z over K belong to $\mathbb{C}[V]$. Indeed, each coefficient of F belongs to K and is integral over $\mathbb{C}[V]$. Since the ring $\mathbb{C}[V]$ is integrally closed, all coefficients of F are in in $\mathbb{C}[V]$. In particular the discriminant D of F also belongs to $\mathbb{C}[V]$. We will use the following well known fact about integral extensions (see [6]): if $z \in \mathbb{C}[V]$ is a generator of L over K then $D\mathbb{C}[X] \subset \mathbb{C}[V][z]$, where D is a discrimininant of the minimal polynomial of z over K.

Proposition 2.3. Let W be a complex vector space of finite dimension n and let X be an irreducible algebraic set of finite codimension in $V \times W$ such that (V, W) is an admissible factorisation for X. Suppose that $Z \in W^*$ is such that $z = Z + I(X) \in \mathbb{C}[X]$ is a generator of the field L over the field K and let $D \in \mathbb{C}[V]$ be the discriminant of the minimal polynomial F of z over the field K. Then $X_D = p^{-1}(V_D)$ is a connected complex submanifold of $V_D \times W$ of codimension n which is dense in X, and $p|_{X_D} : X_D \to V_D$ is a k-sheeted covering map, where $k = [L : K]$.

Proof. Let $e_1, \dots, e_n$ be a basis of W with dual basis $Z_1, \dots, Z_n$ such that

$Z_1 = Z$. Let $z_i = Z_i + I(X) \in \mathbb{C}[X]$, $i = 2, \dots n$. Since $Dz_i \in \mathbb{C}[V][z]$, $i = 2, \dots n$, there are polynomials $F_i(Z) \in \mathbb{C}[V][Z]$, $i = 2, \dots n$ such that $Dz_i = F_i(z)$, $i = 2, \dots, n$. All polynomials $F(Z_1)$ and $DZ_i - F_i(Z)$, $i = 2, \dots, n$, belong to $I(X)$ because $F(z_1) = 0$ and $Dz_i - F_i(z) = 0$, $i = 2, \dots, n$.

We will show first that X_D is the set of all solutions of the equations

$$F(Z_1) = 0\,,\; Z_2 = D^{-1}F_2(Z_1)\,,\; \dots\;,\; Z_n = D^{-1}F_n(Z_1) \tag{2.2}$$

in $V_D \times W$. Let J (resp. $I(X)_D$) be the ideal generated in $\mathbb{C}[V][Z_1, \dots, Z_n]_D$ by $F(Z_1)$ and $Z_i - D^{-1}F_i(Z_1)$, $i = 2, \dots, n$, (resp. by $I(X)$). It is enough to show that $I(X)_D = J$. We observe that the factor-ring $\mathbb{C}[V][Z_1, \dots, Z_n]_D/J \cong \mathbb{C}[V][z]_D$ is both an integral domain and an integral extension of $\mathbb{C}[V]_D$. Furthermore, the prime ideal $I(X)_D/J$ is such that $I(X)_D/J \cap \mathbb{C}[V]_D = (0)$ (since $I(X) \cap \mathbb{C}[V] = (0)$). This implies $I(X)_D/J = (0)$ because if $A \subset B$ is an integral extension of integral domains and P is a prime ideal in B such that $P \cap A = (0)$, then $P = (0)$ (see [5]). Thus $I(X)_D = J$ and X_D is defined by equations (2.2).

Next we find local solutions of the equation $F(v, Z) = 0$ by means of an integral formula. For a given $v_0 \in V_D$, let α_j, $j = 1, \dots, k$, be the roots of the $F(v_0, Z)$. Choose a positive real number δ such that α_j is the only root of $F(v_0, Z)$ in the disk $|Z - \alpha_j| \le \delta$, $j = 1, \dots, k$. Let Γ_j be the circle $|Z - \alpha_j| = \delta$, and let $\Gamma \cup_{j=1}^{k} \Gamma_j$. Since F does not vanish on $\{v_0\} \times \Gamma$, there exists a connected neighbourhood $U \subset A(V)_D$ of v_0 such that F does not vanish on $U \times \Gamma$. For $v \in U$ the number of roots of $F(v, Z)$ (counted with multiplicities) lying inside Γ_j is given by the holomorphic function

$$n_j(v) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{F'_Z(v, Z)}{F(v, Z)}\, \mathrm{d}Z\,, \qquad j = 1, \dots, k\,.$$

Since $n_j(v_0) = 1$, the polynomial $F(v, Z)$ has exactly one root $\alpha_j(v)$ lying inside Γ_j for $v \in U$ and this root is given by the holomorphic function

$$\alpha_j(v) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{F'_Z(v, Z)}{F(v, Z)} Z\, \mathrm{d}Z\,, \qquad j = 1, \dots, k\,.$$

Hence $F|_{U \times W} = (Z - \alpha_1(v)) \dots (Z - \alpha_k(v))$. Let $q_j : U \to U \times W$, $j = 1, \dots, k$, be the graph of the holomorphic map $r_j : U \to W$ given by

$$r_j(v) = \alpha_j(v)\, e_1 + D(v)^{-1} \sum_{i=2}^{n} G_i(\alpha_j(v))\, e_i\,, \qquad v \in U\,.$$

Then $p^{-1}(U)$ is the disjoint union of the complex manifolds $q_j(U)$, $j = 1, \dots, k$, and each restriction $p|_{q_j(U)} : q_j(U) \to U$, $j = 1, \dots, k$, is a biholomorphic map.

The set V_D is connected because for any v_1, $v_2 \in V_D$ the complex line joining v_1 and v_2 intersects $Z(D)$ in a finite set. Let C be a connected component of X_D. Then $p|_C : C \to V_D$ is a covering of degree $k_1 \le k$ and

$$F_1(v, Z) = \prod_{x \in p^{-1}(v) \cap C} (Z - Z(x)) \;\in\; \mathcal{O}(V_D)[Z]\,, \quad v \in V_D\,,$$

is a polynomial of degree k_1 in Z which divides F in the ring $\mathcal{O}(V_D)[Z]$. Since for any $v \in V$ the roots of $F(v, Z)$ are uniformly bounded in some neighbourhood of v, the same is true about the coefficients of F_1, and by the Riemann extension theorem all coefficients of F_1 can be extended to holomorphic functions on V. Moreover, estimate (2.1) shows that the roots of F grow polynomially, i.e. there is a natural number N and a positive real number C such that $|\alpha| \leq C(1 + \|v\|)^N$ for any root α of $F(v, Z)$. Hence the coefficients of F_1 also grow polynomially which shows that they are polynomials. We conclude that $F_1 = F$ because F_1 divides the irreducible polynomial F in $\mathbb{C}[V][Z]$. This immediately yields $C = X_D$, and thus X_D is connected.

It remains to prove that X_D is dense in X. Let $x_0 \in X$ and let $O \subset X$ be a neighbourhood of x_0. Choose a real number $\delta > 0$ such that $|Z(x) - Z(x_0)| > \delta$ for all $x \in p^{-1}(p(x_0))$, $x \neq x_0$. Since $O \cup \{x \in X : |Z(x) - Z(x_0)| > \delta\}$ is a neighbourhood of $p^{-1}(p(x_0))$ and the map p is closed, there exists a neighbourhood $U \subset V$ of $p(x_0)$ such that if $p(x) \in U$, then either $x \in O$, or $|Z(x) - Z(x_0)| > \delta$. After shrinking U we may also assume that for any $v \in U$ the polynomial $F(v, Z)$ has a root α such that $|\alpha - Z(x_0)| < \delta$. Let $v \in U \cap V_D$. Then the fiber $p^{-1}(v)$ contains a point x such that $|Z(x) - Z(x_0)| < \delta$, and it is clear that $x \in O$. Thus the set X_D is dense in X. □

Remark 2. Suppose that (V', W) is an admissible factorisation for an irreducible algebraic set X of finite codimension in V. Denote by $i : \mathbb{C}[V] \to \mathbb{C}[X]$ the natural homomorphism given by $i(f) = f + I(X)$, $f \in \mathbb{C}[V]$. Let $Z_1, \ldots, Z_n \in W^*$ be a basis of the dual space W^*. Since the elements $z_1 = i(Z_1), \ldots, z_n = i(Z_n)$ generate the ring $\mathbb{C}[X]$ over the ring $\mathbb{C}[V']$, they also generate the field of fractions L of $\mathbb{C}[X]$ over the field of fractions K of $\mathbb{C}[V']$. Let $A = \{(a_1, \ldots, a_n) \in \mathbb{C}^n : z = a_1 z_1 + \ldots a_n z_n$ is a generator of L over $K\}$. By the theorem for the primitive element, A is a non-empty Zariski open set in $\mathbb{C}^n$. Hence the set $W_g^* = \{Z \in W^* : i(Z)$ generates L over $K\}$ is a non-empty Zariski open set in W^*.

Remark 3. Let (V', W) be an admissible factorisation for an irreducible algebraic set X of finite codimension in V. Proposition 2.3 and Remark 2 show that there exists an open dense subset V_0' of V' such that for any $y \in V_0'$ the fiber $\pi^{-1}(y) = \{y\} \times W$ intersects X transversely in k regular points of X, where $k = [L : K]$.

Corollary 2.4. For any irreducible algebraic set X of finite codimension in V the set X_{reg} is a connected locally closed submanifold of finite codimension in V, which is dense in X. Moreover, for any pair (V', W) of complex vector subspaces of V that is admissible for X, the dimension of W is equal to the codimension of X_{reg} in V.

Proof. Let (V', W) be an admissible factorisation for X. According to Remark 2, there exists $Z \in W^*$ such that $z = Z + I(X) \in \mathbb{C}[X]$ generates the field of fractions L of $\mathbb{C}[X]$ over the field of fractions K of $\mathbb{C}[V']$. Let $F \in \mathbb{C}[V'][Z]$ be the minimal polynomial of Z over K, and let $D \in \mathbb{C}[V']$ be the discriminant of F. Then $X_D \subset X_{reg}$ by Proposition 2.3. Since X_D is connected and dense in X, so is X_{reg}. Proposition 2.3 implies $\mathrm{codim}_V X_{reg} = \mathrm{codim}_V X_D = \dim W$. □

In view of Corollary 2.4 we define the codimension of an irreducible algebraic subset X in V as the codimension of the submanifold X_{reg} in V. The codimension of X in V will be denoted by $\mathrm{codim}_V X$. The next lemma describes the behaviour of the codimension under finite projections.

Lemma 2.5. Let X be an irreducible algebraic set of finite codimension in V, and let (V', W) be a pair of complex vector subspaces of V such that $\dim W < \infty$, V' is complementary to W, and the projection $p : X \to V'$ is finite. Then the set $X' = p(X)$ is an irreducible algebraic set of finite codimension in V', and $\mathrm{codim}_{V'} X' = \mathrm{codim}_V X - \dim W$.

Proof. Let (V'', W') be an admissible factorisation for X' in V'. Then the pair of subspaces $(V', W' \times W)$ is an admissible factorisation for X in V, and Corollary 2.4 yields $\mathrm{codim}_{V'} X' = \dim W' = \dim W' \times W - \dim W = \mathrm{codim}_V X - \dim W$. □

Let X be an irreducible algebraic set of finite codimension n in V. The next two lemmas will be used to prove that for any $x \in X_{reg}$ there exist n polynomials $f_1, \dots, f_n \in I(X)$ such that their differentials $\mathrm{d}f_1, \dots, \mathrm{d}f_n$ are linearly independent at x.

Lemma 2.6. We keep the notation and the assumptions of Proposition 2.3. Suppose that $x_0 = (v_0, w_0) \in X_{reg}$ is such that:
(i) the fiber $\pi^{-1}(\pi(x_0)) = \{v_0\} \times W$ is transversal to X at x_0;
(ii) $Z(x_0) \neq Z(x)$ for all $x \in p^{-1}(p(x_0))$, $x \neq x_0$.
Then $Z(x_0)$ is a simple root of $F(v_0, Z)$.

Proof. It follows from (i) that there exist neighbourhoods $U \subset V$ of v_0, $B \subset W$ of w_0, and a holomorphic map $f : U \to B$ such that $X \cap U \times B = \Gamma(f)$, where $\Gamma(f) \subset U \times B$ is the graph of f. Let δ be a positive real number such that $|Z(x) - Z(x_0)| > \delta$ for all $x \in p^{-1}(v_0)$, $x \neq x_0$, and let $X_\delta = \{x \in X : |Z(x) - Z(x_0)| > \delta\}$. Since p is a proper map, the fiber $p^{-1}(v)$ is contained in $U \times B \cup X_\delta$ for all v that are close to v_0. Let l be the multiplicity of $Z(x_0)$ as a root of $F(v_0, Z)$. For all $v \in V_D$ that are close to v_0, the fiber $p^{-1}(v)$ contains l distinct points x_i, $i = 1, \dots, l$ such that $|Z(x_i) - Z(x_0)| < \delta$ for $i = 1, \dots, l$. Hence for all $v \in V_D$ that are close to v_0, the graph $\Gamma(f)$ contains l distinct points of the fiber $p^{-1}(v)$. This implies $l = 1$. □

Lemma 2.7 Let X be an irreducible algebraic set of finite codimension in

V, and let $V' \subset V$ be a closed complex vector subspace such that $\operatorname{codim}_V V' = \operatorname{codim}_V X$. Then there exists a finite dimensional complex vector subspace $W \subset V$ such that (V', W) is an admissible factorisation for X.

Proof. The claim is true when $\operatorname{codim}_V X = 0$. Suppose that $\operatorname{codim}_V X = n > 0$, and that the lemma is true for all irreducible algebraic subsets of codimension less than n in a Banach space. Let f be a non-zero polynomial in $I(X)$ with leading term f_d. Since the set $V \setminus V'$ is dense in V, there exists a vector $v \in V \setminus V'$ such that $f_d(v) = 1$. Let T be a bounded linear functional on V such that $U = \operatorname{Ker} T \supset V'$ and $T(v) = 1$. Then $\mathbb{C}[V] \cong \mathbb{C}[U][T]$ and $f = T^d + \sum_{i=1}^{d} a_i T^{d-i}$ with $a_i \in \mathbb{C}[U]$, $i = 1, \ldots, d$. Thus the projection $p_1 : X \to U$ along $\{\mathbb{C}v\}$ is finite, and by Lemma 2.4 the set $X' = p_1(X)$ is an irreducible algebraic subset of codimension $n-1$ in U. Since $\operatorname{codim}_U V' = \operatorname{codim}_U X'$, there exists a finite dimensional complex vector subspace $W' \subset U$ such that (V', W') is an admissible factorisation for X' in U. Then $(V', W' \times \{\mathbb{C}v\})$ is an admissible factorisation for X in V, which finishes the proof. □

Proposition 2.8. For any regular point x_0 of an irreducible algebraic set X of finite codimension n in V, there exist polynomials $f_1, \ldots, f_n \in I(X)$ such that their differentials $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ are linearly independent at x_0.

Proof. By Lemma 2.7 with $V' = T_{x_0}X$, there is an n-dimensional subspace W of V such that (V', W) is an admissible factorisation for X in V. Let $Z_1, \ldots, Z_n$ be a basis of W^* *that satisfies the following two conditions:* (i) $Z_i(x) \neq Z_i(x_0)$ for $x \in p^{-1}(p(x_0))$, $x \neq x_0$, $i = 1, \ldots, n$; (ii) $z_i = Z_i + I(X) \in \mathbb{C}[X]$ generates the field of fractions of $\mathbb{C}[X]$ over the field of fractions of $\mathbb{C}[V']$, $i = 1, \ldots, n$. The existence of such a basis of W^* follows from Remark 2. Let $g_i \in \mathbb{C}[V'][Z_i]$ be the minimal polynomial of z_i over the field of fractions of $\mathbb{C}[V']$, $i = 1, \ldots, n$. Then $\mathrm{d}g_i|_W = g_i'(p(x_0), Z_i(x_0))\, \mathrm{d}Z_i$, $i = 1, \ldots, n$, and since by Lemma 2.6 $g_i'(p(x_0), Z_i(x_0)) \neq 0$, $i = 1, \ldots, n$, the differentials $\mathrm{d}g_1, \ldots, \mathrm{d}g_n$ are linearly independent at x_0. □

3. PROJECTIVE ALGEBRAIC SETS

In this section we first consider a complex Banach space V and briefly describe the complex structure of the corresponding complex projective space $\boldsymbol{P}(V)$. After that we study the properties of algebraic subsets of finite codimension in $\boldsymbol{P}(V)$.

Let V be a complex Banach space with a dual space V^*. The projective space $\boldsymbol{P}(V)$ associated with V consists of all complex lines in V. The set $\boldsymbol{P}(V)$ is given a structure of a complex manifold as follows. For any $v \in V$, $v \neq 0$, the complex line spanned by v will be denoted by $[v]$. For a given bounded linear functional $h \in V^*$, $l \neq 0$, let $\boldsymbol{P}(V)_h = \{[v] \in \boldsymbol{P}(V) : h(v) \neq 0\}$. Denote by A_h the affine hyperplane $A_h = \{v \in V : h(v) = 1\}$, and let $\varphi_h : \boldsymbol{P}(V)_h \to A_h$ be the coordinate map given by $\varphi_h([v]) = h(v)^{-1}v$. The family of sets $\boldsymbol{P}(V)_h$, $h \in V^*$, $h \neq 0$, is a covering of $\boldsymbol{P}(V)$ by the Hahn-Banach theorem, and it is easy to verify that the coordinate maps φ_h, $h \in V^*$, $h \neq 0$, endow $\boldsymbol{P}(V)$ with a structure of a complex manifold.

We note that for any $h \in V^*$, $h \neq 0$, the open set $\boldsymbol{P}(V)_h$ is an affine space. Moreover, for any $F \in \mathbb{C}[V]_d$, the function $f : \boldsymbol{P}(V)_h \to \mathbb{C}$ given by $f([v]) = h(v)^{-d}F(v)$ for $[v] \in \boldsymbol{P}(V)_h$, is a polynomial on $\boldsymbol{P}(V)_h$ because $f \circ \varphi_h^{-1} = F|_{A_h}$ is a polynomial on the affine hyperplane A_h. Since for every polynomial f on A_h there exists a homogeneous polynomial F on V such that $f = F|_{A_h}$, the ring $\mathbb{C}[\boldsymbol{P}(V)_h]$ of all polynomials on $\boldsymbol{P}(V)_h$ is naturally isomorphic to the so called homogeneous localisation $\mathbb{C}[V]_{(h)} = \{F/h^d : F \in \mathbb{C}[V]_d,\ d \geq 0\}$.

A subset $X \subset \boldsymbol{P}(V)$ is an *analytic set of finite codimension* in $\boldsymbol{P}(V)$, if for any $x \in X$ there exist a neighbourhood U and a finite number of holomorphic functions $\varphi_1, \dots, \varphi_s \in \mathcal{O}(U)$ such that $X \cap U = Z(\varphi_1, \dots, \varphi_s)$. We say that the point $x \in X$ is *regular*, if there exist a neighbourhood U of x and a finite number of holomorphic functions $\psi_1, \dots, \psi_n \in \mathcal{O}(U)$ such that $X \cap U = Z(\psi_1, \dots, \psi_n)$ and the differentials $\mathrm{d}\psi_1, \dots, \mathrm{d}\psi_n$ are linearly independent at x. The subset X_{reg}, consisting of of all regular points of X, is open in X, and it is known that X_{reg} is dense in X (see [8]). An analytic set X of finite codimension in $\boldsymbol{P}(V)$ is a *submanifold of finite codimension* in $\boldsymbol{P}(V)$, if every point of X is regular.

A subset X of $\boldsymbol{P}(V)$ is an *algebraic set of finite codimension* in $\boldsymbol{P}(V)$, if there is a finite number of homogeneous polynomials $f_1, \dots, f_r \in \mathbb{C}[V]$ such that X is the set of common zeros of $f_1, \dots, f_r$ in $\boldsymbol{P}(V)$. Every algebraic set of finite codimension in $\boldsymbol{P}(V)$ is a closed analytic set of finite codimension in $\boldsymbol{P}(V)$. The ideal generated by all homogeneous polynomials on V that vanish on X will be denoted by $I(X)$. Let $I(X)_d$ be the vector space $I(X) \cap \mathbb{C}[V]_d$. Since $I(X) = \bigoplus_{d \geq 0} I(X)_d$, the ideal $I(X)$ is a homogeneous ideal in the graded ring $\mathbb{C}[V]$. The factor-ring $\mathbb{C}[V]/I(X)$ is denoted by $S[X]$ and is called the *homogeneous coordinate ring* of X. Since $I(X)$ is a homogeneous ideal in $\mathbb{C}[V]$, the ring $S[X]$ inherits the grading of $\mathbb{C}[V]$; $S(X) \cong \bigoplus_{d \geq 0} S(X)_d$, where $S(X)_d = \mathbb{C}[V]_d / I(X)_d$. The set $Z(I(X)) \subset V$ is a cone in V and will be denoted by $C(X)$. It is clear that $C(X)$ is an algebraic set of finite codimension in V and $\mathbb{C}[C(X)] = S(X)$.

For any $h \in V^*$, $h \neq 0$, the open set $X_h = X \cap \boldsymbol{P}(V)_h$ is an algebraic set of finite codimension in the affine space $\boldsymbol{P}(V)_h$. If $F \in I(X)_d$, then $F/h^d \in I(X_h)$ and, conversely, if $F/h^d \in I(X_h)$ for some $F \in \mathbb{C}[V]_d$, then $Fh \in I(X)_{d+1}$. Hence the coordinate ring $\mathbb{C}[X_h]$ of X_h is isomorphic to the so called homogeneous localisation $S(X)_{(h)}$.

We will say that X is an *irreducible* algebraic set of finite codimension in $\boldsymbol{P}(V)$ if $S[X]$ is an integral domain. If X is an irreducible algebraic set of finite codimension in $\boldsymbol{P}(V)$, then X_h is an irreducible algebraic set of finite codimension in $\boldsymbol{P}(V)_h$ for any $h \in V^*$, $h \notin I(X)$. It is clear that, for every irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V)$, the family of sets $\mathcal{C} = \{\boldsymbol{P}(V)_h\}$, $h \in V^*$, $h \notin I(X)$, is an open covering of X. Moreover, for any $\boldsymbol{P}(V)_{h_1}, \boldsymbol{P}(V)_{h_2} \in \mathcal{C}$ the intersection $P(V)_{h_1} \cap P(V)_{h_2}$ is dense in both $P(V)_{h_1}$ and $P(V)_{h_2}$. Hence all sets $P(V)_h \in \mathcal{C}$ are dense in X.

Proposition 3.1. For any irreducible algebraic subset X of finite codimension in $P(V)$, the set X_{reg} is a connected open subset of X which is dense in X.

Proof. This follows from Corollary 2.4 and the considerations above. □

In view of Proposition 3.1, we define the codimension of an irreducible algebraic subset X in $P(V)$ as the codimension of the locally closed submanifold X_{reg} in $P(V)$. The codimension of X in $P(V)$ will be denoted by $\operatorname{codim}_{P(V)} X$.

Let W be a finite-dimensional complex vector space of dimension n. The projection map $V \times W \to V$ induces a holomorphic map

$$\pi : P(V \times W) \setminus P(W) \to P(V)$$

given by $p([(v,w)]) = [v]$ for $[(v,w)] \in P(V \times W) \setminus P(W)$. For $h \in V^*$, $h \neq 0$, let $\theta_h : \pi^{-1}(P(V)_h) \to P(V)_h \times W$ be the trivialisation given by $\pi_h([v,w]) = ([v], h(v)^{-1}w)$. It is easy to verify that the family of trivialisations θ_h, $h \in V^*$, $h \neq 0$, makes π into a locally trivial vector bundle over $P(V)$ with fiber W.

