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Aims and Scope. The *Annuaire* is the oldest Bulgarian journal, founded in 1904, devoted to pure and applied mathematics, mechanics and computer sciences. It is reviewed by *Zentralblatt für Mathematik*, *Mathematical Reviews* and the Russian *Referativnii Jurnal*. The *Annuaire* publishes significant and original research papers of authors both from Bulgaria and abroad in some selected areas that comply with the traditional scientific interests of the Faculty of Mathematics and Informatics at the "St. Kliment Ohridski" University of Sofia, i.e., algebra, geometry and topology, analysis, mathematical logic, theory of approximations, numerical methods, computer sciences, classical, fluid and solid mechanics, and their fundamental applications.

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UNIVERSAL LEVENSTEIN AUTOMATA FOR A GENERALIZATION OF THE LEVENSHTEIN DISTANCE

PETAR MITANKIN, STOYAN MIHOV, KLAUS U. SCHULZ

The need to efficiently find approximate matches for a given input string in a large background dictionary arises in many areas of computer science. In earlier work we introduced the concept of a universal Levenshtein automaton for a distance bound n. Given two arbitrary strings v and w, we may use a sequence of bitstrings $\chi(v, w)$ obtained from v and w in a trivial way as input for the automaton. The automaton is deterministic. The sequence $\chi(v, w)$ is accepted iff the Levenshtein distance between v and w does not exceed n. We showed how universal Levenshtein automata can be used to efficiently select approximate matches in large dictionaries. In this paper we consider variants of the Levenshtein distance were substitutions may be blocked for specific symbol pairs. The concept of an universal Levenshtein automaton is extended to cope with this larger class of similarity measures.

1. INTRODUCTION

The problem of how to find good correction candidates for a garbled input word is important for many fundamental applications, including spelling correction, speech recognition, OCR-recognition, error-tolerant querying of search engines for the world wide web and other kinds of information systems. Due to its relevance the problem has been considered by many authors (e.g. [2, 13, 21, 1, 18, 19, 7, 25, 4]).

If an electronic dictionary is available that covers the possible input words, a simple procedure may be used for detecting and correcting errors. Given an input word w, it is first checked if the word is in the dictionary. In the negative case, the words of the dictionary that are most similar to w are suggested as correction candidates. If necessary, appropriate statistical data can be used for refinement of ranking. Similarity between two words can be measured in several ways. Popular distance measures are the Levenshtein distance ([8, 23, 12, 22, 15, 11]) or n-gram distances ([1, 12, 20, 5, 6]).

In previous research we have shown that for each bound n there exists a finite state automaton - the so-called *universal Levenshtein automaton*, which represents - in some sense - the set of all couples of words $\langle w, v \rangle$ such that the Levenshtein distance between w and v is at most n: Given two arbitrary strings v and w, we may use a sequence of bitstrings $\chi(v, w)$ obtained from v and w in a trivial way as input for the automaton. The sequence $\chi(v, w)$ is accepted iff the Levenshtein distance between v and w does not exceed n. The fact that the automaton is deterministic and does not depend on the particular words but only on the bound n makes the universal Levenshtein automaton very suitable for practical applications. We can use this automaton to extract very efficiently all words from a dictionary that are sufficiently similar to a given input word.

In this paper we show how to compute universal Levenshtein automata for a generalization of the Levenshtein distance. The usual Levenshtein distance represents the minimal number of edit operations required to transform one of the two words into the other. Edit operations are *substitution* (replacement of one symbol of the word with another), *deletion* or *insertion* of a symbol. Here we restrict the set of possible substitutions, thus obtaining a more general and flexible notion of string distance. A set S is fixed that consists of couples of symbols. When we transform one of the two words into the other, i.e. when we calculate the distance, we allow to replace the symbol a with the symbol b only if $\langle a, b \rangle \in S$. When S contains all possible couples, we have the usual Levenshtein distance.

The research on this generalization is motivated by the fact that in many practical applications some of the symbol substitutions are not possible. For instance in a spell checker it would be relevant to restrict S to those couples of symbols whose corresponding keys are situated close to each other on the keyboard or which can have similar phonetic realizations.

The main result presented in this report is a construction of the finite automaton that represents in some sense the set of all couples of words $\langle w, v \rangle$ for which the generalized (via S) Levenshtein distance between w and v is at most n. This automaton has properties analogous to those of the universal Levenshtein automaton.

Ths paper is structured as follows. In Section 2 we start with formal preliminaries. In Section 3 we introduce a non-deterministic variant of the Levenshtein automaton for the generalized distance with restricted substitutions. Section 4 presents a determinization procedure. In Sections 5 and 6we define the corresponding universal automaton. In Section 7 we represent some properties of the universal automaton for the new distance measure. We also present some statistics on the universal automata for bounds $n \leq 5$. The role of each type of automaton will become clearer after reading the formal preliminaries.

2. PRELIMINARIES

Let Σ be finite alphabet and $S \subseteq \Sigma \times \Sigma$. We define d_L^S - function that generalizes the usual Levenshtein distance.

Definition 2.1. $d_L^S : \Sigma^* \times \Sigma^* \to N$
1) $w = \epsilon$ or $x = \epsilon$
$d_L^S(w,x) \stackrel{def}{=} max(|w|,|x|)$
2) $w \neq \epsilon$ and $x \neq \epsilon$
Let $w = w_1 w_2 ... w_p$ and $x = x_1 x_2 ... x_k$.

$$\begin{aligned} d_L^S(w,x) \stackrel{def}{=} \ min(\ & if(w_1 = x_1, d_L^S(w_2...w_p, x_2...x_k), \infty), \\ & 1 + d_L^S(w_2...w_p, x), \\ & 1 + d_L^S(w, x_2...x_k), \\ & if(\langle w_1, x_1 \rangle \in S, 1 + d_L^S(w_2...w_p, x_2...x_k), \infty) \ ) \end{aligned}$$

The following proposition shows that we may think of d_L^S in the terms of *substitution, deletion* and *insertion.*

Proposition 2.1. Let us consider that Σ and Σ' are equal but the symbols in Σ are black and the symbols in Σ' are red. In other words, let $\Sigma' = \{a' | a \in \Sigma\}$, r be a biection from Σ into Σ' and $\Sigma \cap \Sigma' = \phi$. For each black symbol $a \in \Sigma$ with a' we denote the corresponding red symbol $r(a)$. Let $v \in (\Sigma \cup \Sigma')^*$. We say that v *is transformed into* w *via deletion of a symbol* iff $v = v_1 v_2 ... v_t$ and $w = v_1 v_2 ... v_{i-1} v_{i+1} ... v_t$ for some i such that $1 \leq i \leq t$ and $v_i \in \Sigma$. We say that v *is transformed into* w *via insertion of a symbol* iff $v = v_1 v_2 ... v_t$ and $w = v_1 v_2 ... v_i b' v_{i+1} ... v_t$ for some i such that $0 \leq i \leq t$ and $b' \in \Sigma'$. We say that v *is transformed into* w *via substitution* iff $v = v_1 v_2 ... v_t$ and $w = v_1 v_2 ... v_{i-1} b' v_{i+1} ... v_t$ for some i such that $1 \leq i \leq t$, $v_i \in \Sigma$, $b' \in \Sigma'$ and $\langle v_i, b \rangle \in S$. If, $w, x \in \Sigma^*$, then $d_L^S(w,x)$ is the minimal natural number k for which there exists sequence of words $w_0, w_1, ..., w_k$ such that

1) $w_0 = w$,

2) if $0 \leq i \leq k-1$ then w_i is transformed into w_{i+1} via deletion of a symbol, insertion of a symbol or substitution,

3) if $w_k = a_1 a_2 ... a_t$ then $x = b_1 b_2 ... b_t$ where $b_i = \begin{cases} a_i & \text{if } a_i \in \Sigma \\ c & \text{if } a_i \in \Sigma' \text{ and } c' = a_i \end{cases}$

In fact this proposition shows that the order of applying the operations that transform w into v is not crucial. For eaxample, if $S = \phi$, then $d_L^S(abc, acd) = 2$. We could apply first the deletion and after it the insertion: $w_0 = abc$, $w_1 = ac$, $w_2 = acd'$. But we could also aplly first the insertion and after it the deletion: $w_0 = abc$, $w_1 = abcd'$, $w_2 = acd'$.

Is it true that d_L^S is a distance? It is true that $d_L^S(w,x) = 0 \Leftrightarrow w = x$ and the triangle inequality holds for d_L^S. But d_L^S is not always symmetric, i.e. d_L^S is not always distance. d_L^S is distance only when the relation S is symmetric.

How can we compute $d_L^S(w,x)$? Wagner and Fischer show how dynamic programming scheme can be used to compute the usual Levenshtein distance ([23]).The same tehnique can be used for $d_L^S(w,x)$: we find recursively the values M_{ij} of a $(|w|+1)\times(|x|+1)$ sized matrix M:

1) $M_{1j}=j-1$ for $1\le j\le |x|+1$ and $M_{i1}=i-1$ for $1\le i\le |w|+1$

2) Let us suppose that we have found M_{ij}, $M_{i+1,j}$ and $M_{i,j+1}$. Then

$$\begin{aligned} M_{i+1,j+1} = \ min(\quad & if(w_i=x_j, M_{ij}, \infty), \\ & 1+M_{i,j+1}, \\ & 1+M_{i+1,j}, \\ & if(\langle w_i,x_j\rangle \in S, 1+M_{ij}, \infty) \quad) \end{aligned}$$

After we have found the values of M, we have that $d_L^S(w,x)=M_{|w|+1,|x|+1}$.

Let $n\in N$. We use d_L^S to introduce a criterion for proximity of two words. We consider that the word x is proximate to the word w if $d_L^S(w,x)\le n$. With $L(w,n)$ we denote the set of all words that are proximate to the word w: $L(w,n) \stackrel{def}{=} \{x|d_L^S(w,x)\le n\}$. It turns out that for each word w and each n we can build finite automaton $A_n^D(w)$, such that its language $L(A_n^D(w))=L(w,n)$. We give a definition of $A_n^D(w)$ in section 3. The main result of this report is the so-called *universal automaton* $A_n^\forall$ - deterministic finite automaton that represents in some sense $L(w,n)$ for each word w. We call $A_n^\forall$ 'universal' because, in contrast to $A_n^D(w)$, $A_n^\forall$ does not depend neither on particular word w nor on the set S, but it depends only on n. We give a definition of $A_n^\forall$ in section 4.

How does $A_n^\forall$ represent $L(w,n)$ for each word w? Let w be a word and n be a natural number. For given word $x\in\Sigma^+$ we want to know whether $x\in L(w,n)$. We suppose that $|x|\le|w|+n$. (If $|x|>|w|+n$, then $x\notin L(w,n)$.) $\Sigma_n^\forall$ ($\Sigma_n^\forall$ is the alphabet of $A_n^\forall$) consists of couples of binary vectors, i.e. $\Sigma_n^\forall\subset\{0,1\}^*\times\{0,1\}^*$. By w and x we build in some way a word $\alpha=\alpha_1\alpha_2...\alpha_{|x|}$ whose symbols are couples of binary vectors, i.e. $\alpha_i\in\{0,1\}^*\times\{0,1\}^*$ for $1\le i\le|x|$. Then $\alpha\in L(A_n^\forall)\Leftrightarrow x\in L(w,n)$. We build α in the following way: $\alpha=\alpha_1\alpha_2...\alpha_{|x|}$ as $\alpha_i=\langle\beta_i,(\beta_s)_i\rangle$ for $1\le i\le|x|$ and

1) $\beta_i=\chi(x_i, w_{i-n}w_{i-n+1}...w_k)$, where
$k=min(|w|, i+n+1)$, $w_{-n+1}=w_{-n}=...=w_0=\$$ for $n>0$, $\$$ is such symbol, that $\$\notin\Sigma$,

and $\chi(c, a_1a_2...a_r)=b_1b_2...b_r$ as $b_j=\begin{cases}1, & \text{if } c=a_j\\ 0, & \text{if } c\ne a_j\end{cases}$

2) $(\beta_s)_i=\chi_s(x_i, w_{i-n+1}w_{i-n+1}...w_k)$, where
$k=min(|w|, i+n-1)$, $w_{-n+2}=w_{-n+1}=...=w_0=\$$ for $n>1$

and $\chi_s(c, a_1a_2...a_r)=b_1b_2...b_r$ as $b_j=\begin{cases}1, & \text{if } \langle a_j,c\rangle\in S\\ 0, & \text{if } \langle a_j,c\rangle\notin S\end{cases}$

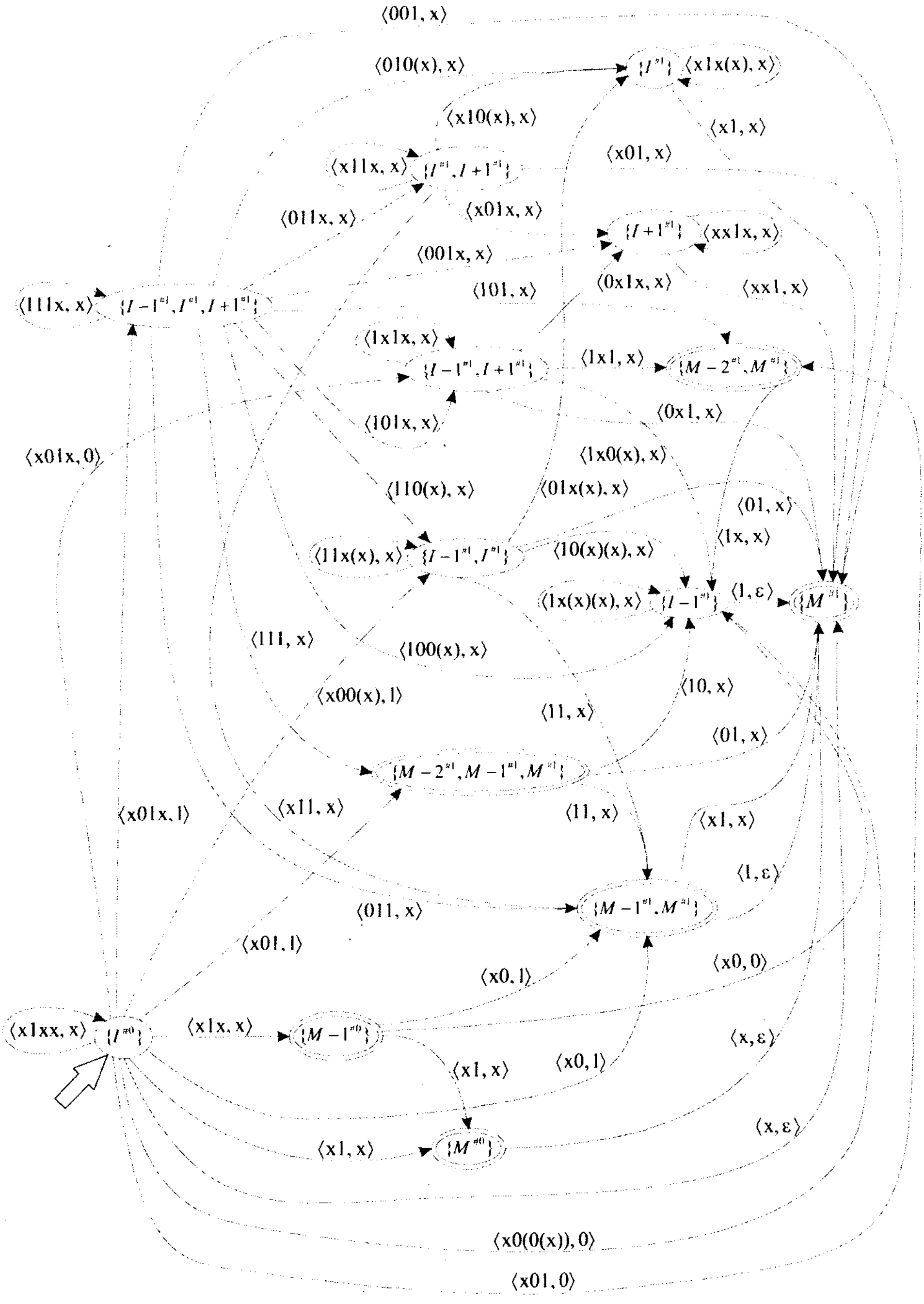

Fig. 1 $A_1^{\forall}$

Example. Let us consider that $\Sigma = \{a, b, c, ..., z\}$ and $S = \{\langle a, d\rangle, \langle d, a\rangle, \langle h, k\rangle, \langle h, n\rangle\}$. Let $w = hahd$ and $x = hand$. We want to know whether $x \in L(hahd, 1)$. We construct the word $\alpha = \alpha_1\alpha_2\alpha_3\alpha_4$:

$\alpha_1 = \langle\beta_1, (\beta_s)_1\rangle = \langle\chi(h, \$hah), \chi_s(h, h)\rangle = \langle 0101, 0\rangle$

$\alpha_2 = \langle\beta_2, (\beta_s)_2\rangle = \langle\chi(a, hahd), \chi_s(a, a)\rangle = \langle 0100, 0\rangle$

$\alpha_3 = \langle\beta_3, (\beta_s)_3\rangle = \langle\chi(n, ahd), \chi_s(n, h)\rangle = \langle 000, 1\rangle$

$\alpha_4 = \langle\beta_4, (\beta_s)_4\rangle = \langle\chi(d, hd), \chi_s(d, d)\rangle = \langle 01, 0\rangle$

The automaton $A_1^\forall$ is depicted on fig. 1. On fig. 1 the notation x means 0 or 1 and the bracketed expressions are optional. For instance from the state $\{I - 1^{\#1}, I^{\#1}, I + 1^{\#1}\}$ with $\langle 010(x), x\rangle$ we can reach the state $\{I^{\#1}\}$. This means that from $\{I - 1^{\#1}, I^{\#1}, I + 1^{\#1}\}$ we can reach $\{I^{\#1}\}$ with $\langle 010, 0\rangle$, $\langle 010, 1\rangle$, $\langle 0100, 0\rangle$, $\langle 0100, 1\rangle$, $\langle 0101, 0\rangle$ and $\langle 0101, 1\rangle$. So we start from the initial state $\{I^{\#0}\}$ and with the symbols $\langle 0101, 0\rangle$, $\langle 0100, 0\rangle$, $\langle 000, 1\rangle$ and $\langle 01, 0\rangle$ we visit the states $\{I^{\#0}\}$, $\{I^{\#0}\}$, $\{I - 1^{\#1}, I^{\#1}\}$ and $\{M^{\#1}\}$. $\{M^{\#1}\}$ is final state. Therefore $hand \in L(hahd, 1)$.

With the notions and notations introduced above, the structure of the paper may now be rephrased as follows. In Section we build nondeterministic finite automaton $A_n^{ND}(w)$, such that its language $L(A_n^{ND}(w))$ is $L(w, n)$. In Section we determinize in a specific way $A_n^{ND}(w)$. As a result we obtain the deterministic automaton $A_n^D(w)$. In Section we define the universal automaton $A_n^\forall$ and show the connection between $A_n^D(w)$ and $A_n^\forall$. In Section we represent some properties of $A_n^\forall$, that are based on our previous research. We also show some final results for $A_n^\forall$ when $n \leq 5$.

3. NONDETERMINISTIC FINITE AUTOMATON $A_N^{ND}(W)$

Definition 3.1. Let $w \in \Sigma^*$ and $n \in N$.

$A_n^{ND}(w) \stackrel{def}{=} \langle\Sigma, Q_n^{ND}, 0^{\#0}, \delta_n^{ND}, F_n^{ND}\rangle$

Let $|w| = p$. The set of states of $A_n^{ND}(w)$ is $Q_n^{ND} \stackrel{def}{=} \{i^{\#e} | 0 \leq i \leq p \,\&\, 0 \leq e \leq n\}$. (With $i^{\#e}$ we denote the couple $\langle i, e\rangle$.) The set of the final states is $F_n^{ND} \stackrel{def}{=} \{i^{\#e} | p - i \leq n - e\}$. The initial state is $0^{\#0}$. $\delta_n^{ND} \subseteq Q_n^{ND} \times (\Sigma \cup \{\epsilon\}) \times Q_n^{ND}$ is the transition relation. Let $c \in \Sigma \cup \{\epsilon\}$ and $q_1, q_2 \in Q_n^{ND}$. Then

$\langle q_1, c, q_2\rangle \in \delta_n^{ND} \stackrel{def}{\Leftrightarrow}$

$q_1 = i^{\#e} \,\&\, c = w_{i+1} \,\&\, q_2 = i + 1^{\#e}$ or

$q_1 = i^{\#e} \,\&\, c \in \Sigma \,\&\, q_2 = i^{\#e+1}$ or

$q_1 = i^{\#e} \,\&\, c = \epsilon \,\&\, q_2 = i + 1^{\#e+1}$ or

$q_1 = i^{\#e} \,\&\, \langle w_{i+1}, c\rangle \in S \,\&\, q_2 = i + 1^{\#e+1}$

The automaton $A_2^{ND}(w_1w_2w_3w_4w_5)$ is depicted on fig 2. where we use $S_{w_i}^\epsilon$ to denote $\{c | \langle w_i, c\rangle \in S\} \cup \{\epsilon\}$.

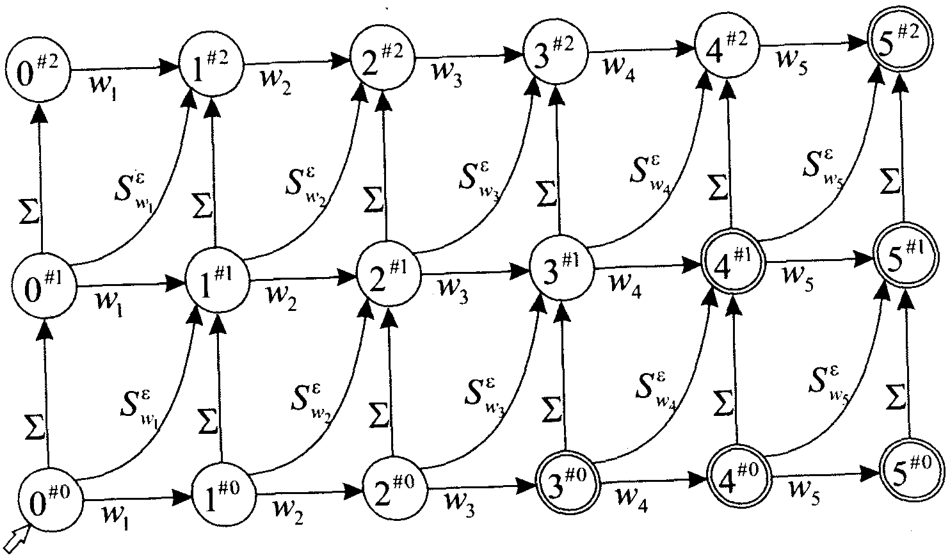

Fig. 2 $A_2^{ND}(w_1w_2w_3w_4w_5)$

If we think of d_L^S in the terms of the operations *substitution*, *deletion* and *insertion*, we can draw the following analogy between $A_n^{ND}(w)$ and d_L^S: each transition $\langle i^{\#e}, \epsilon, i+1^{\#e+1}\rangle$ in $A_n^{ND}(w)$ corresponds to deletion of the symbol w_{i+1}, each transition $\langle i^{\#e}, c, i+1^{\#e+1}\rangle$ for $c \in \Sigma$ corresponds to substitution - replacement of the symbol x_{i+1} with c, each transition $\langle i^{\#e}, c, i^{\#e+1}\rangle$ corresponds to insertion of the symbol c. The e in the state $i^{\#e}$ indicates the number of the operations that have been applied on the way from the initial state $0^{\#0}$ to $i^{\#e}$.

Proposition. $L(A_n^{ND}(w)) = L(w,n)$

Proof. With δ_n^{ND*} we denote the extended transition relation, that is defined by induction as usual:

1) $\langle \pi, \epsilon, \pi\rangle \in \delta_n^{ND*}$

2) $\langle \pi_1, v, \pi'\rangle \in \delta_n^{ND*}$ & $\langle \pi', a, \pi''\rangle \in \delta_n^{ND}$ & $\langle \pi'', \epsilon, \pi_2\rangle \in \delta_n^{ND*} \Rightarrow \langle \pi_1, va, \pi_2\rangle$ for $v \in \Sigma^*$ and $a \in \Sigma \cup \{\epsilon\}$

3) δ_n^{ND*} is the smallest w. r .t. $\subseteq$ relation for which the conditions 1) and 2) are true.

We check that $L(i^{\#e}) \stackrel{def}{=} \{v \in \Sigma^* | \exists \pi \in F_n^{ND} : \langle i^{\#e}, v, \pi\rangle \in \delta_n^{ND*}\} = \{v | d_L^S(w_{i+1}w_{i+2}...w_p, v) \leq n-e\} = L(w_{i+1}w_{i+2}...w_p, n-e)$, where $p = |w|$. When $i^{\#e} = 0^{\#0}$, we have $L(A_n^{ND}(w)) = L(0^{\#0}) = L(w,n)$.

Automata that are similar to $A_n^{ND}(w)$ are used for approximate search of a word w in a text T ([24, 3]). If we add a Σ loop to the initial state $0^{\#0}$, the language of the automaton will be $\Sigma^*.L(w,n)$ and we could use the automaton to traverse the text T.

4. DETERMINISTIC FINITE AUTOMATON $A_N^D(W)$

In this section we determinize the automaton $A_n^{ND}(w)$ in a specific way. In result we get the deterministic automaton $A_n^D(w)$. In the standard subset construction for determinization each state of the deterministic automaton is subset of Q_n^{ND}. We also define each state of $A_n^D(w)$ as a subset of Q_n^{ND}. The difference is that we use the so-called relation of *subsumption* $<_s \subseteq Q_n^{ND} \times Q_n^{ND}$. $<_s$ is defined in such a way that $\pi_1 <_s \pi_2 \Rightarrow L(\pi_1) \supseteq L(\pi_2)$. This allows each state Q of $A_n^D(w)$ to be built such that $\forall \pi', \pi'' \in Q : \pi' \not<_s \pi''$.

The construction that we represent here is analogous to one presented for the usual Levenshtein distance in [17].The main differences are the additional characteristic vector χ_s that depends on S and the additional relevant subword.

4.1. THE RELATION OF SUBSUMPTION $<_S$

Definition 4.1. $Q_n \stackrel{def}{=} \{i^{\#e} | i \in Z \,\&\, 0 \leq e \leq n\}$

Definition 4.2. $<'_s \subseteq Q_n \times Q_n$
$i^{\#e} <'_s j^{\#f} \stackrel{def}{\Leftrightarrow} j^{\#f} \in \{i-1^{\#e+1}, i^{\#e+1}, i+1^{\#e+1}\}$

Proposition 4.1. *Let* $i^{\#e}, j^{\#f} \in Q_n^{ND}$. *Then* $i^{\#e} <'_s j^{\#f} \Rightarrow L(i^{\#e}) \supseteq L(j^{\#f})$.

Definition 4.3. $<_s \subseteq Q_n \times Q_n$
$<_s$ is the transitive closure of $<'_s$.

Corollary 4.1. *Let* $i^{\#e}, j^{\#f} \in Q_n^{ND}$. *Then* $i^{\#e} <_s j^{\#f} \Rightarrow L(i^{\#e}) \supseteq L(j^{\#f})$.

The next proposition gives a direct way to compute whether $i^{\#e} <_s j^{\#f}$.

Proposition 4.2. $i^{\#e} <_s j^{\#f} \Leftrightarrow |j-i| \leq f-e \,\&\, f > e$

The set $\{\pi \in Q_n^{ND} | 3^{\#0} \leq_s \pi\}$ for $n = 2$ and $|w| = 5$ is depicted with bold circles on fig. 3.

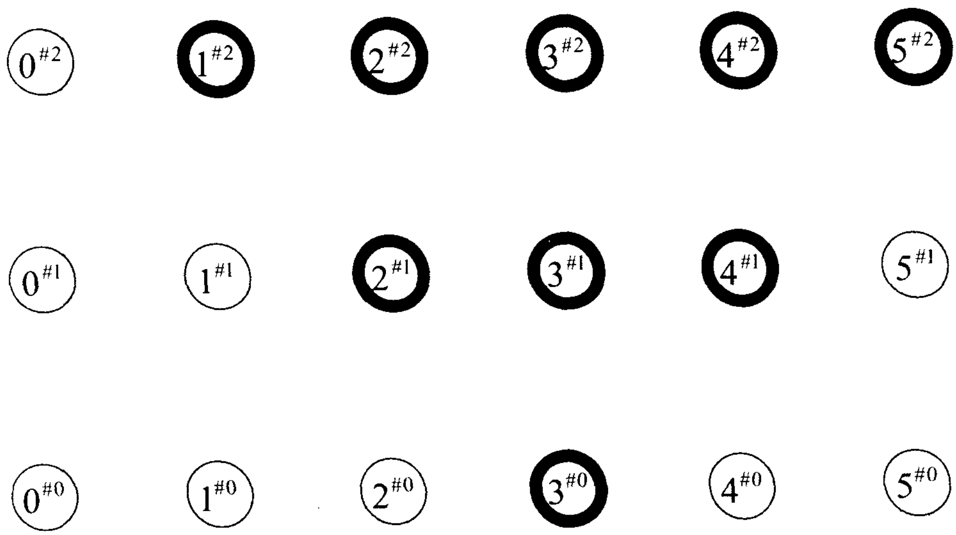

Fig. 3 $\{\pi \in Q_n^{ND} | 3^{\#0} \leq_s \pi\}$

4.2. CHARACTERISTIC VECTORS. RELEVANT SUBWORDS

Let us consider that $A_n^{ND}(w)$ is fixed. Let $\pi \in Q_n^{ND}$ and $a \in \Sigma$.

Definition 4.4. $R(\pi, a) \stackrel{def}{=} \{\pi' \in Q_n^{ND} | \langle \pi, a, \pi' \rangle \in \delta_n^{ND*}\}$
We call $R(\pi, a)$ *the set of all states reachable from* π *with* a.

To determinize $A_n^{ND}(w)$ we have to know $R(\pi, a)$ for each $\pi \in Q_n^{ND}$ and for each $a \in \Sigma$. We introduce $w_{[\pi]}$ and $w^s_{[\pi]}$ - subwords of w. We call $w_{[\pi]}$ and $w^s_{[\pi]}$ resp. *relevant to* π *subword of* w and *s-relevant to* π *subword of* w. For each symbol $a \in \Sigma$ and each word $a_1 a_2 ... a_k \in \Sigma^*$ we introduce also the binary vectors $\chi(a, a_1 a_2 ... a_k)$ and $\chi_s(a, a_1 a_2 ... a_k)$. We call them resp. *characteristic vector* and *s-characteristic vector of* a *w. r. t.* $a_1 a_2 ... a_k$. We define the relevant subwords and the characteristic vectors in such a way that if we know $\chi(a, w_{[\pi]})$ and $\chi_s(a, w_{[\pi]})$ then we know $R(\pi, a)$. In $A_n^{ND}(w)$ there are four types of transitions $\langle \tilde{q}_1, c, q_2 \rangle$:
1) $q_1 = i^{\#e}$, $c = w_{i+1}$ and $q_2 = i + 1^{\#e}$
2) $q_1 = i^{\#e}$, $c \in \Sigma$ and $q_2 = i^{\#e+1}$
3) $q_1 = i^{\#e}$, $c = \epsilon$ and $q_2 = i + 1^{\#e+1}$
4) $q_1 = i^{\#e}$, $\langle w_{i+1}, c \rangle \in S$ and $q_2 = i + 1^{\#e+1}$
To know all states $\pi' \in R(\pi, a)$ means to know all sequences
(*) $\langle \pi, \epsilon, \pi_1 \rangle, \langle \pi_1, \epsilon, \pi_2 \rangle, ... \langle \pi_{r-1}, \epsilon, \pi_r \rangle, \langle \pi_r, a, \pi'_1 \rangle, \langle \pi'_1, \epsilon, \pi'_2 \rangle, ... \langle \pi'_h, \epsilon, \pi' \rangle$.

Of course, if we know π then we know all sequences (*), for which the transition $\langle \pi_r, a, \pi'_1 \rangle$ has the type 2). So we define the relevant subwords and the characteristic vectors in such a way that if we know $\chi(\pi, a)$, then we know each sequence (*) for which the transition $\langle \pi_r, a, \pi'_1 \rangle$ has the type 1) or 4):

Definition 4.5. $\chi : \Sigma \times \Sigma^* \to \{0,1\}^*$

$$\chi(a, a_1 a_2 ... a_k) = b_1 b_2 ... b_k \text{ where } b_i = \begin{cases} 1 & \text{if } a = a_i \\ 0 & \text{if } a \neq a_i \end{cases}$$

Definition 4.6. $\chi_s : \Sigma \times \Sigma^* \to \{0,1\}^*$

$$\chi_s(a, a_1 a_2 ... a_k) = b_1 b_2 ... b_k \text{ where } b_i = \begin{cases} 1 & \text{if } \langle a_i, a \rangle \in S \\ 0 & \text{if } \langle a_i, a \rangle \notin S \end{cases}$$

Definition 4.7. $w_{[\,]} : Q_n^{ND} \to \Sigma^*$

$$w_{[i\#e]} \stackrel{def}{=} w_{i+1} w_{i+2} ... w_{i+k} \text{ where } k = min(n - e + 1, |w| - i).$$

Definition 4.8. $w^s_{[\,]} : Q_n^{ND} \to \Sigma^*$

$$w^s_{[i\#e]} \stackrel{def}{=} w_{i+1} w_{i+2} ... w_{i+k} \text{ where } k = min(n - e, |w| - i).$$

4.3. δ_E^D - THE FUNCTION OF THE ELEMENTARY TRANSITIONS

If we apply the standard subset construction for determinization of $A_n^{ND}(w)$ and A is some state received during the determinization, then $B = \bigcup_{\pi \in A} R(\pi, c)$ will be also state of the deterministic automaton for $c \in \Sigma$. But if $\pi_1, \pi_2 \in B$ and $\pi_1 <_s \pi_2$, we can continue the determinization with $B' = B \backslash \{\pi_2\}$ instead with B, because $\bigcup_{\pi \in B} L(\pi) = \bigcup_{\pi \in B'} L(\pi)$. So we can remove from B each π for which we can find $\pi' \in B$ such that $\pi' <_s \pi$. This means that we can remove from B all states that are not minimal w.r.t. $<_s$. We denote with $\bigsqcup B$ the set of all states that are minimal in B w.r.t. $<_s$. We use also $A \sqcup B$ to denote $\bigsqcup(A \cup B)$ and $\bigsqcup_{\pi \in A} f(\pi)$ to denote $\bigsqcup(\bigcup_{\pi \in A} f(\pi))$.

Definition 4.9. Let $A \subseteq Q_n^{ND}$

$$\bigsqcup A \stackrel{def}{=} \{\pi \in A | \neg \exists \pi' \in A (\pi' <_s \pi)\}$$

Proposition 4.3. Let $A \subseteq Q_n^{ND}$. Then

$$\bigcup_{q \in A} L(q) = \bigcup_{q \in \bigsqcup A} L(q)$$

We define function $\delta_e^D : Q_n^{ND} \times \Sigma \to P(Q_n^{ND})$, such that $\delta_e^D(\pi, a) = \bigsqcup R(\pi, a)$. The function δ_e^D is called *function of the elementary transitions.*

Definition 4.10. $\delta_e^D : Q_n \times \{0,1\}^* \times \{0,1\}^* \to P(Q_n)$

$\delta_c^D(i^{\#e}, \beta, \beta_s) \stackrel{def}{=} A \sqcup A_S$, where

$$A = \begin{cases} \{i + 1^{\#e}\} & \text{if } 1 < \beta \\ \{i^{\#e+1}\} & \text{if } e < n \,\&\, \beta = 0^{k_1} \text{ for some } k_1 \in N \\ \{i^{\#e+1}, i + k_1^{\#e+k_1-1}\}, & \text{if } k_1 \geq 2 \,\&\, 0^{k_1-1}1 < \beta \,\&\, e + k_1 - 1 \leq n \\ \phi & \text{otherwise} \end{cases}$$

$$A_S = \begin{cases} \{i + k_2^{\#e+k_2}\} & \text{if } k_2 > 0 \,\&\, 0^{k_2-1}1 < \beta_s \,\&\, e + k_2 \leq n \\ \phi & \text{ohterwise} \end{cases}$$

Definition 4.11. $\delta_e^D : Q_n^{ND} \times \Sigma \to P(Q_n^{ND})$

$\delta_e^D(\pi, a) \stackrel{def}{=} \delta_c^D(\pi, \chi(a, w_{[\pi]}), \chi_s(a, w_{[\pi]}^s))$

Proposition 4.4. $\bigsqcup R(\pi, a) = \delta_e^D(\pi, a)$

4.4. DETERMINISTIC FINITE AUTOMATON $A_N^D(W)$

Definition 4.12. $A_n^D(w) \stackrel{def}{=} \langle \Sigma, Q_n^D, \{0^{\#0}\}, \delta_n^D, F_n^D \rangle$

The set of states of $A_n^D(w)$ is $Q_n^D \stackrel{def}{=} \{A | A \subseteq Q_n^{ND} \,\&\, \forall \pi_1, \pi_2 \in A(\pi_1 \not<_s \pi_2) \,\&\, \exists i \in [0, |w|] \forall \pi \in A(i^{\#0} \leq_s \pi)\} \backslash \{\phi\}$.

The set of final states is $F_n^D \stackrel{def}{=} \{A \in Q_n^D | A \cap F_n^{ND} \neq \phi\}$.

δ_n^D is partial transition function:

$\delta_n^D : Q_n^D \times \Sigma \to Q_n^D$

1) $\bigcup_{\pi \in A} \delta_e^D(\pi, c) = \phi$

In this case $\delta_n^D(A, a)$ is not defined.

2) $\bigcup_{\pi \in A} \delta_e^D(\pi, a) \neq \phi$

$\delta_n^D(A, c) \stackrel{def}{=} \bigsqcup_{\pi \in A} \delta_e^D(\pi, c)$

The following two propositions give us the correctness of the definition of δ_n^D.

Proposition 4.4. Let $i < |w|$. Then $\forall \pi \in \delta_e^D(i^{\#e}, a) : i + 1^{\#e} \leq \pi$.

Proposition 4.5. $\forall \pi \in \delta_e^D(|w|^{\#e}, a) : |w|^{\#e} \leq \pi$.

From the propositions in 3.3 it follows that $L(A_n^D(w)) = L(A_n^{ND}(w)) = L(w, n)$. The automaton $A_1^D(hahd)$ is depicted on fig. 4. The set S is the one defined in the example in section 1.

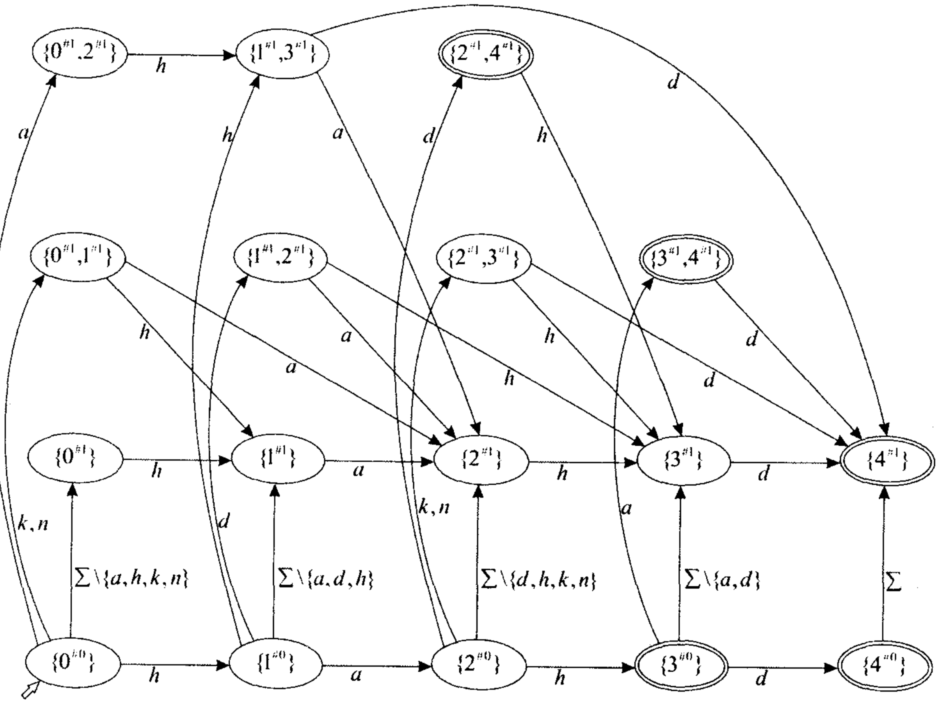

Fig. 4 $A_1^D(hahd)$

5. UNIVERSAL AUTOMATON $A_N^\forall$

We define the universal automaton $A_n^\forall$ in such way that for each execution of $A_n^\forall$ on some word α we can evaluate the execution of $A_n^D(w)$ on x, where w and x are those words from Σ^*, for which we have built the word α in the way described in section 1. The states of $A_n^\forall$ are sets, whose elements have the type $I + a^{\#b}$ or $M + a^{\#b}$ (fig. 1). I and M are parameters. When we replace these parameters with appropriate numbers, the states of $A_n^\forall$ are transformed into the states of $A_n^D(w)$.

Let $q_0^D = \{0^{\#0}\}$, q_1^D,..., q_f^D be those states of $A_n^D(w)$, that we visit with the word $x \in \Sigma^+$. $0 \le f \le |x|$. In some cases we may have $f < |x|$ because δ_n^D (the transition function of $A_n^D(w)$) may not be defined. Let $|x| \le |w| + n$. Let $\alpha = \alpha_1\alpha_2...\alpha_{|x|}$ be built from w and x in the way defined in section 1. Let also $q_0^\forall = \{I^{\#0}\}$, $q_1^\forall$,..., $q_g^\forall$ be those states of $A_n^\forall$, that we visit with the word α. Then:

1) $g = f$,

2) for $0 < i \le f$ it is true that $q_i^\forall$ is final state iff q_i^D is final state,

3) for $0 \le i \le f$ it is true that $q_i^\forall$ is transformed into q_i^D by replacement of each I in $q_i^\forall$ with i and each M with $|w|$.

Example. Let us look the execution of $A_n^D(w)$ on the word $x = hand$ for the example in section 1. The automaton $A_1^D(w)$ is depicted on fig. 4. So $q_0^D = \{0^{\#0}\}$, $q_1^D = \{1^{\#0}\}$, $q_2^D = \{2^{\#0}\}$, $q_3^D = \{2^{\#1}, 3^{\#1}\}$ and $q_4^D = \{4^{\#1}\}$. In section 1 we saw that $q_0^\forall = \{I^{\#0}\}$, $q_1^\forall = \{I^{\#0}\}$, $q_2^\forall = \{I^{\#0}\}$, $q_3^\forall = \{I - 1^{\#1}, I^{\#1}\}$ and $q_4^\forall = \{M^{\#1}\}$. If we replace in $q_i^\forall$ I with i and M with $|w|$ we get q_i^D.

To the end of section 4 we present the formal definition of the universal automaton $A_n^\forall \overset{def}{=} \langle \Sigma_n^\forall, Q_n^\forall, \{I^{\#0}\}, F_n^\forall, \delta_n^\forall \rangle$.

5.1. $\Sigma_N^\forall$, $Q_N^\forall$ AND $F_N^\forall$

Definition 5.1. $\Sigma_n^\forall \subset \{0,1\}^* \times \{0,1\}^*$
$\Sigma_n^\forall \overset{def}{=} \{\langle \beta, \beta_s \rangle | \beta, \beta_s \in \{0,1\}^* \,\&\, 1 \le |\beta| \le 2n+2 \,\&\, 0 \le |\beta_s| \le 2n-1\}$

Definition 5.2. $<_s \subseteq Q_n^I \times Q_n^I$
$Q_n^I \overset{def}{=} \{I + i^{\#e} | i \in Z \,\&\, 0 \le e \le n\}$
$I + i^{\#e} <_s I + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <_s j^{\#f}$

Definition 5.3. $<_s \subseteq Q_n^M \times Q_n^M$
$Q_n^M \overset{def}{=} \{M + i^{\#e} | i \in Z \,\&\, 0 \le e \le n\}$
$M + i^{\#e} <_s M + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <_s j^{\#f}$

Definition 5.4. $I_s \subset Q_n^I, M_s \subset Q_n^M$
$I_s \overset{def}{=} \{I + i^{\#e} \mid |i| \le e \,\&\, e \le n\}$
$M_s \overset{def}{=} \{M + i^{\#e} \mid e \ge -i - n \,\&\, i \le 0 \,\&\, 0 \le e \le n\}$

Definition 5.5. $I_{states} \subset P(I_s), M_{states} \subset P(M_s)$
$I_{states} \overset{def}{=} \{A | A \subseteq I_s \,\&\, \forall q_1, q_2 \in A(q_1 \not<_s q_2)\} \backslash \{\phi\}$
$M_{states} \overset{def}{=} \{A | A \subseteq M_s \,\&\, \forall q_1, q_2 \in A(q_1 \not<_s q_2) \,\&$
$\exists j \exists f (M + j^{\#f} \in A \,\&\, M + j^{\#f} \le_s M^{\#n}) \,\&\, \exists i \in [-n, 0] \forall q \in A(M + i^{\#0} \le_s q)\}$
$Q_n^\forall \overset{def}{=} I_{states} \cup M_{states}$
$F_n^\forall \overset{def}{=} M_{states}$

6. $\delta_N^\forall$ - THE TRANSITION FUNCTION

6.1. $\delta_E^\forall$ - THE FUNCTION OF THE ELEMENTARY TRANSITIONS

Definition 6.1. $r_n : (I_s \cup M_s) \times \{0,1\}^* \to \{0,1\}^*$

1) $r_n(I + i^{\#e}, x_1x_2...x_k) \stackrel{def}{=} \begin{cases} x_{n+i+1}x_{n+i+2}...x_{n+i+h} & \text{if } h \geq 0 \\ \text{not defined} & \text{if } h < 0 \end{cases}$

where $h = min(n - e + 1, k - n - i)$.

2) $r_n(M+i^{\#e}, x_1x_2...x_k) \stackrel{def}{=} \begin{cases} x_{k+i+1}x_{k+i+2}...x_{k+i+h} & \text{if } h \geq 0 \,\&\, k+i+1 > 0 \\ \text{not defined} & \text{otherwise} \end{cases}$

where $h = min(n - e + 1, -i)$.

Definition 6.2. $r_n^s : (I_s \cup M_s) \times \{0,1\}^* \to \{0,1\}^*$

1) $r_n^s(I + i^{\#e}, x_1x_2...x_k) \stackrel{def}{=} \begin{cases} x_{n+i}x_{n+i+1}...x_{n+i+h-1} & \text{if } h \geq 0 \,\&\, n+i > 0 \\ \epsilon & \text{otherwise} \end{cases}$

where $h = min(n - e, k - n - i + 1)$.

2) $r_n^s(M+i^{\#e}, x_1x_2...x_k) \stackrel{def}{=} \begin{cases} x_{k+i+1}x_{k+i+2}...x_{k+i+h} & \text{if } h \geq 0 \,\&\, k+i+1 > 0 \\ \epsilon & \text{otherwise} \end{cases}$

where $h = min(n - e, -i)$.

Definition 6.3. $I : P(Q_n) \to P(Q_n^I)$
$I(A) \stackrel{def}{=} \{I + i - 1^{\#e} | i^{\#e} \in A\}$

Definition 6.4. $M : P(Q_n) \to P(Q_n^M)$
$M(A) \stackrel{def}{=} \{M + i^{\#e} | i^{\#e} \in A\}$

Definition 6.5. $\delta_e^\forall : (I_s \cup M_s) \times \Sigma_n^\forall \to P(I_s) \cup P(M_s)$
Let $A \in I_s \cup M_s$ and $\langle \beta, \beta_s \rangle \in \Sigma_n^\forall$.
1) $r_n(A, \beta)$ is not defined
In this case $\delta_e^\forall(A, \langle \beta, \beta_s \rangle)$ is not defined.
2) $r_n(A, \beta)$ is defined
2.1) $A \in I_s$
Let $q = i^{\#e}$ where i and e are such that $A = I + i^{\#e}$.
$\delta_e^\forall(A, \langle \beta, \beta_s \rangle) \stackrel{def}{=} I(\delta_e^D(q, r_n(A, \beta), r_n^s(A, \beta_s)))$
2.2) $A \in M_s$
Let $q = i^{\#e}$ where i and e are such that $A = M + i^{\#e}$.
$\delta_e^\forall(A, \langle \beta, \beta_s \rangle) \stackrel{def}{=} M(\delta_e^D(q, r_n(A, \beta), r_n^s(A, \beta_s)))$

Definition 6.6. $rm : I_{states} \cup M_{states} \to I_s \cup M_s$
Let $A \in I_{states} \cup M_{states}$.
Let $t = \begin{cases} \mu z[\exists e \exists i(z = e - i \,\&\, I + i^{\#e} \in A & \text{if } A \in I_{states} \\ \mu z[\exists e \exists i(z = e - i \,\&\, M + i^{\#e} \in A & \text{if } A \in M_{states} \end{cases}$.
With $\mu z[X]$ we denote the least z such that the X is true.
$rm(A) \stackrel{def}{=} \begin{cases} I + i^{\#e} & \text{if } I + i^{\#e} \in A \,\&\, e - i = t \\ M + i^{\#e} & \text{if } M + i^{\#e} \in A \,\&\, e - i = t \end{cases}$

$rm(A)$ is called *right most element of A*.

Definition 6.7. $\nabla_a : I_{states} \cup M_{states} \to P(N)$
1) $A = \{I^{\#0}\}$
$\nabla_a(A) \stackrel{def}{=} \{k | n \leq k \leq 2n + 2\}$
2) $A \in I_{states} \,\&\, A \neq \{I^{\#0}\}$
Let $rm(A) = I + i^{\#e}$.
$\nabla_a(A) \stackrel{def}{=} \{k | 2n + i - e + 1 \leq k \leq 2n + 2\}$
3) $A \in M_{states}$
$\nabla_a(A) \stackrel{def}{=} \{k \in N | \forall q \in A(if(k < n, M^{\#n-k}, M + n - k^{\#0}) \leq_s q)\} \backslash \{0\}$

Definition 6.8. $l_n : N \times N \to \{true, false\}$
$l_n(k_1, k_2) = true \stackrel{def}{\Leftrightarrow} (k_1 = 2n + 2 \,\&\, k_2 = 2n - 1)$ or $(k_1 = 2n + 1 \,\&\, k_2 = 2n - 1)$ or $(1 \leq k_1 \leq 2n \,\&\, k_2 = k_1 - 1)$

6.3. SOME OTHER FUNCTIONS AND $\delta_N^{\forall}$

Definition 6.9. $f_n : (I_s \cup M_s) \times N \to \{true, false\}$
1) $f_n(I + i^{\#e}, k) \stackrel{def}{=} \begin{cases} true & \text{if } k \leq 2n + 1 \,\&\, e \leq i + 2n + 1 - k \\ false & \text{otherwise} \end{cases}$
2) $f_n(M + i^{\#e}, k) \stackrel{def}{=} \begin{cases} true & \text{if } e > i + n \\ false & \text{otherwise} \end{cases}$

Definition 6.10. $m_n : (Q_n^I \cup Q_n^M) \times N \to Q_n^I \cup Q_n^M$
$m_n(A, k) \stackrel{def}{=} \begin{cases} M + i + n + 1 - k^{\#e} & \text{if } A = I + i^{\#e} \\ I + i - n - 1 + k^{\#e} & \text{if } A = M + i^{\#e} \end{cases}$
$m_n : (P(Q_n^I) \cup P(Q_n^M)) \times N \to P(Q_n^I) \cup P(Q_n^M)$
$m_n(A, k) \stackrel{def}{=} \{m_n(a, k) | a \in A\}$

Definition 6.11. $\bigsqcup : P(P(I_s)) \cup P(P(M_s)) \to P(I_s) \cup P(M_s)$
$\bigsqcup A \stackrel{def}{=} \{q \in \cup A | \neg \exists q' \in \cup A : q' <_s q\}$

Definition 6.12. $\delta_n^{\forall} : Q_n^{\forall} \times \Sigma_n^{\forall} \to Q_n^{\forall}$

Let $A \in Q_n^\forall$ and $\langle \beta, \beta_s \rangle \in \Sigma_n^\forall$.

1) $|\beta| \notin \nabla_a(A)$ or $l_n(|\beta|, |\beta_s|) = false$

In this case $\delta_n^\forall(A, \langle \beta, \beta_{subs} \rangle)$ is not defined.

2) $|\beta| \in \nabla_a(A)$ & $l_n(|\beta|, |\beta_s|) = true$

2.1) $\bigcup_{q \in A} \delta_e^\forall(q, \langle \beta, \beta_{subs} \rangle) = \phi$

In this case $\delta_n^\forall(A, \langle \beta, \beta_{subs} \rangle)$ is not defined.

2.2) $\bigcup_{q \in A} \delta_e^\forall(q, \langle \beta, \beta_{subs} \rangle) \neq \phi$

Let $\Delta = \bigsqcup_{q \in A} \delta_e^\forall(q, \langle \beta, \beta_{subs} \rangle)$.

$$\delta_n^\forall(A, \langle \beta, \beta_{subs} \rangle) \stackrel{def}{=} \begin{cases} \Delta & \text{if } f_n(rm(\Delta), |\beta|) = false \\ m_n(\Delta, |\beta|) & \text{if } f_n(rm(\Delta), |\beta|) = true \end{cases}$$

7. SOME PROPERTIES OF $A_N^\forall$

When $S = \Sigma \times \Sigma$ d_L^S is the usual Levenshtein distance that we denote with d_L. In [27] and [26] we have shown that for d_L one can build universal automaton which here we denote with $A_n^u = \langle \Sigma_n^u, Q_n^u, \{I^{\#0}\}, \delta_n^u, F_n^u \rangle$. In this section we show the connection between $A_n^\forall$ and A_n^u and some corollaries.

Σ_n^u is the set of the first projections of the elements of $\Sigma_n^\forall$, i.e. $\Sigma_n^u = \{\beta \in \{0,1\}^* | 1 \leq |\beta| \leq 2n+2\}$. To define A_n^u we use the sets I_s and M_s, the relation $<_s$ and the sets I_{states} and M_{states} defined in section 4. So each state in Q_n^u, just like each state in $Q_n^\forall$, is a subset of I_s or subset of M_s. In [26] we have shown the following:

1) for A_n^u it is true that if $q \in I_{states} \cup M_{states}$, then q is useful in the sense that q is reachable from the initial state $\{I^{\#0}\}$ and some final state is reachable from q,

2) A_n^u is minimal.

Proposition 7.1 (for the connection between A_n^u and $A_n^\forall$). *Let $q \in Q_n^u$ and $\beta = \Sigma_n^u$. Then $\exists! \beta_s \in \{1\}^* : \delta_n^u(q, \beta) = \delta_n^\forall(q, \langle \beta, \beta_s \rangle)$ (either both the left expression and the right expression are not defined or both the left expression and the right expression are defined and equal).*

Remark. That β_s, for which $\delta_n^u(q, \beta) = \delta_n^\forall(q, \langle \beta, \beta_s \rangle)$, is $\beta_s = 1^{k_2}$ where k_2 is such that $l_n(|\beta|, k_2) = true$.

It follows from the proposition for the connection that 1) and 2) hold also for $A_n^\forall$. $Q_n^u = Q_n^\forall$. In [26] we have presented rough upper limitation for $|Q_n^u|$:

$|I_{states}| = O(2^{4n - \log_2 \sqrt{2n+1}})$

$|M_{states}| = O(n 2^{4n - \log_2 \sqrt{2n+1}})$

In the table below we show some final results for $A_n^\forall$ when $n \leq 5$. The value of the column 'transitions' is $|\{\langle q_1, b, q_2 \rangle | \langle q_1, b, q_2 \rangle \in \delta_n^\forall\}|$.

Table 1.

n	$\lvert I_{states} \rvert$	$\lvert M_{states} \rvert$	transitions
1	8	6	320
2	50	40	39552
3	322	280	4480416
4	2187	2025	504895904
5	15510	15026	58028259232

8. CONCLUSION

Besides the operations insertion, deletion and substitution in many applications there is a need for adding other operations. For example for spell checker it would be relevant to add *transposition* (swap two adjacent symbols) - mistake that occurs very frequently while typing text on keyboard. To correct text recognized by an OCR program it would be useful to add *merge* (merge of two adjacent symbols into one) and *split* (split one symbol into two others). In [17] and [26] we have shown that in the case of adding transposition as well as in the case of adding merge and split we can build deterministic automaton and universal automaton such that the universal one simulates the deterministic one. The technique developed in this research can be successfully applied if we restrict the allowed operations in these cases. For instance restricting the allowed substitutions, the allowed merges and the allowed splits results in universal automaton whose alphabet consists of fourtuples of binary vectors: besides χ and χ_s we add two other characteristic vectors that depend on the allowed merges and the allowed splits.

Here comes the problem for characterization of all functions $d : \Sigma^* \times \Sigma^* \to N$ for which universal automaton can be built. Our future research will be devoted to this problem.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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MONADIC SECOND-ORDER LOGIC ON EQUIVALENCE RELATIONS [1]

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This paper is devoted to exploring expressible power of monadic second-order sentences over the class of all relational structures containing only finite number of equivalence relations which are in local agreement (i.e. for any point of the universe the corresponding equivalence classes with set theoretic inclusion form linear order). Using the pebble games we prove the finite model property and establish an effective translation of these sentences in the first-order language preserving the models. So, the monadic second-order language over the considered class of relational structures has the same expressible power as the first-order language and the monadic second-order theory of this class of structures is decidable.

Keywords: MSO sentences, equivalence relations, finite model property, elimination of second-order quantifiers, decidability.

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We consider purely relational finite languages for the first-order predicate calculus with only unary and binary predicate symbols. Let $\mathcal{L} = (P_1, \dots P_r, R_1 \dots R_n)$ be such a language, with $P_1, \dots, P_r$ and $R_1, \dots, R_n$ being the unary and the binary predicate symbols, respectively. Take the class of structures where the interpretations of the binary predicate symbols are equivalence relations. In [3] Ershov announces that the monadic second-order logic of this class of structures is decidable for $n = 1$. Furthermore, in [2] Janiczak shows that the first-order logic of this class is undecidable for $n \geq 2$.

We further restrict the equivalence relations and consider the class of structures in which the binary relations are interpreted by equivalence relations in local

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agreement. We then show the decidability of the resulting monadic second order logic and demonstrate that there is a translation of every MSO sentence ψ to a first-order sentence ψ' such that ψ and ψ' have exactly the same models.

From now on, unless explicitly stated otherwise, we consider all languages to be finite and purely relational with only unary and binary predicate symbols. We also fix a language $\mathcal{L} = (P_1, \ldots, P_r, R_1, \ldots, R_n)$, with $P_1, \ldots, P_r$ and $R_1, \ldots, R_n$ being the unary and the binary predicate symbols, respectively. We assume that we always have equality in the structures and for convenience treat the equality as one of the binary predicates of the structures. Thus we always have $n > 0$.

In what follows we extensively use a kind of bisimulation games called "pebble games" to show similarity between structures. For complete details refer to [1]. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathcal{L}$ and let $s \in \mathbf{N}, s > 0$. The infinite pebble game $G^s_\infty(\mathfrak{A}, \mathfrak{B})$ is played by two players on a board which consists of the two structures $\mathfrak{A}$ and $\mathfrak{B}$. Each player has s pebbles, numbered from 1 to s. Players take turns. The first player chooses a pebble from her set of pebbles and a structure ($\mathfrak{A}$ or $\mathfrak{B}$) and places the selected pebble on some element of the structure. The second player answers by placing his pebble with the same number on some element of the other structure. The game continues indefinitely. Each time, after Player II has made his move, there is an even number of pebbles on the board. Half of them are in $\mathfrak{A}$ and the other half – in $\mathfrak{B}$. For example let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be the elements of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, on which the players have pebbles, and let for all $i = 1, \ldots, k$, a_i and b_i are under equal-numbered pebbles. Before each move of Player I the players review the configuration on the board, and if they find that the mapping $f : a_i \to b_i, i = 1, \ldots, k$ is not a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, Player I wins. Player II wins only if Player I does not win at any move.

Definition 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for the language $\mathcal{L}$ and let $s \in \mathbf{N}, s > 0$. We say that Player II has a winning strategy for the infinite pebble game $G^s_\infty(\mathfrak{A}, \mathfrak{B})$ iff Player II can win the game. no matter how Player I plays.*

A similar definition can be given for winning strategy for Player I. Obviously, for a given game exactly one player has a winning strategy.

Definition 2. *We say that two structures $\mathfrak{A}$ and $\mathfrak{B}$ for the language $\mathcal{L}$ are s-partially isomorphic (and write it $\mathfrak{A} \cong^s_{part} \mathfrak{B}$) iff Player II has a winning strategy for the infinite pebble game $G^s_\infty(\mathfrak{A}, \mathfrak{B})$.*

The main result about pebble games is given by the following:

Theorem 1. *([1]) For any two structures $\mathfrak{A}$ and $\mathfrak{B}$ for the language $\mathcal{L}$ and for any $s \in \mathbf{N}, s > 0$, $\mathfrak{A} \cong^s_{part} \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same formulas of no more than s variables of the infinitary logic $\mathcal{L}_{\infty\omega}$.*

Definition 3. *Let $R_1, \ldots, R_n$ be equivalence relations with common domain. $R_1, \ldots, R_n$ are in local agreement iff for all x in the domain of the relations, the set $\{|x|_{R_1}, \ldots, |x|_{R_n}\}$ of the equivalence classes of x according to $R_1, \ldots, R_n$ is linearly ordered according to the set theoretic inclusion.*

Definition 4. *Let $\mathcal{L}$ be a language of the considered type. We denote by $\mathcal{K}_{\mathcal{L}}$ the class of structures for $\mathcal{L}$ in which the binary predicates are interpreted with equivalence relations in local agreement. Sometimes, when the language $\mathcal{L}$ can be determined from the context we write $\mathcal{K}$ instead of $\mathcal{K}_{\mathcal{L}}$*

Definition 5. *Let $\mathfrak{A} \in \mathcal{K}$ and let $C \subseteq |\mathfrak{A}|$. We say that C is a maximal equivalence class in $\mathfrak{A}$ iff*

$$(\exists 1 \leq i \leq n)(|x|_{R_i} = C \,\&\, (\forall x \in C)(\forall 1 \leq j \leq n)(|x|_{R_j} \subseteq |x|_{R_i}))$$

Note that C is a maximal equivalence class iff C is a equivalence class and it is not a proper subset of any other equivalence class.

For brevity we sometimes use the term 'class' instead of the long form 'maximal equivalence class'.

Let $\mathfrak{A} = (|\mathfrak{A}|, P_1, \ldots, P_r, R_1, \ldots R_n), \mathfrak{A} \in \mathcal{K}$ and C is a maximal equivalence class in $\mathfrak{A}$. Let $P'_i \rightleftharpoons P_{i|C}$, for $1 \leq i \leq r$ and $R'_i \rightleftharpoons R_{i|C}$ for $1 \leq i \leq n$. Then $\mathfrak{C} = (C, P'_1, \ldots, P'_r, R'_1, \ldots, R'_n)$ is a substructure of $\mathfrak{A}$. We say that $\mathfrak{C}$ is the substructure of $\mathfrak{A}$ generated by C.

Since the maximal equivalence classes do not intersect and also cover the whole set $|\mathfrak{A}|$, we get that the structure $\mathfrak{A}$ can be represented as a direct sum of the substructures generated by its maximal equivalence classes. As there is one-to-one mapping from maximal equivalence classes and the substructures generated by them we shall use these two terms interchangeably. Whether we speak about an equivalence class or a substructure will be clear from the context.

Definition 6. *Let $s \in \mathbf{N}, s > 0$, and let k and l be cardinals (finite or infinite). We say that k and l are s-equal iff:*

$$k = l \vee (k \geq s \,\&\, l \geq s)$$

Note that for any $s \in \mathbf{N}, s > 0$, any two cardinals greater or equal to s are s-equal.

Proposition 1. *Let $s \in \mathbf{N}, s > 0$. Two structures $\mathfrak{A} \in \mathcal{K}$ and $\mathfrak{B} \in \mathcal{K}$ are s-partially isomorphic if and only if for each maximal equivalence class in one of the structures. the number of the classes s-partially isomorphic to it in $\mathfrak{A}$ is s-equal to the number of the classes s-partially isomorphic to it in $\mathfrak{B}$.*

Proof: We use Pebble games to show the equivalence.

First, let the condition be true. We show that $\mathfrak{A} \cong^s_{part} \mathfrak{B}$ by showing there is a winning strategy for Player II in the infinite pebble game with s pebbles.

Suppose we have k pebbles, $a_1, \ldots, a_k$ in $\mathfrak{A}$ and k pebbles, $b_1, \ldots, b_k$ in $\mathfrak{B}$, $k \leq s$ and the mapping $a_1, \ldots, a_k \to b_1, \ldots, b_k$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. We will show that whatever the first player does, the second player can preserve the partial isomorphism. Obviously, this implies that the two structures are s-partially isomorphic.

When $k < s$, the first player has two choices – she can either move one of her pebbles already placed in one of the structures, or she can place a new pebble in one of the structures. Without loss of generality we shall consider the case when a new pebble is placed in one of the structures.

Thus, let a_{k+1} be the new pebble placed in $\mathfrak{A}$. The pebble goes to one of the classes in $\mathfrak{A}$. Denote that class by A. Clearly, if A contains any other pebbles, then the class B where Player II should place his answer is determined. Otherwise, Player II should pick a class B of $\mathfrak{B}$ which is s-partially isomorphic to A and does not contain any pebbles in it. This can always be done because the number of classes in $\mathfrak{A}$ which are s-partially isomorphic to A is s-equal to the number of class in $\mathfrak{B}$ which are s-partially isomorphic to A. A similar argument can be used when Player I places the pebble in $\mathfrak{B}$. Thus Player II has a winning strategy for the game and therefore $\mathfrak{A} \cong^s_{part} \mathfrak{B}$.

Now suppose $\mathfrak{A} \cong^s_{part} \mathfrak{B}$ and suppose there is a class C in one of the structures, such that the number of classes of $\mathfrak{A}$ which are s-partially isomorphic to C is not s-equal to the number of classes in $\mathfrak{B}$ which are s-partially isomorphic to C. There are several cases, but without loss of generality we shall consider only one of them – when there are finite number of classes s-partially isomorphic to C in both structures. So, let $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ are all the classes in $\mathfrak{A}$ and $\mathfrak{B}$ respectively, which are s-partially isomorphic to C (the class C, of course, is among them). As k is not s-equal to l it follows that either $k < l \,\&\, k < s$ or $l < k \,\&\, l < s$.

1. $k < l \,\&\, k < s$.

   In that case the following is a winning strategy for the first player:

   Start placing one pebble in each class from $B_1, \ldots, B_l$. Since $k < l \,\&\, k < s$ there will be a move in which Player II will place his pebble in some class A after Player I has placed her pebble in some class B and the class A will either has already a pebble in it (in which case Player II immediately loses) or A will not be s-partially isomorphic to C (and B). At this point Player I can restrict the game to the classes A and B only. As these two classes are not s-partially isomorphic Player I has a winning strategy for the rest of the game.

2. $l < k \,\&\, l < s$

   This case is resolved by symmetry (this time Player I starts to place pebbles in classes of $\mathfrak{A}$)

□

Definition 7. *Let φ be a formula of the form $\exists x_1 \ldots \exists x_k \forall y \psi$, where ψ is a formula. Let x be a variable and R be a binary predicate symbol. We call the formula*

$$\varphi^R_x \rightleftharpoons \exists x_1 \ldots \exists x_k \forall y \left(\bigwedge_{i=1}^{k} x_i R x \,\&\, (y R x \Rightarrow \psi) \right)$$

a shallow relativization of φ (w.r.t. x and R).

Theorem 2. *For each structure $\mathfrak{A} \in \mathcal{K}$ and for each $s \in \mathbf{N}, s > 0$, there is a FO-formula $\psi^s_{\mathfrak{A}}$, such that for every structure $\mathfrak{B} \in \mathcal{K}$:*

$$\mathfrak{B} \models \psi^s_{\mathfrak{A}} \iff \mathfrak{A} \cong^s_{part} \mathfrak{B}$$

Moreover, for a given $s \in \mathbf{N}, s > 0$, there are only finite number of different formulae $\psi^s_{\mathfrak{A}}$ (modulo logical equivalence). In other words, for a given $s \in \mathbf{N}, s > 0$, the relation $\cong^s_{part}$ has finitely many equivalence classes.

Proof: Let M be the set of all substructures of $\mathfrak{A}$ which are generated by the maximal equivalence classes of $\mathfrak{A}$. The relation $\cong^s_{part}$ partitions M to equivalence classes. Let J be the factorization of M by $\cong^s_{part}$ and $I_j, j \in J$ be the equivalence classes of M. For $j \in J$ and $i \in I_j$, denote by $\mathfrak{A}^i_j$ the structure i. Then $\mathfrak{A}$ can be represented in the following way:

$$\mathfrak{A} = \bigcup_{j \in J, i \in I_j} \mathfrak{A}^i_j$$

We have:

$$\mathfrak{A}^{i_1}_{j_1} \cong^s_{part} \mathfrak{A}^{i_2}_{j_2} \iff j_1 = j_2$$

Let $m_j = \min\{\overline{\overline{I}}_j, s\}$, for $j \in J$ and $m = \overline{\overline{J}}$.

The proof is by induction on the number n of the binary predicate symbols in the language of the structure $\mathfrak{A}$.

1. $n = 1$

   In this case each of the structures $\mathfrak{A}^i_j$ consists of a single element. Consider the following formula:

   $$\psi^{\mathfrak{A}^i_j}(x) \rightleftharpoons \lambda^1_j P_1(x) \,\&\, \ldots \,\&\, \lambda^r_j P_r(x)$$

   where λ^k_j is the empty word, when $\mathfrak{A}^i_j \models \exists x P_k(x)$, and λ^k_j is the negation sign otherwise. Obviously, if two one-element structures satisfy the same formula of the above mentioned type then the structures are s-partially isomorphic. Note that $\psi^{\mathfrak{A}^i_j}(x)$ depends only on j, but not on i. For that reason we may omit the upper index of $\mathfrak{A}$ and just write $\psi^{\mathfrak{A}_j}(x)$.

   Now take the formula:

   $$\begin{aligned} \psi^s_{\mathfrak{A}} \rightleftharpoons \quad & \exists x^1_1 \ldots \exists x^1_{m_1} \\ & \exists x^2_1 \ldots \exists x^2_{m_2} \\ & \ldots \\ & \exists x^m_1 \ldots \exists x^m_{m_m} \end{aligned}$$

$$
\begin{aligned}
\forall y(&\bigwedge_{i_1 \neq i_2 \vee j_1 \neq j_2} x_{i_1}^{j_1} \neq x_{i_2}^{j_2} \,\& \\
&\& \,\tau_1(x_1^1, \ldots, x_{m_1}^1) \,\& \\
&\ldots \\
&\& \,\tau_m(x_1^m, \ldots, x_{m_m}^m) \,\& \\
&\& \,(\bigvee_{j \in J} \psi^{\mathfrak{A}_j}(y)))
\end{aligned}
$$

where the formulae τ are defined as follows:

$$
\begin{aligned}
&\tau_j(x_1, \ldots, x_k) \\
&\rightleftharpoons \forall z(\bigwedge_{i=1}^{k} z \neq x_i \Rightarrow \neg\psi^{\mathfrak{A}_j}(z) \,\&\, \psi^{\mathfrak{A}_j}(x_1) \,\&\, \ldots \,\&\, \psi^{\mathfrak{A}_j}(x_k)), \text{ when } k < s
\end{aligned}
$$

and

$$
\tau_j(x_1, \ldots, x_k) \rightleftharpoons \psi^{\mathfrak{A}_j}(x_1) \,\&\, \ldots \,\&\, \psi^{\mathfrak{A}_j}(x_k), \text{ when } k = s
$$

The meaning of the formula is evident - it just guarantees the desired condition from Proposition 1.

2. $n > 1$

   Let $j \in J$ is fixed. Since in each of the structures $\mathfrak{A}_j^i$ at least one of the binary relations coincide with the universal relation, we can drop one relation and use the induction hypothesis. Without loss of generality suppose that R_n is interpreted by the universal relation in each structure $\mathfrak{A}_j^i$. Let $\mathcal{L}' = \mathcal{L} \setminus \{R_n\}$. We can use the induction hypothesis and see that for the language $\mathcal{L}'$ there is a formula $\psi_{\mathfrak{A}_j^i}^s$ such that:

$$
\mathfrak{B}_{|\mathcal{L}'} \models \psi_{\mathfrak{A}_j^i}^s \iff \mathfrak{B}_{|\mathcal{L}'} \cong_{part}^s \mathfrak{A}_j^i \text{ for } i \in I_j
$$

   From here we see that $\psi_{\mathfrak{A}_j^i}^s$ does not depend on i and hence we again can use the short notation $\psi_{\mathfrak{A}_j}^s$. From the induction hypothesis we also get that all these formulae are finitely many (modulo logical equivalence), and hence J is finite.

   Note that $\psi_{\mathfrak{A}_j}^s$ is a formula in both $\mathcal{L}$ and $\mathcal{L}'$. Now let the formulae $\overline{\psi_{\mathfrak{A}_j}^s}$ are shallow relativizations of $\psi_{\mathfrak{A}_j}^s$ w.r.t. R_n (The variable w.r.t. which we relativize will be clear from the context).

   The formula we are seeking is:

$$
\begin{aligned}
\psi_{\mathfrak{A}}^s \rightleftharpoons \quad &\exists x_1^1 \ldots \exists x_{m_1}^1 \\
&\exists x_1^2 \ldots \exists x_{m_2}^2 \\
&\ldots
\end{aligned}
$$

$$
\begin{aligned}
&\exists x_1^m \ldots \exists x_{m_m}^m \\
&\forall y(\bigwedge_{\substack{i_1 \neq i_2 \vee j_1 \neq j_2 \\ k=1\ldots n}} \neg x_{i_1}^{j_1} R_k x_{i_2}^{j_2} \,\& \\
&\& \, \tau_1(x_1^1, \ldots, x_{m_1}^1) \,\& \\
&\ldots \\
&\& \, \tau_m(x_1^m, \ldots, x_{m_m}^m) \,\& \\
&\& \, (\bigvee_{j \in J} \overline{\psi_{\mathfrak{A}_j}^s}(y)))
\end{aligned}
$$

where the formulae τ are defined as:

$$
\begin{aligned}
&\tau_j(x_1, \ldots, x_k) \\
&\rightleftharpoons \forall z(\bigwedge_{i=1}^{k} z \neq x_i \Rightarrow \neg\overline{\psi_{\mathfrak{A}_j}^s}(z) \,\&\, \overline{\psi_{\mathfrak{A}_j}^s}(x_1) \,\&\, \ldots \,\&\, \overline{\psi_{\mathfrak{A}_j}^s}(x_k)), \text{ when } k < s
\end{aligned}
$$

and as:

$$
\tau_j(x_1, \ldots, x_k) \rightleftharpoons \overline{\psi_{\mathfrak{A}_j}^s}(x_1) \,\&\, \ldots \,\&\, \overline{\psi_{\mathfrak{A}_j}^s}(x_k), \text{ when } k = s
$$

Since the number of elements of J and the numbers m_j are bounded we get that there are only finite number of formulae $\psi_{\mathfrak{A}}^s$, for all structures of $\mathcal{K}$.

Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and $\mathfrak{B}_i$ is a maximal equivalence class in $\mathfrak{B}$ relatively the binary predicate symbol R. Let $\overline{\psi_{\mathfrak{A}}^s}$ be a shallow relativisation of $\psi_{\mathfrak{A}}^s$ w.r.t. R and x.

Observe that:

$$
\mathfrak{B}_i \models \psi_{\mathfrak{A}}^s \iff \mathfrak{B} \models \overline{\psi_{\mathfrak{A}}^s}[a/x], \text{ for } a \in |\mathfrak{B}_i|
$$

From the observation it is easy to verify that the formula we gave guarantees the condition from Proposition 1.

□

Definition 8. *Let $h(s, n, r)$ denote the number of equivalence classes of the relation $\cong_{part}^s$ for a language with n binary and r unary predicate symbols.*

Corollary 1. *For each structure $\mathfrak{A}$ in $\mathcal{K}$ there exists a finite structure $\mathfrak{A}_{\mathrm{fin}}$, such that:*

$$
\mathfrak{A} \cong_{part}^s \mathfrak{A}_{\mathrm{fin}}
$$

and the structure $\mathfrak{A}_{\mathrm{fin}}$ can be selected to be of cardinality bounded from above by a computable function.

Proof: The proof easy follows from Proposition 1 by induction on the number of binary predicate symbols in the language and the fact that $\cong^s_{part}$ has only finite number of equivalence classes. The cardinality of $\mathfrak{A}_{fin}$ is bounded by $s^n . \prod_{k=1}^{n} h(s,k,r)$. □

Proposition 2. *Let $\mathfrak{A}$ и $\mathfrak{B}$ be structures from $\mathcal{K}$ and let $\mathfrak{A} \cong^{s.h(s,n,r+1)}_{part} \mathfrak{B}$. Then for any $P \subseteq |\mathfrak{A}|$ exists $Q \subseteq |\mathfrak{B}|$, such that: $(\mathfrak{A}, P) \cong^s_{part} (\mathfrak{B}, Q)$*

Proof: The proof is by induction on the number n of the binary predicate symbols in the language.

As in the proof of Theorem 2, let the structure $\mathfrak{A}$ be represented as a direct sum of its maximal equivalence classes:

$$\mathfrak{A} = \bigcup_{j \in J^{\mathfrak{A}}, i \in I_j} \mathfrak{A}^i_j$$

Where $J^{\mathfrak{A}}$ is the factorization of the set of maximal equivalence classes of $\mathfrak{A}$ over the relation $\cong^{s.h(s,n,r+1)}_{part}$. For $j \in J^{\mathfrak{A}}$, I_j is the appropriate equivalence class of $\cong^{s.h(s,n,r+1)}_{part}$ over the set of the maximal equivalence classes of $\mathfrak{A}$ and $\mathfrak{A}^i_j$ is more verbose notation for i.

Similarly, $\mathfrak{B}$ can be represented as:

$$\mathfrak{B} = \bigcup_{j \in J^{\mathfrak{B}}, i \in I^{\mathfrak{B}}_j} \mathfrak{B}^i_j$$

Note that due to the fact that the structures $\mathfrak{A}$ and $\mathfrak{B}$ are $s.h(s,n,r+1)$-partially isomorphic we have $J^{\mathfrak{A}} = J^{\mathfrak{B}}$.

For convenience we assume that the two representations are compatible in the following sense:

$$(\forall j_1 \in J^{\mathfrak{A}})(\forall j_2 \in J^{\mathfrak{B}})(j_1 = j_2 \iff (\forall i_1 \in I^{\mathfrak{A}}_{j_1})(\forall i_2 \in I^{\mathfrak{B}}_{j_2})(\mathfrak{A}^{i_1}_{j_1} \cong^{s.h(s,n,r+1)}_{part} \mathfrak{B}^{i_2}_{j_2}))$$

From Proposition 1 we get

$$\overline{\overline{I^{\mathfrak{A}}_j}} =_{s.h(s,n,r+1)} \overline{\overline{I^{\mathfrak{B}}_j}} \tag{1}$$

for all $j \in J^{\mathfrak{B}}$.

Let now $P \subseteq |\mathfrak{A}|$. We introduce a new predicate symbol P_{r+1}, which will be interpreted in $\mathfrak{A}$ by the predicate P. Thus we obtain a new structure $\mathfrak{C} \rightleftharpoons (\mathfrak{A}, P)$ for the enriched language $\mathcal{L} \cup \{P_{r+1}\}$.

The structure $\mathfrak{C}$ can be represented as a direct sum of maximal equivalence classes as well:

$$\mathfrak{C} = \bigcup_{j \in J^{\mathfrak{C}}, i \in I^{\mathfrak{C}}_j} \mathfrak{C}^i_j$$

Now let $J_j^{\mathfrak{C}} \rightleftharpoons \{\mathfrak{c}_{|\mathcal{L}} \mid \mathfrak{c} \in I_j^{\mathfrak{C}}\}$, for $j \in J^{\mathfrak{C}}$. For the above representation,evidently, for each $j \in J^{\mathfrak{A}}$, there are $j_1, \ldots, j_r \in J^{\mathfrak{C}}$, with $r \leqq h(s,n,r+1)$ such that:

$$I_j^{\mathfrak{A}} = J_{j_1}^{\mathfrak{C}} \cup \ldots \cup J_{j_r}^{\mathfrak{C}}$$

Now fix a $j \in J^{\mathfrak{A}}$. Because of equation (1) and since $r \leqq h(s,n,r+1)$ it is easy to see that $I_j^{\mathfrak{B}}$ can be represented as:

$$I_j^{\mathfrak{B}} = I_{j_1} \cup \ldots \cup I_{j_r}$$

and

$$(\forall 1 \leqq i \leqq r)(\overline{\overline{I_{j_i}}} =_s \overline{\overline{I_{j_i}^{\mathfrak{C}}}})$$

Let $1 \leqq i \leqq r$ be fixed, $\mathfrak{B}_{j_i}^{\ell}$ be arbitrary structure from I_{j_i} and $\mathfrak{C}_{j_i}^{t} \in I_{j_i}^{\mathfrak{C}}$. We shall define a predicate Q in $\mathfrak{B}_{j_i}^{\ell}$, with the desired properties. The definition of Q can be carried in the same way for any structure of I_{j_i}.

We distinguish two cases:

1. $n = 1$:

   Without loss of generality we can assume that the only binary relation in the language is the equality. In this case $\mathfrak{C}_{j_i}^{t}$ and $\mathfrak{B}_{j_i}^{\ell}$ are one-point structures and Q can be defined on $\mathfrak{B}_{j_i}^{\ell}$, in the same way as P is defined in $\mathfrak{C}_{j_i}^{t}$.

2. $n > 1$:

   Let $\mathfrak{A}_j^t \rightleftharpoons \mathfrak{C}_{j_i|\mathcal{L}}^{t}$. From the fact that $\mathfrak{A}_j^t$ and $\mathfrak{B}_{j_i}^{\ell}$ are maximal equivalence classes for $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and from $\mathfrak{A}_j^t \cong_{part}^{s.h(s,n,r+1)} \mathfrak{B}_{j_i}^{\ell}$, it follows that there exists a binary predicate symbol R from the language $\mathcal{L}$, such that it is interpreted in $\mathfrak{A}_j^t$ and $\mathfrak{B}_{j_i}^{\ell}$ with the universal relation. Let $\mathcal{L}' \rightleftharpoons \mathcal{L} \setminus \{R\}$ and consider the restrictions of $\mathfrak{A}_j^t$ and $\mathfrak{B}_{j_i}^{\ell}$ to the language $\mathcal{L}'$. We have:

   $$\mathfrak{A}_j^t|_{\mathcal{L}'} \cong_{part}^{s.h(s,n,r+1)} \mathfrak{B}_{j_i}^{\ell}|_{\mathcal{L}'}$$

   Since $h(s,n,r+1) \geqq h(s,n-1,r+1)$, from the induction hypothesis we get:

   $$\left(\exists Q \subseteq |\mathfrak{B}|_{j_i}^{\ell}\right)\left(\left(\mathfrak{A}_j^t, P\right)_{|\mathcal{L}'} \cong_{part}^{s} \left(\mathfrak{B}_{j_i}^{\ell}, Q\right)_{|\mathcal{L}'}\right)$$

   As R is interpreted in both structures with the universal relation we get:

   $$\left(\mathfrak{A}_j^t, P\right) \cong_{part}^{s} \left(\mathfrak{B}_{j_i}^{\ell}, Q\right)$$

Thus we show that the predicate Q can be defined on any maximal equivalence class. Now, taking the union of these predicates we define the interpretation of Q on the structure $\mathfrak{B}$. □

Definition 9. *Let φ be a monadic second-order formula. We say that φ is first-order definable over $\mathcal{K}$ iff there is a first-order formula ψ such that for every structure $\mathfrak{A} \in \mathcal{K}$:*

$$\mathfrak{A} \models \varphi \iff \mathfrak{A} \models \psi$$

Theorem 3. *Every MSO sentence is first-order definable over $\mathcal{K}$.*

Proof: Let $\varphi \rightleftharpoons K_1P_1 \dots K_mP_m\varphi_1$ be a MSO formula, where $K_1, \dots, K_m$ are quantifiers, $P_1, \dots, P_m$ are monadic second-order variables, and φ_1 is a FO formula. We shall prove that there exists a FO formula ψ, such that for each structure $\mathfrak{A} \in \mathcal{K}$:

$$\mathfrak{A} \models \varphi \iff \mathfrak{A} \models \psi$$

It is sufficient to show how to remove from the formula φ a single quantifier over a second-order variable.

1. Let $\varphi \rightleftharpoons \exists P\varphi_1$, where φ_1 is a first-order formula and let $\mathfrak{A} \in \mathcal{K}$. Let s be the number of the first-order variables that appear in φ_1. Consider the formula:

$$\psi \rightleftharpoons \bigvee_{\substack{\mathfrak{A}\in\mathcal{K} \\ (\exists P\subseteq|\mathfrak{A}|)((\mathfrak{A},P)\models\varphi_1)}} \psi_{\mathfrak{A}}^{s.h(s,n,r+1)}$$

Because of the finite number of the formulae $\psi_{\mathfrak{A}}^m$ (modulo logical equivalence), for all $m \in \mathbf{N}, m > 0$, the disjunction is a finite and hence ψ is a first-order formula.

Note that the above disjunction can be empty. In that case the normal conventions apply: that is, we consider the empty disjunctions to be false (replacing the disjunction with $\neg\forall x(x = x)$, for example)

Apparently, for each $\mathfrak{B} \in \mathcal{K}$ if $\mathfrak{B} \models \exists P\varphi_1$, then $\mathfrak{B} \models \psi$.

In the opposite direction, take $\mathfrak{B} \in \mathcal{K}$ and $\mathfrak{B} \models \psi$. We shall demonstrate that $\mathfrak{B} \models \exists P\varphi_1$. Since $\mathfrak{B} \models \psi$ there exists a structure $\mathfrak{A} \in \mathcal{K}$ such that $\mathfrak{B} \models \psi_{\mathfrak{A}}^{s.h(s,n,k+1)}$ and hence $\mathfrak{A} \cong_{part}^{s.h(s,n,k+1)} \mathfrak{B}$. Moreover, for $\mathfrak{A}$ we have:

$$(\exists P \subseteq |\mathfrak{A}|)\,((\mathfrak{A}, P) \models \varphi_1)$$

From Proposition 2 we obtain that there exists $Q \subseteq |\mathfrak{B}|$ such that:

$$(\mathfrak{A}, P) \cong_{part}^{s} (\mathfrak{B}, Q)$$

Thus we get $(\mathfrak{B}, Q) \models \varphi_1$, from where we conclude the desired $\mathfrak{B} \models \varphi$.

2. Let $\varphi \rightleftharpoons \forall P\varphi_1$, where φ_1 is a first-order formula. Consider the formula:

$$\psi \rightleftharpoons \bigvee_{\substack{\mathfrak{A}\in\mathcal{K} \\ (\forall P\subseteq|\mathfrak{A}|)((\mathfrak{A},P)\models\varphi_1)}} \psi_{\mathfrak{A}}^{s.h(s,n,k+1)}$$

By an argument similar to the above we obtain $\mathfrak{B} \models \psi \iff \mathfrak{B} \models \varphi$.

□

Corollary 2 *The monadic second-order logic of $\mathcal{K}$ is decidable.*

Proof: Immediately from Corollary 1 and Theorem 3. □

Finally, we would like to mention some further directions for research on the subject. One interesting area is adding functional symbols to the language. We already know that adding one functional symbol under some simple restrictions does produce a theory which is not decidable, but have not investigated other interesting cases. Also, one can try to expand this result to arbitrary formulae, not just sentences. We do not know yet if this can be done. Another interesting perspective is research on some possible connections with Data Analysis Logic.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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MODAL LOGIC FOR 3D INCIDENCE GEOMETRY

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Three sorted geometrical space of incidence is represented as equivalent uni-sorted objects structure of incidences, and a modal logic using three equivalence relations and a difference relation is used to axiomatize that class. Completeness theorem is proven.

Keywords: Modal Logic of Irreflexivity, Incidence Geometry, Equivalence Relations, Axiomatization, Completeness Theorem.

1. INTRODUCTION

The incidence 3D geometry consists of 3 different sorts of objects: points, lines and plains and 3 relations called incidences. We introduce an equivalent one-sort geometrical structure, called a structure of incidence, which is suitable for modal considerations. The approach is the same as in the papers of Balbiani et al. [1], [2] but extended to 3D geometry.

In the beginning we present the 3D geometrical space of incidence and the one-sort geometrical structure of incidence. The category of incidence spaces corresponds closely to the geometry and its properties and its semantics is taken from it. One-sort geometrical structure of incidence is a structure which contains only one sort objects and 3 equivalence relations. Each object can play as a point, a line or a plane at the same time, and the incidence relations are expressed as composition of the equivalence relations. The equivalence of the category of the incidence spaces and the category of the structures of incidence is proven by defining functors from the incidence spaces to the structures of incidence and from the structures of incidence to the incidence spaces.

The one-sort objects based structures of incidence with 3 relations of equivalence and the difference relation are suitable for frames of a modal logic. The language of the modal logic contains 3 unary modalities for the equivalence relations and the difference operator representing inequality. The semantics uses the structures of incidence for frames of this logic.

The deductive system uses well-known rules as *Modus Ponens*, *Generalization* for each modal symbol, and the irreflexivity rule proposed by Gabbay [3]. The completeness proof does not use the *irreflexivity* rule directly but replaces it with an *infinitary* rule deductively equivalent to it. For that equivalent deductive system we prove the completeness. The completeness is proved using maximal consistent theories and the canonical frame and model. The completeness of the original system is a consequence of the deductive equivalence between these two rules.

The geometrical modal logic is derived from the minimal one with adding several axioms for each property of the incidence frames. Each property of the incidence frames is axiomatized and it is a canonical property. So the proposed finite axiomatization is the axiomatization of the logic which frames are the structures of incidence.

2. INCIDENCE GEOMETRY AND INCIDENCE FRAMES

First we show briefly the category of 3D geometrical incidence space. It is consisted of points, lines and planes, and the relations: a point belongs to a line, a line lays into a plane, and a point lays into a plane. So these relations are called incidence relations. Another relation which is also important is the difference between 2 points, 2 lines and 3 planes.

The incidence frames are explained afterward and the relations of incidences are replaced with 3 equivalence relations and the difference. The definition of the incidence frames and the equivalence between incidence frames and the category of incidence geometry is the topic of this first chapter.

2.1. THE CATEGORY OF INCIDENCE SPACES AND THE INCIDENCE GEOMETRY

Definition 2.1. Incidence space *we call any multi-sort system of the type* $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$, *where:*

- *Po is a non-empty set which elements are called* ***points****. We note them with upper case Latin letters.*
- *Li is a non-empty set which elements are called* ***lines****. We note them with lower case Latin letters.*

- *Pl is a non-empty set which elements are called* **planes**. *We note them with Greek letters.*
- $\varepsilon_{1,2} \subseteq Po \times Li, \varepsilon_{1,2}$ *is a two-sort relation between points and lines. It says that a point is into a line.*
- $\varepsilon_{1,3} \subseteq Po \times Pl, \varepsilon_{1,3}$ *is a two-sort relation between points and planes. It says that a point lays onto a plane.*
- $Po \cap Li = \emptyset, Po \cap Pl = \emptyset, Li \cap Pl = \emptyset$. *There aren't any objects that are points and lines, points and planes or planes or lines.*
- *Po, Li, Pl,* $\varepsilon_{1,2}$ *and* $\varepsilon_{1,3}$ *must have the properties (geometry axioms) below:*

1. $(\exists X \in Po, \exists Y \in Po)(X \neq Y)$. *There are at least 2 different points.*
2. $(\forall X \in Po, \forall Y \in Po)(\exists z \in Li)(X\varepsilon_{1,2}z \wedge Y\varepsilon_{1,2}z)$. *For each 2 points there is a line which goes through them. The points are incident with the line.*
3. $(\forall X \in Po, \forall Y \in Po)(\forall z \in Li, \forall t \in Li)(X \neq Y \wedge X\varepsilon_{1,2}z \wedge Y\varepsilon_{1,2}z \wedge X\varepsilon_{1,2}t \wedge Y\varepsilon_{1,2}t \Rightarrow z = t)$. *For 2 different points there is maximum one line which is incident with them.*
4. $(\forall z \in Li)(\exists X \in Po, \exists Y \in Po)(X \neq Y \wedge X\varepsilon_{1,2}z \wedge Y\varepsilon_{1,2}z)$. *For each line there are at least 2 different points that are incident with the line.*
5. $(\forall z \in Li)(\exists X \in Po)(\neg X \in z)$. *For each line there is a point that is not incident with the line.*
6. $(\forall X \in Po, \forall Y \in Po, \forall Z \in Po)(\exists \alpha \in Pl)(X\varepsilon_{1,3}\alpha \wedge Y\varepsilon_{1,3}\alpha \wedge Z\varepsilon_{1,3}\alpha)$. *For each 3 points there is a plane that is incident with them.*
7. $(\forall X \in Po, \forall Y \in Po, \forall Z \in Po)(\forall \alpha \in Pl, \forall \beta \in Pl)((\forall l \in Li)((\neg X \in l) \vee (\neg Y \in l) \vee (\neg Z \in l)) \wedge X\varepsilon_{1,3}\alpha \wedge Y\varepsilon_{1,3}\alpha \wedge Z\varepsilon_{1,3}\alpha \wedge X\varepsilon_{1,3}\beta \wedge Y\varepsilon_{1,3}\beta \wedge Z\varepsilon_{1,3}\beta \Rightarrow \alpha = \beta)$. *For each 3 points that is not incident with the same line, there is maximum one plane that is incident with all 3 points.*
8. $(\forall \alpha \in Pl)(\exists X \in Po)(X\varepsilon_{1,3}\alpha)$. *For each plane there is a point that is incident with the plane.*
9. $(\forall \alpha \in Pl)(\exists X \in Po)(\neg X\varepsilon_{1,3}\alpha)$. *For each plane there is a point that is not incident with the plane.*
10. $(\forall l \in Li)(\forall \alpha \in Pl)(\forall X \in Po, \forall Y \in Po)(X \neq Y \wedge X\varepsilon_{1,2}l \wedge Y\varepsilon_{1,2}l \wedge X\varepsilon_{1,3}\alpha \wedge Y\varepsilon_{1,3}\alpha \Rightarrow (\forall Z \in Po)(Z\varepsilon_{1,2}l \Rightarrow Z\varepsilon_{1,3}\alpha))$. *If 2 different points are incident with a line and with a plane at the same time, then each point which is incident with the line is incident with the plane too. So if 2 different points from a line are incident with the plane then the whole line lays onto the plane.*

11. $(\forall\alpha \in Pl)(\forall\beta \in Pl)(\exists X \in Po)(X\varepsilon_{1,3}\alpha \wedge X\varepsilon_{1,3}\beta \Rightarrow (\exists Y \in Po)(X \neq Y \wedge Y\varepsilon_{1,3}\alpha \wedge Y\varepsilon_{1,3}\beta))$ *If 2 planes have a point which is incident with both. then the planes have another point different from the first one which is also incident with the 2 planes.*

These relations $\varepsilon_{1,2}$ and $\varepsilon_{1,3}$ we call the incidence relations.

The relation $\varepsilon_{2,3} \subseteq Li \times Pl$ which says that *a line lays onto a plane*, is expressible with the incidence relations $\varepsilon_{1,2}$ and $\varepsilon_{1,3}$.

Definition 2.2. The line lays onto a plane *if any point that is incident with the line is incident with the plane. We can express that relation $\varepsilon_{2,3} \subseteq Li \times Pl$ with the equivalence:*

$l\varepsilon_{2,3}\alpha \Leftrightarrow (\forall X \in Po)(X\varepsilon_{1,2}l \Rightarrow X\varepsilon_{1,3}\alpha)$, *where l is a line and α is a plane.*

This way the relation if a line is incident with a plane (lays onto a plane) is expressible with the other 2 incidence relations.

The incidence spaces we call them just spaces. And we turn them into categories with defining the notion of homomorphism between spaces.

Let $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ and $S' = (Po', Li', Pl', \varepsilon'_{1,2}, \varepsilon'_{1,3})$ be 2 incidence spaces.

Definition 2.3. Homomorphism between incidence spaces. *We says that the 2 incidence spaces has a homomorphism f from S into S' if the f is a function with domain $Po \cup Li \cup Pl$ and range $Po' \cup Li' \cup Pl'$. $f : Po \cup Li \cup Pl \to Po' \cup Li' \cup Pl'$. which follows the conditions:*

1. $(\forall X \in Po)(\forall y \in Li)(\forall\alpha \in Pl)(f(X) \in Po' \wedge f(y) \in Li' \wedge f(\alpha) \in Pl')$
2. $(\forall X \in Po)(\forall y \in Li)(X\varepsilon_{1,2}y \Rightarrow f(X)\varepsilon'_{1,2}f(y))$
3. $(\forall X \in Po)(\forall\alpha \in Pl)(X\varepsilon_{1,3}\alpha \Rightarrow f(X)\varepsilon'_{1,3}f(\alpha))$

   f is an isomorphism if it follows the additional conditions:
4. $f : Po \to Po'$. $f : Li \to Li'$. $f : Pl \to Pl'$ *all these are bijective.*
5. $(\forall X \in Po)(\forall y \in Li)(f(X)\varepsilon'_{1,2}f(y) \Rightarrow X\varepsilon_{1,2}y)$
6. $(\forall X \in Po)(\forall\alpha \in Pl)(f(X)\varepsilon'_{1,3}f(\alpha) \Rightarrow X\varepsilon_{1,3}\alpha)$

2.2. STRUCTURE OF INCIDENCE

The aim is to introduce a new kind of structures which contains only one sort objects, and it is suitable for frames of modal languages. The new structures are equivalent to the incidence spaces and the properties of the incidence spaces are translated as properties of the structures.

So we introduce a construction with which from an incidence space we can create a new kind of one-sort structure. That structures are the structure of incidence.

Definition 2.4. *Let* $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ *be an incidence space and we call the structure* $\underline{W}(S) = (W(S), \equiv_1, \equiv_2, \equiv_3)$ structure of incidence over an incidence space S. *if:*

1. *The set W(S) is defined as:*

$$W(S) = \{(X, y, \alpha) | (X \in Po) \wedge (y \in Li) \wedge (\alpha \in Pl) \wedge (X \varepsilon_{1,2} y) \wedge (y \varepsilon_{2,3} \alpha)\}$$

2. *The relations* $\equiv_1, \equiv_2, \equiv_3$ *are defined as:*

$$(X_1, y_1, \alpha_1) \equiv_1 (X_2, y_2, \alpha_2) \Leftrightarrow X_1 = X_2$$

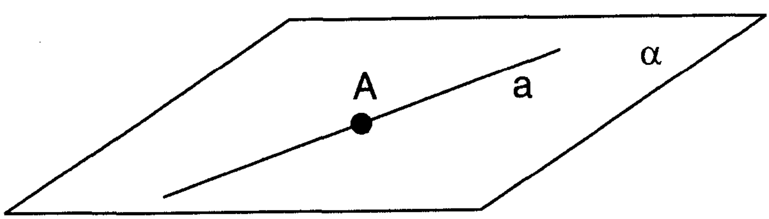
$$(X_1, y_1, \alpha_1) \equiv_2 (X_2, y_2, \alpha_2) \Leftrightarrow y_1 = y_2$$

$$(X_1, y_1, \alpha_1) \equiv_3 (X_2, y_2, \alpha_2) \Leftrightarrow \alpha_1 = \alpha_2$$

where $(X_1, y_1, \alpha_1) \in W(S)$ *and* $(X_2, y_2, \alpha_2) \in W(S)$.

The relations $\equiv_1, \equiv_2, \equiv_3$ are equivalence relations.

Elements of $W(S)$ are triples of a point, a line and a plane, such that the point is incident with the line and the line is incident with the plane. Each triple of $W(S)$ plays as a point, a line and a plane at the same time. See the figure:



As subsequence of that, the point is incident with the plane too - if $(X, y, \alpha) \in W(S)$ then $X \varepsilon_{1,3} \alpha$. We call $W(S)$ only W for shortly.

Definition 2.5. *The relations* $\in_{1,2}, \in_{1,3}, \in_{2,3}$ *into* $W(S)$ *defined with the equations are called* structure incidence relations*:*

$$(X_1, y_1, \alpha_1) \in_{1,2} (X_2, y_2, \alpha_2) \Leftrightarrow X_1 \varepsilon_{1,2} y_2$$

$$(X_1, y_1, \alpha_1) \in_{1,3} (X_2, y_2, \alpha_2) \Leftrightarrow X_1 \varepsilon_{1,3} \alpha_2$$

$$(X_1, y_1, \alpha_1) \in_{2,3} (X_2, y_2, \alpha_2) \Leftrightarrow y_1 \varepsilon_{2,3} \alpha_2$$

where $(X_1, y_1, \alpha_1) \in W(S)$ *and* $(X_2, y_2, \alpha_2) \in W(S)$.

These relations $\in_{1,2}, \in_{1,3}, \in_{2,3}$ which corresponds to the incidence relations are expressible as compositions of the equivalence $S5$ relations $\equiv_1, \equiv_2, \equiv_3$

Lemma 2.1. *The equations are valid:*

$$\in_{1,2} = \equiv_1 \circ \equiv_2$$

$$\in_{1,3} = \equiv_1 \circ \equiv_3$$

$$\in_{2,3} = \equiv_2 \circ \equiv_3$$

Proof. Using the definitions 2.1, 2.4, 2.5 it is easy to prove that:

$$\overline{x} \in_{1,2} \overline{y} \Leftrightarrow (\exists \overline{z} \in W)(\overline{x} \equiv_1 \overline{z} \wedge \overline{z} \equiv_2 \overline{y})$$

$$x \in_{1,3} y \Leftrightarrow (\exists z \in W)(x \equiv_1 z \wedge z \equiv_3 y)$$

$$x \in_{2,3} y \Leftrightarrow (\exists z \in W)(x \equiv_2 z \wedge z \equiv_3 y)$$

The proofs of the implications above in the direction "$\Leftarrow$" are very simple. We show the proof for $(\exists \overline{z} \in W)(\overline{x} \equiv_1 \overline{z} \wedge \overline{z} \equiv_2 \overline{y}) \Rightarrow \overline{x} \in_{1,2} \overline{y}$:

Let $\overline{x} = (X, x, \alpha)$, $\overline{y} = (Y, y, \beta)$, $\overline{z} = (Z, z, \gamma)$.

If for some $\overline{z} \in W$ is true that $\overline{x} \equiv_1 \overline{z}$ and $\overline{z} \equiv_2 \overline{y}$ then from the definition of $\equiv_1$ and $\equiv_2$ it follows that $X = Z$ and $z = y$. And from $Z\varepsilon_{1,3}z$ we conclude that $X\varepsilon_{1,3}y$ which is $(X, x, \alpha) \in_{1,2} (Y, y, \beta)$ and that's the definition of $\overline{x} \in_{1,2} \overline{y}$. The other 2 implications are proved in the same way.

Proofs of the implications of the direction "$\Rightarrow$" are also simple. We show the proof for $\overline{x} \in_{1,3} \overline{y} \Rightarrow (\exists \overline{z} \in W)(\overline{x} \equiv_1 \overline{z} \wedge \overline{z} \equiv_3 \overline{y})$:

Let $\overline{x} = (X, x, \alpha)$, $\overline{y} = (Y, y, \beta)$. We must find a suitable $\overline{z} = (Z, z, \gamma)$. From $\overline{x} \in_{1,3} \overline{y}$ it follows that $X\varepsilon_{1,3}\beta$ and from $\overline{x} \in W(S)$ follows that $X\varepsilon_{1,3}\alpha$. So both planes α and β have a common point X. From the definition 2.1 axiom (12) it follows that there is another point Z different from X, which belongs to both planes. $\exists Z \in Po, X \neq Z \wedge Z\varepsilon_{1,3}\alpha \wedge Z\varepsilon_{1,3}\beta$. For the 2 different points X and Z from the definition 2.1 axiom (3) there is a line z that is incident with the 2 points, $\exists z \in Li, X\varepsilon_{1,2}z \wedge Z\varepsilon_{1,2}z$ We choose $\overline{z} = (X, z, \beta)$ and we have proved so far that $X \neq Z \wedge X\varepsilon_{1,3}\beta \wedge Z\varepsilon_{1,3}\beta \wedge X\varepsilon_{1,2}z \wedge Z\varepsilon_{1,2}z$, then from the definition 2.1 axiom (11) we can conclude that it is true $(\forall T \in Po)(T\varepsilon_{1,2}z \Rightarrow T\varepsilon_{1,3}\beta)$, which is the definition of $z\varepsilon_{2,3}\beta$. So we discovered a triple $\overline{z} = (X, z, \beta)$ such that $X\varepsilon_{1,2}z \wedge z\varepsilon_{2,3}\beta$ so $\overline{z} \in W(S)$, and $(X, x, \alpha) \equiv_1 (X, z, \beta)$, and $(X, z, \beta) \equiv_3 (Y, y, \beta)$, finally the $\overline{z} = (X, z, \beta)$ suffice $x \equiv_1 z \wedge z \equiv_3 y$.

The remaining 2 implications are simple. □

The reverted relations $\in_{1,2}^{-1}, \in_{1,3}^{-1}, \in_{2,3}^{-1}$ of $\in_{1,2}, \in_{1,3}, \in_{2,3}$ are also expressed with a composition of the equivalence relations.

$$\in_{1,2}^{-1} = \equiv_2 \circ \equiv_1$$

$$\in_{1,3}^{-1} = \equiv_3 \circ \equiv_1$$

$$\in_{2,3}^{-1} = \equiv_3 \circ \equiv_1$$

Lemma 2.2. *Let $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ be an incidence space and $W(S)$ is the structure of incidence over S. Then the following conditions are true:*

1. *If X is a point then exist a line x and a plane α such that the triple $(X, x, \alpha) \in W(S)$.*

2. *If x is a line then exist a point X and a plane α such that the triple $(X, x, \alpha) \in W(S)$.*

3. *If α is a plane then exist a point X and a line x such that the triple $(X, x, \alpha) \in W(S)$.*

Proof. For the first one:

Let $X \in Po$ is a point, then from the definition 2.1 axiom (1), there is 2 different points $Y_1 \in Po$, and $Y_2 \in Po$ and $Y_1 \neq Y_2$. So because $Y_1 \neq Y_2$ then if $X = Y_1$ then $X \neq Y_2$. Let's note with Y the point of Y_1 and Y_2 which is different from X.

Apply the definition 2.1 axiom (3) and let y be the line incident with X and Y. $X\varepsilon_{1,2}y \wedge Y\varepsilon_{1,2}y$.

Apply the definition 2.1 axiom (7) for the 3 points X, Y, Y and let γ be the plane that is incident with X and Y.

From the definition 2.1 axiom (11) and from the definition 2.2 for

$X \neq Y \wedge X\varepsilon_{1,2}y \wedge Y\varepsilon_{1,2}y \wedge X\varepsilon_{1,3}\gamma \wedge X\varepsilon_{1,3}\gamma$ we conclude that $y\varepsilon_{2,3}\gamma$.

Similar reasoning proves 2 and 3. □

The meaning of this lemma is that each point, line or plane can be completed with redundant points, lines and planes to produce a triple that belong to the structures of incidences. This way the working with multi-sort points, lines and planes can be replaced with one-sort objects which are points, lines and planes at the same time. All geometrical properties can be translated as properties of these one sort objects.

Lemma 2.3. *Let $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ be an incidence space and $\underline{W}(S)$ $(W(S), \equiv_1, \equiv_2, \equiv_3)$ is the structure of incidence over S. Then for the structure of incidence $W(S)$ the following conditions are true.*

- *The relations $\equiv_1, \equiv_2, \equiv_3$ are equivalence relations and they follow the conditions below:*

$*$ $(\forall x \in W(S))(\forall y \in W(S))((x \equiv_1 y) \wedge (x \equiv_2 y) \wedge (x \equiv_3 y) \Rightarrow x = y)$

$**$ $(\forall x \in W(S))(\forall y \in W(S))(\forall z \in W(S))((x \in_{1,2} y) \wedge (y \in_{2,3} z) \Rightarrow (\exists t \in W(S))((x \equiv_1 t) \wedge (y \equiv_2 t) \wedge (z \equiv_3 t)))$

$***$ $(\forall x \in W(S))(\forall y \in W(S))((\forall z \in W(S))(z \in_{1,2} x \wedge z \in_{1,3} y) \Rightarrow x \in_{2,3} y)$

The next conditions correspond to the geometrical axioms of the incidence space:

1. $(\exists x \exists y \in W(S))(\neg x \equiv_1 y)$

2. $(\forall x \forall y \in W(S))(\exists z \in W(S))((x \in_{1,2} z) \wedge (y \in_{1,2} z))$

3. $(\forall x \forall y \forall z \forall t \in W(S))((\neg x \equiv_1 y) \wedge (x \in_{1,2} z) \wedge (y \in_{1,2} z) \wedge (x \in_{1,2} t) \wedge (y \in_{1,2} t) \Rightarrow (z \equiv_2 t))$

4. $(\forall x \exists y \exists z \in W(S))((\neg y \equiv_1 z) \wedge (y \in_{1,2} x) \wedge (z \in_{1,2} x))$

5. $(\forall x \exists y \in W(S))(\neg y \in_{1,2} x)$

6. $(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))(x \in_{1,3} t \wedge y \in_{1,3} t \wedge z \in_{1,3} t)$

7. $(\forall x \forall y \forall z \in W(S))(\forall u \forall v \in W(S))((x \in_{1,3} u) \wedge (y \in_{1,3} u) \wedge (z \in_{1,3} u) \wedge (x \in_{1,3} v) \wedge (y \in_{1,3} v) \wedge (z \in_{1,3} v) \wedge (\forall l \in W(S))((\neg x \in_{1,2} l) \vee (\neg y \in_{1,2} l) \vee (\neg z \in_{1,2} l)) \Rightarrow (u \equiv_3 v))$

8. $(\forall x \exists y \in W(S))(y \in_{1,3} x)$

9. $(\forall x \exists y \in W(S))(\neg y \in_{1,3} x)$

10. $(\forall x \forall y \forall z \forall t \in W(S))((\neg x \equiv_1 y) \wedge (x \in_{1,2} z) \wedge (y \in_{1,2} z) \wedge (x \in_{1,3} t) \wedge (y \in_{1,3} t) \Rightarrow (z \in_{2,3} t))$

11. $(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))((z \in_{1,3} x) \wedge (z \in_{1,3} y) \Rightarrow (\neg t \equiv_1 z) \wedge (t \in_{1,3} x) \wedge (t \in_{1,3} y))$

Proof. It is a simple check with applying the definitions and use the lemma 2.2. □

Thus the lemma 2.3 gives us the confidence to introduce the next abstract definition od the notion of *structure of incidence*.

Definition 2.6. *We say that the structure* $\underline{\underline{W}} = (W, \equiv_1, \equiv_2, \equiv_3)$ *is an incidence structure if the set is a non empty set* $W \neq \emptyset$ *and the relations* $\equiv_1, \equiv_2, \equiv_3$ *are relations of equivalence. and they follow all the conditions from the* lemma 2.3. *where* $\in_{1,2} = \equiv_1 \circ \equiv_2, \in_{1,3} \equiv_1 \circ \equiv_3, \in_{2,3} = \equiv_2 \circ \equiv_3$.

Now we turn the class of the structures of incidences into a category with introducing the notion of the homomorphism.

Definition 2.7. *Let* $\underline{\underline{W}} = (W, \equiv_1, \equiv_2, \equiv_3)$ *and* $\underline{\underline{W'}} = (W', \equiv'_1, \equiv'_2, \equiv'_3)$ *be two structures of incidence, and* $f : W \to W'$ *is a function. We says that* f *is a* homomorphism *if it follows the next condition:*

1. $(\forall x \in W)(\forall y \in W)(x \equiv_i y \Rightarrow f(x) \equiv'_i f(y))$ *for each* $i = 1, 2, 3$

   We says that f *is a* isomorphism *if it follows the next conditions:*

2. *f is a bijection.*

3. $(\forall x \in W)(\forall y \in W)(f(x) \equiv_i' f(y) \Rightarrow x \equiv_i y)$ *for each* $i = 1, 2, 3$

The category of the structures of incidences we note with Φ_i. And the category of incidence spaces we note with Σ_i.

2.3. Equivalence between the categories of the incidence spaces and the structures of incidences

Similarly to the functional correspondence from definition 2.4 which for each incidence space finds a structure of incidence, we make another functional correspondence which for each structure of incidence finds an incidence space.

Let $\underline{W} = (W, \equiv_1, \equiv_2, \equiv_3)$ be a structure of incidence then we can split the set W into equivalence classes with each equivalence relations.

Definition 2.8. *For each* $x \in W$ *we define the classes:*

$$|x|_1 = \{y \in W | x \equiv_1 y\}, |x|_2 = \{y \in W | x \equiv_2 y\}, |x|_3 = \{y \in W | x \equiv_3 y\}$$

These classes are equivalence classes.

Definition 2.9. *Let* $\underline{W} = (W, \equiv_1, \equiv_2, \equiv_3)$ *be a structure of incidence then we define* $S(\underline{W})$ *to be the structure* $S(\underline{W}) = (Po(\underline{W}), Li(\underline{W}), Pl(\underline{W}), \varepsilon_{1,2}(\underline{W}), \varepsilon_{1,3}(\underline{W}))$ *where:*

$$Po(\underline{W}) = W/\equiv_1 = \{|x|_1 | x \in W\}$$
$$Li(\underline{W}) = W/\equiv_2 = \{|x|_2 | x \in W\}$$
$$Pl(\underline{W}) = W/\equiv_3 = \{|x|_3 | x \in W\}$$
$$\varepsilon_{1,2}(\underline{W}) = \{(|x|_1, |y|_2) | x \epsilon_{1,2} y\}$$
$$\varepsilon_{1,3}(\underline{W}) = \{(|x|_1, |y|_3) | x \epsilon_{1,3} y\}$$

Lemma 2.4. *The following conditions are true:*

1. *The definition of the relations* $\varepsilon_{1,2}(\underline{W})$ *and* $\varepsilon_{1,3}(\underline{W})$ *is correct.*
2. $Po(\underline{W}) \cap Li(\underline{W}) = \emptyset, Po(\underline{W}) \cap Pl(\underline{W}) = \emptyset, Li(\underline{W}) \cap Pl(\underline{W}) = \emptyset$

Proof. For 1 we have to proof that the relations $\varepsilon_{1,2}(\underline{W})$ and $\varepsilon_{1,3}(\underline{W})$ are independent from the concrete representatives of the equivalence classes.

From the definition 2.6 it follows that $\in_{1,2} = \equiv_1 \circ \equiv_2, \in_{1,3} \equiv_1 \circ \equiv_3$ and the relations $\equiv_1, \equiv_2, \equiv_3$ are relations of equivalence.

So let $x \epsilon_{1,2} y \wedge x \equiv_1 x' \wedge y \equiv_2 y'$. From $\in_{1,2} \equiv_1 \circ \equiv_2$ it follows that there is $z \in W$ such that $x \equiv_1 z \wedge z \equiv_2 y$. Because the $\equiv_1$ and $\equiv_2$ are equivalence relations it follows that $x' \equiv_1 z \wedge z \equiv_2 y'$, thus we conclude that $x' \epsilon_{1,2} y'$. The same way we can see that for any x, x', y, y' it is true $x \epsilon_{1,3} y \wedge x \equiv_1 x' \wedge y \equiv_3 y' \Rightarrow x' \epsilon_{1,3} y'$

We can prove 2 with accepting the opposite and proof the contradiction. Let us assume that there are $|x|_1, |x|_2, |y|_2, |y|_3$ equivalence classes such that $|x|_1 |y|_2$ or $|x|_1 = |y|_3$, or $|x|_2 = |y|_3$, and from that it follows — there is $x \in W$ such that $|x|_1 |x|_2$ or $|x|_1 = |x|_3$, or $|x|_2 = |x|_3$.

Let us assume that there is $x \in W$ such that $|x|_1 |x|_2$. From the definition 2.6 of the incidence structure the axiom (4) we know that $(\forall x \in W)(\exists y \in W)(\exists z \in W)((\neg y \equiv_1 z) \wedge y \in_{1,2} x \wedge z \in_{1,2} x)$. Applying that "axiom" for the x we find $y \in W, z \in W$ and $y \in_{1,2} x$ and $z \in_{1,2} x$ and $(\neg y \equiv_1 z)$. From the definition of $\in_{1,2}$ and $\in_{1,3}$, it follows that there are $u \in W$ and $v \in W$ such that $y \equiv_1 u \wedge u \equiv_2 x$ and $z \equiv_1 v \wedge v \equiv_2 x$. So $u \in |x|_2, v \in |x|_2$. Form $(\neg y \equiv_1 z) \wedge z \equiv_1 v \wedge y \equiv_1 u$ we conclude that $(\neg u \equiv_1 v)$. But if $|x|_1 = |x|_2$ and $u \in |x|_2, v \in |x|_2$ then $u \in |x|_1, v \in |x|_1$ and $u \equiv_1 v$ contradiction with $(\neg u \equiv_1 v)$. Finally we proved that for any $x \in W$ it is true that $|x|_1 \neq |x|_2$.

From the definition 2.6 axioms (8) and (11) it follows that:

$((\forall x \exists z \in W)(z \in_{1,3} x)) \wedge ((\forall x \forall x \forall z \in W)(\exists t \in W)(z \in_{1,3} x \wedge z \in_{1,3} x \Rightarrow (\neg t \equiv_1 z) \wedge t \in_{1,3} x \wedge t \in_{1,3} x))$ and from that we can conclude that for any "plane" there are 2 different "points" that belong to the "plane" $(\forall x \in W)(\exists z \in W)(\exists t \in W)((\neg t \equiv_1 z) \wedge z \in_{1,3} x \wedge t \in_{1,3} x)$. From that statement we can proof that $|x|_1 \neq |x|_3$ in the same way as for $|x|_1 \neq |x|_2$.

The proof of the4 statement $|x|_2 \neq |x|_3$ uses the proof of the statement $(\forall x \in W)(\exists y \in W)(\exists z \in W)((\neg y \equiv_3 z) \wedge x \in_{2,3} y \wedge x \in_{2,3} z)$ which speaks that for any "line" there are 2 different "planes" that contain the "line", next using the definition of $\in_{2,3}$ it follows that there are $u \in W$ and $v \in W$ such that $x \equiv_2 u \wedge u \equiv_3 y$ and $x \equiv_2 v \wedge v \equiv_3 z$, because $(\neg y \equiv_3 z)$ then $(\neg u \equiv_3 v)$, but because $u \equiv_2 x \equiv_2 v$ then $u \in |x|_2$ and $v \in |x|_2$ and if we assume that $|x|_2 = |x|_3$ then $u \in |x|_3$ and $v \in |x|_3$ and it follows the contradiction $u \equiv_3 v$ with $\neg u \equiv_3 v$, so $|x|_3 \neq |x|_2$.

The proof of the fact $(\forall x \in W)(\exists y \in W)(\exists z \in W)((\neg y \equiv_3 z) \wedge x \in_{2,3} y \wedge x \in_{2,3} z)$.

Let $x \in W$ is the "line". Using the definition 2.6 axiom (4) there are "points" $x_1 \in W$ and $x_2 \in W$ such that $(\neg x_1 \equiv_1 x_2) \wedge x_1 \in_{1,2} x \wedge x_2 \in_{1,2} x$. From axiom (6) there is $u \in W$ such that $x_1 \in_{1,3} u \wedge x_2 \in_{1,3} u$. For the "plane" u using axiom (9) we find a "point" $x_3 \in W$ such that $\neg x_3 \in_{1,3} u$. Using again the axiom (6) for the points x_1, x_2, x_3 there is a "plane" $v \in W$ which contains that points $x_1 \in_{1,3} v \wedge x_2 \in_{1,3} v \wedge x_3 \in_{1,3} v$. If we assume that $u \equiv_3 v$ then from $x_3 \in_{1,3} v$ it follows that there is $t \in W$ and $x_3 \equiv_1 t \wedge t \equiv_3 v$ and $v \equiv_3 u$, so there is $t \in W$ and $x_3 \equiv_1 t \wedge t \equiv_3 u$ which is $x_3 \in_{1,3} v$ contradiction with $\neg x_3 \in_{1,3} u$. So it is true that $\neg u \equiv_3 v$. And because both "planes" contains the "points" x_1, x_2 and $(\neg x_1 \equiv_1 x_2)$, and the "line" contains x_1, x_2 too, using axiom (10) we conclude that $x \in_{2,3} u \wedge x \in_{2,3} v$. Thus we proved that $(\forall x \in W)(\exists u \in W)(\exists v \in W)((\neg u \equiv_3 v) \wedge x \in_{2,3} u \wedge x \in_{2,3} v)$.

Note! The statement $(\forall x \in W)(\exists y \in W)(\exists z \in W)((\neg y \equiv_2 z) \wedge y \in_{2,3} x \wedge z \in_{2,3} x)$ which says that for any "plane" there is 2 different "lines" that lays onto that "plane" has more complex proof.

Finally we proved that for any $x \in W$ it is true that: $|x|_1 \neq |x|_2$ and $|x|_1 \neq |x|_3$, and $|x|_3 \neq |x|_2$. Which proof that the sets $W/\equiv_1, W/\equiv_2, W/\equiv_3$ have no common elements. □

Lemma 2.5. *If $\underline{W}$ is a structure of incidence then $S(\underline{W})$ defined with the definition 2.9 is an incidence space.*

Proof. To prove that, we need to check each of the statements from definition of incidence space 2.1 for $Po(\underline{W}), Li(\underline{W}), Pl(\underline{W}), \varepsilon_{1,2}(\underline{W}), \varepsilon_{1,3}(\underline{W})$.

From the previous lemma 2.4 it follows that the defined sets and relations at the definition 2.9 are correct. Also no "point" is a "line", no "line" is a "plane" and no "plane" is a "point". So the statement $Po(\underline{W}) \cap Li(\underline{W}) = \emptyset, Po(\underline{W}) \cap Pl(\underline{W}) = \emptyset, Li(\underline{W}) \cap Pl(\underline{W}) = \emptyset$ is exactly (1) from the definition 2.1 of the incidence space. The rest of the statements from definition 2.1 - the statements from (2) until (12), are checked easily with using the corresponding "axiom" from (1) until (11) from the definition of the structure of incidence - the definition 2.6. □

Lemma 2.6. $|x|_2 \varepsilon_{2,3}(\underline{W})|y|_3 \Leftrightarrow x \in_{2,3} y$

Proof. According to the definitions 2.2 and 2.9 the following equivalences are true: $|x|_2\varepsilon_{2,3}(\underline{W})|y|_3 \Leftrightarrow (\forall Z \in Po(\underline{W}))(Z\varepsilon_{1,2}(\underline{W})|x|_2 \rightarrow Z\varepsilon_{1,3}(\underline{W})|y|_3) \Leftrightarrow (\forall z \in W)(|z|_1\varepsilon_{1,2}(\underline{W})|x|_2 \rightarrow |z|_1\varepsilon_{1,3}(\underline{W})|y|_3) \Leftrightarrow (\forall z \in W)(z \in_{1,2} x \rightarrow z \in_{1,3} y)$

From the incidence structure property (***) it follows that $(\forall z \in W)(z \in_{1,2} x \rightarrow z \in_{1,3} y) \Rightarrow x \in_{2,3} y$

Let for any $z \in W$ be true that $z \in_{1,2} x$ and from $x \in_{2,3} y$ according to the incidence structure property (**) it follows $(\exists t \in W)(z \equiv_1 t \wedge x \equiv_2 t \wedge y \equiv_3 t)$. From lemma 2.1 from $z \equiv_1 t \wedge y \equiv_3 t$ it follows $z \in_{1,3} y$. So we proof $x \in_{2,3} y \Rightarrow (\forall z \in W)(z \in_{1,2} x \rightarrow z \in_{1,3} y)$. □

Theorem 2.1. Representing structures of incidences as spaces of incidences.

Let $\underline{W} = (W, \equiv_1, \equiv_2, \equiv_3)$ be a structure of incidence.

Let $S(\underline{W}) = (Po(\underline{W}), Li(\underline{W}), Pl(\underline{W}), \varepsilon_{1,2}(\underline{W}), \varepsilon_{1,3}(\underline{W}))$ be an incidence space over $S(\underline{W})$.

Let $\underline{W}(S(\underline{W})) = (W', \equiv_1', \equiv_2', \equiv_3)'$ be a structure of incidence over $S(\underline{W})$.

Then there is an isomorphism from the structure of incidence $\underline{W}$ to the structure of incidence $\underline{W}(S(\underline{W}))$. The structures of incidences $\underline{W}$ and $\underline{W}(S(\underline{W}))$ are isomorphic.

Proof. We define function $f : W \rightarrow W'$ this way $f(x)(|x|_1, |x|_2, |x|_3)$ for every $x \in W$. First we must check that the definition is correct. For every $x \in W$ we must check that $(|x|_1, |x|_2, |x|_3) \in W'$. So let $x \in W$, and from $\equiv_{1,2,3}$ equivalence relations then: $x \equiv_1 x \wedge x \equiv_2 x$ and $x \equiv_2 x \wedge x \equiv_3 x$. From the definition of 2.6 $x \in_{1,2} x$ and $x \in_{2,3} x$. From the definition 2.9 we conclude that $|x|\varepsilon_{1,2}(\underline{W})|x|$ and from lemma 2.6 we conclude that $|x|\varepsilon_{2,3}(\underline{W})|x|$, so the triple $(|x|_1, |x|_2, |x|_3) \in W'$ according to the definition 2.9.

Check for that f is a homomorphism is also easy:

$x \equiv_{1,2,3} y \leftrightarrow |x|_{1,2,3} = |y|_{1,2,3} \leftrightarrow (|x|_1, |x|_2, |x|_3) \equiv_{1,2,3} (|y|_1, |y|_2, |y|_3) \leftrightarrow f(x) \equiv_{1,2,3} f(y)$, so f is a homomorphism.

If we finally proof that f is a bijection then we can conclude f is an isomorphism.

First we proof that f is a surjective. For any element of $(|x|_1, |y|_2, |z|_3) \in W'$ there is $t \in W$ such that $f(t) = (|x|_1, |y|_2, |z|_3)$. Let $(|x|_1, |y|_2, |z|_3) \in W'$. From the definition 2.4 we know that $|x|_1\varepsilon_{1,2}(\underline{W})|y|_2$ and $|y|_2\varepsilon_{2,3}(\underline{W})|z|_3$. From the definition 2.9 and from lemma 2.6 we conclude that $x \in 1,2y \wedge y \in 2,3z$. Now we apply the statement (**) from the definition of the structure of incidence 2.6, and we find $t \in W$ such that $x \equiv_1 t \wedge y \equiv_2 t \wedge z \equiv_3 t$, and for that t it is true that $(|t|_1, |t|_2, |t|_3) = (|x|_1, |y|_2, |z|_3)$. So f is a surjective.

Let's now proof that f is an injective function. So let $x, y \in W$ and $f(x) = f(y)$. From the definition of f we have that $(|x|_1, |x|_2, |x|_3) = (|y|_1, |y|_2, |y|_3)$ which means that the equivalence classes of x and y for each relation are the same: $|x|_1 = |y|_1$, $|x|_2 = |y|_2$ and $|x|_3 = |y|_3$. And we conclude that the elements x and y are equivalent by each relation: $x \equiv_1 y \wedge x \equiv_2 y \wedge x \equiv_3 y$. Now we use the statement (*) from the definition 2.6 and conclude that x and y are equal, $x = y$, which proofs that f is an injective function.

The defined here function f preserves the relations, and also is a bijection from W to W' So f is an isomorphism. □

Theorem 2.2. Representing spaces of incidences as structures of incidences.

Let $S = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ is an incidence space.

Let $\underline{W}(S) = (W, \equiv_1, \equiv_2, \equiv_3)$ be the structure of incidence over the space S.

Let $S(\underline{W}(S)) = (Po(\underline{W}), Li(\underline{W}), Pl(\underline{W}), \varepsilon_{1,2}(\underline{W}), \varepsilon_{1,3}(\underline{W}))$ be the incidence space over the structure of incidence $\underline{W}(S)$.

Then there is an isomorphism from the incidence space S to the incidence space $S(\underline{W}(S))$. The incidence spaces incidences S and $S(\underline{W}(S))$ are isomorphic.

Proof. We know from the definition 2.9 that:

$$Po(\underline{W}(S)) = W(S)/\equiv_1 = \{|(X,y,\alpha)|_1 | (X,y,\alpha) \in W(S)\}$$

$$Li(\underline{W}(S)) = W(S)/\equiv_2 = \{|(X,y,\alpha)|_2 | (X,y,\alpha) \in W(S)\}$$

$$Pl(\underline{W}(S)) = W(S)/\equiv_3 = \{|(X,y,\alpha)|_3 | (X,y,\alpha) \in W(S)\}$$

$$\varepsilon_{1,2}(\underline{W}(S)) = \{(|(X',y',\alpha')|_1, |(X'',y'',\alpha'')|_2) | (X',y',\alpha')\epsilon_{1,2}(X'',y'',\alpha'')\}$$

$$\varepsilon_{1,3}(\underline{W}(S)) = \{(|(X',y',\alpha')|_1, |(X'',y'',\alpha'')|_3) | (X',y',\alpha')\epsilon_{1,3}(X'',y'',\alpha'')\}$$

And from definition 2.4 we know the equivalences:

$(X',y',\alpha')\epsilon_{1,2}(X'',y'',\alpha'') \leftrightarrow X'\varepsilon_{1,2}y''$ and $(X',y',\alpha')\epsilon_{1,3}(X'',y'',\alpha'') \leftrightarrow X'\varepsilon_{1,3}\alpha''$. This way we see that the relations $\varepsilon_{1,2}(\underline{W}(S)), \varepsilon_{1,3}(\underline{W}(S))$ follows the next conditions:

$$|(X',y',\alpha')|_1\varepsilon_{1,2}(\underline{W}(S))|(X'',y'',\alpha'')|_2 \leftrightarrow X'\varepsilon_{1,2}y''$$

$$|(X',y',\alpha')|_1\varepsilon_{1,3}(\underline{W}(S))|(X'',y'',\alpha'')|_2 \leftrightarrow X'\varepsilon_{1,3}\alpha''$$

Now let's define the function $f : Po \cup Li \cup Pl \to Po(\underline{W}(S)) \cup Li(\underline{W}(S)) \cup Pl(\underline{W}(S))$. For each point $X \in Po$ from lemma 2.2 statement (1) we know that there is a line incident with the point and a plane incident with the line, so there are $y \in Li, \alpha \in Pl$ such that $(X,y,\alpha) \in W(S)$. Then we define $f(X)|(X,y,\alpha)|_1$. For each line $y \in Li$ using lemma 2.2 statement (2) we know that there are incident a point and a plane, so the triple $(X,y,\alpha) \in W(S)$. Then we $f(y) = |(X,y,\alpha)|_2$. For each plane $\alpha \in Pl$ using using lemma 2.2 statement (2) there are a point and a line such that $(X,y,\alpha) \in W(S)$. Then we define $f(\alpha)|(X,y,\alpha)|_3$

f is a function. That means the result of f is independent of the exact choice of the fictive points, lines, and planes which were chosen to make a triple from $W(S)$. We shall prove it for points only for lines and planes it is the same. So let $X \in Po$ and let $y',y'' \in Li$ and $\alpha',\alpha'' \in Pl$ such that the triples: $(X,y',\alpha') \in W(S)$ and $(X,y'',\alpha'') \in W(S)$. It doesn't matter which one we choose for $f(X)$: $|(X,y',\alpha')|_1$ or $|(X,y'',\alpha'')|_1$, because $(X,y',\alpha') \equiv_1 (X,y'',\alpha'')$ then $|(X,y',\alpha')|_1|(X,y'',\alpha'')|_1$.

We have to check that the incidence relations are preserved by the function f.

So let $X \in Po, y \in Li, \alpha \in Pl$ and $f(X)|(X,y',\alpha')|_1$, and $f(y)|(X'',y,\alpha'')|_2$, and $f(\alpha) = |(X''',y''',\alpha)|_3$, where $X'',X''' \in Po$, $y',y''' \in Li$, $\alpha',\alpha'' \in Pl$. Next we use the already proven statements about the relations $\varepsilon_{1,2}(\underline{W}(S))$ and $\varepsilon_{1,3}(\underline{W}(S))$:

$f(X)\varepsilon_{1,2}(\underline{W}(S))f(y) \leftrightarrow |(X,y',\alpha')|_1\varepsilon_{1,2}(\underline{W}(S))|(X'',y,\alpha'')|_2 \leftrightarrow X\varepsilon_{1,2}y$.

$f(X)\varepsilon_{1,3}(\underline{W}(S))f(\alpha) \leftrightarrow |(X,y',\alpha')|_1\varepsilon_{1,3}(\underline{W}(S))|(X''',y''',\alpha)|_3 \leftrightarrow X\varepsilon_{1,3}\alpha$.

It is easy to check that $f \restriction Po$ is a bijection from Po to $Po(\underline{W}(S))$.

$f \restriction Po$ is an injective function. Let $X_1, X_2 \in Po$, and $f(X_1) = f(X_2)$. Let $f(X_1)|(X_1,y',\alpha')|_1$ and $f(X_2) = |(X_2,y'',\alpha'')|_1$, then $|(X_1,y',\alpha')|_1|(X_2,y'',\alpha'')|_1$, and from that we conclude that the triples are equivalent with $\equiv_1$, $(X_2,y'',\alpha'') \equiv_1 (X_2,y'',\alpha'')$, from the definition 2.4 we know that $X_1 = X_2$. So $f \restriction Po$ is an injection.

$f \restriction Po$ is an surjective function. Let $\widetilde{X} \in Po(\underline{W}(S))$, so $\widetilde{X}$ is an equivalence class generated by the triple: $\widetilde{X}|(X,y,\alpha)|_1$, then $f(X) = |(X,y,\alpha)|_1\widetilde{X}$. So $f \restriction Po$ is surjection.

The same way we proof that $f \restriction Li$ and $f \restriction Pl$ are surjective and injective functions. And this way we proof that f is bijection, which with the preserving the relations turns f into isomorphism. □

If we consider $S(\underline{W})$ and $\underline{W}(S)$ as functionals that convert an incidence spaces into structures of incidences, and structures of incidences into incidence spaces, then the conclusion of these 2 theorems is that the 2 functionals behaves as the opposite ones — if S is an incidence space then $S(\underline{W}(S))$ is isomorphic to S, and if $\underline{W}$ is a structure of incidence then $\underline{W}(S(\underline{W}))$ is isomorphic to $\underline{W}$. To proof the equivalence of the category of the incidence spaces and the category of the structures of incidences, we need to turn that functionals into *functors* . To do that we have to extend the functional $S(\underline{W})$ over the class of homomorphisms between structures of incidences and the functional $\underline{W}(S)$ over the class of homomorphisms between incidence spaces.

Definition 2.10. *Let S and S' are incidence spaces and let $\underline{W}(S) = (W, \equiv_1, \equiv_2, \equiv_3)$ and $\underline{W}(S') = (W', \equiv_1', \equiv_2', \equiv_3')$ are the corresponding structures of incidences. For any homomorphism f from S to S', we define the function $\underline{W}(f) : W \to W'$ this way:*

For any triple $(X, x, \alpha) \in W$ we define $\underline{W}(f)((X, x, \alpha)) = (f(X), f(x), f(\alpha))$.

Definition 2.11. *Let $\underline{W}(S) = (W, \equiv_1, \equiv_2, \equiv_3)$ and $\underline{W}(S') = (W', \equiv_1', \equiv_2', \equiv_3')$ are structures of incidences and let $S(\underline{W}) = (Po, Li, Pl, \varepsilon_{1,2}, \varepsilon_{1,3})$ and $S(\underline{W}') = (Po', Li', Pl', \varepsilon_{1,2}', \varepsilon_{1,3}')$ are the corresponding incidence spaces. For any homomorphism f from $\underline{W}$ to $\underline{W}'$, we define the function $S(f) : Po \cup Li \cup Pl \to Po' \cup Li' \cup Pl'$ this way:*

For any $x \in W$ we define $S(f)(|x|_1) = |f(x)|_1$ and $S(f)(|x|_2) = |f(x)|_2$, and $S(f)(|x|_3) = |f(x)|_3$.

Lemma 2.7. *The definitions above 2.10 and 2.11 are correct.*

Proof. For the definition 2.10 it uses the properties of the homomorphism and the definition of $\underline{W}(S)$, see definition 2.4. For the definition 2.11 we use the definition 2.9 that $Po = W/\equiv_1$, $LiW/\equiv_2$, $Pl = W/\equiv_3$ and $Po'W'/\equiv_1'$, $Li'W'/\equiv_2'$, $Pl'W'/\equiv_3'$ and the properties of homomorphisms. The properties of the homomorphism f is used to proof that the definition $S(f)(|x|_i) = |f(x)|_i$ is independent from the concrete example $x \in W$. □

Theorem 2.3. $\underline{W}$ *is a functor.*

Proof. First we shall proof that the defined above $W(f)$ is a homomorphism from $\underline{W}(S)$ to $\underline{W}(S')$. Let $(X', x', \alpha') \in W$ and $(X'', x'', \alpha'') \in W'$, and $(X', x', \alpha') \equiv_1 (X'', x'', \alpha'')$, then from the definition 2.4 we know that the points are the same $X'X''$, and $f(X') = f(X'')$, again with the definition 2.4 we conclude that $(f(X'), f(x'), f(\alpha')) \equiv_1 (f(X''), f(x''), f(\alpha''))$. In the same way we proof that the function f preserves the relations $\equiv_2$ and $\equiv_3$.

The next question is about the composition of homomorphisms. Let f is a homomorphisms from S to S' and g is a homomorphism from S' to S''. Then $g \circ f$ is a homomorphism from S to S''. We must proof that $W(g \circ f) = W(g) \circ W(f)$.

$W(g \circ f)((X, x, \alpha)) = (W(g \circ f)(X), W(g \circ f)(x), W(g \circ f)(\alpha))$
$= (W(g)(W(f)(X)), W(g)(W(f)(x)), W(g)(W(f)(\alpha)))$
$= W(g)((W(f)(X), W(f)(x), W(f)(\alpha)))$
$= W(g)(W(f)((X, x, \alpha))) = (W(g) \circ W(f))((X, x, \alpha))$. □

Theorem 2.4. *S is a functor.*

Proof. We must check that $S(f)$ is a homomorphism from $S(\underline{W})$ to $S(\underline{W}')$. Let $|x|_1 \in Po(\underline{W})$, $|y|_2 \in Li(\underline{W})$, and $|x|_1\varepsilon_{1,2}(\underline{W})|y|_2$. From definition 2.9 we know the equivalence $|x|_1\varepsilon_{1,2}(\underline{W})|y|_2 \leftrightarrow x \in_{1,2} y$. From f homomorphism and the definition of $\in_{1,2}$ we conclude that $f(x) \in'_{1,2} f(y)$. Again applying the definition 2.9 we proof that $|f(x)|_1\varepsilon'_{1,2}(\underline{W}')|f(y)|_2$, which is $S(f)(|x|_1)\varepsilon'_{1,2}(\underline{W}')S(f)(|y|_2)$. In the same way from $|x|_1\varepsilon_{1,3}(\underline{W})|z|_3$ we proof that $S(f)(|x|_1)\varepsilon'_{1,3}(\underline{W}')S(f)(|z|_3)$. So $S(f)$ is a homomorphism.

Let f is a homomorphism from $\underline{W}$ to $\underline{W}'$, and let g is a homomorphism from $\underline{W}'$ to $\underline{W}''$. Then Then $g \circ f$ is a homomorphism from $\underline{W}$ to $\underline{W}''$. Let's check that $S(g \circ f)S(g) \circ S(f)$. Let $|x|_{1,2,3} \in Po(\underline{W}), Li(\underline{W}), Pl(\underline{W})$, then $S(g \circ f)(|x|_{1,2,3}) = |(g \circ f)(x))|_{1,2,3} = |g(f(x))|_{1,2,3}S(g)(|f(x)|_{1,2,3}) = S(g)(S(f)(|x|_{1,2,3}))$. □

With these theorems 2.1, 2.2, 2.3 and 2.4 we conclude that the category of incidence spaces is equivalent to the category of the structures of incidence.

3. MODAL LOGIC FOR INCIDENCE GEOMETRY

3.1. MODAL LANGUAGE

The language is a modal language with 4 different modal operators. 3 of the operators $[\equiv_1], [\equiv_2], [\equiv_3]$ are interpreted with equivalence relations $\equiv_1, \equiv_2, \equiv_3$, and the 4th one $[\neq]$ with the relation difference $\neq$. The language is consisted also of the set of propositional variables $\{p_1, p_2, p_3, ...\}$ and logical operators $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.

3.2. SEMANTICS

The semantics is the Kripke semantics over the frames $\underline{W} = (W, \equiv_1, \equiv_2, \equiv_3, \neq)$ where $W \neq \emptyset$ and $\equiv_1, \equiv_2, \equiv_3, \neq$ are binary relations over W. The relation $\neq$ has a special meaning *difference* between 2 elements of W. The elements of W are called *worlds* .

If we assign binary values to the propositional variables, called valuation, we may assign a truth values to all modal formulas. If v is a valuation of propositional variables and $\underline{W}$ is a frame then $\underline{M} = (\underline{W}, v)$ is called a model for that modal logic.