Let $Z_1, \ldots, Z_n$ be a basis of W^*. For any algebraic set X of finite codimension in $P(V \times W)$, we denote by $p : X \setminus P(W) \to P(V)$ the restriction of π to $X \setminus P(W)$, and by $p^* : \mathbb{C}[V] \to S[X]$ the ring homomorphism given by the composition $\mathbb{C}[V] \hookrightarrow \mathbb{C}[V][Z_1, \ldots, Z_n] \cong \mathbb{C}[V \times W] \to S[X]$. It is clear that the homomorphism p^* respects the grading of $\mathbb{C}[V]$ and $S[X]$, i.e. $p^*(\mathbb{C}[V]_d) \subset S[X]_d$, $d \geq 0$. Furthermore, for any $h \in V^*$, $h \notin I(X)$, the homomorphism $p_h^* : \mathbb{C}[V]_{(h)} \to \mathbb{C}[X_h]$, associated with the projection $p_h = p|_{X_h} : X_h \to P(V)_h$, is induced by the homomorphism $p^* : \mathbb{C}[V] \to S[X]$ (via homogeneous localisation with respect to h). Hence if p^* is finite, then p_h^* is finite for any $h \in V^*$, $h \notin I(X)$.

Remark 1. If p^* is a finite homomorphism, then all fibers of the map $p_C : C(X) \to V$ are finite sets. In particular the cone $p_C^{-1}(0) = C(X) \cap W$ is a finite set. Hence $C(X) = \cap W\{0\}$, which implies $X \cap P(W) = \emptyset$. The next proposition shows that the converse is also true.

Lemma 3.2. Let X be an algebraic set of finite codimension in $P(V \times W)$ such that $X \cap P(W) = \emptyset$. Then the homomorphism $p^* : \mathbb{C}[V] \to S[X]$ is finite, the set $Y = p(X)$ is an algebraic set of finite codimension in $P(V)$, and $I(Y) = I(X) \cap \mathbb{C}[V]$. If the algebraic set X is irreducible, then the algebraic set Y is also irreducible, and $\operatorname{codim}_{P(V')} Y = \operatorname{codim}_{P(V)} X - \dim W$.

Proof. As in the proof of Proposition 2.1, we may assume that $\dim W = 1$, i.e. $W = \{\mathbb{C}e\}$, $e \neq 0$. Let $Z \in W^*$ be a basis of W^*. Since $[(e,0)] \notin X$, the ideal $I(X)$ contains a homogeneous polynomial $g = Z^n + a_1 Z^{n-1} + \ldots + a_n$, $a_1, \ldots, a_n \in \mathbb{C}[V]$, with leading coefficient 1 with respect to Z. Thus $p^* : \mathbb{C}[V] \to S[X]$ is a finite homomorphism. Let $C(X) \subset V \times W$ (resp. $C(Y) \subset V$) be the cone of X (resp. Y). By Proposition 2.1 the set $C(Y)$ is an algebraic set of finite codimension in V and $I(C(Y)) = I(C(X)) \cap \mathbb{C}[V]$. Hence Y is an algebraic set of finite codimension in $\boldsymbol{P}(V)$ and $I(Y) = I(X) \cap \mathbb{C}[V']$. If X is irreducible, then $I(X) \cap \mathbb{C}[V']$ is a prime ideal in $\mathbb{C}[V']$, and Y is also irreducible. Let h be a bounded linear functional on V' such that $Y_h = \boldsymbol{P}(V')_h \cap Y \neq \emptyset$. Since the projection $p|_{X_h} : X_h \to \boldsymbol{P}(V')_h$ is finite, $\operatorname{codim}_{\boldsymbol{P}(V')_h} Y_h = \operatorname{codim}_{\boldsymbol{P}(V)_h} X_h - \dim W$ by Lemma 2.5. This proves the last claim of this lemma because X_h and Y_h are open and dense in X and Y respectively. □

The Normalisation Lemma has a natural analogue for projective spaces:

Proposition 3.3. (The Projective Normalisation Lemma) Let X be an algebraic set of finite codimension in $\boldsymbol{P}(V)$ and W_0 be a finite dimensional complex vector subspace such that $\boldsymbol{P}(W_0) \cap X = \emptyset$. Then there is a finite dimensional complex vector subspace $W \supset W_0$ such that for any complementary complex vector subspace V' the projection map $p : X \to \boldsymbol{P}(V')$ is surjective.

Proof. Let V_0' be a complex vector subspace which is complementary to W_0. Since the homomorphism $p_0^* : \mathbb{C}[V_0'] \to S[X] = \mathbb{C}[C(X)]$ is finite by Lemma 3.2, we can apply Proposition 2.2 to the pair (V_0', W_0) and the cone $C(X)$. Thus there exists a pair of complementary complex vector subspaces (V', W), $\dim W < \infty$, $W \supset W_0$, $V' \subset V_0'$, for which the homomorphism $p^* : \mathbb{C}[V'] \to S(X)$ is finite and $I(X) \cap \mathbb{C}[V'] = (0)$. Remark 1 shows that $\boldsymbol{P}(W) \cap X = \emptyset$, and Lemma 3.2 shows that the projection map $p : X \to \boldsymbol{P}(V')$ is surjective. If V_1' is another complex subspace that is complementary to W, then

$$\mathbb{C}[V_1'] = \{f \in \mathbb{C}[V] : f(v + w) = f(v) \text{ for any } w \in W\} = \mathbb{C}[V'],$$

which shows that the projection map $p_1 : X \to \boldsymbol{P}(V_1')$ is surjective too. □

Definition. Let X be an algebraic set of finite codimension in a projective space $\boldsymbol{P}(V)$. Given a pair (V', W) of complementary complex vector subspaces of V, we say that (V', W) is an *admissible factorisation* for X if $\dim W < \infty$, $\boldsymbol{P}(W) \cap X = \emptyset$, and the projection $p : X \to \boldsymbol{P}(V')$ is surjective.

The Projective Normalisation Lemma shows that admissible factorisations exist for any algebraic set X of finite codimension in $\boldsymbol{P}(V)$. If X is irreducible, then a given pair (V', W) of complementary complex vector subspaces is an admissible factorisation for X if and only if $\boldsymbol{P}(W) \cap X = \emptyset$ and $\dim W = \operatorname{codim}_{\boldsymbol{P}(V)} X$.

We note that if the pair (V, W) is an admissible factorisation for an irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V \times W)$, then $p^* : \mathbb{C}[V] \to S[X]$ is an injective homomorphism, and the field of fractions L of $S(X)$ is a finite extension of the field of fractions K of $\mathbb{C}[V]$. Now we are going to prove an analogue of Proposition 2.3 for irreducible algebraic sets of finite codimension in projective space.

Proposition 3.4. Let W be a complex vector space of finite dimension n and let X be an irreducible algebraic set of finite codimension in $\boldsymbol{P}(V \times W)$ such (V, W) is an admissible factorisation for X. Suppose that $Z \in W^*$ is such that $z = Z + I(X) \in S[X]$ is a generator of the field L over the field K, and let $D \in \mathbb{C}[V]$ be the discriminant of the minimal polynomial of z over K. Then $X_D = p^{-1}(\boldsymbol{P}(V)_D)$ is a complex submanifold of $\boldsymbol{P}(V \times W)_D$ of codimension n and $p|_{X_D} : X_D \to \boldsymbol{P}(V)_D$ is a d-sheeted covering map, where $d = [L : K]$.

Proof. We note that for a given $h \in V^*$, $h \neq 0$, the fraction z/h generates the field of fractions L_h of $S[X]_{(h)}$ over the field of fractions K_h of $\mathbb{C}[V]_{(h)}$. Let D_h be the discriminant of the minimal polynomial of z/h over K_h. According to Proposition 2.3, the set $(X_h)_{D_h}$ is a complex submanifold of $(\boldsymbol{P}(V)_h)_{D_h} \times W$ of codimension n and the map $p_h|_{(X_h)_{D_h}} : (X_h)_{D_h} \to (\boldsymbol{P}(V)_h)_{D_h}$ is a k_h-sheeted covering map, where $k_h = [L_h : K_h]$. Since $K = K_h(h)$, $L = L_h(h)$, and h is transcendental over K_h, we see that $k_h = d$ for $h \in V^*$, $h \neq 0$. A simple calculation shows that $D_h = D/h^{k(k-1)}$ which implies $(\boldsymbol{P}(V)_h)_{D_h}(\boldsymbol{P}(V)_D)_h$. To finish the proof, we observe that the family of sets $(\boldsymbol{P}(V)_D)_h$, $h \in V^*$, $h \neq 0$, is an open covering of $\boldsymbol{P}(V)_D$. □

Definition. Let (V', W) be an admissible factorisation for an irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V)$ and $z \in S(X)_1$. We will say that (W, V', z) is an *admissible triple* for X, if z is a generator of the field of fractions L of $S(X)$ over the field of fractions K of $\mathbb{C}[V]$. Given an admissible triple (W, V', z) for X, the discriminant of the minimal polynomial of z over K is denoted by D.

Remark 2. According to Remark 2 in Section 2, if (V', W) is an admissible factorisation for an irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V)$, then there is an element $z \in S(X)_1$ such that (V', W, z) is an admissible triple for X.

If (W, V', z) is an admissible triple for an irreducible algebraic set X of finite codimension in $\boldsymbol{P}(V)$ and $y \in \boldsymbol{P}(V')_D$, then the fiber $\pi^{-1}(y) = \{y\} \times W$ intersects the manifold X_{reg} transversely in d points, where $d = [L : K]$. Let $U \subset V$ be the complex vector subspace spanned by W and y in V. Then $\dim \boldsymbol{P}(U) = \operatorname{codim}_{\boldsymbol{P}(V)} X$ and $\boldsymbol{P}(U)$ intersects the manifold X_{reg} transversely in d points. The next lemma shows that the number d is the same for all admissible triples for X in $\boldsymbol{P}(V)$.

Lemma 3.5. Let X be an irreducible algebraic set of finite codimension n in $\boldsymbol{P}(V)$, and let U_1, U_2 be $n+1$-dimensional complex vector subspaces of V such that both sets $\boldsymbol{P}(U_1) \cap X$, $\boldsymbol{P}(U_2) \cap X$ consist of regular points of X and both intersections $P(U_1) \cap X_{reg}$, $P(U_2) \cap X_{reg}$ are transversal. Then the cardinality of the finite sets $P(U_1) \cap X$ and $P(U_2) \cap X$ is the same.

Proof. If $\dim V < \infty$, then the lemma is true because the cardinality of both sets $P(U_1) \cap X$ and $P(W_2) \cap X$ is equal to the degree of X in $P(V)$. Let V' be a finite dimensional complex vector subspace of V which contains both U_1 and U_2 and let $X' = X \cap P(V')$. Then both sets $P(U_1) \cap X'$, $P(W_2) \cap X'$ consist of regular points of X' and both intersections $P(U_1) \cap X'_{reg}$, $P(U_2) \cap X'_{reg}$ are transversal. Hence the cardinality of the sets $P(U_1) \cap X$, $P(U_2) \cap X$ is the same. □

In view of Lemma 3.5 the following definition makes sense.

Definition. Let X be an irreducible algebraic set of finite codimension in $P(V)$. The *degree* of X in $P(V)$ is the degree of the field of fractions of $S(X)$ over the field of fractions of $\mathbb{C}[V']$ for any admissible factorisation (V', W) for X. The degree of X in $P(V)$ will be denoted by $\deg X$.

Remark 3. Let X be an irreducible algebraic set X of finite codimension $n < \infty$ and degree d in $P(V)$. Then for any given point $x_0 \in P(V) \setminus X$ there exists an $n+1$-dimensional complex vector subspace U of V such that $P(U)$ passes through x_0, and $P(U)$ intersects X transversely in d distinct regular points of X. Indeed, let (V', W, z) be an admissible triple for X such that $x_0 \in P(W)$ (see Proposition 3.3 and Remark 2). Then for any $y \in P(V')_D$ the complex vector subspace U, spanned by W and y in V, has the required properties.

Now we are going to show that if x_0 is a regular point of an irreducible algebraic subset X of finite codimension n and degree d in $P(V)$, then there exists an $n+1$-dimensional complex vector subspace U of V such that $P(U)$ passes through x_0, and $P(U)$ intersects X transversely in d distinct regular points of X.

Let x_0 be a regular point of an irreducible algebraic set X of finite codimension in $P(V)$ and let $l \subset P(V)$ be a projective line passing through x_0. We say that l is a *tangent line* to X at x_0 if $f|_l$ vanishes of order > 1 at x_0 for any $f \in I(X)$. The union of all tangent lines to X at x_0 will be denoted by $P_{x_0}X$. It is clear that the set $P_{x_0}X$ is a projective subspace of codimension n in $P(V)$. Let V' be a closed hyperplane in V such that $x_0 \notin P(V')$ and let $\pi : P(V) \setminus \{x_0\} \to P(V')$ be the map induced by the projection $p_1 : V \cong V' \times x_0 \to V'$. Let $Z : V \to \mathbb{C}$ be a bounded linear functional on V such that $\operatorname{Ker} Z = V'$. Then for any given homogeneous polynomial $0 \neq f \in \mathbb{C}[V]$ there are unique homogeneous polynomials $a_i(f) \in \mathbb{C}[V']$, $i = 0, \ldots, m$, such that $f = \sum_{i=0}^{m} a_i(f)\, Z^{m-i}$, $a_0 \neq 0$, and $\deg a_i + m - i = \deg f$, $i = 0, \ldots, m$.

Lemma 3.6. Let x_0 be a regular point of an irreducible algebraic set X of finite codimension n in $P(V)$ and V' be a hyperplane in V such that $x_0 \notin P(V')$.

Then

$$\pi(P_{x_0}X \setminus \{x_0\}) = \{ [v'] \in \boldsymbol{P}(V') : a_0(f)(v') = 0 \text{ for all } f \in \cup_{d\geq 0} I(X)_d \setminus \{0\} \}.$$

Proof. Let $v_0 \in x_0$ be such that $Z(x_0) = 1$. Since $\boldsymbol{P}(V)_Z$ is by definition the affine subspace $v_0 + V' \subset V$, we will identify the tangent space $T_{x_0}\boldsymbol{P}(V)$ with V'. Then $P_{x_0}X \setminus \boldsymbol{P}(V') = \{[v', v_0] \in \boldsymbol{P}(V)_Z : v' \in T_{x_0}X\}$ and $\pi(P_{x_0}X \setminus \{x_0\}) = \boldsymbol{P}(T_{x_0}X)$. Thus it is enough to prove that

$$T_{x_0}X = \{ v' \in V' : a_0(f)(v') = 0 \text{ for all } f \in \cup_{d\geq 0} I(X)_d \setminus \{0\} \}. \tag{3.1}$$

Denote by $A_{x_0}X$ the set on the right-hand side of (3.1). For $f \in \cup_{d\geq 0} I(X)_d \setminus \{0\}$, let $g \in I(X_Z)$ be given by $g([v', v_0]) = f(v', v_0) = \sum_{i=0}^{m} a_i(f)(v')$, $v' \in V'$. Then $g(x) = \sum_{i\geq 0} a_i(f)(x - x_0)$ for all $x \in \boldsymbol{P}(V)_Z$, which shows that $a_0(f)$ is the leading term of the holomorphic germ $g_{x_0} \in \mathcal{I}_{x_0}(X_Z)$. Hence $A_{x_0}X \supset C_{x_0}X$, where $C_{x_0}X$ is the tangent cone of X at x_0. According to Proposition 2.8, there exist polynomials $g_j \in I(X_Z)$, $j = 1, \ldots, n$ such that the differentials $\mathrm{d}g_1, \ldots, \mathrm{d}g_n$ are linearly independent at x_0. Let $g_j([v', v_0]) = \sum_{i=0}^{m_j} a_{ij}(v')$, $v' \in V'$, where $a_{ij} \in \mathbb{C}[V']_{i+1}$ for $j = 1, \ldots, n$, $0 \leq i \leq m_j$. Then the linear functionals $a_{01}, \ldots, a_{0n}$ are exactly the differentials of $g_1, \ldots, g_n$ at x_0. Let $f_j \in \mathbb{C}[V]_{m_j+1}$ be given by $f_j(v) = \sum_{i=0}^{m_j} a_{ij}(p_1(v))\, Z(v)^{m_j - i}$, $v \in V$, $j = 1, \ldots, n$. Then $f_j \in I(X)_{m_j+1}$ and $a_0(f_j) = a_{0j}$, $j = 1, \ldots, n$. Hence $T_{x_0}X \supset A_{x_0}X$. Since $T_{x_0}X = C_{x_0}X$ (see Remark 1 in Section 2), we conclude that $A_{x_0}X = T_{x_0}X$. □

In the next lemma we keep the notation and the assumptions of Lemma 3.6.

Lemma 3.7. Let $Y = \pi(X \setminus \{x_0\}) \cup \pi(P_{x_0} \setminus \{x_0\})$. Then Y is an irreducible algebraic set of finite codimension in $\boldsymbol{P}(V')$ such that $I(Y) = I(X) \cap \mathbb{C}[V']$. If $h : V' \to \mathbb{C}$, $h \notin I(Y)$, is a bounded linear functional which vanishes on $\pi(P_{x_0} \setminus \{x_0\})$, then $\pi|_{X_h} : X_h \to \boldsymbol{P}(V')_h$ is a finite map (in the sense that the homomorphism $p_{(h)} : \mathbb{C}[V]_{(h)} \to S(X)_{(h)}$ is finite).

Proof. Let $f_j \in I(X)$, $j = 1, \ldots, n$, be the polynomials which were defined in the proof of Lemma 3.6. Choose homogeneous polynomials $G_j = \sum_{i=0}^{m_j} b_{ij}\, Z^{m_j - i} \in I(X)$, $j = 1, \ldots, r$ such that $G_j = f_j$ for $j = 1, \ldots, n$, and $X = Z(F_1, \ldots, F_r)$. Lemma 3.6 shows that $Z(b_{01}, \ldots, b_{0r}) = \pi(P_{x_0} \setminus \{x_0\})$. Let $D_1, \ldots, D_h \in \mathbb{C}[V']$ be a resultant system of $G_1, \ldots, G_r$ (see the proof of Proposition 2.1). The properties of $D_1, \ldots, D_h$ imply $Z(D_1, \ldots, D_h) = \pi(X \setminus \{x_0\}) \cup Z(b_{01}, \ldots, b_{0r}) = Y$, which proves that Y is an algebraic set of finite codimension in $\boldsymbol{P}(V')$.

We note that $a_0(f) = f$ for any homogeneous polynomial $f \in I(X) \cap \mathbb{C}[V']$. Hence any $f \in I(X) \cap \mathbb{C}[V']$ vanishes on both sets $\pi(X \setminus \{x_0\})$ and $\pi(P_{x_0} \setminus \{x_0\})$, which yields $I(X) \cap \mathbb{C}[V'] \subset I(Y)$. Since any $f \in I(Y)$ vanishes on X, we obtain $I(Y) = I(X) \cap \mathbb{C}[V']$.

Suppose that $h \in V'^*$ vanishes on $\pi(P_{x_0} \setminus \{x_0\}) = Z(b_{01}, \dots, b_{0n})$. Then h belongs to the subspace which is spanned by the linear functionals $h_j = b_{0j}$, $j = 0, \dots, n$, in V'^*, which shows that the ideal generated by h_j/h, $j = 1, \dots, n$, in $\mathbb{C}[V]_{(h)}$ is exactly $\mathbb{C}[V]_{(h)}$. Thus, in order to prove that the homomorphism $p_{(h)}$, is finite, it is enough to prove that all localisations $(p_{(h)})_{hj/h} : (\mathbb{C}[V']_{(h)})_{hj/h} \to (S(X)_{(h)})_{hj/h}$, $j = 1, \dots, n$, of $p_{(h)}$ are finite. In view of the natural isomorphisms $(\mathbb{C}[V']_{(h)})_{hj/h} \cong (\mathbb{C}[V']_{(h_j)})_{h/h_j}$ and $(S(X)_{(h)})_{hj/h} \cong (S(X)_{(h_j)})_{h/h_j}$, we see that it is enough to prove that all homomorphisms $p_{(h_j)} : \mathbb{C}[V']_{(h_j)} \to S(X)_{(h_j)}$, $j = 1, \dots, n$, are finite. To this end we note first, that $\mathbb{C}[V]_{(h_j)} \cong \mathbb{C}[V']_{(h_j)}[Z/h_j]$, $j = 1, \dots, n$. Let $d_j = \deg f_j$ $j = 1, \dots, n$. Then $h_j^{-d_j} f_j \in \mathbb{C}[V']_{(h_j)}[Z/h_j]$ is a monic polynomial which belongs to $I(X_{h_j})$, $j = 1, \dots, n$, whence all homomorphisms $p_{(h_j)} : \mathbb{C}[V']_{(h_j)} \to S(X)_{(h_j)}$, $j = 1, \dots, n$, are finite. □

Lemma 3.8. For any regular point x_0 of an irreducible algebraic subset X of finite codimension n and degree d in $\boldsymbol{P}(V)$, there exists an $n+1$-dimensional complex subspace U of V such that $\boldsymbol{P}(U)$ passes through x_0 and $\boldsymbol{P}(U)$ intersects X transversely in d regular points of X.

Proof. Since the claim is obvious when X is a linear projective subspace of finite codimension in $\boldsymbol{P}(V)$, we will assume that $\deg X > 1$. Let V' be a closed hyperplane in $\boldsymbol{P}(V)$ such that $x_0 \notin \boldsymbol{P}(V')$ and let $\pi : \boldsymbol{P}(V) \setminus \{x_0\} \to \boldsymbol{P}(V')$ be the map induced by the projection $p_1 : V \cong V' \times x_0 \to V'$. Then according to Lemma 3.7, the set $Y = \pi(X \setminus \{x_0\}) \cup \pi(P_{x_0} \setminus \{x_0\})$ is an irreducible algebraic set of codimension $n-1$ in $\boldsymbol{P}(V')$, and the set $\pi(P_{x_0}X \setminus \{x_0\})$ is a linear projective subspace of codimension n in $\boldsymbol{P}(V)$. Let (V'', W') be an admissible factorisation for Y in $\boldsymbol{P}(V')$ and let W be the n-dimensional vector subspace of V which is spanned by W'' and x_0. Then $\boldsymbol{P}(W) \cap (X \cup P_{x_0}X) = \{x_0\}$ because $\boldsymbol{P}(W') = \cap Y\emptyset$. Denote by $\pi' : \boldsymbol{P}(V) \setminus \boldsymbol{P}(W) \to \boldsymbol{P}(V'')$ (resp. $\pi'' : \boldsymbol{P}(V') \setminus \boldsymbol{P}(W') \to \boldsymbol{P}(V'')$) the map induced by the projection$V = V'' \times W \to V''$ (resp. $V' = V'' \times W' \to V''$). Since the set $\pi'(P_{x_0}X \setminus \{x_0\})$ is a projective hyperplane in $\boldsymbol{P}(V'')$, there exists a linear functional $h' \in V''^*$ such that $Z(h') = \pi'(P_{x_0}X \setminus \{x_0\})$. Let $h \in V'^*$ be given by the composition $V' = V'' \times W' \to V'' \xrightarrow{h'} \mathbb{C}$. Then h vanishes on $\pi(P_{x_0} \setminus \{x_0\})$ and the map $\pi|_{X_h} : X_h \to \boldsymbol{P}(V')_h$ is finite by Lemma 3.7. Taking into account that the map $\pi''|_Y : Y \to \boldsymbol{P}(V'')$ is finite and surjective, we conclude that $\pi'|_{X_h} : X_h \to \boldsymbol{P}(V'')_{h'}$ is a finite surjective map. Hence $\boldsymbol{P}(V)_h \cong \boldsymbol{P}(V'')_{h'} \times W$ is an admissible factorisation for X_h in $\boldsymbol{P}(V)_h$. According to Remark 3 in Section 2, there is a point $y \in \boldsymbol{P}(V'')_h$ such that the set $\pi'^{-1}(y) \cap X_h$ consists of regular points of X_h, and the intersection of $\pi'^{-1}(y)$ and X_h is transversal at each of these points. We also note that $\pi'^{-1}(y) \cap X_h \pi'^{-1}(y) \cap X$ and $\pi'^{-1}(y) \cap P_{x_0}X = \emptyset$ because $\pi'^{-1}(y) \subset \boldsymbol{P}(V)_h$. Let U be the $n+1$-dimensional complex vector subspace spanned by W and y in V. Since $\boldsymbol{P}(U) = \pi'^{-1}(y) \cup \boldsymbol{P}(W)$, we obtain $\boldsymbol{P}(U) \cap X = (\pi'^{-1}(y) \cap X_h) \cup \{x_0\}$ and $\boldsymbol{P}(U) \cap P_{x_0}X = \{x_0\}$. Hence the set $\boldsymbol{P}(U) \cap X$ consists of regular points of X

and the intersection of $\boldsymbol{P}(U)$ and X is transversal at each of those points. □

Proposition 3.9. If X is a submanifold of finite codimension in $\boldsymbol{P}(V)$, then for any $x_0 \in \boldsymbol{P}(V)$ there exists an admissible triple (V', W, z) for X such that $x_0 \notin \boldsymbol{P}(W)$ and $\pi(x_0) \in \boldsymbol{P}(V')_D$.

Proof. Let $n = \operatorname{codim}_{\boldsymbol{P}(V)} X$ and $d = \deg X$. Choose an $n+1$-dimensional complex vector subspace U of V which passes through x_0 and intersects X transversely in d regular points x_i, $i = 1, \ldots, d$, of X (see Remark 3 and Lemma 3.8). Choose an n-dimensional complex vector subspace W of U such that $x_i \notin \boldsymbol{P}(W)$, $i = 0, \ldots, d$. Let V' be a vector subspace of V which is complementary to W. Then (V', W) is an admissible factorisation for X and $\pi^{-1}(\pi(x_0)) = \boldsymbol{P}(U) \setminus \boldsymbol{P}(W)$. Let $y = \pi(x_0) = [v_0']$, $v_0' \in V'$, and let $x_i = [v_0' + w_i]$, $w_i \in W$, $i = 1, \ldots, d$. Choose a linear functional $Z : W \to \mathbb{C}$ such that $Z(w_i) \neq Z(w_j)$ for $i \neq j$, and set $z = Z + I(X) \in S(X)$. Let $F(v', Z) \in \mathbb{C}[V'][Z]$ be the minimal polynomial of z over the field of fractions of $\mathbb{C}[V']$. Since $F(v_0', Z(w_i)) = 0$, $i = 1, \ldots, d$, the degree of the polynomial F is equal to d. Hence z is a generator of the field of fractions of $S(X)$ over the field of fractions of $\mathbb{C}[V]$ and $\pi(x_0) \in \boldsymbol{P}(V')_D$. □

Corollary 3.10. For any submanifold X of finite codimension in $\boldsymbol{P}(V)$, there exists a family of admissible triples $\{(V_i', W_i, z_i)\}_{i \in I}$, for which the family of open sets $\{\boldsymbol{P}(V)_{D_i}\}_{i \in I}$ is an open covering of $\boldsymbol{P}(V)$.

Proof. This is just a rephrasing of the previous proposition. □

4. A REPRESENTATION THEOREM FOR DIFFERENTIAL FORMS

In this section, we consider a submanifold X of finite codimension in $\boldsymbol{P} = \boldsymbol{P}(V)$ and an admissible triple (V', W, z) for X in $\boldsymbol{P}$. We note that $p^*\mathcal{O}_{\boldsymbol{P}(V')}(k) = \mathcal{O}_X(k)$, where $\mathcal{O}_X(k)$ is the restriction of the line bundle $\mathcal{O}_{\boldsymbol{P}}(k)$ to X, $k \in \mathbb{Z}$. According to Proposition 3.4, the map $p|_{X_D} : X_D \to \boldsymbol{P}(V')_D$ is a finite covering of degree $d = \deg X$. We will show that, for any differential form $g \in C^r_{p,q}(X_D, \mathcal{O}_X(k))$, there exist unique differential forms $g_j \in C^r_{p,q}(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')}(k-j))$, $j = 0, \ldots, d-1$, such that

$$g = \sum_{j=0}^{d-1} (p|_{X_D})^* g_j \otimes z^j . \tag{4.1}$$

Representation (4.1) will be derived in a more general setting. Let Y and Z be complex manifolds and let $\pi : Y \to Z$ be a covering map of finite degree d. Let $L \to Z$ be a given holomorphic line bundle over Z and let $M \to Y$ be the line bundle $\pi^* L$. The ring $\oplus_{n \in \mathbb{Z}} H^0(Z, L^n)$ will be denoted by S.

Proposition 4.1. Let $s \in H^0(Y, M)$ and $a_i \in H^0(Z, L^i)$, $i = 1, \ldots, d$, be such that $s^d + (\pi^* a_1) s^{d-1} + \cdots + (\pi^* a_{d-1}) s + \pi^* a_d = 0$. If the discriminant $D \in$

$H^0(Z, L^{d(d-1)})$ of the polynomial $Z^d + a_1 Z^{d-1} + \cdots + a_{d-1} Z + a_d \in S[Z]$ vanishes nowhere on Z then for any given differential form $g \in C^r_{p,q}(Y, M^k)$, $n \in \mathbb{Z}$, there exist unique differential forms $g_j \in C^r_{p,q}(Z, L^{k-j})$, $j = 0, \dots d-1$, such that

$$g = \sum_{j=0}^{d-1} \pi^* g_j \otimes s^j . \tag{4.2}$$

Furthermore, the differential form g is $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact) if and only if all differential forms g_j, $j = 0, \dots, d-1$, are $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact).

Proof. Since the claim is local with respect to Z, we may assume that $L = \mathcal{O}_Z$ and that π has d distinct right inverses $r_i : Z \to Y$, $qr_i = \mathrm{id}_Z$, $i = 1, \dots, d$. Let $s_i = r_i^* s \in H^0(Z, \mathcal{O}_Z)$, $i = 1, \dots, d$. Then $s_{i_1}(b) \neq s_{i_2}(b)$ for $i_1 \neq i_2$ and $b \in Z$ because D vanishes nowhere on Z. Eq. (4.2) is equivalent to the linear system

$$\sum_{j=0}^{d-1} s_i^j g_j = r_i^* g , \qquad i = 1, \dots, d .$$

Its determinant is

$$\Delta = \prod_{1 \le i_1 < i_2 \le d} (s_{i_2} - s_{i_1}) .$$

Since $\Delta^2 = D$, the holomorphic function Δ vanishes nowhere on Z. Thus the differential forms g_j, $j = 1, \dots, d$, are determined uniquely by Cramer's formulae: $g_j = \Delta^{-1} \det(A_j)$, $j = 0, \dots, d-1$, where A_j, $j = 0, \dots, d-1$, is the matrix

$$A_j = \begin{pmatrix} 1 & s_1 & \cdots & s_1^{j-1} & r_1^* g & s_1^{j+1} & \cdots & s_1^{d-1} \\ 1 & s_2 & \cdots & s_2^{j-1} & r_2^* g & s_2^{j+1} & \cdots & s_2^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & s_d & \cdots & s_d^{j-1} & r_d^* g & s_d^{j+1} & \cdots & s_d^{d-1} \end{pmatrix} .$$

Since $r_i^* g \in C^r_{p,q}(Z)$ for $i = 1, \dots, d$, all differential forms g_j, $j = 0, \dots, d-1$, also belong to $C^r_{p,q}(Z)$. We note that $\overline{\partial} g = \sum_{j=0}^{d-1} \pi^*(\overline{\partial} g_j) \otimes s^j$ because the section s is holomorphic. Since the representation (4.2) is unique and the homomorphism π^* is injective, the differential forms g_j, $j = 0, \dots, d-1$, are $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact) if and only if the differential form g is $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact). □

Now we can deal with Representation (4.1).

Proposition 4.2. Let (V', W, z) be an admissible triple for X in $\boldsymbol{P}$. Then for any differential form $g \in C^r_{p,q}(X_D, \mathcal{O}_X(k))$, $k \in \mathbb{Z}$, there exist unique differential forms $g_j \in C^r_{p,q}(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')}(k-j))$, $j = 0, \dots, d-1$, such that

$$g = \sum_{j=0}^{d-1} (p|_{X_D})^* g_j \otimes z^j .$$

Furthermore, the differential form g is $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact) if and only if all differential forms g_j, $j = 0, \ldots, d-1$, are $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact).

Proof. By Proposition 3.4, the holomorphic map $p|_{X_D} : X_D \to \boldsymbol{P}(V')_D$ is a finite covering of degree $d = \deg X$. Proposition 4.2 now follows from Proposition 4.1 with $Y = X_D$, $Z = \boldsymbol{P}(V')_D$, $L = \mathcal{O}_{\boldsymbol{P}(V')}(1)|_{\boldsymbol{P}(V')_D}$, and $s = z|_{X_D}$. □

We also need a version of Proposition 4.2 for an algebraic submanifold X of finite codimension n in a Banach space V. Let (V', W) be an admissible factorisation for X as in Section 2, and let $Z_1, \ldots, Z_n$ be a basis of W^* such that $z = Z_1 + I(X) \in \mathbb{C}[X]$ generates the field of fractions of $\mathbb{C}[X]$ over the field of fractions of $\mathbb{C}[V']$. Let $D \in \mathbb{C}[V']$ be the discriminant of the minimal polynomial F of z over the field of fractions of $\mathbb{C}[V']$. By Proposition 2.3, the holomorphic map $p_D = p|_{X_D} : X_D \to V'_D$ is a covering of degree $d = \deg F$. We note that the vector bundle $T^{p,q}V' \to V'$ is canonically isomorphic to the trivial bundle $V' \times \wedge^p V' \wedge^q \overline{V'} \to V'$.

Proposition 4.3. For any differential form $g \in C^r_{p,q}(X_D)$, there exist unique differential forms $g_j \in C^r_{0,q}(V'_D)$, $j = 0, \ldots, d-1$, such that

$$g = \sum_{j=0}^{d-1} z^j \, p_D^* \, g_j \,.$$

If U is an open subset of V'_D such that p_D has d distinct right inverses $r_i : U \to X_D$ on U, $\pi \circ r_i = \mathrm{id}_U$, $i = 1, \ldots, d$, then

$$g_j(b, \xi, \overline{\xi}) = D(b)^{-1} \, \Delta(b) \, \det A_j(b, \xi, \overline{\xi}) \,, \qquad j = 0, \ldots, d-1 \,,$$

for $b \in U$, $\xi \in \wedge^p V'$, $\overline{\xi} \in \wedge^q \overline{W'}$, where

$$\Delta(b) = \prod_{1 \le i_1 < i_2 \le d} (z(r_{i_2}(b)) - z(r_{i_1}(b)))$$

and $A_j(b, \xi, \overline{\xi})$ is the $d \times d$ matrix

$$\begin{pmatrix} 1 & z(r_1(b)) & \cdots & z(r_1(b))^{j-1} & r_1^* g\,(b, \xi, \overline{\xi}) & z(r_1(b))^{j+1} & \cdots & z(r_1(b))^{d-1} \\ 1 & z(r_2(b)) & \cdots & z(r_2(b))^{j-1} & r_2^* g\,(b, \xi, \overline{\xi}) & z(r_2(b))^{j+1} & \cdots & z(r_2(b))^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(r_d(b)) & \cdots & z(r_d(b))^{j-1} & r_d^* g\,(b, \xi, \overline{\xi}) & z(r_d(b))^{j+1} & \cdots & z(r_d(b))^{d-1} \end{pmatrix}.$$

Proof. We use Proposition 4.2 with $Y = X_D$, $Z = V'_D$, $L = \mathcal{O}_{V'_D}$, and $s = z$. Since $s_i = r_i^*(s) = z \circ r_i$, we have $s_i(b) = z(r_i(b))$, $i = 1, \ldots, d$. Hence

$$\begin{aligned} \Delta(b)^2 &= \prod_{1 \le i_1 < i_2 \le d} (s_{i_2}(b) - s_{i_1}(b))^2 \\ &= \prod_{1 \le i_1 < i_2 \le d} (z(r_{i_2}(b)) - z(r_{i_1}(b)))^2 = D(b) \,, \end{aligned}$$

which gives $D(b)^{-1}\Delta(b) = \Delta(b)^{-1}$. The claim now follows from Cramer's formulae as in the proof of Proposition 4.2. □

Propositions 4.2 and 4.3 are used in full generality in [3]. Here we use them only for $p = q = 0$. Let us consider first the affine case.

Lemma 4.4. We keep the assumptions and the notation of Proposition 4.3. Let $g \in H^0(X, \mathcal{O}_X)$ and $g_j \in H^0(V'_D, \mathcal{O}_{V'})$, $j = 0, \ldots, d-1$, be such that

$$g = \sum_{j=0}^{d-1} z^j p_D^* g_j .$$

Then there exist $\tilde{g}_j \in H^0(V', \mathcal{O}_{V'})$, $j = 0, \ldots, d-1$ such that $Dg_j = \tilde{g}_j|_{V'_D}$ for $j = 0, \ldots, d-1$.

Proof. By virtue of Riemann's removable singularity theorem it is enough to show that, for any $b_0 \in Z(D)$, there is a neighbourhood U of b_0 such that all functions Dg_j, $j = 0, \ldots, d-1$, are bounded on $U \cap V'_D$. Let $G_j : X^d \to \mathbb{C}$, $j = 0, \ldots, d-1$, be the holomorphic function given by

$$G_j(x_1, x_2, \ldots, x_d) = \Delta(x_1, x_2, \ldots, x_d) \det A_j(x_1, x_2, \ldots, x_d)$$

where

$$\Delta(x_1, x_2, \ldots, x_d) = \prod_{1 \le i_1 < i_2 \le d} (z(x_{i_2}) - z(x_{i_1}))$$

and $A_j(x_1, x_2, \ldots, x_d)$ is the matrix

$$\begin{pmatrix} 1 & z(x_1) & \cdots & z(x_1)^{j-1} & g(x_1) & z(x_1)^{j+1} & \cdots & z(x_1)^{d-1} \\ 1 & z(x_2) & \cdots & z(x_2)^{j-1} & g(x_2) & z(x_2)^{j+1} & \cdots & z(x_2)^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(x_d) & \cdots & z(x_d)^{j-1} & g(x_d) & z(x_d)^{j+1} & \cdots & z(x_d)^{d-1} \end{pmatrix}.$$

Since the fiber $p^{-1}(b_0)$ is a finite set, there is a neighbourhood N of $p^{-1}(b_0)$ such that the functions G_j, $j = 0, \ldots, d-1$, are bounded on N^d and since $p : X \to V'$ is a proper map, there is a neighbourhood U of b_0 in V' such that $p^{-1}(U) \subset N$. By Proposition 4.3, $D(b)g_j(b) = G_j(x_1(b), x_2(b), \ldots, x_d(b))$, $j = 0, \ldots, d-1$, for $b \in V'_D$, where $\{x_1(b), x_2(b), \ldots, x_d(b)\} = p^{-1}(b)$. Hence all functions g_j, $j = 0, \ldots, d-1$, are bounded on $U \cap V'_D$ because $\{x_1(b)\} \times \{x_2(b)\} \times \ldots \times \{x_d(b)\} \in N^d$ for any $b \in U \cap V'_D$. □

Lemma 4.5. We keep the assumptions and the notation of Proposition 4.2. Let $g \in H^0(X, \mathcal{O}_X(k))$, $k \in \mathbb{Z}$, and $g_j \in H^0(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')}(k-j))$, $j = 0, \ldots, d-1$, be such that

$$g|_{X_D} = \sum_{j=0}^{d-1} (p|_{X_D})^* g_j \otimes z^j .$$

Then there exist $\tilde{g}_j \in H^0(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(k - j + \deg D))$, $j = 0, \ldots, d-1$, such that $Dg_j = \tilde{g}_j|_{\boldsymbol{P}(V')_D}$ for $j = 0, \ldots, d-1$.

Proof. Let $s = \deg D = d(d-1)$. It is enough to show that for any $h \in V'^*$, $h \neq 0$, the holomorphic functions $Dg_j h^{j-k-s} \in H^0(\boldsymbol{P}(V')_D \cap \boldsymbol{P}(V')_h, \mathcal{O}_{\boldsymbol{P}(V')_h})$, $j = 0, \ldots, d-1$, can be extended to holomorphic functions on $\boldsymbol{P}(V')_h$. Since

$$gh^{-k}|_{X_D \cap X_h} = \sum_{j=0}^{d-1} (z/h)^j \, (p|_{X_D \cap X_h})^* (g_j h^{j-k}),$$

this follows from Lemma 4.4. □

Proposition 4.6. If X is a submanifold of finite codimension in $\boldsymbol{P} = \boldsymbol{P}(V)$ and (V', W, z) is an admissible triple for X in $\boldsymbol{P}$, then for any $g \in H^0(X, \mathcal{O}_X(k))$, $k \in \mathbb{Z}$, there exists $\tilde{g} \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k + \deg D))$ such that $Dg|_{X_D} = \tilde{g}|_{X_D}$.

Proof. Let $g_j \in H^0(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')})(k-j)$, $j = 0, \ldots, d-1$, and $\tilde{g}_j \in H^0(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(k - j + \deg D))$, $j = 0, \ldots, d-1$, be as in Lemma 4.5. Choose a bounded linear functional $Z : V \to \mathbb{C}$ such that $Z|X = Z + I(X) = z$, and set $\tilde{g}_0 = \sum_{j=0}^{d-1} \pi^* \tilde{g}_j \otimes Z^j \in H^0(\boldsymbol{P} \setminus \boldsymbol{P}(W), \mathcal{O}_{\boldsymbol{P}}(k + \deg D))$, where π is the vector bundle $\boldsymbol{P} \setminus \boldsymbol{P}(W) \to \boldsymbol{P}(V')$. Since $\operatorname{codim}_{\boldsymbol{P}} \boldsymbol{P}(W) < 1$, there exists a $\tilde{g} \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k + \deg D))$ such that $\tilde{g}|_{\boldsymbol{P} \setminus \boldsymbol{P}(W)} = \tilde{g}_0$. According to Lemma 4.5, we have $\tilde{g}_0|_{X_D} = Dg|_{X_D}$ which implies $\tilde{g}|_{X_D} = Dg|_{X_D}$. □

5. A DOLBEAULT ISOMORPHISM FOR INFINITE-DIMENSIONAL PROJECTIVE SPACES

In this section, we will assume that V is a Banach space which admits smooth partitions of unity. In general we say that manifold X admits smooth partitions of unity if for any open cover $\{U_i\}_{i \in I}$ of X there are $\theta_i \in C^\infty(X)$, supported in U_i such that $\sum_{i \in I} \theta_i = 1$, the sum being locally finite. Hilbert spaces are examples of such manifolds. Separable and reflexive Banach spaces that localise are other examples. Paracompact manifolds modeled on spaces that admit smooth partitions of unity also admit smooth partitions of unity. We refer to [2] for more details. In particular if V is a Banach space that admits smooth partitions of unity, then the associated projective space $\boldsymbol{P} = \boldsymbol{P}(V)$ also admits smooth partitions of unity.

For a finite-dimensional complex manifold X, the Dolbeault cohomology groups and the Čech cohomology groups of a holomorphic vector bundle on X are the same by the Dolbeault isomorphism. Let X be a submanifold of finite codimension in $\boldsymbol{P}$ and let $E \to X$ be a holomorphic vector bundle over X. We will consider a covering $\{X_i\}_{i \in I}$ of X with Zariski open sets and define a complex $\mathcal{C}(X, E)$ which is a subcomplex of the usual Čech complex associated with $\{X_i\}_{i \in I}$ and E. In this section, we will prove that $H^q(\mathcal{C}(\boldsymbol{P}, E)) \cong H^{0,q}(\boldsymbol{P}, E)$ for $q \geq 0$. Since

$H^{0,q}(\boldsymbol{P}, E) = 0$ for $q \geq 1$ (see [4, Theorem 7.3]), we obtain $H^q(\mathcal{C}(\boldsymbol{P}, E)) = 0$ for $q \geq 1$. The vanishing of the higher cohomology groups of the complex $\mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$, $n \in \mathbb{Z}$, is used in the next section.

Let $\mathcal{D} = \{D_i \in C[V]\}_{i \in I}$ be a collection of homogeneous polynomials such that $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i \in I}$ is a covering of $\boldsymbol{P}$. The degree of the polynomial D_i will be denoted by d_i, $i \in I$. The open set

$$X_{D_{i_0}} \cap \ldots \cap X_{D_{i_q}} = \{x \in X \ : \ D_{i_m}(x) \neq 0 \text{ for } m = 0, \ldots, q\}$$

will be denoted by $X_{i_0 \ldots i_q}$. The complex $\mathcal{C}(X, E)$, corresponding to the collection $\mathcal{D}$, is defined in the following way. For any natural number N, we define first a complex of abelian groups

$$\mathcal{C}_N(X, E) = \{\mathcal{C}_N^q(X, E), \delta\}_{q \geq 0}$$

as follows: Let $\mathcal{C}_N^q(X, E)$ be the subgroup of $\prod_{i_0, \ldots, i_q \in I} H^0(X_{i_0 \ldots i_q}, E)$ that consists of all $\varphi = \{\varphi_{i_0 \ldots i_q} \in H^0(X_{i_0 \ldots i_q}, E)\}_{i_0, \ldots, i_q \in I}$ such that for any $i_0, \ldots, i_q \in I$ there exists a global section $\tilde{\varphi}_{i_0 \ldots i_q} \in H^0(X, E \otimes \mathcal{O}_X(Nd_{i_0} + \ldots + Nd_{i_q}))$ for which

$$\varphi_{i_0 \ldots i_q} = (D_{i_0} \cdots D_{i_q})^{-N} (\tilde{\varphi}_{i_0 \ldots i_q})|_{X_{i_0 \ldots i_q}} .$$

We use the standard convention for alternate cochains: if $\{m_0 \ldots m_q\}$ is a permutation of $\{0 \ldots q\}$, then $\varphi_{i_{m_0} \ldots i_{m_q}} = (-1)^{\epsilon(m_0 \ldots m_q)} \varphi_{i_0 \ldots i_q}$, where $\epsilon(m_0 \ldots m_q)$ is the parity of the permutation $\{m_0 \ldots m_q\}$. The differential $\delta : \mathcal{C}_N^q(X, E) \to \mathcal{C}_N^{q+1}(X, E)$ is the Čech coboundary operator:

$$(\delta\varphi)_{i_0 \ldots i_{q+1}} = \sum_{m=0}^{q+1} (-1)^m \, \varphi_{i_0 \ldots \widehat{i_m} \ldots i_{q+1}}|_{X_{i_0 \ldots i_{q+1}}} . \tag{5.1}$$

Since

$$(D_{i_0} \cdots D_{i_{q+1}})^N (\delta\varphi)_{i_0 \ldots i_{q+1}} = \left\{ \sum_{m=0}^{q+1} (-1)^m \, (D_{i_m}^N \, \tilde{\varphi}_{i_0 \ldots \widehat{i_m} \ldots i_{q+1}}) \right\} \Big|_{X_{i_0 \ldots i_{q+1}}} , \tag{5.2}$$

δ is well defined. We note that for any $N \in \mathbb{N}$, there is a natural injective chain map

$$\mathcal{C}_N(X, E) \to \mathcal{C}_{N+1}(X, E) .$$

The complex $\mathcal{C}(X, E)$ is now defined as the union of all complexes $\mathcal{C}_N(X, E)$:

$$\mathcal{C}(X, E) = \bigcup_{N=0}^{\infty} \mathcal{C}_N(X, E) . \tag{5.3}$$

Remark 1. The definition of the complex $\mathcal{C}(X, E)$ was suggested by the proof of [4, Theorem 8.2]. We note, however, that the proof of [4, Theorem 8.2] is not rigorous since it assumes implicitly that $\boldsymbol{P}$ is paracompact with the Zariski topology. We also note that the complex $\mathcal{C}(X, E)$ depends not only on the covering $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i\in I}$ but also on the collection of polynomials $\mathcal{D} = \{D_i\}_{i\in I}$. For example, let $I = \mathbb{N}$ and let $\mathcal{D} = \{D_i \in C[V]\}_{i\in I}$ be a collection of homogeneous polynomials such that $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i\in I}$ is a covering of $\boldsymbol{P}$. Let $D'_i = (D_i)^i$, $i \in I$. Then the covering $\mathcal{U}' = \{\boldsymbol{P}_{D'_i}\}_{i\in I}$ is the same as the covering $\mathcal{U}$, but the complex $\mathcal{C}'(X, E)$, corresponding to the collection of homogeneous polynomials $\mathcal{D}' = \{D'_i\}_{i\in I}$, is not necessarily the same as the complex $\mathcal{C}(X, E)$.