If we have a model $\underline{M} = (\underline{W}, v)$ we can extend the truth value over all formulas using next definition.

Definition 3.1. *The truth value of a modal formula.*

$x \models_v A$ *means the formula A is* true *in the world x according to the valuation v.*

$x \not\models_v A$ *means the formula A is* false *in the world x according to the valuation v.*

1. $x \models_v p \leftrightarrow v(x,p) = true$. *for any* $p \in \{p_1, p_2, ...\}$ *propositional variable.*

2. $x \models_v \neg A \leftrightarrow x \not\models_v A$.

3. $x \models_v (A \wedge B) \leftrightarrow x \models_v A$ *and* $x \models_v B$.

4. $x \models_v [R]A \leftrightarrow (\forall y \in W)(xRy \rightarrow y \models_v A)$. *where* $R \in \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.

Definition 3.2. *We say that* A *is true into the frame* $\underline{W}$. $\underline{W} \models A$ *if for each valuation* v *and for each world* $x \in W$ *it is satisfied* $x \models_v A$.

We say that A *is true into a class* Σ *of frames if* A *is true into each of frames from that class.*

If Σ *is a class of frames then the set of all formulas* A *which are true at that class* Σ *forms the logic over that class, and it is noted with* $L(\Sigma)$.

Frames which are interesting for us are frames with $\equiv_1, \equiv_2, \equiv_3$ being equivalence relations and the relation $\neq$ the difference relation. So the class of such frames we note with Σ_0. And the logic over that class $L(\Sigma_0)$ is the minimal logic L_0. First we make the axiomatization of that logic L_0. Next we proceed with axiomatization of the logic over the class of structure of incidences 2.6. That is an extension of the minimal one L_0 with adding axioms for base geometrical properties of the structure of incidences, see 2.6, and next we proof that each of those geometrical properties is a *canonical* property. We note the class of frames which are structures of incidences with Σ_{gsi}. And the geometrical logic is noted with $L(\Sigma_{gsi})$.

3.3. DEFINABLE MODALITIES IN Σ_0

Some relations has modal operators that are definable with these modal operators : $\equiv_1, \equiv_2, \equiv_3, \neq$.

1. Incidence Relations, Like $\in_{1,2}, \in_{1,3}, \in_{2,3}$ and $\in^{-1}_{1,2}, \in^{-1}_{1,3}, \in^{-1}_{2,3}$. The modal operators associated with these relations are $[\in_{1,2}], [\in_{1,3}], [\in_{2,3}], [\in^{-1}_{1,2}], [\in^{-1}_{1,3}], [\in^{-1}_{2,3}]$.

2. Universal Relation. Universal relation U means that every two world are in relation, it connects any with any. So $U = \{< x,y > | x \in W \wedge y \in W\}$. The modality for it is $\blacksquare$.

   The relations below uses the semantics attached with the structures of incidences.

3. Two lines has an incident point.

4. Two planes has an incident point.

5. Two planes has an incident line.

Lemma 3.1. *The modalities with* incidence relations $\in_{1,2}, \in_{1,3}, \in_{2,3}$ *and* $\in_{1,2}^{-1}, \in_{1,3}^{-1}, \in_{2,3}^{-1}$ *are definable as:*

$$[\in_{1,2}] = [\equiv_1][\equiv_2], [\in_{1,3}] = [\equiv_1][\equiv_3], [\in_{2,3}] = [\equiv_2][\equiv_3]$$

$$[\in_{1,2}^{-1}] = [\equiv_2][\equiv_1], [\in_{1,3}^{-1}] = [\equiv_3][\equiv_1], [\in_{2,3}^{-1}] = [\equiv_3][\equiv_2]$$

Lemma 3.2. *The modality with* universal relation *is expressible with:*

$$\blacksquare A = A \wedge [\neq]A$$

Lemma 3.3. *If* $\underline{W}$ *is a structure of incidence then the relations are expressed:*

$$\cap_2 = \equiv_2 \circ \equiv_1 \circ \equiv_2$$

$$\cap_{3,1} = \equiv_3 \circ \equiv_1 \circ \equiv_3, \cap_{3,2} = \equiv_3 \circ \equiv_2 \circ \equiv_3, \cap_{3,2} = \cap_{3,1}$$

Lemma 3.4. *The relations about two lines has a common point and two planes has a common point, line are definable:*

$$[\cap_2] = [\equiv_2][\equiv_1][\equiv_2]$$

$$[\cap_{3,1}] = [\equiv_3][\equiv_1][\equiv_3], [\cap_{3,2}] = [\equiv_3][\equiv_2][\equiv_3]$$

3.4. AXIOMATIZATION FOR THE MINIMAL LOGIC $L(\Sigma_0)$

Axioms

1. All Propositional Tautologies.
2. $[R](A \Rightarrow B) \Rightarrow ([R]A \Rightarrow [R]B)$, where $R \in \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.
3. $S5$ axioms for modalities $[\equiv_1], [\equiv_2], [\equiv_3]$.
   $[\equiv_i]A \Rightarrow A$ noted with (T_i).
   $<\equiv_i> [\equiv_i]A \Rightarrow A$ noted with (B_i).
   $[\equiv_i]A \Rightarrow [\equiv_i][\equiv_i]A$ noted with (4_i).
4. $<\neq> [\neq]A \Rightarrow A$ noted with $(B_{\neq})$. The relation $\neq$ is a symmetric.
5. $<\neq><\neq> A \Rightarrow (A \vee <\neq> A)$.
6. $\blacksquare A \Rightarrow [\equiv_i]A$.

Deductive Rules

1. Modus Ponens (MP) $A, A \Rightarrow B \vdash B$.

2. Normality (N_R) $A \vdash [R]B$, where $R \in \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.

3. Irreflexivity (Irr) $(p \wedge [\neq]\neg p) \Rightarrow A$, the variable p does not enter $A \vdash A$.

Definition 3.3. *Formal proof with the deductive system we call any sequence of formulas each of them could be a variant of one of the axiom schemas or produced by some rule from the previous formulas.*

$\vdash A$ There is a formal proof of A without using the rule (Irr).
$\vdash_{Irr} A$ There is a formal proof of A with using the rule (Irr).

Lemma 3.5. $\vdash \blacksquare A \Rightarrow A, \vdash \blacklozenge \blacksquare A \Rightarrow A, \vdash \blacksquare A \Rightarrow \blacksquare\blacksquare A$

To proof the completeness we will use another rule for irreflexivity which will offer simpler completeness proof, and next the deductive equivalence of the two rules will proof completeness of the logic $L(\Sigma_0)$.

3.5. DIFFERENT IRREFLEXIVITY RULES AND THEIR DEDUCTIVE EQUIVALENCE

The all mentioned deductive systems will contain the axioms 1 - 6 and the rules MP and $N_{\equiv_1, \equiv_2, \equiv_3, \neq}$, they will only differ with the irreflexivity rule.

The new infinite irreflexivity rule is:

(Irr^*) $(p \wedge [\neq]\neg p) \Rightarrow A$, for each variable $p \vdash^* A$.

This rule makes the ordinary definition of formal proof not appropriate and requires a new definition. The set of axioms 1 - 6 we note with A_0.

Definition 3.4. *Infinite formal proof of the formula ϕ is the ordered pair (Γ, ρ), where Γ is a tree with a* finite *path from the root to any leaf. and ρ is the correspondence between each tree node and a formula from the modal language. For Γ and ρ it is true that:*

1. *if v is a leaf from the tree then $\rho(v) \in A_0$*

2. *if v is not a leaf then $\rho(v)$ is a formula which is a conclusion of some of the rules.*

3. *if v is the root of the tree then $\rho(v) = \phi$.*

We can note such infinite proof of ϕ with $\rhd \phi$. The "triangle" sign symbolizes the infinite tree with finite path to the leaves. if the rule (Irr) is used then we use $\rhd \phi$, else if the rule (Irr^*) is used we use $\rhd^* \phi$. So the formula ϕ is proved with (Irr), $\vdash_{Irr} \phi$ if and only if there is $\rhd \phi$. Also ϕ is proved with (Irr^*), $\vdash_{Irr^*} \phi$ if and only if $\rhd^* \phi$.

Lemma 3.6. $\vdash_{Irr} \phi$ *if and only if* $\vdash_{Irr^*} \phi$.

Proof. If $\vdash_{Irr} \phi$ then there is a infinite proof $\rhd \phi$. Next with induction over the max tree path length we can create infinite proof $\rhd^* \phi$. The only interesting case is when ϕ is a conclusion of irreflexivity rule. Let the proof looks like $\rhd ((p\wedge[\neq]\neg p) \Rightarrow \phi) \vdash \phi$, where the variable p does not enter ϕ. Then everywhere inside the infinite proof $\rhd ((p\wedge[\neq]\neg p) \Rightarrow \phi)$ we can replace the variable p with any other variable q, so the result will be the infinite proof $\rhd ((q\wedge[\neq]\neg q) \Rightarrow \phi)$, ϕ remains unchanged because p does not enter ϕ. So for each infinite proof by induction assumption we have $\rhd^* ((q\wedge[\neq]\neg q) \Rightarrow \phi)$, for each variable q. Now applying of the infinite rule Irr^* we construct the infinite proof $\rhd^* \phi$.

In the other direction if we have the infinite proof $\rhd^* \phi$, again induction over the height of the tree, also the interesting case is with ϕ is being conclusion of the infinite rule Irr^*. The proof for ϕ looks like: $\rhd_1^* ((q_1\wedge[\neq]\neg q_1) \Rightarrow \phi), ..., \rhd_n^* ((q_n\wedge[\neq]\neg q_n) \Rightarrow \phi), ... \vdash \phi$. Because ϕ has a finite number of variables we can choose one variable p which does not enter ϕ and for which there is a proof $\rhd_n^* ((p\wedge[\neq]\neg p) \Rightarrow \phi)$, now applying the inductional assumption and applying the finite Irr rule we receive the proof $\rhd^* \phi$.

The obvious observation that if (Γ, ρ) is an infinite proof but if it uses only finite rules then it is equivalent to the ordinary finite proof. □

The conclusion of this lemma 3.6 is that the formal systems with the infinite rule $Irr*$ is deductive equivalent with the formal system with the finite rule Irr.

Because all our relations are symmetric for all the modalities $[\equiv_1], [\equiv_2], [\equiv_3], [\neq]$, there are the axioms for symmetric relation: $< R > [R]\phi \Rightarrow \phi$.

Lemma 3.7. *(Rijke [5]) If for one modal operator we have the symmetric axiom* $< R > [R]\phi \Rightarrow \phi$. *then:* $\vdash \phi \Rightarrow [R]\psi$ *if and only if* $\vdash < R > \phi \Rightarrow \psi$

Proof. It is used contraposition, the normality rule N_R, the axiom 2 $[R](\phi \Rightarrow \psi) \Rightarrow ([R]\phi \Rightarrow [R]\psi)$, and the above axiom B_R $< R > [R]\phi \Rightarrow \phi$, and propositional tautologies.

Let $\vdash < R > \phi \Rightarrow \psi$, then using N_R, we have $\vdash [R](< R > \phi \Rightarrow \psi)$, using axiom 2 and MP, we have $\vdash [R] < R > \phi \Rightarrow [R]\psi$. Now with contraposition we have $\vdash \neg[R]\psi \Rightarrow \neg[R] < R > \phi$, which is the same as $\vdash < R > \neg\psi \Rightarrow < R > [R]\neg\phi$. Now using the B_R axiom for $\neg\phi$, $< R > [R]\neg\phi \Rightarrow \neg\phi$ and tautology we have that $\vdash < R > \neg\psi \Rightarrow \neg\phi$, contraposition, $\vdash \phi \Rightarrow [R]\psi$. □

This lemma gives us the opportunity to use an infinite many rules instead of a single irreflexivity rule:

Definition 3.5. *Long Irreflexive Rules. For each natural number* $n \geqq 0$:
$(Adm_0 Irr^*)$ *is* (Irr^*).
$(Adm_n Irr^*)$ $A_1 \Rightarrow [R_1](A_2 \Rightarrow [R_2](A_3 ... \Rightarrow [R_n]((p\wedge[\neq]\neg p) \Rightarrow A)...))$. *for each variable* $p \vdash A_1 \Rightarrow [R_1](A_2 \Rightarrow [R_2](A_3 \Rightarrow ... \Rightarrow [R_n](A)...))$, *where* $n > 0$, $\{R_1, R_2, ...R_n\} \subseteq \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.

Lemma 3.8. *If into the proof of the formula ϕ the rule Adm_nIrr^* is used for some n, then that proof can be reworked into a proof of ϕ with the rule Adm_nIrr^* eliminated and replaced with Irr^*.*

Proof. Induction over n can show the elimination and the replacement of the rule Adm_nIrr^* with the rule Irr^*, also we choose to eliminate the Adm_nIrr^* rule closest to the leaves of the infinite proof.

If $n = 0$ then Adm_nIrr^* is Irr^*.

If $n = 1$ then Adm_1Irr^* is : $A_1 \Rightarrow [R_1]((p\wedge[\neq]\neg p) \Rightarrow A)$, for each $p \vdash A_1 \Rightarrow [R_1]A$. Because $\vdash_{Irr^*} A_1 \Rightarrow [R_1]((p\wedge[\neq]\neg p) \Rightarrow A)$, for each p, and using the lemma 3.7 $\vdash_{Irr^*} < R_1 > A_1 \Rightarrow ((p\wedge[\neq]\neg p) \Rightarrow A)$, according to the tautology : $\vdash_{Irr^*} ((< R_1 > A_1\wedge(p\wedge[\neq]\neg p)) \Rightarrow A)$, and $\vdash_{Irr^*} (p\wedge[\neq]\neg p) \Rightarrow (< R_1 > A_1 \Rightarrow A))$, for each p, then applying the rule Irr^*, we get $\vdash_{Irr^*} < R_1 > A_1 \Rightarrow A$, again using the lemma 3.7 we proof that $\vdash_{Irr^*} A_1 \Rightarrow [R_1]A$.

Let's for some n the assumption is true.

Let's have the occurrence $Adm_{n+1}Irr^*$, the closest to the tree-proof leaves, and it is look like:

$(A_1 \Rightarrow [R_1](A_2 \Rightarrow [R_2](A_3 \Rightarrow ... \Rightarrow [R_{n+1}]((p\wedge[\neq]\neg p) \Rightarrow A)...)))$, for each variable $p \vdash (A_1 \Rightarrow [R_1](A_2 \Rightarrow [R_2](A_3 \Rightarrow ... \Rightarrow [R_{n+1}](A)...)))$.

Let's note with $\psi(p) = [R_2](A_3 \Rightarrow ... \Rightarrow [R_{n+1}]((p\wedge[\neq]\neg p) \Rightarrow A)...))$, for each p. Let's note with $\chi = [R_2](A_3 \Rightarrow ... \Rightarrow [R_{n+1}](A)...))$. Now the rule is written as : $A_1 \Rightarrow [R_1](A_2 \Rightarrow \psi(p))$, for each $p \vdash A_1 \Rightarrow [R_1](A_2 \Rightarrow \chi)$.

Because $\vdash_{Irr^*} A_1 \Rightarrow [R_1](A_2 \Rightarrow \psi(p))$ then using the lemma 3.7 we get that $\vdash_{Irr^*} < R_1 > A_1 \Rightarrow (A_2 \Rightarrow \psi(p))$, propositional tautology, $\vdash_{Irr^*} (< R_1 > A_1\wedge A_2) \Rightarrow \psi(p)$, for each p. Now we can apply the rule Adm_nIrr^* and also the inductive assumption, and the result is : $\vdash_{Irr^*} (< R_1 > A_1\wedge A_2) \Rightarrow \chi$, that formula is proved with Irr^* only. Using propositional tautology : $\vdash_{Irr^*} (< R_1 > A_1 \Rightarrow (A_2 \Rightarrow \chi))$. Again from the lemma 3.7 we receive $\vdash_{Irr^*} A_1 \Rightarrow [R_1](A_2 \Rightarrow \chi)$. This shows how to eliminate the $Adm_{n+1}Irr^*$ rule. □

Theorem 3.1. *The rules Irr, Irr^*, and the set Adm_nIrr^* of rules are deductive equivalent:* $\{\phi | \vdash_{Irr} \phi\} = \{\phi | \vdash_{Irr^*} \phi\} = \{\phi | \vdash_{Adm_nIrr^*} \phi\}$.

Note: The rule is needed only to proof the lemma 3.14, neither deduction lemma 3.11 nor Lindenbaum's lemma 3.13 needs that rule and they can be proofed with the finite irreglexivity rule, but the rules Adm_nIrr^* change the nature of the ω-theories.

3.6. COMPLETENESS THEOREM FOR THE MINIMAL LOGIC $L(\Sigma_0)$.

We proof now the completeness of the minimal logic for axioms from 1 to 6 and the rules MP, $N_{\equiv_1,\equiv_2,\equiv_3,\neq}$ and the set of long infinite irreflexive rules Adm_nIrr^* instead of the finite irreflexive rule Irr.

Let Φ be the set of all modal formulas.

Let with $L = \{\phi | \vdash_{Irr} \phi\} = \{\phi | \vdash_{Irr^*} \phi\} = \{\phi | \vdash_{Adm_nIrr^*} \phi\}$ we note the set of all logical theorems.

Definition 3.6. *ω-theory — Any set of modal formulas $X \subseteq \Phi$ such that:*

1. *$L = \{\phi | \vdash_{Adm_n Irr^*} \phi\} \subseteq X$ All logical theorems belongs to X.*
2. *X closed under (MP).*
3. *X closed under $(Adm_n Irr^*)$.*

Note: The ω-theories are not closed under the normality rules $N_{\equiv_1,\equiv_2,\equiv_3,\neq}$.

Definition 3.7. *The ω-theory X is inconsistent if and only if $X = \Phi$.*

Definition 3.8. *The ω-theory X is consistent if and only if $X \neq \Phi$.*

Also X is inconsistent ω-theory if and only if $\perp \in X$, and X is consistent if and only if $\perp \notin X$.

Lemma 3.9. *Intersection of two ω-theories is a ω-theory.*

This lemma 3.9 gives us the opportunity to give the next definition:

Definition 3.9. *For each set of formulas $Y \in \Phi$, the set of formulas $Th(Y)$ is the smallest ω-theory that contains Y.*

Lemma 3.10. $Th(Y) = \bigcap\{X | X$ *is a ω-theory, and* $Y \subseteq X\}$.

So for each set Y it is true that:

1. $L \subseteq Th(Y)$,
2. $Y \subseteq Th(Y)$,
3. $Th(Y)$ is closed under (MP),
4. $Th(Y)$ is closed under all rules $(Adm_n Irr^*)$.

Lemma 3.11. Deduction Lemma. *Let ϕ formula and X ω-theory, then:*

$$(\phi \Rightarrow \psi) \in X \leftrightarrow \psi \in Th(X \cup \{\phi\})$$

Proof. Let's choose $Y = \{\psi | (\phi \Rightarrow \psi) \in X\}$.

1. Let's proof that $X \subseteq Y$. Let $\psi \in X$. Because $(\psi \Rightarrow (\phi \Rightarrow \psi))$ is a classical axiom, and X closed under (MP) then $(\phi \Rightarrow \psi) \in X$, and according the definition of Y, $\psi \in Y$, so $X \subseteq Y$. Also from $L \subseteq X$ we conclude that $L \subseteq Y$, all logical theorems belong to Y.

2. $(\phi \Rightarrow \phi) \in L$, it is a classical theorem, this way $(\phi \Rightarrow \phi) \in X$, so from the definition of Y, $\phi \in Y$.

3. Let $\psi \in Y$ and $(\psi \Rightarrow \chi)$, so $(\phi \Rightarrow \psi) \in X$ and $(\phi \Rightarrow (\psi \Rightarrow \chi)) \in X$, now using the classical axiom $(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi))$, and X closed

under (MP), then $(\phi \Rightarrow \chi) \in X$, and it follows that $\chi \in Y$, so Y is closed under (MP).

4. Finally we proof that Y is closed under Irr^* the infinite irreflexivity rule. Let $((p \wedge [\neq] \neg p) \Rightarrow \psi) \in Y$, for each variable p. Then $(\phi \Rightarrow ((p \wedge [\neq] \neg p) \Rightarrow \psi)) \in X$, for each variable p. Using propositional tautologies we conclude that for each variable p the formulas $((p \wedge [\neq] \neg p) \Rightarrow (\phi \Rightarrow \psi)) \in X$. X is closed under (Irr^*), so $(\phi \Rightarrow \psi) \in X$, $\psi \in Y$ and we conclude that Y is closed under (Irr^*).

Y is an ω-theory, $(X \cup \{\phi\}) \subseteq Y$, and $Th(X \cup \{\phi\})$ is the minimal ω-theory containing $(X \cup \{\phi\})$. That's why $Th(X \cup \{\phi\}) \subseteq Y$.

Now if $\psi \in Th(X \cup \{\phi\})$, then $\psi \in Y$, and from the definition of Y, $(\phi \Rightarrow \psi) \in X$. □

Lemma 3.12. *If X is an ω-theory, and $\phi \notin X$ then $Th(X \cup \{\neg\phi\})$ is a consistent ω-theory.*

Proof. Let's assume that $Th(X \cup \{\neg\phi\})$ is inconsistent, then $\bot \in Th(X \cup \{\neg\phi\})$, now using deduction lemma 3.11 we have that $(\neg\phi \Rightarrow \bot) \in X$, now apply classical tautology we get that $\phi \in X$, which contradicts with $\phi \notin X$. □

Long infinite rules are hard to write that's why we can specify some short writing form. If the formula ϕ is graphically equal to: $(\psi_0 \Rightarrow [R_0](\psi_1 \Rightarrow [R_1](....(\psi_m \Rightarrow [R_m](\psi))...)))$, then we note with $\Psi_\phi(p,i) = (\psi_0 \Rightarrow [R_0](\psi_1 \Rightarrow [R_1](....(\psi_i \Rightarrow [R_i]((p \wedge [\neq] \neg p) \Rightarrow (\psi_{i+1} \Rightarrow [R_{i+1}](\psi_{i+2} ... \Rightarrow [R_m](\psi)...)))...)))$, where $i > 0$, and $\Psi_\phi(p,0) = ((p \wedge [\neq] \neg p) \Rightarrow \phi)$. If the formula ϕ is not of the above form then only $\Psi(p,0)$ makes sense. As a conclusion we can say that for each formula ϕ, if we can specify $\Psi_\phi(p,i)$, then ϕ can be a conclusion of the rule $Adm_i Irr^*$. The rule $Adm_i Irr^*$ looks like: $\Psi_\phi(p,i)$, for each $p \vdash \phi$.

Lemma 3.13. *Lindenbaum's lemma. Let X is an ω-theory and $\phi \notin X$ then there is a maximal consistent ω-theory Y such that $X \subseteq Y$ and $\phi \notin Y$.*

Proof. Let's order all modal formulas into a sequence starting with $\neg\phi$ as the first formula: $\neg\phi = \phi_0, \phi_1, \phi_2, ...\phi_n, ...$ Now we define a sequence of consistent ω-theories : $X_0 \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq X_n \subseteq ...$ inductively. For X_0 we choose $Th(X \cup \{\neg\phi\})$, according lemma 3.12 it is a consistent ω-theory. Also $X \subseteq X_0$. Let's assume for some n the sequence $X_0, ..., X_n$ is created. Now we must define X_{n+1}, and there are two cases:

(i). If $Th(X_n \cup \{\phi_{n+1}\})$ is a consistent ω-theory, then we choose : $X_{n+1} = Th(X_n \cup \{\phi_{n+1}\})$. It is easy seen that $X_n \subseteq X_{n+1}$ and $\phi_{n+1} \in X_{n+1}$.

(ii). If $Th(X_n \cup \{\phi_{n+1}\})$ is a *inconsistent* ω-theory, then $\bot \in Th(X_n \cup \{\phi_{n+1}\})$, and according the deduction lemma 3.11, $(\phi_{n+1} \Rightarrow \bot) \in X_n$, and ω-theories are closed under propositional tautologies, so $\neg\phi_{n+1} \in X_n$. The right choice for X_{n+1} could be X_n , but we must do something more to guarantee that the infinite sequence union would not accumulate all formulas for the infinite rules Irr^* and

$Adm_m Irr^*, m > 1$ that can produce ϕ_{n+1}, and this way the infinite union becomes inconsistent. If the formula ϕ_{n+1} is graphically equal to: $\psi_0 \Rightarrow [R_0](\psi_1 \Rightarrow [R_1](....(\psi_m \Rightarrow [R_m](\psi))...))$, then this formula can be a result of any of the rules $Irr^* Adm_0 Irr^*, Adm_1 Irr^*, ..., Adm_m Irr^*$. If the formula ϕ_{n+1} is not of that form then it can be a result of the rule $Irr^* = Adm_0 Irr^*$ only.

Now we define the sequence $Y_{-1}, Y_0, Y_1, ..., Y_m$, of ω-theories, where $Y_{-1} = X_n$, and the aim is for each of Y_i to prevent ability to produce ϕ_{n+1} from the rule $Adm_i Irr^*$. Inductively. $Y_{-1} = X_n$ is a consistent and $\neg\phi_{n+1} \in Y_{-1}$. Let's we have defined already some Y_{i-1}, now we can define Y_i.

Let's assume that for each variable p, $\Psi_{\phi_{n+1}}(p, i) \in Y_{i-1}$. Then ϕ_{n+1} is a conclusion of the rule $Adm_i Irr^*$ and Y_{i-1} is an ω-theory, thus it is closed under $Adm_i Irr^*$, then the formula $\phi_{n+1} \in Y_i$, but $\neg\phi_{n+1} \in Y_i$, contradiction with Y_{i-1} is a consistent ω-theory. So we conclude that there must be a variable p_i such that $\Psi_{\phi_{n+1}}(p_i, i) \notin Y_{i-1}$. From the lemma 3.12 it follows that the ω-theory $Th(Y_{i-1} \cup \{\neg\Psi_{\phi_{n+1}}(p_i, i)\})$ is a consistent. Thus we choose for $Y_i = Th(Y_{i-1} \cup \{\neg\Psi_{\phi_{n+1}}(p_i, i)\})$. This way the sequence $Y_{-1}, Y_0, Y_1, ..., Y_m$ is defined. It is not hard to see that $X_n \subseteq Y_0 \subseteq Y_1 \subseteq ... \subseteq Y_m$, and $\neg\phi_n \in Y_m$, and for each of the chosen variables $p_0, p_1, ...p_m$ during the inductive definition, it is true that: $(\neg\Psi_{\phi_{n+1}}(p_i, i)) \in Y_m$.

We choose for $X_{n+1} = Y_m$. This way the inductive definition for consistent ω-theories is complete.

Let's $X_\omega = \bigcup_{n=0}^{\infty} X_n$.

1. X_ω is a consistent. Let's assume that X_ω is inconsistent, then $\perp \in X_\omega$ and according definition of X_ω, then there is $m : \perp \in X_m$, and X_m is inconsistent which is a contradiction with the build of X_m.

2. From $L \subseteq X \subseteq X_0 \subseteq X_\omega$, we get that $L \subseteq X_\omega$ and $X \subseteq X_\omega$.

3. Because the sequence is monotonic $X_0 \subseteq X_1 \subseteq ... \subseteq X_n \subseteq ...$ and each set is closed under MP then X_ω is closed under MP.

4. Because of the inductive construction, for each formula ψ, it is true: $\psi \in X_\omega$ or $\neg\psi \in X_\omega$. Also $\neg\phi \in X_0 \subseteq X_\omega$, then $\phi \notin X_\omega$.

5. Let's assume that X_ω is not closed under some $Adm_n Irr^*$ rule, then there is a formula ϕ such that for each variable p, $\Psi_\phi(p, n) \in X_\omega$, and $\phi \notin X_\omega$. According to the previous point, there is an index m: $\phi_m = \phi$, and $\neg\phi \in X_m$. According to the inductive construction, the case (ii) was chosen, and there is a variable p_n^m, such that the formula $\neg\Psi_\phi(p_n^m, n) \in X_m$, and $X_m \subseteq X_\omega$, it follows that $\neg\Psi_\phi(p_n^m, n) \in X_\omega$. This way we conclude that $\perp \in X_\omega$, contradiction with 1. X_ω is closed under any $Adm_n Irr^*$ rule, and it is ω-theory.

6. X_ω is a maximal according the subset relation "$\subseteq$". $\square$

Definition 3.10. $[R]X = \{\phi | [R]\phi \in X\}$, *where* $R \in \{\equiv_1, \equiv_2, \equiv_3, \neq\}$.

Lemma 3.14. *If X is an ω-theory then $[R]X$ is an ω-theory.*

Proof. 1. Let $\phi \in L$, logical theorem, then by normality rule N_R it is a theorem $[R]\phi \in L$, $L \subseteq X$, then $[R]\phi \in X$, and $\phi \in [R]X$, so $L \subseteq [R]X$.

2. Let $\phi \in [R]X$ and $(\phi \Rightarrow \psi) \in [R]X$, then $[R]\phi \in X$ and $[R](\phi \Rightarrow \psi) \in X$. It is an axiom $[R](\phi \Rightarrow \psi) \Rightarrow ([R]\phi \Rightarrow) \in L \subseteq X$. Because X is closed under MP, $[R]\psi \in X$, and finally $\psi \in [R]X$, $[R]X$ is closed under MP.

3. Let $(\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n]((p\wedge[\neq]\neg p) \Rightarrow \phi)...))) \in [R]X$, for each variable p. Then $[R](\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n]((p\wedge[\neq]\neg p) \Rightarrow \phi)...))) \in X$, it can be written into equivalent form $(T \Rightarrow [R](\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n]((p\wedge[\neq]\neg p) \Rightarrow \phi)...)))) \in X$, for each variable p. The ω-theory X is closed under $Adm_{n+1}Irr^*$, then $(T \Rightarrow [R](\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n](\phi)...)))) \in X$, and back using the definition of $[R]X$, we get that $(\phi_1 \Rightarrow [R_1](\phi_2 \Rightarrow [R_2](\phi_3 ... \Rightarrow [R_n](\phi)...))) \in [R]X$. So the set $[R]X$ is closed under Adm_nIrr^* rule, and it is closed under the whole set of rules Adm_nIrr^*, $n \geqslant 0$. □

Now we can define *canonical frame* using ω-theories. Canonical frame would not be perfect for difference $\neq$ relation and will be improved to *generic canonical frame* in which the difference is a real difference.

Definition 3.11. *Canonical Frame.* $\underline{W_k} = (W_k, \equiv_{1k}, \equiv_{2k}, \equiv_{3k}, \neq_k)$, *where* $W_k = \{X | X$ *is a maximal consistent* ω *theory* $\}$, $X \equiv_{1k} Y \leftrightarrow [\equiv_1]X \subseteq Y$. $X \equiv_{2k} Y \leftrightarrow [\equiv_2]X \subseteq Y$. $X \equiv_{3k} Y \leftrightarrow [\equiv_3]X \subseteq Y$. $X \neq_k Y \leftrightarrow [\neq]X \subseteq Y$.

Definition 3.12. *Canonical Evaluation.* $V_k(X,p) = true \leftrightarrow p \in X$, *for each variable p.*

Definition 3.13. *Canonical Model.* $M_k = (\underline{W_k}, V_k)$.

Lemma 3.15. *Truth Lemma*

Let $M_k = (\underline{W_k}, V_k)$ *canonical model, then for each formula* ϕ, *and for each maximal* ω*-theory* X *it is true that:* $V_k(X,\phi) = true \leftrightarrow \phi \in X$.

The relations $\equiv_{1k}, \equiv_{2k}, \equiv_{3k}$ are equivalence relations because of $S5$ axioms for each of them. Alas the relation $\neq_{1k}$ is not exactly the difference relation, that's why we will rework this canonical model into a new one.

Definition 3.14. *Let's* X *and* Y *are maximal consistent* ω*-theories. Then we say that* Y *is a* finite reachable *from* X, $X \curvearrowright Y$, *if there is a finite sequence of maximal consistent* ω*-theories* $X = Z_0, Z_1, ... Z_{m-1}, Z_m = Y$, *and relations* $R_{1k}, R_{2k}, ..., R_{mk}$, $R_{ik} \in \{\equiv_{1k}, \equiv_{2k}, \equiv_{3k}, \neq_k\}$,
such that $X = Z_0 R_{1k} Z_1 R_{2k} ... Z_{m-1} R_{mk} Z_m = Y$.

Definition 3.15. *Generic Canonical Frame. Let* X *is a maximal consistent* ω*-theory. The generic canonical frame is* $\underline{W'_k} = (W'_k, \equiv'_{1k}, \equiv'_{2k}, \equiv'_{3k}, \neq'_k)$, *where* $W'_k = \{Y | X \curvearrowright Y\}$, *the set of finite reachable from* X. *The relations* $\equiv'_{1k}, \equiv'_{2k}, \equiv'_{3k}$, $\neq'_k$ *are restrictions of* $\equiv_{1k}, \equiv_{2k}, \equiv_{3k}, \neq_k$ *over the set* W'_k.

The generic canonical model $M'_k = (W'_k, V'_k)$ is defined in the same way, and also the truth lemma 3.15 is true for the generic canonical frames and generic models.

Lemma 3.16. *The relations $\equiv'_{1k}, \equiv'_{2k}, \equiv'_{3k}$ are equivalence relations and the relation $\neq'_{1k}$ is a symmetric relation.*

Analogically to the definition 3.10 we can define $\blacksquare X$, where X is an ω-theory. $\blacksquare X = \{\phi | \blacksquare\phi \in X\}$, from the definition of $\blacksquare\phi = (\phi \wedge [\neq]\phi)$, we see that: $\blacksquare X = \{\phi | \phi \wedge [\neq]\phi \in X\} = \{\phi | \phi \in X\} \cap \{\phi | [\neq]\phi \in X\} = X \cap [\neq]X$, intersection of ω-theories is a ω-theory, so $\blacksquare X$ is an ω-theory. Now we can define the relation $Y \blacksquare'_k Z \leftrightarrow \blacksquare Y \subseteq Z$, and we can see that it is the universal relation into W'_k, any 2 objects are with relation $\blacksquare'_k$.

Lemma 3.17. *For any two maximal ω-theories $X \in W'_k$ and $Y \in W'_k$, then $\blacksquare X \subseteq Y$.*

Proof. First we proof that if $[R]X \subseteq Y$ then $\blacksquare X \subseteq Y$, and for that we use that it is an axiom: $\blacksquare\phi \Rightarrow [\equiv_i]\phi$. When R is $\neq$ then we use the tautology $(\phi \wedge [\neq]\phi) \Rightarrow [\neq]\phi$. Second from the lemma 3.5 we know that $\vdash \blacksquare\phi \Rightarrow \blacksquare\blacksquare\phi$, and using that theorem we can proof that if $\blacksquare X \subseteq Y$, and $\blacksquare Y \subseteq Z$, then $\blacksquare X \subseteq Z$, so $\blacksquare'_k$ is a transitive. Thus finally if $X, Y \in W'_k$, then there is a finite sequence $X = Z_0 R'_{1k} Z_1 R'_{2k} ... R'_{mk} Zm = Y$, now applying what we have proof, leads to $\blacksquare X \subseteq Y$. □

The formulas $p \wedge [\neq]\neg p$, where p is a variable, have special meaning, they "lock" the variable to be true at exactly one world and false in any other. So we can call them constants.

Definition 3.16. *If p is a variable then the formula noted with Op. $Op = (p \wedge [\neq]\neg p)$ is called a* constant.

Lemma 3.18. *If X is a maximal consistent ω-theory then there is a variable p. such that the constant $Op \in X$.*

Proof. Let'a assume that for each variable p, $Op \notin X$, because X is a maximal consistent ω-theory, then $\neg Op \in X$, the equivalent from is $(Op \Rightarrow \bot) \in X$, for each p, X is closed under $Irr^* = Adm_0 Irr^*$, then $\bot \in X$, contradiction with X consistent. □

Lemma 3.19. *For any variable p there is a maximal consistent ω-theory X, such that $Op \in X$, it contains the constant of p.*

Proof. $\neg Op \notin L$. If it were $\neg Op \in L$, then the formula $\neg Op$ must be true in any frame in any evaluation, it is simple to show a frame and an evaluation, and a world x such that $x \nvDash_v \neg Op$, thus $\neg Op \notin L$.

L is a consistent ω-theory, and $\neg Op \notin L$, according to lemma 3.13 there is a maximal consistent ω-theory X such that $L \subseteq X$, and $\neg Op \notin X$, thus $Op \in X$. □

Lemma 3.20. *Canonically defined $\neq_k$ is irreflexive. For any maximal consistent ω-theory X. $[\neq]X \not\subseteq X$.*

Proof. Let's assume that for some X maximal consistent ω-theory, $[\neq]X \subseteq X$. From lemma 3.18, there is p, and the constant $Op \in X$, $(p \wedge [\neq]\neg p) \in X$, so $p \in X$ and $[\neq]\neg p \in X$, $\neg p \in [\neq]X$, and from $[\neq]X \subseteq X$, we conclude that $\neg p \in X$ and from $p \in X$, X is an inconsistent, which is the contradiction. □

Lemma 3.21. *Let $W'_k = (W'_k, \equiv'_{1k}, \equiv'_{2k}, \equiv'_{3k}, \neq'_k)$ be a generic canonical frame, then for each $X \in W'_k$ and $Y \in W'_k$, and $X \neq Y$, then $X \neq'_k Y$, $[\neq]X \subseteq Y$.*

Proof. Because $X, Y \in W'_k$ from lemma 3.17, then $\blacksquare X \subseteq Y$. From $X \neq Y$ the there is a formula ϕ: $\phi \in X$ and $\phi \notin Y$. Let's now assume that $[\neq]X \not\subseteq Y$, so there is a formula ψ: $\psi \in [\neq]X$ and $\psi \notin Y$. For Y maximal consistent ω-theory we have $\neg(\phi \vee \psi) \in Y$ or $(\phi \vee \psi) \notin Y$. From $\blacksquare X \subseteq Y$ it follows that $\blacksquare(\phi \vee \psi) \notin X$, next we have $\neg((\phi \vee \psi) \wedge [\neq](\phi \vee \psi)) \in X$.

From the classical axiom $(\phi \Rightarrow (\phi \vee \psi)) \in X$, and $\phi \in X$, we conclude that $(\phi \vee \psi) \in X$. From the classical axiom $\vdash (\psi \Rightarrow (\phi \vee \psi))$, now applying the normality rule $N_{\neq}$, $\vdash [\neq](\psi \Rightarrow (\phi \vee \psi))$, and from the monotonic axiom $\vdash [\neq](\psi \Rightarrow (\phi \vee \psi)) \Rightarrow ([\neq]\psi \Rightarrow [\neq](\phi \vee \psi)))$, we get the theorem $([\neq]\psi \Rightarrow [\neq](\phi \vee \psi)) \in L \subseteq X$, and $[\neq]\psi \in X$, then $[\neq](\phi \vee \psi) \in X$, thus we get that $(\phi \vee \psi) \wedge [\neq](\phi \vee \psi) \in X$. Contradiction with X consistent. □

The conclusion of the last lemma is that the generic canonical frames belongs to the class of frames Σ_0, and it gives us the completeness theorem.

Theorem 3.2. *Completeness theorem for the minimal logic. Each formula ϕ that is true at the class of frames Σ_0 is provable, $\vdash_{Irr} \phi$.*

Proof. Contraposition. Let $\nvdash_{Irr} \phi$, ϕ is not a theorem, then $\phi \notin L$, using the Lindenbaum's lemma 3.13, there is a maximal consistent ω-theory X, such that $\phi \notin X$. Let's get the generic canonical frame and model, in which $W'_k = \{Y | X \frown Y\}$, generated from X. That frame belongs to the class Σ_0. In that model using the truth lemma 3.15, we get that $V_k(X, \phi) = false$, because $\phi \notin X$. And the deductive equivalence of the rules makes no difference between Irr and $Adm_n Irr^*$. □

In the end, some properties about constants and maximal consistent ω-theories that are useful and reveals the character of the maximal consistent ω-theories are expressed:

Lemma 3.22. *Let X and Y are maximal consistent ω-theories such that they are finite reachable, $\blacksquare X \subseteq Y$. If there is a variable p such that $Op \in X$ and $p \in Y$ then $X = Y$.*

Proof. Let's assume that $X \neq Y$, because $X \frown Y$, $\blacksquare X \subseteq Y$, then from lemma 3.21, we conclude the key fact that $[\neq]X \subseteq Y$. From $Op \in X$ it follows: $p \wedge [\neq]\neg p \in X$, so $p \in X$ and $[\neq]\neg p \in X$, $\neg p \in [\neq]X$. From $[\neq]X \subseteq Y$, then $\neg p \in Y$, and $p \in Y$, contradiction with Y consistent. □

If two maximal consistent ω-theories possess the same constant Op, they are the equal ω-theories.

Lemma 3.23. *$[R]X \subseteq Y$ if and only if there is a variable p such that $Op \in Y$ and $< R > Op \in X$, where $R \in \{\equiv_1, \equiv_2, \equiv_3, \blacksquare, \in_{ij}\}$.*

Proof. Let $[R]X \subseteq Y$, then from lemma 3.18, there is a constant $Op \in Y$, then $\neg Op \notin Y$, and from $[R]X \subseteq Y$, $\neg Op \notin [R]X$, then $[R]\neg Op \notin X$, and finally $\neg[R]\neg Op \in X$, which is $< R > Op \in X$.

Let there is a constant $Op \in Y$ and $< R > Op \in X$. From $< R > Op \in X$ it follows that $[R]\neg Op \notin X$, and $\neg Op \notin [R]X$. From lemma 3.14 then $[R]X$ is an ω-theory. Because $\neg Op \notin [R]X$, then $[R]X$ we know that it is a consistent theory. Now applying Lindenbaum's lemma 3.13 we get that there is Z maximal consistent ω-theory such that $[R]X \subseteq Z$, and $\neg Op \notin Z$. Z is a maximal, then $Op \in Z$, but $Op \in Y$, from lemma 3.22, $Y = Z$, and from $[R]X \subseteq Z$, then $[R]X \subseteq Y$. □

Next three lemmas are related with expressible modalities as the incidences: $[\in_{12}]$, $[\in_{13}]$, $[\in_{23}]$, $[\in_{12}^{-1}]$, $[\in_{13}^{-1}]$, $[\in_{23}^{-1}]$, or simply about $[\in_{ij}]$. Actually $\in_{ij} = \equiv_i \circ \equiv_j$.

Lemma 3.24. *If X is an ω-theory then $[\in_{ij}]X$ is an ω-theory.*

Proof. It uses that $[\in_{ij}][\equiv_i][\equiv_i]$, thus $[\in_{ij}]X = \{\phi | [\equiv_i][\equiv_j]\phi \in X\}$, so $[\in_{ij}]X[\equiv_i][\equiv_j]X$, and now applying lemma 3.14. □

Lemma 3.25. *The expressible relations are compliant with the canonical model.*

$\in_{ij} X \subseteq Y \leftrightarrow X \in_{ijk} Y \leftrightarrow (\exists Z)(X \equiv_{ik} Z \wedge Z \equiv_{jk} Y)$

Lemma 3.26. $X \in_{ijk} Y \leftrightarrow Y \in_{jik}^{-1} X$ *or* $[\in_{ij}]X \subseteq Y \leftrightarrow [\in_{ji}^{-1}]Y \subseteq X$

3.7. AXIOMATIZATION FOR THE STRUCTURES OF INCIDENCES LOGIC $L(\Sigma_{GSI})$.

The axioms of the logic $L(\Sigma_{gsi})$ will contain all axioms of the minimal logic $L(\Sigma_0)$, and the rules are MP and Irr, the finite one, and also several other axioms specific for the geometrically related properties of the structures of incidences. Each axiomatic property of the structures of incidences have a corresponding modal axiom, which modally expresses it, and also it makes that property a property of the generated canonical frame — canonical property. The new axioms are:

A_0^* $(<\equiv_1> Op \wedge <\equiv_2> Op \wedge <\equiv_3> Op) \Rightarrow Op$, axiom for the property:

$(\forall x \in W(S))(\forall y \in W(S))(x \equiv_1 y \wedge x \equiv_2 y \wedge x \equiv_3 y \Rightarrow x = y)$

A_0^{**} $<\in_{12}> (Op \wedge <\in_{12}> Oq) \Rightarrow <\equiv_1> (<\equiv_2> Op \wedge <\equiv_3> Oq)$, axiom for the property:

$(\forall x \in W(S))(\forall y \in W(S))(\forall z \in W(S))(x \in_{1,2} y \wedge y \in_{2,3} z \Rightarrow (\exists t \in W(S))(x \equiv_1 t \wedge y \equiv_2 t \wedge z \equiv_3 t))$

A_0^{***} $\blacklozenge Op \wedge [\in_{12}^{-1}] <\in_{13}> Op \Rightarrow <\in_{23}> Op$, axiom for the property:

$(\forall x \in W(S))(\forall y \in W(S))((\forall z \in W(S))(z \in_{1,2} x \wedge z \in_{1,3} y) \Rightarrow x \in_{2,3} y)$

- The property $(\exists x \exists y \in W(S))(\neg x \equiv_1 y)$ does not need an axiom.

A_2 $\blacklozenge A \Rightarrow <\in_{12}><\in_{12}^{-1}> A$, axiom for the property:

$(\forall x \forall y \in W(S))(\exists z \in W(S))(x \in_{1,2} z \wedge y \in_{1,2} z)$

A_3 $\blacklozenge Op \wedge [\equiv_1]\neg Op \wedge <\in_{12}> (Oq \wedge <\in_{12}^{-1}> Op) \Rightarrow [\in_{12}](<\in_{12}^{-1}> Op \Rightarrow <\equiv_2> Oq)$, axiom for the property:

$(\forall x \forall y \forall z \forall t \in W(S))(\neg x \equiv_1 y \wedge x \in_{1,2} z \wedge y \in_{1,2} z \wedge x \in_{1,2} t \wedge y \in_{1,2} t \Rightarrow z \equiv_2 t)$

A_4 $Op \Rightarrow <\equiv_2> [\equiv_1]\neg Op$, axiom for the property:

$(\forall x \exists y \exists z \in W(S))(\neg y \equiv_1 z \wedge y \in_{1,2} x \wedge z \in_{1,2} x)$

A_5 $Op \Rightarrow \blacklozenge([\in_{12}]\neg Op)$, axiom for the property:

$(\forall x \exists y \in W(S))(\neg y \in_{1,2} x)$

A_6 $\blacklozenge Op \wedge \blacklozenge Oq \Rightarrow <\in_{13}> (<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq)$, axiom for the property:

$(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))(x \in_{1,3} t \wedge y \in_{1,3} t \wedge z \in_{1,3} t)$

A_7 $\blacklozenge Op \wedge \blacklozenge Oq \wedge [\in_{12}]([\in_{12}^{-1}]\neg Op \vee [\in_{12}^{-1}]\neg Oq) \wedge <\in_{13}> (Or \wedge <\in_{13}^{-1}] > \neg Op \wedge <\in_{13}^{-1}> \neg Oq) \Rightarrow [\in_{13}](<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}] > Oq \Rightarrow <\equiv_3> Or)$, axiom for the property:

$(\forall x \forall y \forall z \in W(S))(\forall u \forall v \in W(S))(x \in_{1,3} u \wedge y \in_{1,3} u \wedge z \in_{1,3} u \wedge x \in_{1,3} v \wedge y \in_{1,3} v \wedge z \in_{1,3} v \wedge (\forall l \in W(S))(\neg x \in_{1,2} l \vee \neg y \in_{1,2} l \vee \neg z \in_{1,2} l) \Rightarrow u \equiv_3 v)$

- The property $(\forall x \exists y \in W(S))(y \in_{1,3} x)$ does not need an axiom.

A_9 $Op \Rightarrow \blacklozenge[\in_{13}]\neg Op$, axiom for the property:

$(\forall x \exists y \in W(S))(\neg y \in_{1,3} x)$

A_{10} $\blacklozenge Op \wedge [\equiv_1]\neg Op \wedge <\in_{12}> (Or \wedge <\in_{12}^{-1}> Op) \wedge <\in_{13}> (Oq \wedge <\in_{13}^{-1}> Op) \Rightarrow \blacksquare(Or \Rightarrow <\in_{23}> Oq)$, axiom for the property:

$(\forall x \forall y \forall z \forall t \in W(S))(\neg x \equiv_1 y \wedge x \in_{1,2} z \wedge y \in_{1,2} z \wedge x \in_{1,3} t \wedge y \in_{1,3} t \Rightarrow z \in_{2,3} t)$

A_{11} $\blacklozenge Op \wedge <\in_{13}^{-1}> (Oq \wedge <\in_{13}> Op) \Rightarrow <\in_{13}^{-1}> (<\in_{13}> Op \wedge [\equiv_1]\neg Oq)$, axiom for the property:

$(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))(z \in_{1,3} x \wedge z \in_{1,3} y \Rightarrow (\neg t \equiv_1 z) \wedge t \in_{1,3} x \wedge t \in_{1,3} y)$

Lemma 3.27. *All that modal formulas are true at the class of structures of incidences* Σ_{gsi}

Proof. Simple check that each modal formula is true at frames with the corresponding property, using contraposition. If the formula is not true then the frame does not have the corresponding property. □

Lemma 3.28. *All the modal formulas above modally expresses their correspondent properties of the structures of incidences.*

Proof. Simple check for each modal formula using contraposition. If the frame does not possess its correspondent property then there is an evaluation in which the the formula is not true. □

Lemma 3.29. *Adding each formula from above list as an axiom, makes the generic canonical frame to posses the same property, which the axiom modally expresses — generic canonical frame is a structure of incidence.*

Proof. Check that adding each modal formula, makes its property a property of the generic canonical frame, using properties of the *constant* formulas and maximal consistent ω-theories — lemmas 3.19 and 3.18. We can demonstrate it for 2 formulas, for A_4 and A_6:

A_4: Let we have added the axiom A_4, so $Op \Rightarrow <\equiv_2> [\equiv_1]\neg Op \in L$, for any variable p. Let X is a maximal consistent ω-theory, according to lemma 3.18, then there is a variable p_1, such that the constant $Op_1 \in X$, and also $Op_1 \Rightarrow <\equiv_2> [\equiv_1]\neg Op_1 \in L \subseteq X$. X closed under MP, then $<\equiv_2> [\equiv_1]\neg Op_1 \in X$, equivalent to $\neg[\equiv_2]\neg[\equiv_1]\neg Op_1 \in X$, X is a maximal, then $[\equiv_2]\neg[\equiv_1]\neg Op_1 \notin X$, and then according to 3.10, $\neg[\equiv_1]\neg Op_1 \notin [\equiv_2]X$, from the lemma 3.14 it follows that $[\equiv_2]X$ is an ω-theory, from Lindenbaum's lemma 3.13, there is a maximal consistent ω-theory Z, such that $[\equiv_2]X \subseteq Z$ and $\neg[\equiv_1]\neg Op_1 \notin Z$, $<\equiv_1> Op_1 \notin Z$. From $X \equiv_{2k} Z$, reachable from Z, then Z is into the domain of the generic canonical model, see definition 3.15. Now if we assume that $X \equiv_{1k} Z$ from lemma 3.23 $<\equiv_1> Op_1 \in Z$, contradiction, so $\neg(X \equiv_{1k} Z)$. From $X \equiv_{2k} Z$, and $\equiv_{2k}$ equivalence relation we have $Z \equiv_{2k} X$, and $\equiv_{1k}$ equivalence relation we get that $Z \equiv_{1k} X \equiv_{2k} X$ and $X \equiv_{1k} X \equiv_{2k} X$, from the lemma 3.25 $Z \in_{12k} X$, and $X \in_{12k} X$. As conclusion we can say that for each maximal consistent ω-theory X we found a maximal consistent ω-theories $Y = X$ and Z such that $\neg(Y \equiv_{1k} Z)$ and $Z \in_{12k} X$ and $Y \in_{12k} X$.

Also it is seen that the property $(\forall x \exists y \exists z \in W(S))(\neg y \equiv_1 z \wedge y \in_{1,2} x \wedge z \in_{1,2} x)$ is equivalent to the property $(\forall x \exists z \in W(S))(\neg x \equiv_1 z \wedge x \equiv_2 z)$.

A_6: Let we have added the axiom A_6. Let X, Y and Z are maximal consistent ω-theories such that they belong to the generic canonical frame, thus $\blacksquare X \subseteq Y$, $\blacksquare X \subseteq Z$. From lemma 3.18 there is a variables p, q such that $Op \in Y$ and $Oq \in Z$, now from lemma 3.23, it follows that $\blacklozenge Op \in X$ and $\blacklozenge Oq \in X$. A_6 axiom is $\blacklozenge Op \wedge \blacklozenge Oq \Rightarrow <\in_{13}> (<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq)$, and X is closed under MP, then $<\in_{13}> (<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \in X$, so $[\in_{13}]\neg(<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \notin X$, from lemma 3.25 we get that $\neg(<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \notin [\in_{13}]X$, and $[\in_{13}]X$ ω-theory. Lindenbaum's lemma 3.13 found that there is a maximal consistent ω-theory T such that $[\in_{13}]X \subseteq T$ and $\neg(<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \notin T$, or $(<\in_{13}^{-1}> Op \wedge <\in_{13}^{-1}> Oq) \in T$. From the lemma 3.23 we conclude for T that $[\in_{13}^{-1}]T \subseteq Y$ and $[\in_{13}^{-1}]T \subseteq Z$, and finally from the lemma 3.26 we conclude for T that: $Y \in_{13k} T$ and $Y \in_{13k} Z$. And from $[\in_{13}]X \subseteq T$ then $X \in_{13k} T$, which shows that we found T maximal consistent ω-theory from generic canonical frame that suffices the property: $(\forall x \forall y \forall z \in W(S))(\exists t \in W(S))(x \in_{1,3} t \wedge y \in_{1,3} t \wedge z \in_{1,3} t)$. □

Theorem 3.3. *The logic with the axiomatization above is complete for the class of structures of incidences.*

This completes the axiomatization of $L(\Sigma_{gsi})$ logic of the class of structures of incidences.

3.8. OPEN QUESTIONS

Besides the axiomatization with finite number of axiom schemas for the logic of the structures of incidences, and ability to proof geometrically related properties with it. There are several open question unsolved up to now.

Q_1 Is it decidable? It is not clean if there is an algorithm about checking if a formula is a theorem or not.

Q_2 Is it useful to proof some interesting? Some simple properties that are easy proofed with first order logic are not seen how to be proof with this modal logic. For example the property that: "for each line there is another line that is crossed to the first one, the 2 lines has no common point". That property is modally definable with the formula: $Op \Rightarrow \blacklozenge[\equiv_2][\equiv_1][\equiv_2]\neg Op$ or it's semantically equivalent form $Op \Rightarrow \blacklozenge[\equiv_2][\equiv_1][\equiv_2]\neg p$. Desarque's Theorem is also modally definable. Both modal formulas should be theorems, but the proof is not seen.

Q_3 Is it has a simpler axiomatization? For example is it possible to eliminate the rule Irr. Also it is not known if the rule Irr is useful in any proofs with the current axiomatization.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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SOME RESULTS FOR IDENTIFICATION FOR SOURCES AND ITS EXTENSION TO LIAR MODELS [1]

ZLATKO VARBANOV

1. INTRODUCTION

The classical transmission problem deals with the question how many possible messages can we transmit over a noisy channel? Transmission means there is an answer to the question "What is the actual message?" In the identification problem we deal with the question how many possible messages the receiver of a noisy channel can identify? Identification means there is an answer to the question "Is the actual message u?". Here u can be any member of the set of possible messages.

Let $(\mathcal{U}, P)$ be a source, where $\mathcal{U} = \{1, 2, \ldots, N\}, P = \{P_1, P_2, \ldots, P_N\}$, and let $\mathcal{C} = \{c_1, c_2, \ldots, c_N\}$ be a binary prefix code (PC) for this source with $||c_u||$ as length of c_u. Introduce the random variable U with $\text{Prob}(U = u) = p_u$ for $u = 1, 2, \ldots, N$ and the random variable C with $C = c_u = (c_1, c_2, \ldots, c_{u||c_u||})$ if $U = u$.

We use the PC for noiseless identification, that is user u wants to know whether the source output equals u, that is, whether C equals c_u or not. The user iteratively checks whether C coincides with c_u in the first, second, etc. letter and stops when the first different letter occurs or when $C = c_u$.

What is the expected number $L_{\mathcal{C}}(P, u)$ of checkings?

In order to calculate this quantity we introduce for the binary tree $T_{\mathcal{C}}$, whose leaves are the codewords $c_1, c_2, \ldots, c_N$, the sets of leaves $\mathcal{C}_{ik} (1 \leq i \leq N; 1 \leq k)$, where $\mathcal{C}_{ik} = \{c \in \mathcal{C} : c$ coincides with c_i exactly until the k'th letter of $c_i\}$. If C

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takes a value in $\mathcal{C}_{uk}, 0 \le k \le ||c_u|| - 1$, the answers are k times "Yes" and 1 time "No". For $C = c_u$ the

$$L_{\mathcal{C}}(P, u) = \sum_{k=0}^{||c_u||-1} P(C \in \mathcal{C}_{uk})(k+1) + ||c_u||P_u.$$

For a code $\mathcal{C}$

$$L_{\mathcal{C}}(P) = \max_{1 \ge u \ge N} L_{\mathcal{C}}(P, u)$$

is the expected number of checkings in the worst case and

$$L(P) = \min_{\mathcal{C}} L_{\mathcal{C}}(P)$$

is this number for the best code.

2. RESULTS FOR UNIFORM DISTRIBUTION

Let $P^N = \{\frac{1}{N}, \dots, \frac{1}{N}\}$. We construct a prefix code $\mathcal{C}$ in the following way. In each node (starting at the root) we split the number of remaining codewords in proportion as close as possible to $(\frac{1}{2}, \frac{1}{2})$.

It is known [1] that

$$\lim_{N \to \infty} L_{\mathcal{C}}(P^N) = 2 \tag{2.1}$$

Also, in [2] was stated the problem to estimate an universal constant $A = \sup L(P)$ for general $P = (P_1, \dots, P_N)$. We compute this constant for uniform distribution and this code $\mathcal{C}$.

Using decomposition formula for trees, we obtain the following recursion

$$L_{\mathcal{C}}(P^N) = \frac{\lceil \frac{N}{2} \rceil}{N} L_{\mathcal{C}}(P^{\lceil \frac{N}{2} \rceil}) + 1, L_{\mathcal{C}}(P^2) = 1 \tag{2.2}$$

From (2) follows that the worst case for $L_{\mathcal{C}}(P^N)$ is when $N = 2^k + 1$, for any integer k. We compute the exact value for $L_{\mathcal{C}}(P^N)$ in this case and obtain

$$\sup_N L_{\mathcal{C}}(P^N) = 2 + \frac{log_2(N-1) - 2}{N}$$

Also, we consider the case where not only the source outputs but the users occur at random. In addition to the source $(\mathcal{U}, P)$ and random variable U, we are given $(\mathcal{V}, Q), \mathcal{V} \equiv \mathcal{U}$ with random variable V independent of U and defined by $\text{Prob}(V = v) = Q_v$ for $v \in V$. The source encoder knows the value u of U but not that of V, which chooses the user v with probability Q_v. Again let $\mathcal{C} = \{c_1, \dots, c_N\}$ be a binary prefix code and let $L_{\mathcal{C}}(P, u)$ be the expected number of checkings on code $\mathcal{C}$ for user u. Instead of $L_{\mathcal{C}}(P) = \max_{u \in \mathcal{U}} L_{\mathcal{C}}(P, u)$ we can consider the average number of expected checkings (also called *average identification length*):

$$L_{\mathcal{C}}(P, Q) = \sum_{v \in \mathcal{V}} Q_v L_{\mathcal{C}}(P, v); \quad L(P, Q) = \min_{\mathcal{C}} L_{\mathcal{C}}(P, Q)$$

Special case is the case $Q = P$. Here

$$L_{\mathcal{C}}(P, P) = \sum_{u \in \mathcal{U}} P_u L_{\mathcal{C}}(P, u); \quad L(P, P) = \min_{\mathcal{C}} L_{\mathcal{C}}(P, P)$$

and for uniform distribution we have

$$L_{\mathcal{C}}(P^N, P^N) = \frac{1}{N} \sum_{u \in \mathcal{U}} L_{\mathcal{C}}(P^N, u)$$

We calculate exact values of $L_{\mathcal{C}}(P^N)$ and $L_{\mathcal{C}}(P^N, P^N)$ for some N and summarize them in Table 1 (for $N = 2^k, L_{\mathcal{C}}(P^N) = L_{\mathcal{C}}(P^N, P^N) = 2 - \frac{2}{N}$ [1]).

TABLE 1 - some exact values for uniform distribution, $2^k < N < 2^{k+1}, k \geq 3$

N	$L_{\mathcal{C}}(P^N)$	$L_{\mathcal{C}}(P^N, P^N)$
$2^k + 1$	$2 + \frac{log_2(N-1)-2}{N}$	$2 + \frac{log_2(N-1)-2}{N^2}$
$2^k + 2^{k-1} - 1$	2	$2 - \frac{5(N+1)-3log_2(\frac{2N+2}{3})}{3N^2}$
$2^k + 2^{k-1}$	$2 - \frac{1}{N}$	$2 - \frac{5}{3N}$
$2^k + 2^{k-1} + 1$	$2 + \frac{log_2(\frac{N-1}{12})}{N}$	$2 - \frac{(5N-2)-3log_2(\frac{N-1}{12})}{3N^2}$
$2^{k+1} - 1$	$2 - \frac{1}{N}$	$2 - \frac{2N-log_2(N+1)+1}{N^2}$

3. EXTENSION TO LIAR MODELS

Suppose that when user u iteratively checks whether C coincides with c_u in the first, second, etc. letter, for some reasons he obtains wrong information in any position. Then there is a lie(error) in this position of the codeword. In this model with lies, the user knows only that the general number of lies is at most e and no information for the positions of lies.

Let $L_{\mathcal{C}}(P,u) = L_{\mathcal{C}}(P)$ for any $u \in \mathcal{U}$. In this case, we denote by $L_{\mathcal{C}}(P;e)$ the expected number of checkings if there are at most e lies. Then, to be sure for the correct answer in any position the user needs of $e+1$ the same answers ("Yes" or "No"). If the user has done $2e+1$ questions for any position he gets exact information for the value in this position. Therefore, there exists trivial upper bound

$$L_{\mathcal{C}}(P;e) \leq (2e+1)L_{\mathcal{C}}(P)$$

Clearly, this upper bound can be improved by decreasing the number of remaining lies. The following algorithm can be used for any $u \in \mathcal{U}$:

Step 0: BEGIN $i := 1, Checkings := 0$, actual message $:= v$;

Step 1: If $i > ||c_v||$ then Step 3. Otherwise, check codeword position i until $e+1$ the same answers. Let t be the number of obtained answers "Yes" and f be the number of obtained answers "No";

Step 2: $Checkings := Checkings + (t+f)$. If $t > f$, then $e := e - f$, $i := i+1$, Step 1. Otherwise, the actual message $v \neq u$;

Step 3: END.

Let v be the current checked codeword and let i be the first position in which c_u and c_v differ (if $c_u = c_v$ then $i = ||c_u||$). We can see that the worst case with respect by e is when all lies(errors) occur in position i. In this case

$Checkings = (e+1)(i-1) + (2e+1).1 = e(i+1) + i$.

If there is a lie in any position m $(1 \leq m \leq i-1)$, for every position j $(m+1 \leq j \leq i)$ the user needs of e the same answers. Then

$Checkings = (m-1)(e+1) + (e+2) + (i-m-1)e + (2e-1) = e(i+1) + m < e(i+1) + i$

Therefore, if $k = ||c_u||$ and $P_{ui} = P(C \in \mathcal{C}_{ui})$, for the worst case we obtain the following upper bound

$$\begin{aligned} L_{\mathcal{C}}(P;e) &\leq \sum_{i=0}^{k-1} P_{ui}(e(i+2)+i+1) + (e(k+1)+k)P_u \\ &= e\sum_{i=0}^{k-1} P_{ui}(i+2) + e(k+1)P_u + \sum_{i=0}^{k-1} P_{ui}(i+1) + kP_u \\ &= e\sum_{i=0}^{k-1}(P_{ui}(i+1) + P_{ui}) + e(k+1)P_u + L_{\mathcal{C}}(P) \\ &= e\left(\sum_{i=0}^{k-1} P_{ui}(i+1) + kP_u\right) + e\left(\sum_{i=0}^{k-1} P_{ui} + P_u\right) + L_{\mathcal{C}}(P) \\ &= eL_{\mathcal{C}}(P) + e + L_{\mathcal{C}}(P) = \underline{(e+1)L_{\mathcal{C}}(P) + e} \end{aligned}$$

Let $M_C(P;e) = (e+1)L_C(P) + e$. Then from (1) follows that for uniform distribution P^N

$$\lim_{N\to\infty} M_C(P^N;e) = 3e+2$$

Let consider other distribution P when all individual probabilities are powers of $\frac{1}{2}$

$$P_u = \frac{1}{2^{\ell_u}}, \quad u \in \mathcal{U} = \{1,2,\ldots,N\}$$

We know that there is a prefix code $\mathcal{C}$ with codeword lengths $||c_u|| = \ell_u$ and for such code $L_C(P,u) = 2(1-P_u)$ [2]. Therefore

$$\lim_{N\to\infty} L_C(P) = 2$$

and again for $M_C(P;e)$ we obtain

$$\lim_{N\to\infty} M_C(P;e) = 3e+2$$

Also, for general distribution $P = (P_1, P_2, \ldots, P_N)$ we know that $L(P) \leq 3$([1], Theorem 3). Therefore, for $L(P;e)$ (the expected number of checkings for the best code $\mathcal{C}$ and at most e lies) we obtain that

$$L(P;e) \leq 4e+3$$

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SOME THEOREMS ON THE CONVERGENCE OF SERIES IN BESSEL-MAITLAND FUNCTIONS [1]

Jordanka Paneva-Konovska

Some important properties of the power series in complex domain are given by the classical Cauchy-Hadamard, Abel and Tauber theorems. In this paper we prove same type theorems for series in the Bessel-Maitland functions.

Keywords: Cauchy-Hadamard, Abel and Tauber theorems, Bessel-Maitland function
2000 MSC: 30B10, 30B30; 33C10, 33C20

1. INTRODUCTION

Some important properties of the power series $\sum_{n=0}^{\infty} a_n z^n$ in a complex domain are given by the classical Cauchy-Hadamard, Abel and Tauber theorems.

In general, by the classical Abel theorem, from the convergence of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ at a point z_0, follows the existence of the limit $\lim_{z \to z_0} f(z) = f(z_0)$, when z belongs to a suitable angle domain with a vertex at a point z_0. The geometrical series [6, p.92]: $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$ at $z_0 = 1$ gives an example that, in general, the inverse proposition is not true, i.e. the existence of this limit

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does not imply the convergence of the series $\sum_{n=0}^{\infty} a_n z_0^n$ without additional conditions on the growth of the coefficients.

The corresponding classical result is given by the following theorem.

Theorem (Tauber). *If the coefficients of the power series satisfy the condition* $\lim_{n\to\infty} na_n = 0$ *and if* $\lim_{z\to 1} f(z) = S$ *(*$z \to 1$ *radially), then the series* $\sum a_n$ *is convergent and* $\sum_{n=0}^{\infty} a_n = S$.

It turns out that Abel's theorem fails even for series of the type $\sum_{k=1}^{\infty} a_{n_k} z^{n_k}$, where $(n_1, n_2, \dots, n_k, \dots)$ is a suitable permutation of nonnegative integers [6, p.92]. Therefore, it is interesting to know if for series in a given sequence of holomorphic functions, a statement like Abel's theorem is available. A positive answer to this question for series in Laguerre and Hermite polynomials is given in [5, §11.3], [1], and for Bessel functions - in [4].

Let $J_\nu^\mu(z)$ be the so-called Bessel-Maitland function, see [2, p.336, 352], [3, p.110]:

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(1+\nu+\mu k)}, \quad z \in \mathbb{C}, \quad \mu > -1.$$

Let us consider series of the form

$$\sum_{n=0}^{\infty} a_n z^n J_n^\mu(z), \quad z \in \mathbb{C}, \quad \mu > 0. \tag{1}$$

We prove in this paper the corresponding Cauchy-Hadamard, Abel and Tauber type theorems for series in Bessel-Maitland functions of form (1).

2. A CAUCHY-HADAMARD TYPE THEOREM

Denote for shortness:

$$\widetilde{J}_n^\mu(z) = z^n J_n^\mu(z), \quad n = 0, 1, 2, \dots.$$

The following asymptotic formula can be easily verified for the Bessel-Maitland functions:

$$\widetilde{J}_n^\mu(z) = z^n(1 + \theta_n^\mu(z)) \,/\, \Gamma(n+1), \;\; z \in \mathbb{C}, \quad \mu > 0, \tag{2}$$
$$\theta_n^\mu(z) \to 0 \;\text{ as }\; n \to \infty \quad (n \in \mathbb{N}).$$

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Theorem 1 (Cauchy-Hadamard type). *The domain of convergence of the series* (1) *is the circle domain* $|z| < R$ *with a radius of convergence* $R = 1/\Lambda$, *where*

$$\Lambda = \limsup_{n\to\infty}(\ |a_n|\ /\ \Gamma(n+1)\ )^{1/n}. \tag{3}$$

The cases $\Lambda = 0$ *and* $\Lambda = \infty$ *are incorporated in the common case, if* $1/\Lambda$ *means* ∞, *respectively 0.*

Proof. Let us denote

$$u_n(z) = a_n \widetilde{J}_n^{\mu}(z), \quad b_n = (\ |a_n|\ /\ \Gamma(n+1)\ )^{1/n}.$$

Using the asymptotic formula (2), we get

$$u_n(z) = a_n z^n (1 + \theta_n^{\mu}(z))\ /\ \Gamma(n+1).$$

The proof goes in three cases.

1. $\Lambda = 0$, then $\lim_{n\to\infty} b_n = \limsup_{n\to\infty} b_n = 0$. Let us fix $z \neq 0$. Obviously, there exists a number N_1 such that for every $n > N_1$: $|1 + \theta_n^{\mu}(z)| < 2$ and $2b_n < 1/|z|$ which is equivalent to $|u_n(z)| = b_n^n |z|^n |1+\theta_n^{\mu}(z)| < 2^{1-n}$. The absolute convergence of (1) follows immediately from this inequality.

2. $0 < \Lambda < \infty$. First, let z be inside the domain $|z| < R$ ($z \in \mathbb{C}$), i.e. $|z|/R < 1$. Then $\limsup_{n\to\infty} |z| b_n < 1$. Therefore, there exists a number $q < 1$ such that $\limsup_{n\to\infty} |z| b_n \leq q$, whence $|z|^n b_n^n \leq q^n$. Using the asymptotic formula for the common member $u_n(z)$ of the series (1), we obtain $|u_n(z)| = b_n^n |z|^n |1 + \theta_n^{\mu}(z)| \leq q^n |1+\theta_n^{\mu}(z)|$. Since $\lim_{n\to\infty} \theta_n^{\mu}(z) = 0$, there exists N_2: for every $n > N_2$ $|1+\theta_n^{\mu}(z)| < 2$ and hence $|u_n(z)| \leq 2q^n$. Since the series $\sum_{n=0}^{\infty} 2q^n$ is convergent, the series (1) is also convergent, even absolutely.