The next theorem is the main result of this section.

Theorem 5.1. Let V be a complex Banach space that admits smooth partitions of unity and let $\boldsymbol{P} = \boldsymbol{P}(V)$. Let $\{D_i\}_{i\in I}$ be a collection of homogeneous polynomials such that $\boldsymbol{P} = \cup_{i\in I}\boldsymbol{P}_{D_i}$. Then $H^q(\mathcal{C}(\boldsymbol{P}, E)) = 0$, $q \geq 1$, for any holomorphic vector bundle $E \to \boldsymbol{P}$ of finite rank over $\boldsymbol{P}$.

To prove Theorem 5.1, it is enough to show that the group $H^q(\mathcal{C}_N(\boldsymbol{P}, E)) = 0$ for $N \in \mathbb{N}$, $q \geq 1$. We will prove that $H^q(\mathcal{C}_N(\boldsymbol{P}, E)) \cong H^{0,q}(\boldsymbol{P}, E)$ for $N \in \mathbb{N}$, $q \geq 0$. To this end we define a double complex $\mathcal{B}_N(\boldsymbol{P}, E) = \{\mathcal{B}_N^{pq}(\boldsymbol{P}, E, \delta, \overline{\partial}\}_{p,q\geq 0}$ as follows. Let $\mathcal{B}_N^{pq}(\boldsymbol{P}, E)$ be the subgroup of the group $\prod\limits_{i_0,\ldots,i_q\in I} C^\infty_{0,p}(\boldsymbol{P}_{i_0\ldots i_q}, E)$ that consists of all $\varphi \in \prod\limits_{i_0,\ldots,i_q\in I} C^\infty_{0,p}(\boldsymbol{P}_{i_0\ldots i_q}, E)$ such that for any $i_0, \ldots, i_q \in I$ there exists a global section

$$\tilde{\varphi}_{i_0\ldots i_q} \in C^\infty_{0,p}(\boldsymbol{P}, E \otimes \mathcal{O}_{\boldsymbol{P}}(Nd_{i_0} + \ldots + Nd_{i_q}))$$

for which

$$\varphi_{i_0\ldots i_q} = (D_{i_0}\cdots D_{i_q})^{-N}\tilde{\varphi}_{i_0\ldots i_q}|_{\boldsymbol{P}_{i_0\ldots i_q}}.$$

The differential

$$\delta : \mathcal{B}_N^{pq}(\boldsymbol{P}, E) \to \mathcal{B}_N^{p\,q+1}(\boldsymbol{P}, E)$$

is given by formula (5.1). Formula (5.2) shows that δ is well defined. The differential

$$\overline{\partial} : \mathcal{B}_N^{pq}(\boldsymbol{P}, E) \to \mathcal{B}_N^{p+1\,q}(\boldsymbol{P}, E)$$

is given by

$$(\overline{\partial}\varphi)_{i_0\ldots i_q} = \overline{\partial}\varphi_{i_0\ldots i_q}, \qquad i_0, \ldots, i_q \in I,$$

for $\varphi \in \mathcal{B}_N^{pq}(\boldsymbol{P}, E)$. Since

$$(D_{i_0}\cdots D_{i_q})^N(\overline{\partial}\varphi)_{i_0\ldots i_q} = \overline{\partial}((D_{i_0}\cdots D_{i_q})^N\varphi_{i_0\ldots i_q}) = \overline{\partial}(\tilde{\varphi}_{i_0\ldots i_q}|_{\boldsymbol{P}_{i_0\ldots i_q}}),$$

we see that $\overline{\partial}$ is well defined, too.

Lemma 5.2. Let V be a Banach space that admits smooth partitions of unity and let $\boldsymbol{P} = \boldsymbol{P}(V)$. Then, for any $n \in \mathbb{Z}$ and $N \in \mathbb{N}$:

i) $H^p_{\overline{\partial}}(\mathcal{B}_N^{\cdot q}(\boldsymbol{P}, E)) = 0$ for $p \geq 1$;
ii) $H^q_\delta(\mathcal{B}_N^{p\cdot}(\boldsymbol{P}, E)) = 0$ for $q \geq 1$.

Proof. Suppose $p \geq 1$, and let $\varphi \in \mathcal{B}_N^{pq}(\boldsymbol{P}, E)$ be such that $\overline{\partial}\varphi = 0$. For $i_0, \ldots, i_q \in I$, let $\tilde{\varphi}_{i_0 \ldots i_q} \in C^\infty_{0,p}(\boldsymbol{P}, E \otimes \mathcal{O}_{\boldsymbol{P}}(Nd_{i_0} + \ldots + Nd_{i_q}))$ be the unique form such that

$$(D_{i_0} \cdots D_{i_q})^N \varphi_{i_0 \ldots i_q} = \tilde{\varphi}_{i_0 \ldots i_q}|_{\boldsymbol{P}_{i_0 \ldots i_q}}.$$

Since the form $\varphi_{i_0 \ldots i_q}$ is closed, the form $\tilde{\varphi}_{i_0 \ldots i_q}$ is closed, too. By [4, Theorem 7.3] there exists a form $\tilde{\psi}_{i_0 \ldots i_q} \in C^\infty_{0,p-1}(\boldsymbol{P}, E \otimes \mathcal{O}_{\boldsymbol{P}}(Nd_{i_0} + \ldots + Nd_{i_q}))$ such that $\overline{\partial}\tilde{\psi}_{i_0 \ldots i_q} = \tilde{\varphi}_{i_0 \ldots i_q}$. Let $\psi \in \mathcal{B}_N^{p-1\,q}(\boldsymbol{P}, E)$ be the cochain given by

$$\psi_{i_0 \ldots i_q} = (D_{i_0} \cdots D_{i_q})^{-N} (\tilde{\psi}_{i_0 \ldots i_q}|_{\boldsymbol{P}_{i_0 \ldots i_q}}) \in C^\infty_{0,p-1}(\boldsymbol{P}_{i_0 \ldots i_q}, E), \; i_0, \ldots, i_q \in I.$$

Since $\overline{\partial}\psi_{i_0 \ldots i_q} = \varphi_{i_0 \ldots i_q}$ for all $i_0, \ldots i_q \in I$, we obtain $\overline{\partial}\psi = \varphi$. This proves (i).

Suppose $q \geq 1$, and let $\varphi \in \mathcal{B}_N^{pq}(\boldsymbol{P}, E)$ be such that $\delta\varphi = 0$. Let $\{\theta_i\}$ be a smooth partition of unity which is subordinated to the open covering $\{\boldsymbol{P}_i\}_{i \in I}$, and let $\overline{\varphi} \in \prod_{i_0, \ldots, i_{q-1} \in I} C^\infty_{0,p}(\boldsymbol{P}_{i_0 \ldots i_{q-1}}, E)$ be the cochain given by

$$\overline{\varphi}_{i_0 \ldots i_{q-1}} = \sum_{i \in I} \theta_i \varphi_{i i_0 \ldots i_{q-1}} \tag{5.4}$$

for $i_0, \ldots, i_{q-1} \in I$. (On the right-hand side of (5.4) we use alternate cochains.) A simple calculation shows that $\delta\overline{\varphi} = \varphi$ (see [1, Proposition 8.5]). Let us verify that $\overline{\varphi} \in \mathcal{B}^{p\,q-1}(\boldsymbol{P}, E)$.

$$(D_{i_0} \cdots D_{i_{q-1}})^N \overline{\varphi}_{i_0 \ldots i_{q-1}} =$$

$$= \left\{ \sum_{i \in I} \theta_i \, (D_i|_{\boldsymbol{P}_i})^{-N} \left[(D_i D_{i_0} \cdots D_{i_{q-1}})^N \varphi_{i i_0 \ldots i_{q-1}} \right] \right\} \Big|_{V_{i_0 \ldots i_{q-1}}}$$

$$= \left\{ \sum_{i \in I} \theta_i \, (D_i|_{\boldsymbol{P}_i})^{-N} \, \tilde{\varphi}_{i i_0 \ldots i_{q-1}} \right\} \Big|_{V_{i_0 \ldots i_{q-1}}}$$

Since $\theta_i (D_i|_{\boldsymbol{P}_i})^{-N} \in C^\infty(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(-Nd_i))$ and $\operatorname{supp} \theta_i (D_i|_{\boldsymbol{P}_i})^{-N} \subset \operatorname{supp} \theta_i$ for all $i \in I$, we obtain

$$\sum_{i \in I} \theta_i \, (D_i|_{\boldsymbol{P}_i})^{-N} \, \tilde{\varphi}_{i i_0 \ldots i_{q-1}} \; \in \; C^\infty_{0,p}(\boldsymbol{P}, E \otimes \mathcal{O}_{\boldsymbol{P}}(Nd_{i_0} + \ldots + Nd_{i_{q-1}})).$$

Hence $\overline{\varphi} \in \mathcal{B}^{p\,q-1}(\boldsymbol{P}, E)$. Since $\delta\overline{\varphi} = \varphi$, part (ii) has been proved. □

Remark 2. We note that we were able to prove part (ii) of Lemma 5.2 because the same N "worked" for all cochains $\varphi_{i_0 \ldots i_q}$, $i_0, \ldots, i_q \in I$, (cf. Remark 1.)

Now we can give a proof of Theorem 5.1.

Proof of Theorem 5.1. As it has already been mentioned, it is enough to show that $H^q(\mathcal{C}_N(\boldsymbol{P}, E)) = 0$ for all $q \geq 1$, $n \in \mathbb{Z}$, and $N \in N$. It is well known (see for example [1, Proposition 8.8]) that if conditions (i) and (ii) of Lemma 5.1 hold for a double complex $B = \{\mathcal{B}^{pq}, d', d''\}_{p,q\geq 0}$, then the groups $H^q_{d'}(H^0_{d''}(B))$ and $H^q_{d''}(H^0_{d'}(B))$ are naturally isomorphic for all $q \geq 0$. We note that the complex $H^0_\delta(B(\boldsymbol{P}, E))$ is the Dolbeault complex of the vector bundle E on $\boldsymbol{P}$, and the complex $H^0_{\bar\partial}(B(\boldsymbol{P}, E))$ is just the complex $\mathcal{C}_N(\boldsymbol{P}, E)$. Since the higher Dolbeault cohomology groups of the vector bundle E on $\boldsymbol{P}$ vanish by [4, Theorem 7.3], we obtain $H^q(\mathcal{C}_N(\boldsymbol{P}, E)) = 0$ for $q \geq 1$, $N \in \mathbb{N}$. □

6. A DOLBEAULT ISOMORPHISM FOR COMPLETE INTERSECTIONS IN INFINITE-DIMENSIONAL PROJECTIVE SPACES

In this section we assume that X is a complete intersection in $\boldsymbol{P}$, and that $\{(V_i', W_i, z_i)\}_{i\in I}$ is a collection of admissible triples for X such that $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i\in I}$ is a covering of $\boldsymbol{P}$. According to Corollary 3.10, such collections exist for every submanifold X of finite codimension in $\boldsymbol{P}$. Let $\mathcal{C}(X, \mathcal{O}_X(k))$ be the complex (5.3) corresponding to the collection of homogeneous polynomials $\mathcal{D} = \{D_i\}_{i\in I}$ and to the line bundle $\mathcal{O}_X(k)$, $k \in \mathbb{Z}$. In this setup all polynomials D_i, $i \in I$, are of the same degree $d_i = d(d-1)$, where $d = \deg X$. We will show that if $\boldsymbol{P}$ admits smooth partitions of unity, then $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q \geq 1$, $k \in \mathbb{Z}$.

Before dealing with the general case, let us outline the argument in the case of a hypersurface. Suppose X is the set of zeros of a homogeneous polynomial $P \in C[V]$ of degree d. Then multiplication by P yields the exact sequence of sheaves

$$0 \leftarrow \mathcal{O}_X(k) \leftarrow \mathcal{O}_{\boldsymbol{P}}(k) \xleftarrow{P} \mathcal{O}_{\boldsymbol{P}}(k-d) \leftarrow 0\,. \tag{6.1}$$

The sequence (6.1) induces an exact sequence of complexes

$$0 \leftarrow \mathcal{C}(X, \mathcal{O}_X(k)) \leftarrow \mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)) \xleftarrow{P} \mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k-d)) \leftarrow 0 \tag{6.2}$$

Since by Theorem 5.1 $H^q(\mathcal{C}(P, \mathcal{O}_{\boldsymbol{P}}(k))) = 0$ for $q \geq 1$, $k \in \mathbb{Z}$, the long exact sequence of cohomology groups that is associated with the short exact sequence (6.2) yields $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q \geq 1$, $k \in \mathbb{Z}$. This argument carries over to complete intersections in $\boldsymbol{P}$ because the exact sequence (6.1) is a special case of the Koszul complex.

To define complete intersections in $\boldsymbol{P}$, we need the notion of a regular sequence from commutative algebra.

Let A be a commutative ring and let $a_1, \ldots, a_n \in A$. Let $I_j, j = 1, \ldots, n$, be the ideal generated by $a_1, \ldots, a_j$ in A. The sequence $a_1, \ldots, a_j$ is called *regular* if $I_k \neq A$ and $a_j + I_{j-1}$ is not a zero divisor in the factor-ring A/I_{j-1} for $j = 1, \ldots, n$.

Given a commutative ring A and $a_1, \ldots, a_n \in A$, we define a complex K as follows:

$$K_0 \leftarrow \cdots \leftarrow K_{p-1} \xleftarrow{\mathrm{d}} K_p \leftarrow \cdots \leftarrow K_n \leftarrow 0$$

(cf. [5]) Set $K_0 = A$. For $1 \leq p \leq k$, let $K_p = \oplus A e_{j_1 \dots j_p}$ be the free A-module of rank $\binom{k}{p}$ with basis $\{e_{j_1 \dots j_p}\}_{1 \leq j_1 < \dots < j_p \leq k}$. The differential $\mathrm{d} : K_p \to K_{p-1}$ is given by

$$\mathrm{d}(e_{j_1 \dots j_p}) = \sum_{r=1}^{p} (-1)^{r-1} a_{j_r} e_{j_1 \dots \hat{j}_r \dots j_p} ;$$

(for $p = 1$, set $\mathrm{d}(e_j) = a_j$). One checks easily that $\mathrm{dd} = 0$. The complex K is called the Koszul complex corresponding to $a_1, \dots, a_n$. We note that $d(K_1) = I_n \subset A$ and $\operatorname{coker}\{K_0 \leftarrow K_1\} = A/I_n$.

The proof of the following important theorem can be found in [5].

Theorem 6.1 Let A be a commutative ring and let $a_1, \dots, a_n$ be a regular sequence in A. Then $H_p(K) = 0$ for $p > 0$. If A is an $\mathbb{N}$-graded ring and $a_1, \dots, a_n$ are homogeneous elements of positive degree, then the converse is also true.

It follows from Theorem 6.1 that if A is an $\mathbb{N}$-graded ring and $a_1, \dots, a_n$ is a regular sequence that consists of homogeneous elements of positive degree then any permutation of $a_1, \dots, a_n$ is also a regular sequence.

Definition. A submanifold X of finite codimension in $\boldsymbol{P}$ is called a *complete intersection* if there exists a regular sequence of homogeneous polynomials $P_1, \dots, P_n$ that generates the ideal $I(X)$.

From now on we assume that X is a complete intersection in $\boldsymbol{P}$, and $P_1, \dots, P_n$ is a given regular sequence of homogeneous polynomials in $C[V]$ that generates $I(X)$. We will denote by K the Koszul complex corresponding to $P_1, \dots, P_n$.

Let $\mathcal{C}(X)$, $\mathcal{C}(\boldsymbol{P})$, and $\mathcal{C}_N(\boldsymbol{P})$, $N \in \mathbb{N}$, be the complexes

$$\mathcal{C}(X) = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}(X, \mathcal{O}_X(k)), \qquad \mathcal{C}(\boldsymbol{P}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)),$$

$$\text{and} \quad \mathcal{C}_N(\boldsymbol{P}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}_N(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)), \qquad N \in \mathbb{N}.$$

Let us note that $\mathcal{C}(\boldsymbol{P}) = \bigcup_{N \in \mathbb{N}} \mathcal{C}_N(\boldsymbol{P})$. We are going to construct a resolution of the complex $\mathcal{C}(X)$

$$0 \leftarrow \mathcal{C}(X) \overset{r}{\leftarrow} \mathcal{C}_0 \leftarrow \cdots \leftarrow \mathcal{C}_{p-1} \overset{\mathrm{d}'}{\leftarrow} \mathcal{C}_p \leftarrow \cdots \leftarrow \mathcal{C}_n \leftarrow 0 \tag{6.3}$$

such that $H^q(\mathcal{C}_p) = 0$ for $q > 0$ and $p = 0, \dots, k$. This will immediately imply $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q > 0$ and any $n \in \mathbb{Z}$.

In the construction of the resolution (6.3) we will use the existence of a natural C[V]-module structure on the complexes $\mathcal{C}_N(\boldsymbol{P})$, $N \in \mathbb{N}$, and $\mathcal{C}(\boldsymbol{P})$. To exhibit this module structure, we consider homogeneous polynomials of degree m as sections of the line bundle $\mathcal{O}_{\boldsymbol{P}}(m)$. If $P \in C[V]_m$ and $\varphi \in \mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, then it is easy to verify that the collection

$$(P\varphi)_{i_0 \dots i_q} = P\varphi_{i_0 \dots i_q} \in H^0(\boldsymbol{P}_{i_0 \dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k+m)) \tag{6.4}$$

is in $\mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k+m))$. Thus the group $\mathcal{C}_N^q(\boldsymbol{P}) = \oplus_{k \in \mathbb{Z}} \mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, $q \geq 0$, has a structure of a $C[V]$-module that is given by (6.4). It follows from (5.1) and (6.4) that the coboundary operator $\delta : \mathcal{C}_N^q(\boldsymbol{P}) \to \mathcal{C}_N^{q+1}(\boldsymbol{P})$ is a homomorphism of $C[V]$-modules. Since $\mathcal{C}(\boldsymbol{P}) = \cup_{N \in \mathbb{N}} \mathcal{C}_N(\boldsymbol{P})$ and since the $C[V]$-module structures on $\mathcal{C}_N(\boldsymbol{P})$ and $\mathcal{C}_{N+1}(\boldsymbol{P})$ agree for all $N \in \mathbb{N}$, the complex $\mathcal{C}(\boldsymbol{P})$ also has a $C[V]$-module structure.

Remark 1. For $N \in \mathbb{N}$ and $i_0, \dots, i_q \in I$, let $C[V](D_{i_0} \cdots D_{i_q})^{-N}$ be the $C[V]$-module generated by $(D_{i_0} \cdots D_{i_q})^{-N}$ in the field of fractions of $C[V]$. It is clear that $C[V](D_{i_0} \cdots D_{i_q})^{-N}$ is a free $C[V]$-module of rank 1. It follows from the definition of the groups $\mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$ that, for any $\varphi \in \mathcal{C}_N^q(\boldsymbol{P}) = \oplus_{k \in \mathbb{Z}} \mathcal{C}_N^q(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, there exist unique $\tilde{\varphi}_{i_0 \dots i_q} \in \oplus_{k \in \mathbb{Z}} H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, $i_0, \dots, i_q \in I$ such that

$$\varphi_{i_0 \dots i_q} = (D_{i_0} \cdots D_{i_q})^{-N} (\tilde{\varphi}_{i_0 \dots i_q})|_{\boldsymbol{P}_{i_0 \dots i_q}}$$

for any $i_0, \dots, i_q \in I$. Since $\oplus_{k \in \mathbb{Z}} H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)) = C[V]$, there exists an isomorphism of $C[V]$-modules

$$\mathcal{C}_N^q(\boldsymbol{P}) \cong \prod_{i_0, \dots, i_q \in I} C[V](D_{i_0} \cdots D_{i_q})^{-N}. \tag{6.5}$$

The resolution (6.3) is constructed as follows: for $p = 0, \dots, n$, let $\mathcal{C}_p$ and $\mathcal{C}_{p,N}$, $N \in \mathbb{N}$, be the complexes

$$\mathcal{C}_p = K_p \otimes_{C[V]} \mathcal{C}(\boldsymbol{P}) \quad \text{and} \quad \mathcal{C}_{p,N} = K_p \otimes_{C[V]} \mathcal{C}_N(\boldsymbol{P}), \quad N \in \mathbb{N}. \tag{6.6}$$

The differential $\mathrm{d} : K_p \to K_{p-1}$ induces chain maps

$$\mathrm{d}' = d \otimes \mathrm{id} : \mathcal{C}_p \to \mathcal{C}_{p-1} \quad \text{and} \quad \mathrm{d}' = d \otimes \mathrm{id} : \mathcal{C}_{p,N} \to \mathcal{C}_{p-1,N}, \quad N \in \mathbb{N}, \tag{6.7}$$

for $p = 1, \dots, k$.

Proposition 6.1. The sequence of complexes

$$\mathcal{C}_0 \leftarrow \cdots \leftarrow \mathcal{C}_{p-1} \xleftarrow{\mathrm{d}'} \mathcal{C}_p \leftarrow \cdots \leftarrow \mathcal{C}_n \leftarrow 0$$

is exact.

Proof. Since $\mathcal{C}(\boldsymbol{P}) = \cup_{N \in \mathbb{N}} \mathcal{C}_N(\boldsymbol{P})$ it is enough to check that for any $N \in \mathbb{N}$ and any $q \geq 0$ the sequence of $C[V]$-modules

$$\mathcal{C}_{0,N}^q \leftarrow \cdots \leftarrow \mathcal{C}_{p-1,N}^q \xleftarrow{\mathrm{d}'} \mathcal{C}_{p,N}^q \leftarrow \cdots \leftarrow \mathcal{C}_{n,N}^q \leftarrow 0 \tag{6.8}$$

is exact. It follows from (6.6), (6.6) and (6.7) that the complex (6.8) is isomorphic to the complex

$$K \otimes \mathcal{C}_N^q(P) = K \otimes \prod_{i_0,\dots,i_q \in I} C[V](D_{i_0}\cdots D_{i_q})^{-N}.$$

Since each K_p, $p = 1,\dots,k$, is a *finitely generated* $C[V]$-module, there is an isomorphism of complexes

$$K \otimes \prod_{i_0,\dots,i_q \in I} C[V](D_{i_0}\cdots D_{i_q})^{-N} \cong \prod_{i_0,\dots,i_q \in I} K \otimes C[V](D_{i_0}\cdots D_{i_q})^{-N}. \qquad (6.9)$$

Now we note that the complex on the right-hand side of (6.9) is exact because the Koszul complex is exact by Theorem 5.1, and $C[V](D_{i_0}\cdots D_{i_q})^{-N}$ is a free $C[V]$-module for any $i_0,\dots,i_q \in I$. □

It remains to define a surjective chain map $r : \mathcal{C}_0 \to \mathcal{C}(X)$ such that $\ker r = \operatorname{im}\{\mathcal{C}_1 \xrightarrow{d'} \mathcal{C}_0\}$. We note that the complex $\mathcal{C}_0$ coincides with the complex $\mathcal{C}(\boldsymbol{P})$ because $K_0 = C[V]$. Then the map $r : \mathcal{C}_0 \to \mathcal{C}(X)$ is given by the restriction of the sections of the line bundles $\mathcal{O}_{\boldsymbol{P}}(k)$, $k \in \mathbb{Z}$, to the submanifold X. More precisely, the restriction homomorphisms

$$\{H^0(\boldsymbol{P}_{i_0\dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k)) \to H^0(X_{i_0\dots i_q}, \mathcal{O}_X(k))\}_{i_0,\dots,i_q \in I},$$
$$H^0(\boldsymbol{P}_{i_0\dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k)) \ni \varphi_{i_0\dots i_q} \mapsto (\varphi_{i_0\dots i_q})|_{X_{i_0\dots i_q}} \in H^0(X_{i_0\dots i_q}, \mathcal{O}_X(k))$$

induce chain maps $r_N : \mathcal{C}_N(P, \mathcal{O}_{\boldsymbol{P}}(k)) \to \mathcal{C}_N(X, \mathcal{O}_X(k))$ for $k \in \mathbb{Z}$, $N \in \mathbb{N}$. The collection of chain maps $\{r_N\}_{N\in\mathbb{N}}$ then induces a natural chain map

$$r_k : \mathcal{C}(P, \mathcal{O}_{\boldsymbol{P}}(k)) \to \mathcal{C}(X, \mathcal{O}_X(k)) \qquad (6.10)$$

for any $k \in \mathbb{Z}$. The next lemma is instrumental in the proof of the surjectivity of the chain map (6.10).