Now, let z lie outside this domain. Then $|z|/R > 1$ and $\limsup_{n\to\infty} |z| b_n > 1$. Therefore there exist infinite number of values n_k of n: $|z|^{n_k} b_{n_k}^{n_k} > 1$. Because $\lim_{n\to\infty} \theta_n^{\mu}(z) = 0$, there exists N_3 so that for $n_k > N_3$; $|1 + \theta_{n_k}^{\mu}(z)| \geq 1/2$, i.e. $|u_{n_k}(z)| \geq 1/2$ for infinite number of values of n. The necessary condition for convergence is not satisfied. Therefore the series (1) is divergent.

3. $\Lambda = \infty$. Let $z \in \mathbb{C}\backslash\{0\}$. Then $b_{n_k} > 1/|z|$ for infinite number of values n_k of n. But, from here $|u_{n_k}(z)| = |z|^{n_k}\ b_{n_k}^{n_k}\ |1 + \theta_{n_k}^{\mu}(z)| \geq 1/2$ and the necessary condition for the convergence of the series (1) is not satisfied and we conclude that the series (1) is divergent for every $z \neq 0$. □

3. AN ABEL TYPE THEOREM

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angle domain with size $2\varphi < \pi$ and vertex at the point $z = z_0$, which is symmetric in the straight line defined by the points 0 and z_0. The following theorem is valid:

Theorem 2 (Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, Λ be defined by (3), $0 < \Lambda < \infty$. Let $K = \{|z| < R, R = 1/\Lambda\}$. If $f(z)$ is the sum of the series (1) on the domain K and this series is convergent at the point z_0 of the boundary of K, then $\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^\mu(z_0)$, for $|z| < R$ and $z \in g_\varphi$, i.e.*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^\mu(z_0), \quad z \in g_\varphi. \tag{4}$$

Proof. Consider the difference

$$\Delta(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^\mu(z_0) - f(z) = \sum_{n=0}^{\infty} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)), \tag{5}$$

representing it in the form

$$\Delta(z) = \sum_{n=0}^{k} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)) + \sum_{n=k+1}^{\infty} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)).$$

Let $p > 0$. By using the notations

$$\beta_m = \sum_{n=k+1}^{m} a_n \widetilde{J}_n^\mu(z_0), \quad m > k, \quad \beta_k = 0,$$

$$\gamma_n(z) = 1 - \widetilde{J}_n^\mu(z)/\widetilde{J}_n^\mu(z_0),$$

and the Abel transformation [1], we obtain consequently:

$$\sum_{n=k+1}^{k+p} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1})\gamma_n(z)$$

$$= \beta_{k+p}\gamma_{k+p}(z) - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1}(z) - \gamma_n(z)),$$

i.e.

$$\sum_{n=k+1}^{k+p} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z)) = (1 - \widetilde{J}_{k+p}^\mu(z)/\widetilde{J}_{k+p}^\mu(z_0)) \sum_{n=k+1}^{k+p} a_n \widetilde{J}_n^\mu(z_0)$$

$$-\sum_{n=k+1}^{k+p-1}\Big(\sum_{s=k+1}^{n} a_s \widetilde{J}_s^{\mu}(z_0)\Big)\Big(\frac{\widetilde{J}_n^{\mu}(z))}{\widetilde{J}_n^{\mu}(z_0))} - \frac{\widetilde{J}_{n+1}^{\mu}(z))}{\widetilde{J}_{n+1}^{\mu}(z_0))}\Big).$$

From the asymptotic formula (2), it follows that there exists a natural number M such that $\widetilde{J}_n^{\mu}(z_0) \neq 0$ for $n > M$. Let $k > M$. Then, for every natural $n > k$:

$$\widetilde{J}_n^{\mu}(z)/\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_{n+1}^{\mu}(z)/J_{n+1}^{\mu}(z_0) = (z/z_0)^n \tag{6}$$

$$\times \frac{(1+\theta_n^{\mu}(z))(1+\theta_{n+1}^{\mu}(z_0)) - (z/z_0)(1+\theta_{n+1}^{\mu}(z))(1+\theta_n^{\mu}(z_0))}{(1+\theta_n^{\mu}(z_0))(1+\theta_{n+1}^{\mu}(z_0))}.$$

For the right hand side of (6) we apply the Schvartz lemma. Then we get that there exists a constant C:

$$|\widetilde{J}_n^{\mu}(z)/\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_{n+1}^{\mu}(z)/J_{n+1}^{\mu}(z_0)| \le C|z-z_0||z/z_0|^n.$$

Analogously there exists a constant B:

$$|1 - \widetilde{J}_{k+p}^{\mu}(z)/\widetilde{J}_{k+p}^{\mu}(z_0)| \le B|z-z_0| \le 2B|z_0|.$$

Let ε be an arbitrary positive number and choose $N(\varepsilon)$ so large that for $k > N(\varepsilon)$ the inequality

$$|\sum_{s=k+1}^{n} a_s \widetilde{J}_s^{\mu}(z_0)| < \min(\varepsilon\cos\varphi/(12B|z_0|), \varepsilon\cos\varphi/(6C|z_0|))$$

holds for every natural $n > k$. Therefore, for $k > \max(M, N(\varepsilon))$:

$$|\sum_{s=k+1}^{\infty} a_s J_s^{\mu}(z_0)| \le \min(\varepsilon\cos\varphi/(12B|z_0|), \varepsilon\cos\varphi/(6C|z_0|),$$

and

$$|\sum_{n=k+1}^{\infty} a_n(\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z))| \le (\varepsilon\cos\varphi/6)(1 + \sum_{n=k+1}^{\infty} |z_0|^{-1}|z-z_0||z/z_0|^n)$$

$$\le (\varepsilon\cos\varphi/6)(1 + |z-z_0|/(|z_0|-|z|)).$$

But near the vertex of the angle domain g_φ in the part d_φ closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle we have $|z-z_0|/(|z_0|-|z|) < 2/\cos\varphi$, i.e. $|z-z_0|\cos\varphi < 2(|z_0|-|z|)$. That is why the inequality

$$|\sum_{n=k+1}^{\infty} a_n(\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z))| < (\varepsilon\cos\varphi)/6 + \varepsilon/3 \le \varepsilon/2 \tag{7}$$

holds for $z \in d_\varphi$ and $k > \max(M, N(\varepsilon))$. Fix some $k > \max(M, N(\varepsilon))$ and after that choose $\delta(\varepsilon)$ such that if $|z - z_0| < \delta(\varepsilon)$ then the inequality

$$|\sum_{n=0}^{k} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z))| < \varepsilon/2 \tag{8}$$

holds inside d_φ. We get

$$|\Delta(z)| = |\sum_{n=0}^{\infty} a_n(\widetilde{J}_n^\mu(z_0) - \widetilde{J}_n^\mu(z))|$$

for the module of the difference (5). From (7) and (8) it follows that the equality (4) is satisfied. □

4. A TAUBER TYPE THEOREM

Let us consider the series $\sum\limits_{n=0}^{\infty} a_n$, $a_n \in \mathbb{C}$. Let
$z_0 \in \mathbb{C}$, $|z_0| = R$, $0 < R < \infty$, $J_n^\mu(z_0) \neq 0$ for $n = 0, 1, 2, \dots$.
For shortness, denote

$$J_{n,\mu}^*(z; z_0) = \frac{\widetilde{J}_n^\mu(z)}{\widetilde{J}_n^\mu(z_0)}.$$

Let the series $\sum\limits_{n=0}^{\infty} a_n J_{n,\mu}^*(z; z_0)$ be convergent for $|z| < R$ and

$$F(z) = \sum_{n=0}^{\infty} a_n J_{n,\mu}^*(z; z_0), \quad |z| < R.$$

Theorem 3 (Tauber type). *If $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with*

$$\lim\{n a_n\} = 0, \tag{9}$$

and there exists

$$\lim_{z \to z_0} F(z) = S \quad (|z| < R, z \to z_0 \text{ radially}),$$

then the series $\sum\limits_{n=0}^{\infty} a_n$ is convergent and

$$\sum_{n=0}^{\infty} a_n = S.$$

Proof. For a point z of the segment $[0, z_0]$ we have

$$\sum_{n=0}^{k} a_n - F(z) = \sum_{n=0}^{k} a_n - \sum_{n=0}^{\infty} a_n J^*_{n,\mu}(z; z_0)$$

$$= \sum_{n=0}^{k} a_n \frac{\widetilde{J}^\mu_n(z_0)}{\widetilde{J}^\mu_n(z_0)} - \sum_{n=0}^{\infty} a_n \frac{\widetilde{J}^\mu_n(z)}{\widetilde{J}^\mu_n(z_0)}$$

$$= \sum_{n=0}^{k} a_n \frac{\widetilde{J}^\mu_n(z_0) - \widetilde{J}^\mu_n(z)}{\widetilde{J}^\mu_n(z_0)} - \sum_{n=k+1}^{\infty} a_n J^*_{n,\mu}(z; z_0)$$

and therefore

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| \le \sum_{n=0}^{k} |a_n| \left| \frac{\widetilde{J}^\mu_n(z_0) - \widetilde{J}^\mu_n(z)}{\widetilde{J}^\mu_n(z_0)} \right| + \sum_{n=k+1}^{\infty} |a_n| \left|J^*_{n,\mu}(z; z_0)\right|. \qquad (10)$$

By using the asymptotic formula (2) for the Bessel-Maitland functions, we obtain:

$$a_n J^*_{n,\mu}(z; z_0) = a_n \left(\frac{z}{z_0}\right)^n \frac{1 + \theta^\mu_n(z)}{1 + \theta^\mu_n(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_{n,\mu}(z; z_0)\right).$$

Let ε be an arbitrary positive number. We choose a number N_1 so large that the inequalities $|1 + \widetilde{\theta}_{k,\mu}(z; z_0)| < 2$, $|ka_k| < \frac{\varepsilon}{6}$ hold as $k \ge N_1$. If $k > N_1$ and z is on the segment $[0, z_0]$, then for the second summand in (10) the following estimate is valid:

$$\sum_{n=k+1}^{\infty} |a_n| \left|J^*_{n,\mu}(z; z_0)\right| = \sum_{n=k+1}^{\infty} |a_n| \left|\frac{z}{z_0}\right|^n |1 + \widetilde{\theta}_{n,\mu}(z; z_0)| \qquad (11)$$

$$\le 2 \left|\frac{z}{z_0}\right|^{k+1} \sum_{n=k+1}^{\infty} |a_n| \left|\frac{z}{z_0}\right|^{n-k-1} \le 2 \sum_{n=0}^{\infty} |a_{n+k+1}| \left|\frac{z}{z_0}\right|^n$$

$$= 2 \sum_{n=0}^{\infty} \frac{|(n+k+1) a_{n+k+1}|}{n+k+1} \left|\frac{z}{z_0}\right|^n < 2 \sum_{n=0}^{\infty} \frac{\varepsilon/6}{n+k+1} \left|\frac{z}{z_0}\right|^n$$

$$< \frac{2}{k} \frac{\varepsilon}{6} \frac{1}{1 - |z/z_0|} = \frac{\varepsilon}{3} \frac{1}{k} \frac{|z_0|}{|z_0| - |z|}.$$

Now let us consider the first summand in (10). We have:

$$\sum_{n=0}^{k} |a_n| \left| \frac{\widetilde{J}^\mu_n(z_0) - \widetilde{J}^\mu_n(z)}{\widetilde{J}^\mu_n(z_0)} \right|$$

$$= \sum_{n=0}^{m} |a_n| \left| \frac{\widetilde{J}^\mu_n(z_0) - \widetilde{J}^\mu_n(z)}{\widetilde{J}^\mu_n(z_0)} \right| + \sum_{n=m+1}^{k} |a_n| \left| \frac{\widetilde{J}^\mu_n(z_0) - \widetilde{J}^\mu_n(z)}{\widetilde{J}^\mu_n(z_0)} \right|.$$

According to Schwarz's lemma, there exists a constant C such that

$$\left|\frac{\widetilde{J}_n^{\mu}(z_0)-\widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right| < C|z-z_0|.$$

Moreover, there exists a number N_2 such that the following inequality

$$\sum_{n=0}^{m}|a_n|\left|\frac{\widetilde{J}_n^{\mu}(z_0)-\widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right| \le C\,|z-z_0|\,k\,\frac{\sum\limits_{n=0}^{m}|a_n|}{k} \tag{12}$$

$$< C\,|z-z_0|\,k\,\frac{\varepsilon}{3RC} = |z-z_0|\,k\,\frac{\varepsilon}{3R}.$$

holds as $k > N_2$. It remains to estimate the sum

$$\sum_{n=m+1}^{k}|a_n|\left|\frac{\widetilde{J}_n^{\mu}(z_0)-\widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right|.$$

To this end, using asymptotic formula (2) for the Bessel-Maitland functions, we find consequently:

$$\frac{\widetilde{J}_n^{\mu}(z_0)-\widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} = \frac{(z_0)^n(1+\theta_n^{\mu}(z_0))-z^n(1+\theta_n^{\mu}(z))}{z_0^n(1+\theta_n^{\mu}(z_0))} = 1-\left(\frac{z}{z_0}\right)^n\frac{1+\theta_n^{\mu}(z)}{1+\theta_n^{\mu}(z_0)}$$

$$= 1-\left(\frac{z}{z_0}\right)^n\left[1+\frac{\theta_n^{\mu}(z)-\theta_n^{\mu}(z_0)}{1+\theta_n^{\mu}(z_0)}\right] = 1-\left(\frac{z}{z_0}\right)^n-\left(\frac{z}{z_0}\right)^n\frac{\theta_n^{\mu}(z)-\theta_n^{\mu}(z_0)}{1+\theta_n^{\mu}(z_0)}.$$

Therefore,

$$\left|\frac{\widetilde{J}_n^{\mu}(z_0)-\widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)}\right| \le \left|1-\left(\frac{z}{z_0}\right)^n\right| + \left|\frac{z}{z_0}\right|^n\left|\frac{\theta_n^{\mu}(z)-\theta_n^{\mu}(z_0)}{1+\theta_n^{\mu}(z_0)}\right|. \tag{13}$$

We obtain the following inequalities

$$\left|1-\left(\frac{z}{z_0}\right)^n\right| = \left|1-\frac{z}{z_0}\right|\left|1+\frac{z}{z_0}+\left(\frac{z}{z_0}\right)^2+\cdots+\left(\frac{z}{z_0}\right)^{n-1}\right| \le n\left|1-\frac{z}{z_0}\right|$$

for the first summand of (13). According to Schwarz's lemma, there exists a constant ρ such that

$$\left|\frac{\theta_n^{\mu}(z)-\theta_n^{\mu}(z_0)}{1+\theta_n^{\mu}(z_0)}\right| \le 1 \qquad \text{as} \qquad |z-z_0| < \rho.$$

Then, for such $|z|$, we obtain for the second summand of (14):

$$\left|\frac{z}{z_0}\right|^n\left|\frac{\theta_n^{\mu}(z)-\theta_n^{\mu}(z_0)}{1+\theta_n^{\mu}(z_0)}\right| \le \left|\frac{z}{z_0}\right|^n|z-z_0|.$$

From (9) it follows that

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} n|a_n|}{k} = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} |a_n|}{k} = 0.$$

Then a number N_3 exists such that

$$\frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{and} \quad \frac{\sum\limits_{n=m+1}^{k} |a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{as} \quad k > N_3.$$

Therefore,

$$\sum_{n=m+1}^{k} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right| \le \sum_{n=m+1}^{k} n|a_n| \left|1 - \frac{z}{z_0}\right| \tag{14}$$

$$+ \sum_{n=m+1}^{k} |a_n| \left|\frac{z}{z_0}\right|^n |z - z_0| \le k \frac{|z - z_0|}{R} \frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} + k\,|z - z_0| \frac{\sum\limits_{n=m+1}^{k} |a_n|}{k}$$

$$< k\,|z - z_0| \frac{1+R}{R} \frac{\varepsilon}{3(1+R)} = k\,|z - z_0| \frac{\varepsilon}{3R}.$$

Finally, let us note that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| \le \sum_{n=0}^{m} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right|$$

$$+ \sum_{n=m+1}^{k} |a_n| \left| \frac{\widetilde{J}_n^{\mu}(z_0) - \widetilde{J}_n^{\mu}(z)}{\widetilde{J}_n^{\mu}(z_0)} \right| + \sum_{n=k+1}^{\infty} |a_n| \left|J^*_{n,\mu}(z; z_0)\right| .$$

Let $N = \max(N_1, N_2, N_3), k > N$ and $|z - z_0| < \rho$. Then by using (11),(12),(14), we can conclude that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| < |z - z_0|\, k \frac{\varepsilon}{3R} + k\,|z - z_0| \frac{\varepsilon}{3R} + \frac{\varepsilon}{3} \frac{1}{k} \frac{|z_0|}{|z_0| - |z|}$$

$$= \frac{\varepsilon}{3} \left[\frac{2k}{R} |z - z_0| + \frac{1}{k} \frac{|z_0|}{|z_0| - |z|} \right] .$$

If we substitute z by $z_0(1 - \frac{1}{k})$, then

$$\left|\sum_{n=0}^{k} a_n - F\left(z_0(1 - \frac{1}{k})\right)\right| < \frac{\varepsilon}{3} 3 = \varepsilon.$$

This proves that $\lim\limits_{k\to\infty}\sum\limits_{n=0}^{k} a_n$ exists and equals $\lim\limits_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right)$, i.e.

$$\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right) = S.$$

Thus the theorem is proved. □

Remark. Putting $\mu = 1$ in above considerations leads to the corresponding results (see [4]) for series in the Bessel functions $J_\nu(z) = (z/2)^\nu J_\nu^1(z^2/4)$, namely for

$$\sum_{n=0}^{\infty} a_n J_n(z), \quad z \in \mathbb{C}.$$

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ERROR ESTIMATES OF HIGH-ORDER DIFFERENCE SCHEMES FOR ELLIPTIC EQUATIONS WITH INTERSECTING INTERFACES

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In the work [1] high-order difference schemes (numerical experiments show second and fourth order of convergence) were derived, but with 1-st and 3-d order local truncation error, respectively, compact difference schemes for elliptic equations with intersecting interfaces. Here, for these difference schemes, we provide error estimates in discrete Sobolev norms.

Keywords: High-order finite difference schemes, Compact stencils, Elliptic problems, Intersected interfaces, Error estimates, Discrete Sobolev norms

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1. INTRODUCTION

Many important physical and industrial applications involve material interface problems, described by differential equations with discontinuous coefficients and concentrated sources. A good example of a problem of this type is the two-dimensional elliptic equation with discontinuous coefficients and Dirac-delta right side.

When the interface is smooth, the singularity is not severe , with smooth solutions inside each region. Various methods have been developed for this case and they work well, at least for moderately large contrast [4], [8],[13], [14], [17]. A method that is the simplest conceptually but nontrivial in implementation is aligning the grid with discontinuity. In [4], it is proved that if the boundary is at least C^2 this FEM converges nearly the same optimal way, in both the L^2 and energy norm, as the problems without interfaces.

An efficient method that uses regular grids is the Immersed Interface Method (IIM), [10], [13], [14]. The essence of the IIM includes using uniform or adaptive Cartesian grids and introducing non-zero correction forms in the starting difference approximations near the interfaces. The role of the jump (interface) relations is very important. In fact the idea for using the jump relations was first proposed for an elliptic problem with line interface in [16]. The construction of our difference schemes also relies on this idea.

The standard strategy for generating higher-order difference schemes is to expand the stencil (see for example [15], [16]). It was applied very successfully recently to elliptic interface problems [3]. A such an approach has the obvious disadvantages of creating large matrix bandwidths, complicating the numerical treatment near the boundaries.

Our method of discretization is based on aligning the grid with discontinuities. The construction uses the differential equation and the jump (interface) relations as additional identities which can be differentiated to eliminate lower order local truncation errors.

Our schemes are compact, i.e. they use minimal stencils: 5-points for the second-order scheme and 9-points for the fourth-order one. In order the maximum principle to be satisfied for the 4-th order difference scheme the mesh steps must satisfied very restrictive inequalities. Here we apply the energy method to obtain error estimates for the difference schemes.

The rest of this paper is organized as follows. In Subsection 1.1 the boundary value problem is stated; the notation used is introduced in Subsection 1.2, a second-order difference scheme on five points stencil, while a fourth-order ones derived in [1] are presented. In Section 3 error estimates for the schemes are obtained.

It is noted that there also exist some analytical and numerical studies about problems on intersecting interfaces [5], [7], [11]. Also, almost second order difference scheme for singularly perturbed problem with a line interface parallel to the axis Oy is constructed and studied for uniform convergence in [2]. Convergence of finite difference schemes for elliptic problems with curvilinear interfaces intersected the domain boundary is studied in [6].

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1.1. BOUNDARY VALUE PROBLEM

The more difficult interface problem is with singularities that arise due a non-smooth or an intersecting interface. As a typical example we consider the elliptic equation.

$$Lu := -\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + r(x,y)u = F(x,y), \tag{1.1}$$
$$(x,y) \in \Omega \equiv (0,X) \times (0,Y),$$

where

$$F(x,y) = f(x,y) - \delta(x-\xi,y)K_x(y) - \delta(x,y-\eta)K_y(x), \quad (\xi,\eta) \in \Omega.$$

We assume that the functions p,q,r,f,K_x,K_y are piecewise continuous (with possible discontinuity on the segments $\Gamma_x \equiv \{(x,y) : x=\xi,\ 0<y<1\}$, $\Gamma_y \equiv \{(x,y) : 0<x<1,\ y=\eta\}$ for p,q,r,f while in $y=\eta$ for K_x and in $x=\xi$ for K_y) and

$$0 < c_0 \le p(x,y), q(x,y) \le C_0, \quad 0 \le r_0 \le r(x,y) \le C_0 \ \text{ on } \bar{\Omega}. \tag{1.2}$$

We shall solve (1.1) for continuous solution subjected with Dirichlet boundary condition

$$u|_{\partial\Omega} = \varphi(x,y). \tag{1.3}$$

Then the equation (1.1) is equivalent to the: equation

$$Lu := f(x,y), \quad (x,y) \in \Omega \backslash \Gamma = \bigcup_{s=1}^{4} \Omega_s, \quad \Gamma = \Gamma_x \cup \Gamma_y, \text{ see Fig.1} \tag{1.4}$$

and the interface relations

$$[u]_{\Gamma_x} \equiv u(\xi+,y) - u(\xi-,y) = 0, \ y \in (0,1), \tag{1.5}$$
$$[u]_{\Gamma_y} \equiv u(x,\eta+) - u(x,\eta-) = 0, \ x \in (0,1), \tag{1.6}$$

$$\left[p\frac{\partial u}{\partial x}\right]_{\Gamma_x \backslash \{\eta\}} = K_x(y), \ y \in (0,1)\backslash\{\eta\}, \tag{1.7}$$
$$\left[q\frac{\partial u}{\partial y}\right]_{\Gamma_y \backslash \{\xi\}} = K_y(x), \ x \in (0,1)\backslash\{\xi\}, \tag{1.8}$$

$$\left\{\left[p\frac{\partial u}{\partial x}\right]_{\xi}\right\}_{\eta} + \left\{\left[q\frac{\partial u}{\partial y}\right]_{\eta}\right\}_{\xi} = \{K_x\}_{\eta} + \{K_y\}_{\xi}, \tag{1.9}$$

where, for example, $\{K_x\}_{\eta} = \frac{1}{2}\left(K_x(\eta-) + K_x(\eta+)\right)$.

Similar interface relations can be derived if a finite number of line interfaces $x=\xi_i,\ i=1,...,I$ and $y=\eta_j,\ j=1,...,J$ are assumed (see the numerical results in Table 6 and Figure 5, 6 for $I=J=2$ in [1]).

Let m be a nonnegative integer and $\alpha \in (0,1)$. The standard notation $C^m(\bar{\Omega})$ is known for the space of functions where derivatives up to order m are continuous on $\bar{\Omega}$ with maximum norm ([9], [12]). The notation $C^{m+\alpha}(\bar{\Omega})$ is used for the space of

Holder continuous functions with corresponding norm [9],[12]. Finally, by $C^{m+\alpha}(\Omega)$ we denote the space of functions, which belong to $C^{m+\alpha}(\bar{\Omega}')$, where $\Omega' \subset \Omega$.

We also will use the functional space

$$\mathcal{H}_{\Omega}^{s+\alpha} = \bigcap_{k=1}^{4} C^{s+\alpha}(\overline{\Omega}_k), \ s \in N. \tag{1.10}$$

The regularity of the solutions (i.e. the belonging of the solution to appropriate Holder space) depending on the input data smoothness and various compatibility conditions satisfied by the input data. Interface problems with smooth interface curves that do not intersect the domain boundary have been widely investigated in the literature, [4], [12], [14] and the references there. But for problems of type (1.1)-(1.3)(or (1.4)-(1.9)) such results are not known. We will further assume for (1.1)-(1.3) that the solutions have the necessary smoothness (see Propositions 2.1, 3.1).

1.2. GRIDS AND GRID FUNCTIONS

Let introduce the non-uniform mesh $\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_k, \quad \omega = \bar{\omega} \cap \Omega, \quad \gamma = \bar{\omega} \backslash \omega$, where

$\bar{\omega}_h = \{x_0 = 0, \ x_i = x_{i-1} + h_i, \quad i = 1, \dots, N_1 - 1, \ x_{N_1} = x_{N_1 - 1} + h = \xi, \ x_{N_1+1} = \xi + h, \ x_i = x_{i-1} + h_i, \ i = N_1 + 2, \dots, N, \ x_N = X\}$,

$\bar{\omega}_k = \{y_0 = 0, \ y_j = y_{j-1} + k_j, \ j = 1, \dots, M_1 - 1, \ y_{M_1} = y_{M_1 - 1} + h = \eta, \ y_{M_1+1} = \eta + h, \ y_j = y_{j-1} + k_j, \ j = M_1 + 2, \dots, M, \ y_M = Y\}$.

A such mesh is designed on Fig.2.

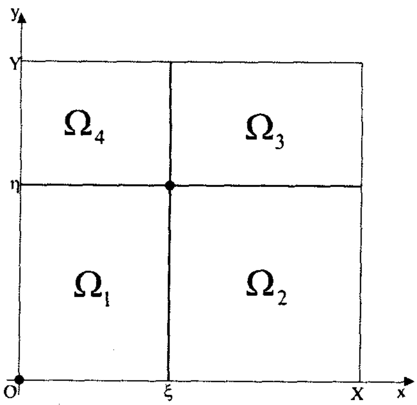

Fig.1. The Domain Ω

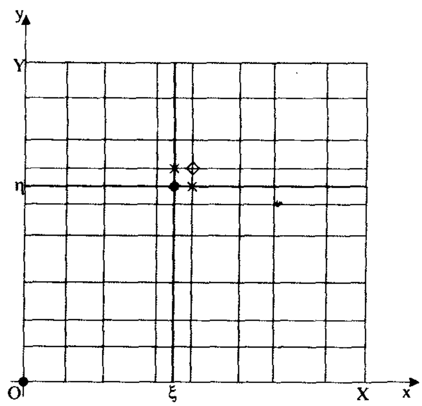

Fig.2. Non-uniform mesh

The finite-difference operators are defined in standard manner by $U(x, y)$:

$$U_{\bar{x}} = U_{\bar{x},i} = (U(x_i, y_j) - U(x_{i-1}, y_j))/h_i, \ U_x = U_{x,i} = U_{\bar{x},i+1},$$

$$U_{\bar{y}} = U_{\bar{y},j} = (U(x_i, y_j) - U(x_i, y_{j-1}))/k_j, \ U_y = U_{y,j} = U_{\bar{y},j+1},$$

$$U_{\hat x} = U_{\hat x,i} = (U(x_{i+1/2}, y_j) - U(x_{i-1/2}, y_j))/\hbar_i,$$

$$\hbar_i = \frac{1}{2}(h_i + h_{i+1}),\ \hbar_0 = h_1/2,\ \hbar_N = h_N/2,$$

$$U_{\hat y} = U_{\hat y,j} = (U(x_i, y_{j+1/2}) - U(x_i, y_{j-1/2}))/\bar k_j,$$

$$\bar k_j = \frac{1}{2}(k_j + k_{j+1}),\ \bar k_0 = k_1/2,\ \bar k_M = k_M/2,$$

$$U_{\bar x \hat x} = U_{\bar x \hat x,i} = \frac{1}{\hbar_i}(U_{x,i} - U_{\bar x,i}),\ U_{\bar y \hat y} = U_{\bar y \hat y,j} = \frac{1}{\bar k_j}(U_{y,j} - U_{\bar y,j}).$$

$$U_{\mathring x} = U_{\mathring x,i} = \frac{h_{i+1}U_{\bar x,i} + h_i U_{x,i}}{h_i + h_{i+1}},\ U_{\mathring y} = U_{\mathring y,j} = \frac{k_{j+1}U_{\bar y,j} + k_j U_{y,j}}{k_j + k_{j+1}},$$

$$U_{\breve x \hat x} = U_{\breve x \hat x,i} = \frac{1}{\hbar_i}(U_{x,i} - U_{\breve x,i}),\ U_{\breve y \hat y} = U_{\breve y \hat y,j} = \frac{1}{\bar k_j}(U_{y,j} - U_{\breve y,j}).$$

$$U_{\mathring x \mathring y} = \left(U_{\mathring x}\right)_{\mathring y} = \left(U_{\mathring y}\right)_{\mathring x}.$$

Here U_{ij} is any discrete function. Note that when it is clear that $u(x, y)$ is a continuous function, we shall sometimes use the notation $u_{ij} := u(x_i, y_j)$, while when it is clear that U_{ij} is a discrete function, we shall sometimes use the notation $U(x_i, y_j) := U_{ij}$.

Let $g(x, y)$ be a piecewise continuous function define in Ω.

$$g_{\breve x} = g_{\breve x_i}(y) = \frac{h_i g(x_i-, y) + h_{i+1} g(x_i+, y)}{h_i + h_{i+1}},$$

$$g_{\breve y} = g_{\breve y_j}(x) = \frac{k_j g(x, y_j-) + k_{j+1} g(x, y_j+)}{k_j + k_{j+1}},$$

$$g_{\breve x \breve y} = g_{\breve x_i \breve y_j} = (g_{\breve x_i})_{\breve y_j} = (g_{\breve y_j})_{\breve x_i}.$$

If $h_i = h_{i+1}$, then $g_{\breve x} = \{g\}_{x_i}$. If $k_j = k_{j+1}$, then $g_{\breve y} = \{g\}_{y_j}$.

Throughout the paper, C sometimes subscribed, denotes a generic positive constant that is independent of any mesh used.

2. THE DIFFERENCE SCHEMES

In this section we present the finite difference schemes for the problem (1.1)-(1.9) derived in [1].

The following difference scheme is derived in [1]:

$$\begin{aligned} \Lambda U_{ij} &= -(aU_{\bar{x}})_{\hat{x}_i\breve{y}_j} - (bU_{\bar{y}})_{\hat{y}_j\breve{x}_i} + c_{\breve{x}_i\breve{y}_j}U_{ij} \qquad (2.1)\\ &= \varphi_{\breve{x}_i\breve{y}_j} - \frac{1}{\hbar_i}\left(\left[p\frac{\partial u}{\partial x}\right]_{x_i}\right)_{\breve{y}_j} - \frac{1}{\bar{k}_j}\left(\left[q\frac{\partial u}{\partial y}\right]_{y_j}\right)_{\breve{x}_i} \\ &= \begin{cases} \varphi, & x_i \neq \xi,\ y_j \neq \eta; \\ \varphi_{\breve{\xi}} - \frac{1}{h}K_x(y), & x_i = \xi,\ y_j \neq \eta; \\ \varphi_{\breve{\eta}} - \frac{1}{h}K_y(x), & x_i \neq \xi,\ y_j = \eta; \\ \varphi_{\breve{\xi}\breve{\eta}} - \frac{1}{h}\left\{K_x(y)\right\}_{\eta} - \frac{1}{h}\left\{K_y(x)\right\}_{\xi}, & x_i = \xi,\ y_j = \eta. \end{cases} \end{aligned}$$

The following assertion was proved in [1].

Proposition 2.1. *Suppose that* $p,\ q \in \mathcal{H}_{\Omega}^{3+\alpha}$, $r,\ f \in \mathcal{H}_{\Omega}^{2+\alpha}$, $K_x \in C^{\alpha}[0,\eta]\bigcup[\eta,1]$, $K_y \in C^{\alpha}[0,\xi]\bigcup[\xi,1]$, $\alpha \in (0,1)$, $u \in C(\overline{\Omega})\bigcap\mathcal{H}_{\Omega}^{4+\alpha}$. *Then the truncation error of the scheme (2.1) is of order one.*

2.2. FOURTH ORDER DIFFERENCE SCHEME

In this subsection we consider the problem (1.4)-(1.9) in the case of piecewise constant coefficients

$$p(x,y) = p_s,\ q(x,y) = q_s,\ r(x,y) = r_s,\ (x,y) \in \Omega_s;\ s = 1,2,3,4.$$

Then (1.4) reads as follows:

$$-p_s\frac{\partial^2 u}{\partial x^2} - q_s\frac{\partial^2 u}{\partial y^2} + r_s u = f_s(x,y),\ \ (x,y) \in \Omega_s,\ s = 1,2,3,4, \qquad (2.2)$$

Further, for simplicity we shall describe our construction in the case

$$\left[\frac{r}{p}\right]_{\Gamma_x} = \left[\frac{r}{q}\right]_{\Gamma_y} = \left[\frac{q}{p}\right]_{\Gamma_x} = \left[\frac{p}{q}\right]_{\Gamma_y} = 0. \qquad (2.3)$$

In this case the following difference scheme is derived in [1].

Case 2.1. *Point of type* $\diamond$

$$\begin{aligned} \Lambda' U &= -\left(pU_{\bar{x}}\right)_{\hat{x}} - \left(qU_{\bar{y}}\right)_{\hat{y}} + rU - \frac{1}{12}\left(\left(k^2\left(pU_{\bar{x}}\right)_{\hat{x}\bar{y}}\right)_{\hat{y}} + \left(h^2\left(qU_{\bar{y}}\right)_{\hat{y}\bar{x}}\right)_{\hat{x}}\right) \\ &- \frac{1}{6}\left([k]_{y_j}\left(pU_{\bar{x}}\right)_{\hat{x}\overset{\circ}{y}} + [h]_{x_i}\left(qU_{\bar{y}}\right)_{\hat{y}\overset{\circ}{x}}\right) + \frac{1}{12}\left(\left(h^2rU_{\bar{x}}\right)_{\hat{x}} + \left(k^2rU_{\bar{y}}\right)_{\hat{y}}\right) \\ &+ \frac{1}{6}r\left([h]_{x_i}U_{\overset{\circ}{x}} + [k]_{y_j}U_{\overset{\circ}{y}} + \frac{2}{3}[h]_{x_i}[k]_{y_j}U_{\overset{\circ}{x}\overset{\circ}{y}}\right) \qquad (2.4) \\ = f &+ \frac{1}{12}\left((h^2f_{\bar{x}})_{\hat{x}\check{y}} + (k^2f_{\bar{y}})_{\hat{y}\check{x}}\right) + \frac{1}{6}\left([h]_{x_i}f_{\overset{\circ}{x}} + [k]_{y_j}f_{\overset{\circ}{y}} + \frac{2}{3}[h]_{x_i}[k]_{y_j}f_{\overset{\circ}{x}\overset{\circ}{y}}\right). \end{aligned}$$

If $[h]_{x_i} = [k]_{y_j} = 0$ (2.4) reduces to the Samarskii's famous scheme [15]:

$$\begin{aligned}\Lambda' U &= -(pU_{\bar{x}})_x - (qU_{\bar{y}})_y + rU - \frac{k^2}{12}(pU_{\bar{x}})_{x\bar{y}y} - \frac{h^2}{12}(qU_{\bar{y}})_{y\bar{x}x}\\ &+ \frac{h^2}{12}(rU_{\bar{x}})_x + \frac{k^2}{12}(rU_{\bar{y}})_y = f + \frac{h^2}{12}f_{\bar{x}x} + \frac{k^2}{12}f_{\bar{y}y}.\end{aligned}$$

Case 2.2. *Points $x_i = \xi,\ y_j \neq \eta$ of type $*$.*

$$\begin{aligned}\Lambda' U_{ij} &= -(pU_{\bar{x}})_x - (qU_{\bar{y}})_{\hat{y}\check{x}} + r_{\check{x}}U - \frac{1}{12}\left(k^2(pU_{\bar{x}})_{x\bar{y}}\right)_{\hat{y}} - \frac{h^2}{12}(qU_{\bar{y}})_{\hat{y}\bar{x}x}\\ &+ \frac{h^2}{12}(rU_{\bar{x}})_x + \frac{1}{12}(k^2 rU_{\bar{y}})_{\hat{y}\check{x}} + \frac{[k]_{y_j}}{6}\left(\{r\}_{x_i} U_{\overset{\circ}{y}} - (pU_{\bar{x}})_{x\overset{\circ}{y}}\right) \qquad (2.5)\\ &= f_{\check{x}} + \frac{h^2}{12}f_{\bar{x}x} + \frac{1}{12}(k^2 f_{\bar{y}})_{\hat{y}\check{x}} + \frac{[k]_{y_j}}{6}f_{\overset{\circ}{y}\check{x}} + \frac{h}{12}\left[\frac{\partial f}{\partial x}\right]_{x_i}\\ &- \left(\frac{h}{12}\left\{\frac{r}{p}\right\}_{x_i} + \frac{1}{h}\right)K_x(y_j) - \frac{1}{12h}\left(k^2 K_x(y)\right)_{\bar{y}\hat{y}} + \frac{h}{12}\left\{\frac{q}{p}\right\}_{x_i}(K_x(y))_{\bar{y}\hat{y}}\\ &- \frac{[k]_{y_j}}{18}\left(\frac{3}{h}(K_x(y))_{\overset{\circ}{y}} + h\left\{\frac{r}{p}\right\}_{x_i} K'_x(y_j) - h\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{x_i,j}\right).\end{aligned}$$

If $[k]_{y_j} = 0,\ k_j = k_{j+1} = k$, i.e. the mesh in y is uniform, we have

$$\begin{aligned}\Lambda' U_{ij} &= -(pU_{\bar{x}})_x - (qU_{\bar{y}})_{y\check{x}} + r_{\check{x}}U - \frac{k^2}{12}(pU_{\bar{x}})_{x\bar{y}y} - \frac{h^2}{12}(qU_{\bar{y}})_{y\bar{x}x}\\ &+ \frac{h^2}{12}(rU_{\bar{x}})_x + \frac{k^2}{12}(rU_{\bar{y}})_{y\check{x}} = f_{\check{x}} + \frac{h^2}{12}f_{\bar{x}x} + \frac{k^2}{12}f_{\bar{y}y\check{x}} + \frac{h}{12}\left[\frac{\partial f}{\partial x}\right]_{x_i} \qquad (2.6)\\ &- \left(\frac{h}{12}\left\{\frac{r}{p}\right\}_{x_i} + \frac{1}{h}\right)K_x(y_j) + \frac{1}{12}\left(h\left\{\frac{q}{p}\right\}_{x_i} - \frac{k^2}{h}\right)(K_x(y))_{\bar{y}\hat{y}_j}.\end{aligned}$$

Similarly if $y_j = \eta,\ x_i \neq \xi$ we obtain

$$\begin{aligned}\Lambda' U_{ij} &= -(pU_{\bar{x}})_{x\check{y}} - (qU_{\bar{y}})_y + r_{\check{y}}U - \frac{h^2}{12}(pU_{\bar{x}})_{\hat{x}\bar{y}y} - \frac{1}{12}\left(h^2(qU_{\bar{y}})_{y\bar{x}}\right)_{\hat{x}}\\ &+ \frac{1}{12}(h^2 rU_{\bar{x}})_{\hat{x}\check{y}} + \frac{h^2}{12}(rU_{\bar{y}})_y + \frac{[h]_{x_i}}{6}\left(\left\{ru_{\overset{\circ}{x}}\right\}_{y_j} - (qu_{\bar{y}})_{y\overset{\circ}{x}}\right) \qquad (2.7)\\ &= f_{\check{y}} + \frac{1}{12}\left(h^2 f_{\bar{x}}\right)_{\hat{x}\check{y}} + \frac{h^2}{12}f_{\bar{y}y} + \frac{[h]_{x_i}}{6}f_{\overset{\circ}{x}\check{y}} + \frac{h}{12}\left[\frac{\partial f}{\partial y}\right]_{y_j,i}\\ &- \left(\frac{h}{12}\left\{\frac{r}{q}\right\}_{y_j} + \frac{1}{h}\right)K_y(x_i) - \frac{1}{12h}\left(h^2 K_y(x)\right)_{\bar{x}\hat{x}} + \frac{h}{12}\left\{\frac{p}{q}\right\}_{y_j}(K_y(x))_{\bar{x}\hat{x}}\end{aligned}$$

$$- \frac{[h]_{x_i}}{18}\left(\frac{3}{\hbar}\left(K_y(x)\right)_{\overset{\circ}{x}} + h\left\{\frac{r}{q}\right\}_{y_j} K'_y(x_i) - h\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{y_j,i}\right).$$

In the particular case $[h]_{x_i} = 0,\ h_i = h_{i+1} = \hbar_i = \hbar$ we obtain

$$\begin{aligned}
\Lambda' U_{ij} &= -(pU_{\bar{x}})_{x\breve{y}} - (qU_{\bar{y}})_y + r_{\breve{y}}U - \frac{h^2}{12}(pU_{\bar{x}})_{x\bar{y}y} - \frac{\hbar^2}{12}(qU_{\bar{y}})_{y\bar{x}x} \qquad (2.8)\\
&+ \frac{\hbar^2}{12}(rU_{\bar{x}})_{x\breve{y}} + \frac{h^2}{12}(rU_{\bar{y}})_y = f_{\breve{y}} + \frac{\hbar^2}{12}f_{\bar{x}x\breve{y}} + \frac{h^2}{12}f_{\bar{y}y} + \frac{h}{12}\left[\frac{\partial f}{\partial y}\right]_{y_j}\\
&- \left(\frac{h}{12}\left(\frac{r}{q}\right)_{\breve{y}} + \frac{1}{h}\right)K_y(x_i) + \frac{1}{12}\left(h\left(\frac{p}{q}\right)_{\breve{y}} - \frac{\hbar^2}{h}\right)\left(K_y(x)\right)_{\bar{x}\hat{x}}.
\end{aligned}$$

Case 2.3. *Point of type •*

$$\begin{aligned}
\Lambda' U_{ij} &= -(pU_{\bar{x}})_{x\breve{y}} - (qU_{\bar{y}})_{y\breve{x}} + r_{\breve{x}\breve{y}}U - \frac{h^2}{12}(pU_{\bar{x}})_{x\bar{y}y} - \frac{h^2}{12}(qU_{\bar{y}})_{y\bar{x}x}\\
&+ \frac{h^2}{12}(rU_{\bar{x}})_{x\breve{y}} + \frac{h^2}{12}(rU_{\bar{y}})_{y\breve{x}} = f_{\breve{x}\breve{y}} + \frac{h^2}{12}\left(f_{\bar{x}x\breve{y}} + f_{\bar{y}y\breve{x}}\right)\\
&+ \frac{h}{12}\left(\left[\frac{\partial f}{\partial x}\right]_{x\breve{y}} + \left[\frac{\partial f}{\partial y}\right]_{y\breve{x}}\right) - \frac{h}{12}\left((K_x)_{\bar{y}y} + (K_y)_{\bar{x}x}\right) \qquad (2.9)\\
&- \left(\left(\frac{h}{12}\left\{\frac{r}{p}\right\}_{\breve{x}} + \frac{1}{h}\right)K_x(y)\right)_{\breve{y}} - \left(\left(\frac{h}{12}\left\{\frac{r}{q}\right\}_{\breve{y}} + \frac{1}{h}\right)K_y(x)\right)_{\breve{x}}\\
&+ \frac{h}{12}\left(\left\{\frac{q}{p}\right\}_{\breve{x}} K''_x(y)\right)_{\breve{y}} + \frac{h}{12}\left(\left\{\frac{p}{q}\right\}_{\breve{y}} K''_y(x)\right)_{\breve{x}}\\
&+ \frac{h^2}{72}\left[\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{x_i} - \left\{\frac{r}{p}\right\}_{x_i} K'_x(y)\right]_{y_j}.
\end{aligned}$$

Remark 2.1. *If one omits the last term in (2.9) the resulting scheme is already of order lower than four. see also Table 5 in [1].*

The following assertion was proved in [1].

Proposition 2.2. *Suppose that* $p,\ q \in \mathcal{H}_\Omega^{5+\alpha}$, $r,\ f \in \mathcal{H}_\Omega^{4+\alpha}$, $K_x \in C^{2+\alpha}[0,\eta]\bigcup[\eta,1]$, $K_y \in C^{2+\alpha}[0,\xi]\bigcup[\xi,1]$, $u \in C(\overline{\Omega})\bigcap\mathcal{H}_\Omega^{6+\alpha}$, $\alpha \in (0,1)$. *Then the truncation error of the scheme (2.4)-(2.9) is of third order.*

3. CONVERGENCE AND ERROR ESTIMATES

Let us introduce the scalar products and the corresponding norms:

$$
\begin{aligned}
(U,V)_{\bar\omega} &= \sum_{i=0}^{N}\sum_{j=0}^{M} \hbar_i \bar k_j (UV)_{\breve x_i \breve y_j}, \ \|U\|_0^2 = (U,U)_{\bar\omega}, \\
(U,V)_{\omega_1^+\times\bar\omega_2} &= \sum_{i=1}^{N}\sum_{j=0}^{M} h_i \bar k_j (UV)_{\breve y_j, i}, \ \|U_{\bar x}\|_0^2 = (U_{\bar x}, U_{\bar x})_{\omega_1^+\times\bar\omega_2}, \\
(U,V)_{\bar\omega_1\times\omega_2^+} &= \sum_{i=0}^{N}\sum_{j=1}^{M} \hbar_i k_j (UV)_{\breve x_i, j}, \ \|U_{\bar y}\|_0^2 = (U_{\bar y}, U_{\bar y})_{\bar\omega_1\times\omega_2^+}, \\
(U,V)_{\omega_1^+\times\omega_2^+} &= \sum_{i=1}^{N}\sum_{j=1}^{M} h_i k_j (UV)_{i,j}, \ \|U_{\bar x\bar y}\|_0^2 = (U_{\bar x}, U_{\bar x})_{\omega_1^+\times\omega_2^+}, \\
(U,V)_{\omega_1\times\bar\omega_2} &= \sum_{i=1}^{N-1}\sum_{j=0}^{M} \hbar_i \bar k_j (UV)_{\breve x_i \breve y_j}, \ \|U_{\bar x\hat x}\|_0^2 = (U_{\bar x\hat x}, U_{\bar x\hat x})_{\omega_1\times\bar\omega_2}, \\
(U,V)_{\bar\omega_1\times\omega_2} &= \sum_{i=0}^{N}\sum_{j=1}^{M-1} \hbar_i \bar k_j (UV)_{\breve x_i \breve y_j}, \ \|U_{\bar y\hat y}\|_0^2 = (U_{\bar y\hat y}, U_{\bar y\hat y})_{\bar\omega_1\times\omega_2}, \\
\|\nabla U\|_0^2 &= \|U_{\bar x}\|_0^2 + \|U_{\bar y}\|_0^2, \\
\|U\|_1^2 &= \|U\|_0^2 + \|\nabla U\|_0^2, \\
\|U\|_2^2 &= \|U_{\bar x\hat x}\|_0^2 + \|U_{\bar y\hat y}\|_0^2 + 2\|U_{\bar x\bar y}\|_0^2,
\end{aligned}
$$

where $\bar\omega_1 = \bar\omega_h$, $\bar\omega_2 = \bar\omega_k$, $\bar\omega = \bar\omega_1 \times \bar\omega_2$, $\gamma = \partial\Omega \cap \bar\omega$.

Using Green's formula it is easily to check the identity [18]

$$\|U\|_2^2 = \|\Delta U\|_0^2 = \|U_{\bar x\hat x} + U_{\bar y\hat y}\|_0^2.$$

When deriving a priori error estimates of the difference schemes the following negative mesh norm will be often used

$$\|\Psi\|_{-1} = \sup_{v|_\gamma = 0} \frac{|(\Psi, v)_\omega|}{\|v\|_1}.$$

Lemma 3.1. *For every mesh function $v(x,y)$ with zero boundary values Friedrichs inequality holds*

$$\|\nabla v\|_0^2 \geq 16\|v\|_0^2. \tag{3.1}$$

The following equalities will also be used:

$$-\left((pU_{\bar{x}})_{\hat{x}\breve{y}}, U\right)_\omega = \left(p_{\breve{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2}, \tag{3.2}$$

$$-\left((qU_{\bar{y}})_{\hat{y}\breve{x}}, U\right)_\omega = \left(q_{\breve{x}}, U_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+}. \tag{3.3}$$

3.1. SECOND ORDER DIFFERENCE SCHEME

Theorem 3.1. *The problem*

$$\Lambda U = -\left(pU_{\bar{x}}\right)_{\hat{x}\breve{y}} - \left(qU_{\bar{y}}\right)_{\hat{y}\breve{x}} + r_{\breve{x}\breve{y}}U = \varphi,\ U|_\gamma = 0 \tag{3.4}$$

has a unique solution that satisfies the estimate

$$\|U\|_1 \le C\|\varphi\|_{-1}.$$

Proof. We take the scalar product of (3.4) and U and sum on up the mesh w. Using (3.2), (3.3) we obtain

$$\begin{aligned}(\Lambda U, U)_\omega &= \left(p_{\breve{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2} + \left(q_{\breve{x}}, U_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+} + \left(r_{\breve{x}\breve{y}}, U^2\right)_\omega \\ &\ge \left(p_{\breve{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2} + \left(q_{\breve{x}}, U_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+} \ge c_1\|\nabla U\|_0^2, \\ &\text{where } c_1 = \min\{p, q\}.\end{aligned}$$

Therefore

$$\|\nabla U\|_0^2 \le \frac{1}{c_1}(\Lambda U, U)_\omega. \tag{3.5}$$

From (3.1) and (3.5) we get

$$\|U\|_1^2 \le \frac{17}{16c_1}(\Lambda U, U)_\omega = \frac{17}{16c_1}(\varphi, U)_\omega, \tag{3.6}$$

which implies the existence and uniqueness of the solution of (3.4). Further, from the inequality $|(\varphi, U)_\omega| \le \|\varphi\|_{-1}\|U\|_1$ and (3.6), follows the estimate in the theorem □.

Theorem 3.2. *Suppose that the assumptions in Proposition 2.1 are fulfilled. Then for the error $z_{ij} = U_{ij} - u(x_i, y_j)$ of the difference scheme (2.1) the error estimate holds*

$$\|z\|_1 \le C\left(\|h^2\|_0 + \|k^2\|_0\right). \tag{3.7}$$

Proof. We have

$$\Lambda z_{ij} = \Psi_{ij} = \left((pu_{\bar{x}})_{\hat{x}_i} - \frac{1}{\hbar_i}[w_1]_{x_i} - \left(\frac{\partial w_1}{\partial x}\right)_{\hat{x}_i} \right)_{\breve{y}_j} + \left((qu_{\bar{y}})_{\hat{y}_j} - \frac{1}{\bar{k}_j}[w_2]_{y_j} - \left(\frac{\partial w_2}{\partial y}\right)_{\breve{y}_j} \right)_{\breve{x}_i} = (\varphi_i(y))_{\breve{y}_j} + (\psi_j(x))_{\breve{x}_i}.$$

$$\varphi_i(y) = \eta^x_{\hat{x}_i} + O(\hbar_i^2), \quad \text{where } \eta^x_i(y) = \frac{h_i^2}{6}\left(\frac{1}{4}p\frac{\partial^3 u}{\partial x^3} + \frac{3}{4}\frac{\partial^2}{\partial x^2}\left(p\frac{\partial u}{\partial x}\right)\right)(x_{i-1/2}, y),$$

$$\psi_j(x) = \eta^y_{\hat{y}_j} + O(\bar{k}_i^2), \quad \text{where } \eta^y_j(x) = \frac{k_j^2}{6}\left(\frac{1}{4}q\frac{\partial^3 u}{\partial y^3} + \frac{3}{4}\frac{\partial^2}{\partial y^2}\left(q\frac{\partial u}{\partial y}\right)\right)(x, y_{j-1/2}).$$

$$\Lambda z_{ij} = \eta^x_{\hat{x}_i\breve{y}_j} + \eta^y_{\hat{y}_j\breve{x}_i} + \Psi^*_{ij}, \tag{3.8}$$

where $\Psi^*_{ij} = O(\hbar_i^2 + \bar{k}_j^2)$. (3.8) whit z_{ij} and . Using Green's formula, we obtain

$$(\Psi, z)_\omega = -\left(\eta^x_{\breve{y}}, z_{\bar{x}}\right)_{\omega_1^+\times\omega_2} - \left(\eta^y_{\breve{x}}, z_{\bar{y}}\right)_{\omega_1\times\omega_2^+} + (\Psi^*, z)_{\bar{\omega}}. \tag{3.9}$$

Hence

$$\begin{aligned} |(\Psi, z)_\omega| &\le C\left(\|h^2\|_0\|z_{\bar{x}}\|_0 + \|k^2\|_0\|z_{\bar{y}}\|_0 + (\|h^2\|_0 + \|k^2\|_0)\|z\|_0\right) \\ &\le C\left(\|h^2\|_0 + \|k^2\|_0\right)\|z\|_1, \end{aligned} \tag{3.10}$$

where $\|h^2\|_0^2 = \sum_{i=0}^N \hbar_i^5$, $\|k^2\|_0^2 = \sum_{j=0}^M \bar{k}_j^5$, and C is a constant independent of h, k. Therefore

$$\|z\|_1 \le C\left(\|h^2\|_0 + \|k^2\|_0\right). \quad \square \tag{3.11}$$

3.2. FOURTH ORDER DIFFERENCE SCHEME

Lemma 3.2. *Let the mesh ω_h satisfy $\frac{1}{\alpha} \le \frac{h_i+1}{h_i}, \frac{k_j+1}{k_j} \le \alpha$, $i = 1, ..., N-1$; $j = 1, .., M-1$, where $1 \le \alpha \le 1.45$ and V is a mesh function defined on the mesh $\overline{\omega}$ with zero boundary conditions $V|_\gamma = 0$. Then*

$$\left|\left([k]\,(pV)_{\bar{x}\overset{\circ}{y}}, V_{\bar{x}}\right)_{\omega_1^+\times\omega_2}\right| \le \left(p_{\breve{y}}, V_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2},$$

$$\left|\left([h]\,(qV)_{\bar{y}\overset{\circ}{x}}, V_{\bar{y}}\right)_{\omega_1\times\omega_2^+}\right| \le \left(q_{\breve{x}}, V_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+},$$

$$\left|\left([h]\,(rV)_{\overset{\circ}{x}\breve{y}} + [k]\,(rV)_{\overset{\circ}{y}\breve{x}} + \frac{2}{3}[h]\,[k]\,(rV)_{\overset{\circ}{x}\overset{\circ}{y}}, V\right)_\omega\right| \le 2\left(r_{\breve{x}\breve{y}}, V^2\right)_\omega.$$

Proof. Let us denote
$[h]_{x_i} = [h]_i,\ [k]_{y_j} = [k]_j,\ [h]_0 = 0,\ [h]_N = 0,\ [k]_0 = 0,\ [k]_M = 0$. Then

$$\begin{aligned}
&\left([h]\,(pV)_{\overset{\circ}{x}}, U\right)_{\omega_1} \\
= &\ \frac{1}{2}\sum_{i=1}^{N-1}[h]_i V_i\left(\frac{h_{i+1}}{h_i}p(x_i-)\,(V_i - V_{i-1}) + \frac{h_i}{h_{i+1}}p(x_i+)\,(V_{i+1}-V_i)\right) \\
= &\ \sum_{i=1}^{N-1}\left(\left([h]_{i-1}\frac{h_{i-1}}{h_i} - [h]_i\frac{h_{i+1}}{h_i}\right)p(x_i-)V_{i-1}V_i\right. \\
+ &\ \left.\frac{[h]_i}{2}\left(\frac{h_{i+1}}{h_i}p(x_i-) - \frac{h_i}{h_{i+1}}p(x_i+)\right)V_i^2\right)
\end{aligned}$$

The inequalities

$$\begin{aligned}
|V_{i-1}V_i| &\le \frac{1}{2}\sqrt{\frac{\hbar_{i-1}}{\hbar_i}}V_{i-1}^2 + \frac{1}{2}\sqrt{\frac{\hbar_i}{\hbar_{i-1}}}V_i^2, \\
|V_iV_{i+1}| &\le \frac{1}{2}\sqrt{\frac{\hbar_{i+1}}{\hbar_i}}V_{i+1}^2 + \frac{1}{2}\sqrt{\frac{\hbar_i}{\hbar_{i+1}}}V_i^2.
\end{aligned}$$

imply

$$\begin{aligned}
\left|\left([h]\,(pV)_{\overset{\circ}{x}}, V\right)_{\omega_1}\right| \le &\ \max_i\left\{\frac{[h]_i^2}{h_ih_{i+1}} + \frac{1}{4}\sqrt{\frac{\hbar_{i-1}}{\hbar_i}}\left|\frac{[h]_{i-1}}{\hbar_{i-1}}\frac{h_{i-1}}{h_i} - \frac{[h]_i}{\hbar_i}\frac{h_{i+1}}{h_i}\right|\right. \\
+ &\ \left.\frac{1}{4}\sqrt{\frac{\hbar_{i+1}}{\hbar_i}}\left|\frac{[h]_i}{\hbar_i}\frac{h_i}{h_{i+1}} - \frac{[h]_{i+1}}{\hbar_{i+1}}\frac{h_{i+2}}{h_{i+1}}\right|\right\}\left(p_{\check{x}}, V^2\right)_{\omega_1}.
\end{aligned}$$

Since

$$\frac{[h]_i}{2}\left(\frac{h_{i+1}}{h_i}r(x_i-) - \frac{h_i}{h_{i+1}}r(x_i+)\right) = \begin{cases} 0, & r(x_i-) \ne r(x_i+), \\ \hbar_i\frac{[h]_i^2}{h_ih_{i+1}}r(x_i), & [r]_{x_i} = 0. \end{cases}$$

$$\frac{[h]_i^2}{h_ih_{i+1}} \le \alpha - 2 + \frac{1}{\alpha},\quad \frac{\hbar_i}{\hbar_{i-1}} \le \alpha,\quad \frac{|[h]_i|}{\hbar_i} \le \frac{2(\alpha-1)}{\alpha+1},$$

we have

$$\left|\left([h]\,(pV)_{\overset{\circ}{x}\check{y}}, V\right)_{\omega}\right| \le \left(\alpha - 2 + \frac{1}{\alpha} + 2\alpha\sqrt{\alpha}\frac{\alpha-1}{\alpha+1}\right)\left(p_{\check{x}\check{y}}, V^2\right)_{\omega} \le \left(p_{\check{x}\check{y}}, V^2\right)_{\omega}.$$

Analogously one can prove that

$$\left|\left([k]\,(qV)_{\mathring{y}\check{x}}, V\right)_\omega\right| \leq \left(q_{\check{x}\check{y}}, V^2\right)_\omega .$$

Next, we have

$$\left([h]\,[k]\,(rV)_{\mathring{x}\mathring{y}}, V\right)_\omega = \frac{1}{4}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}[h]_i[k]_jV_{ij}\,\big(h_{i+1}k_{j+1}r(x_i-,y_j-)V_{\check{x}_i\check{y}_j}$$

$$+\ h_ik_{j+1}r(x_i+,y_j-)V_{\check{x}_{i+1}\check{y}_j} + h_ik_jr(x_i+,y_j+)V_{\check{x}_{i+1}\check{y}_{j+1}} + h_{i+1}k_jr(x_i-,y_j+)V_{\check{x}_i\check{y}_{j+1}}\big)$$

$$= \sum_{i=1}^{N-1}\sum_{j=1}^{M-1}\left(\frac{[h]_i^2}{h_ih_{i+1}}\frac{[k]_j^2}{k_jk_{j+1}}\hbar_i\bar{k}_jr_{\check{x}_i\check{y}_j}V_{ij}^2\right.$$

$$+\hbar_i\frac{[h]_i^2}{h_ih_{i+1}}\left([k]_{j-1}\frac{k_{j-1}}{k_j} - [k]_j\frac{k_{j+1}}{k_j}\right)r_{\check{x}_i\check{y}_j}V_{ij-1}V_{ij}$$

$$+\bar{k}_j\frac{[k]_j^2}{k_jk_{j+1}}\left([h]_{i-1}\frac{h_{i-1}}{h_i} - [h]_i\frac{h_{i+1}}{h_i}\right)r_{\check{x}_i\check{y}_j}V_{i-1j}V_{ij}$$

$$+\frac{1}{2}\left([h]_{i-1}[k]_{j-1}\frac{h_{i-1}}{h_i}\frac{k_{j-1}}{k_j} + [h]_i[k]_j\frac{h_{i+1}}{h_i}\frac{k_{j+1}}{k_j}\right)r_{\check{x}_i\check{y}_j}V_{i-1j-1}U_{ij}$$

$$-\frac{1}{4}\left([h]_{i-1}[k]_{j+1}\frac{h_{i-1}}{h_i}\frac{k_{j+2}}{k_{j+1}} + [h]_i[k]_j\frac{h_{i+1}}{h_i}\frac{k_j}{k_{j+1}}\right)r_{\check{x}_i\check{y}_j}V_{i-1j+1}U_{ij}$$

$$\left.-\frac{1}{4}\left([h]_i[k]_j\frac{h_i}{h_{i+1}}\frac{k_{j+1}}{k_j} + [h]_{i+1}[k]_{j-1}\frac{h_{i+2}}{h_{i+1}}\frac{k_{j-1}}{k_j}\right)r_{\check{x}_i\check{y}_j}V_{i+1j-1}V_{ij}\right)$$

$$\leq \max_i\left\{\frac{[h]_i^2}{h_ih_{i+1}}\frac{[k]_j^2}{k_jk_{j+1}} + \sqrt{\frac{\hbar_i\bar{k}_j}{\hbar_{i+1}\bar{k}_{j+1}}}\left|\frac{[h]_i[k]_j}{\hbar_i\bar{k}_j}\frac{h_ik_j}{h_{i+1}k_{j+1}}\right|\right\}\left(r_{\check{x}\check{y}}, V^2\right)_\omega$$

$$\leq \left(\left(\alpha - 2 + \frac{1}{\alpha}\right)^2 + 4\alpha^3\frac{(\alpha-1)^2}{(\alpha+1)^2}\right)\left(r_{\check{x}\check{y}}, V^2\right)_\omega .$$

Therefore

$$\left|\left([h]\,(rV)_{\mathring{x}\check{y}} + [k]\,(rV)_{\mathring{y}\check{x}} + \frac{2}{3}\,[h]\,[k]\,(rV)_{\mathring{x}\mathring{y}}, V\right)_\omega\right|$$

$$\leq \left(2\left(\alpha - 2 + \frac{1}{\alpha}\right) + 4\alpha\sqrt{\alpha}\frac{\alpha-1}{\alpha+1} + \frac{2}{3}\left(\alpha - 2 + \frac{1}{\alpha}\right)^2 + \frac{8}{3}\alpha^3\frac{(\alpha-1)^2}{(\alpha+1)^2}\right)\left(r_{\check{x}\check{y}}, V^2\right)_\omega$$

$$\leq 2\left(r_{\check{x}\check{y}}, V^2\right)_\omega . \qquad \square$$

Theorem 3.3. *The solution of the problem*

$$\begin{aligned}
\Lambda' U &= -(pU_{\bar{x}})_{\hat{x}\check{y}} - (qU_{\bar{y}})_{\hat{y}\check{x}} + r_{\check{x}\check{y}}U - \frac{1}{12}\left(\left(k^2(pU_{\bar{x}})_{\hat{x}\check{y}}\right)_{\hat{y}} + \left(h^2(qU_{\bar{y}})_{\hat{y}\check{x}}\right)_{\hat{x}}\right) \\
&- \frac{1}{6}\left([k]_{y_j}(pU_{\bar{x}})_{\hat{x}\mathring{y}} + [h]_{x_i}(qU_{\bar{y}})_{\hat{y}\mathring{x}}\right) + \frac{1}{12}\left((h^2rU_{\bar{x}})_{\hat{x}\check{y}} + (k^2rU_{\bar{y}})_{\hat{y}\check{x}}\right) \quad (3.12) \\
&+ \frac{1}{6}\left([h]_{x_i}(rU)_{\mathring{x}\check{y}} + [k]_{y_j}(rU)_{\mathring{y}\check{x}} + \frac{2}{3}[h]_{x_i}[k]_{y_j}(rU)_{\mathring{x}\mathring{y}}\right) = \varphi',\ U|_\gamma = 0,
\end{aligned}$$

where φ' stands for the right hand-sides in (2.4)-(2.9), respectively, exists and is unique. It satisfies the a priori estimates:

$$\|U\|_1 \le C\|\varphi'\|_{-1}, \quad (3.13)$$

$$\|U\|_2 \le C\|\varphi'\|_0, \quad (3.14)$$

where the constant C doesn't depend on the mesh.

Proof. Let us organize the scalar product

$$\begin{aligned}
(\Lambda' U, U)_\omega &= -\left((pU_{\bar{x}})_{\hat{x}\check{y}}, U\right)_\omega - \left((qU_{\bar{y}})_{\hat{y}\check{x}}, U\right)_\omega + (r_{\check{x}\check{y}}U, U)_\omega \\
&-\frac{1}{12}\left(\left(k^2(pU_{\bar{x}})_{\hat{x}\check{y}}\right)_{\hat{y}}, U\right)_\omega - \frac{1}{12}\left(\left(h^2(qU_{\bar{y}})_{\hat{y}\check{x}}\right)_{\hat{x}}, U\right)_\omega \\
&+\frac{1}{12}\left((h^2rU_{\bar{x}})_{\hat{x}\check{y}}, U\right)_\omega + \frac{1}{12}\left((k^2rU_{\bar{y}})_{\hat{y}\check{x}}\right)_\omega \quad (3.15) \\
&-\frac{1}{6}\left([k](pU_{\bar{x}})_{\hat{x}\mathring{y}}, U\right)_\omega - \frac{1}{6}\left([h](qU_{\bar{y}})_{\hat{y}\mathring{x}}, U\right)_\omega \\
&+\frac{1}{6}\left([h]_{x_i} rU_{\mathring{x}\check{y}}, U\right)_\omega + \frac{1}{6}\left([k]_{y_j} rU_{\mathring{y}\check{x}}, U\right)_\omega \\
&+\frac{1}{9}\left([h]_{x_i}[k]_{y_j} rU_{\mathring{x}\mathring{y}}, U\right)_\omega .
\end{aligned}$$

In view of formulas (3.2), (3.3)

$$\begin{aligned}
\left(\left(k^2(pU_{\bar{x}})_{\hat{x}\check{y}}\right)_{\hat{y}}, U\right)_\omega &= (k^2p, U^2_{\bar{x}\bar{y}})_{\omega_1^+\times\omega_2^+}, \\
\left(\left(h^2(qU_{\bar{y}})_{\hat{y}\bar{x}}\right)_{\hat{x}}, U\right)_\omega &= (h^2q, U^2_{\bar{x}\bar{y}})_{\omega_1^+\times\omega_2^+} \\
-\left((h^2rU_{\bar{x}})_{\hat{x}\check{y}}, U\right)_\omega &= (k^2r_{\check{x}}, U^2_{\bar{y}})_{\omega_1\times\omega_2^+}, \\
-\left((k^2rU_{\bar{y}})_{\hat{y}\check{x}}, U\right)_\omega &= (h^2r_{\check{y}}, U^2_{\bar{x}})_{\omega_1^+\times\omega_2}, \\
-\left([k](pU_{\bar{x}})_{\hat{x}\mathring{y}}, U\right)_\omega &= \left([k](pU)_{\bar{x}\mathring{y}}, U^2_{\bar{x}}\right)_{\omega_1^+\times\omega_2}, \\
-\left([h](qU)_{\mathring{x}\check{y}}, U\right)_\omega &= \left([h](qU)_{\bar{y}\mathring{x}}, U^2_{\bar{y}}\right)_{\omega_1\times\omega_2^+};
\end{aligned}$$

we can cultivate (3.18) in the form:

$$
\begin{aligned}
&(\Lambda' U, U)_\omega \\
= \;& (p_{\check{y}}, U_{\bar{x}}^2)_{\omega_1^+ \times \omega_2} + (q_{\check{x}}, U_{\bar{y}}^2)_{\omega_1 \times \omega_2^+} + (r_{\check{x}\check{y}} U, U)_\omega \\
- \;& \frac{1}{12} (k^2 p + h^2 q, U_{\bar{x}\bar{y}}^2)_{\omega_1^+ \times \omega_2^+} - \frac{1}{12} (h^2 r_{\check{y}}, U_{\bar{x}}^2)_{\omega_1^+ \times \omega_2} - \frac{1}{12} (k^2 r_{\check{x}}, U_{\bar{y}}^2)_{\omega_1 \times \omega_2^+} \\
+ \;& \frac{1}{6} \left([k] (pU)_{\bar{x}\mathring{y}}, U_{\bar{x}} \right)_{\omega_1^+ \times \omega_2} + \frac{1}{6} \left([h] (qU)_{\bar{y}\mathring{x}}, U_{\bar{y}} \right)_{\omega_1 \times \omega_2^+} \\
+ \;& \frac{1}{6} \left([h]_{x_i} (rU)_{\mathring{x}\check{y}} + [k]_{y_j} (rU)_{\mathring{y}\check{x}} + \frac{2}{3} [h]_{x_i} [k]_{y_j} (rU)_{\mathring{x}\mathring{y}}, U \right)_\omega .
\end{aligned}
$$

Further, we will use the inequalities

$$
\begin{aligned}
(k^2 p, U_{\bar{x}\bar{y}}^2)_{\omega_1^+ \times \omega_2^+} &\le 4 (p_{\check{y}}, U_{\bar{x}}^2)_{\omega_1^+ \times \omega_2}, \\
(h^2 q, U_{\bar{x}\bar{y}}^2)_{\omega_1^+ \times \omega_2^+} &\le 4 (q_{\check{x}}, U_{\bar{y}}^2)_{\omega_1 \times \omega_2^+} \\
(h^2 r_{\check{y}}, U_{\bar{x}}^2)_{\omega_1^+ \times \omega_2} &\le 4 (r_{\check{x}\check{y}}, U^2)_\omega \\
(k^2 r_{\check{x}}, U_{\bar{y}}^2)_{\omega_1 \times \omega_2^+} &\le 4 (r_{\check{x}\check{y}}, U^2)_\omega
\end{aligned}
$$

We will check only the first one.The others can proved analogously.

$$
\begin{aligned}
0 \;\le\; & (k^2 p, U_{\bar{x}\bar{y}}^2)_{\omega_1^+ \times \omega_2^+} = \sum_{i=1}^{N} \sum_{j=1}^{M} h_i k_j p(x_i -, y_j -) \left(U_{\bar{x}_{i,j}} - U_{\bar{x}_{i,j-1}} \right)^2 \\
\le \;& 2 \sum_{i=1}^{N} \sum_{j=1}^{M} h_i k_j p(x_i -, y_j -) \left(U_{\bar{x}_{i,j}}^2 + U_{\bar{x}_{i,j-1}}^2 \right) = 4 (p_{\check{y}}, U_{\bar{x}}^2)_{\omega_1^+ \times \omega_2} .
\end{aligned}
$$

Using these inequalities and Lemma(3.2), we obtain

$$
(\Lambda' U, U)_\omega \;\ge\; \frac{1}{2} \left((p_{\check{y}}, U_{\bar{x}}^2)_{\omega_1^+ \times \omega_2} + (q_{\check{x}}, U_{\bar{y}}^2)_{\omega_1 \times \omega_2^+} \right) \ge \frac{c_0}{2} \|\nabla U\|_0^2 .
$$

Therefore

$$
\|\nabla U\|_0^2 \le \frac{2}{c_0} (\Lambda' U, U)_\omega . \tag{3.16}
$$

It follows from (3.1) and (3.16) that

$$
\|U\|_1^2 \le \frac{17}{8 c_0} (\Lambda U, U)_\omega = \frac{17}{8 c_0} (\varphi, U)_\omega , \tag{3.17}
$$

which implies existence and uniqueness of the solution of (3.12).Next, from the inequality $|(\varphi, U)_\omega| \le \|\varphi\|_{-1} \|U\|_1$ and (3.17) follows the desired estimate (3.13).

Let us organize the scalar product

$$
\begin{aligned}
-\left(\Lambda' U, \triangle U\right)_\omega = & \left((pU_{\bar{x}})_{\hat{x}\check{y}} + (qU_{\bar{y}})_{\hat{y}\check{x}}, \triangle U\right)_\omega - \left(r_{\check{x}\check{y}} U, \triangle U\right)_\omega \\
& + \frac{1}{12}\left(\left(k^2 (pU_{\bar{x}})_{\hat{x}\bar{y}}\right)_{\hat{y}} + \left(h^2 (qU_{\bar{y}})_{\hat{y}\bar{x}}\right)_{\hat{x}}, \triangle U\right)_\omega \\
& - \frac{1}{12}\left(\left(h^2 r U_{\bar{x}}\right)_{\hat{x}\check{y}} + \left(k^2 r U_{\bar{y}}\right)_{\hat{y}\check{x}}, \triangle U\right)_\omega \\
& + \frac{1}{6}\left([k] (pU_{\bar{x}})_{\hat{x}\mathring{y}} + [h] (qU_{\bar{y}})_{\hat{y}\mathring{x}}, \triangle U\right)_\omega \\
& - \frac{1}{6}\left([h]_{x_i} r U_{\mathring{x}\check{y}} + [k]_{y_j} r U_{\mathring{y}\check{x}} + \frac{2}{3}[h]_{x_i}[k]_{y_j} r U_{\mathring{x}\mathring{y}}, \triangle U\right)_\omega .
\end{aligned} \tag{3.18}
$$

From the inequalities:

$$
\begin{aligned}
\left((pU_{\bar{x}})_{\hat{x}\check{y}}, U_{\bar{x}\hat{x}}\right)_\omega &\geq c_0 \|U_{\bar{x}\hat{x}}\|_0^2, \\
\left((qU_{\bar{y}})_{\hat{y}\check{x}}, U_{\bar{y}\hat{y}}\right)_\omega &\geq c_0 \|U_{\bar{y}\hat{y}}\|_0^2, \\
\left((pU_{\bar{x}})_{\hat{x}\check{y}}, U_{\bar{y}\hat{y}}\right)_\omega &\geq c_0 \|U_{\bar{x}\bar{y}}\|_0^2, \\
\left((qU_{\bar{y}})_{\hat{y}\check{x}}, U_{\bar{x}\hat{x}}\right)_\omega &\geq c_0 \|U_{\bar{x}\bar{y}}\|_0^2, \\
\left((pU_{\bar{x}})_{\hat{x}\check{y}} + (qU_{\bar{y}})_{\hat{y}\check{x}}, \triangle U\right)_\omega &\geq c_0 \|U\|_2^2, \\
-\left(r_{\check{x}\check{y}} U, U_{\bar{x}\hat{x}}\right)_\omega &= \left(r_{\check{y}}, U_{\bar{x}}^2\right)_{\omega_1^+ \times \omega_2}, \\
-\left(r_{\check{x}\check{y}} U, U_{\bar{y}\hat{y}}\right)_\omega &= \left(r_{\check{x}}, U_{\bar{y}}^2\right)_{\omega_1 \times \omega_2^+}, \\
-\left(r_{\check{x}\check{y}} U, \triangle U\right)_\omega &\geq r_0 \|U\|_1^2, \\
\left(\left(k^2 (pU_{\bar{x}})_{\hat{x}\bar{y}}\right)_{\hat{y}} + \left(h^2 (qU_{\bar{y}})_{\hat{y}\bar{x}}\right)_{\hat{x}}, \triangle U\right)_\omega &\geq -4c_0 \|U\|_2^2, \\
-\left(\left(h^2 r U_{\bar{x}}\right)_{\hat{x}\check{y}} + \left(k^2 r U_{\bar{y}}\right)_{\hat{y}\check{x}}, \triangle U\right)_\omega &\geq -8r_0 \|U\|_1^2, \\
\left([k] (pU_{\bar{x}})_{\hat{x}\mathring{y}} + [h] (qU_{\bar{y}})_{\hat{y}\mathring{x}}, \triangle U\right)_\omega &\geq -c_0 \|U\|_2^2, \\
-\left([h]_{x_i} r U_{\mathring{x}\check{y}} + [k]_{y_j} r U_{\mathring{y}\check{x}} + \frac{2}{3}[h]_{x_i}[k]_{y_j} r U_{\mathring{x}\mathring{y}}, \triangle U\right)_\omega &\geq -2r_0 \|U\|_1^2,
\end{aligned}
$$

we obtain

$$
-\left(\Lambda' U, \triangle U\right)_\omega \geq \frac{c_0}{2} \|U\|_2^2. \tag{3.19}
$$

Therefore

$$
\|U\|_2^2 \leq -\frac{2}{c_0}\left(\varphi', \triangle U\right)_\omega \leq \frac{2}{c_0} \|\varphi'\|_0 \|U\|_2. \tag{3.20}
$$

Now (3.14) follow from (3.20). □

Now we turn our attention to the convergence of the scheme.

Theorem 3.4. *Suppose that the assumptions in Proposition 2.2 are fulfilled. Then for the error $z_{ij} = U_{ij} - u(x_i, y_j)$ of the difference scheme (3.12) the error estimates hold:*

$$\begin{aligned} &\textit{if the mesh is uniform} \\ &\|z\|_1 \le Ch^4, \\ &\textit{else} \\ &\|z\|_1 \le Ch^3. \end{aligned} \tag{3.21}$$

Proof. We will derive the approximation error, considering several cases.

Case 3.1. Mesh points of type $\diamond$.

In this case

$$\begin{aligned} \Psi &= f - ru + (pu_{\bar{x}})_{\hat{x}} + (qu_{\bar{y}})_{\hat{y}} + \frac{1}{12}\left(h^2(f-ru)_{\bar{x}}\right)_{\hat{x}} + \frac{1}{12}\left(k^2(f-ru)_{\bar{y}}\right)_{\hat{y}} \\ &+ \frac{1}{12}\left(k^2\left(pu_{\bar{x}}\right)_{\hat{x}\bar{y}}\right)_{\hat{y}} + \frac{1}{12}\left(h^2\left(qu_{\bar{y}}\right)_{\hat{y}\bar{x}}\right)_{\hat{x}} + \frac{1}{9}[h]_{x_i}[k]_{y_j}(f-ru)_{\overset{\circ}{x}\overset{\circ}{y}} \\ &+ \frac{[h]_{x_i}}{6}\left(f - ru + (qu_{\bar{y}})_{\hat{y}}\right)_{\overset{\circ}{x}} + \frac{[k]_{y_j}}{6}\left(f - ru + (pu_{\bar{x}})_{\hat{x}}\right)_{\overset{\circ}{y}} \end{aligned}$$

Using the differential equation (1.1), and formulas (20), (21), (31), (32) in [1], we obtain

$$\begin{aligned} (pu_{\bar{x}})_{\hat{x}} &= p\frac{\partial^2 u}{\partial x^2} + \frac{[h^2]_i}{6\hbar_i}p\frac{\partial^3 u}{\partial x^3} + \frac{(h^2)_{\check{x}_i}}{12}p\frac{\partial^4 u}{\partial x^4} + \frac{[h^4]_i}{120\hbar_i}p\frac{\partial^5 u}{\partial x^5} + O(\hbar_i^4), \\ (qu_{\bar{y}})_{\hat{y}} &= q\frac{\partial^2 u}{\partial y^2} + \frac{[k^2]_j}{6\bar{k}_j}q\frac{\partial^3 u}{\partial y^3} + \frac{(k^2)_{\check{y}_j}}{12}q\frac{\partial^4 u}{\partial y^4} + \frac{[k^4]_j}{120\bar{k}_j}q\frac{\partial^5 u}{\partial y^5} + O(\bar{k}_j^4), \\ \left(h^2 g_{\bar{x}}\right)_{\hat{x}} &= \frac{[h^2]_i}{\hbar_i}\frac{\partial g}{\partial x} + (h^2)_{\check{x}_i}\frac{\partial^2 g}{\partial x^2} + \frac{[h^4]_i}{6\hbar_i}\frac{\partial^3 u}{\partial x^3} + O(\hbar_i^4), \\ \left(k^2 g_{\bar{y}}\right)_{\hat{y}} &= \frac{[k^2]_j}{\bar{k}_j}\frac{\partial g}{\partial y} + (k^2)_{\check{y}_j}\frac{\partial^2 u}{\partial y^2} + \frac{[k^4]_j}{6\bar{k}_j}\frac{\partial^3 g}{\partial y^3} + O(\bar{k}_j^4), \\ \left(k^2\left(pu_{\bar{x}}\right)_{\hat{x}\bar{y}}\right)_{\hat{y}} &= \frac{[k^2]_j}{\bar{k}_j}p\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{[h^2]_i[k^2]_j}{6\hbar_i\bar{k}_j}p\frac{\partial^4 u}{\partial x^3 \partial y} + (k^2)_{\check{y}_j}p\frac{\partial^4 u}{\partial x^2 \partial y^2} \\ &+ \frac{[k^4]_j}{6\bar{k}_j}p\frac{\partial^5 u}{\partial x^2 \partial y^3} + \frac{(h^2)_{\check{x}_i}[k^2]_j}{12\bar{k}_j}p\frac{\partial^5 u}{\partial x^4 \partial y} \\ &+ \frac{[h^2]_i(k^2)_{\check{y}_j}}{6\hbar_i}p\frac{\partial^5 u}{\partial x^3 \partial y^2} + O(\hbar_i + \bar{k}_j)^4, \\ \left(h^2\left(qu_{\bar{y}}\right)_{\hat{y}\bar{x}}\right)_{\hat{x}} &= \frac{[h^2]_i}{\hbar_i}q\frac{\partial^3 u}{\partial x \partial y^2} + \frac{[h^2]_i[k^2]_j}{6\hbar_i\bar{k}_j}q\frac{\partial^4 u}{\partial x \partial y^3} + (h^2)_{\check{x}_i}q\frac{\partial^4 u}{\partial x^2 \partial y^2} \\ &+ \frac{[h^4]_j}{6\hbar_i}q\frac{\partial^5 u}{\partial x^3 \partial y^2} + \frac{[h^2]_i(k^2)_{\check{y}_j}}{12\hbar_i}q\frac{\partial^5 u}{\partial x \partial y^4} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{(h^2)_{\check{x}_i}[k^2]_j}{6\bar{k}_j} q \frac{\partial^5 u}{\partial x^2 \partial y^3} + O(\hbar_i + \bar{k}_j)^4, \\
(pu_{\bar{x}})_{\hat{x}\mathring{y}} &= p\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{[h^2]_i}{6\hbar_i} p \frac{\partial^4 u}{\partial x^3 \partial y} + \frac{(h^2)_{\check{x}_i}}{12} p \frac{\partial^5 u}{\partial x^4 \partial y} \\
&+ \quad \frac{k_j k_{j+1}}{3} p \frac{\partial^5 u}{\partial x^2 \partial y^3} + O(\hbar_i + \bar{k}_j)^3, \\
(qu_{\bar{y}})_{\hat{y}\mathring{x}} &= q\frac{\partial^3 u}{\partial x \partial y^2} + \frac{[k^2]_j}{6\bar{k}_j} q \frac{\partial^4 u}{\partial x \partial y^3} + \frac{(k^2)_{\check{y}_j}}{12} q \frac{\partial^5 u}{\partial x \partial y^4} \\
&+ \quad \frac{h_i h_{i+1}}{3} q \frac{\partial^5 u}{\partial x^3 \partial y^2} + O(\hbar_i + \bar{k}_j)^3, \\
g_{\mathring{x}} &= \frac{\partial g}{\partial x} + \frac{h_i h_{i+1}}{6} \frac{\partial^3 u}{\partial x^3} + O(\hbar_i^3), \\
g_{\mathring{y}} &= \frac{\partial g}{\partial y} + \frac{k_j k_{j+1}}{6} \frac{\partial^3 u}{\partial y^3} + O(\bar{k}_j^3), \\
g_{\mathring{x}\mathring{y}} &= \frac{\partial^2 u}{\partial x \partial y} + O(\hbar_i + \bar{k}_j)^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
(pu_{\bar{x}})_{\hat{x}} &= p\frac{\partial^2 u}{\partial x^2} + \frac{[h^2]_i}{6\hbar_i} p \frac{\partial^3 u}{\partial x^3} + \frac{(h^2)_{\check{x}_i}}{12} p \frac{\partial^4 u}{\partial x^4} + \frac{[h^4]_i}{120\hbar_i} p \frac{\partial^5 u}{\partial x^5} + O(\hbar_i^4), \\
(qu_{\bar{y}})_{\hat{y}} &= q\frac{\partial^2 u}{\partial y^2} + \frac{[k^2]_j}{6\bar{k}_j} q \frac{\partial^3 u}{\partial y^3} + \frac{(k^2)_{\check{y}_j}}{12} q \frac{\partial^4 u}{\partial y^4} + \frac{[k^4]_j}{120\bar{k}_j} q \frac{\partial^5 u}{\partial y^5} + O(\bar{k}_j^4), \\
\left(h^2 g_{\bar{x}}\right)_{\hat{x}} &= \frac{[h^2]_i}{\hbar_i} \frac{\partial g}{\partial x} + (h^2)_{\check{x}_i} \frac{\partial^2 g}{\partial x^2} + \frac{[h^4]_i}{6\hbar_i} \frac{\partial^3 u}{\partial x^3} + O(\hbar_i^4), \\
\left(k^2 g_{\bar{y}}\right)_{\hat{y}} &= \frac{[k^2]_j}{\bar{k}_j} \frac{\partial g}{\partial y} + (k^2)_{\check{y}_j} \frac{\partial^2 u}{\partial y^2} + \frac{[k^4]_j}{6\bar{k}_j} \frac{\partial^3 g}{\partial y^3} + O(\bar{k}_j^4), \\
\left(k^2 \left(pu_{\bar{x}}\right)_{\hat{x}\mathring{y}}\right)_{\hat{y}} &= \frac{[k^2]_j}{\bar{k}_j} p \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{[h^2]_i [k^2]_j}{6\hbar_i \bar{k}_j} p \frac{\partial^4 u}{\partial x^3 \partial y} + (k^2)_{\check{y}_j} p \frac{\partial^4 u}{\partial x^2 \partial y^2} \\
&+ \quad \frac{[k^4]_j}{6\bar{k}_j} p \frac{\partial^5 u}{\partial x^2 \partial y^3} + \frac{(h^2)_{\check{x}_i}[k^2]_j}{12\bar{k}_j} p \frac{\partial^5 u}{\partial x^4 \partial y} \\
&+ \quad \frac{[h^2]_i (k^2)_{\check{y}_j}}{6\hbar_i} p \frac{\partial^5 u}{\partial x^3 \partial y^2} + O(\hbar_i + \bar{k}_j)^4, \\
\left(h^2 \left(qu_{\bar{y}}\right)_{\hat{y}\mathring{x}}\right)_{\hat{x}} &= \frac{[h^2]_i}{\hbar_i} q \frac{\partial^3 u}{\partial x \partial y^2} + \frac{[h^2]_i [k^2]_j}{6\hbar_i \bar{k}_j} q \frac{\partial^4 u}{\partial x \partial y^3} + (h^2)_{\check{x}_i} q \frac{\partial^4 u}{\partial x^2 \partial y^2} \\
&+ \quad \frac{[h^4]_j}{6\hbar_i} q \frac{\partial^5 u}{\partial x^3 \partial y^2} + \frac{[h^2]_i (k^2)_{\check{y}_j}}{12\hbar_i} q \frac{\partial^5 u}{\partial x \partial y^4} \\
&+ \quad \frac{(h^2)_{\check{x}_i}[k^2]_j}{6\bar{k}_j} q \frac{\partial^5 u}{\partial x^2 \partial y^3} + O(\hbar_i + \bar{k}_j)^4,
\end{aligned}
$$

$$
\begin{aligned}
(pu_{\bar{x}})_{\hat{x}\mathring{y}} &= p\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{[h^2]_i}{6\hbar_i} p\frac{\partial^4 u}{\partial x^3 \partial y} + \frac{(h^2)_{\breve{x}_i}}{12} p\frac{\partial^5 u}{\partial x^4 \partial y} \\
&+ \frac{k_j k_{j+1}}{3} p\frac{\partial^5 u}{\partial x^2 \partial y^3} + O(\hbar_i + \bar{k}_j)^3, \\
(qu_{\bar{y}})_{\hat{y}\mathring{x}} &= q\frac{\partial^3 u}{\partial x \partial y^2} + \frac{[k^2]_j}{6\bar{k}_j} q\frac{\partial^4 u}{\partial x \partial y^3} + \frac{(k^2)_{\breve{y}_j}}{12} q\frac{\partial^5 u}{\partial x \partial y^4} \\
&+ \frac{h_i h_{i+1}}{3} q\frac{\partial^5 u}{\partial x^3 \partial y^2} + O(\hbar_i + \bar{k}_j)^3, \\
g_{\mathring{x}} &= \frac{\partial g}{\partial x} + \frac{h_i h_{i+1}}{6}\frac{\partial^3 u}{\partial x^3} + O(\hbar_i^3), \\
g_{\mathring{y}} &= \frac{\partial g}{\partial y} + \frac{k_j k_{j+1}}{6}\frac{\partial^3 u}{\partial y^3} + O(\bar{k}_j^3), \\
g_{\mathring{x}\mathring{y}} &= \frac{\partial^2 u}{\partial x \partial y} + O(\hbar_i + \bar{k}_j)^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Psi_{ij} &= \frac{1}{72\hbar_i}\left[h^2 k^2 \left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)\right]_{x_i \breve{y}_j} \\
&+ \frac{1}{72\bar{k}_j}\left[h^2 k^2 \left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)\right]_{y_j \breve{x}_i} \\
&- \frac{1}{180\hbar_i}\left[h^4 p\frac{\partial^5 u}{\partial x^5}\right]_{x_i \breve{y}_j} - \frac{1}{180\bar{k}_j}\left[k^4 q\frac{\partial^5 u}{\partial y^5}\right]_{y_j \breve{x}_i} - \Psi_{ij}^* + \Psi_{ij}^{**} \\
&= \frac{\left(k^2\right)_{\breve{y}_j}}{72}\left(h^2 \mu^x\right)_{\hat{x}_i, j} + \frac{\left(h^2\right)_{\breve{x}_i}}{72}\left(k^2 \mu^y\right)_{\hat{y}_j, i} - \frac{1}{180}\left(h^4 \eta^x\right)_{\hat{x}_i, j} \\
&- \frac{1}{180}\left(k^4 \eta^y\right)_{\hat{y}_j, i} - \Psi_{ij}^* + \Psi_{ij}^{**},
\end{aligned}
$$

where

$$
\mu_i^x(y) = \left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)(x_{i-1/2}, y),\ \mu_j^y(x) = \left(r\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 f}{\partial x^2 \partial y}\right)(x, y_{j-1/2}),
$$

$$
\eta_i^x(y) = \left(p\frac{\partial^5 u}{\partial x^5}\right)(x_{i-1/2}, y),\ \eta_j^y(x) = \left(q\frac{\partial^5 u}{\partial y^5}\right)(x, y_{j-1/2}),
$$

$$
\Psi_{ij}^* = \frac{h_i h_{i+1}}{72\hbar_i}\left[h^2 p\frac{\partial^5 u}{\partial x^5}\right]_{x_i} + \frac{k_j k_{j+1}}{72\bar{k}_j}\left[k^2 p\frac{\partial^5 u}{\partial y^5}\right]_{y_j},\ \Psi_{ij}^{**} = O(\hbar_i + \bar{k}_j)^4.
$$

Case 3.2. Points $x_i = \xi$, $y_j \neq \eta$ of type $*$. Now

$$\begin{aligned}
\Psi &= f_{\breve{x}} - r_{\breve{x}} u + (pu_{\bar{x}})_x + (qu_{\bar{y}})_{\hat{y}\breve{x}} + \frac{h^2}{12}(f - ru)_{\bar{x}x} + \frac{1}{12}\left(k^2 (f - ru)_{\bar{y}}\right)_{\hat{y}\breve{x}} \\
&+ \frac{1}{12}\left(k^2 (pu_{\bar{x}})_{\hat{x}\bar{y}}\right)_{\hat{y}} + \frac{h^2}{12}(qu_{\bar{y}})_{\hat{y}\bar{x}x} + \frac{[k]_{y_j}}{6}\left(f_{\breve{x}} - r_{\breve{x}} u + (pu_{\bar{x}})_{\hat{x}}\right)_{\overset{\circ}{y}} \\
&- \left(\frac{h}{12}\left\{\frac{r}{p}\right\}_{x_i} + \frac{1}{h}\right) K_x(y_j) - \frac{1}{12h}\left(k^2 K_x(y)\right)_{\bar{y}\hat{y}} + \frac{h}{12}\left\{\frac{q}{p}\right\}_{x_i} (K_x(y))_{\bar{y}\hat{y}} \\
&+ \frac{h}{12}\left[\frac{\partial f}{\partial x}\right]_{x_i} - \frac{[k]_{y_j}}{18}\left(\frac{3}{h}(K_x(y))_{\overset{\circ}{y}} + h\left\{\frac{r}{p}\right\}_{x_i} K'_x(y_j) - h\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{x_i,j}\right)
\end{aligned}$$

Using the differential equation (1.1), and formulas (20), (21), (31), (32) in [1], we obtain

$$\begin{aligned}
(pu_{\bar{x}})_{\hat{x}} &= \frac{1}{h} K_x(y_j) + \left(p\frac{\partial^2 u}{\partial x^2}\right)_{x_i} + \frac{h}{6}\left[p\frac{\partial^3 u}{\partial x^3}\right] + \frac{h^2}{12}\left(p\frac{\partial^4 u}{\partial x^4}\right)_{x_i} \\
&+ \frac{h^3}{120}\left[p\frac{\partial^5 u}{\partial x^5}\right]_{x_i} + O(h^4), \\
(qu_{\bar{y}})_{\hat{y}\breve{x}} &= \{q\}_{x_i}\frac{\partial^2 u}{\partial y^2} + \frac{[k^2]_j}{6\bar{k}_j}\{q\}_{x_i}\frac{\partial^3 u}{\partial y^3} + \frac{(k^2)_{\breve{y}_j}}{12}\{q\}_{x_i}\frac{\partial^4 u}{\partial y^4} \\
&+ \frac{[k^4]_j}{120\bar{k}_j}\{q\}_{x_i}\frac{\partial^5 u}{\partial y^5} + O(\bar{k}_j^4), \\
h^2 g_{\bar{x}x} &= h\left[\frac{\partial g}{\partial x}\right]_{x_i} + h^2\left\{\frac{\partial^2 g}{\partial x^2}\right\}_{x_i} + \frac{h^3}{6}\left[\frac{\partial^3 g}{\partial x^3}\right]_{x_i} + O(h^4), \\
\left(k^2 g_{\bar{y}}\right)_{\hat{y}\breve{x}} &= \frac{[k^2]_j}{\bar{k}_j}\left\{\frac{\partial g}{\partial y}\right\}_{x_i} + (k^2)_{\breve{y}_j}\left\{\frac{\partial^2 u}{\partial y^2}\right\}_{x_i} + \frac{[k^4]_j}{6\bar{k}_j}\left\{\frac{\partial^3 g}{\partial y^3}\right\}_{x_i} + O(\bar{k}_j^4), \\
\left(k^2 (pu_{\bar{x}})_{x\bar{y}}\right)_{\hat{y}} &= \frac{[k^2]_j}{\bar{k}_j}\left\{p\frac{\partial^3 u}{\partial x^2 \partial y}\right\}_{x_i} + \frac{h[k^2]_j}{6\bar{k}_j}\left[p\frac{\partial^4 u}{\partial x^3 \partial y}\right]_{x_i} + (k^2)_{\breve{y}_j}\left\{p\frac{\partial^4 u}{\partial x^2 \partial y^2}\right\}_{x_i} \\
&+ \frac{[k^4]_j}{6\bar{k}_j}\left\{p\frac{\partial^5 u}{\partial x^2 \partial y^3}\right\}_{x_i} + \frac{h^2[k^2]_j}{12\bar{k}_j}\left\{p\frac{\partial^5 u}{\partial x^4 \partial y}\right\}_{x_i} \\
&+ \frac{h(k^2)_{\breve{y}_j}}{6}\left[p\frac{\partial^5 u}{\partial x^3 \partial y^2}\right]_{x_i} + \frac{[k^2]_j}{h\bar{k}_j} K'_x(y_j) + \frac{(k^2)_{y_j}}{h} K''_x(y_j) \\
&+ \frac{[k^4]_j}{6h\bar{k}_j} K'''_x(y_j) + \frac{(k^4)_{y_j}}{12h} K^{IV}_x(y_j) + O(h + \bar{k}_j)^4, \\
h^2 (qu_{\bar{y}})_{\hat{y}\bar{x}x} &= h\left\{\frac{q}{p}\right\}_{x_i}\left(K''_x(y_j) + \frac{[k^2]_j}{6\bar{k}_j} K'''_x(y_j) + \frac{(k^2)_{y_j}}{12} K^{IV}_x(y_j)\right) \\
&+ h^2\left\{q\frac{\partial^4 u}{\partial x^2 \partial y^2}\right\}_{x_i} + \frac{h^3}{6}\left[q\frac{\partial^5 u}{\partial x^3 \partial y^2}\right]_{x_i}
\end{aligned}$$

$$+ \quad \frac{h^2[k^2]_j}{6\bar{k}_j}\left\{q\frac{\partial^5 u}{\partial x^2 \partial y^3}\right\}_{x_i} + O(h+\bar{k}_j)^4,$$

$$\begin{aligned}
(pu_{\bar{x}})_{x\mathring{y}} &= \frac{1}{h}\left(K_x(y)\right)_{\mathring{y}} + \left\{p\frac{\partial^3 u}{\partial x^2 \partial y}\right\}_{x_i} + \frac{h}{6}\left[p\frac{\partial^4 u}{\partial x^3 \partial y}\right]_{x_i} + \frac{h^2}{12}\left\{p\frac{\partial^5 u}{\partial x^4 \partial y}\right\}_{x_i} \\
&+ \frac{k_j k_{j+1}}{3}\left\{p\frac{\partial^5 u}{\partial x^2 \partial y^3}\right\}_{x_i} + O(h+\bar{k}_j)^3, \\
g_{\mathring{y}\breve{x}} &= \left\{\frac{\partial g}{\partial y}\right\}_{x_i} + \frac{k_j k_{j+1}}{6}\left\{\frac{\partial^3 u}{\partial y^3}\right\}_{x_i} + O(\bar{k}_j^3).
\end{aligned}$$

Then

$$\begin{aligned}
\Psi_{ij} &= \frac{1}{72h}\left[h^2k^2\left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)\right]_{x_i\breve{y}_j} \\
&+ \frac{1}{72\bar{k}_j}\left[h^2k^2\left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)\right]_{y_j\breve{x}_i} \\
&- \frac{1}{180\hbar_i}\left[h^4 p\frac{\partial^5 u}{\partial x^5}\right]_{x_i\breve{y}_j} - \frac{1}{180\bar{k}_j}\left[k^4 q\frac{\partial^5 u}{\partial y^5}\right]_{y_j\breve{x}_i} - \Psi_{ij}^* + \Psi_{ij}^{**} \\
&= \frac{\left(k^2\right)_{\breve{y}_j}}{72}\left(h^2\mu^x\right)_{\hat{x}_i,j} + \frac{\left(h^2\right)_{\breve{x}_i}}{72}\left(k^2\mu^y\right)_{\hat{y}_j,i} - \frac{1}{180}\left(h^4\eta^x\right)_{\hat{x}_i,j} - \frac{1}{180}\left(k^4\eta^y\right)_{\hat{y}_j,i} \\
&- \Psi_{ij}^* + \Psi_{ij}^{**},
\end{aligned}$$

where

$$\mu_i^x(y) = \left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)(x_{i-1/2}, y),\ \mu_j^y(x) = \left(r\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 f}{\partial x^2 \partial y}\right)(x, y_{j-1/2}),$$

$$\eta_i^x(y) = \left(p\frac{\partial^5 u}{\partial x^5}\right)(x_{i-1/2}, y),\ \eta_j^y(x) = \left(q\frac{\partial^5 u}{\partial y^5}\right)(x, y_{j-1/2}),$$

$$\Psi_{ij}^* = \frac{k_j k_{j+1}}{72\bar{k}_j}\left[k^2 q\frac{\partial^5 u}{\partial y^5}\right]_{y_j\breve{x}_i},\quad \Psi_{ij}^{**} = O(h+\bar{k}_j)^4.$$

One can analogously obtain the error in the case $y_j = \eta,\ x_i \neq \xi$.

$$\begin{aligned}
\Psi_{ij} &= \frac{\left(k^2\right)_{\breve{y}_j}}{72}\left(h^2\mu^x\right)_{\hat{x}_i,j} + \frac{\left(h^2\right)_{\breve{x}_i}}{72}\left(k^2\mu^y\right)_{\hat{y}_j,i} - \frac{1}{180}\left(h^4\eta^x\right)_{\hat{x}_i,j} - \frac{1}{180}\left(k^4\eta^y\right)_{\hat{y}_j,i} \\
&- \Psi_{ij}^* + \Psi_{ij}^{**},
\end{aligned}$$

where

$$\mu_i^x(y) = \left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)(x_{i-1/2}, y),\ \mu_j^y(x) = \left(r\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 f}{\partial x^2 \partial y}\right)_{\breve{y}}(x, y_{j-1/2}),$$

$$\eta_i^x(y) = \left(p\frac{\partial^5 u}{\partial x^5}\right)(x_{i-1/2}, y),\ \eta_j^y(x) = \left(q\frac{\partial^5 u}{\partial y^5}\right)(x, y_{j-1/2}),$$

$$\Psi_{ij}^* = \frac{h_i h_{i+1}}{72\bar{k}_j}\left[h^2 p\frac{\partial^5 u}{\partial x^5}\right]_{x_i \breve{y}_j},\ \Psi_{ij}^{**} = O(\hbar_i + h)^4.$$

Case 3.3. Points of type •. Now we have

$$\begin{aligned}
\Psi \ = \ & f_{\breve{x}\breve{y}} - r_{\breve{x}\breve{y}}u + (pu_{\bar{x}})_{x\breve{y}} + (qu_{\bar{y}})_{y\breve{x}} + \frac{h^2}{12}(f - ru)_{\bar{x}x\breve{y}} + \frac{h^2}{12}(f - ru)_{\bar{y}y\breve{x}} \\
+ \ & \frac{h^2}{12}(pu_{\bar{x}})_{x\bar{y}y} + \frac{h^2}{12}(qu_{\bar{y}})_{y\bar{x}x} + \frac{h}{12}\left[\frac{\partial f}{\partial x}\right]_{x_i\breve{y}} + \frac{h}{12}\left[\frac{\partial f}{\partial y}\right]_{y_j\breve{x}} \\
- \ & \left(\frac{h}{12}\left(\left\{\frac{r}{p}\right\}_{x_i} + \frac{1}{h}\right)K_x(y)\right)_{\breve{y}} - \left(\frac{h}{12}\left(\left\{\frac{r}{q}\right\}_{y_j} + \frac{1}{h}\right)K_y(x)\right)_{\breve{x}} \\
- \ & \frac{h}{12}(K_x(y))_{\bar{y}y} - \frac{h}{12}(K_y(x))_{\bar{x}x} + \frac{h}{12}\left(\left\{\frac{q}{p}\right\}_{x_i} K_x''(y)\right)_{\breve{y}} \\
+ \ & \frac{h}{12}\left(\left\{\frac{p}{q}\right\}_{y_j} K_y''(x)\right)_{\breve{x}} + \frac{h^2}{12}\left[\left[\frac{\partial^2 f}{\partial x \partial y} - r\frac{\partial^2 u}{\partial x \partial y}\right]_{x_i}\right]_{y_j}.
\end{aligned}$$

Using the equation (1.1) and the equalities

$$\begin{aligned}
(pu_{\bar{x}})_{x\breve{y}} \ = \ & \frac{1}{h}K_x(y)_{\breve{y}} + \left(p\frac{\partial^2 u}{\partial x^2}\right)_{\breve{x}\breve{y}} + \frac{h}{6}\left[p\frac{\partial^3 u}{\partial x^3}\right]_{x_i\breve{y}} + \frac{h^2}{12}\left(p\frac{\partial^4 u}{\partial x^4}\right)_{\breve{x}\breve{y}} \\
+ \ & \frac{h^3}{120}\left[p\frac{\partial^5 u}{\partial x^5}\right]_{x_i\breve{y}} + O(h^4), \\
(qu_{\bar{y}})_{\hat{y}\breve{x}} \ = \ & \frac{1}{h}K_y(x)_{\breve{x}} + \left(q\frac{\partial^2 u}{\partial y^2}\right)_{\breve{x}\breve{y}} + \frac{h}{6}\left[q\frac{\partial^3 u}{\partial y^3}\right]_{y_j\breve{x}} + \frac{h^2}{12}\left(q\frac{\partial^4 u}{\partial y^4}\right)_{\breve{x}\breve{y}} \\
+ \ & \frac{h^3}{120}\left[q\frac{\partial^5 u}{\partial y^5}\right]_{y_j\breve{x}} + O(h^4), \\
h(K_y)_{\bar{x}x} \ = \ & [K_y']_{x_i} + h\{K_y''\}_{x_i} + \frac{h^2}{6}[K_y''']_{x_i} + \frac{h^3}{12}\{K_y^{IV}\}_{x_i} + O(h^4), \\
h(K_x)_{\bar{y}y} \ = \ & [K_x']_{y_j} + h\{K_x''\}_{y_j} + \frac{h^2}{6}[K_x''']_{y_j} + \frac{h^3}{12}\{K_x^{IV}\}_{y_j} + O(h^4), \\
h^2(pu_{\bar{x}})_{x\bar{y}y} \ = \ & \left[\left[p\frac{\partial^u}{\partial x \partial y}\right]_{x_i}\right]_{y_j} + h\left[p\frac{\partial^3 u}{\partial x \partial y^2}\right]_{x_i\breve{y}} + h\left[p\frac{\partial^3 u}{\partial x^2 \partial y}\right]_{y_j\breve{x}}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad h^2 \left\{ p \frac{\partial^4 u}{\partial x^2 \partial y^2} \right\}_{x_i y_j} + \frac{h^2}{6} \left[\left[p \frac{\partial^4 u}{\partial x^3 \partial y} + p \frac{\partial^4 u}{\partial x \partial y^3} \right]_{x_i} \right]_{y_j} \\
&+ \quad \frac{h^3}{12} \left[p \frac{\partial^5 u}{\partial x \partial y^4} \right]_{x_i \breve{y}} + \frac{h^3}{12} \left[p \frac{\partial^5 u}{\partial x^4 \partial y} \right]_{y_j \breve{x}} + \frac{h^3}{6} \left[p \frac{\partial^5 u}{\partial x^3 \partial y^2} \right]_{x_i \breve{y}} \\
&+ \quad \frac{h^3}{6} \left[p \frac{\partial^5 u}{\partial x^2 \partial y^3} \right]_{y_j \breve{x}} + O(h^4),
\end{aligned}
$$

$$
\begin{aligned}
h^2 \left(q u_{\bar{y}} \right)_{\hat{y} \bar{x} x} &= \left[\left[q \frac{\partial^u}{\partial x \partial y} \right]_{x_i} \right]_{y_j} + h \left[q \frac{\partial^3 u}{\partial x \partial y^2} \right]_{x_i \breve{y}} + h \left[q \frac{\partial^3 u}{\partial x^2 \partial y} \right]_{y_j \breve{x}} \\
&+ \quad h^2 \left\{ q \frac{\partial^4 u}{\partial x^2 \partial y^2} \right\}_{x_i y_j} + \frac{h^2}{6} \left[\left[q \frac{\partial^4 u}{\partial x^3 \partial y} + q \frac{\partial^4 u}{\partial x \partial y^3} \right]_{x_i} \right]_{y_j} \\
&+ \quad \frac{h^3}{12} \left[q \frac{\partial^5 u}{\partial x \partial y^4} \right]_{x_i \breve{y}} + \frac{h^3}{12} \left[q \frac{\partial^5 u}{\partial x^4 \partial y} \right]_{y_j \breve{x}} \\
&+ \quad \frac{h^3}{6} \left[q \frac{\partial^5 u}{\partial x^3 \partial y^2} \right]_{x_i \breve{y}} + \frac{h^3}{6} \left[q \frac{\partial^5 u}{\partial x^2 \partial y^3} \right]_{y_j \breve{x}} + O(h^4),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\Psi_{ij} &= \frac{h^3}{72} \left[r \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2} \right]_{x_i \breve{y}_j} + \frac{h^3}{72} \left[r \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2} \right]_{y_j \breve{x}_i} \\
&- \quad \frac{h^3}{180} \left[p \frac{\partial^5 u}{\partial x^5} \right]_{x_i \breve{y}_j} - \frac{h^3}{180} \left[q \frac{\partial^5 u}{\partial y^5} \right]_{y_j \breve{x}_i} - \Psi^*_{ij} + \Psi^{**}_{ij} \\
&= \frac{h^4}{72} \left(\mu^x \right)_{\hat{x}_i \breve{y}_j} + \frac{h^4}{72} \left(\mu^y \right)_{\hat{y}_j \breve{x}_i} - \frac{h^4}{180} \left(\eta^x \right)_{\hat{x}_i \breve{y}_j} - \frac{h^4}{180} \left(\eta^y \right)_{\hat{y}_j \breve{x}_i} \\
&- \quad \Psi^*_{ij} + \Psi^{**}_{ij},
\end{aligned}
$$

where

$$
\mu_i^x(y) = \left(r \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2} \right) (x_{i-1/2}, y), \ \mu_j^y(x) = \left(r \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 f}{\partial x^2 \partial y} \right) (x, y_{j-1/2})
$$

$$
\eta_i^x(y) = \left(p \frac{\partial^5 u}{\partial x^5} \right) (x_{i-1/2}, y), \ \eta_j^y(x) = \left(q \frac{\partial^5 u}{\partial y^5} \right) (x, y_{j-1/2}),
$$

$$
\Psi^*_{ij} = \frac{h^3}{144} \left[q \frac{\partial^5 u}{\partial x \partial y^4} \right]_{x_i \breve{y}_j} + \frac{h^3}{144} \left[p \frac{\partial^5 u}{\partial x^4 \partial y} \right]_{y_j \breve{x}_i}, \ \Psi^{**}_{ij} = O(h^4).
$$

Therefore

$$
\begin{aligned}
(\Psi, z)_\omega &= -\frac{1}{72} \left(h^2 k^2 \mu^x, z_{\bar{x}} \right)_{\omega_1^+ \times \omega_2} - \frac{1}{72} \left(h^2 k^2 \mu^y, z_{\bar{y}} \right)_{\omega_1 \times \omega_2^+} \\
&+ \quad \frac{1}{180} \left(h^4 \eta^x, z_{\bar{x}} \right)_{\omega_1^+ \times \omega_2} + \frac{1}{180} \left(k^4 \eta^y, z_{\bar{y}} \right)_{\omega_1 \times \omega_2^+} - (\Psi^*, z)_\omega + (\Psi^{**}, z)_\omega .
\end{aligned}
$$

Hence

$$\begin{aligned}|(\Psi, z)_\omega| &\leq C\|h^2\|_0\|k^2\|_0\left(\|z_{\bar{x}}\|_0 + \|z_{\bar{y}}\|_0\right) + C\|h^4\|_0\|z_{\bar{x}}\|_0 + C\|k^4\|_0\|z_{\bar{y}}\|_0 \\ &+ \|\Psi^*\|_{-1}\|z\|_1 + \|\Psi^{**}\|_0\|z\|_0 \\ &\leq C\left(\|h^4\|_0 + \|k^4\|_0 + \|\Psi^*\|_{-1}\right)\|z\|_1,\end{aligned}$$

where C is a constant, independent of the mesh. Therefore if the mesh is uniform

$$\begin{aligned}\|z\|_1 &\leq Ch^4, \\ &\text{else} \\ \|z\|_1 &\leq Ch^3. \quad \square\end{aligned} \tag{3.22}$$

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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COMPARISON PRINCIPLE FOR LINEAR NON-COOPERATIVE ELLIPTIC SYSTEMS

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This paper presents some sufficient conditions for the validity of the comparison principle for the weak solutions of non - cooperative weakly coupled systems of elliptic second-order PDE.

Keywords: Elliptic systems, eigenvalue problem, comparison principle.
2000 MSC: 35J65, 35K60, 35B05, 35R05

In this paper are considered weakly coupled elliptic systems of the form

$$L_M u = 0 \text{ in a bounded domain } \Omega \in R^n \tag{1}$$

where $L_M = L + M$, L is a matrix operator with null off-diagonal elements

$$L = diag.(L_1, L_2, ...L_n),$$

$$L_k u_k = -\sum_{i,j=1}^n D_j\left(a_k^{ij}(x) D_i u_k\right) + \sum_{i=1}^n b_k^i(x) D_i u_k + c_k u_k \text{ in } \Omega, \text{ for } k = 1, 2, ...N,$$

and $M = \{m_{ij}(x)\}_{i,j=1}^N$.

Operators L_k are supposed to be uniformly elliptic ones, i.e. there are constants $\lambda, \Lambda > 0$ such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^n a_k^{ij}(x) \xi_i \xi_j \le \Lambda |\xi|^2 \tag{2}$$

for every k and any $\xi = (\xi_1, ...\xi_n) \in R^n$.

As for the smoothness of the coefficients $a_k^{ij}(x), b_k^i(x), c_k$ and m_{ij}, we suppose $a_k^{ij}(x), b_k^i(x) \in W^{1,\infty}(\Omega)$, c_k and m_{ij} are continuous in $\overline{\Omega}$.

Hereafter by $f^-(x) = min(f(x), 0)$ and $f^+(x) = max(f(x), 0)$ are denoted the non-negative and, respectively, the non-positive part of the function f. The same convention is valid for matrixes as well. For instance, we denote by M^+ the non-negative part of M, i.e.$M^+ = \{m_{ij}^+(x)\}_{i,j=1}^N$.

This paper concerns the validity of the comparison principle for weakly-coupled elliptic systems. Let us briefly recall the definition of the comparison principle in a weak sense.

The comparison principle holds in a weak sense for the operator L_M if $(L_M u, v) \le 0$ and $u|_{\partial\Omega} \le 0$ imply $(u, v) \le 0$ in Ω for every $v \in W^{1,\infty}(\Omega)$. (3)

As it is well-known, there is no comparison principle for an arbitrary elliptic system (see Theorem 5 below). On the other hand, there are broad classes of elliptic systems, such that the comparison principle holds frue. One of these classes is constructed using condition (4) (see Theorem 1 below):

There is an eigenvalue λ of L_M and its adjoint operator $L*_M$ and the corresponding eigenfunctions $\tilde{w}, w \in \left(W_{loc}^{2,n}(\Omega) \bigcap C_0(\overline{\Omega})\right)^n$ are positive ones. (4)

Note. By adjoint operator we mean $L^*{}_M = L^* + M^t$, $L^* = diag\,(L_1^*, L_2^*, ..., L_n^*)$, and L_k^* are L^2-adjoint operators to L_k.

More precisely, the class is $C^4 = \{L_M$ satisfies (4) and $\lambda > 0\}$, i.e. C^4 contains linear elliptic systems possessing a positive principal eigenvalue with positive corresponding eigenfunction. In C^4 the necessary and sufficient condition for the validity of the comparison principle for systems (Theorem 1 below) is the same as the one for a single equation (See [1]).

Theorem 1. *Assume that (2), (3) and (4) are satisfied. The comparison principle holds for system (1) if and only if $\lambda > 0$.*

Proof. 1. Assume that the comparison principle does not hold for L_M. Let $\underline{u}, \overline{u} \in W^{1,\infty}(\Omega)$ be an arbitrary weak sub- and super-solution of L_M. Then $u = \underline{u} - \overline{u} \in W^{1,\infty}(\Omega)$ is a weak sub-solution of L_M as well, i.e. $(L_M(u), v) \le 0$ in Ω for any $v \in W^{1,\infty}, v > 0$ and $u \equiv 0$ on $\partial\Omega$. Suppose $u^+ \not\equiv 0$. Then

$$0 \ge \left(L_M u^+, w\right) = \left(u^+, L_M^* w\right) = \lambda \left(u^+, w\right) > 0$$

for λ, w defined in (4).

Therefore $u^+ \equiv 0$, i.e for an arbitrary couple sub- and super-solution of L_M we obtain $\underline{u} \le \overline{u}$.

2. Suppose $\lambda < 0$ and $\tilde{w}$ is the corresponding positive eigenfunction of L_M. Then $L_M(\tilde{w}) = \lambda\tilde{w} \leq 0$ but $\tilde{w} > 0$. Therefore there is no comparison principle for (1). □

Unfortunately, the application of this general theorem faces some odds, all about the fact that condition (4) is uneasy to check. First of all, the existence of the principal eigenvalue does not hold for every system (1) (See [9]). The second obstacle is related to the computation of λ even when it exists.

Another broad class, such that the comparison principle holds true, is the class of so-called cooperative elliptic systems, i.e. the systems with $m_{ij}(x) \geq 0$ for $i \neq j$ (See [8]). Most results on the positivity of the classical solutions of linear elliptic systems with non-negative boundary data are obtained for the cooperative systems (See [5,5,12,13,14,16,17,19]). Comparison principle for the diffraction problem for weakly coupled elliptic and parabolic systems is proved in [2].

The spectrum properties of the cooperative L_M are studied as well. A powerful tool in the cooperative case is the theory of the positive operators (See [15]) since the inverse of the cooperative operator L_{M^-} is positive in weak sense. Unfortunately, this approach cannot be applied to the general case $M \neq M^-$ since L_M is not a positive operator at all. Nevertheless in [18] is given a prove for the validity of the comparison principle for non-cooperative systems obtained by small perturbations of cooperative ones.

In [11] are studied existence and local stability of positive solutions of systems with $L_k = -d_k\Delta$, linear cooperative and non-linear competitive part, and Neumann boundary conditions. Theorem 2.4 in [*] is similar to Theorem 2 in the present article for the case $L_k = -d_k\Delta$ and shares the same idea in the proof of adding a big constant.

Let us recall that the comparison principle was proved in [10] for the viscosity sub-and super-solutions of general fully non-linear elliptic systems

$$G^l(x, u^1, \dots u^N, Du^l, D^2u^l) = 0, \quad l = 1, \dots N$$

(see also the references there). The systems considered in [10] are degenerate elliptic ones and satisfy the same structure-smoothness condition as the one for a single equation. The first main assumption in [10] guarantees the quasi-monotonicity of the system. Quasi-monotonicity in the non-linear case is an equivalent condition to the cooperativeness in the linear one.

The second main assumption in [10] comes from the method of doubling of the variables in the proof.

Note. For linear equations the positiveness and the comparison principle are equivalent. As for the non-linear case, the positiveness of the solutions is an weaker statement than the comparison result for arbitrary sub-and super-solutions; positiveness can hold without comparison and uniqueness of the solutions at all.

This work extends the results obtained for cooperative systems to the non-cooperative ones. The general idea is the separation the cooperative and competitive part of system (1). Then using the appropriate spectral properties of the cooperative part are derived conditions on the general system. In particular we employee the fact that irreducible cooperative system possesses a principal eigenvalue and the corresponding eigenfunction is a positive one, i.e. condition (4) holds. This way we derive some sufficient conditions for validity of the comparison principle for non-cooperative systems as well.

As a preliminary statement we need the following extension of Theorem 1.1.1 [16]:

Theorem 2. *Every cooperative system L_{M^-} has unique principal eigenvalue with positive corresponding eigenfunction.*

Proof. Let us consider the operator $L_c = L_{M^-} + cI$ where c is a real constant and I is the identity matrix in R^n. Then L_c satisfies the conditions of Theorem 1.1.1 [16] if c is large enough, namely

1. L_c is a cooperative one;
2. L_c is a fully coupled;
3. There is a super-solution φ of $L_c\varphi = 0$.

Conditions 1 and 2 above are obviously fulfilled by L_c, since L_{M^-} is a cooperative and a fully coupled one, and L_c inherits this properties from L_{M^-}.

As for the condition 3, we construct the super solution φ using the principal eigenfunctions of the operators $L_k - c_k$. More precisely, $\varphi = (\varphi_1, \varphi_2, , \varphi_n)$, where $(L_k - c_k)\,\varphi_k = \lambda_k\varphi_k$, and $\lambda_k, \varphi_k > 0$ in Ω. Existence of φ_k is a well - known fact.

We claim that φ is a super solution of L_c if c is large enough, i.e. $\varphi \in \left(W^{2,n}_{loc}(\Omega)\bigcap C(\overline{\Omega})\right)^n$ and $\varphi \geq 0$, $L_c\varphi \geq 0$ and φ is not identical to null in Ω.

Since we have chosen φ_k being the principal eigenfunctions of $L_k - c_k$, we have $\varphi_k \in \left(C^2(\Omega)\bigcap C(\overline{\Omega})\right)$ and $\varphi_k > 0$. The last (remaining) condition to prove is $L_c\varphi \geq 0$.

Let

$$A_k = (L_c\varphi)_k = -\sum_{i,j=1}^{n} D_j\left(a^{ij}_k(x)D_i\varphi_k\right) + \sum_{i=1}^{n} b^i_k(x)D_i\varphi_k + \sum_{i=1}^{n} m_{ki}(x)\varphi_i + (c_k + c)\varphi_k =$$

$$= (\lambda_k + c_k + c)\varphi_k + \sum_{i=1}^{n} m_{ki}(x)\varphi_i$$

We claim that $A_k \geq 0$ for every i.

First of all, if we denote by n the the outer unitary normal vector, then

$$\frac{dA_k}{dn}|_{\partial\Omega} = (\lambda_k + c_k + c)\frac{d\varphi_k}{dn} + \sum_{i=1}^{n} m_{ki}(x)\frac{d\varphi_i}{dn}$$

since $\varphi_i|_{\partial\Omega} = 0$. Therefore $\frac{dA_k}{dn}|_{\partial\Omega} < 0$ for $c > c'$ since $\frac{d\varphi_i}{dn} < 0$ on $\partial\Omega$ (See [14], Theorem 7, p.65) and λ_i is independent on c.

Hence there is a neighbourhood $\Omega_\varepsilon = \{x \in \overline{\Omega} : dist(x, \partial\Omega) < \varepsilon\}$ for some $\varepsilon > 0$, such that

$$\frac{dA_k}{dn}|_{\Omega_\varepsilon} < 0$$

Since $A_k = 0$ on $\partial\Omega$, then $A_k > 0$ in Ω_ε

The set $\Omega \setminus \Omega_\varepsilon$ is compact, therefore there is $c" > 0$ such that $A_k > 0$ in the compact set $\Omega \setminus \Omega_\varepsilon$ for $c > c"$, since $\varphi_k > 0$ in $\Omega \setminus \Omega_\varepsilon$.

Considering $c > max(c', c")$ we obtain $A_k > 0$ in Ω, therefore φ is indeed a super - solution of L_c.

The rest of the proof follows the proof of Theorem 1.1.1 [16]. □

Theorem 3. *Let (1) be a weakly coupled system with irreducible cooperative part of $L^*_{M^-}$ such that (2) and (3) are satisfied. Then the comparison principle holds for system (1) if*

$$\left(\lambda + \sum_{k=1}^n m^+_{kj}(x)\right) > 0 \ for\ j = 1...n\ and\ x \in \Omega, \tag{5}$$

$$\lambda + m^+_{jj}(x) \geq 0 \ for\ j = 1...n\ and\ x \in \Omega, \tag{5'}$$

where λ is the principal eigenvalue of the operator L_{M^-}.

Proof. Suppose all conditions of Theorem 3 are satisfied by L_M but the comparison principle does not hold for L_M. Let $\underline{u}, \overline{u} \in W^{1,\infty}(\Omega)$ be an arbitrary weak sub- and super-solution of L_M. Then $u = \underline{u} - \overline{u} \in W^{1,\infty}(\Omega)$ is a weak sub-solution of L_M as well, i.e. $(L_M(u), v) \leq 0$ in Ω for any $v \in W^{1,\infty}, v > 0$ and $u \equiv 0$ on $\partial\Omega$.

Assume $u^+ \not\equiv 0$. Then for any $v > 0$, $v \in W^{1,\infty}(\Omega)$

$$0 \geq (\mathrm{L}_M u^+, v) = (u^+, \mathrm{L}^*_{M^-} v) + (M^+ u^+, v) \tag{6}$$

is satisfied since $L_M(u^+) \leq 0$.

As L_{M^-} is a cooperative operator, such is $(L_{M^-})^* = L^* + (M^-)^t$ as well. According to Theorem 2 above, there is a unique positive eigenfunction $w \in \left(W^{2,n}_{loc}(\Omega) \bigcap C_0(\overline{\Omega})\right)^n$ such that $w > 0$ and $L^*_{M^-} w = \lambda w$ for some $\lambda > 0$.

Then w is a suitable test-function for (6). Inequality (6) reads for $v = w$ as

$$0 \geq (u^+, \mathrm{L}^*_{M^-} w) + (M^+ u^+, w) = (u^+, \lambda w) + (M^+ u^+, w)$$

or componentwise

$$0 \geq (u^+_k, \lambda w_k) + \left(\sum_{j=1}^n m^+_{kj} u^+_j, w_k\right) \tag{7}$$

for $k = 1, ...n$.

The sum of inequalities (7) is

$$0 \geq \sum_{k=1}^n \left(\left(u^+_k, \tilde{L}^*_k w_k\right) + \left(\sum_{j=1}^n m^+_{kj} u^+_j, w_k\right)\right) =$$

$$= \sum_{k=1}^{n} \left(u_k^+, \lambda w_k\right) + \sum_{k,j=1}^{n} \left(u_j^+, m_{kj}^+ w_k\right) =$$

$$= \sum_{j=1}^{n} \left(u_j^+, \sum_{k=1}^{n} \left(\delta_{jk}\lambda + m_{kj}^+\right) w_k\right) > 0$$

since $u^+ > 0$, $w_k > 0$, (5) and (5').

The above contradiction proves that $u^+ \equiv 0$ and therefore the comparison principle holds for operator L_M. □

Since in [17] are considered only systems with irreducible cooperative part, the ones with reducible L_{M^-} are excluded of the range of Theorem 3. Nevertheless the same idea is applicable to such systems as well, as it is given it Theorem 4.

Theorem 4. *Assume $m_{ij}^- \equiv 0$ for $i \neq j$ and (2), (3) are satisfied. Then the comparison principle holds for system (1) if*

$$\left(\lambda_j + \sum_{k=1}^{n} m_{kj}^+(x)\right) > 0 \text{ for } j = 1...n \text{ and } x \in \Omega, \tag{8}$$

$$\lambda_j + m_{jj}^+(x) \geq 0 \text{ for } j = 1...n \text{ and } x \in \Omega, \tag{9}$$

where λ_j is the principal eigenvalue of the operator L_j.

Proof. Let all conditions of Theorem 4 be satisfied by L_M but the comparison principle does not hold for $\tilde{L}_{M^+}$. Let $\underline{u}, \overline{u} \in W^{1,\infty}(\Omega)$ be an arbitrary weak sub- and super-solution of $\tilde{L}_{M^+}$. Then $u = \underline{u} - \overline{u} \in W^{1,\infty}(\Omega)$ is a weak sub-solution of $\tilde{L}_{M^+}$ as well, i.e. $(\tilde{L}_{M^+}(u), v) \leq 0$ in Ω for any $v \in W^{1,\infty}, v > 0$ and $u \equiv 0$ on $\partial\Omega$.

Suppose that $u^+ \neq 0$. Then for any $v > 0$, $v \in W_2^{1,\infty}(\Omega)$

$$0 \geq \left(\tilde{L}_{M^+} u^+, v\right) = \left(u^+, \tilde{L}^* v\right) + (M^+ u^+, v) \tag{10}$$

is satisfied since $\tilde{L}_{M^+} u^+ \leq 0$.

According to Theorem 2.1 in [1], there is a positive principal eigenfunction for the operator $\tilde{L}_k^*$, i.e. $\exists \quad w_k(x) \in C^2(\Omega \bigcap R^1)$ such that $\tilde{L}_k^* w_k(x) = \lambda_k w_k(x)$ and $w_k(x) > 0$. Note that w_k are even classical solutions.

Then the vector-function $w(x) = (w_1(x), ..., w_n(x))$, composed of the principal eigenfunctions $w_k(x)$, is suitable as a test-function in (10).

Componentwise, inequality (10) reads for $v = w$ as

$$0 \geq \left(u_k^+, \tilde{L}_k^* w_k\right) + \left(\sum_{j=1}^{n} m_{kj}^+ u_j^+, w_k\right) \tag{11}$$

for $k = 1, ...n$.

The sum of inequalities (11) is

$$0 \geq \sum_{k=1}^{n} \left(\left(u_k^+, \tilde{L}_k^* w_k\right) + \left(\sum_{j=1}^{n} m_{kj}^+ u_j^+, w_k\right)\right) =$$

$$= \sum_{k=1}^{n} \left(u_k^+, \lambda_k w_k\right) + \sum_{k,j=1}^{n} \left(u_j^+, m_{kj}^+ w_k\right) =$$

$$= \sum_{j=1}^{n} \left(u_j^+, \sum_{k=1}^{n} \left(\delta_{jk}\lambda_j + m_{kj}^+ \right) w_k \right) > 0$$

since $u^+ > 0$, $w_k > 0$, (8) and (9).

The above contradiction proves that $u^+ \equiv 0$ and therefore the comparison principle holds for operator L_M^+. □

Condition (9) is useful for construction of contra-example for the non-validity of comparison principle in general.

Theorem 5. *Let (1) be a weakly coupled system with reducible cooperative part L_{M^-} such that (2) and (3) are satisfied. Suppose that (9) is not true, i.e there is some $j \in \{1...n\}$ such that $(\lambda_j + m_{jj}^+(x)) < 0$ for any $x \in \Omega$, and $m_{jl}^+ = 0$ for $l \neq l$, $l = 1, ...n$. Then the comparison principle does not hold for system (1).*

Note. In Theorem 5 we need violation of the condition (9) in all Ω.

Proof. Let us suppose for simplicity that $j = 1$. We consider vector-function $w(x) = w_1(x), 0, ..., 0$, where $w_1(x)$ is the principal eigenfunction of L_1.

Then for the first component $(L_M)_1$ of L_M is valid

$$(L_M w)_1 = \lambda w_1(x) + m_{11}^+ w_1(x) < 0 \quad \text{in} \quad \Omega$$

where λ_j is the principal eigenvalue of L_1, and $(L_M w)_k = 0$ for $k = 1, ...n$. Therefore, $L_M w \leq 0$ but $w(x) \geq 0$ and comparison principle fails. □

Analogous to Theorem 5 statement is valid for for irreducible systems as well.

Theorem 6. *Let (1) be a weakly coupled system with irreducible cooperative part L_{M^-} such that (2) and (3) are satisfied. Suppose that (5) is not true, i.e there is some $j \in \{1...n\}$ such that $(\lambda + m_{jj}^+(x)) < 0$ for any $x \in \Omega$, and $m_{jl}^+ = 0$ for $l \neq l$, $l = 1, ...n$. Then the comparison principle does not hold for system (1).*

Note. In Theorem 5 we need violation of the condition (5) in all Ω.

The proof of Theorem 6 follows the proof of Theorem 5 with the obvious corrections.

The sufficient conditions in Theorems 3 and 4 are derived from the spectral properties of the cooperative part of (1) - the operator L_{M^-}, or, in other words, comparing the principal eigenvalue of L_{M^+} with the quantities in M^+. In fact the positive matrix M^+ causes a migration of the principal eigenvalue of L_{M^-} to the left.

Theorems 3 and 4 provide a huge class of non-cooperative systems such that the comparison principle is valid for. The idea of migrating the spectrum of a positive operator on the right works in this case, though the spectrum itself is not studied in this article. The results for non-cooperative systems in this paper are not sharp and the validity of the comparison principle is to be determined more precisely in the future.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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A CLASSIFICATION OF THE UNIFORM COVERINGS [1]

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A classification of the uniform coverings over a given uniformly locally path-wise connected and semi-one connected space is made, by the set of the classes conjugated by themselves subgroups of the fundamental group of the base. We use some well-known theorems in the topological case, proving that they are available in the category of the uniform spaces. We supply the covering space with a suitable uniformity, very closely connected with the uniformity of the base and use it for our investigations.

1. CONNECTION BETWEEN THE U-COVERINGS AND THE FUNDAMENTAL GROUP OF THEIR BASE. AUTOMORFISMS OF THE U-COVERINGS

Definition 1. By an uniform covering we mean a covering $p\colon (\tilde{X},\tilde{U}) \to (X,U)$ over an uniform space (X,U), which is trivial over every element U_α of an uniform cover $\delta = \{U_\alpha\}_{\alpha\in A}$ of X, but the family $\{p/\tilde{U}_{\alpha\lambda}\}_{\lambda\in A,\alpha\in A}$ of the uniform isomorfisms is equicontinuous [2].

As a particular case of the topological coverings, the uniform coverings satisfy some well-known theorems. For example, the homomorphism $p_\sharp\colon \pi(\tilde{X},\tilde{x}_0) \to \pi(X,x_0)$ is a monomorphism. If we replace the point $\tilde{x}_0$ by x'_0 and connect the two points by a path $\tilde{\omega}$, the monomorphism $p_\sharp$ commutes with the isomorphism of conjugatness $h_{[\omega]}$.

That is: $p_\sharp\pi(\tilde{X},\tilde{x}_0) = h_{[\omega]}p_\sharp\pi(\tilde{X},x'_0)$. $(\omega = p\tilde{\omega})$

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Theorem. [4] *Let* $p\colon (\tilde{X},\tilde{U}) \to (X,U)$ *be a U-covering and* $x_0 \in X$. *Then the set* $\{p_\sharp\pi(\tilde{X},\tilde{x}_0) \mid \tilde{x}_0 \in p^{-1}(x_0)\}$ *is a class of conjugated by themselves subgroups of* $\pi(X,x_0)$. *The isomorphism* $h_{[\omega]}$ *maps the class of conjugated subgroups of* $\pi(X,x_1)$ *onto the corresponding class in the group* $\pi(X,x_0)$.

Furthermore, the group $\pi(X,p(\tilde{x}_0))$ acts as a group of the right transformations on the set $p^{-1}(p(\tilde{x}_0))$ as follows: For every $\alpha \in \pi(X,p(\tilde{x}_0))$ let $\tilde{\alpha}$ be the unique lifting of α with $\tilde{\alpha}(0) = \tilde{x}_0$. Then by definition $\tilde{x}_0.\alpha = \tilde{\alpha}(1)$. If $\tilde{X}$ and X are linear connected spaces, this action is transitive. Obviously, the isotropy subgroup of the point $\tilde{x}_0$ is just $p_\sharp\pi(\tilde{X},\tilde{x}_0)$. It turns out (from the algebraic considerations) that there exists one to one correspondence between the set of the right classes.

$$\pi(X,p(\tilde{x}_0))/p_\sharp\pi(\tilde{X},\tilde{x}_0) \tag{1.1}$$

and the fibre over point $p(\tilde{x}_0) - p^{-1}(p(\tilde{x}_0))$ [1].

Let us denote by $G_u(p)$ the group of the uniform automorphisms f of the U-covering $(\tilde{X},\tilde{U}) \xrightarrow{p} (X,U)$ (such, that $pf \equiv p$). Then for every $\varphi \in G_u(p)$ the multiplication $\tilde{x}.\alpha$ satisfies the equality $f(\tilde{x}.\alpha) = (f(\tilde{x}))\alpha$. That is, $f/p^{-1}(x)$ is an automorphism of the set $p^{-1}(x)$, treating as a right $\pi(X,x)$ space. We shall prove that if (X,U) is a uniformly locally connected space (ULC), the converse also is true.

Theorem 1. *Every automorphism* $f \in G_u(p)$ *is quite defined by its restriction on* $p^{-1}(x)$.

We need some preparations before establishing that the group $G_u(p)$ is isomorphic to a subgroup $\dfrac{N(p_\sharp\pi(\tilde{X},\tilde{x}_0))}{p_\sharp\pi(\tilde{X},\tilde{x}_0)}$ of (1) (Theorem 2). The homomorphism

$$\varphi\colon G_u(p) \to N(p_\sharp\pi(\tilde{X},\tilde{x}_0))/p_\sharp\pi(\tilde{x},\tilde{x}_0) \tag{1.2}$$

is defined as follows: Let $f \in G_u(p)$ and $\tilde{\omega}$ be a curve in $\tilde{X}$ connecting $\tilde{x}_0$ and $f(\tilde{x}_0)$.

Then $\psi(f) = \varphi([p.\tilde{\omega}])$, where φ is the factor map

$$N(p_\sharp\pi(\tilde{X},\tilde{x}_0)) \to N(p_\sharp\pi(\tilde{X},\tilde{x}_0))/p_\sharp\pi(\tilde{X},\tilde{x}_0).$$

Of course, we need the lemma:

Lemma 1. *Let* $p\colon (\tilde{X},\tilde{U}) \to (X,U)$ *be a U-covering,* $p(\tilde{x}_0) = x_0$ *and the map* $f \in G_u(p)$. *If the path* $\tilde{\omega}$ *connects* $\tilde{x}_0$ *and* $f(\tilde{x}_0)$, *then* $[p\tilde{\omega}] \in N(p_\sharp\pi(\tilde{X},\tilde{x}_0))$.

Proof. (see [3]). Since f is a homeomorphism, we can write the following equalities:

$$\begin{aligned}[p\tilde{\omega}]^{-1}p_\sharp\pi(\tilde{X},\tilde{x}_0)[p\tilde{\omega}] &= h_{[p\tilde{\omega}]}(p_\sharp\pi(\tilde{X},\tilde{x}_0)) = p_\sharp(h_{[\tilde{\omega}]}\pi(\tilde{X},\tilde{x}_0)) = \\ &= p_\sharp\pi(\tilde{X},f(\tilde{x}_0)) = p_\sharp(f_\sharp\pi(\tilde{X},\tilde{x}_0)) = p_\sharp\pi(\tilde{X},\tilde{x}_0). \quad \square\end{aligned}$$

Theorem 2. *Let* p: $(\tilde{X}, \tilde{U}) \to (X, U)$ *be a uniform covering, its base is being a connected and ULC-space. Then the map (2) is a group isomorphism.*

Proof. Analogous theorem is known about the topologycal case ([3]). We shall prove only that ψ is an epimorphism, which is new. Let the class of the loop $\omega - [\omega]$ belong to $N(p_\sharp \pi(\tilde{X}, \tilde{x}_0))$:

$[\omega]^{-1} p_\sharp \pi(\tilde{X}, \tilde{x}_0)[\omega] = p_\sharp \pi(\tilde{X}, \tilde{x}_0)$. We lift ω to $\tilde{\omega}$ such, that $\tilde{\omega}(0) = \tilde{x}_0$ and put $\tilde{x}'_0 = \tilde{\omega}$ (1). Writing the equalities

$$p_\sharp \pi(\tilde{X}, \tilde{x}_0) = p_\sharp h_{[\tilde{\omega}]} \pi(\tilde{X}, \tilde{x}'_0) = h_{[\omega]} p_\sharp \pi(\tilde{X}, \tilde{x}'_0) = [\omega^{-1}] p_\sharp \pi(\tilde{X}, \tilde{x}'_0)[\omega] = p_\sharp \pi(\tilde{X}, \tilde{x}'_0),$$

we get $p_\sharp \pi(\tilde{X}, \tilde{x}'_0)[\omega] = p_\sharp \pi(\tilde{X}, \tilde{x}_0)$. Now we make use of the theorem of the uniformly continuous lifting of the map p ([2]). Denoting the corresponding liftings by f and g, we obtain the diagrams

$$\begin{array}{ccc} (\tilde{X}, \tilde{x}_0) & \overset{f}{- - \to} & (\tilde{X}, \tilde{x}'_0) \\ p \downarrow & & \downarrow p \\ (X, x_0) & = & (X, x_0) \end{array} \qquad \begin{array}{ccc} (\tilde{X}, \tilde{x}'_0) & \overset{g}{- - \to} & (\tilde{x}, \tilde{x}_0) \\ p \downarrow & & \downarrow p \\ (X, x_0) & = & (X, x_0) \end{array} \tag{1.3}$$

It follows (by the uniqueness of f and g) that f and g are multually reverse uniform isomorphisms, i.e. $f \in G_u(p)$. □

Remark. If we combine the isomorphism ψ and the diagrams (3), we see that every map $f \in G_u(p)$ is well defined by its restriction on $p^{-1}(x)$.

Of course, the space (X, U) must be connected and uniformly locally linear connected.

2. REGULAR UNIFORM COVERINGS

In this point we shall assume, that (X, U) is a connected, uniformly locally connected space.