Lemma 6.2. Suppose that

$$\Phi \in H^0(X_{i_0\dots i_q}, \mathcal{O}_X(k)) \quad \text{and} \quad \widetilde{\Phi} \in H^0(X, \mathcal{O}_X(k + Nd_{i_0} + \cdots + Nd_{i_q}))$$

are such that $\Phi \doteq (D_{i_0}\cdots D_{i_q})^{-N}(\widetilde{\Phi}|_{X_{i_0\dots i_q}})$ for some $N \in \mathbb{N}$. Then there exist sections

$$\varphi \in H^0(\boldsymbol{P}_{i_0\dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k)) \text{ and } \tilde{\varphi} \in H^0(P, \mathcal{O}_{\boldsymbol{P}}(k + (N+1)d_{i_0} + \cdots + (N+1)d_{i_q}))$$

such that $\Phi = \varphi|_{X_{i_0\dots i_q}}$ and $\varphi = (D_{i_0}\cdots D_{i_q})^{-N-1}\tilde{\varphi}|_{\boldsymbol{P}_{i_0\dots i_q}}$.

Proof. By Proposition 4.6, there exists a homogeneous polynomial $P \in C[V]$ such that $\deg P = k + Nd_{i_0} + \cdots + Nd_{i_q} + d_{i_0}$ and $(D_{i_0}\widetilde{\Phi})|_{X_{i_0}} P|_{X_{i_0}}$. Let

$$\tilde{\varphi} = D_{i_1}\cdots D_{i_q} P \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k + (N+1)d_{i_0} + \cdots + (N+1)d_{i_q}))$$

and

$$\varphi = (D_{i_0} \cdots D_{i_q})^{-N-1} \tilde{\varphi}|_{\boldsymbol{P}_{i_0 \dots i_q}} \in H^0(\boldsymbol{P}_{i_0 \dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k)).$$

Then $\Phi = \varphi|_{X_{i_0 \dots i_q}}$ which proves the lemma. □

Lemma 6.3. The natural chain map

$$r_k : \mathcal{C}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k)) \to \mathcal{C}(X, \mathcal{O}_X(k))$$

is surjective for any $k \in \mathbb{Z}$.

Proof. Let Φ be a cochain in $\mathcal{C}^q_N(X, \mathcal{O}_X(k))$. It follows from Lemma 6.2 that for any $i_0, \dots, i_q \in I$ there are $\tilde{\varphi}_{i_0 \dots i_q} \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k + (N+1)d_{i_0} + \cdots + (N+1)d_{i_q}))$ and $\varphi_{i_0 \dots i_q} \in H^0(\boldsymbol{P}_{i_0 \dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k))$ such that

$$\varphi_{i_0 \dots i_q}|_{X_{i_0 \dots i_q}} = \Phi_{i_0 \dots i_q} \qquad \text{and} \qquad \varphi_{i_0 \dots i_q} = (D_{i_0} \cdots D_{i_q})^{-N-1} (\tilde{\varphi}_{i_0 \dots i_q}|_{\boldsymbol{P}_{i_0 \dots i_q}}).$$

By the definition of the group $\mathcal{C}^q_{N+1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$, the cochain $\varphi = \{\varphi_{i_0 \dots i_q}\}_{i_0 \dots i_q \in I}$ belongs to $\mathcal{C}^q_{N+1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(k))$ and $r_k(\varphi) = \Phi$. □

Let $r : \mathcal{C}_0 = \mathcal{C}(\boldsymbol{P}) \to \mathcal{C}(X)$ be the chain map

$$r = \oplus_{k \in \mathbb{Z}} r_k \ : \ \mathcal{C}_0 = \oplus_{k \in \mathbb{Z}} \mathcal{C}(P, \mathcal{O}_{\boldsymbol{P}}(k)) \to \oplus_{k \in \mathbb{Z}} \mathcal{C}(X, \mathcal{O}_X(k)) = \mathcal{C}(X),$$

where $r_k : \mathcal{C}(P, \mathcal{O}_{\boldsymbol{P}}(k)) \to \mathcal{C}(X, \mathcal{O}_X(k))$, $k \in \mathbb{Z}$, is the chain map (6.10).

Lemma 6.4. The sequence of complexes $0 \leftarrow \mathcal{C}(X) \xleftarrow{r} \mathcal{C}_0 \xleftarrow{\mathrm{d}'} \mathcal{C}_1$ is exact.

Proof. By Lemma 6.3 the chain map r is surjective. It remains to show that $\ker r = \operatorname{im} \mathrm{d}'$. Let

$$\varphi = \{\varphi_{i_0 \dots i_q} \in \oplus_{k \in \mathbb{Z}} H^0(\boldsymbol{P}_{i_0 \dots i_q}, \mathcal{O}_{\boldsymbol{P}}(k))\}_{i_0, \dots, i_q \in I} \in \mathcal{C}^q_{0,N}(\boldsymbol{P}) = \mathcal{C}^q_N(\boldsymbol{P})$$

be such that $r(\varphi) = 0$. For any $i_0, \dots, i_q \in I$, there exists a polynomial $\tilde{\varphi}_{i_0 \dots i_q} \in C[V]$ such that $\varphi_{i_0 \dots i_q} = (D_{i_0} \cdots D_{i_q})^{-N} (\tilde{\varphi}_{i_0 \dots i_q}|_{X_{i_0 \dots i_q}})$. Each polynomial $\tilde{\varphi}_{i_0 \dots i_q}$, $i_0, \dots, i_q \in I$, vanishes on X because $\varphi_{i_0 \dots i_q}$ vanishes on $X_{i_0 \dots i_q}$ and the set $X_{i_0 \dots i_q}$ is dense in X. Since the ideal $I(X)$ is generated by $P_1, \dots, P_n$, for any polynomial $\tilde{\varphi}_{i_0 \dots i_q}$, $i_0, \dots, i_q \in I$, there exist polynomials $P_{1, i_0 \dots i_q}, \dots, P_{n, i_0 \dots i_q}$, such that

$$\tilde{\varphi}_{i_0 \dots i_q} = \sum_{j=1}^{n} P_j P_{j, i_0 \dots i_q}.$$

Let $\varphi_j = \{(D_{i_0} \cdots D_{i_q})^{-N} P_{j, i_0 \dots i_q}\}_{i_0, \dots, i_q \in I} \in \mathcal{C}^q_N(\boldsymbol{P})$, $j = 1, \dots, n$. Then $\varphi = \sum_{j=1}^k P_j \varphi_j$. Set

$$\overline{\varphi} = \sum_{j=1}^{k} e_j \otimes \varphi_j \in K_1 \otimes \mathcal{C}^q_N(\boldsymbol{P}) = \mathcal{C}^q_{1,N}(\boldsymbol{P}).$$

Then

$$\mathrm{d}'(\overline{\varphi}) = \mathrm{d}'\left(\sum_{j=1}^{n} e_j \otimes \varphi_j\right) = \sum_{j=1}^{n} \mathrm{d}'(e_j) \otimes \varphi_j = \sum_{j=1}^{n} P_j \otimes \varphi_j = \varphi.$$

Thus for any $N \in \mathbb{N}$, any $q \geq 0$, and any $\varphi \in C^q_{0,N}(\boldsymbol{P})$ such that $r(\varphi) = 0$, there exists $\overline{\varphi} \in C^q_{1,N}(\boldsymbol{P})$ such that $\mathrm{d}'(\overline{\varphi}) = \varphi$. □

Theorem 6.5. Let V be an infinite-dimensional Banach space that admits smooth partitions of unity and let $\boldsymbol{P} = \boldsymbol{P}(V)$. If X is a complete intersection in $\boldsymbol{P}$, then $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q \geq 1$ and any $k \in \mathbb{Z}$.

Proof. The sequence of complexes

$$0 \leftarrow \mathcal{C}(X) \xleftarrow{r} \mathcal{C}_0 \leftarrow \cdots \leftarrow \mathcal{C}_{p-1} \xleftarrow{\mathrm{d}'} \mathcal{C}_p \leftarrow \cdots \leftarrow \mathcal{C}_n \leftarrow 0$$

is exact by Proposition 6.1 and Lemma 6.4. Since each K_p is a free $C[V]$-module, we have $H^q(\mathcal{C}_p) = K_p \otimes H^q(\mathcal{C}(\boldsymbol{P}))$ for $p = 0, \ldots, k$, and $q \geq 0$. By Theorem 5.1, $H^q(\mathcal{C}(\boldsymbol{P})) = 0$ for $q \geq 1$. Hence $H^q(\mathcal{C}_p) = 0$ for $p = 0, \ldots, n$ and $q \geq 1$. Let B_j be the complex $\operatorname{coker}\{\mathcal{C}_j \xleftarrow{\mathrm{d}'} \mathcal{C}_{j+1}\}$, $j = 0, \ldots, n$. We note that $B_0 = \mathcal{C}(X)$ and $B_n = \mathcal{C}_n$. For any $j = 1, \ldots, n$, we have a short exact sequence

$$0 \leftarrow B_{j-1} \xleftarrow{d'} \mathcal{C}_j \leftarrow B_j \leftarrow 0$$

Using the long exact sequence of cohomology groups, we derive by descending induction on j that $H^q(B_{j-1}) = 0$ for $q \geq 1$ and $j = n, \ldots, 1$. Hence $H^q(\mathcal{C}(X)) = 0$ for $q \geq 1$ and this implies $H^q(\mathcal{C}(X, \mathcal{O}_X(k))) = 0$ for $q \geq 1$ and any $k \in \mathbb{Z}$. □

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: bkotzev@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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VANISHING OF THE FIRST DOLBEAULT COHOMOLOGY GROUP OF HOLOMORPHIC LINE BUNDLES ON COMPLETE INTERSECTIONS IN INFINITE DIMENSIONAL PROJECTIVE SPACE

BORIS KOTZEV

We consider a complex submanifold X of finite codimension in an infinite-dimensional complex projective space $\boldsymbol{P}$ and prove that the first Dolbeault cohomology group of all line bundles $\mathcal{O}_X(n)$, $n \in \mathbb{Z}$, vanishes when X is a complete intersection and $\boldsymbol{P}$ admits smooth partitions of unity.

Keywords: Dolbeault cohomology groups, infinite-dimensional complex manifolds, projective manifolds, vanishing theorems

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1. INTRODUCTION

In this paper, we prove a vanishing theorem for the first Dolbeault cohomology group of the line bundles $\mathcal{O}_X(n)$, $n \in \mathbb{Z}$, where X is a complete intersection in an infinite-dimensional projective space $\boldsymbol{P}$ which admits smooth partitions of unity.

For a given complex Banach space V, the associated complex projective space $\boldsymbol{P}(V)$ consists of all complex lines in V. The set $\boldsymbol{P}(V)$ has a natural structure of complex manifold which is described in detail in [3]. For a submanifold X of finite codimension in $\boldsymbol{P}(V)$ the complexified tangent bundle $T_{\mathbb{C}}X$, the holomorphic tangent bundle $T^{1,0}X$, and the antiholomorphic tangent bundle $T^{0,1}X$ of X can be defined as in finite dimensions. Given a vector bundle $E \to X$, we define $(0,q)$-forms on X with values in E as bundle maps from $T^{0,q}X = \bigwedge^q T^{0,1}X$ to E. For any open set $U \subset X$, we denote by $C^r_{0,q}(U,E)$ the vector space of all r-times continuously differentiable $(0,q)$-forms with values in E, $0 \le r \le \infty$. We will also write $C^r(U,E)$ instead of $C^r_{0,0}(U,E)$, $C_{0,q}(U,E)$ instead of $C^0_{0,q}(U,E)$, and $C(U,E)$ instead of $C^0_{0,0}(U,E)$. When the vector bundle E is holomorphic, the $\overline{\partial}$-operator, $\overline{\partial} : C^r_{0,q}(U,E) \to C^{r-1}_{0,q+1}(U,E)$, $r \ge 1$, is defined by means of Cartan's formula for the exterior derivative. The Dolbeault cohomology groups $H^{0,q}(X,E)$, $q \ge 0$, of a holomorphic vector bundle $E \to X$ are defined as in finite dimensions:

$$H^{0,q}(X,E) = \frac{\{\text{closed smooth } (0,q)\text{-forms with values in } E\}}{\{\text{exact smooth } (0,q)\text{-forms with values in } E\}}.$$

We refer to [5] for a detailed treatment of (p,q)-forms with values in vector bundles and the $\overline{\partial}$-operator on infinite dimensional complex manifolds.

L. Lempert has proved in [5, Theorem 7.3] that if $E \to \boldsymbol{P}(V)$ is a holomorphic vector bundle of finite rank over localising infinite-dimensional complex projective space $\boldsymbol{P}(V)$, then $H^{0,q}(\boldsymbol{P}(V),E) = 0$, $q \ge 1$. The extra condition on the projective space $\boldsymbol{P}(V)$ has to do with the existence of bump functions. A differentiable manifold M localises if, for every nonempty open set $W \subset M$, there exists a smooth not identically zero function $\phi : M \to \mathbb{R}$ that is supported in W. Every Hilbert space localises whereas the space l^1 does not [4]. A projective space $\boldsymbol{P}(V)$ associated with a locally convex topological vector space V localises if and only if V localises [5, p. 509].

In this paper, we partially extend some of the results in [5] to complete intersections in infinite-dimensional complex projective space. The methods we use require that even stronger conditions should be imposed on the projective space $\boldsymbol{P}(V)$. Namely we have to assume that $\boldsymbol{P}(V)$ admits smooth partitions of unity. A differentiable manifold X admits smooth partitions of unity if, for any open cover $\{U_i\}_{i\in I}$ of X, there are $\theta_i \in C^\infty(X)$, supported in U_i such that $\sum_{i\in I}\theta_i = 1$, the sum being locally finite. Hilbert spaces are examples of such manifolds. Separable and reflexive Banach spaces that localise are other examples. Paracompact manifolds modeled on spaces that admit smooth partitions of unity also admit smooth partitions of unity. In particular, if V is a Banach space that admits smooth partitions of unity, then the associated projective space $\boldsymbol{P} = \boldsymbol{P}(V)$ admits smooth partitions of unity. We refer to [1] for more details about smooth partitions of unity.

Here is a brief outline of the contents of the paper.

In Section 2, we consider a closed form $f \in C^r_{0,1}(\boldsymbol{P}(V), \mathcal{O}_{\boldsymbol{P}(V)}(n))$, $1 \le r \le \infty$, $n \in \mathbb{Z}$. In Proposition 2.2.1 and Proposition 2.2.2, we prove that if V localises and $f|_W \in C^\infty_{0,1}(W, \mathcal{O}_{\boldsymbol{P}(V)}(n))$ for some none-empty open set $W \subset \boldsymbol{P}(V)$, then f is exact. Both propositions are generalisations of [5, Theorem 7.3] for $(0,1)$-forms. The difference is that the differential form f is assumed to be smooth in [5], whereas for our purposes we have to give a proof for differential forms that are smooth on a proper open subset of $\boldsymbol{P}(V)$. Let us emphasise that these results are global.The local solvability of the $\overline{\partial}$-equation can not be taken for granted in infinite dimensions - see [6] for an example of a complex Banach space V and a closed form $f \in C^{\infty}_{0,1}(V)$ which is not exact on any nonempty open subset U of V. In the proofs we use Lempert's idea to solve the $\overline{\partial}$-equation on the blow up $Bl_x\boldsymbol{P}(V)$ of $\boldsymbol{P}(V)$ at a point $x \in \boldsymbol{P}(V)$.

In Section 3, we prove the main result of this paper. The proof consists of two parts. The first part is to find local solutions to the $\overline{\partial}$-equation: we consider an arbitrary submanifold X of finite codimension in $\boldsymbol{P}(V)$ and a closed form $f \in C^{\infty}_{0,1}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, and construct an open covering $\{X_i\}_{i\in I}$ of X and a collection of sections $\{u_i \in C^{\infty}(X_i, \mathcal{O}_{X_i})\}_{i\in I}$ such that $\overline{\partial} u_i = f|_{X_i}$ for $i \in I$. For this part of the proof we need to assume only that the projective space $\boldsymbol{P}(V)$ localises. The second part of the proof is to solve the Cousin problem for the cocycle $\{u_j - u_i \in H^0(U_i \cup U_j, \mathcal{O}_X)\}_{i,j\in I}$. That this is possible is proved in [3]. For the second part we have to assume that X is a complete intersection and $\boldsymbol{P}(V)$ admits smooth partitions of identity.

This paper is based on the author's Ph.D. thesis (Purdue University, 2001).

2. THE $\overline{\partial}$-EQUATION ON INFINITE-DIMENSIONAL PROJECTIVE SPACES

Let X be a complex manifold and let Y be a submanifold of X of codimension 1. We recall that there exists a holomorphic line bundle L_Y over X, and a section $u_Y \in H^0(X, L_Y)$ such that $Y u_Y^{-1}(0)$ and $\mathrm{d}u_Y(y) \not\equiv 0$ for any $y \in Y$ (where $\mathrm{d}u_Y(y)$ is calculated in some local trivialisation of L_Y at y). Let $L \to X$ be a complex line bundle. Given a section $u \in C(U, L)$ on an open set $U \subset X$, we say that u is locally bounded at a subset $X' \subset X$ if for any $x' \in X'$ there exist an open set $W \ni x'$ and a local trivialisation $\phi : L|_W \to W \times \mathbb{C}$ such that the function $p_2\phi u|_{W\cap U} : W \cap U \to \mathbb{C}$ is bounded on $W \cap U$. We say that u vanishes at X' if for any $x' \in X'$ and any real number $\epsilon > 0$ there exist a neighbourhood W of x' and a local trivialisation $\phi : L|_W \to W \times \mathbb{C}$ such that $|p_2\phi u(x)| < \epsilon$ for all $x \in W \cap U$. Given a submanifold Y of codimension 1 in X, and an integer $n \in \mathbb{Z}$, we write $u = O(|u_Y|^n)$ at Y (resp. $u = o(|u_Y|^n)$ at Y) if the restriction of $u \otimes u_Y^{-n}$ to $U \backslash Y$ is locally bounded at Y (resp. vanishes at Y). Given a differential form $f \in C_1(U, L)$, we write $f = O(|u_Y|^n)$ at Y (resp. $f = o(|u_Y|^n)$ at Y) if $f(\Omega) = O(|u_Y|^n)$ at Y (resp. $f(\Omega) = o(|u_Y|^n)$ at Y) for any vector field $\Omega \in C^{\infty}(X, T_{\mathbb{C}}X)$.

In 2.1.1 we will need the concept of a *weak solution* of the $\overline{\partial}$-equation. Let A be an open subset of a complex Banach space V. Suppose that $u \in C(A)$ and $f \in C_{0,1}(A)$. We say that $\overline{\partial} u = f$ in the *weak sense* if for any finite dimensional affine subspace $F \subset V$, $\overline{\partial}(u|_{F\cap A}) = f|_{F\cap A}$ holds in the sense of distribution theory. For example, if for any $x \in A$ and any $\xi \in V$ the directional derivative

$$\overline{\partial} u(x; \overline{\xi}) = \frac{1}{2}\, \mathrm{d}/\mathrm{d}t|_{t=0}\{u(x + t\xi) + iu(x + it\xi)\}$$

exists and $\overline{\partial}u(x;\overline{\xi}) = f(x;\overline{\xi})$, then $\overline{\partial}u = f$ in the weak sense. We will use the following fact from "elliptic regularity theory":

Proposition 2.1 If $u \in C(A)$, $f \in C^r_{0,1}(U)$, $1 \le r \le \infty$, and $\overline{\partial}u = f$ in the weak sense, then $u \in C^r(A)$ and $\overline{\partial}u = f$ holds according to the original definition of the $\overline{\partial}$-operator.

Proof. See [5, Proposition 9.3]. □

2.1. THE $\overline{\partial}$-EQUATION FOR $(0,1)$-FORMS ON $\boldsymbol{P}^1$-BUNDLES

In this section, we consider first a trivial $\boldsymbol{P}^1$-bundle $\pi : X = B \times \boldsymbol{P}^1 \to B$ over a complex manifold B. Let q be the projection $X = B \times \boldsymbol{P}^1 \to \boldsymbol{P}^1$. For an integer $n \in \mathbb{Z}$, we denote by $\mathcal{O}_X(n)$ the holomorphic line bundle $q^*(\mathcal{O}_{\boldsymbol{P}^1}(n))$ over X.

Let $z : \boldsymbol{P}^1 \setminus \{\infty\} \to \mathbb{C}$ and $w = z^{-1} : \boldsymbol{P}^1 \setminus \{0\} \to \mathbb{C}$ be the local coordinates on $\boldsymbol{P}^1 \setminus \{\infty\}$ and $\boldsymbol{P}^1 \setminus \{0\}$, respectively. A section $u \in C^r(W, \mathcal{O}_X(n))$ on an open set $W \subset X$ is represented by a pair of functions $u_1 \in C^r(W \setminus B \times \{\infty\})$ and $u_2 \in C^r(U \setminus B \times \{0\})$ such that

$$u_2(b,w) = w^n u_1(b, w^{-1}), \qquad (b,w) \in W,\ w \neq 0. \tag{2.1.1}$$

Proposition 2.1.1. If $f \in C^r_{0,1}(\boldsymbol{P}^1 \setminus \{y\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$, $y \in \boldsymbol{P}^1$, $n \in \mathbb{Z}$, $0 \le r \le \infty$, is such that $f = O(|u_{\{y\}}|^n)$, then there is a unique section $u \in C^r(\boldsymbol{P}^1 \setminus \{y\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$ such that $\overline{\partial}u = f$ and $u = o(|u_{\{y\}}|^n)$ at $\{y\}$.

Proof. The section u is unique because if $v \in H^0(\boldsymbol{P}^1 \setminus \{y\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$ is such that $v = o(|u_{\{y\}}|^n)$ at $\{y\}$, then $v = 0$.