Definition 2. The uniform covering $(\tilde{X}, \tilde{U}) \to (X, U)$ is called regular at the point $x_0 \in X$, iff for every $\tilde{x}_0 \in p^{-1}(x_0)$ the group $p_\sharp \pi(\tilde{X}, \tilde{x}_0)$ is a normal dividor of $\pi(X, x_0)$, i.e. it coincides with all its conjugated subgroup of $\pi(X, x_0)$

As it is known, this definition does not depend on the choise of the point x_0. For regular uniform coverings the isomorphism ψ looks as follows:

$$\psi \colon G_u(p) \to \pi(X, x_0)/p_\sharp \pi(\tilde{X}, \tilde{x}_0). \tag{2.1}$$

Given two points $\tilde{x}_0$ and $\tilde{x}'_0$ of $\tilde{x}$ with $p(\tilde{x}_0) = p(\tilde{x}'_0)$, we can write the diagrams (3) and get that there exists an isomorphism $f \in G_u(p)$ such that $f \in (\tilde{x}_0) = \tilde{x}'_0$. This fact often is accepted as a definition of regularity.

Now, let us connect $\tilde{x}_0$ and $\tilde{x}'_0 = f(\tilde{x}_0)$ by a path and recall the action of the group $\pi(X, x_0)$ on the layer $p^{-1}(x_0)$. We obtain that the group $\pi(X, x_0)$ acts transitively on the layer $p^{-1}(x_0)$.

Let us recall the important particular case of the regular coverings ∐ the universal coverings.

Definition 3. The uniform covering $p\colon (\tilde{X}, \tilde{U}) \to (X, U)$ is called universal uniform covering if $\pi(\tilde{X}, \tilde{x}_0) = 0$.

In this case $p_\sharp \pi(\tilde{X}, \tilde{x}_0) = 0$ and hence the group $\pi(X, x_0)$ acts on $p^{-1}(x_0)$ without fixed points. We immediately obtain

Theorem 3. *If the uniform covering $p\colon (\tilde{X}, \tilde{U}) \to (X, U)$ is an universal covering, the groups $G_u(p)$ and $\pi(X, x_0)$ are isomorphic. The order of the group $\pi(X, x_0)$ is equal to the number of the leafs of p.*

Now we go into details in the action of the group $G_u(p)$ over an regular U-covering p. The regular U-covering p is defined by a uniform cover $\{U_\alpha\}_{\alpha \in A}$ of (X, U) that consists of the fundamental neighborhoods of the points, i.e. for each $\alpha \in A \quad p^{-1}(U_\alpha) = \bigcup_{\lambda \in A} \tilde{U}_{\alpha\lambda}$ and all isomorphisms $p/\tilde{U}_{\alpha\lambda}\colon \tilde{U}_{\alpha\lambda} \to \tilde{U}_\alpha$ are equicontinuous.

Let the points $\tilde{x}_{\alpha\lambda} \in \tilde{U}_{\alpha\lambda}$ and $\tilde{x}_{\alpha\lambda'} \in \tilde{U}_{\alpha\lambda'}$ satisfy $p(\tilde{x}_{\alpha\lambda}) = p(\tilde{x}_{\alpha\lambda'})$.

Then the automorphism $f_\lambda^{\lambda'}$ maps $\tilde{x}_{\alpha\lambda}$ into $\tilde{x}_{\alpha\lambda'}$ also maps a connected neighborhood $\tilde{U}_{\alpha\lambda}$ of $\tilde{x}_{\alpha\lambda}$ into $\tilde{U}_{\alpha\lambda'}$ uniformly isomorphic. We obtain the next theorem.

Theorem 4. *For arbitrary $\alpha \in A$ and a couple of points $\tilde{x}_{\alpha\lambda}$, $\tilde{x}_{\alpha\lambda'}$ with $p(\tilde{x}_{\alpha\lambda}) = p(\tilde{x}_{\alpha\lambda'})$ there exists $f_\lambda^{\lambda'} \in G_u(p)$, $f_\lambda^{\lambda'}\colon (\tilde{X}, \tilde{x}_{\alpha\lambda}) \to (\tilde{X}, \tilde{x}_{\alpha\lambda'})$, that maps a neighborhood $\tilde{U}_{\alpha\lambda}$ uniformly isomorphic. onto $\tilde{U}_{\alpha\lambda'}$. The family $\{f_\lambda^{\lambda'}\}$ is equicontinuous on (out of) $(\lambda, \lambda' \in \Lambda)$, and even on $\alpha \in A$.*

Before we proceed to the construction of a uniform regular covering, we give the following

Definition 4. *Let (Y, V) be a uniform space and G be a group of its equicontinuous uniform isomorphisms. We say that the group G acts uniformly discretely over (Y, V), iff there exists a uniform cover $\{V_\lambda\}_{\lambda \in \Lambda}$ of Y such that: if $gV_\lambda \cap g'V_\lambda \neq \emptyset \to g = g'$ (V_λ is an arbitrary element of $\{V_\lambda\}_{\lambda \in \Lambda}$).*

Theorem 5. *Let (Y, V) be a connected and uniformly locally linear connected space and G is a group of its isomorphisms, which acts uniformly discretely over (Y, V). Then the natural projection p of Y on the space of orbits Y/G is a regular uniform covering with a group of automorphisms $G_u(p)$.*

Proof. First we shall supply Y/G with a factor-uniformity $\overline{V}$. If W belongs to the uniformity V then two orbits yG and y_1G we shall call $\overline{W}$-near if they have representatives yg and y_1g, which are W-near. This definition satisfies the axioms of uniformity as the action of G on (Y, V) is uniformly equicontinuous. V is the strongest uniformity at which p is uniformly continuous.

Now, let $\{V_\lambda\}_{\lambda\in\Lambda}$ be a uniform covering of Y, such that for $g_1 \neq g_2 \in G$ we have $g_1 V_\lambda \cap g_2 V_\lambda \neq \phi$ and the sets V_λ are linear connected. We put $U_\lambda = p(V_\lambda)$. Obviously $p/V_\lambda \colon V_\lambda \to U_\lambda$ is an uniform isomorphism. If V_μ is another component $p^{-1}(U_\lambda)$, then there exists an automorphism $h \in G$, such that $V_\mu = V_\lambda.h$. Hence $\dfrac{p}{V_\mu} = \dfrac{p}{V_\lambda}.h$ is also automorphism. The family $\{p/V_\lambda\}_{\lambda\in\Lambda}$ is equicontinuous at the given condition.

The group of automorphisms $A(Y, p)$ coincides with G and as it acts transitively on $p^{-1}(yG)$, the constructed covering is regular. □

3. CLASSIFICATION OF THE UNIFORM COVERINGS

Let (X, U) be a uniform space and $\langle k\rangle$ be a class of selfconjugated subgroups of the group $\pi(X, x)$. We shall prove that there exists an uniform covering $p\colon \tilde{X} \to X$, such that the group $p_\sharp\pi(\tilde{X}, \tilde{x})$ belongs to the class $\langle k\rangle$. It is known that topologically such unique covering $\tilde{X}$ exists in some additional suppositions about the space X. It is necessary to supply the space $\tilde{X}$ by a suitable uniform structure such that we get a uniform covering. We need to increase the suppositions on (X, U) for this purpose.

In Theorem 6 we solve this task, when there exists a universal covering (Y, q) over (X, U). The construction on this covering (Y, q) is done in Theorem 7.

Theorem 6. *Let the uniform space (X, U) be uniformly locally linear connected and uniformly locally semione-connected. If $\langle k\rangle$ is an arbitrary class of conjugated subgroups of $\pi(X, x)$, there exists a uniform covering $(\tilde{X}, \tilde{U}, p)$: $p_\sharp\pi(\tilde{X}, \tilde{x})$ belongs to the class $\langle k\rangle$.*

Let $(Y, V) \xrightarrow{q} (X, U)$ is the universal uniform covering over (X, U) (see theorem 7). As we know, the group $\pi(X, x)$ acts on $q^{-1}(x)$ transitively and without fixed points. We take $y \in q^{-1}(x)$ and $k \subset \pi(X, x)$. Then the following subgroup $H \subset G_u(q)$ corresponds to K by the isomorphism (4):

$$\varphi \in H \Leftrightarrow \text{ there exist } \alpha \in K \colon \varphi(y) = y\alpha\,.$$

As H is a subgroup of $G_u(q)$, it acts uniformly discretely on Y and we can introduce a factor uniformity in the space of orbits Y/H. Let $\tilde{X} = Y/H$ and $p\colon Y/H \to X$ is the map defined by q. We got the commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\;r\;} & Y/H \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & = & X \end{array}$$

which shows, that p is a uniform covering. The isotropy group of the point $\tilde{x} = p^{-1}(x) = r(y)$ is obviously K. Hence $p_\sharp\pi(\tilde{X}, \tilde{x}) = K$. □

Before we construct the universal uniform covering q, we need the following

Definition 5. The space (X, U) is called uniformly semilocally oneconnected (USL1) if there exist arbitrary little open uniform covers δ with the property: every loop $S^1 \to X$, whose image consists of some element of δ, is contractable.

Theorem 7. *Let (X, U) be a USL1-space. Then it has a unique (precisely to a uniform isomorphism) universal covering.*

Proof. Although the proof reminds the traditional in the topological case, we shall expose it, because it is specific in the ushering an uniform structure $\tilde{U}$ in X. The space $\tilde{X}$ is constructed as a space of the classes $[\alpha_x]$ homotopic curves in X, beginning at x_0. The map $p\colon \tilde{X} \to X$ is $p[\alpha] = \alpha(1)$. To usher an uniform structure in $\tilde{X}$ first of all we choose a basic open uniform cover δ of X, satisfying the USL1 Ц condition. For every $U \in \delta$ and a class $[\alpha]\colon p(\alpha) \in U$, we put $\langle \alpha, U\rangle = \{\beta\colon \beta = \alpha.\alpha'$, where $\alpha'(I) \subset U\}$.

We got a cover $\tilde{\delta}$ of $\tilde{x}$, consisting of the sets $\langle \alpha, U\rangle$ the base of $u \in \delta, \alpha(1) \in U$. If δ varies trough U, then the family $\tilde{\delta}$ defines the base of a uniform structure $\tilde{U}$ on $\tilde{X}$. $\tilde{U}$ is the coarsest uniform structure in $\tilde{X}$ with which p is uniform continuous. Some important, but easily proved properties of sets $\langle \alpha, U\rangle$ are available.

I. The map $p/\langle \alpha, U\rangle$ is an isomorphism of $\langle \alpha, U\rangle$ on U.

II. Let $U \in \delta$, $x_0 \in X$, and $x \in U$ are fixed. Let $\langle \alpha_\lambda\rangle_{\lambda\in\Lambda}$ is the set of all classes of paths, beginning at x_0 and ending at x. If x varies through U we get that $p^{-1}(U) = \bigcup_{\lambda\in\Lambda} \langle [\alpha_\lambda], U\rangle$ and the sets $\langle [\alpha_\lambda], U\rangle$ do not intersect as U is one-connected set.

III. The family of isomorphisms $\{p/\langle [\alpha_\lambda], U\rangle\}_{\lambda\in\Lambda}$ is equicontinuous.

Hence we got a uniform covering $p\colon (\tilde{X}, \tilde{U}) \to (X, U)$.

We shall not repeat the known fact that $\tilde{x}$ is a linear connected space, but we shall prove that it is one-connected. For this purpose we shall recall how the curves in X can be lifted in $\tilde{x}$.

Let $\alpha\colon I \to X$ be a curve, beginning at $x_0 \in X$. We denote by $x_0 \in X$ the class of the constant curve, i.e. $\tilde{x}_0 = [c_{x_0}]$. For an arbitrary $t \in I$ let α^t be the curve $\alpha^t(s) = \alpha(st)$. Then, for $\tilde{\alpha}$ we have $\tilde{\alpha}(t) = [\alpha^t]$. Obviously $\tilde{\alpha}(0) = [\alpha^0] = [c_{x_0}] = \tilde{x}_0$. As $p_\sharp$ is a mononiorphism, we have to prove that $p_\sharp(\tilde{X}, \tilde{x}_0) = 0$, i.e., if α is a loop in (X, x_0), whose lifting is a loop, then $\alpha \in c_{x_0}$. But this follows from the definition. The equality $\tilde{\alpha}(1) = [\alpha^1] = \tilde{x}_0 = [c_{x_0}]$ holds iff the curves α and c_{x_0} are homotopic. The existence of the universal uniform covering over each uniform L1C-space (X, U) is proved.

We shall not prove the uniqueness of $(\tilde{X}, \tilde{U})$, although it does not follow automatically from those in the topological case. □

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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GEOMETRY AND SOLUTIONS OF THE PLANAR PROBLEM OF TWO CENTERS OF GRAVITATION

ASSEN LASHKOV, ANGEL ZHIVKOV

The planar problem of two centers of gravitation was studied by Euler, who found a second "momentum-like" integral and thus the problem turned out to be completely integrable. We present some effective solutions of the motion of the free particle under the influence of the two centers. These solutions are expressed by elliptic theta functions. We also classify all types of such motions from topological point of view. There exist exactly 16 types of motions. Ten of them are unbounded and six are bounded.

Keywords: integrability, general solution, topological classification

2000 MSC: 37J35

1. INTRODUCTION

One of the famous integrable problems of classical mechanics is the problem of two centers of gravitation, i. e. the problem of determining the motion of a particle in a plane, attached by two fixed centers of force in the plane. Its integrability was discovered by Euler in 1760 [4].

In his "Vorlesungen über Dynamik" [5], Jacobi separated the variables and integrated the equations of motion in terms of elliptic coordinates. Solution in four Jacobi's theta functions is due to Köningsberger [6]. However, both above mentioned solutions contain complete Abelian integrals and that's why they are not convenient for use.

In Theorem 2 below we present effective solutions of the problem of two centers of gravitation. These solutions are expressed in terms of four Jacobi's theta functions and depend on six arbitrary complex constants of motion, including the masses of the centers.

The present article is devoted to the topological classification of the real motions. They depend on six real constants of motion. This problem was studied by Charlier [1] and later by Deprit [2], see also [8, 9]. The case of equal masses of the centers was studied in [3].

In order to define topological invariants of each real solution, it is necessary to define the so–called bifurcation set $\mathbb{B}$. This set $\mathbb{B}$ includes all singular solutions and separates the phase space into connected parts, namely into topologically different types of solutions.

According to our definition, a solution is singular if and only if some Jacobi's theta function[1] degenerates, i. e. became exponent, sinh or cosh. We shall consider a solution $u(t)$ to be topologically equal to the solution $u(-t)$, t being the time. Indeed, the change $t \mapsto -t$ just turns on the opposite the direction on each trajectory.

We also should not make difference between any two solutions, symmetrical according to the line, joining the centers.

The main result of the article is those from Theorem 1: there exist exactly 16 topologically different solutions. Let us remark that any clear definition of topologically equal or different solutions could not be found in the relevant articles, which makes impossible to compare in details our with others' results.

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2. SEPARATION OF VARIABLES

Let $2c$ denote the distance between the two centers. Take the point in the middle of the interval between them as origin and the line joining them as axis of x, so their coordinates will be $(c, 0)$ and $(-c, 0)$.

Denote also by $(x, y) = (x(t), y(t))$ the coordinates of the particle. According Newton's law, the motion of the particle is governed by the equations

$$\begin{aligned} m\,\frac{d^2x}{dt^2} &= -\frac{Gmm_1\,(x-c)}{\left[(x-c)^2+y^2\right]^{\frac{3}{2}}} - \frac{Gmm_2\,(x+c)}{\left[(x+c)^2+y^2\right]^{\frac{3}{2}}}\,, \\ m\,\frac{d^2y}{dt^2} &= -\frac{Gmm_1\,y}{\left[(x-c)^2+y^2\right]^{\frac{3}{2}}} - \frac{Gmm_2\,y}{\left[(x+c)^2+y^2\right]^{\frac{3}{2}}}\,, \end{aligned}$$

where m_1, m_2, m are the masses of the centers and the particle respectively, G is the constant of gravity.

[1] Which takes part on that solution.

By properly choosing the units of distance and mass, we can achieve $c = 1$ and $G = 1$, so that Newton's equations become

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{\mu_1 (x-1)}{\left[(x-1)^2+y^2\right]^{\frac{3}{2}}} - \frac{\mu_2 (x+1)}{\left[(x+1)^2+y^2\right]^{\frac{3}{2}}} , \\ \frac{d^2y}{dt^2} &= -\frac{\mu_1 y}{\left[(x-1)^2+y^2\right]^{\frac{3}{2}}} - \frac{\mu_2 y}{\left[(x+1)^2+y^2\right]^{\frac{3}{2}}} , \end{aligned}$$

with x, y, μ_1 and μ_2 dimensionless.

Any ellipse or hyperbola with the two centers as foci is a possible orbit when any of the centers acts alone. It is therefore natural, in defining the position of the particle, to replace the rectangular coordinates (x, y) by elliptic coordinates (p, q):

$$x = p\,q\,, \qquad y = \pm\sqrt{(p^2-1)(1-q^2)}$$

and the inverse

$$\begin{aligned} 2p &= \sqrt{(x+1)^2+y^2} + \sqrt{(x-1)^2+y^2} , \\ 2q &= \sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2} . \end{aligned}$$

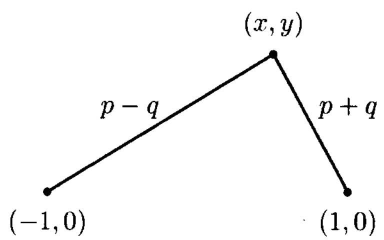

Fig.1. Elliptic coordinates (p, q)

The equations $p =$ *constant* and $q =$ *constant* then represent respectively ellipses and hyperbolas whose foci are at the centers $(\pm 1, 0)$. These ellipses and hyperbolas are a particular family of confocal orbits, namely if $m_1 = 0$ or $m_2 = 0$.

In addition to the integral of the total energy

$$\begin{aligned} h &= \frac{\dot{x}^2+\dot{y}^2}{2} - \frac{\mu_1}{\sqrt{(x-1)^2+y^2}} - \frac{\mu_2}{\sqrt{(x+1)^2+y^2}} \\ &= \frac{p^2-q^2}{2}\left(\frac{\dot{p}^2}{p^2-1}+\frac{\dot{q}^2}{1-q^2}\right) - \frac{\mu_1}{p-q} - \frac{\mu_2}{p+q} \\ &= \mathit{constant}\ , \end{aligned}$$

there also exists an extraordinary "momentum–like" integral [4]

$$\begin{aligned}\gamma &= \frac{(p^2-q^2)^2\,\dot p^2}{2\,(p^2-1)} - h\,p^2 - (\mu_1+\mu_2)\,p \\ &= \frac{(p^2-q^2)^2\,\dot q^2}{2\,(q^2-1)} - h\,q^2 - (-\mu_1+\mu_2)\,q \\ &= \mathit{constant}\ .\end{aligned}$$

For details see [10]. The last equations can be rewritten as

$$\begin{aligned}\tfrac{1}{2}\left(p^2-q^2\right)^2\dot p^2 &= (p^2-1)\left[\,h\,p^2+(\,\mu_1+\mu_2)\,p+\gamma\,\right]\ ,\\ \tfrac{1}{2}\left(p^2-q^2\right)^2\dot q^2 &= (q^2-1)\left[\,h\,q^2+(-\mu_1+\mu_2)\,q+\gamma\,\right]\ .\end{aligned}$$

In order to separate finally the variables p and q, we introduce an appropriate scaling of time $s = s(t)$ as follows:

$$dt = \left(p^2-q^2\right)ds \qquad \text{or, equivalently,} \qquad s = \int_0^t \frac{dt}{p^2-q^2}\ .$$

Now the equations of motion

$$\begin{aligned}\frac{1}{2}\left(\frac{dp}{ds}\right)^2 &= (p^2-1)\left[\,h\,p^2+(\,\mu_1+\mu_2)\,p+\gamma\,\right]\ ,\\ \frac{1}{2}\left(\frac{dq}{ds}\right)^2 &= (q^2-1)\left[\,h\,q^2+(-\mu_1+\mu_2)\,q+\gamma\,\right]\ .\end{aligned} \tag{2.1}$$

separate into motions of p– and q–variables.

3. TOPOLOGY OF THE SOLUTIONS

We shall discuss the topological types of the solutions of (2.1) in terms of the zeros p_1, p_2, q_1 and q_2 of the polynomials

$$\begin{aligned}L(p) &= h\,p^2+(\,\mu_1+\mu_2)\,p+\gamma \;=\; h\,(p-p_1)(p-p_2)\ ,\\ M(q) &= h\,q^2+(-\mu_1+\mu_2)\,q+\gamma \;=\; h\,(q-q_1)(q-q_2)\ ,\end{aligned}$$

where h and γ are arbitrary real numbers, μ_1 and μ_2 are real and positive.

Definition 3.1. Two solutions of (2.1),

$$\begin{matrix} p^* = p^*(\,s\,;\,h^*,\gamma^*,\,\mu_1^*,\mu_2^*\,) & & p^{**} = p^{**}(\,s\,;\,h^{**},\gamma^{**},\,\mu_1^{**},\mu_2^{**}\,) \\ & \text{and} & \\ q^* = q^*(\,s\,;\,h^*,\gamma^*,\,\mu_1^*,\mu_2^*\,) & & q^{**} = q^{**}(\,s\,;\,h^{**},\gamma^{**},\,\mu_1^{**},\mu_2^{**}\,) \end{matrix}\ ,$$

are *topologically equivalent* provided there exist some continuous functions

$$h(\lambda)\ ,\ \gamma(\lambda)\ ,\ \mu_1(\lambda)\ ,\ \mu_2(\lambda)\ ,\ p_0(\lambda)\ ,\ q_0(\lambda)\ , \qquad \lambda \in [0,1]$$

which connect respectively h^* and h^{**}, γ^* and γ^{**}, ..., $p^*{}_{|s=0}$ and $p^{**}{}_{|s=0}$, $q^*{}_{|s=0}$ and $q^{**}{}_{|s=0}$. Moreover, the four-tuples $\big(h(\lambda),\gamma(\lambda),\mu_1(\lambda),\mu_2(\lambda)\big)$ should never belong to the bifurcation set

$$\mathbb{B} \;=\; \left\{ \begin{array}{l} \big(h,\gamma,\mu_1,\mu_2\big) \;\in\; \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ \;: \\ \qquad h\,(p_1^2-1)\,(p_2^2-1)\,(p_1-p_2)\,(q_1^2-1)\,(q_2^2-1)\,(q_1-q_2) \;=\; 0 \end{array} \right\}.$$

Equivalently, the roots of the quartics $(p^2-1)L(p)$ and $(q^2-1)M(q)$ remain simple when λ varies from 0 to 1.

By definition, $\mathbb{B}$ consists of all singular solutions of the problem. We consider each possible position of the roots p_1, p_2, q_1, q_2, as well as the initial conditions $p(0)$ and $q(0)$ to prove the main result of the paper.

Theorem 3.1. *There exist exactly 16 topologically different types of solutions in the problem of two centers of gravitation. Six types of solutions are unbounded, another ten types of solutions are bounded :*

Orbits with positive energy $(h>0)$

1.	$p_1<-1<p_2<1$	$1\le p$	$q_1<-1<q_2<1$	$q\in[-1,q_1]$
2.	$p_1<-1<p_2<1$	$1\le p$	$-1<q_1<q_2<1$	$q\in[q_1,q_2]$
3.	$p_1<-1<p_2<1$	$1\le p$	$-1<q_1<1<q_2$	$q\in[q_2,1]$
4.	$p_1<-1<p_2<1$	$1\le p$	$q_1<-1<1<q_2$	$q\in[-1,1]$
5.	$-1<p_1<p_2<1$	$1\le p$	$-1<q_1<q_2<1$	$q\in[q_1,q_2]$
6.	$p_1<-1<1<p_2$	$p_2\le p$	$q_1<-1<1<q_2$	$q\in[-1,1]$

Orbits with negative energy $(h<0)$

7.	$-1<p_1<1<p_2$	$p\in[1,p_2]$	$q_1<q_2<-1$	$q\in[-1,1]$
8.	$-1<p_1<1<p_2$	$p\in[1,p_2]$	$q_1<-1<q_2<1$	$q\in[q_2,1]$
9.	$-1<p_1<1<p_2$	$p\in[1,p_2]$	$-1<q_1<q_2<1$	$q\in[-1,q_1]$
10.	$-1<p_1<1<p_2$	$p\in[1,p_2]$	$-1<q_1<q_2<1$	$q\in[q_2,1]$
11.	$-1<p_1<1<p_2$	$p\in[1,p_2]$	$-1<q_1<1<q_2$	$q\in[-1,q_1]$
12.	$-1<p_1<1<p_2$	$p\in[1,p_2]$	$1<q_1<q_2$	$q\in[-1,1]$
13.	$-1<p_1<1<p_2$	$p\in[1,p_2]$	$q_{1,2}\in\mathbb{C}\setminus\mathbb{R}$	$q\in[-1,1]$
14.	$1<p_1<p_2$	$p\in[p_1,p_2]$	$q_1<q_2<-1$	$q\in[-1,1]$
15.	$1<p_1<p_2$	$p\in[p_1,p_2]$	$1<q_1<q_2$	$q\in[-1,1]$
16.	$1<p_1<p_2$	$p\in[p_1,p_2]$	$q_{1,2}\in\mathbb{C}\setminus\mathbb{R}$	$q\in[-1,1]$.

4. EXPLICIT SOLUTIONS

We remind the reader that the Jacobi theta functions are given by their Fourier series

$$\begin{aligned}
\theta_{00}(z,\tau) &= 1 + 2\,e^{\pi i\tau}\cos 2\pi z + 2\,e^{4\pi i\tau}\cos 4\pi z + 2\,e^{9\pi i\tau}\cos 6\pi z + \cdots\ ,\\
\theta_{01}(z,\tau) &= 1 - 2\,e^{\pi i\tau}\cos 2\pi z + 2\,e^{4\pi i\tau}\cos 4\pi z - 2\,e^{9\pi i\tau}\cos 6\pi z + \cdots\ ,\\
\theta_{10}(z,\tau) &= \quad 2\,e^{\frac{\pi i\tau}{4}}\cos \pi z + 2\,e^{\frac{9\pi i\tau}{4}}\cos 3\pi z + 2\,e^{\frac{25\pi i\tau}{4}}\cos 5\pi z + \cdots\ ,
\end{aligned}$$

for $z \in \mathbb{C}$ and $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$. The fourth Jacobi's theta will not take part in the solutions. Introduce also the notations

$$\begin{aligned}
\Theta_{00}(z) &= \theta_{00}(0,\tau)\,\theta_{00}(z,\tau)\ , \qquad & \Theta^*_{00}(z^*) &= \theta_{00}(0,\tau^*)\,\theta_{00}(z^*,\tau^*)\ ,\\
\Theta_{01}(z) &= \theta_{01}(0,\tau)\,\theta_{01}(z,\tau)\ , \qquad & \Theta^*_{01}(z^*) &= \theta_{01}(0,\tau^*)\,\theta_{01}(z^*,\tau^*)\ ,\\
\Theta_{10}(z) &= \theta_{10}(0,\tau)\,\theta_{10}(z,\tau)\ , \qquad & \Theta^*_{10}(z^*) &= \theta_{10}(0,\tau^*)\,\theta_{10}(z^*,\tau^*)\ .
\end{aligned}$$

Theorem 4.2. *The general solution of the planar problem of two centers of gravitation reads*

$$x = pq = \frac{\Theta_{00}(z) + \operatorname{th}\alpha.\Theta_{10}(z)}{\Theta_{10}(z) + \operatorname{th}\alpha.\Theta_{00}(z)} \cdot \frac{\Theta^*_{00}(z^*) + \operatorname{th}\alpha^*.\Theta_{10}(z^*)}{\Theta^*_{10}(z^*) + \operatorname{th}\alpha^*.\Theta_{00}(z^*)}$$

$$\begin{aligned}
y &= \pm\sqrt{(p^2-1)(1-q^2)}\\
&= \frac{i\,\Theta_{01}(z)\Theta^*_{01}(z^*)}{\left[\operatorname{sh}\alpha.\Theta_{00}(z) + \operatorname{ch}\alpha.\Theta_{10}(z)\right]\left[\operatorname{sh}\alpha^*.\Theta^*_{00}(z)^* + \operatorname{ch}\alpha^*.\Theta^*_{10}(z)^*\right]}\ .
\end{aligned}$$

The elliptic coordinates

$$p = \frac{\Theta_{00}(z) + \operatorname{th}\alpha.\Theta_{10}(z)}{\Theta_{10}(z) + \operatorname{th}\alpha.\Theta_{00}(z)} \qquad \text{and} \qquad q = \frac{\Theta^*_{00}(z^*) + \operatorname{th}\alpha^*.\Theta_{10}(z^*)}{\Theta^*_{10}(z^*) + \operatorname{th}\alpha^*.\Theta_{00}(z^*)}$$

satisfy equations (2.1).

The constants which enter in the above formulas depend on the constants of *motion* μ_1, μ_2, h, γ. z_0 *and* z_0^* *as follows* :

$$\kappa^2 = \frac{h-\gamma+\sqrt{(h+\gamma)^2-(\mu_1+\mu_2)^2}}{h-\gamma-\sqrt{(h+\gamma)^2-(\mu_1+\mu_2)^2}}, \quad \kappa^{*2} = \frac{h-\gamma+\sqrt{(h+\gamma)^2-(\mu_1-\mu_2)^2}}{h-\gamma-\sqrt{(h+\gamma)^2-(\mu_1-\mu_2)^2}},$$

$$K = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-\kappa^2 s^2)}}, \qquad K^* = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-\kappa^{*2} s^2)}},$$

$$\tau = \frac{1}{K}\int_1^{\kappa} \frac{ds}{\sqrt{(1-s^2)(1-\kappa^2 s^2)}}, \qquad \tau^* = \frac{1}{K^*}\int_1^{\kappa^*} \frac{ds}{\sqrt{(1-s^2)(1-\kappa^{*2} s^2)}},$$

$$z = z_0 + \frac{2s}{K\sqrt{h-\gamma-\sqrt{(h+\gamma)^2-(\mu_1+\mu_2)^2}}}, \qquad \tanh 2\alpha = -\frac{\mu_1+\mu_2}{h+\gamma},$$

$$z^* = z_0^* + \frac{2s}{K^*\sqrt{h-\gamma-\sqrt{(h+\gamma)^2-(\mu_2-\mu_1)^2}}}, \qquad \tanh 2\alpha^* = \frac{\mu_1-\mu_2}{h+\gamma}.$$

The proof of the theorem above reduces to a straightforward check of certain relations between elliptic theta functions. These relations are in fact well-known, see for example [7].

In general, the solution are complex for $h, \gamma, \mu_{1,2}, z_0, z_0^* \in \mathbb{C}$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 99

NON-INTEGRABILITY OF A HAMILTONIAN SYSTEM, BASED ON A PROBLEM OF NONLINEAR VIBRATION OF AN ELASTIC STRING

PETYA BRAYNOVA, O. CHRISTOV

In this paper we study the problem for non-integrability of a Hamiltonian system, based on the nonlinear vibrations of an elastic string. We have the following hamiltonian:

$$H(q,p) = \frac{1}{2}\sum_{k=1}^{N} {p_k}^2(t) + \frac{c_1}{2}\sum_{k=1}^{N} k^2{q_k}^2(t) - \frac{c_2}{2}\sum_{k=1}^{N} {q_k}^2(t) +$$
$$+\frac{h1}{8}\left(\sum_{k=1}^{N} k^2{q_k}^2(t)\right)^2 - \frac{h_2}{8}\left(\sum_{k=1}^{N} {q_k}^2(t)\right)^2 = const$$

The main result is that the responding Hamiltonian system is non-integrable, except in the cases $N > 2$ and $h_1 = 0$ and $N = 2$ and $h_1 = 0$ or $h_2 = 4h_1$. In the proof we use the Morales - Ramis theorem based on Differential Galois Theory.

Keywords: Nonlinear elastic string, Hamiltonian system, Morales-Ramis theory
2000 MSC: 70J50, 70H08

1. INTRODUCTION

Free lateral "finite" vibrations of uniform beam with the ends restrained can be described by the equation

$$\rho h\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = \left(P_0 + \frac{Eh}{2L}\int_0^L (\frac{\partial w}{\partial x})^2 dx\right)\frac{\partial^2 w}{\partial x^2}, \qquad (1.1)$$

where $w(t,x)$ is the lateral deflection of the string, E – the Young's modulus, EI – the flexural rigidity, ρ – the mass density, h is the thickness of a beam of unit width, L – the string's length, P_0 is the initial axial tension. Suppose the following initial and boundary conditions

$$w(0,x) = w_0(x), \qquad \frac{\partial w}{\partial t}(0,x) = w_1(x)$$

$$w(t,0) = \frac{\partial^2 w}{\partial x^2}(t,0) = w(t,L) = \frac{\partial^2 w}{\partial x^2}(t,L) = 0.$$

In 1971 Nishida [1] examined the problem of the elastic string's vibration, in the case there is no resistance ($EI = 0$) and the equation (1.1) changes into

$$\rho h \frac{\partial^2 w}{\partial t^2} = \left(P_0 + \frac{Eh}{2L} \int_0^L (\frac{\partial w}{\partial x})^2 dx \right) \frac{\partial^2 w}{\partial x^2}$$

If there is such a natural number N, that the initial and boundary conditions look like

$$w_0(x) = \sum_{k=1}^{N} a_k \sin(k\frac{\pi}{L}x), \quad w_1(x) = \sum_{k=1}^{N} b_k \sin(k\frac{\pi}{L}x),$$

where $a_k, b_k, k = 1, \dots, N$ are real constants, then there exists a solution

$$w(t,x) = \sum_{k=1}^{N} u_k(t) \sin(k\frac{\pi}{L}x), \tag{1.2}$$

which is unique in a certain class of functions. Having put (1.2) in (1.1), Nishida got a Hamiltonian system of differential equations for $u_k(t), k = 1, \dots, N$ and proved that conditional-periodic motions are preserved around equilibrium using the KAM theorem.

Another kind of problems on the vibrations of the nonlinear string were studied by Dickey [2].

In 1994 Iliev [3] studied a more general integro-differential equation

$$\frac{\partial^2 \omega}{\partial t^2} - \left(c_1 + h_1 \int_0^{\pi} (\frac{\partial \omega}{\partial x})^2 dx \right) \frac{\partial^2 \omega}{\partial x^2} = \left(c_2 + h_2 \int_0^{\pi} \omega^2 dx \right) \omega \tag{1.3}$$

and under the same assumptions as Nishida, he brought it to a Hamiltonian system with N degrees of freedom, namely

$$H(q,p) = \frac{1}{2} \sum_{k=1}^{N} p_k{}^2(t) + \frac{c_1}{2} \sum_{k=1}^{N} k^2 q_k{}^2(t) - \frac{c_2}{2} \sum_{k=1}^{N} q_k{}^2(t) +$$

$$+\frac{h_1}{8}\left(\sum_{k=1}^{N} k^2 {q_k}^2(t)\right)^2 - \frac{h_2}{8}\left(\sum_{k=1}^{N} {q_k}^2(t)\right)^2 = const \tag{1.4}$$

Then Iliev focused himself on the integrability problem in analytic functions in the case $N = 2$. Using the Ziglin's theory, he has proved the following result:

Theorem 1. *The Hamiltonian system with Hamiltonian (1.4) is not integrable for $N = 2$. if we have*

$$\frac{c_2 - 4c_1}{c_2 - c_1} < 0 \qquad \sqrt{1 + 8\frac{h_2 - 4h_1}{h_2 - h_1}} \text{ is not odd.}$$

In 2003 Yagasaki [4] studied the same model of unforced and undamped beam as equation (1.1) with $EI = 1$. He proved non-integrability of the corresponding Hamiltonian system after the same truncation as the solution (1.2) using Differential Galois Theory for Hamiltonian systems.

One should note that considering the model (1.3) without resistance ($EI = 0$) there is no lost of generality. Having in mind the concrete form of the solution (1.2), the contribution of the forth derivative with respect to x will change the coefficients of the Hamiltonian (1.4).

Here we study the Hamiltonian system

$$\left|\begin{array}{l} \dot{q_j} = \dfrac{\partial H}{\partial p_j} = p_j \\ \dot{p_j} = -\dfrac{\partial H}{\partial q_j} = -\left(c_1 + \dfrac{h_1}{2}\sum_{k=1}^{N} k^2 {q_k}^2\right) j^2 q_j + \left(c_2 + \dfrac{h_2}{2}\sum_{k=1}^{N} {q_k}^2\right) q_j, \end{array}\right. \tag{1.5}$$

$$j = 1, \ldots, N$$

with Hamiltonian (1.4) for N degrees of freedom and generalize the result of the Theorem 1 as follows. Consider the complexified system (1.5) on the phase space $M := \{(q(t), p(t)) \in \mathbb{C}^{2N}\}$ with standard symplectic structure, $t \in \mathbb{C}$. We are interested in the question at which values of the parameters c_1, c_2, h_1, h_2, the system (1.5) is integrable (of course the case $N = 1$ is trivial).

Theorem 2. *The Hamiltonian system with Hamiltonian (1.4) is non-integrable. excluding the following two cases*

a) $N > 2$ and $h_1 = 0$.

b) $N = 2$ and $h_1 = 0$ or $h_2 = 4h_1$.

Remark. 1) If $N > 1$ and $h_1 = 0$, we have

$$H(q,p) = \frac{1}{2}\sum_{k=1}^{N} {p_k}^2 + \frac{c_1}{2}\sum_{k=1}^{N} k^2 {q_k}^2 - \frac{c_2}{2}\sum_{k=1}^{N} {q_k}^2 - \frac{h_2}{8}\left(\sum_{k=1}^{N} {q_k}^2\right)^2;$$

whence the starting Hamiltonian system is equivalent to so-called 'anharmonic oscillator' with Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{k=1}^{N} {p_k}^2 + \sum_{k=1}^{N} a_k {q_k}^2 + \left(\sum_{k=1}^{N} {q_k}^2\right)^2,$$

which is integrable in Liouville sense [5].

2) If $N = 2$ and $h_2 = 4h_1$, the variables in the system with Hamiltonian

$$H(q_1,q_2,p_1,p_2) = \frac{1}{2}({p_1}^2 + {p_2}^2) + \frac{(c_1 - c_2)}{2}{q_1}^2 + \frac{(4c_1 - c_2)}{2}{q_2}^2 - \frac{3}{8}h_1{q_1}^4 + \frac{3}{2}h_1{q_2}^4$$

can be separated, hence in this case it is integrable.

Comment. The proof of Theorem 2 is based on "Differential Galois Theory", which gives a *necessary* condition for integrability. Moreover, in a view of the last remark, it follows this condition is also a *sufficient* one. So Theorem 2 gives a complete answer, when the system is integrable and when it is not.

The paper is organized as follows. In section 2 we summarize the theoretical results about Ziglin's and Morales-Ramis's theories. The proof of the Theorem 2 is given in section 3. In the last section some numerical experiment, confirming the theoretical results, are presented.

2. THEORY

In this section we summarize briefly some results on integrability of Hamiltonian systems. For more detailed description on Differential Galois theory see [6], [7].

Let (M^{2n}, ω) be a complex symplectic manifold. H is an analytic function over M^{2n} and the respective Hamiltonian system is

$$\dot{x} = X_H(x).$$

A Hamiltonian system is integrable in *Liouville sense* if there exist n independent first integrals $F_1 = H$, F_2,.., F_n in involution, namely $\{F_i, F_j\} = 0$ for all i and j, where $\{,\}$ is the Poisson's bracket [9]. Let $z = z(t)$ is a solution (not equilibrium) of the Hamiltonian system and $\Gamma := \{z = z(t)\}$ is its integral curve. The *variational equations* (VE) responding to $z = z(t)$ are

$$\dot{\eta} = \frac{\partial X_H}{\partial x}(z(t))\eta.$$

Reducing (VE) by the first integral dH, we get so called *normal variational equations* (NVE)

$$\dot{\xi} = A(t)\xi \qquad \text{with dimention } 2(n-1).$$

One of the first, who gave a criterion for having non-integrability, based on (VE) was Poincaré. Let M^{2n} be real and $z = z(t)$ be a periodic solution of the Hamiltonian system. Poincaré has studied the monodromy matrix, corresponding to (VE) [10]and he has proved that if the Hamiltonian system has k first integrals, then k characteristic exponents must be zero.

In 1982 Ziglin [11]proved the following result for integrability of a complex-analytical Hamiltonian systems:

Theorem 3. *Let a Hamiltonian system have n first integrals, independent around Γ, but not necessary on Γ. Suppose that there is a nonresonant element g in the monodromy group of (NVE). Then every other element g' of the monodromy group transforms the set of eigendirections of g into itself.*

Let us remind of $g \in Sp(2n, \mathbb{C})$ (the monodromy group is a subgroup of the symplectic group) is a resonant if $l_1^{r_1} \dots l_n^{r_n} = 1$, where r_i are nonzero integers and l_i are the eigenvalues of g.

Note that in the Ziglin's result, there is no assumption that the integrals are in involution, in addition it refers to the case $n = 2$, because in higher dimensions there are resonanses.

Another method for proving non-integrability is based on the *Galois group* of (VE). In result of the efforts of Ramis, Morales-Ruiz, Simo, Chirchil and Rod, the following result has appeared in the end of the last century [6]:

Theorem 4. *Let a Hamiltonian system has n meromorphic first integrals in involution around Γ, but not necessary on Γ. Then the identity component G^0 of the Galois group G of (VE) with respect to Γ is abelian.*

In applications is used the next algorithm:

1) to find out a solution $z = z(t)$ of the hamiltonian system

2) to write the variational equations (VE) and (NVE), corresponding to $z = z(t)$

3) to check for commutativity of the Galois group of (VE), (NVE)

If once is proved, that G^0 is not abelian, than the respective system is non-integrable in Liouville sense. But the fact that G^0 is abelian doesn't imply integrability.

3. PROOF OF THEOREM 2

The proof of Theorem 2 is divided in several lemmas.

Lemma 1. *The system (1.5) has a particular solution*

$$\left|\begin{array}{l} \widetilde{q_r} = \sqrt{\lambda_1} sn\left(\dfrac{\sqrt{(h_2 - r^4 h_1)\lambda_2}}{2} t, \kappa\right) \\ \widetilde{p_r} = \dot{\widetilde{q_r}} \\ \widetilde{q_j} = 0 \\ \widetilde{p_j} = 0 \end{array}\right. \qquad j = 1, \ldots, N, j \neq r, \tag{3.1}$$

where λ_1, $\lambda_2 \in \mathbb{C}$, $|\lambda_1| < |\lambda_2|$ *and* λ_1 *and* λ_2 *are roots of the equation*

$$\frac{h_2 - r^4 h_1}{4}\lambda^2 + \frac{c_2 - r^2 c_1}{2}\lambda + 2f = 0.$$

and $\kappa = \sqrt{\frac{\lambda_1}{\lambda_2}}$.

Proof. There exists r, such that $h_2 - r^4 h_1 \neq 0$. Putting in the Hamiltonian system (1.5) $(\widetilde{q}, \widetilde{p}) = (0, .., 0, \widetilde{q_r}, 0, .., 0, \widetilde{p_r}, 0, .., 0)$ we get

$$\left|\begin{array}{l} \dot{\widetilde{q_r}} = \widetilde{p_r} \\ \dot{\widetilde{p_r}} = -\widetilde{q_r}\left(r^2 c_1 - c_2 + \dfrac{r^4 h_1 - h_2}{2}(\widetilde{q_r})^2\right) \end{array}\right.$$

The corresponding to this system Hamiltonian H is obtained from (1.4) after putting $(\widetilde{q}, \widetilde{p}) = (0, .., 0, \widetilde{q_r}, 0, .., 0, \widetilde{p_r}, 0, .., 0)$

$$\tilde{p}_r^2 = \frac{h_2 - r^4 h_1}{4}\tilde{q}_r^4 + (c_2 - r^2 c_1)\tilde{q}_r^2 + 2f,$$

Taking into account that $p_r = \dot{\widetilde{q_r}}$ we obtain the family of curves

$$\Gamma(f) : (\dot{\widetilde{q_r}})^2 = \frac{h_2 - r^4 h_1}{4}(\widetilde{q_r})^4 + (c_2 - r^2 c_1)(\widetilde{q_r})^2 + 2f$$

from where after some transformations we reach

$$\left(\frac{\dot{\widetilde{q_r}}}{\sqrt{\lambda_1}}\right)^2 = \frac{h_2 - r^4 h_1}{4}\lambda_2\left(1 - \left(\frac{\widetilde{q_r}}{\sqrt{\lambda_1}}\right)^2\right)\left(1 - \left(\sqrt{\frac{\lambda_1}{\lambda_2}}\frac{\widetilde{q_r}}{\sqrt{\lambda_1}}\right)^2\right)$$

where λ_1, $\lambda_2 \in \mathbb{C}$ and $|\lambda_1| < |\lambda_2|$. This is precisely the definition of Jacobi's elliptic sn [14], so we get the particular solution as the lemma states. □

The function $sn(\tau, \kappa)$ is double periodic meromorphic function with periods $4K(\kappa)$ and $i2K'(\kappa)$. In the parallelogram of the periods $sn(\tau, \kappa)$ has two simple poles $iK'(\kappa)$ and $2K(\kappa) + iK'(\kappa)$ [14].

Therefore $sn\left(\dfrac{\sqrt{(h_2 - r^4 h_1)\lambda_2}}{2} t, \kappa\right)$ has periods $T_1 = \dfrac{8K(\kappa)}{\sqrt{(h_2 - r^4 h_1)\lambda_2}}$, $T_2 = \dfrac{i4K'(\kappa)}{\sqrt{(h_2 - r^4 h_1)\lambda_2}}$ and poles $t_1 = \dfrac{i2K'(\kappa)}{\sqrt{(h_2 - r^4 h_1)\lambda_2}}$, $t_2 = \dfrac{4K(\kappa) + i2K'(\kappa)}{\sqrt{(h_2 - r^4 h_1)\lambda_2}}$.

Geometrically, $\Gamma(f)$ are tori with two points removed.

Next, in order to reduce the domain of the solution (3.1) consider the involution

$$R : (q_1, \ldots, q_r, \ldots, q_n, p_1, \ldots, p_r, \ldots, p_n) \to (q_1, \ldots, -q_r, \ldots, q_n, p_1, \ldots, -p_r, \ldots, p_n)$$

The involution R leaves the Hamiltonian system invariant and changes the places of the two missing points. Let us denote with F_R the set of the fixed points of the involution R,

$$F_R := \{(q_1, \ldots, q_{r-1}, 0, q_{r+1}, \ldots, q_n, p_1, \ldots, p_{r-1}, 0, p_{r+1}, \ldots, p_n).$$

Then factoring $M \backslash F_R$ in R we get the smooth symplectic manifold $\hat{M} = (M \backslash F_R)/R$. The Hamiltonian H is transformed to the Hamiltonian $\hat{H}$ for the same Hamiltonian system (1.5) defined on $\hat{M}$. It is clear that if the system (1.5) has enough independent first integrals they will be transformed into independent first integrals on $\hat{M}$. Then factorizing $\hat{\Gamma}(f) = \Gamma(f)/R$ and having in mind that

$$sn(\tau + 2K(\kappa)) = -sn(\tau), \tau \in \mathbb{C}.$$

we obtain that the domain of the family of the curves is mapped as tori with one point removed.

Lemma 2. *The normal variational equations (NVE) of the system with hamiltonian (1.4) around the particular solution (3.1) are*

$$\left| \begin{array}{l} \dot{\xi}_j = \eta_j \\ \dot{\eta}_j = \xi_j \left((c_2 - j^2 c_1) + \dfrac{(h_2 - j^2 r^2 h_1)}{2} (\tilde{q}_1)^2 \right) \end{array} \right. \quad j = 1, \ldots, N, j \neq r. \tag{3.2}$$

The proof is straightforward and therefore is omitted.

In view of Lemma 2, (NVE) breaks into $N-1$ separate systems, as each of them consists of two first-order linear differential equations. So each of these $N-1$ systems can be written as a second-ordered linear differential equation denoted with $(NVE_j), j = 1, \ldots, N, j \neq r$, namely

$$\ddot{\xi}_j + \left((j^2 c_1 - c_2) + \frac{(j^2 r^2 h_1 - h_2)}{2} \lambda_1 sn^2 \left(\frac{\sqrt{(h_2 - r^4 h_1)\lambda_2}}{2} t, \sqrt{\frac{\lambda_1}{\lambda_2}} \right) \right) \xi_j = 0. \tag{3.3}$$

In our problem, because of the specific kind of (NVE), the Galois group G looks like a direct product

$$G = G_1 \otimes G_2 \otimes \ldots \otimes G_{r-1} \otimes G_{r+1} \otimes \ldots \otimes G_N,$$

where the missing part G_r corresponds to the tangent equations.

Therefore, in order to prove non-integrability, it is sufficient one part G_j - corresponding to the equation (NVE_j) to be nonabelian.

The equation (3.3) is Fuchsian one. It is known that in this case the monodromy group M topologically generates the Galois group G [8], [6].The monodromy group M has the same specific structure as G.

$$M = M_1 \otimes M_2 \otimes \ldots \otimes M_{r-1} \otimes M_{r+1} \otimes \ldots \otimes M_N, \tag{3.4}$$

Again if one M_j (corresponding to the equation (NVE_j)) is nonabelian, then this will imply that G_j is non-abelian and therefore due to Morales-Ramis theorem (. Theorem 4) we have non-integrability.

Now we shall study the monodromy group M_j for the equation (NVE_j).

Let g_1 and g_2 are the generators of the monodromy group M_j. The element g_1 is associated with a path along the parallel of the torus $\hat{\Gamma}$, which corresponds to adding the period $\frac{T_1}{2}$. Similarly, g_2 is associated with a path along the meridian of $\hat{\Gamma}$ or adding the period T_2 of the function $sn^2(\tau)$.

Lemma 3. *The commutator* $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ *has the following eigenvalues*

$$\exp\left(\pi i\left(1 \pm \sqrt{1 + 8\frac{j^2r^2h_1 - h_2}{r^4h_1 - h_2}}\right)\right).$$

Proof. The commutator corresponds to one winding around the regular singular point $t_1 = \dfrac{i2K'(\kappa)}{\sqrt{(h_2 - r^4h_1)\lambda_2}}$ of the equation (3.3).

It is known that for a linear differential equation [12]

$$\ddot{\xi_j} + \frac{P(t)}{(t - t_1)}\dot{\xi_j} + \frac{Q(t)}{(t - t_1)^2}\xi_j = 0$$

where $P(t)$ and $Q(t)$ are holomorphic in a neighborhood of $t = t_1$, then the eigenvalues of the monodromy transformation, corresponding to one circle around the regular singular point $t = t_1$, are $\exp(2\pi i\rho_{1,2})$, where $\rho_{1,2}$ are the roots of the indicial equation

$$\rho(\rho - 1) + P(t_1)\rho + Q(t_1) = 0 \tag{3.5}$$

The analytical theory of the differential equations is described in details in [13].Hence we have

$$sn\left(\frac{\sqrt{(h_2 - r^4h_1)\lambda_2}}{2}t, \sqrt{\frac{\lambda_1}{\lambda_2}}\right) = -\frac{2}{\sqrt{(h_2 - r^4h_1)\lambda_1}}\frac{1}{(t - t_1)} + O(1),$$

so

$$Q(t_1) = -2\frac{(j^2r^2h_1 - h_2)}{(r^4h_1 - h_2)}, \qquad P(t) \equiv 0$$

and the roots of the quadratic equation (3.5) are exactly

$$\rho_{1.2} = \frac{1}{2}\left(1 \pm \sqrt{1 + 8\frac{j^2r^2h_1 - h_2}{r^4h_1 - h_2}}\right). \qquad \square$$

Taking into account Lemma 3 we conclude that if the eigenvalues of the j-th commutator are not units, than M_j is not abelian. Let us denote

$$\mu_j := 1 + 8\frac{j^2r^2h_1 - h_2}{r^4h_1 - h_2}.$$

Then the sufficient condition for non-integrability is the existence of j, such that the number μ_j is not equal to a square of some odd integer.

Lemma 4. *The monodromy group (3.4) is not abelian for $N > 2$ and $h_1 \neq 0$.*

Proof. Suppose that there exists $j \neq r$ such that $\rho_1 \in \mathbb{Z}$, so $\mu_j = (2k-1)^2$ for some $k \in \mathbb{Z}$. Hence when $h_2 \neq 0$ we get

$$\frac{h_1}{h_2} = \frac{1 - s_j}{j^2r^2 - s_jr^4}, \tag{3.6}$$

where $k(k-1) = 2s_j, s_j \in \mathbb{Z}$. We notice that for the numbers $s_j, 1 \leq j \leq N, j \neq r$ we have $s_j \geq 1$ or $s_j = 0$. From (3.6), if some $s_j = 0$, that can happen for only one j, namely j: $j^2r^2h_1 = h_2$.

The aim is to show, there exists a number l, such that $\mu_l \neq (2p-1)^2$ for all $p \in \mathbb{Z}$. For the purpose we examine cases according to r.

Case 1. *Let $1 < r < N$, $j < r$ and none of the numbers s_j is zero.*

Then there exists $l = r + 1$ and we assume $\mu_l = \mu_{r+1} = (2p-1)^2$ for some $p \in \mathbb{Z}$. Again we express

$$\frac{h_1}{h_2} = \frac{1 - s_{r+1}}{(r+1)^2r^2 - s_{r+1}r^4}, \tag{3.7}$$

where $p(p-1) = 2s_{r+1}, s_{r+1} \in \mathbb{Z}$. From both (3.6) and (3.7), after some computations we get

$$(s_j - 1)(2r + 1) + (s_{r+1} - 1)(r - j)(r + j) = 0,$$

which is true if and only if $s_j = s_{r+1} = 1$, exactly when $h_1 = 0$ – the integrable case called "anharmonic oscillator" [5]. The last is in contradiction with the current lemma, so we have proved that the monodromy group M_{r+1} is not abelian.

Case 2. *Let $1 < r < N$, $r < j$ and none of the numbers s_j is zero.*

Then we take $l = r - 1$ and we obtain the following equation

$$(s_j - 1)(2r - 1)(-1) + (s_{r-1} - 1)(r - j)(r + j) = 0,$$

and similarly as in the previous case, we obtain that M_{r-1} is not abelian.

Case 3. *Let* $N = 3$.

The cases $\{r = 1, j = 2, l = 3\}$ and $\{r = 1, l = 2, j = 3\}$ are equivalent, if one transposes j and l and again from (3.6) and (3.7) we have

$$\frac{1 - s_2}{4 - s_2} = \frac{1 - s_3}{9 - s_3},$$

going back, to that we have noted above, we obtain

$$3(p + 1)(p - 2) = 8(k + 1)(k - 2),$$

and there is an equivalent case too, in transposing p and k. The last is true only when $p, k \in \{-1, 2\}$, which implies the integrable case - $h_1 = 0$. From the another pair of equivalent cases $\{l = 1, j = 2, r = 3\}$ and $\{j = 1, l = 2, r = 3\}$, analogously we get

$$5(p + 1)(p - 2) = 8(k + 1)(k - 2),$$

– the next contradiction with the current lemma. The cases $\{j = 1, r = 2, l = 3\}$ and $\{l = 1, r = 2, j = 3\}$ lead to

$$5(s_1 + 1) = -3(s_3 + 1),$$

which is not possible.

Case 4. *Let* $N > 3$, $1 < r < N$ *and* $s_j = 0$.

Then we can choose another number j_0 instead of j and to fall in case 1) or case 2), because only one s_j can be zero.

Case 5. *Let* $N > 3$, $r = 1$ *or* $r = N$.

If $r = 1$, then there exist at least two equations, the j-th and the l-th for $N > 3$, such that $s_j \neq 0 \neq s_l$. Without lost generality we can focus on the case $j = 2$, $l = 3$, so we obtain

$$3(s_3 - 1) = 8(s_2 - 1),$$

and similarly as the previous cases it leads to a contradiction. For $r = N$, we take the variational equations with numbers $j = N - 1$ and $l = N - 2$, therefore we get

$$4(s_{N-1} - 1)(N - 1) = (s_{N-2} - 1)(2N - 1),$$

which is available only when $h_1 = 0$.

Case 6. *The last case left is* $h_1 \neq 0$ *and* $h_2 = 0$.

Here

$$\mu_j = 1 + 8\frac{j^2}{r^2}.$$

Let first $r > 1$, then there exists an equation with number $j = r - 1$. If $\mu_j = \mu_{r-1}$ is equal to a square of an odd

$$1 + 8\frac{(r-1)^2}{r^2} = (2k-1)^2$$

we get

$$r^2(k+1)(k-2) = 2(1-2r),$$

which is possible only when $k = 0, 1$ and $r = 1$, which is in contradiction with the case. So we conclude that the group M_{r-1} is not abelian. Now let $r = 1$. Then $\mu_j = 1 + 8j^2$, which, for example, for $j = 2$, is not equal to a square of an odd number, so the monodromy group M_2 is not abelian and that proves Lemma 4. □

The first four lemmas prove part a) of Theorem 2. Let us formulate the last two lemmas, proving the second part of the theorem.

Lemma 5. *The system with Hamiltonian (1.4) is non-integrable for* $N = 2$, $h_2 \neq 4h_1$ *and* $h_1 \neq 0$.

Proof. The monodromy group for $j = 2$, corresponding to variations around the particular solution $(\tilde{q_1}, \tilde{p_1})$ is not abelian when

$$1 + 8\frac{4h_1 - h_2}{h_1 - h_2} \neq (2k-1)^2, \qquad k \in \mathbb{Z}. \tag{3.8}$$

First, we write the particular solution of the system $(\tilde{q_2}, \tilde{p_2})$, then the respective (NVE) around it and the following condition for non-integrability

$$1 + 8\frac{4h_1 - h_2}{16h_1 - h_2} \neq (2m-1)^2, \qquad m \in \mathbb{Z}. \tag{3.9}$$

Let us assume that for some values of the parameters $h_{1,2}$ such that $h_1 \neq 0$ and $h_2 \neq 4h_1$ there exist integers k and m, that we have equalities in (3.8) and (3.9).

$$1 + 8\frac{4h_1 - h_2}{h_1 - h_2} = (2k-1)^2, \qquad 1 + 8\frac{4h_1 - h_2}{16h_1 - h_2} = (2m-1)^2.$$

After some transformations in the first equality, like in the proof of Lemma 4, we express

$$h_2 = 4h_1\frac{4s-1}{s-1}$$

Here $s \neq 1$ since $s = 1$ implies $h_1 = 0$. Putting this h_2 in the second equality, we obtain

$$\frac{4s}{5s-1} \in \mathbb{Z}$$

which is true only in the case $s = 0$, that is exactly the separable case – $h_2 = 4h_1$.□

The last case we haven't examined yet is $h_1 = h_2$.

Lemma 6. *The system with Hamiltonian (1.4) is non-integrable for $N = 2$, $h_2 = h_1$ and $h_1 \neq 0$.*

Proof. In this case, the hamiltonian is

$$H_0 = \frac{1}{2}({p_1}^2 + {p_2}^2) + \frac{(c_1 - c_2)}{2}{q_1}^2 + \frac{(4c_1 - c_2)}{2}{q_2}^2 + \frac{3}{4}h_1{q_1}^2{q_2}^2 + \frac{15}{8}h_1{q_2}^4$$

and we find a particular solution $(\tilde{q_2},\tilde{p_2})$, of respective Hamiltonian system, namely

$$\left| \begin{array}{l} \tilde{q_2} = \sqrt{\mu_1} sn\left(\frac{i\sqrt{(15h_1\mu_2)}}{2}t, \sqrt{\frac{\mu_1}{\mu_2}}\right) \\ \tilde{p_2} = \dot{\tilde{q_2}} \\ \tilde{q_1} = 0 \\ \tilde{p_1} = 0 \end{array} \right. \tag{3.10}$$

where μ_1, $\mu_2 \in \mathbb{C}$ and $|\mu_1| < |\mu_2|$, μ_1 and μ_2 are the roots of the equation

$$-\frac{15h_1}{4}\mu^2 + (c_2 - 4c_1)\mu + 2H_0 = 0.$$

The normal variational equations (NVE) for $j = 1$ around the solution (3.10) is

$$\left| \begin{array}{l} \dot{\xi_1} = \eta_1 \\ \dot{\eta_1} = -\xi_1\left((c_1 - c_2) + \frac{3}{2}h_1(\tilde{q_2})^2\right) \end{array} \right. .$$

Hence we get the second-ordered differential equation

$$\ddot{\xi_1} + \xi_1\left((c_1 - c_2) + \frac{3}{2}h_1(\tilde{q_2})^2\right) = 0$$

and having in mind the Laurant's expansion of (3.10), we write the indicial equation

$$\rho^2 - \rho - \frac{2}{5} = 0,$$

whose roots are not integers, therefore the monodromy group is not abelian, which proves nonintegrabitity in the case $h_1 = h_2$. □

This concludes the proof of Theorem 2.

4. NUMERICAL EXPERIMENTS

Practically the integrability of a Hamiltonian system can be examined with so called "Poicaré sections". Let there be a Hamiltonian system in $\mathbb{R}^{2n}$

$$\dot{z} = X_H(z)$$

with Hamiltonian H and a periodic solution – $z = z(t)$, let Γ be the respective phase curve. We build a transversal intersection S to Γ and the solution $z = z(t)$ crosses S in a point z_0. In a sufficiently small neighborhood U of S, containing z_0, we look at those solutions of the Hamiltonian system, which have initial conditions in U. We always take solutions, whose initial conditions lay on the same energy level $H = E$. We draw the consecutive intersection points where these paths cross S. This mapping $P : S \to S$ is called "Poincaré mapping".

If the intersection points form regular curves, then we suppose integrability. If a chaotic picture is obtained, then we conclude that the Hamiltonian system is non-integrable.

In practice we examine two-dimensional intersection S and here are some Poincaré sections for our system, drawn by Maple.

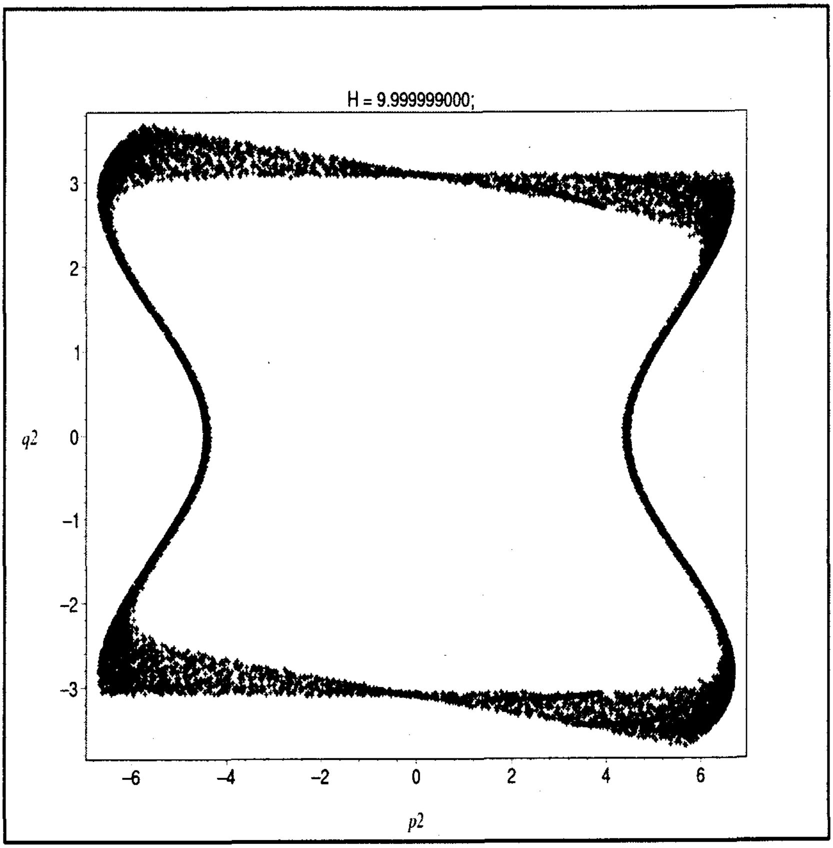

Fig. 1. $c_1 = -1.2, c_2 = 1.3, h_1 = 0, h_2 = -1.5, S = (p_1, q_1)$

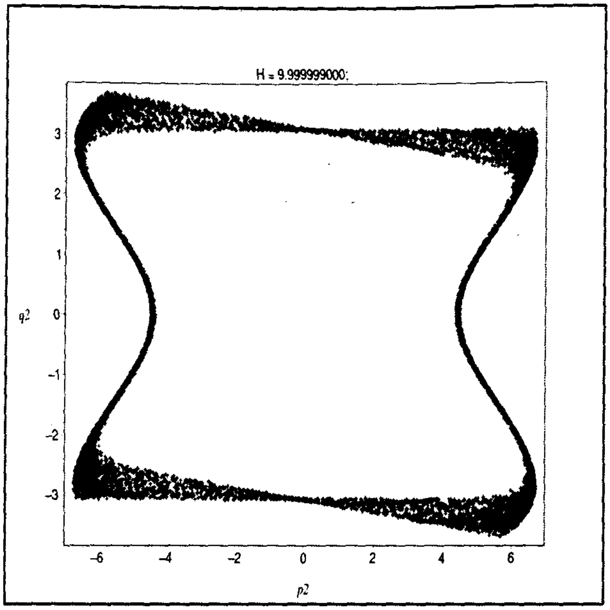

Fig. 2. $c_1 = -1.2, c_2 = 1.3, h_1 = 0, h_2 = -1.5, S = (p_2, q_2)$

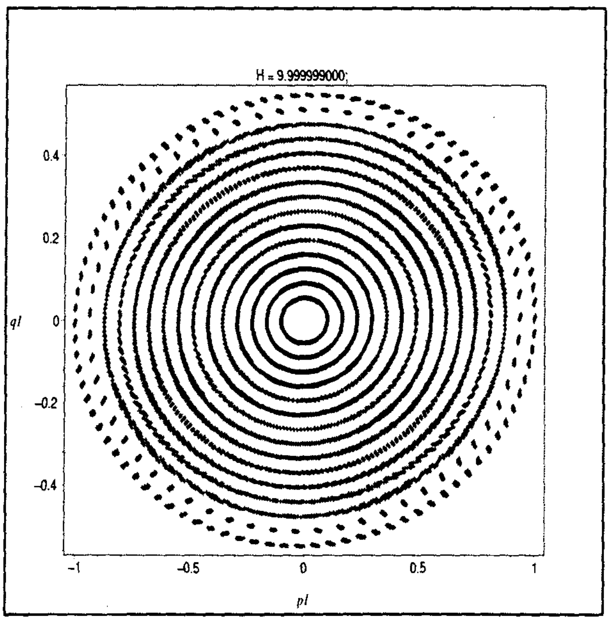

Fig. 3. $c_1 = 2.2, c_2 = -2.3, h_1 = 0, h_2 = 1.5, S = (p_1, q_1)$

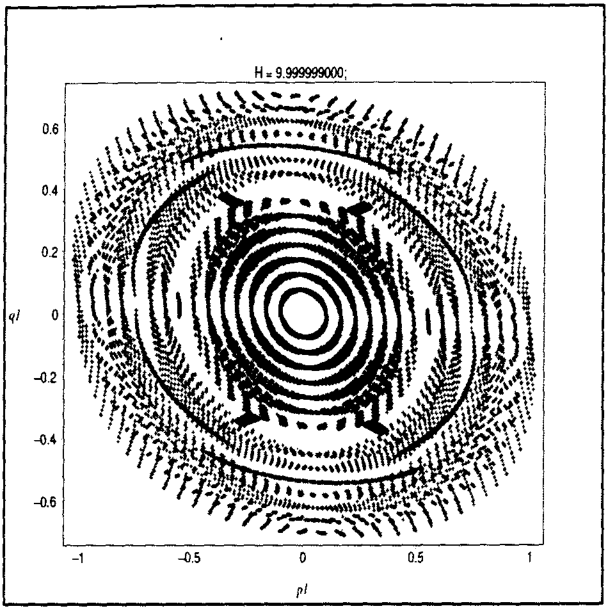

Fig. 4. $c_1 = -1.2, c_2 = 1.3, h_1 = 1, h_2 = -1.5, S = (p_1, q_1)$

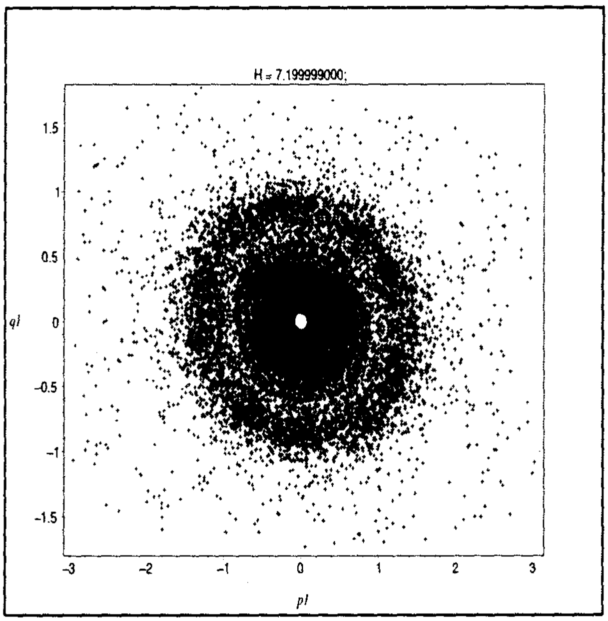

Fig. 5. $c_1 = -1.2, c_2 = 1.3, h_1 = 1.5, h_2 = -1.5, S = (p_1, q_1)$

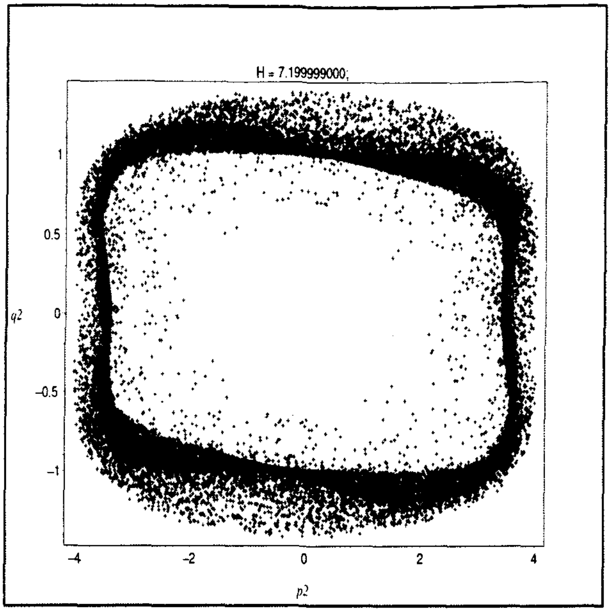

Fig. 6. $c_1 = -1.2, c_2 = 1.3, h_1 = 1.5, h_2 = -1.5, S = (p_2, q_2)$

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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NUMERICAL SOLUTION OF HEAT-CONDUCTION PROBLEMS ON A SEMI-INFINITE STRIP WITH NONLINEAR LOCALIZED FLOW SOURCES [1]

MIGLENA KOLEVA

We consider semi-linear heat problems on a semi-infinite interval. They model systems of temperature regulation in isotropic media with non-uniform source terms which can provide cooling or heating effects. Numerical method for overcoming the nonlinearities in the equation and boundary conditions as well as the unbounded domain is discussed.

Keywords: Parabolic problems, difference schemes, convergence rate, semi-infinite strip, artificial boundary conditions.

2000 MSC: main 35K05, secondary 65M60

1. INTRODUCTION

This paper is concerned with the following initial-boundary value problems for the one-dimensional heat equation on a semi-infinite interval:

$$(P_1) \qquad \begin{aligned} &u_t - u_{xx} = -F_1(u_x(0,t)), \quad 0 < x < \infty, \quad 0 < t \le T, \quad T < \infty, \\ &u(0,t) = 0, \quad 0 \le t \le T, \\ &u(x,0) = h_1(x), \quad 0 \le x < \infty, \end{aligned}$$

[1] The results of this paper were reported at International Conference "Pioneers of Bulgarian Mathematics", 8 - 10.07.2006, Sofia.

$$(P_2) \qquad \begin{aligned} & u_t - u_{xx} = \Phi(x).F_2(u_x(0,t),t), \quad 0 < x < \infty, \quad 0 < t \le T, \ T < \infty, \\ & u(0,t) = g(t), \quad 0 \le t \le T, \\ & u(x,0) = h_2(x), \quad 0 \le x < \infty, \end{aligned}$$

where $u = u(x,t)$ denotes the temperature distribution (the unknown), x and t are the spatial and time coordinate respectively; T is a given positive constant. The data functions $h_1(x)$, $h_2(x)$, $g(t)$ (representing initial and boundary conditions) and $\Phi(x)$ are real, defined on R^+. F_1 and F_2 are sink sources of heat energy, uniform in x. Such problems can be thought as by the modelling of a system of temperature regulation in isotropic media, with non-uniform source term $F_1(u_x(0,t))$ for (P_1) and $\Phi(x).F_2(u_x(0,t),t)$ for (P_2), which provides a cooling or heating effect depending upon the properties of F_1 or F_2, related to the source of the localized heat flux $u_x(0,t)$, see [13, 14].

In [3, 15] results on existence, uniqueness and asymptotic behavior of the solution have been proved for problem (P_1). Some results on the behavior of the solution and explicit formula for the solution in special cases are obtained in [13].

The existence, uniqueness and asymptotic behavior of the solution of problem (P_2) are explicated in [14]. Also, the validation of the maximum principle for (P_1) and (P_2) is shown in [13, 14, 15].

The goal of this paper is to solve numerically problems (P_1) and (P_2) with effective and accurate methods.

The remainder part of this work is organized as follows. In Section 2 properties of the solutions to the considered problems are described. The original problems are written in new, equivalent forms, more appropriate for numerical treatment. In the next section we construct exact artificial boundary conditions of the new formulated problems. Also, full discretizations are derived. In Section 4 we present some numerical results, demonstrating the accuracy of the algorithms.

2. PRELIMINARY RESULTS

As observed in [13, 14], the heat flux

$$v(x,t) = u_x(x,t), \tag{2.1}$$

for problems (P_1) and (P_2) satisfies the classical heat conduction problems with a nonlinear convective condition at $x = 0$, which can be written in the forms:

$$(V_1) \qquad \begin{aligned} & v_t - v_{xx} = 0, \quad 0 < x < \infty, \quad 0 < t \le T, \quad T < \infty, \\ & v_x(0,t) = F_1(v(0,t)), \quad 0 \le t \le T, \\ & v(x,0) = h_1'(x), \quad 0 \le x < \infty, \end{aligned}$$

$$(V_2) \qquad \begin{aligned} & v_t - v_{xx} = \Phi'(x).F_2(v(0,t),t), \quad 0 < x < \infty, \quad 0 < t \le T, \quad T < \infty, \\ & v_x(0,t) = g'(t) - \Phi(0).F_2(v(0,t),t), \quad 0 \le t \le T, \\ & v(x,0) = h_2'(x), \quad 0 \le x < \infty, \end{aligned}$$

For problem (V_1), it has been proved in [13] that the maximum principle holds. A qualitative analysis of the problem (V_2) is given in [14]. The authors show that under some assumptions for $F_2(v(0,t),t)$, $g'(t)$, $\Phi'(x)$ and $h_2'(x)$, we have $v(0,t) > 0$ for $t > 0$ and $u_x(x,t) \geq 0$ for $x \geq 0$, $t \geq 0$. Also the monotonicity properties of the solution of the problem (V_2) are proved.

It's worth to note that if $h_1'(x) \geq 0$ in R^+, $v(0,t).F_1(v(0,t)) > 0$, $v(0,t) > 0$, $t \in [0,T]$ (for problem (V_1)) or for problem (V_2): $h_2(x) > 0$, $\Phi(x) \leq 0$, $\Phi'(x) \geq 0$ in R^+, $v(0,t).F_2(v(0,t),t) > 0$, $\forall\, v(0,t) \neq 0$, $\forall\, t > 0$ and $g(t) \geq 0$ or $\lim\limits_{t\to\infty} g(t) = 0$, $\forall\, t > 0$, together with some other hypotheses, then the corresponding solutions of (P_1) or (P_2), $u(x,t) \to 0$, as $t \to \infty$ uniformly for $x \geq 0$, see [14, 15].

In the work done, we restrict our considerations to the case: $h_i'(x) \geq 0$, supp $h_i'(x) < \infty$, $i = 1,2$, $\Phi'(x) \geq 0$ and supp $\Phi'(x) < \infty$. The last constraint we shall remove later. Many physical processes lead to models with compact supported initial datum. Such kind of problems are well studied in [4, 5]. From supp $h_i'(x) < \infty$, $i = 1,2$ follows that there exists L_i, $i = 1,2$ and $0 < L_i < +\infty$: $h_i'(x) = 0$ for $x > L_i$, $i = 1,2$. Then $v(x,t) \to 0$ when $x \to +\infty$, i.e. $v(+\infty,t) = 0$, $\forall\, t > 0$.

3. NUMERICAL METHOD

We focuss our attention to problems (V_1) and (V_2). Having obtained their numerical solutions, it's very easy to find the solutions of (P_1) and (P_2), using (2.1) and the well-known numerical integration formulas (by Trapezoidal rule, for example).

The most widely used methods are the finite element and the finite difference schemes. Since the grids are finite, then on the grid boundary the same type boundary conditions as on the infinity in the differential problem, are often imposed, see for example [1, 2]. This, however, leads to the loss of accuracy, especially in the case, when the solution does not go to zero as $x \to \infty$ or the compact support of the solution become large in time. More accurate are artificial boundary conditions. For linear parabolic problems with linear boundary conditions such results can be found in [6, 16] and for semi-linear one and two-dimensional heat problems, see [7, 10]. Also, the comparison with other methods is available.

Having in mind all those results, our approach will be to use an artificial boundary method. Generally, it means to introduce artificial boundaries, construct exact boundary conditions on the artificial boundaries and reduce the original problem to an equivalent or approximate problem, defined on a bounded domain. In general, the boundary conditions on the artificial boundaries are obtained by considering the exterior problems outside the artificial boundaries.

The idea is employed from [10, 16].

Let supp $h_i'(x) = [0, L_i]$, $i = 1,2$ and supp $\Phi'(x) \in [0, L_2]$. For computing the numerical solutions of any of the problems (V_1) and (V_2), we introduce an artificial boundary: $\Gamma_i = \{(x,t)|x = l, \quad l > L_i, \quad i = 1,2; \quad 0 \le t \le T\}$. Then the domain $\Omega = \{(x,t)|0 \le x < +\infty, \quad 0 < t \le T\}$ is divided into the bounded part Ω^0 and unbounded part $\Omega^e = \{(x,t)|l < x < +\infty, \quad 0 < t \le T\}$. On the domain Ω^e, $h_i'(x) \equiv 0$, $i = 1,2$ and $\Phi'(x) \equiv 0$. We first consider the restriction of the solution of the considered problems on the domain Ω^e (counterpart domain). In this domain, the solutions of both problems (V_1) and (V_2) satisfy one and the same initial-boundary value problem (V^e).

$$(V^e) \qquad \begin{aligned} & v_t - v_{xx} = 0, \quad (x,t) \in \Omega^e, \\ & v(x,0) = 0, \quad l \le x < +\infty, \\ & v(x,t) \to 0, \quad \text{when} \quad x \to +\infty. \end{aligned}$$

If $v(l,t)$ is given, then (V^e) is a properly posed problem. We can get the solution $(v(x,t))$ for given $(v(l,t))$, see [12].

$$v(x,t) = \frac{x-l}{2\sqrt{\pi}} \int_0^t v(l,\lambda)(t-\lambda)^{-\frac{3}{2}} e^{-\frac{(x-l)^2}{4(t-\lambda)}} d\lambda, \tag{3.1}$$

Next, we shall obtain the artificial boundary condition, using (3.1). Setting $\rho = (x-l)/(2\sqrt{t-\lambda})$, then we have

$$v(x,t) = \frac{2}{\sqrt{\pi}} \int_{\frac{x-l}{2\sqrt{t}}}^{\infty} v\left(l, t - \frac{(x-l)^2}{4\rho^2}\right) e^{-\rho^2} d\rho,$$

$$\frac{\partial v(x,t)}{\partial x} = \frac{1}{\sqrt{\pi t}} v(l,0) e^{-\frac{(x-l)^2}{4t}} + \frac{1}{\sqrt{\pi}} \int_{\frac{x-l}{2\sqrt{t}}}^{\infty} \frac{\partial v}{\partial t}\left(l, t - \frac{(x-l)^2}{4\rho^2}\right) \frac{l-x}{\rho^2} e^{-\rho^2} d\rho.$$

Returning to the variable λ, we get

$$\frac{\partial v(x,t)}{\partial x} = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial v(l,\lambda)}{\partial \lambda} \frac{1}{\sqrt{t-\lambda}} e^{-\frac{(x-l)^2}{4(t-\lambda)}} d\lambda.$$

Taking the limit $x \to +l$, we obtain the following exact boundary condition, satisfied by the solution $v(x,t)$ on the artificial boundary $x = l$.

$$\frac{\partial v(l,t)}{\partial x} = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial v(l,\lambda)}{\partial \lambda} \frac{1}{\sqrt{t-\lambda}} d\lambda, \quad 0 \le t < +\infty. \tag{3.2}$$

Using (3.2), we reduce original problems (V_1) and (V_2) to problems on the bounded domain $\Omega^0 = \{(x,t)|0 < x < l, \ \ 0 \le t \le T\}$.

$$v_t - v_{xx} = \begin{cases} 0, & (x,t) \in \Omega^0, \quad \text{for } (V_1), \\ \Phi'(x).F_2(v(0,t),t), & (x,t) \in \Omega^0, \quad \text{for } (V_2), \end{cases}$$

$$v_x(0,t) = \begin{cases} F_1(v(0,t)), & 0 \le t \le T, \quad \text{for } (V_1), \\ g'(t) - \Phi(0).F_2(v(0,t),t), & 0 \le t \le T, \quad \text{for } (V_2), \end{cases}$$

$$(Ri) \qquad v_x(l,t) = -\frac{1}{\sqrt{\pi}} \int_0^t v_\lambda(l,\lambda) \frac{1}{\sqrt{t-\lambda}} d\lambda, \quad 0 \le t \le T,$$

$$v(x,0) = \begin{cases} h_1'(x), & 0 \le x \le l, \quad \text{for } (V_1), \\ h_2'(x), & 0 \le x \le l, \quad \text{for } (V_2). \end{cases}$$

The solutions of the problems (V_1) and (V_2) in Ω^e can be computed by formula (3.1) for $v(l,t)$ already known.

If supp $\Phi'(x) = \infty$, then the problem (V^e), corresponding to (V_2) becomes

$$(\tilde{V}^e) \qquad \begin{aligned} & v_t - v_{xx} = \Phi'(x).F_2(v(0,t),t), \quad (x,t) \in \Omega^e, \\ & v(x,0) = 0, \quad l \le x < +\infty. \end{aligned}$$

Now, for the solution $v(x,t)$ of problem $(\tilde{V}^e)$ for given $v(l,t)$ we have (see [12]):

$$\begin{aligned} v(x,t) &= \frac{x-l}{2\sqrt{\pi}} \int_0^t v(l,\lambda)(t-\lambda)^{-\frac{3}{2}} e^{-\frac{(x-l)^2}{4(t-\lambda)}} d\lambda \\ &+ \frac{1}{2\sqrt{\pi}} \int_0^t \frac{F_2(v(0,\lambda),\lambda)}{\sqrt{t-\lambda}} \int_0^\infty \Phi'(\xi) \left[e^{-\frac{(x-l-\xi)^2}{4(t-\lambda)}} - e^{-\frac{(x-l+\xi)^2}{4(t-\lambda)}} \right] d\xi d\lambda \qquad (3.3) \end{aligned}$$

As before, applying the technique: 1° change of variable; 2° differentiating with respect to x; 3° returning to the original variable; 4° taking a limit $x \to l$, for the first addend of (3.3) and only 2° and 4° for the second one, we obtain the following artificial boundary condition at $x = l$:

$$\begin{aligned} \frac{\partial v(l,t)}{\partial x} &= -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\partial v(l,\lambda)}{\partial \lambda} \frac{1}{\sqrt{t-\lambda}} d\lambda \\ &+ \frac{1}{2\sqrt{\pi}} \int_0^t \frac{F_2(v(0,\lambda),\lambda)}{(t-\lambda)^{3/2}} \int_0^\infty \xi \Phi'(\xi) e^{-\frac{\xi^2}{4(t-\lambda)}} d\xi d\lambda \qquad (3.4) \end{aligned}$$

Again, the assumption $v(l,0) = 0$ is an essential one.

Let V_h is a piecewise linear finite element space, defined on an uniform mesh with size h in $\bar{D} = [0, l]$: $\bar{\omega}_h = \{x_i,\ x_i = (i-1)h,\ i = 1, 2, ..., N;\ (N-1)h = l\}$. For a discrete function, defined on $\bar{\omega}_h$, we introduce the following norms:

$$||v||_{L_\infty(\bar{\omega}_h)} = \max_{\bar{\omega}_h} |v(x_i)| \quad \text{and} \quad ||v||_{L_2(\bar{\omega}_h)} = \left(\sum_{i=1}^{N} hv^2(x_i)\right)^{\frac{1}{2}}.$$

The standard finite element discretization of the problems (Ri), $i = 1, 2$ is to find

$$v^h \in V_h, \quad v^h = \sum_{i=1}^{N} V_i(t)\varphi_i(x),$$

satisfying the weak forms of the problems (R_i).

Now, after doing a mass lumping, we obtain for $V_i = V_i(t)$, $i = 1, ..., N$ and $0 \le t \le T$ the following system of ordinary integro-differential equations

$$\dot{V}_1 = \frac{2}{h}\left[\frac{V_2 - V_1}{h} - \begin{cases} F_1(V_1), & \text{for } (R1) \\ g'(t) - \Phi(0).F_2(V_1, t) - \frac{h}{2}\Phi'(0).F_2(V_1, t), & \text{for } (R2) \end{cases}\right], \quad (3.5)$$

$$\dot{V}_i = \frac{1}{h^2}[V_{i-1} - 2V_i + V_{i+1}] + \begin{cases} 0, & \text{for } (R1) \\ \Phi'(x_i).F_2(V_1, t), & \text{for } (R2) \end{cases}, i = 2, ..., N-1, \quad (3.6)$$

$$\dot{V}_N = -\frac{2}{h}\left[\frac{1}{\sqrt{\pi}}\int_0^t \frac{\dot{V}_N(\lambda)}{\sqrt{t-\lambda}}d\lambda + \frac{V_N - V_{N-1}}{h}\right]. \quad (3.7)$$

In the case supp $\Phi'(x) = \infty$ (concerning the problem (V_2)), using (3.4), the equation (3.7) becomes

$$\begin{aligned} \dot{V}_N \quad = \quad & -\frac{2}{h}\left[\frac{1}{\sqrt{\pi}}\int_0^t \frac{\dot{V}_N(\lambda)}{\sqrt{t-\lambda}}d\lambda + \frac{V_N - V_{N-1}}{h} - \frac{h}{2}\Phi'(0).F_2(V_1, t)\right. \\ & \left.-\frac{1}{2\sqrt{\pi}}\int_0^t \frac{F_2(V_1(\lambda), \lambda)}{(t-\lambda)^{3/2}}\int_0^\infty \xi\Phi'(\xi)e^{-\frac{\xi^2}{4(t-\lambda)}}d\xi d\lambda\right]. \end{aligned} \quad (3.8)$$

Next, in order to obtain the full discretization of (3.5)-(3.7)(or (3.8)) we define an uniform mesh in time:

$$t_n = n\tau, \quad n = 0, 1, ...M, \quad M\tau = T.$$

The following lemma we also need

Lemma 3.1 ([16]). *Suppose $f(t) \in C^2[0, t_n]$. Then*

$$\left|\int_0^{t_n} \frac{f'(t)dt}{\sqrt{t_n - t}} - \sum_{k=1}^{n} \frac{f(t_k) - f(t_{k-1})}{\tau}\int_{t_{k-1}}^{t_k} \frac{dt}{\sqrt{t_n - t}}\right| \le \frac{(10\sqrt{2} - 11)}{6} \max_{0 \le t \le t_n} |f''(t)|\, \tau^{\frac{3}{2}}.$$

Lemma 3.1 ground on the approximations of the integrals in the formulas (3.7) and (3.8). The integrals (obtained applying this lemma) are calculated exactly. This semi-analytical integration rule has better accuracy and stability properties than the Trapezoidal rule yet involves about the same computational effort. This is the best possible integration rule, since no additional information on $V_N(\lambda)$ is available in the interval $[t_{n-1}, t_n]$, see [11].

Consequently we obtain the full discretization of the system (3.5)-(3.7), $V_i^n = v^h(x_i, t_n)$, $i = 2, \dots, N-1$, $n = 1, \dots, M$

$$\left[1 + \frac{2\tau}{h^2}\right] V_1^n - \frac{2\tau}{h^2} V_2^n +$$

$$\frac{2\tau}{h}\left\{\begin{array}{ll} F_1(V_1^n), & \text{for } (R1) \\ g'(t_n) - \Phi(0).F_2(V_1^n, t_n) - \frac{h}{2}\Phi'(0).F_2(V_1^n, t_n), & \text{for } (R2) \end{array}\right\} = V_1^{n-1},$$

$$\left(1 + \frac{2\tau}{h^2}\right) V_i^n - \frac{\tau}{h^2}\left(V_{i-1}^n + V_{i+1}^n\right) - \left\{\begin{array}{ll} 0, & (R1) \\ \tau\Phi'(x_i).F_2(V_1^n, t_n), & (R2) \end{array}\right\} = V_i^{n-1},$$

$$\left(\frac{h}{2} + \frac{2\sqrt{\tau}}{\sqrt{\pi}} + \frac{\tau}{h}\right) V_N^n - \frac{\tau}{h} V_{N-1}^n = \left(\frac{h}{2} + \frac{2\sqrt{\tau}}{\sqrt{\pi}}\right) V_N^{n-1}$$

$$-\frac{2}{\sqrt{\pi}} \sum_{k=1}^{n-1} \left(\sqrt{t_n - t_{k-1}} - \sqrt{t_n - t_k}\right)\left(V_N^k - V_N^{k-1}\right). \qquad (3.9)$$

To obtain the full discretization of (3.8) we need, in addition to (3.9) (approximation of the first addend of (3.8)), the approximation of the second addend of (3.8)

Lemma 3.2 ([16]). *Let $f(t) \in C^2[0, t_n]$ and $g(t) = (t_n - t)^{-3/2} e^{-\frac{a^2}{4(t_n - t)}}$ with $a > 0$. Then*

$$\left| \int_0^{t_n} f(t)g(t)dt - \sum_{k=1}^{n} \frac{1}{2}\left[f(t_k) + f(t_{k-1})\right] \int_{t_{k-1}}^{t_k} g(t)dt \right|$$

$$\leq \left(\frac{2c}{a^3} \max_{0 \leq t \leq t_n} |f'(t)| + \frac{\sqrt{\pi}}{4a} \max_{0 \leq t \leq t_n} |f''(t)|\right) \tau^2, \text{ where } c = \int_0^{\infty} \left|\frac{3}{2} - \mu^2\right| \mu^2 e^{-\mu^2} d\mu.$$

Calculation of the integrals $\int_{t_{k-1}}^{t_k} g(t)dt$ exactly, leads to the semi-analytical integration approximation, which was commented earlier. Lemma 3.2 is essential for deriving the following discretization:

$$\frac{1}{2\sqrt{\pi}} \int_0^t \frac{F_2(V_1(\lambda), \lambda)}{(t - \lambda)^{3/2}} \int_0^{\infty} \xi\Phi'(\xi) e^{-\frac{\xi^2}{4(t-\lambda)}} d\xi d\lambda$$

$$\approx h\sum_{i=1}^{\infty}\Phi'(x_i)\sum_{k=1}^{n}\frac{F_2(V_1(t_{k-1}),t_{k-1})+F_2(V_1(t_k),t_k)}{2}\cdot[I(t_k)-I(t_{k-1})], \quad (3.10)$$

where $I(t_k)=\text{erf}\left(\frac{x_i}{2\sqrt{t_n-t_k}}\right)$ and $\text{erf}\,(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\rho^2}d\rho.$

The calculations are tedious, but standard, and we shall outline the main steps:

1. Use Lemma 3.1 for integral with respect to time;
2. Rearrange the integrals and integrand functions;
3. Change the variable λ: $\rho=\frac{\xi}{2\sqrt{t_n-\lambda}}$;
4. Involve the erf integrals;
5. Approximate the integral $\int_0^\infty$ by Rectangular (Trapezoidal or some other

quadrature is also possible) rule. At this stage, using Lemma 3.1 would lead to unduly complication of the approximation. Moreover, in contrast to F_2, the integrand function Φ' is a known function of the known argument, thus the usage of Lemma 3.1 can be avoided.

Finally, the full discretization of (3.8) is

$$\left(\frac{h}{2}+\frac{2\sqrt{\tau}}{\sqrt{\pi}}+\frac{\tau}{h}\right)V_n^N-\frac{\tau}{h}V_{n-1}^N-\frac{h\tau}{2}F_2(V_1^n,t_n)\left[\Phi'(0)+\sum_{i=1}^{\infty}\Phi'(x_i)\left(1-erf\left(\frac{x_i}{2\sqrt{\tau}}\right)\right)\right]$$

$$=\left(\frac{h}{2}+\frac{2\sqrt{\tau}}{\sqrt{\pi}}\right)V_{n-1}^N-\frac{2}{\sqrt{\pi}}\sum_{k=1}^{n-1}(V_N^k-V_n^{k-1})(\sqrt{t_n-t_{k-1}}-\sqrt{t_n-t_k})$$

$$+\frac{h\tau}{2}\sum_{i=1}^{\infty}\Phi'(x_i)\left[F_2(V_1^{n-1},t_{n-1})\left(1-erf\left(\frac{x_i}{2\sqrt{\tau}}\right)\right)\right.$$

$$\left.+\sum_{k=1}^{n-1}\left[F_2(V_1^k,t_k)+F_2(V_1^{k-1},t_{k-1})\right][I(t_k)-I(t_{k-1})]\right].$$

For numerical implementations, the infinite series is truncated at a large number of terms, say S, depending on the function $\Phi'(x)$.

The solutions in counterpart domain ($x>l$) can be computed, using integral formulas

$$v(x,t_n)=\frac{1}{2}\sum_{k=1}^{n}\left[erf\left(\frac{x-l}{2\sqrt{t_n-t_{k-1}}}\right)-erf\left(\frac{x-l}{2\sqrt{t_n-t_k}}\right)\right](V_N^k-V_N^{k-1})$$

$$+\begin{cases}0, & \text{for problems } (V_1) \text{ and } (V_2) \text{ with supp}\,\Phi'(x)<\infty,\\ A, & \text{for problem } (V_2) \text{ with supp}\,\Phi'(x)=\infty,\end{cases}$$

where A is the second addend of (3.3), if it can be calculated exactly, otherwise we use it's approximation, obtained in the similar way as (3.10). We trace the main points of the calculations:

1. Present as a difference of two integrals (the difference between two integrals is only in the exponential power). Both integrals we treat in mush the same manner;
2. Apply Lemma 3.2;
3. Rearrange the integrals and integrand functions;
4. Apply Green formula in order to generate the term $(t-\lambda)^{-3/2}$.

Next, we follow the same steps as before, but the change of the variable λ is $\rho = \frac{x-t\pm\xi}{2\sqrt{t_n-\lambda}}$ for the corresponding integrals. Thus we obtain:

$$A = \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{S} \Phi'(x_i) \sum_{k=1}^{n} \frac{F_2(V_1(t_{k-1}),t_{k-1}) + F_2(V_1(t_k),t_k)}{2} \cdot \bar{I}(t_{k-1},t_k),$$

$$\bar{I}(t_{k-1},t_k) = I^+(t_{k-1},t_k) - I^-(t_{k-1},t_k), \quad P^{\pm}(t_s) = \frac{(x-l\pm x_i)^2}{4(t_n-t_s)},$$

$$I^{\pm}(t_{k-1},t_k) = 2\sqrt{t_n-t_{k-1}}e^{-P^{\pm}(t_{k-1})} - 2\sqrt{t_n-t_k}e^{-P^{\pm}(t_k)} + \frac{(x-l\pm x_i)\sqrt{\pi}}{2}\left\{\operatorname{erf}\left(\sqrt{P^{\pm}(t_{k-1})}\right) - \operatorname{erf}\left(\sqrt{P^{\pm}(t_k)}\right)\right\}.$$

Remark 3.1. The discrete maximum principle and convergence of the numerical schemes can be proved, as it's already done for similar problems in our previous works [8, 9].

4. NUMERICAL EXAMPLES

In this section we verify numerically the efficiency, convergency and accuracy of the algorithm, based on the construction of artificial boundary condition (ABCM). The results are compared with ones, obtained by standard method: solving numerically the original problem on a large enough finite interval $\bar{D} \in [0, L]$ and imposing zero Dirichlet boundary condition on the remote boundary.

Example 1. The test problem is (P_2), $\Phi(x) = -e^{-x}$, $F_2(u_x(0,t),t) = -u_x(0,t) - \frac{2e^{-\frac{1}{t^2}}}{t^2\sqrt{\pi}}$, $g(t) = \operatorname{erfc}(\frac{1}{t})$ and $h_2(x) \equiv 0$. Then, the exact solution is $u(x,t) = e^{-x}\operatorname{erfc}(\frac{1}{t})$. This example is favourable for standard method, because the solution goes to zero ($x\to\infty$, $t\to\infty$) rapidly. On the other hand, since $v(x,0)\equiv 0$ in the equivalent problem (V_2), we may take any line $x = l > 0$ as the artificial boundary. Let $l = 1$, $S = 20$ and the ratio $\frac{\tau}{h^2} = 1$ is fixed. In Table 1 we present the errors under different discrete norms, convergence rates and CPU times (in seconds) of the algorithms - ABCM and standard method at $t = 0.5$. The errors are defined as follows:

$$E^h_\infty = \|u-U\|_{L_\infty(\bar{\omega}_h)} \quad \text{and} \quad E^h_2 = \|u-U\|_{L_2(\bar{\omega}_h)}.$$

The convergence rate is computed, using the formula

$$CR = \log_2 \frac{E^h_{2(or\ \infty)}}{E^{\frac{h}{2}}_{2(or\ \infty)}}.$$

TABLE 1. Errors in different norms, convergence rates (CR) and CPU times (sec)

h / CR / CPU	ABCM, $l=1$, $L=5$		Standard Method, $u(L,t)=0$			
			$L=5$		$L=10$	
	E^h_∞	E^h_2	E^h_∞	E^h_2	E^h_∞	E^h_2
0.1	7.23721e-4	6.923436e-4	7.96384e-4	7.43435e-4	7.53891e-4	7.02529e-4
CPU	*3.3441*		*0.5940*		*0.9372*	
0.05	1.97566e-4	1.87719e-4	2.41093e-4	2.20404e-4	2.11617e-4	1.95553e-4
CR	**1.8731**	**1.8829**	**1.7238**	**1.7541**	**1.8329**	**1.8450**
CPU	*48.3750*		*3.4380*		*5.7970*	
0.025	5.31497e-5	4.96847e-5	8.21020e-5	7.45691e-5	6.70987e-5	6.14957e-5
CR	**1.8942**	**1.9177**	**1.5541**	**1.5635**	**1.6571**	**1.6690**
CPU	*6.5449e+2*		*25.6561*		*41.7352*	
0.0125	1.37550e-5	1.26190e-5	3.95401e-5	3.57905e-5	2.45255e-5	2.18145e-5
CR	**1.9501**	**1.9772**	**1.0540**	**1.0590**	**1.4520**	**1.4951**
CPU	4.5971e+3		*1.9002e+2*		*3.4294e+2*	
0.00625	3.47663e-6	3.17846e-6	3.55344e-5	3.21402e-5	9.80292e-6	8.58736e-6
CR	**1.9842**	**1.9892**	**0.1541**	**0.1552**	**1.3230**	**1.3452**
CPU	7.6963e+4		*1.3051e+3*		*2.9707e+3*	

Even for a fast vanished solution, L is not large enough and we "lose" convergence of the standard method. The reason is that the main source of error: Dirichlet boundary condition, $u(L,t)=0$, remain one and the same, independently of the mesh step size. If L is bigger, the computational efforts become unjustifiable large. The convergence rate is $O(\tau + h^2)$, if we compute the solution with ABCM. For problem (V_2), supp $\Phi'(x) = \infty$ (just as in this case), the algorithm of ABCM implicate two type convolution integrals: concerning V_1^{n-1} and V_N^{n-1}. Due to this terms, which makes the problem nonlocal in time and the interaction of the integrals and different terms, the solution process involves, at any given time step, the history of V_1^{n-1}, V_N^{n-1} and t. For problems (V_1) and (V_2), supp $\Phi'(x) < \infty$, the convolution integrals concern only V_N^{n-1}. Also the summation by S is missing. Thus the CPU time of the computations is approximately two times less, than this, shown in Table 1,2.

Note that the choice of time step: $\tau = h^2$ (Table 1) leads to long time computations and CPU time of ABCM is large. If for example $\tau = h$, the CPU time of ABCM and standard method is $\sim \rho$: $\frac{2^{k+1}}{h} \leq \rho \leq \frac{2^{k+2}}{h}$ ($k=1$ for $h=0.1$, $k=2$ for $h=0.05,\ldots$) and $\sim \frac{1}{h}$ times less, respectively, but the accuracy of ABCM ia still better (in comparison with standard method). For example: CPU time for computations with ABCM, $\tau = h$, $t=0.5$, $L=5$ and $h=0.025$ is 0.719 and the corresponding one with standard method is 0.625 (also for $L=5$).

TABLE 2. Max errors and CPU times

	$l = 0.5$	$l = 1$	$l = 1.5$	$l = 2$	
E^h_∞ CPU	6.71998e-4 *0.359*	6.75316e-4 *0.469*	6.77910e-4 *0.594*	6.82012e-4 *0.609*	$S = 10$
E^h_∞ CPU	6.70188e-4 *0.610*	6.73815e-4 *0.719*	6.75822e-4 *0.875*	6.79556e-4 *0.891*	$S = 20$
E^h_∞ CPU	6.69881e-4 *1.031*	6.73579e-4 *1.093*	6.75601e-4 *1.266*	6.79273e-4 *1.390*	$S = 40$
E^h_∞ CPU	6.69876e-4 *2.047*	6.73573e-4 *2.109*	6.75592e-4 *2.172*	6.79266e-4 *2.281*	$S = 80$
E^h_∞ CPU	6.69876e-4 *19.766*	6.73573e-4 *19.844*	6.75592e-4 *20.125*	6.79266e-4 *20.657*	$S = 1000$

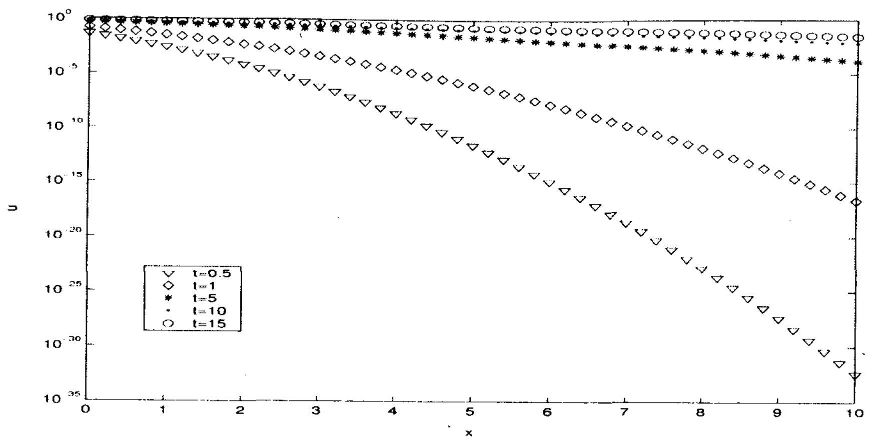

Fig. 1. Exact solution at different time

In Table 2 we give the max errors and CPU times of the numerical solution at $t = 0.5$, computed with ABCM for different values of l and S, $\tau = h$, $h = 0.025$ and $L = 5$.

Example 2. Let $\Phi(x) \equiv 1$, $F_2(u_x(0,t),t) = u_x(0,t) + \frac{1}{\sqrt{\pi t}} e^{-\frac{1}{t}}$, $g(t) = \operatorname{erfc} \frac{1}{\sqrt{t}}$ and $h_2(x) \equiv 0$. Then the exact solution of problem (P_2) is $u(x,t) = \operatorname{erfc} \frac{x+2}{2\sqrt{t}}$. The shape of the solution's profile stretches in x (as $t \to \infty$), see Figure 1. Using the standard method, we take a risk: to compute the solution in large enough (for some t) interval and then it turns out that this interval is not enough large for bigger t. In this situation the ABCM is still effective. On Figure 2, 3, 4 are plotted exact solution and numerical one, obtained by standard method and ABCM for different time levels, $\tau = h = 0.025$, $l = 0.5$ and $L = 5$.

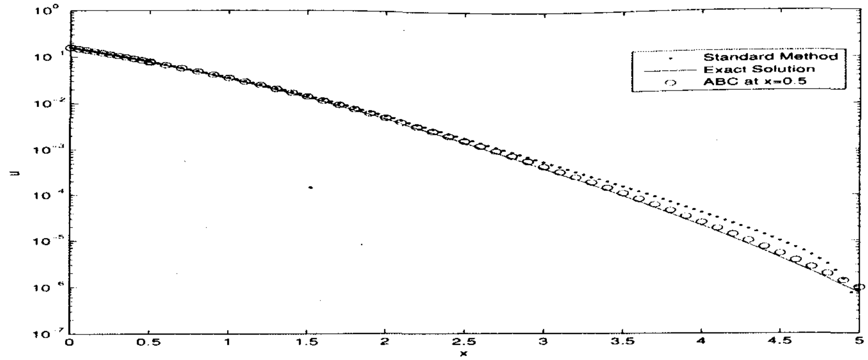

Fig. 2. Numerical and exact solution at $t = 1$

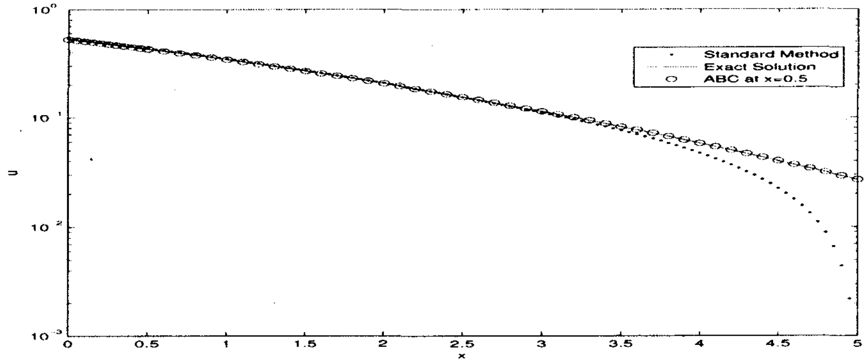

Fig. 3. Numerical and exact solution at $t = 5$

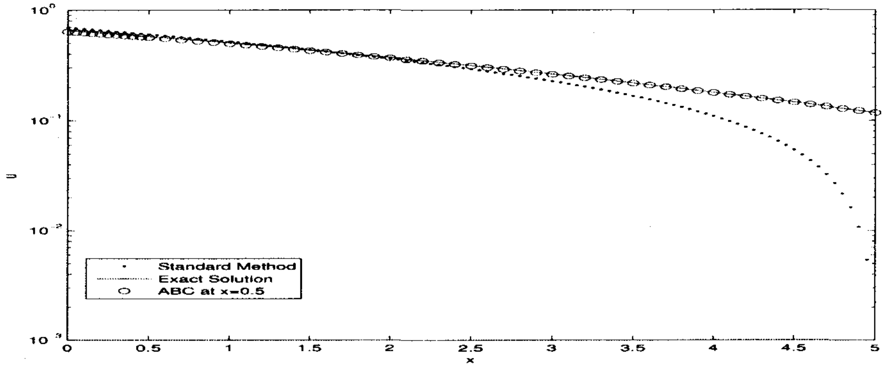

Fig. 4. Numerical and exact solution at $t = 10$

Below we summarize advantages and disadvantages of our numerical algorithms in comparison with known ones: using approximate finite boundary or infinite quasi-uniform grids (uniform or quasi-uniform), etc.

On the positive side are the following features of the schemes:

▷ The high accuracy of the numerical solution.

▷ The scheme on the bounded sub-domain has second order local truncation error in space and first in time. It is not difficult to construct the scheme from high order accuracy as well in time, using three level time scheme, see [1, 16];

▷ The solution in counterpart domains can be computed at any point directly, using formula.

▷ The computations can be performed on a very small region.

▷ The artificial boundary, say l, can be chosen in a very simple way: *supp* $h_i'(x) < l$. Since our boundary condition is exact, the smaller the l, the smaller the computational domain and consequently the less the computational amount.

On the negative side are the following three main disadvantages:

▷ The construction of artificial boundary condition is possible for a restricted class of problems and it's derivation is often not easy;

▷ Geometrically not universal.

▷ Algorithmically simple, but numerically expensive, because of involving the convolution integral with 'memory property', but related only with one point: $x = l$. We could not succeeded to cope with this problem because of the singularity of the integrals kernels. Straightforward evaluation of those convolution requires storing the information along the artificial boundary for all times since $t = 0$ and re-processing this information a each time step. Nevertheless, the performance of the presented schemes (in terms of CPU time or number of operations) is less than that of a standard finite element scheme with no artificial boundary condition, but with sufficiently long domain and zero Dirichlet boundary condition on the remote boundary, such that the the accuracy of both schemes is about the same (the mesh density is one and the same), see [10]. The CPU time of the erf (x) integral is the same as the one of the function $\sin x$ or $\cos x$.

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EQUALITY CASES FOR TWO POLYNOMIAL INEQUALITIES

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A complete characterization of the equality cases for two recent polynomial inequalities is given. The proofs are based on simple interpolation and quadrature techniques. We discuss also the meaning and the sharpness of these inequalities.

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Let $\mathcal{P}_n$ be the linear space of polynomials $p(z) := \sum_{k=0}^{n} a_k z^k$ of degree at most n with complex coefficients. The following two polynomial inequalities have been a subject of extensive research.

Bernstein polynomial inequality. Let $\mathcal{P}_n$ be equipped with the norm $\|p\|_{\mathbb{D}} := \max_{z\in\partial\mathbb{D}} |p(z)|$ with $\mathbb{D} := \{z : |z| < 1\}$, $p \in \mathcal{P}_n$. Then

$$\| p' \|_{\mathbb{D}} \leq n \| p \|_{\mathbb{D}} \tag{1}$$

with equality only for the monomials $p_n(z) := Kz^n$, where $K \in \mathbb{C}$.

Markov polynomial inequality. Let $\mathcal{P}_n$ be equipped with the norm $\| p \|_{[-1,1]} := \max_{x\in[-1,1]} |p(x)|$, $p \in \mathcal{P}_n$. Then

$$\| p' \|_{[-1,1]} \leq n^2 \| p \|_{[-1,1]} \tag{2}$$

with equality only for multiples of the n^{th} Chebyshev polynomial $T_n \in \mathcal{P}_n$ defined by $T_n(x) := \cos(n \arccos(x))$, $x \in [-1, 1]$.

We refer the reader to the survey paper [1], and to the books [2], [9], [10] for up-to-date references concerning (1) and (2) and their extensions. One of the most striking results [7] on polynomial inequalities is the following discrete improvement of (2).

Duffin and Schaeffer polynomial inequality. Let $\mathcal{P}_n$ be equipped with the norm $\|p\|_{[-1,1]} := \max_{-1\le x\le 1}|p(x)|$, $p\in\mathcal{P}_n$ and let $x_j = \cos(j\pi/n)$, $0\le j\le n$, be the extremal points of T_n in $[-1,1]$. Then

$$\|p'\|_{[-1,1]} \le n^2 \max_{0\le j\le n} |p(x_j)| \tag{3}$$

with equality only for multiples of the n^{th} Chebyshev polynomial $T_n \in \mathcal{P}_n$.

In this article we give a complete characterization of the equality cases for two polynomial inequalities (see Theorem A and Theorem B), recently published in [5]. The proofs are based on simple interpolation and quadrature techniques. We also discuss the meaning and the sharpness of these inequalities.

We consider the following two inequalities:

Theorem A. *Let $p\in\mathcal{P}_n$ and $\theta\in\mathbb{R}$. Then*

$$\left|\frac{p\left(e^{\mathrm{i}\theta}\right)-p\left(e^{-\mathrm{i}\theta}\right)}{e^{\mathrm{i}\theta}-e^{-\mathrm{i}\theta}}\right| \le n \max_{0\le j\le n}\left|\frac{p\left(e^{\mathrm{i}j\pi/n}\right)+p\left(e^{-\mathrm{i}j\pi/n}\right)}{2}\right|, \tag{4}$$

where the inequality is strict for each $\theta\notin\{0,\pi\}$ (mod 2π) and any polynomial $p\not\equiv 0$.

Theorem B. Let $p\in\mathcal{P}_n$ and $\theta\in\mathbb{R}$. Then

$$\left|p'\left(e^{\mathrm{i}\theta}\right)\right| \le n \max_{j\in J_n}\left|\frac{p\left(e^{\mathrm{i}(\theta+j\pi/n)}\right)+p\left(e^{\mathrm{i}(\theta-j\pi/n)}\right)}{2}\right|, \tag{5}$$

where $J_n := \{0\}\cup\{j : 1\le j\le n,\ j \text{ odd}\}$.

Theorem A is a Duffin and Schaeffer type result in the spirit of (3). It gives an upper bound for the uniform norm of the divided difference

$$\frac{p\left(e^{\mathrm{i}\theta}\right)-p\left(e^{-\mathrm{i}\theta}\right)}{e^{\mathrm{i}\theta}-e^{-\mathrm{i}\theta}}$$

of a polynomial $p\in\mathcal{P}_n$.

Remark. Theorem B gives a pointwise estimate for the first derivative of a given polynomial $p\in\mathcal{P}_n$ of degree at most n by using $(n+1)$ functional values of p. Note that $(n+1)$ is the minimal number of functional values for which such an estimate holds. Assume, on the contrary, that for a fixed point $z_0\in\partial\mathbb{D}$ there exist n distinct complex numbers $z_1,\dots,z_n$ in $\bar{\mathbb{D}} := \{z : |z|\le 1\}$ such that $|p'(z_0)| \le \sum_{k=1}^{n}\beta_k|p(z_k)|$ $(\beta_k\ge 0)$ for any polynomial p of degree at most n. Applying Gauss-Lucas Theorem, the polynomial $p(z) := (z-z_1)(z-z_2)\cdots(z-z_n)$

satisfies $p'(z_0) \neq 0$ and we are led to a contradiction. Furthermore, *Theorem B contains an improvement of Bernstein's inequality (1).* It follows from (5) that

$$\left|p'(e^{\mathrm{i}\theta})\right| \leq n \max_{j\in J_n} \left|\frac{p\left(e^{\mathrm{i}(\theta+j\pi/n)}\right)+p\left(e^{\mathrm{i}(\theta-j\pi/n)}\right)}{2}\right| \leq n\,\|p\|_{\mathbb{D}}$$

for any $p \in \mathcal{P}_n$ and $\theta \in \mathbb{R}$.

Remark. The following polynomial inequality

$$\|\, p'\,\|_{\mathbb{D}} \leq n \max_{0\leq j\leq 2n-1} \left|p\left(e^{\mathrm{i}j\pi/n}\right)\right| \tag{6}$$

has been published in [8]. The above inequality may be thought as an analogue of (3) on the unit disk $\mathbb{D}$. It is seen from (6) that for $p \in \mathcal{P}_n$ and any $\gamma \in \mathbb{R}$,

$$|\,z\,p'(z)\,| \leq n \max_{0\leq j\leq 2n-1} \left|p\left(|z|\,e^{\mathrm{i}(\gamma+j\pi/n)}\right)\right| \qquad (\,|z| \leq 1\,)$$

However, for a given $z := re^{\mathrm{i}\,\theta}$ with $|z| \leq 1$, it is not clear at all how to choose $\gamma = \gamma(z)$ in order to minimize the right hand-side in the above inequality. On the other hand, it follows from (5) that for any $r \in (0,1]$

$$\begin{aligned} |\,z\,p'(z)\,| &\leq n \max_{j\in J_n} \left|\frac{p\left(r\,e^{\mathrm{i}(\theta+j\pi/n)}\right) + p\left(r\,e^{\mathrm{i}(\theta-j\pi/n)}\right)}{2}\right| \\ &\leq n \max_{0\leq j\leq 2n-1} \left|p(r\,e^{\mathrm{i}(\theta+j\pi/n)})\right| \end{aligned}$$

and, because the number of the functional values used in (5) is $(n+1)$, hence smaller than $2n$, the estimate (5) can be considerably better than the estimate (6). We show in this paper that (5) has many extremal polynomials including all extremals of (6). Hence *(5) is more sensitive than (6).* Let us point out that the strength of (6) lies in the fact that it gives an upper-bound for the *uniform* norm $\|p'\|_{\mathbb{D}}$ of a polynomial. However, it is not true that for all $p \in \mathcal{P}_n$

$$\|p'\|_{\mathbb{D}} \leq n \max_{0\leq j\leq n} \left|\frac{p\left(e^{\mathrm{i}j\pi/n}\right)+p\left(e^{-\mathrm{i}j\pi/n}\right)}{2}\right|.$$

This can be seen by taking the polynomials $p_{n,k}(z) := z^n + \mathrm{i}z^k,\ \ 0 < k < n$. Obviously $\|p'_{n.k}\|_{\mathbb{D}} = n + k$ while

$$n \max_{0\leq j\leq n} \left|\frac{p_{n,k}\left(e^{\mathrm{i}j\pi/n}\right)+p_{n,k}\left(e^{-\mathrm{i}j\pi/n}\right)}{2}\right| = \sqrt{2}n.$$

The proof of Theorem A is based on the following

Representation Formula 1. Let $\theta \in \mathbb{R}$ be fixed. Then there exist $(n+1)$ numbers $\alpha_0(\theta)$, $\alpha_1(\theta)$, ..., $\alpha_n(\theta)$ such that

$$\frac{p\left(e^{\mathrm{i}\theta}\right) - p\left(e^{-\mathrm{i}\theta}\right)}{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}} = \sum_{j=0}^{n} (-1)^j \alpha_j(\theta) \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2}$$

holds for all $p \in \mathcal{P}_n$ and $\sum_{j=0}^{n} |\alpha_j(\theta)| \le n$. More precisely, we have the following explicit expressions for the numbers $\alpha_0(\theta)$, $\alpha_1(\theta)$, ..., $\alpha_n(\theta)$:

$$\alpha_0(\theta) = \frac{1}{2n}\frac{1 - \cos n\theta}{1 - \cos\theta}; \quad \alpha_n(\theta) = \frac{(-1)^{n-1}}{2n}\frac{1 - (-1)^n \cos n\theta}{1 + \cos\theta}$$

and

$$\alpha_j(\theta) = \frac{(-1)^j - \cos n\theta}{n\left(\cos\frac{j\pi}{n} - \cos\theta\right)}, \qquad 1 \le j \le n-1.$$

On the other hand, the proof of Theorem B is based on the next representation formula which amounts to the particular case $\theta = 0$ in the representation formula 1.

Representation Formula 2. For all $p \in \mathcal{P}_n$ and $\theta \in \mathbb{R}$,

$$e^{\mathrm{i}\theta}p'(e^{\mathrm{i}\theta}) = \beta_0\, p(e^{\mathrm{i}\theta}) - \sum_{j \in J_n,\, j \ge 1} \beta_j \frac{p(e^{\mathrm{i}(\theta + j\pi/n)}) + p(e^{\mathrm{i}(\theta - j\pi/n)})}{2}$$

where $\beta_0 = n/2$, $\beta_j = (n\sin^2(j\pi/2n))^{-1}$, $j \in J_n$, $1 \le j < n$, $\beta_n = \frac{1+(-1)^{n-1}}{4n}$, and $\sum_{j \in J_n,\, j\ge 1} \beta_j = n/2$.

Although the representation formula 2 follows easily from the representation formula 1, it is an interesting result by its own. It implies for example that

$$\left|e^{\mathrm{i}\theta}p'(e^{\mathrm{i}\theta}) - \frac{n}{2}p(e^{\mathrm{i}\theta})\right| \le \frac{n}{2}\max_{j\in J_n,\, j\ge 1}\left|\frac{p(e^{\mathrm{i}(\theta+j\pi/n)}) + p(e^{\mathrm{i}(\theta-j\pi/n)})}{2}\right|,$$

$(p \in \mathcal{P}_n,\ \theta \in \mathbb{R})$. This is clearly a Duffin-Schaeffer type extension of the following classical result:

$$\left|zp'(z) - \frac{n}{2}p(z)\right| \le \frac{n}{2}\|p\|_{\mathbb{D}} \quad (p \in \mathcal{P}_n,\ z \in \mathbb{D}).$$

The representation formula 2 can be used also to obtain *a refinement of Bernstein trigonometric inequality* in the form

$$|t'(\theta)| \le n \max_{1\le k\le n}\left|\frac{t(\theta + (2k-1)\pi/(2n)) - t(\theta - (2k-1)\pi/(2n))}{2}\right|$$

for $\theta \in \mathbb{R}$ and any trigonometric polynomial t of degree $\le n$ with complex coefficients. It is easily seen that $2n$ is the minimal number of functional values for which such pointwise estimate for the first derivative of a trigonometric polynomial t of degree $\leq n$ is possible (see [6, Theorem 4.1]). For any such t, we define an algebraic polynomial $p_t \in \mathcal{P}_{2n}$ by $p_t(e^{i\theta}) := e^{in\theta}t(\theta)$, $\theta \in \mathbb{R}$. Then simple calculations show that $t'(\theta) = ie^{-in\theta}[e^{i\theta}p_t'(e^{i\theta}) - np_t(e^{i\theta})]$. Applying representation formula 2 to $p_t \in \mathcal{P}_{2n}$ yields

$$t'(\theta) = \sum_{k=1}^{n} \lambda_{n,k} \frac{t\left(\theta + (2k-1)\pi/(2n)\right) - t\left(\theta - (2k-1)\pi/(2n)\right)}{2}, \quad \theta \in \mathbb{R},$$

where $\lambda_{n,k} = (-1)^{k-1}[2n \sin^2\left((2k-1)\pi/(4n)\right)]^{-1}$, $1 \leq k \leq n$ with $\sum_{k=1}^{n} |\lambda_{n,k}| = n$. This is *a variant of M. Riesz interpolation formula* that implies the above *refinement of Bernstein trigonometric inequality.*

We present a complete characterization of the equality cases in (4) and (5). The following quadrature formula is useful in studying polynomial inequalities: Let $\mathcal{T}_n$ denote the linear space of all complex trigonometric polynomials of degree at most n, $n \in \mathbb{N}$. The quadrature formula (we mention [6, Theorem 2.1] as a ready reference)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} t(\theta)\, d\theta = \frac{1}{m} \sum_{j=0}^{m-1} t\left(\frac{2j\pi}{m} + \gamma\right) \quad (\gamma \in \mathbb{R}) \tag{7}$$

holds for all $t \in \mathcal{T}_{m-1}$. The quadrature (7) is the unique, up to a real translation γ of the nodes, quadrature formula based on m nodes which is exact in $\mathcal{T}_{m-1}$, i.e., a quadrature formula with trigonometric degree of precision $m-1$. There is no quadrature formula with m nodes and having a trigonometric degree of precision greater than $m-1$.

The equality cases in Theorem B. Let a polynomial $p_{\theta_0} \in \mathcal{P}_n$ be extremal for (5) at a fixed number $\theta = \theta_0 \in \mathbb{R}$. Then, for an arbitrarily chosen $\theta_1 \in \mathbb{R}$, the polynomial $p_{\theta_1}(z) := p_{\theta_0}\left(e^{i(\theta_0-\theta_1)}z\right)$ $(z = e^{i\theta})$ is extremal for (5) at $\theta = \theta_1$.

Let $E_{\theta_0,n}$ denote the class of all polynomials from $\mathcal{P}_n$ extremal for (5) at θ_0. Then

$$E_{\theta_0,n} := \left\{ p\left(e^{-i\theta_0} z\right) \; : \; p \in E_{0,n} \right\} .$$

Hence, in order to determine all extremal polynomials in Theorem B, it is sufficient to describe the class $E_{0,n}$ of all polynomials that are extremal for (5) at $\theta = 0$.

Now, suppose that $p \in \mathcal{P}_n$, $p(z) := \sum_{k=0}^{n} a_k z^k$ is extremal for (5) at $\theta = 0$, i.e., $p \in E_{0,n}$. Let $h_n(z)$ be the Lagrange interpolating polynomial of degree at most n which is uniquely determined by the interpolation conditions

$$h_n(1) = 2n\, a_0,\ h_n\left(\cos(l\pi/n)\right) = n\, a_l,\ (1 \leq l \leq n-1),\ h_n(-1) = 2n\, a_n.$$

Then, $r_n(\theta) := h_n(\cos\theta)$ is the unique *even* trigonometric polynomial of degree at most n which satisfies the interpolation conditions

$$r_n(0) = 2n\, a_0,\ r_n(l\pi/n) = n\, a_l\ (1 \leq l \leq n-1),\ r_n(\pi) = 2n\, a_n .$$

Let

$$M := \max_{j \in J_n} \left| \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2} \right|.$$

The representation formula 2 implies that equality in (5) holds for $\theta = 0$ and the polynomial $p \in \mathcal{P}_n$ if and only if for some $\gamma \in \mathbb{R}$

$$(-1)^j \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2} = Me^{\mathrm{i}\gamma}, \qquad j \in J_n. \tag{8}$$

The linear system (8) whose unknowns are the coefficients of the extremal polynomial p is in general greatly undetermined because of the small cardinality of J_n.

By using the interpolating trigonometric polynomial r_n, the linear system (8) can be represented in the following equivalent form

$$\begin{cases} \frac{1}{2n} r_n(0) + \frac{1}{n} \sum_{l=1}^{n-1} r_n\left(l\pi/n\right) \cos\left(jl\pi/n\right) + \frac{1}{2n} r_n(\pi) \cos\left(jn\pi/n\right) \\ = -Me^{\mathrm{i}\gamma} \quad (j \text{ odd}, \ j \leq n) \\ \frac{1}{2n} r_n(0) + \frac{1}{n} \sum_{l=1}^{n-1} r_n\left(l\pi/n\right) + \frac{1}{2n} r_n(\pi) = M e^{\mathrm{i}\gamma} \\ M > 0 \text{ and } \gamma \in \mathbb{R}. \end{cases} \tag{9}$$

Let n be even. Define $E^e_{0,n} := E_{0,n}$. Then $J_n = \{0\} \cup \{j = 1, 3, \ldots, n-1\}$. By using the quadrature (7) with $m = 2n$, the system (9) is equivalent to the following integral system:

$$\begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} r_n(\theta) \cos j\theta \, d\theta = -M e^{\mathrm{i}\gamma}, \qquad j = 1, 3, \ldots, n-1 \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} r_n(\theta) \, d\theta = M e^{\mathrm{i}\gamma}. \end{cases}$$

Hence, the interpolating trigonometric polynomial r_n must have the form

$$r_n(\theta) = Me^{\mathrm{i}\gamma} \left(1 - 2 \sum_{l=0}^{(n-2)/2} \cos((2l+1)\theta) \right) + \sum_{k=1}^{n/2} b_{2k} \cos(2k\theta) \tag{10}$$

where $b_{2k} \in \mathbb{C}$.

Let us denote by Ω^e_n the class of all *even* trigonometric polynomials $r_n(\theta)$ of the form (10), where the parameters $M \geq 0$, γ real, b_{2k} complex, $k = 0, 1, \ldots, n/2$, are arbitrary. We describe the class $E^e_{0,n}$ of all polynomials, extremal for (5) at $\theta = 0$, n even, through Ω^e_n. The following holds:

$$E^e_{0,n} \equiv \left\{ p \in \mathcal{P}_n \, : \, p(z) = \frac{1}{2n} r_n(0) + \frac{1}{n} \sum_{l=1}^{n-1} r_n\left(l\pi/n\right) z^l \right. \\ \left. + \frac{1}{2n} r_n(\pi) z^n, \ r_n \in \Omega^e_n, \ n \text{ even} \right\}.$$

In view of this, for n even, the class $E^e_{0,n}$ of all polynomials that are extremal for (5) at $\theta = 0$ is completely determined by the trigonometric polynomial class Ω^e_n via a simple interpolation procedure:

Let $a_0, \dots, a_n$ be the coefficients of an extremal polynomial $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ from $E^e_{0,n}$. Then, the $(n+1)$ numbers in the second row of the table

0	π/n	$2\pi/n$	$\dots$	$(n-2)\pi/n$	$n-1)\pi/n$	$n\pi/n$
$(2n)\,a_0$	$n\,a_1$	$n\,a_2$	$\dots$	$n\,a_{n-2}$	$n\,a_{n-1}$	$(2n)\,a_n$

are interpolation functional values at the interpolation nodes given in the first row for the even trigonometric polynomial $r_n \in \Omega^e_n$. The trigonometric polynomial $r_n \in \Omega^e_n$ is uniquely determined by the above $(n+1)$ interpolation conditions.

Conversely, let $r_n \in \Omega^e_n$ with arbitrary $M \geq 0$, γ real, and complex b_{2k}, $k = 1, \dots, n/2$. Then

$$a_0 = \frac{r_n(0)}{2n},\ a_1 = \frac{r_n(\pi/n)}{n},\ \dots,\ a_{n-1} = \frac{r_n((n-1)\pi/n)}{n},\ a_n = \frac{r_n(\pi)}{2n}$$

are the coefficients of an extremal polynomial $p(z) := a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ from the class $E^e_{0,n}$. In other words, $E_{0,n}$ is in one-to-one correspondence to Ω^e_n through $(n+1)$ interpolation conditions at the equally spaced points $k\pi/n$, $k = 0, 1, \dots, n$.

Example 1. Let $n = 2$. Then, the class Ω^e_2 consists of only one even trigonometric polynomial $r_2(\theta) = Me^{i\gamma}(1 - 2\cos\theta) + b_2\cos(2\theta)$ and $r_2(0) = b_2 - Me^{i\gamma}$, $r_2(\pi/2) = Me^{i\gamma} - b_2$, $r_2(\pi) = 3Me^{i\gamma} + b_2$. Following our description of the extremal polynomial set $E^e_{0,n}$ we conclude that the class $E^e_{0,2}$ consists of the three-parametric (M, b_2, γ) set of polynomials

$$p_{M,b_2,\gamma}(z) := \left(3Me^{i\gamma} + b_2\right)\frac{z^2}{4} + \left(Me^{i\gamma} - b_2\right)\frac{z}{2} + \left(-Me^{i\gamma}/4 + b_2/4\right),$$

where $M \geq 0$, γ real, b_2 complex, are arbitrary.

Let $n = 4$. Then, the class Ω^e_4 consists of the even trigonometric polynomials $r_4(\theta) = Me^{i\gamma}(1 - 2\cos\theta - 2\cos(3\theta)) + b_2\cos(2\theta) + b_4\cos(4\theta)$ and $r_4(0) = b_2 + b_4 - 3Me^{i\gamma}$, $r_4(\pi/4) = Me^{i\gamma} - b_4$, $r_2(\pi/2) = Me^{i\gamma} - b_2 + b_4$, $r_4(3\pi/4) = Me^{i\gamma} - b_4$, $r_4(\pi) = 5Me^{i\gamma} + b_2 + b_4$. Following our description of the extremal polynomial set $E^e_{0,n}$ we conclude that the class $E^e_{0,4}$ consists of the four-parametric (M, b_2, b_4, γ) set of polynomials

$$\begin{aligned} p_{M,b_2,b_4,\gamma}(z) \ &:= \ \left(5Me^{i\gamma} + b_2 + b_4\right)\frac{z^4}{8} + \left(Me^{i\gamma} - b_4\right)\frac{z^3}{4} \\ &\quad + \left(Me^{i\gamma} - b_2 + b_4\right)\frac{z^2}{4} + \left(Me^{i\gamma} - b_4\right)\frac{z}{4} + \frac{b_2 + b_4 - 3Me^{i\gamma}}{8} \end{aligned}$$

where $M \geq 0$, γ real, b_2, b_4 complex, are arbitrary.

Example 2. By Theorem B

$$\left|p'\left(e^{\mathrm{i}k\pi/n}\right)\right| \le n \max_{j\in J_n}\left|p\left(e^{\mathrm{i}((k\pm j)\pi/n)}\right)\right| \quad (k\in\mathbb{Z}).$$

We take for simplicity $k=0$. Let p^* be extremal for the above inequality when $k=0$. By the representation formula 2, the polynomial $p^*\in\mathcal{P}_n$ must satisfy $p^*(e^{\mathrm{i}j\pi/n}) = p^*(e^{-\mathrm{i}j\pi/n}) = -Me^{\mathrm{i}\gamma}$ $(j\in J_n,\ j\ge 1)$, $p^*(1)=Me^{\mathrm{i}\gamma}$ for some $M\ge 0$ and $\gamma\in\mathbb{R}$. Taking into account that the cardinality of $\{e^{\pm \mathrm{i}j\pi/n},\ j\in J_n\}$ is $(n+1)$, we conclude that the unique polynomial of degree at most n which satisfies the above $(n+1)$ interpolation conditions is $p^*(z)=Me^{\mathrm{i}\gamma}z^n$, i.e., the only equality cases in the above inequality are constant multiples of z^n. It is easily seen that the same holds for arbitrary $k\in\mathbb{Z}$. From here, the only extremals of the inequality

$$\left|p'\left(e^{\mathrm{i}k\pi/n}\right)\right| \le n \max_{0\le j\le 2n-1}\left|p\left(e^{\mathrm{i}j\pi/n}\right)\right| \quad (k\in\mathbb{Z})$$

are constant multiples of z^n. *Now, taking into account that (5) has many extremal polynomials including the constant multiples of z^n which are the only extremals (see [5] for details) of (6), we conclude that (5) is a much more sensitive estimate than (6).* Following our description for the extremal polynomials in (5), the polynomial $p^*\in E^e_{0,n}$ corresponds to the *even* trigonometric polynomial

$$r_n^*(\theta) := Me^{\mathrm{i}\gamma}(-1)^{n-1}\frac{\sin n\theta}{\cos\theta/2}\sin\theta/2 \in \Omega_n^e$$

which satisfies the interpolation conditions $r_n^*(l\pi/n):=0$, $l=0,\dots,n-1$, $r_n^*(\pi):=(2n)Me^{\mathrm{i}\gamma}$ and this agrees with our description of $E^e_{0,n}$.

Remark. From the fact that the monomials z^k, $0\le k\le n-1$ are evidently not extremal for (5), in other words they do not belong to $E^e_{0,n}$, one may conclude that for fixed k, $0\le k\le n-1$, there is no trigonometric polynomial $r_n\in\Omega_n^e$ which satisfies the following interpolation conditions: $r_n(l\pi/n)=\delta_{k,l}$, $0\le l\le n$.

Remark. It deserves to be mentioned that *there are (many for $n\ge 4$) extremal polynomials for (5) of degree strictly less than n.* It is easily seen that $p\in E^e_{0,n}\cap\mathcal{P}_{n-1}$ if and only if the trigonometric polynomial $r_n\in\Omega_n^e$ corresponding to p satisfies $\sum_{k=1}^{n/2} b_{2k} = -Me^{\mathrm{i}\gamma}(n+1)$ $\quad(n\,\text{even})$. In Example 1 for $n=2$, the above equality is $b_2=-3Me^{\mathrm{i}\gamma}$ and an extremal polynomial in $E^e_{0,2}$ of degree less than 2 is $p(z)=2Me^{\mathrm{i}\gamma}z-Me^{\mathrm{i}\gamma}$. Analogously, for $n=4$ we have $b_2+b_4=-5Me^{\mathrm{i}\gamma}$ and the extremal polynomials in $E^e_{0,4}$ of degree less than 4 are given by the three-parametric (M,b_2,γ) set

$$p(z)=\left(\frac{3Me^{\mathrm{i}\gamma}}{2}+\frac{b_2}{4}\right)z^3-\left(Me^{\mathrm{i}\gamma}+\frac{b_2}{2}\right)z^2+\left(\frac{3Me^{\mathrm{i}\gamma}}{2}+\frac{b_2}{4}\right)z-Me^{\mathrm{i}\gamma}$$

where $M\ge 0$, γ real, and b_2 complex, are arbitrary.

Let n be odd. Define $E_{0,n} := E^o_{0,n}$ and let $r_n(\theta) := t_{n-1}(\theta) + A\cos n\theta \in \mathcal{T}_{n,e}$, where $t_{n-1} \in \mathcal{T}_{n-1,e}$ is of degree at most $n-1$. Then, applying the quadrature (7), we see that the system (9) is equivalent to the following system:

$$\begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} r_n(\theta)\cos k\theta\, d\theta = -M e^{i\gamma} \qquad (k = 1, 3, \dots, n-2) \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} r_n(\theta)\, d\theta = M e^{i\gamma} \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} t_{n-1}(\theta)\cos n\theta\, d\theta + A = -M e^{i\gamma} \Rightarrow A = -M e^{i\gamma} \end{cases}$$

and therefore

$$r_n(\theta) = M e^{i\gamma}\left(1 - 2\sum_{l=0}^{(n-3)/2}\cos\left((2l+1)\theta\right) - \cos n\theta\right) + \sum_{j=1}^{(n-1)/2} b_{2j}\cos(2j\,\theta) \tag{11}$$

with $b_{2j} \in \mathbb{C}$, $M > 0$, and $\gamma \in \mathbb{R}$. Let us denote by Ω^o_n the class of all *even* trigonometric polynomials of the form (11). Then

$$E^o_{0,n} \equiv \left\{ p \in \mathcal{P}_n \ : \ p(z) = \frac{1}{2n} r_n(0) + \frac{1}{n}\sum_{l=1}^{n-1} r_n(l\pi/n)\, z^l + \frac{1}{2n} r_n(\pi)\, z^n,\ r_n \in \Omega^o_n,\ n \text{ odd} \right\}.$$

In view of this and as in the case n even, the extremal set $E^o_{0,n}$ is in one-to-one *interpolation* correspondence with the class of trigonometric polynomials Ω^o_n.

Example 3. Let $n = 3$. Then $E^o_{0,3}$ is the three-parametric (M, b_2, γ) set of polynomials

$$p_{M,b_2,\gamma}(z) := \frac{2Me^{i\gamma} + b_2/2}{3} z^3 + \frac{Me^{i\gamma} - b_2/2}{3} z^2 + \frac{Me^{i\gamma} - b_2/2}{3} z - \frac{Me^{i\gamma} - b_2/2}{3},$$

for arbitrary $M \geq 0$, $\gamma \in \mathbb{R}$ and a complex number b_2.

Remark. It is easily seen that $p \in E^o_{0,n} \cap \mathcal{P}_{n-1}$ if and only if the unique $r_n \in \Omega^o_n$, which corresponds to p, satisfies $\sum_{j=1}^{(n-1)/2} b_{2j} = -Me^{i\gamma}(n+1)$. In the particular case of Example 3 we have $b_2 = -4Me^{i\gamma}$. Hence, an extremal polynomial in $E_{0,3}$ of degree less than 3 is $p(z) = Me^{i\gamma}\left(z^2 + z - 1\right)$. In the case n odd, the *even* trigonometric polynomial r^*_n from Example 2 belongs to Ω^o_n and this agrees with the fact that $Me^{i\gamma}z^n \in E^o_{0,n}$.

The equality case in Theorem A. First of all, let us mention that we have extremal polynomials in (4) only for $\theta \in \{0, \pi\}$ (mod 2π). In view of the explicit form of (5) and (4), the class of all extremal polynomials in (4) is a sub-set of the class of all extremal polynomials in (5) for $\theta = 0$ (mod 2π) and $\theta = \pi$ (mod 2π). Let $\tilde{E}_{0,n}$ denote the class of all polynomials extremal for (4) in the case $\theta = 0$. Then, $\tilde{E}_{\pi,n} = \left\{p(-z) : p \in \tilde{E}_{0,n}\right\}$, and all extremal polynomials in Theorem A are given by $\tilde{E}_{0,n} \cup \tilde{E}_{\pi,n}$. Hence, in order to determine the class of all extremal polynomials in Theorem A, it is enough to describe the subclass $\tilde{E}_{0,n}$ of all extremal polynomials in (5) for $\theta = 0$ satisfying the following additional inequalities on the set $\{0, 1, \ldots, n\} \setminus J_n$:

$$\left| \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2} \right| \le M \quad (j \notin J_n,\, 1 \le j \le n). \tag{12}$$

Surprisingly, there are also many extremal polynomials for the inequality (4) which amounts to

$$|p'(1)| \le n \max_{0 \le j \le n} \left| \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2} \right|,$$

in spite of the fact that these extremals must satisfy not only (8) but also (12).