To prove the existence of u, we can assume that $y = \infty$ and write $f = F(z)\,\mathrm{d}\overline{z}$ with $F \in C^r(\mathbb{C})$. Relation (2.1.1) yields $f = -w^n \overline{w}^{-2} F(w^{-1})\,\mathrm{d}\overline{w}$, $w \neq 0$. Since $f = O(|s_{\{\infty\}}|^n)$, there is a constant $C \ge 0$ such that

$$|F(z)| \le C(1 + |z|)^{-2}, \qquad z \in \mathbb{C}. \tag{2.1.2}$$

We set

$$u_1(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{F(\lambda)}{\lambda - z}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}, \qquad z \in \mathbb{C}. \tag{2.1.3}$$

Integral (2.1.3) converges by estimate (2.1.2). Moreover $u_1 \in C^r(\mathbb{C})$ and $\partial u_1/\partial \overline{z} = F$ [2, Theorem 1.2.2]. Let $u_2 \in C^r(\boldsymbol{P}^1 \setminus \{0, \infty\})$ be given by $u_2(w) = w^n u_1(w^{-1})$, $w \neq 0$. Let $u \in C^r(\boldsymbol{P}^1 \setminus \{\infty\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$ be represented by the pair $u_1(z), u_2(w)$. Then $u \in C^r(\boldsymbol{P}^1 \setminus \{\infty\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$ and $\overline{\partial}u = f$. To complete the proof, we have to show that

$$\lim_{w \to 0} u_1(w^{-1}) = 0.$$

Let $G \in C^r(\mathbb{C} \setminus \{0\})$ be given by $G(w) = -\overline{w}^{-2} F(w^{-1})$, $w \neq 0$. Estimate (2.1.2) yields

$$|G(w)| \leq C(1 + |w|)^{-2}, \qquad w \neq 0. \tag{2.1.4}$$

Making the substitutions $z = w^{-1}$ and $\lambda = \mu^{-1}$ in (2.1.3), we obtain

$$u_1(w^{-1}) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{G(\mu)}{\mu - w} \, \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu} - \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{G(\mu)}{\mu} \, \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}.$$

Let $U : \mathbb{C} \to \mathbb{C}$ be given by

$$U(w) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{G(\mu)}{\mu - w} \, \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{G(\nu + w)}{\nu} \, \mathrm{d}\nu \wedge \mathrm{d}\overline{\nu}.$$

We claim that U is continuous on $\mathbb{C}$. It is easy to see this when G can be extended to a continuous function $\widetilde{G}$ on $\mathbb{C}$ because estimate (2.1.4) yields

$$|\nu^{-1} \widetilde{G}(\nu + w)| \leq C|\nu|^{-1}(1 + |\nu + w|)^{-2} \leq C(1 + |w|)^2 |\nu|^{-1}(1 + |\nu|)^{-2},$$

and the function $|\nu|^{-1}(1 + |\nu|)^{-2}$ is integrable on $\mathbb{C}$. To deal with the general case, we use continuous bump functions at 0 to construct a sequence of functions $G_m \in C(\mathbb{C})$, $m = 1, 2, \ldots$, such that $|G_m(\mu)| \leq |G(\mu)|$ for $\mu \neq 0$, $G_m(\nu) = G(\nu)$ for $|\mu| \geq m^{-1}$, and $G_m(\mu) = 0$ for $|\mu| \leq (2m)^{-1}$. Let

$$U_m(w) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{G_m(\mu)}{\mu - w} \, \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}, \qquad m = 1, 2, \ldots.$$

Now each U_m is continuous on $\mathbb{C}$ and it is not difficult to check that the sequence U_m, $m = 1, 2 \ldots$, converges uniformly to U as $m \to \infty$. Hence U is also continuous on $\mathbb{C}$. Since $u_1(w^{-1}) = U(w) - U(0)$ for $w \neq 0$, we see that $\lim_{w \to 0} u_1(w^{-1}) = 0$. □

Corollary 2.1.2. Suppose that $u \in C^r(\boldsymbol{P}^1 \setminus \{\infty\}, \mathcal{O}_{\boldsymbol{P}^1}(n))$, $n \in \mathbb{Z}$, $1 \leq r \leq \infty$, is such that $u = o(|s_{\{\infty\}}|^n)$ at ∞ and $\overline{\partial} u = O(|s_{\{\infty\}}|^n)$. Then

$$u_1(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial u_1 / \partial \overline{\lambda}}{\lambda - z} \, \mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}.$$

Proof. This follows immediately from Proposition 2.1.1 □

In the next proposition, we denote by σ a holomorphic section $\sigma : B \to X$ of a trivial $\boldsymbol{P}^1$-bundle $X = B \times \boldsymbol{P}^1 \to B$. The submanifold $\sigma(B) \subset X$ will be denoted by Y.

Proposition 2.1.3. Suppose that $f \in C^r_{0,1}(X \setminus Y, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, $1 \leq r \leq \infty$, is a closed form that satisfies the following conditions:

(i) $f = O(|u_Y|^n)$ at Y;
(ii) if $\Omega \in C(X, T_{\mathbb{C}}X)$ is a vector field that is tangent to Y, then $f(\Omega) = o(|u_Y|^n)$ at Y and $\overline{\partial}(f(\Omega)) = O(|u_Y|^n)$ at Y.
Then there exists unique $u \in C^r(X \setminus Y, \mathcal{O}_X(n))$ such that $\overline{\partial} u = f$ and $u = o(|u_Y|^n)$ at Y.

Proof. The uniqueness of u is established as in the proof of Proposition 2.1.2. To prove the existence of u, we can assume that $\sigma : B \to X$ is the section given by $\sigma(b) = (b, \infty)$, $b \in B$ and thus $Y = B \times \{\infty\}$. Then we write $f|_{\{b\}\times\mathbb{C}} = F(b,z)\,\mathrm{d}\overline{z}$ and set

$$u_1(b,z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{F(b,\lambda)}{\lambda - z}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}, \qquad (2.1.5)$$

where condition (i) makes sure that the integral converges. One verifies, as in the proof of Proposition 2.1.1, that u_1 is a continuous function on $X \setminus Y$ and that u_1 vanishes on Y. Let u_2 be given by

$$u_2(b,w) = w^n u_1(b, w^{-1}), \qquad w \neq 0,$$

and let u be the section of $\mathcal{O}_X(n)$ on $X \setminus Y$ that is represented by the pair $u_1(b,z), u_2(b,w)$. It is clear that $u = o(|u_Y|^n)$ at Y.

According to Proposition 2.1.1, we have $\overline{\partial}(u|_{\{b\}\times\mathbb{C}}) = f|_{\{b\}\times\mathbb{C}}$, $b \in B$. To prove that $u \in C^r(X \setminus Y, \mathcal{O}_X(n))$ and $\overline{\partial} u = f$, it is enough to show that $\overline{\partial}(u_1|_{B\times\{z\}}) = f|_{B\times\{z\}}$ weakly for any $z \in \mathbb{C}$. This implies $\overline{\partial} u = f$ weakly and then the claim follows from Proposition 2.1. Let $z \in \mathbb{C}$ and let $\omega \in C^\infty(B \times \{z\}, T^{0,1}(B \times \{z\}))$. Define a vector field $\Omega \in C^\infty(X, T^{0,1}X)$ by $\Omega(b,p) = \omega(b,z)$, $p \in P^1$. It is clear that Ω commutes with the vector field $\partial/\partial\overline{z} \in C^\infty(X \setminus Y, T^{0,1}X)$, i.e $[\Omega, \partial/\partial\overline{z}] = 0$ on $X \setminus Y$. Since f is closed, Cartan's formula yields

$$0 = \overline{\partial} f(\Omega, \partial/\partial\overline{z}) = \Omega(f(\partial/\partial\overline{z})) - \partial/\partial\overline{z}(f(\Omega)) - f([\Omega, \partial/\partial\overline{z}]).$$

Hence $\Omega F = \Omega(f(\partial/\partial\overline{z})) = \partial/\partial\overline{z}(f(\Omega))$. Formal differentiation in (2.1.5) yields

$$\Omega u_1(b,z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\Omega F(b,\lambda)}{\lambda - z}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial/\partial\overline{\lambda}(f(\Omega))(b,\lambda)}{\lambda - z}\,\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}. \quad (2.1.6)$$

Since the vector field Ω is tangent to $Y = B \times \{\infty\}$, condition (ii) holds for $f(\Omega)$ and it follows, from Corollary 2.1.2, that $\Omega u = f(\Omega)$. Hence $\overline{\partial}(u|_{B\times\{z\}}) = f|_{B\times\{z\}}$ weakly.

Formal differentiation in (2.1.5) is justified as follows. It follows from the growth estimate $\overline{\partial}(f(\Omega)) = O(|s_Y|^n)$ that for any $b_0 \in B$ there is a neighbourhood $U \ni b_0$ and a constant $C \geq 0$ such that

$$|\partial/\partial\overline{\lambda}(f(\Omega))(b,\lambda)| \leq C(1 + |\lambda|)^{-2}, \qquad b \in U,\ \lambda \in \mathbb{C}.$$

Since the function $|\lambda - z|(1+|\lambda|)^{-2}$ is integrable on $\mathbb{C}$, integral (2.1.6) converges uniformly in $b \in U$. Thus formal differentiation in (2.1.5) is justified. $\square$

In the next two propositions, we consider a (not necessarily trivial) $\boldsymbol{P}^1$-bundle $\pi : X \to B$ over a complex manifold B which has a holomorphic section $\sigma : B \to X$. The codimension 1 submanifold $\sigma(B) \subset X$ will be denoted by Y.

Proposition 2.1.4. Let $L \to X$ be a holomorphic line bundle such that for any $b \in B$ there is a neighbourhood $U \ni b$ for which $L|_{\pi^{-1}(U)} \cong \mathcal{O}_{\pi^{-1}(U)}(n)$ for some fixed integer $n < 0$. Then for any closed form $f \in C^1_{0,1}(X, L) \cap C^r_{0,1}(X \setminus Y, L)$, $1 \leq r \leq \infty$, there exists a unique section $u \in C^r(X \setminus Y, L)$ such that $\overline{\partial} u = f$ and $u = o(|u_Y|^n)$ at Y.

Proof. Let $\{U_i\}_{i \in I}$ be an open covering of B such that $L|_{\pi^{-1}(U_i)} \cong \mathcal{O}_{\pi^{-1}(U_i)}(n)$, $i \in I$. Denote $Y \cap \pi^{-1}(U_i)$ by Y_i. Conditions (i) and (ii) of Proposition 2.1.3 hold trivially for $f|_{\pi^{-1}(U_i)}$ and u_{Y_i} because $n < 0$. For each $i \in I$ Proposition 2.1.3 yields a unique section $u_i \in C^r(\pi^{-1}(U_i) \setminus Y, L)$ such that $\overline{\partial} u_i = f$ on $\pi^{-1}(U_i) \setminus Y_i$ and $u_i = o(|u_{Y_i}|^n)$. For $i, j \in I$ the restrictions $u_i|_{\pi^{-1}(U_i) \cap \pi^{-1}(U_j)}$ and $u_j|_{\pi^{-1}(U_i) \cap \pi^{-1}(U_j)}$ are the same because $u_i|_{Y_i \cap Y_j} = o(|u_{Y_i \cap Y_j}|^n)$ and $u_j|_{Y_i \cap Y_j} = o(|u_{Y_i \cap Y_j}|^n)$. Hence the sections u_i, $i \in I$, paste together to a section $u \in C^r(X \setminus Y, L)$ such that $\overline{\partial} u = f$ on $X \setminus Y$ and $u = o(|u_Y|^n)$ at Y. The section u is unique because if a holomorphic section $s \in H^0(X \setminus Y, L)$ is such that $s = o(|u_Y|^n)$ at Y, then $s = 0$. $\square$

We recall that a smooth vector-valued function u on a real differentiable manifold X vanishes at $x \in X$ of order $k+1$ if all differentials $\mathrm{d}^0 u, \mathrm{d}^1 u, \ldots, \mathrm{d}^k u$ vanish at x. Given a vector bundle $E \to X$ and a section $u \in C^\infty(X, E)$, we say that u vanishes at $x \in X$ of order $k+1$ if for some (or any) local trivialisation $\phi : E|_U \to U \times \mathbb{R}^n$ of E about x the vector-valued function $p_2 \phi u|_U : U \to \mathbb{R}^n$ vanishes at x of order $k+1$. Given a differential form $f \in C^\infty_1(X, E)$, we say that f vanishes of order $k+1$ at $x \in X$, if for any neighbourhood U of x and any vector field $\Omega \in C^\infty(U, TX)$ the section $f(\Omega) \in C^\infty(U, E)$ vanishes of order $k+1$ at x. Let X' be a subset of X. We will say that $f \in C^\infty_1(X, E)$ vanishes of order $k+1$ at X' if f vanishes of order $k+1$ at x for any $x \in X'$.

Proposition 2.1.5. Let $L \to X$ be a holomorphic line bundle such that for any $b \in B$ there is a neighbourhood $U \ni b$ for which $L|_{\pi^{-1}(U)} \cong \mathcal{O}_{\pi^{-1}(U)}(n)$ for some fixed integer $n \geq 0$. Suppose that $f \in C^r_{0,1}(X, L)$, $1 \leq r \leq \infty$, is a closed form such that

(i) $f \in C^\infty_{0,1}(W, L)$ for some open set $W \supset Y$;

(ii) f vanishes of order n at Y;

(iii) $f(\Omega)$ vanishes of order $n+1$ at Y for any vector field $\Omega \in C^\infty(X, T_{\mathbb{C}}(X))$ that is tangent to Y.

Then there is a unique $u \in C^r(X \setminus Y, L) \cap C^\infty(W \setminus Y, L)$ such that $\overline{\partial} u = f$ on $X \setminus Y$ and $u = o(|u_Y|^n)$ at Y.

Proof. Condition (ii) yields $f = O(|u_Y|^n)$. Condition (iii) yields $f(\Omega) = o(|u_Y|^n)$ and $\overline{\partial}(f(\Omega)) = O(|u_Y|^n)$ for any vector field $\Omega \in C^\infty(X, T_{\mathbb{C}}(X))$ that is tangent to Y. Let $\{U_i\}_{i \in I}$ and Y_i, $i \in I$, be as in the proof of Proposition 2.1.4. Then conditions (i) and (ii) of Proposition 2.1.3 hold for $f|_{\pi^{-1}(U_i)}$ and u_{Y_i}, $i \in I$, and the rest of the proof is analogous to the proof of Proposition 2.1.4. □

2.2. THE $\overline{\partial}$-EQUATION FOR $(0,1)$-FORMS ON PROJECTIVE SPACE

In this subsection, we consider a projective space $\boldsymbol{P}(V)$, corresponding to a complex Banach space V, and apply the results from the previous subsection to construct a solution of the equation $\overline{\partial} u = f$ for $(0,1)$-forms on $\boldsymbol{P}(V)$ with values in the line bundle $\mathcal{O}_{\boldsymbol{P}(V)}(n)$, $n \in \mathbb{Z}$. For a description of the complex structure of $\boldsymbol{P}(V)$, we refer to [3, Sec. 3] . In the proofs we will use the blow up manifold $Bl_x(\boldsymbol{P}(V))$ of $\boldsymbol{P}(V)$ at a point $x \in \boldsymbol{P}(V)$, which is described as follows. For a given $x = [v_0] \in \boldsymbol{P}(V)$, we denote by V' the factor-space $V/[v_0]$, and by q the factoring linear map from V to V'. To simplify the notation, we will write $\boldsymbol{P}$ (resp. $\boldsymbol{P}'$) instead of $\boldsymbol{P}(V)$ (resp. $\boldsymbol{P}(V')$). The blow up $Bl_x(\boldsymbol{P})$ of $\boldsymbol{P}$ at x is the set

$$Bl_x(\boldsymbol{P}) = \{([v], [v']) \in \boldsymbol{P} \times \boldsymbol{P}' : q(v) \in [v']\}.$$

Let π (resp. ρ) be the restriction of the projection $\boldsymbol{P} \times \boldsymbol{P}' \to \boldsymbol{P}'$ (resp. $\boldsymbol{P} \times \boldsymbol{P}' \to \boldsymbol{P}$) to $Bl_x(\boldsymbol{P})$. To make $Bl_x(\boldsymbol{P})$ into a complex manifold, we first choose a bounded linear functional l on V such that $x \in \boldsymbol{P}_l$ and then, for any $l' \in V'^*$, $l' \neq 0$, define a coordinate map $\phi_{l'} : \pi^{-1}(\boldsymbol{P}'_{l'}) \to \boldsymbol{P}'_{l'} \times \boldsymbol{P}^1$ by the formula $\phi_{l'}([v], [v']) = ([v'], [l'(p(v)) : l(v)])$. The family of coordinate maps $\phi_{l'}$, $l' \in V'^*$, $l' \neq 0$, endows $Bl_x(\boldsymbol{P})$ with a structure of a complex manifold such that the maps π and ρ are holomorphic. Furthermore, the map π is a locally trivial projective line bundle over $\boldsymbol{P}'$, and the map ρ is biholomorphic outside the *exceptional divisor* $E = \rho^{-1}(x)$ of $Bl_x(\boldsymbol{P})$. We note that the map $\sigma : \boldsymbol{P}' \to Bl_x(\boldsymbol{P})$, given by the formula $\sigma([v']) = ([v_0], [v'])$ for $[v'] \in \boldsymbol{P}'$, is a holomorphic section of π such that $\sigma(\boldsymbol{P}') = E$.

Proposition 2.2.1. Let $f \in C^r_{0,1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$, $r \geq 1$, $n < 0$, be a closed form. If $\dim \boldsymbol{P} > 1$, then there exists a unique section $u \in C^r(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} u = f$.

Proof. Since $H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n)) = 0$ for $n < 0$, the equation $\overline{\partial} u = f$ cannot have two distinct solutions. To prove the existence of a solution u, it is enough to show that, for any $x \in \boldsymbol{P}$, there exists a section $u_x \in C^r(\boldsymbol{P} \setminus \{x\}, \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} u_x = f$ on $\boldsymbol{P} \setminus \{x\}$. Indeed, let $y \in \boldsymbol{P}$, $x \neq x$, and $u_y \in C^r(\boldsymbol{P} \setminus \{y\}, \mathcal{O}_{\boldsymbol{P}}(n))$ be such that $\overline{\partial} v_y = f$ on $\boldsymbol{P} \setminus \{y\}$. Then $s = u_x - u_y \in H^0(\boldsymbol{P} \setminus \{x, y\}, \mathcal{O}_{\boldsymbol{P}}(n))$ extends to a global holomorphic section $\tilde{s}$ of $\mathcal{O}_{\boldsymbol{P}}(n)$ by Hartogs' theorem. Let

$u \in C^r(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$ be given by u_x on $\boldsymbol{P} \setminus \{x\}$ and by $u_y + \tilde{s}$ on $\boldsymbol{P} \setminus \{y\}$. Then $\overline{\partial} u = f$.

So let $x \in \boldsymbol{P}$ and let $\tilde{f} = \rho^* f \in C^r_{0,1}(Bl_x(\boldsymbol{P}), \rho^* \mathcal{O}_{\boldsymbol{P}}(n))$. By Proposition (2.1.4), there is a section $\tilde{u} \in C^r(Bl_x \boldsymbol{P} \setminus E, \rho^* \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} \tilde{u} = \tilde{f}$. Now the map ρ is biholomorphic on $Bl_x(\boldsymbol{P}) \setminus E$, and $u_x = (\rho^{-1})^* \tilde{f} \in C^r(\boldsymbol{P} \setminus \{p\}, \mathcal{O}_{\boldsymbol{P}}(n))$ is such that $\overline{\partial} u_x = f$ on $\boldsymbol{P} \setminus \{x\}$. □

To deal with (0,1)-forms with values in the line bundles $\mathcal{O}_{\boldsymbol{P}}(n)$, $n \geq 0$, we consider a special class of Banach spaces. They have the property that for any nonempty open subset $W \subset V$, there exists a not identically zero function $\omega \in C^\infty(V)$ that is supported in U. A differentiable manifold M that has this property is called localising (cf. [5, Sec. 7]), or we will say that M localises. The projective space $\boldsymbol{P}(V)$ localises if and only if the Banach space V localises [5, Sec. 7]. All Hilbert spaces localise because the square of the norm is a smooth function. The Banach space l^1 is an example of a space that is not localising [4].

Proposition 2.2.2. Let V be a Banach space that localises. Suppose that $f \in C^r_{0,1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n))$, $n \geq 0$, $r \geq 1$, is a closed form such that $f \in C^\infty_{0,1}(W, \mathcal{O}_{\boldsymbol{P}}(n))$ for some nonempty open set $W \subset \boldsymbol{P}$. Then there exists a $u \in C^r(P, \mathcal{O}_{\boldsymbol{P}}(n)) \cap C^\infty(W, \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} u = f$.

Proof. We will assume that V is an infinite dimensional Banach space (for $\dim V < \infty$ the proposition is well known under much weaker conditions on the regularity of f). Then it is enough to show that for any $x \in W$ there is a $u_x \in C^r(\boldsymbol{P} \setminus \{x\}, \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} u_x = f$ on $\boldsymbol{P} \setminus \{x\}$ (cf. the proof of Proposition (2.2.1)

So let $x \in W$ and $\boldsymbol{P}'$ be a hyperplane in $\boldsymbol{P}$ which does not contain x. Let W' be a neighbourhood of x such that $W' \subset W$ and $W' \cap \boldsymbol{P}' = \emptyset$. Since the line bundle $\mathcal{O}_{\boldsymbol{P}}(n)$ trivialises on W', there exists $u' \in C^\infty(W', \mathcal{O}_{\boldsymbol{P}}(n))$ such that $f|_{W'} - \overline{\partial} u'$ vanishes of order n at x (see [5, Theorem 3.6]). Let $\omega \in C^\infty(\boldsymbol{P})$ be a cut-off function that is supported in W', and equal to 1 in a neighbourhood of x and let $g = f - \overline{\partial}(\omega v)$. Then $g \in C^r_{0,1}(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(n)) \cap C^\infty_{0,1}(W, \mathcal{O}_{\boldsymbol{P}}(n))$ is a closed form that vanishes of order n at x. Consequently, $\tilde{g} = \rho^* g \in C^r_{0,1}(Bl_p(\boldsymbol{P}), \rho^* \mathcal{O}_{\boldsymbol{P}}(n))$ is a closed form that is smooth on $\rho^{-1}(W)$ and vanishes of order n at E. Moreover, if Ω is a smooth vector field on $Bl_x(\boldsymbol{P})$ that is tangent to E, then $\tilde{g}(\Omega)$ vanishes of order $n+1$ at E because $\rho_*(T_{\mathbf{C}} E) = 0$. By Proposition 2.1.5, there is a $\tilde{u} \in C^r(Bl_x(\boldsymbol{P}) \setminus E, \rho^* \mathcal{O}_{\boldsymbol{P}}(n))$ such that $\overline{\partial} \tilde{u} = \tilde{g}$ on $Bl_x(\boldsymbol{P}) \setminus E$. Set $u_x = (\rho^{-1})^* \tilde{u} + \omega v \in C^r(\boldsymbol{P} \setminus \{x\}, \mathcal{O}_{\boldsymbol{P}}(n))$. Then $\overline{\partial} u_x = f$ on $\boldsymbol{P} \setminus \{x\}$, which completes the proof. □

3. ANALYSIS OF REGULARITY

Let X be a submanifold of finite codimension n and degree d in $\boldsymbol{P} = \boldsymbol{P}(V)$ and (V', W, z) be an admissible triple for X (see [3, Section 3]). Let $p : X \to$

$P(V')$ be the map induced by the projection $V = W \times V' \to V'$. According to [3, Proposition 4.2], for any given $f \in C^r_{0,1}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, there exist unique forms $f_j \in C^r_{0,1}(P(V')_D, \mathcal{O}_{P(V')}(n-j))$, $j = 0, \ldots, d-1$ such that $f|_{X_D} = \sum_{j=0}^{d-1} (p|_{X_D})^* g_j \otimes z^j$.

The differential forms f_j, $j = 0, \ldots, d-1$, should not be expected to behave in a regular manner along the divisor $\mathcal{D} = Z(D)$ (see Example 1 below). However, their behaviour can be improved to a degree by twisting with powers of D. Thus for any natural number N, we define new forms

$$\tilde{f}_j^N \in C_{0,1}(P(V'), \mathcal{O}_{P(V')}(n - j + N \deg D)), \quad j = 0, \ldots, d-1,$$

in the following way:

$$\tilde{f}_j^N(b, \xi) = \begin{cases} D(b)^N f_j(b, \xi) & \text{for } b \in P(V')_D \text{ and } \xi \in T_b^{0,1} P(V'), \\ 0 & \text{for } b \notin P(V')_D \text{ and } \xi \in T_b^{0,1} P(V'). \end{cases}$$

The following proposition is the main result in this section.

Proposition 3.1. Let $f \in C^r_{0,1}(X, \mathcal{O}_X(n))$, $1 \leq r < \infty$, $n \in \mathbb{Z}$. If $N > 4r + 3$, then

$$\tilde{f}_j^N \in C^r_{0,q}(P(V'), \mathcal{O}_{P(V')}(n - j + N \deg D)), \quad j = 0, \ldots, d-1.$$

The proof of Proposition 3.1 will be given later in this section.

Since the vector bundle $\pi : P \backslash P(W) \to P(V')$ trivialises over every affine open set $P(V')_h$, $0 \neq h \in V'^*$, we will prove first an affine version of Proposition 3.1.

From now on we will assume that X is an algebraic manifold of finite codimension n in a Banach space V. Let (W, V') be an admissible factorisation for X, and let $Z_1, \ldots, Z_n$ be a basis of W^* such that $z = Z_1 + I(X) \in C[X]$ generates the field of fractions of $C[X]$ over the field of fractions of the $C[V']$. We denote by D the discriminant of the minimal polynomial of z over the field of fractions of $C[V']$. The restriction of the projection $p : X \to V'$ to X_D will be denoted by p_D. By [3, Proposition 2.3] the holomorphic map $p_D : X_D \to V_D$ is a covering of degree $d = \deg F$. For a given $f \in C^r_{0,1}(X)$, let $g = f|_{X_D}$. By [3, Proposition 4.3] there exist unique forms $g_j \in C^r_{0,1}(V'_D)$ $j = 0, \ldots, d-1$ such that $g = \sum_{j=0}^{d-1} z^j \pi^* g_j$. For any natural number N we define new forms $\tilde{g}_j^N \in C_{0,1}(V')$, $j = 0, \ldots, d-1$, in the following way:

$$\tilde{g}_j^N(b, \xi) = \begin{cases} D(b)^N g_j(b, \xi) & \text{for } b \in V'_D \text{ and } \xi \in T_b^{0,1} V' \\ 0 & \text{for } b \notin V'_D \text{ and } \xi \in T_b^{0,1} V'. \end{cases}$$

Proposition 3.2. Let $f \in C^r_{0,1}(X)$, $1 \leq r < \infty$, and let $g = f|_{X_D}$. If $N > 4r + 3$, then $\tilde{g}_j^N \in C^r_{0,1}(V')$, $j = 0, \ldots, d-1$,

The proof of this proposition will be given later in the section.

The following example shows a typical behaviour of the forms g_j, along the divisor $\mathcal{D} = Z(D)$. Let $V = \mathbb{C}^2$ and let $X = \{(Y,Z) \in \mathbb{C}^2 : Z^2 = Y/4\}$. Set $z = Z|_X \in C[X]$. Let $W = \{(0,Z) \in \mathbb{C}^2 : Z \in \mathbb{C}\}$ and $V' = \{(Y,0) \in \mathbb{C}^2 : Y \in \mathbb{C}\}$. Then the projection $p : X \to \mathbb{C}$ given by $p(Y,Z) = Y$ is finite and surjective, and z generates $C[X]$ over $C[V'] = \mathbb{C}[Y]$. The discriminant of the minimal polynomial $F = Z^2 - Y/4$ is $D = Y$. Thus $\mathcal{D} = \{0\}$, $V'_D = \mathbb{C} \setminus \{0\}$, and $X_D = X \setminus \{(0,0)\}$.

Example 1. Let $X \subset \mathbb{C}^2$ be the quadric described above and $f = \mathrm{d}\overline{z}|_X$. Then $8\overline{z}\,\mathrm{d}\overline{z} = p^*(\mathrm{d}\overline{Y})$, and solving for $\mathrm{d}\overline{z}$, we obtain

$$g = \frac{1}{8\overline{z}}\, p_D^*(\mathrm{d}\overline{Y}) = \frac{z}{8z\overline{z}}\, p_D^*(\mathrm{d}\overline{Y}) = z\, p_D^*(\frac{1}{2|Y|}\mathrm{d}\overline{Y}).$$

Hence $g_0 = 0$ and $g_1 = 2^{-1}|Y|^{-1}\mathrm{d}\overline{Y}$. It is easy to see that, for every natural number r, there exists a natural number N_r such that $\tilde{g}_1^N = 2^{-1}|Y|^{-1}Y^N\,\mathrm{d}\overline{Y} \in C^r(\mathbb{C})$ for $N > N_r$. However, there is no natural number N such that $\tilde{g}_1^N \in C^\infty(\mathbb{C})$.

Suppose that $r : U \to X$ is a right inverse to p_D on some open set $U \subset V'_D$, i.e. $p_D \circ r = \mathrm{id}_U$. Let $e_1, \ldots, e_n$ be the basis of W which is dual to the basis $Z_1, \ldots, Z_n$. Let $R_j = z_j \circ r \in H^0(U, \mathcal{O}_U)$, $j = 1, \ldots, n$. Then

$$r(b) = (\, b\,,\ \sum_{j=1}^{n} R_j(b)e_j\,) \ \in\ X \subset V' \times W$$

for all $b \in U$. Since $F(z) = 0$, where F is the minimal polynomial of z, we obtain $F(R_1) = 0$. As in [3, Lemma 2.3], there exist polynomials $F_j \in C[V'][Z]$, $j = 2, \ldots, n$, such that

$$R_j = D^{-1}F_j(R_1)\,, \qquad j = 2, \ldots, n\,. \tag{3.1}$$

The holomorphic map $r : U \to X \subset V$ induces a complex linear map r_* from the complexified tangent space of $b \in U$ to the complexified tangent space of $r(b) \in X$, $r_* : T_b^{\mathbb{C}}U \to T_{r(b)}^{\mathbb{C}}X \subset T_{r(b)}^{\mathbb{C}}V$, for all $b \in U$. For $\xi \in T_b^{\mathbb{C}}U$, we denote by $r_*(b,\xi)$ the image of ξ in $T_{r(b)}^{\mathbb{C}}X$. Since V' and V are vector spaces, we can naturally identify $T_b^{\mathbb{C}}U$ and $T_{r(b)}^{\mathbb{C}}V$ with $V' \oplus \overline{V'}$ and $V \oplus \overline{V} = (V' \oplus \overline{V'}) \times (W \oplus \overline{W})$, respectively. Since the map r is holomorphic, $r_*(T_b^{1,0}U) \subset T_{r(b)}^{1,0}X$ and $r_*(T_b^{0,1}U) \subset T_{r(b)}^{0,1}X$. The restriction of r_* to $T_b^{1,0}U$ will be denoted by dr, and the restriction of r_* to $T_b^{0,1}U$ will be denoted by $d\overline{r}$. For any vector $\xi \in T_b^{1,0}U$, its conjugate vector $\overline{\xi}$ is in $T_b^{0,1}U$, and $d\overline{r}(b,\overline{\xi}) = \overline{dr(b,\xi)}$. It is clear that for $\xi \in T_b^{1,0}U$ we have

$$r_*(b,\xi) = (\,\xi\,,\ \sum_{j=1}^{k} dR_j(b,\xi)\,\frac{\partial}{\partial Z_j}\,) \in V' \times W\,, \tag{3.2}$$

where dR_j is the differential of the holomorphic function R_j, $j = 1, \ldots, n$.

The next lemma is the main step in the proof of Proposition 3.2.

Lemma 3.3. There exists a smooth function $H : V \times V' \to V$ such that:
(i) for any open set U in V'_D and any right inverse $r : U \to X$ to p_D on U we have $dr(b,\xi) = D(b)^{-2}H(r(b),\xi)$ for $b \in U$ and $\xi \in V'$;
(ii) $H(x,\xi) \in T^{1,0}_x X$ for $x \in X$ and $\xi \in V'$;
(iii) H is linear in $\xi \in V'$.

Proof. We want to find the coefficients $dR_j(b,\xi)$, $j = 1,\dots,n$, in (3.2). Since the function R_1 satisfies the equation $F(R_1) = 0$, we use implicit differentiation to find $dR_1(b,\xi)$. Then we differentiate (3.1) to find $dR_j(b,\xi)$, $j = 2,\dots,n$.

Let $F = Z^d + a_1 Z^{d-1} + \cdots + a_d$, where $a_m \in C[V']$, $m = 1,\dots,d$, and let $F' \in C[V'][Z]$ be the derivative of F with respect to Z.

Since $F(R_1) = 0$, we obtain

$$F'(R_1(b))\, dR_1(b,\xi) + \sum_{m=1}^{d} da_m(b,\xi)\, R_1(b)^{d-m} = 0\,. \tag{3.3}$$

It is well known (see for example [7]) that there exist polynomials $A, B \in C[V'][Z]$ such that

$$AF + BF' = D\,.$$

Hence

$$F'(R_1(b))^{-1} = D(b)^{-1}B(R_1(b)) \tag{3.4}$$

for $b \in U$ and $\xi \in V'$. Let $H_1 \in C^\infty(V \times V')$ be the function given by

$$H_1(v,\xi) = -B(Z_1(v)) \sum_{m=1}^{d} da_m(\pi(v),\xi) Z_1(v)^{d-m}\,.$$

It follows from (3.3) and (3.4) that

$$dR_1(b,\xi) = D(b)^{-1} H_1(r(b),\xi) \tag{3.5}$$

for $b \in U$ and $\xi \in V'$.

Let $F_j = \sum_{m=0}^{d-1} a_{mj} Z^{d-m-1} \in C[V'][Z]$, $j = 2,\dots,n$, where $a_{mj} \in C[V']$ for $j = 2,\dots,n$, and $m = 0,\dots,d-1$. Let $F'_j \in C[V'][Z]$, $j = 2,\dots,n$, be the derivative of F_j with respect to Z. Since $R_j = D^{-1}F_j(R_1)$, $j = 2,\dots,n$, (by equation (3.1)), we obtain

$$\begin{aligned} dR_j(b,\xi) &= -D(b)^{-2} dD(b,\xi) F_j(R_1(b)) + D(b)^{-1} dR_1(b,\xi) F'_j(R_1(b)) + \\ &\quad + D(b)^{-1} \sum_{m=0}^{d-1} da_{mj}(b,\xi) R_1(b)^{d-m-1} \qquad \text{(by (3.5))} \\ &= -D(b)^{-2} dD(b,\xi) F_j(R_1(b)) + D(b)^{-2} H_1(r(b),\xi) F'_j(R_1(b)) + \\ &\quad + D(b)^{-1} \sum_{m=0}^{d-1} da_{mj}(b,\xi) R_1(b)^{d-m-1}\,. \end{aligned} \tag{3.6}$$

Let $H_j \in C^\infty(V \times V')$, $j = 2, \dots, k$, be the function given by

$$H_j(v,\xi) = -dD(\pi(v),\xi)\, F_j(Z_1(v)) + H_1(w,\xi)\, F_j'(Z_1(v)) + \\ + D(p(v)) \sum_{m=0}^{d-1} da_{mj}(p(v),\xi) Z_1(v)^{d-m-1}.$$

It follows from (3.6) that $dR_j(b,\xi) = D(b)^{-2} H_j(r(b),\xi)$ for $j = 2, \dots, n$.

Finally, let $H : V \times V' \to V$ be the smooth function given by

$$H(v,\xi) = (\, D(\pi(v))^2 \xi\, ,\, D(\pi(v)) H_1(v,\xi) \frac{\partial}{\partial Z_1} + \sum_{j=2}^{k} H_j(v,\xi) \frac{\partial}{\partial Z_j}\,).$$

It follows from (3.1), (3.5), and (3.6), that (i) holds for H.

To prove that (ii) holds for H, we notice first that if $x \in X_D$ then there exists an open neighbourhood $U \subset V_D'$ of $p(x)$ and a right inverse $r : U \to X$ to p_D on U such that $r(p(x)) = x$. By part (i) we have $H(x,\xi) = D(p(x))^2\, r_*(p(x),\xi) \in T_x^{1,0}X$ for all $\xi \in V'$. Thus (ii) holds for H if $x \in X_D$. Let $\widetilde{H} : X \times V' \to T^{1,0}V = V \times V$ be given by the formula $\widetilde{H}(x,\xi) = (x, H(x,\xi))$ for $(y,\xi) \in X \times V'$. It is clear that $\tilde{H}$ is a continuous map and $\widetilde{H}(X_D \times V') \subset T^{1,0}X$. Since $T^{1,0}X$ is a closed subset of $T^{0,1}V$, and $X_D \times V'$ is a dense subset of $X \times V'$, we see that $\widetilde{H}(X \times V') \subset T^{1,0}X$. Hence condition (ii) holds for H. Finally, condition (iii) also holds for H because all functions H_j, $1 = 2, \dots, n$, are linear in ξ. □

We will need a similar result for the restriction of r_* to the bundle $T^{0,1}U$.

Lemma 3.4. There exists a smooth function $\overline{H} : V \times \overline{V'} \to \overline{V}$ such that:
(i) for any open set U in V_D' and any right inverse $r : U \to X$ to p_D on U we have $d\overline{r}(b,\overline{\xi}) = \overline{D(b)}^{-2}\, \overline{H}(r(b),\overline{\xi})$ for $b \in U$ and $\overline{\xi} \in \overline{V'}$;
(ii) $\overline{H}(x,\overline{\xi}) \in T_x^{0,1}X$ for $x \in X$ and $\overline{\xi} \in \overline{V'}$;
(iii) $\overline{H}$ is linear in $\overline{\xi} \in \overline{V'}$.

Proof. Let $\overline{H} : V \times \overline{V'} \to \overline{V}$ be given by $\overline{H}(w,\overline{\xi}) = \overline{H(w,\xi)}$ for $w \in W$, $\xi \in W'$. It follows from Lemma 3.3 that (i), (ii) and (iii) hold for the map $\overline{H}$ because $d\overline{r}(b,\overline{\xi}) = \overline{dr(b,\xi)}$. □

We denote by $X_{\mathbb{R}}$ the real manifold associated with the complex manifold X. For any $x \in X$ there is a natural inclusion

$$T_x X_{\mathbb{R}} \to T_x^{\mathbb{C}} X = \mathbb{C} \otimes_{\mathbb{R}} T_x X_{\mathbb{R}} = T_x^{1,0}X \oplus T_x^{0,1}X$$

given by

$$T_x X_{\mathbb{R}} \ni \eta \mapsto \frac{1}{2}(1 \otimes \eta + \frac{1}{i} \otimes i\eta) \oplus \frac{1}{2}(1 \otimes \eta - \frac{1}{i} \otimes i\eta) \in T_x^{1,0}X \oplus T_x^{0,1}X.$$

For a given vector $\eta \in T_x X_{\mathbb{R}}$, we denote by $\eta^{1,0}$ and $\eta^{0,1}$ the vectors $\frac{1}{2}(1 \otimes \eta + \frac{1}{i} \otimes i\eta)$ and $\frac{1}{2}(1 \otimes \eta - \frac{1}{i} \otimes i\eta)$, respectively.

If E is a complex vector space, then the real tangent bundle $TE_{\mathbb{R}} \to E_{\mathbb{R}}$ is naturally isomorphic to the trivial bundle $E_{\mathbb{R}} \times E_{\mathbb{R}} \to E_{\mathbb{R}}$. Thus, for any $b \in E_{\mathbb{R}}$, we can canonically identify $T_b E_{\mathbb{R}}$ with $E_{\mathbb{R}}$.

Lemma 3.3 and Lemma 3.4 are combined in the next lemma, to prove a similar result for the map $r_* : TU_{\mathbb{R}} \to TV_{\mathbb{R}}$.

Lemma 3.5 *There exists a smooth function* $R : V_{\mathbb{R}} \times V'_{\mathbb{R}} \to V_{\mathbb{R}}$ *such that:*
(i) for any open set U in V'_D and any right inverse $r : U \to X$ to p_D on U we have $r_*(b,\eta) = D(b)^{-2}\,\overline{D(b)}^{-2}\,R(r(b),\eta)$ for $b \in U_{\mathbb{R}}$ and $\eta \in V'_{\mathbb{R}}$;
(ii) $R(x,\eta) \in T_x X_{\mathbb{R}}$ for $x \in X_{\mathbb{R}}$ and $\eta \in V'_{\mathbb{R}}$;
(iii) R is $\mathbb{R}$-linear in $\eta \in V'_{\mathbb{R}}$.

Proof. For any $b \in U_{\mathbb{R}}$ and any $\eta \in V'_{\mathbb{R}}$

$$\begin{aligned} r_*(b,\eta) &= r_*(b,\eta^{1,0}) + r_*(b,\eta^{0,1}) \\ &= dr(b,\eta^{1,0}) + d\overline{r}(b,\eta^{0,1}) \\ &= D(b)^{-2}H(r(b),\eta^{1,0}) + \overline{D(b)}^{-2}\overline{H}(r(b),\eta^{0,1}) \\ &= D(b)^{-2}\overline{D(b)}^{-2}\left\{\overline{D(b)}^{2}H(r(b),\eta^{1,0}) + D(b)^{2}\overline{H}(r(b),\eta^{0,1})\right\}. \end{aligned} \tag{3.7}$$

Let $R : V_{\mathbb{R}} \times V'_{\mathbb{R}} \to V_{\mathbb{R}}$ be the map given by

$$R(v,\eta) = \overline{D(\pi(v))}^{2}\,H(v,\eta^{1,0}) + D(\pi(v))^{2}\,\overline{H}(v,\eta^{0,1}), \quad v \in V_{\mathbb{R}}, \eta \in V'_{\mathbb{R}}. \tag{3.8}$$

Eq. (3.7) shows that condition (i) holds for R. If $x \in X_{\mathbb{R}}$, then the vectors $\overline{D(p(x))}^{2}H(x,\eta^{1,0})$ and $D(p(x))^{2}\overline{H}(x,\eta^{0,1})$ are conjugate to each other. Hence $R(x,\eta) \in T_x X_{\mathbb{R}}$ for $x \in X_{\mathbb{R}}$. Thus condition (ii) holds for R. It is clear from (3.8) that R is $\mathbb{R}$-linear in η. □

Lemma 3.6. Suppose that $g \in C^r_{0,1}(X_D)$ is such that $g = f|_{X_D}$ for some $f \in C^r_{0,1}(X)$. Then there exists a function $G \in C^r(X \times \overline{V'})$ such that:
(i) for any open set U in V'_D and any right inverse $r : U \to X$ to p_D on U we have $r^*g\,(b,\overline{\xi}) = \overline{D(b)}^{-2}G(r(b),\overline{\xi})$ for $b \in U$ and $\overline{\xi} \in \overline{V'}$;
(ii) G is linear in $\overline{\xi} \in \overline{V'}$.

Proof. For $x \in X$ and $\overline{\xi} \in \overline{V'}$, let $G(x,\overline{\xi}) = f(x,\overline{H}(x,\overline{\xi}))$, where $\overline{H}$ is the map defined in Lemma 3.4. We note that the right-hand side makes sense because $\overline{H}(x,\overline{\xi}) \in T^{0,1}_x X$ by part (ii) of Lemma 3.4. Let us verify that (i) holds for G:

$$r^*g\,(b,\overline{\xi}) = f(r(b),d\overline{r}(b,\overline{\xi})) = f(r(b),\overline{D(b)}^{-2}\,\overline{H}(r(b),\overline{\xi})) = \overline{D(b)}^{-2}\,G(r(b),\overline{\xi}).$$

The definition of G shows that it is linear in $\overline{\xi} \in \overline{V'}$. $\square$

Proposition 3.7. Let $f \in C^r_{0,1}(X)$ and $g = f|_{X_D}$. Suppose that $g_j \in C^r_{0,1}(V'_D)$, $j = 0, \dots, d-1$, are such that $g = \sum_{j=0}^{d-1} z^j\, \pi^* g_j$. Then there exist functions $G_j \in C^r(X^d \times \overline{V'})$, $j = 0, \dots d-1$, such that:
(i) each function G_j is symmetric in $(x_1, \dots, x_d) \in X^d$ and linear in $\overline{\xi} \in \overline{V'}$;
(ii) $g_j(b, \overline{\xi}) = D(b)^{-1}\overline{D(b)}^{-2} G_j(r_1(b), \dots, r_d(b), \overline{\xi})$ for $j = 0, \dots, d-1$, $b \in V'_D$, $\overline{\xi} \in \overline{V'}$, where $\{r_1(b), \dots, r_d(b)\}$ is the fiber of $p_D : X_D \to V'_D$ over $b \in V'_D$.

Proof. Let $b \in V'_D$ and let $U \subset V'_D$ be a neighbourhood of b such that the covering map $p_D : X_D \to V'_D$ has d distinct right inverses $r_i : U \to X_D$ on U, $p \circ r_i = \mathrm{id}_U$, $i = 1, \dots, d$. Then, according to [3, Proposition 4.3],

$$g_j(b, \overline{\xi}) = D(b)^{-1} \Delta(b) \det A_j(b, \overline{\xi})$$

where

$$\Delta(b) = \prod_{1 \le i_1 < i_2 \le d} (z(r_{i_2}(b)) - z(r_{i_1}(b)))\,,$$

and $A_j(b, \overline{\xi})$ is the $d \times d$ matrix

$$\begin{pmatrix} 1 & z(r_1(b)) & \cdots & z(r_1(b))^{j-1} & r_1^* g\,(b, \overline{\xi}) & z(r_1(b))^{j+1} & \cdots & z(r_1(b))^{d-1} \\ 1 & z(r_2(b)) & \cdots & z(r_2(b))^{j-1} & r_2^* g\,(b, \overline{\xi}) & z(r_2(b))^{j+1} & \cdots & z(r_2(b))^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(r_d(b)) & \cdots & z(r_d(b))^{j-1} & r_d^* g\,(b, \overline{\xi}) & z(r_d(b))^{j+1} & \cdots & z(r_d(b))^{d-1} \end{pmatrix}.$$

According to Lemma 3.6 $r_i^* g\,(b, \overline{\xi}) = \overline{D(b)}^{-2} G(r_i(b), \overline{\xi})$ for $i = 1, \dots, d$. Hence $\det A_j(b, \overline{\xi}) = \overline{D(b)}^{-2} \det B_j(b, \overline{\xi})$, where $B_j(b, \overline{\xi})$, $j = 0, \dots, d-1$, is the matrix

$$\begin{pmatrix} 1 & z(r_1(b)) & \cdots & z(r_1(b))^{j-1} & G(r_1(b), \overline{\xi}) & z(r_1(b))^{j+1} & \cdots & z(r_1(b))^{d-1} \\ 1 & z(r_2(b)) & \cdots & z(r_2(b))^{j-1} & G(r_2(b), \overline{\xi}) & z(r_2(b))^{j+1} & \cdots & z(r_2(b))^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(r_d(b)) & \cdots & z(r_d(b))^{j-1} & G(r_d(b), \overline{\xi}) & z(r_d(b))^{j+1} & \cdots & z(r_d(b))^{d-1} \end{pmatrix}.$$

Let $\delta : X^d \to \mathbb{C}$ be the smooth function given by

$$\delta(x_1, \dots, x_d) = \prod_{1 \le i_1 < i_2 \le d} (z(x_{i_2}) - z(x_{i_1}))\,.$$

Let $C_j(x_1, \dots, x_d, \overline{\xi})$, $j = 0, \dots, d-1$, be the matrix

$$\begin{pmatrix} 1 & z(x_1) & \cdots & z(x_1)^{j-1} & G(x_1, \overline{\xi}) & z(x_1)^{j+1} & \cdots & z(x_1)^{d-1} \\ 1 & z(x_2) & \cdots & x(y_2)^{j-1} & G(x_2, \overline{\xi}) & x(y_2)^{j+1} & \cdots & z(x_2)^{d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z(x_d) & \cdots & z(x_d)^{j-1} & G(x_d, \overline{\xi}) & z(x_d)^{j+1} & \cdots & z(x_d)^{d-1} \end{pmatrix}.$$

Finally, let $G_j : X^d \times \overline{V'} \to \mathbb{C}$, $j = 0, \dots, d-1$, be given by

$$G_j(x_1, \dots, x_d, \overline{\xi}) \;=\; \delta(x_1, \dots, x_d) \det C_j(x_1, \dots, x_d, \overline{\xi})$$

for $j = 0, \dots, d-1$. The definition of G_j shows that it satisfies (i). Since $\Delta(b) = \delta(r_1(b), \dots, r_d(b))$ and $B_j(b, \overline{\xi}) = C_j(r_1(b), \dots, r_d(b), \overline{\xi})$, we see that (ii) also holds for G_j, $j = 0, \dots, d-1$. □

Let $\omega \in C^r_{0,1}(V'_D)$ and let $\eta_1, \dots, \eta_m$ be vectors in $V'_{\mathbb{R}}$, $(m \le r)$. The derivative of order m of ω at the point $(b, \overline{\xi}) \in (V'_D)_{\mathbb{R}} \times \overline{V'}$ in the directions $\eta_1, \dots, \eta_m$ will be denoted by $d^m_b \omega(b, \overline{\xi}; \eta_1, \dots, \eta_m)$. For any $\omega \in C^r_{0,1}(V'_D)$ and any natural number N, we will denote by ω^N the form $D^N \omega \in C^r_{0,1}(V'_D)$. The next lemma is an extension of Proposition 3.7 to the derivatives of the forms g_j, $j = 0, \dots, d-1$.

Lemma 3.8. Let $f \in C^r_{0,1}(X)$, $1 \le r < \infty$, and $g = f|_{X_D}$. Suppose that $g_j \in C^r_{0,1}(V'_D)$, $j = 0, \dots, d-1$, are such that $g = \sum_{j=0}^{d-1} z^j \pi^* g_j$. Then for any $0 \le m \le r$, $0 \le j \le d-1$ and any natural number N there exists a function $G^N_{jm} \in C^{r-m}(X^d \times \overline{V'} \times {V'}^m_{\mathbb{R}})$ such that:
(i) G^N_{jm} is symmetric in $x_1, \dots, x_d \in X^d$ and linear in $\overline{\xi} \in \overline{V'}$;
(ii) for any $b \in V'_D$, $\overline{\xi} \in \overline{V'}$, and $\eta_1, \dots, \eta_m \in T_b V'_{\mathbb{R}} = V'_{\mathbb{R}}$ we have

$$d^m_b g^N_j(b, \overline{\xi}; \eta_1, \dots, \eta_m) =$$
$$D(b)^{N-2m-1} \overline{D(b)}^{-2m-2} G^N_{jm}(r_1(b), \dots, r_d(b), \overline{\xi}, \eta_1, \dots, \eta_m).$$

Proof. The proof is by induction on m. For $m = 0$ the lemma is true by Proposition 3.7. We are going to show that if $1 \le m \le r$ and there exists a function $G^N_{j\,m-1} \in C^{r-m+1}(X^d \times \overline{V'} \times {V'}^{m-1}_{\mathbb{R}})$ such that (i) and (ii) hold for $G^N_{j\,m-1}$, then there exists a function $G^N_{jm} \in C^{r-m}(Y^d \times \overline{V'} \times {V'}^m_{\mathbb{R}})$ such that (i) and (ii) hold for G^N_{jm}.

Let $b \in V'_D$ and let $U \subset V'_D$ be a neighbourhood of b such that the covering map $p_D : X_D \to V'_D$ has d distinct right inverses $r_i : U \to X_D$, $i = 1, \dots, d$, on U, $\pi \circ r_i = \mathrm{id}_U$, $i = 1, \dots, d$.

To find $d^m_b g^N_j(b, \overline{\xi}; \eta_1, \dots, \eta_m)$, we differentiate the function

$$D(b)^{N-2m+1} \overline{D(b)}^{-2m} G^N_{j\,m-1}(r_1(b), \dots, r_d(b), \overline{\xi}, \eta_1, \dots, \eta_{m-1})$$

in the direction $\eta_m \in T_b V'_{\mathbb{R}} = V'_{\mathbb{R}}$. After applying the product rule and the chain rule, we obtain the following terms:
I. $A_{Nm}\, D(b)^{N-2m}\, dD(b, \eta_m)\, \overline{D(b)}^{-2m}\, G^N_{j\,m-1}(r_1(b), \dots, r_d(b), \overline{\xi}, \eta_1, \dots, \eta_{m-1})$, where $A_{Nm} = N - 2m - 1$ and $dD(b, \eta_m)$ is the derivative of D in the direction $\eta_m \in T_b V'_{\mathbb{R}} = V'_{\mathbb{R}}$.

Let $\mathcal{F}_{jm}^N \in C^{r-m+1}(X^d \times \overline{V'} \times V_{\mathbb{R}}'^m)$ be the function given by

$$\mathcal{F}_{jm}^N(x_1, \dots, x_d, \overline{\xi}, \eta_1, \dots, \eta_m) = \\ A_{Nm}\, D(b)\, \overline{D(b)}^2\, dD(b, \eta_m)\, G_{j\,m-1}^N(x_1, \dots, x_d, \overline{\xi}, \eta_1, \dots, \eta_{m-1}). \tag{3.9}$$

II. $B_m\, D(b)^{N-2m+1}\, \overline{D(b)}^{-2m-1}\, d\overline{D}(b, \eta_m)\, G_{j\,m-1}^N(r_1(b), \dots, r_d(b), \overline{\xi}, \eta_1, \dots, \eta_{m-1})$, where $B_m = -2m - 1$ and $d\overline{D}(b, \eta_m)$ is the derivative of $\overline{D}$ in the direction $\eta_m \in T_b V_{\mathbb{R}}' = V_{\mathbb{R}}'$.

Let $\mathcal{G}_{jm}^N \in C^{r-m+1}(X^d \times \overline{V'} \times V_{\mathbb{R}}'^m)$ be the function given by

$$\mathcal{G}_{jm}^N(x_1, \dots, x_d, \overline{\xi}, \eta_1, \dots, \eta_m) = \\ B_m\, D(b)^2\, \overline{D(b)}\, d\overline{D}(b, \eta_m)\, G_{j\,m-1}(x_1, \dots, x_d, \overline{\xi}, \eta_1, \dots, \eta_{m-1}). \tag{3.10}$$

III. $D(b)^{N-2m+1}\, \overline{D(b)}^{-2m}\, \dfrac{\partial G_{j\,m-1}^N}{\partial y_i}(r_1(b), \dots, r_d(b), \overline{\xi}, \eta_1, \dots, \eta_{m-1})(r_{i*}(b, \eta_m))$, $i = 1, \dots, d$, where

$$\frac{\partial G_{j\,m-1}^N}{\partial y_i}(r_1(b), \dots, r_d(b), \overline{\xi}, \eta_1, \dots, \eta_{m-1}) : T_{r_i(b)} X_{\mathbb{R}} \to \mathbb{C}$$

is the "the partial derivative" of $G_{j\,m-1}^N$ with respect to x_i, $i = 1, \dots, d$, and

$$r_{i*}(b, \cdot) : T_b V_{\mathbb{R}}' \to T_{r_i(b)} X_{\mathbb{R}}, \qquad i = 1, \dots, d,$$

is the $\mathbb{R}$-linear map from $T_b V_{\mathbb{R}}'$ to $T_{r_i(b)} X_{\mathbb{R}}$ that is induced by $r_i : U \to X_D$ for $i = 1, \dots, d$. By Lemma 3.5,

$$r_{i*}(b, \eta_m) = D(b)^{-2}\, \overline{D(b)}^{-2}\, R(r_i(b), \eta_m), \qquad i = 1, \dots, d.$$

Let $\mathcal{H}_{jmi}^N \in C^{r-m}(X^d \times \overline{V'} \times V_{\mathbb{R}}'^m)$ be the function given by

$$\mathcal{H}_{jmi}^N(x_1, \dots, x_d, \overline{\xi}, \eta_1, \dots, \eta_m) = \\ = \frac{\partial G_{j\,m-1}^N}{\partial y_i}(x_1, \dots, x_d, \overline{\xi}, \eta_1, \dots, \eta_{m-1})(R(y_i, \eta_m)) \tag{3.11}$$

for $i = 1, \dots, d$.

Finally, let $G_{jm}^N \in C^{r-m}(X^d \times \overline{V'} \times V_{\mathbb{R}}'^m)$ be the function

$$G_{jm}^N = \mathcal{F}_{jm}^N + \mathcal{G}_{jm}^N + \sum_{i=1}^d \mathcal{H}_{jmi}^N. \tag{3.12}$$

It follows from (3.9), (3.10), and (3.11) that (ii) holds for G_{jm}. We note that the functions $\mathcal{F}_{jm}^N$ and $\mathcal{G}_{jm}^N$ are symmetric in $x_1, \dots, x_d$. Since the function $\sum_{i=1}^d \mathcal{H}_{jmi}^N$ is also symmetric in $x_1, \dots, x_d$, we see that G_{jm}^N is symmetric in $x_1, \dots, x_d$. The function G_{jm}^N is linear in $\bar{\xi} \in \overline{V'}$ because all terms on the right-hand side of (3.12) are linear, too. Thus (i) holds for G_{jm}^N. □

In the proof of Proposition 3.2 we will need the following simple lemma.

Lemma 3.9. Let X, Y, and W be metric spaces and let $p : Y \to Z$ be a proper map. Let d be a natural number and suppose that G is a continuous function on $Y^d \times W$. Then, for any $z_0 \in Z$ and any $w_0 \in W$, there are neighbourhoods $\mathcal{U}$ and $\mathcal{W}$ of z_0 and w_0, respectively such that the function G is bounded on the set $p^{-1}(\mathcal{U})^d \times \mathcal{W}$.

Proof. Since p is a proper map, the fiber $F = p^{-1}(z_0)$ is compact. Since $F^d \times \{w_0\}$ is a compact subset of $Y^d \times W$, there exists an open set $\mathcal{A} \subset Y^d \times W$ which contains $F^d \times \{w_0\}$, and is such that G is bounded on $\mathcal{A}$. By the tube lemma from topology there exist an open set $\mathcal{V} \subset Y$ that contains F and a neighbourhood $\mathcal{W}$ of w_0 such that $\mathcal{V}^d \times \mathcal{W} \subset \mathcal{A}$. Since p is a proper map, there exists a neighbourhood $\mathcal{U}$ of z_0 such that $p^{-1}(\mathcal{U}) \subset \mathcal{V}$. The function G is bounded on $p^{-1}(\mathcal{U})^d \times \mathcal{W}$ because $p^{-1}(\mathcal{U})^d \times \mathcal{W} \subset \mathcal{A}$. □

Proof of Proposition 3.2. Let $(b_0, \bar{\xi}_0, \eta_0) \in \mathcal{D} \times \overline{V'} \times V'^m$ for $0 \le m \le r$. We will prove that if $N > 4r + 3$, then:

(i) For any $0 \le m \le r - 1$ and any sequence $\{b_n\}_{n=1}^{\infty} \subset V'_D$ such that $\lim_{n\to\infty} b_n = b_0$

$$\lim_{n\to\infty} \frac{d_b^m g_j^N(b_n, \bar{\xi}_0, \eta_0)}{\|b_n - b_0\|} = 0\,, \qquad j = 0, \ldots, d-1\,.$$

This shows that $\tilde{g}_j^N$ has a derivative of order $m+1$ at $(b_0, \bar{\xi}_0, \eta_0)$, and that this derivative vanishes at $(b_0, \bar{\xi}_0, \eta_0)$.

(ii) For any $0 \le m \le r$ and any sequence $\{(b_n, \bar{\xi}_n, \eta_n)\}_{n=1}^{\infty} \subset V'_D \times \overline{V'} \times V'^m$ such that $\lim_{n\to\infty}(b_n, \bar{\xi}_n, \eta_n) = (b_0, \bar{\xi}_0, \eta_0)$

$$\lim_{n\to\infty} d_b^m g_j^N(b_n, \bar{\xi}_n, \eta_n) = 0, \qquad j = 0, \ldots, d-1\,.$$

This shows that all derivatives $d_b^m g_j^N$, $m = 0, \ldots, r$, are continuous.

Let us prove (i). Let $H_{jm}^N \in C^{r-m}(X^d)$ be the function given by

$$H_{jm}^N(x_1, \ldots, x_d) = G_{jm}^N(x_1, \ldots, x_d, \bar{\xi}_0, \eta_0)$$

for $(x_1, \ldots, x_d) \in X^d$. By Lemma 3.9, there exists a neighbourhood $\mathcal{U}$ of b_0 such that H_{jm}^N is bounded on $p^{-1}(\mathcal{U})^d$. Since $\lim_{n\to\infty} b_n = b_0$, the sequence

$$\{G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \bar{\xi}_0, \eta_0)\}_{n=1}^{\infty}$$

is bounded. By Lemma 3.8,

$$\frac{d_b^m g_j^N(b_n, \overline{\xi}_0, \eta_0)}{\|b_n - b_0\|} =$$

$$= \frac{D(b_n)^{N-2m-1}\overline{D(b_n)}^{-2m-2}}{\|b_n - b_0\|} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_0, \eta_0) =$$

$$= \frac{D(b_n)^{N-4m-3}}{\|b_n - b_0\|} \times \left\{ \left[D(b_n) \,/\, \overline{D(b_n)} \right]^{2m+2} \cdot G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_0, \eta_0) \right\}.$$

We note that $N - 4m - 3 = N - 4r - 3 + 4(r - m) \geq 2$ because $N > 4r + 3$ and $0 \leq m \leq r - 1$. Hence

$$\lim_{n\to\infty} \frac{D(b_n)^{N-4m-3}}{\|b_n - b_0\|} = 0.$$

Since the sequence

$$\left\{ \left[D(b_n) \,/\, \overline{D(b_n)} \right]^{2m+2} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_0, \eta_0) \right\}_{n=1}^{\infty}$$

is bounded, we see that

$$\lim_{n\to\infty} \frac{d_b^m g_j^N(b_n, \overline{\xi}_0, \eta_0)}{\|b_n - b_0\|} = 0.$$

The proof of part (ii) is similar. By Lemma 3.9 there exist neighbourhoods $\mathcal{U}$ and $\mathcal{W}$ of b_0 and $(\overline{\xi}_0, \eta_0)$, respectively such that $G_{jm}^N \in C^{r-m}(X^d \times \overline{V'} \times V'^m)$ is bounded on the set $p^{-1}(\mathcal{U})^d \times \mathcal{W}$. Since $\lim_{n\to\infty} = (b_n, \overline{\xi}_n, \eta_n)(b_0, \overline{\xi}_0, \eta_0)$ the sequence

$$\{G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_n, \eta_n)\}_{n=1}^{\infty}$$

is bounded. By Lemma 3.8,

$$d_b^m g_j^N(b_n, \overline{\xi}_n, \eta_n) =$$

$$= D(b_n)^{N-2m-1}\overline{D(b_n)}^{-2m-2} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_n, \eta_n)$$

$$= D(b_n)^{N-4m-3} \left\{ \left[D(b_n)/\overline{D(b_n)} \right]^{2m+2} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_n, \eta_n) \right\}.$$

We note that $N - 4m - 3 \geq 1$ because we assume that $0 \leq m \leq r$ and $N > 4r + 3$. Since the sequence

$$\left\{ \left[D(b_n)/\overline{D(b_n)} \right]^{2m+2q} G_{jm}^N(r_1(b_n), \ldots, r_d(b_n), \overline{\xi}_n, \eta_n) \right\}_{n=1}^{\infty}$$

is bounded, we obtain $\lim_{n\to\infty} d_b^m g_j^N(b_n, \overline{\xi}_n, \eta_n) = 0$ which finishes the proof. □

Proposition 3.1 is derived from Proposition 3.2 as follows.

Proof of Proposition 3.1. For any $h \in V'^*$, $h \neq 0$, we set $D_h = D/h^{d(d-1)} \in C[\boldsymbol{P}(V')_h]$ and $g = f/h^n \in C^r_{0,1}(X_h)$. Let $g_j \in C^r_{0,1}(\boldsymbol{P}(V')_h \cap \boldsymbol{P}(V')_D)$, $j = 0, \ldots, d-1$, be such that

$$g|_{X_{D_h}} = \sum_{j=0}^{d-1} (z/h)^j \, (p|_{X_{D_h}})^* g_j \, .$$

It is easily seen that $\tilde{g}_j^N = h^{-n+j-N \deg D} \tilde{f}_j^N|_{\boldsymbol{P}(V')_h}$, $j = 0, \ldots, d-1$. According to Proposition 3.2, we have $\tilde{g}_j^N \in C^r_{0,1}(\boldsymbol{P}(V')_h)$ for $N > 4r + 3$ and $j = 0, \ldots, d-1$. Hence $\tilde{f}_j^N|_{\boldsymbol{P}(V')_h} \in C^r_{0,q}(\boldsymbol{P}(V')_h, \mathcal{O}_{\boldsymbol{P}(V')}(n - j + N \deg D))$ for $N > 4r + 3$ and $j = 0, \ldots, d-1$. Since the open sets $\{\boldsymbol{P}(V')_h : 0 \neq h \in V'^*\}$ cover $\boldsymbol{P}(V')$, Proposition 3.1 has been proved. □

Corollary 3.10. If (V', W, z) is an admissible triple for a submanifold X of finite codimension in $\boldsymbol{P}$, and $f \in C^\infty_{0,1}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, is a closed form, then there exists $u \in C^\infty(X_D, \mathcal{O}_X(n))$ such that:
(i) $\overline{\partial} u = f|X_D$;
(ii) $D^8 u = v|_{X_D}$ for some $v \in C^1(X, \mathcal{O}_X(n + 8 \deg D))$.

Proof. Let $f_j \in C^\infty_{0,1}(\boldsymbol{P}(V')_D, \mathcal{O}_{\boldsymbol{P}(V')}(n - j))$, $j = 0, \ldots, d-1$, be such that $f|_{X_D} = \sum_{j=0}^{d-1} (p|_{X_D})^* f_j \otimes z^j$. Then $\tilde{f}_j^8 \in C^1_{0,1}(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(n + 8 \deg D))$ for $j = 0, \ldots, d-1$, by Proposition 3.1, and $\tilde{f}_j^8$, $j = 0, \ldots, d-1$, is an exact form that is smooth on the open set $\boldsymbol{P}(V')_D$. By Proposition 2.2.1 (if $n + 8 \deg D < 0$) and Proposition 2.2.2 (if $n + 8 \deg D \geq 0$), there exist sections

$$u_j \in C^1(\boldsymbol{P}(V'), \mathcal{O}_{\boldsymbol{P}(V')}(n + 8 \deg D))$$

such that $\overline{\partial} u_j = \tilde{f}_j^8$ for $j = 0, \ldots, d-1$. Proposition 2.1 shows that all sections u_j, $j = 0, \ldots, d-1$, are smooth on $\boldsymbol{P}(V')_D$. Let

$$u = D^{-8} \sum_{j=0}^{d-1} (p|_{X_D})^* (u_j|_{\boldsymbol{P}(V')_D}) \otimes z^j \in C^\infty(X_D, \mathcal{O}_X(n))$$

and

$$v = \sum_{j=0}^{d-1} (p|_X)^* u_j \otimes z^j \in C^1(X, \mathcal{O}_X(n + 8 \deg D)) \, .$$

Then u and v satisfy (i) and (ii). □

Now we can prove the main result of this paper. In the proof we use the complex $C(X, \mathcal{O}_X(n))$ which was defined in [3, Section 5].

Theorem 3.11. Let V be a Banach space that admits smooth partitions of unity and $\boldsymbol{P} = \boldsymbol{P}(V)$. Then $H^{0,1}(X, \mathcal{O}_X(n)) = 0$, $n \in \mathbb{Z}$, for any complete intersection X in $\boldsymbol{P}$.

Proof. Let $f \in \mathcal{C}^\infty_{0,1}(X, \mathcal{O}_X(n))$, $n \in \mathbb{Z}$, be a given closed form. According to [3, Corollary 3.10], there is a collection $\{(V_i', W_i, z_i)\}_{i\in I}$ of admissible triples for X in $\boldsymbol{P}$ such that $\mathcal{U} = \{\boldsymbol{P}_{D_i}\}_{i\in I}$ is a covering of $\boldsymbol{P}$. By Corollary 3.10 in this paper, for any $i \in I$ there exist $u_i \in C^\infty(X_i, \mathcal{O}_X(n))$ and $v_i \in C^1(X, \mathcal{O}_X(n + 8d_i))$ such that $\overline{\partial} u_i = f|_{X_i}$ and $u_i = D_i^{-8}(v_i|_{X_i})$. For any $i, j \in I$, let

$$\varphi_{ij} = u_j|_{X_{ij}} - u_i|_{X_{ij}} \in \mathcal{C}^\infty(X_{ij}, \mathcal{O}_X(n)),$$
$$\tilde{\varphi}_{ij} = D_i^8 \tilde{\varphi}_j - D_j^8 \tilde{\varphi}_i \in \mathcal{C}^1(X, \mathcal{O}_X(n + 8d_i + 8d_j)).$$

Then $\overline{\partial}\varphi_{ij} = 0$ for any $i, j \in I$, which implies that $\varphi_{ij} \in H^0(X_{ij}, \mathcal{O}_X(n))$ for any $i, j \in I$. Furthermore, the global section $\tilde{\varphi}_{ij}$ is holomorphic on X_{ij} for any $i, j \in I$ because $(D_iD_j)^8\varphi_{ij} = \tilde{\varphi}_{ij}|_{X_{ij}}$. Since $\tilde{\varphi}_{ij}$ is continuous on X, the Riemann removable singularity theorem yields $\tilde{\varphi}_{ij} \in H^0(X, \mathcal{O}_X(n + 8d_i + 8d_j))$ for any $i, j \in I$. Therefore the cocycle $\varphi = \{\varphi_{ij}\}_{i,j\in I}$ belongs to the group $\mathcal{C}^1_8(X, \mathcal{O}_X(n))$ (see [3, Section 6]). Since φ is a closed cocycle and $H^1(\mathcal{C}(X, \mathcal{O}_X(n)) = 0$, $n \in \mathbb{Z}$, by [3, Theorem 6.5], there exists a collection of holomorphic sections

$$\psi = \{w_i \in H^0(X_i, \mathcal{O}_X(n))\}_{i\in I} \in \mathcal{C}^0(X, \mathcal{O}_X(n))$$

such that $\delta\psi = \varphi$. This means that $(u_i - w_i)_{X_i \cap X_j} = (u_j - w_j)_{X_i\cap X_j}$ for all $i, j \in I$. Let $u \in C^\infty(X, \mathcal{O}_X(n))$ be given by $u|_{X_i} = u_i - w_i$, $i \in I$. Since $(\overline{\partial} u)|_{X_i} = \overline{\partial}(u_i - w_i) = \overline{\partial} u_i = f|_{X_i}$ for any $i \in I$, we obtain $\overline{\partial} u = f$. □

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: bkotzev@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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LETTER TO THE EDITOR

ADDENDUM TO MY PAPER "SOME SHORT HISTORICAL NOTES ON DEVELOPMENT OF MATHEMATICAL LOGIC IN SOFIA"

DIMITER SKORDEV

The above-mentioned paper, published in *Ann. Univ. Sofia, Fac. Math. Inf.*, **96**, 2004, 11–21, presents a talk given at the Conference on Mathematical Logic and its Applications, Gyulechitsa, December 14-16, 2002. It was expected that the volume would contain some general information about the conference, and the information would mention, in particular, the fact in question. Therefore no indication to the fact was included in the paper itself, but, unfortunately, the expected general information did not appear in the volume.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: skordev@fmi.uni-sofia.bg
http://www.fmi.uni-sofia.bg/fmi/logic/skordev/

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For journal articles: Frisch, H. L. Statistics of random media. *Trans. Soc. Rheology*, **9**, 1965, 293–312.

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