Let n be even. Let $\tilde{E}^e_{0,n} := \tilde{E}_{0,n}$ and let $r_n \in \Omega^e_n$. Then, applying the quadrature (7) we see that (12) is equivalent to

$$|b_{2k}| \le 2M,\ k = 1, \ldots, (n-2)/2 \ (n > 2) \quad \text{and} \quad |b_n| \le M \ (n \ge 2).$$

Let $\tilde{\Omega}^e_n := \{r_n \in \Omega^e_n,\ |b_{2k}| \le 2M,\ 1 \le k \le (n-2)/2,\ |b_n| \le M\}$. Then we have

$$\tilde{E}^e_{0,n} \equiv \left\{ p \in \mathcal{P}_n \ : \ p(z) = \frac{1}{2n} r_n(0) + \frac{1}{n} \sum_{l=1}^{n-1} r_n(l\pi/n) z^l \right. \\ \left. + \frac{1}{2n} r_n(\pi) z^n,\ r_n \in \tilde{\Omega}^e_n,\ n \text{ even} \right\}.$$

Let n be odd. Let $\tilde{E}^o_{0,n} := \tilde{E}_{0,n}$ and let $r_n \in \Omega^o_n$. Then (13) is equivalent to $|b_{2j}| \le 2M,\ j = 1, \ldots, (n-1)/2$. In view of this we define

$$\tilde{\Omega}^o_n := \{r_n \in \Omega^o_n,\ |b_{2k}| \le 2M,\ 1 \le k \le (n-1)/2\}$$

to conclude that

$$\tilde{E}^o_{0,n} \equiv \left\{ p \in \mathcal{P}_n \ : \ p(z) = \frac{r_n(0)}{2n} + \sum_{l=1}^{n-1} \frac{r_n(l\pi/n)}{n} z^l \right. \\ \left. + \frac{r_n(\pi)}{2n} z^n,\ r_n \in \tilde{\Omega}^o_n,\ n \text{ odd} \right\}.$$

Remark. We point out that *all extremal polynomials $p \not\equiv 0$ in Theorem A are of exact degree n.* Let n be even and let us assume to the contrary. If $p \in \mathcal{P}_{n-1} \cap \tilde{E}_{0,n}$ is extremal, then the corresponding $r_n \in \tilde{\Omega}_n^e$ must satisfy $r_n(\pi) = 0$ which amounts to $\sum_{k=1}^{n/2} b_{2k} = -Me^{i\gamma}(n+1)$ together with $|b_{2k}| \le 2M,\ k = 1, \dots, (n-2)/2,\ |b_n| \le M$. Obviously, this is impossible for $M > 0$ and $n \ge 2$. Analogously, the same conclusion holds for $n \ge 3$ odd (the case $n = 1$ is a trivial one).

An extension of Theorem A. Let $p \in \mathcal{P}_n$ and define a sequence of $\{p_k\} \in \mathcal{P}_n$ by $p_0 := p$ and $p_{k+1}(z) = zp_k'(z)$, $k \ge 0$. The following generalization of Theorem A was obtained in [3]:

Theorem C. *Let $p \in \mathcal{P}_n$, $k \ge 0$, and $\theta \in \mathbf{R}$. Then*

$$\left| \frac{p_k\left(e^{i\theta}\right) - p_k\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}} \right| \le n^{1+k} \max_{0 \le j \le n} \left| \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \right| \tag{13}$$

where the inequality is strict for each $\theta \notin \{0, \pi\}$ (mod 2π) and for any polynomial $p \not\equiv 0$.

Clearly, Theorem C amounts to Theorem A for $k = 0$. We now discuss cases of equality in (13) for $k \ge 1$. It is readily seen that for $k = 1$, (13) is equivalent to the Duffin and Schaeffer result and in particular [7] equality holds in (13) for $k = 1$ if and only if $\theta = 0, \pi$ (mod 2π) and $p(z) = Kz^n,\ K \in \mathbf{C}$.

It has been proved in [3] that for $k \ge 0$, there exist real numbers $\beta_{l,k}(\theta)$, $0 \le l \le n$, such that for all $p \in \mathcal{P}_n$

$$\frac{p_k\left(e^{i\theta}\right) - p_k\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}} = \sum_{l=0}^{n} \beta_{l,k}(\theta) \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2}$$

with $\sum_{l=0}^{n} |\beta_{l,k}(\theta)| < n^{1+k}$ for $\theta \notin \{0, \pi\}$ (mod 2π). Moreover, the following representation formula (see [3] for details) holds:

$$\begin{aligned} p_{k+1}'(1) &= \sum_{j=0}^{n} (-1)^j \alpha_j(0) \left\{ \sum_{l=0}^{n} \frac{\beta_{l,k}(0)}{2} \left[\frac{p\left(e^{i(j+l)\pi/n}\right) + p\left(e^{-i(j+l)\pi/n}\right)}{2} \right.\right. \\ &\quad \left.\left. + \frac{p\left(e^{i(j-l)\pi/n}\right) + p\left(e^{-i(j-l)\pi/n}\right)}{2} \right] \right\}. \end{aligned} \tag{14}$$

Let us assume that for some $p \in \mathcal{P}_n$, ${p_{k+1}}'(1) = n^{2+k}M$, where

$M := \max_{0 \le j \le n} \left| \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \right|$.

Then, by (14)

$$n^{2+k}M = \left| \sum_{j=0}^{n} (-1)^j \alpha_j(0) \left\{ \sum_{l=0}^{n} \frac{\beta_{l,k}(0)}{2} \left[\frac{p\left(e^{i(j+l)\pi/n}\right) + p\left(e^{-i(j+l)\pi/n}\right)}{2} \right.\right.\right.$$

$$
\left. \left. \left. + \frac{p\left(e^{\mathrm{i}(j-l)\pi/n}\right)+p\left(e^{-\mathrm{i}(j-l)\pi/n}\right)}{2} \right] \right\} \right|
$$

$$
\leq \sum_{j=0}^{n} |\alpha_j(0)| \left\{ \frac{1}{2} \sum_{l=0}^{n} |\beta_{l,k}(0)| \left| \frac{p\left(e^{\mathrm{i}(j+l)\pi/n}\right)+p\left(e^{-\mathrm{i}(j+l)\pi/n}\right)}{2} \right| \right.
$$

$$
\left. + \frac{1}{2} \sum_{l=0}^{n} |\beta_{l,k}(0)| \left| \frac{p\left(e^{\mathrm{i}(j-l)\pi/n}\right)+p\left(e^{-\mathrm{i}(j-l)\pi/n}\right)}{2} \right| \right\}
$$

$$
\leq M n^{2+k}
$$

and equality must hold everywhere above. In particular, the modulus of

$$
\sum_{l=0}^{n} \beta_{l,k}(0) \frac{p\left(e^{\mathrm{i}l\pi/n}\right)+p\left(e^{-\mathrm{i}l\pi/n}\right)}{2}
$$

must be equal to $M\,n^{1+k}$, i.e., $|p_k'(1)| = M\,n^{1+k}$. This shows by induction on $k \geq 1$ that equality can hold in (13) for $\theta = 0$ only when $p(z) = Kz^n$ with $K \in \mathbf{C}$. The case $\theta = \pi$ can be treated in a similar way. *Hence, for $k = 0$, the inequality (13), being equivalent to (4), has many extremal polynomials. However, for $k \geq 1$, the only extremal polynomials in (13) are constant multiples of z^n.*

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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EXTENSIONS OF CERTAIN PARTIAL AUTOMORPHISMS OF $\mathcal{L}^*(V_\infty)$

RUMEN DIMITROV

The automorphisms of the lattice $\mathcal{L}(V_\infty)$ have been completely characterized. However, the question about the number of automorphisms of the lattice $\mathcal{L}^*(V_\infty)$ has been open for almost thirty years. We use some of our recent results about the structure of $\mathcal{L}^*(V_\infty)$ to answer questions related to automorphisms of $\mathcal{L}^*(V_\infty)$. We prove that any finite number of partial automorphisms of filters of closures of quasimaximal sets can be extended to an automorphism of $\mathcal{L}^*(V_\infty)$. As a corollary we obtain that closures of quasimaximal sets of the same type are elements of the same orbit in $\mathcal{L}^*(V_\infty)$.

1. INTRODUCTION

The vectors in the space V_∞ are codes of finitely nonzero infinite sequences of elements of the underlying computable field F. The computably enumerable (c.e.) subspaces of V_∞ are the closures of c.e. subsets of V_∞. The c.e. subspaces of V_∞ with the operations of intersection and closure of union form a lattice that is denoted $\mathcal{L}(V_\infty)$. The lattice $\mathcal{L}(V_\infty)$ modulo finite dimension is denoted $\mathcal{L}^*(V_\infty)$. Both $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$ are nondistributive modular lattices. In this respect the study of the structure and automorphisms of $\mathcal{L}^*(V_\infty)$ is an interesting, modular counterpart of the study of the lattice $\mathcal{E}^*$ of c.e. sets modulo $=^*$. Friedberg proved the existence maximal sets as part of Post's program. Maximal sets are c.e. sets with "thin" complements. The complements of maximal sets are called cohesive sets. A set R is cohesive if for every c.e. set W either $W \cap R$ or $\overline{W} \cap R$ is finite. From a lattice theoretic point of view however, the $=^*$ equivalence classes of maximal sets are co-atoms in $\mathcal{E}^*$. According to Sacks [7] it was the Friedberg's construction of maximal sets that "ignited interest" in the lattice $\mathcal{E}^*$. The structure of the filters and the automorphisms of $\mathcal{E}^*$ have then been extensively studied (see [8]).

An interesting class of filters in $\mathcal{E}^*$ are the principal filters of quasimaximal sets[1]. These are exactly the finite Boolean algebras. In [1], [2], and [3] we studied the structure of the principal filters of closures of quasimaximal subsets of a fixed computable basis I_0 of V_∞. Throughout the paper $A =^* B$ will mean that $(A - B) \cup (B - A)$ is finite. If $A =^* B$, then we will also say that A and B are almost equal. By $cl(A)$ we will denote the linear span of the vectors in the set A. The relation $V =^* W$ between vector spaces will mean that there are finite sets A and B such that $cl(V \cup A) = cl(W \cup B)$. The relation $\subseteq^*$ between sets (spaces) is defined similarly. For any c.e. set A the set of elements enumerated into A by the end of stage s will be denoted as A^s. If the partial computable function φ halts on input x by stage s we will denote this fact by $\varphi^s(x)\downarrow$. Otherwise we will write $\varphi^s(x)\uparrow$. To simplify the notation in equalities used for defining partial computable functions we will assume that the function on the left side is defined when all of the elements on the right hand side are defined and the expression is acceptable. For the same reason we will use the same notation (F) for a field $\mathcal{F}$ as a structure and its underlying set F.

Before stating the main result of [3] we give some definitions.

Definition 1.1. *Two sets A and B have the same 1-degree up to $=^*$ (denoted $A \equiv_1^* B$) if there are $A_1 =^* A$ and $B_1 =^* B$ such that $A_1 \equiv_1 B_1$.*

Definition 1.2. *Let R be a cohesive set. The R−cohesive power of the computable field F is a structure $\widetilde{F}$ in the language of fields such that:*

1. *$\widetilde{F} = \{\varphi : \varphi$ is a p.c. function, $R \subseteq^* dom(\varphi) \wedge rng(\varphi) \subseteq F\}/ =_R$. Here $\varphi_1 =_R \varphi_2$ if $R \subseteq^* \{x : \varphi_1(x) = \varphi_2(x)\}$ The equivalence class of φ w.r.t. $=_R$ will be denoted by $[\varphi]_R$ or simply $[\varphi]$ when the set R is fixed.*

2. *$[\varphi_1] + [\varphi_2] = [\varphi_1 + \varphi_2]$, and $[\varphi_1] \cdot [\varphi_2] = [\varphi_1 \cdot \varphi_2]$*

3. *$0^{\widetilde{F}}$ and $1^{\widetilde{F}}$ are the equivalence classes of the recursive functions with constant values 0^F and 1^F respectively.*

It is not difficult to see that $\widetilde{F}$ is a field. See [4] about cohesive powers of general first order structures.

Theorem 1.1. *[3]. Let $I_1, \dots, I_p$ be maximal subsets of I_0 and let $Q = \bigcap_{j=1}^{p} I_j$.*

[1] Intersections of finitely many maximal sets are called quasimaximal.

1. *If $I_1, \ldots, I_p$ have the same 1-degree up to $=^*$, then*

$$\mathcal{L}^* \cong \mathcal{L}(p, \widetilde{F}).^2$$

2. *If I_i are partitioned into m equivalence classes w.r.t. $\equiv_1^*$ and n_i is the number of elements in the $i-th$ class, then*

$$\mathcal{L}^*(cl(Q), \uparrow) \cong \prod_{i=1}^{m} \mathcal{L}(n_i, \widetilde{F_i}).^3$$

The isomorphism established in the proof of (1) is based on the idea that the spaces in $\mathcal{L}^*(cl(Q), \uparrow)$ are spaces spanned by the union of the the c.e. set I and a finite number c.e. set which we formally denote $\sum_{j=1}^{p} \alpha_j \overline{I_j}$ where $[\alpha_j] \in \widetilde{F}$ for $j \leq p$. The set that is formally denoted $\sum_{j=1}^{p} \alpha_j \overline{I_j}$ is in fact a c.e. set of linear combinations $v = \alpha_1(y_1)y_1 + \alpha_2(y_1)y_2 + \ldots + \alpha_p(y_1)y_p$ where $(y_1, y_2, \ldots, y_p)$ is an orbit. The orbit, a notion defined in a different context below, is an element of $\overline{I_1} \times \overline{I_2} \times \ldots \times \overline{I_p}$ at the time the vector v is enumerated into the set $\sum_{j=1}^{p} \alpha_j \overline{I_j}$.

2. AUTOMORPHISMS OF $\mathcal{L}^*(V_\infty)$

Theorem 2.1. *Let $J_1, \ldots, J_m$ be quasimaximal subsets of I_0. Suppose that for $k \leq m$ $J_k = \bigcap_{j=1}^{n_k} I_{kj}$ where I_{kj} (for $j = 1, \ldots, n_k$) are maximal subsets of I_0 of the same 1-degree up to $=^*$. Suppose also that the equivalence classes w.r.t. $\equiv_1^*$ of $I_{k_1 1}$ and $I_{k_2 1}$ are different for each $k_1, k_2 \leq m$ such that $k_1 \neq k_2$. For $k \leq m$ let W_k be an n_k dimensional vector space over the field $\widetilde{F_k} = \prod_{\overline{I_{k1}}} F$ such that the lattice L_k of subspaces of W_k is isomorphic to $\mathcal{L}^*(cl(J_k), \uparrow)$. Finally let f_k be an automorphisms of $\mathcal{L}^*(cl(J_k), \uparrow)$ that is induced by a linear transformation of W_k.*

We claim that there is an automorphism f of $\mathcal{L}^*(V_\infty)$ such that $f|_{\mathcal{L}^*(cl(J_k), \uparrow)} \equiv f_k$ for all $k \leq m$.

---

[2]The field $\widetilde{F}$ is the $\overline{I_1}$-cohesive power of the field F and $\mathcal{L}(m, \widetilde{F})$ is the lattice of subspaces of an m-dimensional space over the field $\widetilde{F}$. Note that in [3] the notion of cohesive power of a structure has not yet been developed.

[3]$\widetilde{F_i}$ is the cohesive power of F w.r.t. a cohesive set that is the complement of a maximal set from the $i-th$ equivalence class w.r.t. $\equiv_1^*$.

Proof. Before we construct a computable linear transformation Φ such that $dom(\Phi) =^* V_\infty =^* rng(\Phi)$ that induces the automorphism f with the desired properties we will introduce some notions.

Suppose we have a fixed simultaneous enumeration of the c.e. sets I_{kj} and let p_{kj} be computable permutations such that $I_{kj} =^* p_{kj}(I_{k1})$ for all $k = 1, \ldots, m$ and $j = 1, \ldots, n_k$. The existence of such computable permutations with the property that $\forall x[p_{kj}^2(x) = x]$ was proved in [3]. There we also introduced the notion of an orbit with respect to our fixed enumeration.

Definition 2.1. *Let $k \leq m$ be fixed. An n-tuple $(y_1, y_2, \ldots, y_{n_k})$ such that $y_j = p_{kj}(y_1)$ is called an $\overline{J_k}$ – orbit at stage s if*

$$\forall i \leq n_k \forall j \leq n_k[(j \neq i) \rightarrow (y_i \notin I_{ki}^s \wedge y_i \in I_{kj}^s)].$$

We now outline the idea behind this definition. At stage s the $\overline{J_k}$ – *orbit* $(y_1, y_2, \ldots, y_{n_k})$ is an element of $\overline{I_{k1}} \times \overline{I_{k2}} \times \ldots \times \overline{I_{kn_k}}$. In the process of describing the structure of a space $V \in \mathcal{L}^*(cl(J_k), \uparrow)$ in [3] we enumerate $\overline{J_k}$ – *orbits* as they appear into a c.e. set O_k. The set O_k is such that $\overline{I_{kj}} \subset^* pr_j(O_k)$ for every $j = 1, \ldots, n_k$. The underlying set $\{y_1, y_2, \ldots, y_{n_k}\}$ of almost every $\overline{J_k}$ orbit that is enumerated at some stage into O_k will eventually be either a subset of J_k or the orbit $(y_1, y_2, \ldots, y_{n_k})$ itself will remail an element of $\overline{I_{k1}} \times \overline{I_{k2}} \times \ldots \times \overline{I_{kn_k}}$. If the latter happens, then we call such orbit a $\overline{J_k}$ orbit. Additionally, the underlying sets of almost every two different $\overline{J_k}$ – *orbits* are disjoint.

We now introduce some notation to describe the isomorphism between the lattice L_k of subspaces of an n_k dimensional space W_k over $\widetilde{F_k}$ and $\mathcal{L}^*(cl(J_k), \uparrow)$. Following the proof of Theorem 1.1 in [3] we can select a basis $\{w_1^k, \ldots, w_{n_k}^k\}$ of W_k in such a way that the vector $w_i^k \in W_k$ "corresponds" to the partial computable function $p_{ki}|_{B_k}$ that is the restriction of the permutation p_{ki} to the c.e. set $B_k = pr_1(O_k)$. The set B_k has the properties that $\overline{I_{k_1}} \subset^* B_k$ and $\overline{I_{k_i}} \subset^* p_{ki}(B_k)$. For each vector $\overline{\beta} = (\beta_1, \beta_2, \ldots, \beta_{n_k}) \in W_k$ define a c.e. set of linear combinations

$$I_{\overline{\beta}} = \{\sum_{i=1}^{n_k} \beta_i(y_1)p_{ki}(y_1) : (y_1 \in B_k) \wedge \forall i \leq n_k(\beta_i(y_1) \downarrow)\}.$$

It is important to note that

$$cl(J_k) \vee cl(I_{\overline{\beta}}) = cl(J_k) \vee cl(\{\sum_{i=1}^{n_k} \beta_i(y_1)p_{ki}(y_1) : y_1 \in \overline{I_{k1}}\}).$$

For each $W \in L_k$ such that $W = cl\{\overline{\beta_1}, \ldots, \overline{\beta_n}\}$ define a c.e. $V_W \in \mathcal{L}^*(cl(J_k), \uparrow)$ such that

$$V_W = cl(J_k) \vee cl(\bigcup_{i=1}^{n} I_{\overline{\beta_i}}).$$

In [3] we proved that the function that maps $W \in L_k$ to $V_W \in \mathcal{L}^*(cl(J_k), \uparrow)$ is an isomorphism between L_k and $\mathcal{L}^*(cl(J_k), \uparrow)$.

Suppose that the automorphism f_k of $\mathcal{L}^*(cl(J_k), \uparrow)$ is induced by a computable linear transformation Φ_k of W_k such that $\Phi_k(w_j^k) = \overline{\alpha_j^k}$ where $\overline{\alpha_j^k} = (\alpha_{j1}^k, \dots, \alpha_{jn_k}^k)$ and $\alpha_{ji}^k \in \widetilde{F_k}$ (for $i \leq n_k$) are the coordinates of the image of w_j^k with respect to the basis $\{w_1^k, \dots, w_{n_k}^k\}$.

We assume that the automorphism f_k of $\mathcal{L}^*(cl(J_k), \uparrow)$ corresponds, via the isomorphism $W \to V_W$, to the automorphism of L_k that is induced by Φ_k. We then have

$$f_k(V_W) = cl(J_k) \vee cl(\bigcup_{i=1}^{m} \Phi_k(I_{\overline{\beta_i}}))$$

where

$$\begin{aligned}\Phi_k(I_{\overline{\beta}}) &= \{\sum_{j=1}^{n_k}\sum_{i=1}^{n_k} \beta_j(y_1)\alpha_{ji}^k(y_1)p_{ki}(y_1) : \\ (y_1 &\in B_k) \wedge \forall i \leq n_k(\beta_i(y_1)\downarrow \wedge \forall j \leq n_k(\alpha_{ji}^k(y_1)\downarrow))\}\end{aligned}$$

We will define a computable linear transformation Φ with co-finite dimensional domain and co-finite dimensional range in V_∞. In the construction below $\Phi(y)$ will be defined for almost every $y \in I_0$. Then Φ will be extended to a liner map. For the construction we will need the following

Definition 2.2. *$(y_1, y_2, \dots, y_{n_k})$ is a generalized $\overline{J_k}$ orbit at stage s if:*
(i) $(y_1, y_2, \dots, y_{n_k})$ is a $\overline{J_k}$ – orbit at stage s,
(ii) $\forall i \leq n_k \forall j \leq m[j \neq k \to y_i \in J_j^s]$

Construction:

Stage 0: $\Phi^0 = \emptyset$.

Stage s+1:

(A) If there is $y \in I^s = \bigcap_{j=1}^{m} J_j^s$ such that $\Phi^{s+1}(y)$ has not yet been defined, then let $\Phi^{s+1}(y) = y$.

(B) See if for some $k \leq m$ there is a tuple $(y_1, y_2, \dots, y_{n_k})$ such that:
- (b1) $(y_1, y_2, \dots, y_{n_k})$ is a generalized $\overline{J_k}$ orbit at stage s,
- (b2) $\alpha_{ij}^{k,s}(y_1)\downarrow$ for every $i, j \leq n_k$,
- (b3) $\forall i \leq n_k[\Phi^{s+1}(y_i)\uparrow]$

In this case for every $j \leq n_k$ let

$$\Phi^{s+1}(y_j) = \alpha_{j1}^k(y_1)y_1 + \dots + \alpha_{jn_k}^k(y_1)y_{n_k}.$$

(C) go to the next stage.

End of Construction.

In the lemmas that follow we will prove that the linear extension of the map Φ induces an automorphism of $\mathcal{L}^*(V_\infty)$ with the desired properties. □

Lemma 2.1. *$\Phi(y)$ is defined for almost every $y \in I_0$.*

Proof. We assumed that I_{kj} (for $k \leq m$ and $j = 1, \ldots, n_k$) are different maximal sets. Using the fact that these sets are maximal we can prove that for almost every $y \in I_0$ either $y \in I$ or there are unique $k_y \leq m$ and $j_y \leq n_{k_y}$ such that $y \in \overline{I_{k_y j_y}}$.

Case 1: If $y \in I = \bigcap_{j=1}^{m} J_j$ and $\Phi(y)$ has not defined by the stage s when $y \in I^s = \bigcap_{j=1}^{m} J_j^s$, then $\Phi(y) = y$ at stage $s+1$.

Case 2: Suppose $y \notin I$ and let $k_y \leq m$ and $j_y \leq n_{k_y}$ be such that $y \in \overline{I_{k_y j_y}}$. Let $(y_1, y_2, \ldots, y_{n_{k_y}})$ be such that $y_1 = p^{-1}_{k_y j_y}(y)$ and $y_j = p_{k_y j}(y_1)$ for $j \leq n_{k_y}$ (notice that in this setting $y = y_{j_y}$). By the definition of the permutations $p_{k_y j}$ we notice that for almost every such $y \notin I$ we will have

(1) $\{y_1, y_2, \ldots, y_{n_{k_y}}\} \cap I = \emptyset$, and

(2) $\forall j \neq k_y [\{y_1, y_2, \ldots, y_{n_{k_y}}\} \subset J_j]$.

This means that $(y_1, y_2, \ldots, y_{n_{k_y}})$ will be identified as a a generalized $\overline{J_{k_y}}$ orbit at some stage s when (b2) in the construction above will also be satisfied for $k = k_y$. Using the fact that the underlying sets of different $\overline{J_k}$ orbits are disjoint we conclude that (b3) above will also be satisfied for $k = k_y$ at stage s and therefore $\Phi(y)$ will be defined. □

Lemma 2.2. *The linear span of $rng(\Phi)$ is cofinite dimensional in V_∞.*

Proof. Notice that either $\Phi(y) = y$ or $\Phi(y)$ is defined by means of part (B) of the construction. In the latter case, it may happen that all the elements of the underlying set $\{y_1, y_2, \ldots, y_{n_k}\}$ of the generalized orbit $(y_1, y_2, \ldots, y_{n_k})$ of y will be later enumerated into I. In all cases Φ is a linear transformation such that $\Phi(cl\{y_1, y_2, \ldots, y_{n_k}\}) = cl\{y_1, y_2, \ldots, y_{n_k}\}$. Using also the previous lemma we conclude that

$$V_\infty =^* cl\{\Phi(y) : y \in I_0 \wedge \Phi(y)\downarrow\}.$$

Lemma 2.3. *If f is the automorphism of $\mathcal{L}^*(V_\infty)$ that is induced by the linear extension Φ^E of Φ, then $f|_{L_k} \equiv f_k$.*

Proof. By the previous two lemmas Φ is computable map such that $cl(dom(\Phi)) =^* V_\infty =^* cl(rng(\Phi))$[4] and therefore Φ^E is a computable linear map that induces an

[4]C. Ash conjectured that all automorphisms of $\mathcal{L}^*(V_\infty)$ are induced by computable semilinear maps that satisfy this property. For more information see [5].

automorphism of $\mathcal{L}^*(V_\infty)$. Fix $k \leq m$. We know that if $W \in L_k$ is such that $W = cl\{\overline{\beta_1}, \ldots, \overline{\beta_n}\}$, then $f_k(V_W) =^* cl(J_k) \vee cl(\bigcup_{i=1}^{n} \Phi_k(I_{\overline{\beta_i}}))$. Also, for almost every $y_1 \in B_k$ such that $y_1 \in I_{k1}$ (an therefore $y_1 \in J_k$ by the definition of orbit) we will have $\{y_1, y_2, \ldots, y_{n_k}\} = \{p_{k1}(y_1), p_{k2}(y_1), \ldots, p_{kn_k}(y_1)\} \subset J_k$. This means that

$$f_k(V_W) =^* cl(J_k) \vee cl(\bigcup_{i=1}^{n} \Phi_k(I_{\overline{\beta_i}})^-) \qquad (\#)$$

where

$$\begin{aligned} \text{for } \overline{\beta} &= (\beta_1, \beta_2, \ldots, \beta_{n_k}) \text{ we let} \\ \Phi_k(I_{\overline{\beta}})^- &= {}^*\{\sum_{j=1}^{n_k}\sum_{i=1}^{n_k} \beta_j(y_1)\alpha_{ji}^k(y_1)p_{ki}(y_1) : y_1 \in \overline{I_{k1}}\}. \end{aligned}$$

Notice that every $\overline{J_k}$ orbit $(y_1, y_2, \ldots, y_{n_k})$ will be identified as a generalized $\overline{J_k}$ orbit at some stage s_1 of the construction of the map Φ and without loss of generality assume that $\alpha_{ij}^{k,s_1}(y_1)\downarrow$. At such stage we define $\Phi(y_j) = \sum_{i=1}^{n_k} \alpha_{ji}^k(y_1)y_i = \sum_{i=1}^{n_k} \alpha_{ji}^k(y_1)p_{ki}(y_1)$ for every $j \leq n_k$.

That means that $f(V_W) =^* cl(\Phi(J_k)) \vee cl(\bigcup_{i=1}^{n} \Phi(I_{\overline{\beta_i}}))$ where

$$\begin{aligned} \Phi(I_{\overline{\beta_i}}) &= \{\sum_{j=1}^{n_k}\sum_{i=1}^{n_k} \beta_j(y_1)\alpha_{ji}^k(y_1)p_{ki}(y_1) : \\ (y_1 &\in C_k) \wedge \forall i \leq n_k(\beta_i(y_1)\downarrow \wedge \forall j \leq n_k(\alpha_{ji}^k(y_1)\downarrow))\}, \text{ and} \end{aligned}$$

$$\begin{aligned} C_k &= \{y_1 : \exists s[(b1), (b2), \text{and } (b3) \\ & \qquad \text{from the construction are satisfied at s}]\}. \end{aligned}$$

Finally using that (1) $cl(\Phi(J_k)) =^* cl(J_k)$, (2) $\overline{I_{k1}} \subset^* C_k \subseteq^* B_k$, as well as identity (#) above, we can now conclude that $f(V_W) =^* f_k(V_W)$. □

Definition 2.3. *Two quasimaximal subsets* $Q_1 = \bigcap_{k=1}^{n}\bigcap_{j=1}^{n_k} I_{kj}$ *and* $Q_2 = \bigcap_{k=1}^{m}\bigcap_{j=1}^{m_k} J_{kj}$ *of* I_0 *have the same type if* I_{kj} *and* J_{kj} *are maximal subsets of* I_0 *and the following hold:*

1. $m = n$ *and* $\forall k \leq n(m_k = n_k)$
2. $I_{kj} \equiv_1^* I_{k_1 j_1}$ *iff* $k = k_1$ *and* $J_{kj} \equiv_1^* J_{k_1 j_1}$ *iff* $k = k_1$
3. $I_{kj} \equiv_1^* J_{kj}$

Corollary 2.1. *Suppose that the quasimaximal Q_1 and Q_2 have the same type. Then there is an automorphism f of $\mathcal{L}^*(V_\infty)$ such that $f(cl(Q_1)) =^* cl(Q_2)$ and $f(cl(Q_2)) =^* cl(Q_1)$.*

Proof. Let $Q_1 = \bigcap_{k=1}^{n} \bigcap_{j=1}^{n_k} I_{kj}$ and $Q_2 = \bigcap_{k=1}^{n} \bigcap_{j=1}^{n_k} J_{kj}$ where I_{kj} and J_{kj} are as in the definition above. Let $J_1, \dots, J_n$ are quasimaximal subsets of I_0 such that for $k \leq n$ $J_k = \bigcap_{j=1}^{n_k} I_{kj} \cap \bigcap_{j=1}^{n_k} J_{kj}$. Let the automorphisms f_k of $\mathcal{L}^*(cl(J_k), \uparrow)$ in the statement of Theorem 2.1 be such that $f_k(cl(I_{kj})) = cl(J_{kj})$ and $f_k(cl(J_{kj})) = cl(I_{kj})$. Notice that it is easy to construct a linear transformation Φ_k of W_k that induces such corresponding automorphism f_k of $\mathcal{L}^*(cl(J_k), \uparrow)$. Let f be the automorphism from the conclusion of Theorem 2.1. Notice that $cl(Q_1) = \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} cl(I_{kj})$ and $cl(Q_2) = \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} cl(J_{kj})$. Then

$$\begin{aligned} f(cl(Q_1)) &= \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} f_k(cl(I_{kj})) = \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} f(cl(I_{kj})) \\ &= \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n_k} cl(J_{kj}) = cl(Q_2). \end{aligned}$$

We similarly observe that $f(cl(Q_2)) =^* cl(Q_1)$. □

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COHESIVE POWERS OF COMPUTABLE STRUCTURES

RUMEN DIMITROV

We develop the notion of cohesive power $\mathcal{B}$ of a computable structure $\mathcal{A}$ over a cohesive set R. In the main theorem of this paper we prove certain connections between satisfaction of different formulas and sentences in the original model $\mathcal{A}$ and its cohesive power $\mathcal{B}$. We also prove various facts about cohesive powers, isomorphisms between them and consider an example in which the structure $\mathcal{A}$ is a computable field.

1. INTRODUCTION

In the study of the structure of the lattice $\mathcal{L}^*(V_\infty)$ we came upon a field with elements that are partial computable functions. We noticed that the construction of the field had certain similarities with the classical model theoretic ultrapower construction. We are now studying similar structures in a more general setting. We introduce the notion of cohesive power of a computable structure and prove an analogue of the fundamental theorem for ultraproducts [1]) for cohesive powers. The connection of cohesive powers of computable fields and the structure of $\mathcal{L}^*(V_\infty)$ is described in the concluding remarks.

A set R is cohesive if for every computably enumerable (c.e.) set W either $W \cap R$ or $\overline{W} \cap R$ is finite. There are continuum many cohesive subsets of ω. There are cohesive sets with computably enumerable complements. The c.e. complements of such cohesive sets are called maximal. For a fixed computable structure $\mathcal{A}$ and a cohesive set R we define the $R-$cohesive power $\mathcal{B}$ of $\mathcal{A}$. The satisfaction of sentences in $\mathcal{B}$ is connected to the existence of decision procedures for different segments of the complete diagram of $\mathcal{A}$. If $\mathcal{A}$ is a decidable structure, then $\mathcal{A}$ and $\mathcal{B}$ will be elementarily equivalent. If $\mathcal{A}$ is computable then $\mathcal{A}$ and $\mathcal{B}$ satisfy the same Π_2 and Σ_2 sentences.

We will use $\varphi_0, \varphi_1, \ldots$ to refer to arbitrary partial computable (p.c.) functions. Also, we assume a fixed enumeration $\phi_0, \phi_1, \ldots$ of the (unary) partial computable functions. We will write $\phi_{e,s}(x) = y$ if $e, x, y < s$ and y is the result of the $e - th$ computation on input n in less than s steps. In this case we will also write $\phi_{e,s}(x) \downarrow$. By $\phi_e(x) \downarrow$ we mean that $\exists s[\phi_{e,s}(x) = y]$. The enumeration of the e-th c.e. set $W_e = dom(\phi_e)$ is given as $W_{e,s} = dom(\phi_{e,s})$. We let use normal equality symbol $=$ (instead of $\simeq$) between partial computable functions. In definitions of p.c. functions we will assume that the function on the left side is defined when all of the elements on the right hand side are defined and the expression is acceptable for the particular values of the functions. For example, $\varphi = \frac{\psi_1}{\psi_2}$ means that

$$\varphi(x) = \begin{cases} \frac{\psi_1(x)}{\psi_2(x)} & \text{if } \psi_1(x) \downarrow,\ \psi_2(x) \downarrow, \text{ and } \psi_2(x) \neq 0 \\ undefined & \text{otherwise} \end{cases}.$$

2. MAIN RESULT

Let $\mathcal{A}$ be a computable structure over a fixed computable language L and let $R \subset \omega$ be a cohesive set. If Ψ is a formula in L, then we will use $\{x : \mathcal{A} \models \Psi(\varphi_1(x), \ldots, \varphi_n(x))\}$ as a shorthand for

$$\{x : \exists s \exists t_1 \ldots \exists t_n (\bigwedge_{i=1}^{n} (\varphi_{i,s}(x) = t_i) \wedge \mathcal{A} \models \Psi(t_1, \ldots, t_n))\}.$$

Definition 2.1. *The cohesive power of $\mathcal{A}$ over R is a structure $\mathcal{B}$ (denoted $\prod_R \mathcal{A}$) in L such that:*

1. $B = \{\varphi : \varphi$ *is a p.c. function,* $R \subseteq^* dom(\varphi)$, $rng(\varphi) \subseteq A\} / =_R$

   Here $\varphi_1 =_R \varphi_2$ *if* $R \subseteq^* \{x : \varphi_1(x) \downarrow = \varphi_2(x) \downarrow\}$. *The equivalence class of* φ *w.r.t.* $=_R$ *will be denoted by* $[\varphi]_R$ *or simply* $[\varphi]$ *when the set* R *is fixed.*

2. *If* $f \in L$ *is an n-ary functional symbol, then* $[f^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])]$ *is the equivalence class of a p.c. function such that*

$$f^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])(x) = f^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x)).$$

3. *If* $P \in L$ *is an m-ary predicate symbol, then* $P^{\mathcal{B}}$ *is a relation such that*

$$P^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_m]) \text{ iff } R \subseteq^* \{x : P^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_m(x))\}.$$

4. *If* $c \in L$ *is a constant symbol, then the interpretation of* c *in* $\mathcal{B}$ *is the equivalence class of the total computable function with constant value* c^A.

The domains of the partial computable functions in the definition above contain the set R and form a filter in the lattice $\mathcal{E}$. The role that the cohesiveness of R plays in the theorem below is similar to the role the maximality of the ultrafilter plays in the ultraproduct construction.

Theorem 2.1. *(Fundamental theorem of cohesive powers)*

1. *If $\tau(y_1, \ldots, y_n)$ is a term in L and $[\varphi_1], \ldots, [\varphi_n] \in B$, then $[\tau^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])]$ is the equivalence class of a p.c. function such that $\tau^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])(x) = \tau^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x))$.*

2. *If $\Phi(y_1, \ldots, y_n)$ is a formula in L that is a boolean combination of Σ_1 and Π_1 formulas and $[\varphi_1], \ldots, [\varphi_n] \in B$, then*

$$\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n]) \text{ iff } R \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \ldots, \varphi_n(x))\}.$$

3. *If Φ is a Π_3 sentence in L, then $\mathcal{B} \models \Phi$ implies $\mathcal{A} \models \Phi$.*

4. *If Φ is a Π_2 (or Σ_2) sentence in L, then $\mathcal{B} \models \Phi$ iff $\mathcal{A} \models \Phi$.*

Proof. (1) The proof is straightforward but we note that we essentially use the fact that the operations in $\mathcal{A}$ are computable.

(2) We proceed by induction:

(2.1) Let $\Phi(y_1, \ldots, y_n) = P(\tau_1(y_1, \ldots, y_n), \ldots, \tau_m(y_1, \ldots, y_n))$ be an atomic formula and suppose $[\psi_i] = \tau_i^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n])$. Then

$\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n])$

iff

$\mathcal{B} \models P([\psi_1], \ldots, [\psi_m])$

iff

$R \subseteq^* \{x : \mathcal{A} \models P(\psi_1(x), \ldots, \psi_m(x))\}$

iff

$R \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \ldots, \varphi_m(x))\}$

(2.2) Suppose $\Phi(y_1, \ldots, y_n) = \Phi_1(y_1, \ldots, y_n) \wedge \Phi_2(y_1, \ldots, y_n)$ and the claim is true for $\Phi_i(y_1, \ldots, y_n)$ $i = 1, 2$. Then

$\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n])$

iff

$\mathcal{B} \models \Phi_1([\varphi_1], \ldots, [\varphi_n])$ and $\mathcal{B} \models \Phi_2([\varphi_1], \ldots, [\varphi_n])$

iff

$R \subseteq^* \{x : \mathcal{A} \models \Phi_1(\varphi_1(x), \ldots, \varphi_n(x))\}$ and
$R \subseteq^* \{x : \mathcal{A} \models \Phi_2(\varphi_1(x), \ldots, \varphi_n(x))\}$

iff

$R \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \ldots, \varphi_n(x))\}$.

(2.3) Suppose $\Phi(y_1, \ldots, y_n) = \exists y \Psi(y, y_1, \ldots, y_n)$ and $\Psi(y, y_1, \ldots, y_n)$ is a quantifier free formula for which the claim is true.

(2.3a) Suppose $\mathcal{B} \models \exists y \Psi(y, [\varphi_1], \ldots, [\varphi_n])$ and suppose that the p.c. function φ is such that $\mathcal{B} \models \Psi([\varphi], [\varphi_1], \ldots, [\varphi_n])$. By the inductive hypothesis $R \subseteq^* \{x : \mathcal{A} \models \Psi(\varphi(x), \varphi_1(x), \ldots, \varphi_n(x))\}$ and so

$$R \subseteq^* \{x : \mathcal{A} \models \exists y \Psi(y, \varphi_1(x), \ldots, \varphi_n(x))\}.$$

(2.3b) Suppose $R \subseteq^* \{x : \mathcal{A} \models \exists y \Psi(y, \varphi_1(x), \ldots, \varphi_n(x))\}$. Since the structure $\mathcal{A}$ is computable and $\Psi(y, y_1, \ldots, y_n)$ is quantifier free we can define a partial computable

$$\varphi(x) = \mu y \in A[\mathcal{A} \models \Psi(y, \varphi_1(x), \ldots, \varphi_n(x))].$$

Then

$$\begin{aligned} \{x \;&:\; \mathcal{A} \models \exists y \Psi(y, \varphi_1(x), \ldots, \varphi_n(x))\} = \\ \{x \;&:\; \mathcal{A} \models \Psi(\varphi(x), \varphi_1(x), \ldots, \varphi_n(x))\} \end{aligned}$$

and $R \subseteq^* \{x : \mathcal{A} \models \Psi(\varphi(x), \varphi_1(x), \ldots, \varphi_n(x))\}$. By the inductive hypothesis $\mathcal{B} \models \Psi([\varphi], [\varphi_1], \ldots, [\varphi_n])$ and so $\mathcal{B} \models \exists y \Psi(y, [\varphi_1], \ldots, [\varphi_n])$.

(2.4) Suppose $\Phi(y_1, \ldots, y_n) = \neg \Psi(y_1, \ldots, y_n)$ and $\Psi(y_1, \ldots, y_n)$ is a Σ_1 formula for which the hypothesis is true.

(2.4a) Suppose $\mathcal{B} \models \Phi([\varphi_1], \ldots, [\varphi_n])$ and let

$$D = \{x : \mathcal{A} \models \Psi(\varphi_1(x), \ldots, \varphi_n(x))\}.$$

Since $\mathcal{B} \not\models \Psi([\varphi_1], \ldots, [\varphi_n])$, then $R \not\subseteq^* D$. Because $\Psi(y_1, \ldots, y_n)$ is a Σ_1 formula and φ_i for $i \leq n$ are p.c., then D is a c.e. set. Since R is cohesive we have $R \cap D =^* \emptyset$. Also, since $R \subseteq^* \bigcap_{i=1}^{n} dom(\varphi_i)$, then for almost all $x \in R$ we have $\mathcal{A} \not\models \Psi(\varphi_1(x), \ldots, \varphi_n(x))$. Therefore $R \subseteq^* \{x : \mathcal{A} \models \neg \Psi(\varphi_1(x), \ldots, \varphi_n(x))\}$.

(2.4b) Suppose $R \subseteq^* \{x : \mathcal{A} \models \neg \Psi(\varphi_1(x), \ldots, \varphi_n(x))\}$. Then

$$R \cap \{x : \mathcal{A} \models \Psi(\varphi_1(x), \ldots, \varphi_n(x))\} =^* \emptyset$$

and by the inductive hypothesis $\mathcal{B} \not\models \Psi([\varphi_1], \ldots, [\varphi_n])$. Therefore

$$\mathcal{B} \models \neg \Psi([\varphi_1], \ldots, [\varphi_n]).$$

(3) Let $\Phi = \forall y \exists z \forall t \Psi(y, z, t)$ where $\Psi(y, z, t)$ is a quantifier free formula. Let $c \in A$ be arbitrary and let $\varphi_c(x) = c$ for every $x \in \omega$. Let $[\varphi] \in B$ be such that $\mathcal{B} \models \forall t \Psi([\varphi_c], [\varphi], t)$. By (2) above we have $R \subseteq^* \{x : \mathcal{A} \models \forall t \Psi(\varphi_c(x), \varphi(x), t)\}$.

Then $R \subseteq^* \{x : \mathcal{A} \models \exists z \forall t \Psi(c, z, t)\}$. The set is R is nonempty and x is not a free variable of $\exists z \forall t \Psi(c, z, t)$. Therefore $\mathcal{A} \models \exists z \forall t \Psi(c, z, t)$ and so $\mathcal{A} \models \Phi$.

(4) Let $\Phi = \forall y \exists z \Psi(y, z)$ where $\Psi(y, z)$ is a quantifier free formula.

(4a) The fact, that $\mathcal{A} \models \Phi$ whenever $\mathcal{B} \models \Phi$, follows from (3).

(4b) Suppose that $\mathcal{A} \models \Phi$ and let $[\varphi] \in B$ be arbitrary. We have that $R \subseteq^* dom(\varphi) = \{x : \mathcal{A} \models \exists z \Psi(\varphi(x), z)\}$. By (2), $\mathcal{B} \models \exists z \Psi([\varphi], z)$ and so $\mathcal{B} \models \Phi$. □

Note that if the structure $\mathcal{A}$ is decidable, then we can similarly prove the following:

Theorem 2.2. *If $\mathcal{A}$ is a decidable structure, then*

1. *If $\Phi(y_1, \dots, y_n)$ is a formula in L, and $[\varphi_1], \dots, [\varphi_n] \in B$, then*

$$\mathcal{B} \models \Phi([\varphi_1], \dots, [\varphi_n]) \text{ iff } R \subseteq^* \{x : \mathcal{A} \models \Phi(\varphi_1(x), \dots, \varphi_n(x))\}.$$

2. *If Φ is a sentence, then*

$$\mathcal{B} \models \Phi \text{ iff } \mathcal{A} \models \Phi.$$

Proof. (1) The proof is almost identical to the proof of part (2) of the main theorem. We note only that for any formula $\Psi(y_1, \dots, y_n)$ the set $\{(a_1, \dots, a_n) : \mathcal{A} \models \Psi(a_1, \dots, a_n)\}$ is computable. Then the set $\{x : \mathcal{A} \models \Psi(\varphi_1(x), \dots, \varphi_n(x))\}$ is c.e. and steps 2.3 and 2.4 of the proof above can be carried for any formula Ψ.

(2) Follows directly from (1). □

Definition 2.2. *For $c \in A$ let $[\varphi_c] \in B$ be the equivalence class of the total function φ_c such that $\varphi_c(x) = c$ for every $x \in \omega$. The map $d : A \to B$ such that $d(c) = [\varphi_c]$ is called the canonical embedding of $\mathcal{A}$ into $\mathcal{B}$.*

Proposition 2.1. *The following hold:*

1. *If the structure $\mathcal{A}$ is finite, then $\mathcal{B} \cong \mathcal{A}$.*

2. *If the structure $\mathcal{A}$ is decidable, then the canonical map d is an elementary embedding of $\mathcal{A}$ into $\mathcal{B}$.*

3. *If $\Phi(y_1, \dots, y_n)$ is a Π_2 or a Σ_2 formula in L and $c_1, \dots, c_n \in A$, then*

$$\mathcal{A} \models \Phi(c_1, \dots, c_n) \text{ iff } \mathcal{B} \models \Phi(d(c_1), \dots, d(c_n)).$$

Proof. (1) Let $[\varphi] \in B$ be arbitrary. For any $c \in A$ let $X_c = \{x : \varphi(x) = c\}$ and notice that X_c is a c.e. set. Since $dom(\varphi) = \bigcup_{c \in A} X_c$ and A is finite, then for some $c_1 \in A$ the set $X_{c_1} \cap R$ is infinite. Since R is cohesive we have $R \subseteq^* X_{c_1}$ and therefore $[\varphi] = [\varphi_{c_1}]$. Therefore all equivalence classes in B correspond to the constants in A and the canonical embedding of $\mathcal{A}$ into $\mathcal{B}$ is a 1-1 map. So $\mathcal{B} \cong \mathcal{A}$ follows directly from the definition of $\mathcal{B}$.

(2) Let $\Phi(y_1, \ldots, y_n)$ be a Σ_2 (or Π_2) formula and let $c_1, \ldots, c_n \in A$. If $\mathcal{A}$ is decidable, then

$\mathcal{B} \models \Phi([\varphi_{c_1}], \ldots, [\varphi_{c_n}])$ iff
$R \subseteq^* \{x : \mathcal{A} \models \Phi(c_1, \ldots, c_n)\}$ iff
$\mathcal{A} \models \Phi(c_1, \ldots, c_n)$.

(3) Let $c_1, \ldots, c_n \in A$ and let $L_C = L \cup \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be the language L expanded by adding a constant symbol for each c_i. Let $\mathcal{A}_C$ be the structure $\mathcal{A}$ with the constant symbols $\mathbf{c}_1, \ldots, \mathbf{c}_n$ interpreted as $c_1, \ldots, c_n$ correspondingly. Let $\mathcal{B}_C$ be the $R-$cohesive power of $\mathcal{A}_C$. Then $\Phi(c_1, \ldots, c_n)$ will be a Σ_2 (or Π_2) sentence in L_C and by the Fundamental theorem part (4)

$$\mathcal{A}_C \models \Phi(\mathbf{c}_1, \ldots, \mathbf{c}_n) \text{ iff } \mathcal{B}_C \models \Phi(\mathbf{c}_1, \ldots, \mathbf{c}_n)$$

which is equivalent to

$$\mathcal{A} \models \Phi(c_1, \ldots, c_n) \text{ iff } \mathcal{B} \models \Phi(d(c_1), \ldots, d(c_n)).\square$$

Definition 2.3. *Two sets A, B have the same 1-degree up to $=^*$ (denoted $A \equiv_1^* B$) if there are $A_1 =^* A$ and $B_1 =^* B$ such that $A_1 \equiv_1 B_1$.*

Proposition 2.2. *If $M_1 \equiv_1^* M_2$ are maximal sets, $\mathcal{B}_1 = \prod_{\overline{M_1}} \mathcal{A}$. and $\mathcal{B}_2 = \prod_{\overline{M_2}} \mathcal{A}$. then $\mathcal{B}_1 \cong \mathcal{B}_2$.*

Proof. Let $M_i' =^* M_i$ for $i = 1, 2$ be such that $M_1' \equiv_1 M_2'$. Let $\mathcal{B}_i' = \prod_{\overline{M_i'}} \mathcal{A}$ and notice that $\mathcal{B}_i' \cong \mathcal{B}_i$ for $i = 1, 2$. Using Myhill Isomorphism Theorem (see [6, p.24]) we let σ be a computable permutation of ω such that $\sigma(M_1') = M_2'$. Define a map $\Phi : B_2' \to B_1'$ as follows:

$\Phi([\psi]) = [\varphi]$ where $\varphi(x) = \psi(\sigma(x))$. We now prove that Φ is an isomorphism of $\mathcal{B}_2'$ and $\mathcal{B}_1'$:

(1) Notice that $\psi_1 =_{\overline{M_2'}} \psi_2$ iff $\overline{M_2'} \subseteq^* \{x : \psi_1(x) = \psi_2(x)\}$ iff $\overline{M_1'} \subseteq^* \{x : \psi_1(\sigma(x)) = \psi_2(\sigma(x))\}$ iff $\Phi(\psi_1) =_{\overline{M_1'}} \Phi(\psi_2)$. So Φ is correctly defined and injective. Finally, if $[\varphi] \in B_1'$ and $\psi(x) = \varphi(\sigma^{-1}(x))$, then $\Phi([\psi]) = [\varphi]$.

(2) Let $f \in L$ be an n-ary functional symbol.

Then $\Phi([f^{\mathcal{B}_2'}([\psi_1], \dots, [\psi_n])])$ is the equivalence class of a p.c. function such that $\Phi([f^{\mathcal{B}_2'}([\psi_1], \dots, [\psi_n])])(x) = f^{\mathcal{A}}(\psi_1(\sigma(x)), \dots, \psi_n(\sigma(x)))$.

That means that $\Phi([f^{\mathcal{B}_2'}([\psi_1], \dots, [\psi_n])]) = [f^{\mathcal{B}_1'}(\Phi([\psi_1]), \dots, \Phi([\psi_n]))]$.

(3) If $P \in L$ is an m-ary predicate symbol, then
$P^{\mathcal{B}_2'}([\psi_1], \dots, [\psi_n])$ iff
$\overline{M_2'} \subseteq^* \{x : P^{\mathcal{A}}(\psi_1(x), \dots, \psi_m(x))\}$ iff
$\overline{M_1'} \subseteq^* \{x : P^{\mathcal{A}}(\psi_1(\sigma(x)), \dots, \psi_m(\sigma(x)))\}$ iff
$P^{\mathcal{B}_1'}(\Phi([\psi_1]), \dots, \Phi([\psi_n]))$.□

Proposition 2.3. *Every computable automorphism of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{B}$.*

Proof. Let σ be a computable automorphism of $\mathcal{A}$. Define a map $\widetilde{\sigma}$ on B as follows:

$$\widetilde{\sigma}([\varphi]) = [\psi]$$

where $\psi(n) = \sigma(\varphi(n))$. The proof that $\widetilde{\sigma}$ is an automorphism of $\mathcal{B}$ is straightforward. Notice also that if $c \in A$ and $\varphi(n) = c$ for almost every $n \in R$, then $\widetilde{\sigma}([\varphi])(n)$ is the constant $\sigma(c)$ for almost every $n \in R$. □

Example. Let F be a computable field and let I be a maximal set. Then $\widetilde{F} = \prod_{\overline{I}} F$ is a field such that:

1. $\widetilde{F} \cong F$ if F is finite,
2. If $[\varphi] \in \widetilde{F}$ is algebraic over F, then φ is a constant function on $\overline{I}$.
3. Every computable automorphism σ of F can be extended naturally to an automorphism $\widetilde{\sigma}$ of $\widetilde{F}$.

Proof. We will prove only (2), (1) and (3) follow directly from the propositions above. Suppose $[\varphi] \in \widetilde{F}$ is root of a polynomial $g(x) \in F[x]$. Extend the language of F by adding new constants for each coefficient of the polynomial g. Let $\widetilde{F_1}$ be the cohesive power of F over $\overline{I}$ in the extended language. By the fundamental theorem of cohesive powers we have

$$\widetilde{F_1} \models (g([\varphi]) = 0^{\widetilde{F_1}}) \text{ iff } \overline{I} \subseteq^* \{x : F \models (g(\varphi(x)) = 0^F)\}.$$

This means that $\varphi(x) \in F$ is a root of the polynomial $g(x)$ for almost every $x \in \overline{I}$. Since $g(x)$ can have finitely many roots, then $C = \{c : \exists x[(\varphi(x) = c) \wedge (g(c) = 0)]\}$ is finite. For each $c \in C$ let $X_c = \{x : \varphi(x) = c\}$. Notice that X_c is c.e.. Using the fact that $\overline{I}$ is cohesive we notice that

$$\forall c_1, c_2 \in C[c_1 \neq c_2 \rightarrow (|X_{c_1} \cap \overline{I}| < \infty \text{ or } |X_{c_2} \cap \overline{I}| < \infty)].$$

Since C is finite this implies that for some $c \in C$ we will have $\overline{I} \subseteq^* X_c$. This means that $[\varphi]$ is the equivalence class of a function that has value c on $\overline{I}$. □

3. CONCLUDING REMARKS

As we mentioned an example of cohesive powers appears naturally in the study of the structure of the lattice of subspaces of the fulliy effective vector space V_∞ over a computable field F. The lattice of computably enumerable subspaces of V_∞ modulo finite dimension is denoted $\mathcal{L}^*(V_\infty)$. The study of V_∞ was initiated by Metakides and Nerode in [5]. The lattice $\mathcal{L}^*(V_\infty)$ is an interesting modular analog of $\mathcal{E}^*$, the extensively studied (see [6]) lattice of c.e. sets modulo finite sets. Different cohesive powers of the field F appear (see [3]) in the characterization of principal filters of closures of quasimaximal sets.

Let $Q = \bigcap_{i=1}^{n} I_i$ where I_i $(i \leq n)$ are maximal subsets of I_0–a fixed computable basis of V_∞. Suppose that I_i $(i \leq n)$ are partitioned into k equivalence classes with respect to the relation $\equiv_1^*$ of having the same 1-degree up to $=^*$. Suppose that the $i-th$ equivalence class has n_i elements. In [3] we proved that the principal filter in $\mathcal{L}^*(V_\infty)$ of the linear span of Q is isomorphic to the product of the lattices $(\mathcal{L}(n_i, \widetilde{F}_i))_{i=1}^k$. Here $\mathcal{L}(n_i, \widetilde{F}_i)$ is the lattice of subspaces of an n_i-dimensional vector space over a field $\widetilde{F}_i$. The field $\widetilde{F}_i$ is the cohesive power of F w.r.t. a cohesive set R_i that is the complement of a maximal set from the i-th equivalence class described above.

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ON MUSIELAK–ORLICZ SEQUENCE SPACES WITH AN ASYMPTOTIC ℓ_∞ DUAL [1]

B. ZLATANOV

We investigate Mushielak–Orlicz sequence spaces ℓ_Φ with a dual ℓ_Φ^*, which is stabilized asymptotic ℓ_∞ with respect to the unit vector basis. We give a complete characterization of the bounded relatively weakly compact subsets $K \subset \ell_\Phi$. We prove that ℓ_Φ is saturated with asymptotically isometric copies of ℓ_1 and thus ℓ_Φ fails the fixed point property for closed, bounded convex sets and non–expansive (or contractive) maps on them.

Keywords: Mushielak–Orlicz sequence spaces, asymptotically isometric copy of ℓ_1, asymptotic ℓ_∞ space, fixed point property, weakly compact.

2000 MSC: 46B20, 46B45, 46E30, 46A45, 47H10.

1. INTRODUCTION

The notion of asymptotic ℓ_p spaces first appeared in [14], where the collection of spaces that are now known as stabilized asymptotic ℓ_p spaces were introduced. Later in [13] more general collection of spaces, known as asymptotic ℓ_p spaces, were introduced. Characterization of the stabilized asymptotic ℓ_∞ MO sequence space was given in [5]. It is found in [17] that if the dual of a MO sequence space ℓ_Φ is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis then ℓ_Φ is saturated with complemented copies of ℓ_1 and has the Schur property.

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A characterization of the relatively weakly compact sets in an Orlicz spaces $L_M[0,1]$, such that the function N complementary to M satisfies $\lim_{t\to\infty}\frac{N(\lambda t)}{N(t)}=\infty$ for some $1<\lambda<\infty$ is given in [2]. Using the technique of [2] and [17] we generalize this result for MO sequence spaces. More precisely we characterize the relatively weakly compact sets of a MO sequence space ℓ_Φ, and its dual ℓ_Φ^* is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis.

In the second part of this note we prove that MO spaces ℓ_Φ with stabilized asymptotic ℓ_∞ dual are saturated with asymptotically isometric copies of ℓ_1. The notion of asymptotically isometric copy of ℓ_1 in a Banach space appeared in [7] and is used to investigate the fpp for non–expansive mappings of the non–reflexive subspaces of $L_1[0,1]$. Using the ideas of [1], [7] and [17] we show that any subspace of ℓ_Φ contains an asymptotically isometric copy of ℓ_1, provided that ℓ_Φ^* is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis and as a consequence of [7] this class of MO sequence spaces fails the fpp for closed, bounded, convex sets in ℓ_Φ and non–expansive maps on them. Let us mention that such a conclusion could have been drawn directly by using the recent characterization of the MO sequence spaces ℓ_Φ having fpp given in [16]: An MO sequence space has fpp for closed bounded convex sets and non–expansive maps on them iff it is reflexive. The examples at the end show that sometimes to check reflexivity is more difficult than to check that ℓ_Φ^* is stabilized asymptotic ℓ_∞ with respect to the unit vector basis, due to the engagement of several constants in the definition of the δ_2–condition for a MO function Φ.

2. PRELIMINARIES

We use the standard Banach space terminology from [11], Let us recall that an Orlicz function M is even, continuous, non-decreasing convex function such that $M(0)=0$ and $\lim_{t\to\infty}M(t)=\infty$. We say that M is non–degenerate Orlicz function if $M(t)>0$ for every $t>0$. A sequence $\Phi=\{\Phi_i\}_{i=1}^\infty$ of Orlicz functions is called a Musielak–Orlicz function or MO function in short.

The MO sequence space ℓ_Φ, generated by a MO function Φ is the set of all real sequences $\{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty \Phi_i(\lambda x_i)<\infty$ for some $\lambda>0$. The Luxemburg's norm in ℓ_Φ is defined by

$$\|x\|_\Phi=\inf\left\{r>0:\sum_{i=1}^\infty \Phi_i(x_i/r)\le 1\right\}.$$

We denote by h_Φ the closed linear subspace of ℓ_Φ, generated by all $x=\{x_i\}_{i=1}^\infty\in\ell_\Phi$, such that $\sum_{i=1}^\infty\Phi_i(\lambda x_i)<\infty$ for every $\lambda>0$.

If the MO function Φ consists of one and the same function M one obtains the Orlicz sequence spaces ℓ_M and h_M.

Let $1\le p_i$, $i\in\mathbb{N}$ be a sequence of reals. The MO sequence space ℓ_Φ, where $\Phi=\{t^{p_i}\}_{i=1}^\infty$ is called Nakano sequence space and is denoted by $\ell_{\{p_i\}}$. In [4] it was proved that two Nakano sequence spaces $\ell_{\{p_i\}}$, $\ell_{\{q_i\}}$ are isomorphic iff there exists $0 < C < 1$ such that

$$\sum_{i=1}^{\infty} C^{1/|p_i - q_i|} < \infty .$$

An extensive study of Orlicz and MO spaces can be found in [11] and [15].

Definition 2.1. *We say that the MO function Φ satisfies the δ_2 condition at zero if there exist constants $K, \beta > 0$ and a non-negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$ such that for every $n \in \mathbb{N}$*

$$\Phi_n(2t) \leq K\Phi_n(t) + c_n$$

provided $t \in [0, \Phi_n^{-1}(\beta)]$.

The spaces ℓ_Φ and h_Φ coincide iff Φ satisfies the δ_2 condition at zero.

Recall that given MO functions Φ and Ψ the spaces ℓ_Φ and ℓ_Ψ coincide with equivalence of norms iff Φ is equivalent to Ψ, i.e. there exist constants $K, \beta > 0$ and a non-negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$, such that for every $n \in \mathbb{N}$ the inequalities

$$\Phi_n(Kt) \leq \Psi_n(t) + c_n \quad \text{and} \quad \Psi_n(Kt) \leq \Phi_n(t) + c_n$$

hold for every $t \in [0, \min(\Phi_n^{-1}(\beta), \Psi_n^{-1}(\beta))]$, [9] and [12].

Throughout this paper M will always denote Orlicz function while Φ - an MO function. As the properties we are dealing with are preserved by isomorphisms without loss of generality we may assume that Φ consists entirely of non–degenerate Orlicz functions, such that for every $i \in \mathbb{N}$ the Orlicz function Φ_i is differentiable, $\Phi_i'(0) = 0$ and $\Phi_i(1) = 1$ [17]

Definition 2.2. *For an Orlicz function M, such that* $\lim_{t\to 0} M(t)/t = 0$ *the function*

$$N(x) = \sup\{t|x| - M(t) : t \geq 0\},$$

is called function complementary to M.

Definition 2.3. *The MO function $\Psi = \{\Psi_j\}_{j=1}^{\infty}$, defined by*

$$\Psi_j(x) = \sup\{t|x| - \Phi_j(t) : t \geq 0\}, j = 1, 2, ..., n, ...$$

is called complementary to Φ.

Let us note that the condition $\lim_{t\to 0} M(t)/t = 0$ secures that the complementary function N is always non-degenerate. Observe that if N is function complementary to M, then M is complementary to N and if the MO function Ψ is complementary to the MO function Φ, then Φ is function complementary to Ψ. Throughout this paper the function complementary to the MO function Φ is denoted by Ψ.

It is well known that $h_M^* \cong \ell_N$ and $h_\Phi^* \cong \ell_\Psi$. Well known equivalent norm in ℓ_Φ is the Orlicz norm $\|x\|_\Phi^O = \sup\left\{\sum_{j=1}^\infty x_j y_j : \sum_{j=1}^\infty \Psi_j(y_j) \leq 1\right\}$, which satisfies the inequalities (see e.g.[10])

$$\|\cdot\|_\Phi \leq \|\cdot\|_\Phi^O \leq 2\|\cdot\|_\Phi .$$

We will use the Hölder's inequality: $\sum_{j=1}^\infty |x_j y_j| \leq \|x\|_\Phi^O \|y\|_\Psi$, which holds for every $x = \{x_j\}_{j=1}^\infty \in \ell_\Phi$ and $y = \{y_j\}_{j=1}^\infty \in \ell_\Psi$, where Φ and Ψ are complementary MO functions.

By $\{e_j\}_{j=1}^\infty$ and $\{e_j^*\}_{j=1}^\infty$ we denote the unit vector basis in h_Φ and h_Ψ respectively. For a Banach space X with a basis $\{v_i\}_{i=1}^\infty$ and element $x \in X$, $x = \sum_{i=1}^\infty x_i v_i$ we define $\mathrm{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$. We write $n \leq x$ if $n \leq \min\{\mathrm{supp} x\}$ and $x < y$ if $\max\{\mathrm{supp} x\} < \min\{\mathrm{supp} y\}$. We say that x is a block vector with respect to the basis $\{v_i\}_{i=1}^\infty$ if $x = \sum_{i=p}^q x_i v_i$ for some finite p and q and we say that x is a normalized block vector if it is a block vector and $\|x\| = 1$.

Definition 2.4. *A Banach space X is said to be stabilized asymptotic ℓ_∞ with respect to a basis $\{v_i\}_{i=1}^\infty$, if there exists a constant $C \geq 1$, such that for every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$, so that whenever $N \leq x_1 < \ldots < x_n$ are successive normalized block vectors, then $\{x_i\}_{i=1}^n$ are C-equivalent to the unit vector basis of ℓ_∞^n, i.e.*

$$\frac{1}{C} \max_{1\leq i\leq n} |a_i| \leq \left\| \sum_{i=1}^n a_i x_i \right\| \leq C \max_{1\leq i\leq n} |a_i|.$$

The following characterization of the stabilized asymptotic ℓ_∞ MO sequence spaces is due to Dew:

Proposition 2.1.*(Proposition 4.5.1 [5]) Let $\Phi = \{\Phi_j\}_{j=1}^\infty$ be a MO function. Then the following are equivalent:*
(i) h_Φ is stabilized asymptotic ℓ_∞ (with respect to its natural basis $\{e_j\}_{j=1}^\infty$);
(ii) there exists $\lambda > 1$ such that for all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $N \leq p \leq q$ and $\sum_{j=p}^q \Phi_j(a_j) \leq 1$, then

$$\sum_{j=p}^q \Phi_j(a_j/\lambda) \leq \frac{1}{n}.$$

An easy sufficient condition for h_Φ to be stabilized asymptotic ℓ_∞ with respect to the unit vector basis is the following

Proposition 2.2. *(Proposition 4.5.3 [5]) Let $\varphi_\lambda(j) = \inf\{\Phi_j(\lambda t)/\Phi_j(t) : t > 0\}$. If $\lim\limits_{j\to\infty} \varphi_\lambda(j) = \infty$ for some $\lambda > 1$ then h_Φ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis.*

Let X be a Banach space. By $Y \hookrightarrow X$ we denote that Y is isomorphic to a subspace of X.

Definition 2.5. *We say that a collection $K \subset h_\Phi$ has equi-absolutely continuous norms if*
for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\sup\{\| \sum_{k=n}^{\infty} x_k e_k \| : x = \{x_k\}_{k=1}^{\infty} \in K\} < \varepsilon$ for every $n \geq N$.

Definition 2.6. *We say that a Banach space $(X, \|\cdot\|)$ is asymptotically isometric to ℓ_1 if it has a normalized basis $\{v_n\}_{n=1}^{\infty}$ such that for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ increasing to 1 we have*

$$\sum_{n=1}^{\infty} \lambda_n |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n v_n \right\| \tag{1}$$

for all $x = \sum_{n=1}^{\infty} t_n v_n \in X$.

Whenever $(X, \|\cdot\|)$ contains a normalized sequence $\{x^{(n)}\}_{n=1}^{\infty}$ satisfying (1) then the closed linear span of $\{x^{(n)}\}_{n=1}^{\infty}$ is asymptotically isometric to ℓ_1

We say that X is saturated with subspaces with the property (*) if in every infinite dimensional subspace Z of X there is an infinite dimensional subspace Y of Z isomorphic to a space with the property (*).

3. WEAKLY COMPACT SETS OF MO SEQUENCE SPACES

Lemma 3.1. *Let Φ be a MO function. which has δ_2 condition at zero and $K \subset h_\Phi$. Suppose that K fails to have equi-absolutely continuous norms. Then there are $\varepsilon_0 > 0$ and sequences $\{x^{(n)}\}_{n=1}^{\infty} \subset K$. $\{p_n, q_n\}_{n=1}^{\infty}$, $p_n, q_n \in \mathbb{N}$. $p_n \leq q_n < p_{n+1}$, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$ such that*

$$\left\| \sum_{i=p_n}^{q_n} x_i^{(n)} e_i \right\| > \varepsilon_0 \tag{1}$$

for every $n \in \mathbb{N}$.

Proof. Since K does not have equi-absolutely continuous norms there are $\varepsilon > 0$, $\{\alpha_n\}_{n\in\mathbb{N}}$, $\alpha_n \in \mathbb{N}$ and $\{z^{(n)}\} \subset K$ such that

$$\left\| \sum_{i=\alpha_n}^{\infty} z_i^{(n)} e_i \right\| > \varepsilon.$$

Let $n_1 = 1$. We choose $n_2 > n_1$ such that

$$\left\| \sum_{i=\alpha_{n_1}}^{\alpha_{n_2}-1} z_i^{(n_1)} e_i \right\| > \varepsilon/2.$$

Put $p_1 = \alpha_{n_1}$, $q_1 = \alpha_{n_2} - 1$, $x^{(1)} = z^{(n_1)}$. We choose $n_3 > n_2$ such that

$$\left\| \sum_{i=\alpha_{n_2}}^{\alpha_{n_3}-1} z_i^{(n_2)} e_i \right\| > \varepsilon/2.$$

Put $p_2 = \alpha_{n_2}$, $q_2 = \alpha_{n_3} - 1$, $x^{(2)} = z^{(n_2)}$.

If we have selected $x^{(1)}, x^{(2)}, \dots, x^{(k)}$ by $x^{(s)} = z^{(n_s)}$, $p_s = \alpha_{n_s}$, $q_s = \alpha_{n_{s+1}} - 1$ for $1 \le s \le k$, then we choose $n_{k+1} > n_k$ such that

$$\left\| \sum_{i=\alpha_{n_{k+1}}}^{\alpha_{n_{k+2}}-1} z_i^{(n_{k+1})} e_i \right\| > \varepsilon/2.$$

Now we put $p_{k+1} = \alpha_{n_{k+1}}$, $q_{k+1} = \alpha_{n_{k+2}} - 1$ $x^{(k+1)} = z^{(n_{k+1})}$.

Obviously the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ verifies (1) with $\varepsilon_0 = \varepsilon/2$. □

Lemma 3.2. *([2]) Let X be a Banach space. Suppose that $\{x_n\} \subset X$ is weakly null and $\{x_n^*\} \subset X^*$ is weakly* null. Then for each $\varepsilon > 0$ there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of positive integers so that for each $k \in \mathbb{N}$ holds:*

$$\sum_{j \neq k} \left| x_{n_j}^*(x_{n_k}) \right| < \varepsilon.$$

Theorem 1. *Let Φ be a MO function, which has δ_2 condition at zero and with a complementary function Ψ such that h_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$. Then any weakly null sequence in ℓ_Φ has equi-absolutely continuous norms.*

Proof. Suppose the contrary. There is a weakly null sequence $\{x^{(n)}\}_{n=1}^{\infty} \subset \ell_\Phi$ that fails to have equi-absolutely continuous norms. By Lemma 3.1 there exist $\varepsilon_0 > 0$ and strongly increasing sequences $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$, $p_n, q_n \in \mathbb{N}$, $p_n \le q_n < p_{n+1}$ such that

$$\left\| \sum_{i=p_n}^{q_n} x_i^{(n)} e_i \right\| > \varepsilon_0.$$

Choose $y^{(n)} \in h_\Psi$ such that $\operatorname{supp} y^{(n)} = \{i\}_{i=p_n}^{q_n}$, $\sum_{k=p_n}^{q_n} \Psi_k(y_k^{(n)}) \le 1$ and $\left| \sum_{k=p_n}^{q_n} y_k^{(n)} x_k^{(n)} \right| > \frac{3}{4}\varepsilon_0$. For a fixed $x \in \ell_\Phi$ by Holder's Inequality:

$$\left| \sum_{k=1}^{\infty} x_k y_k^{(n)} \right| = \left| \sum_{k=p_n}^{q_n} x_k y_k^{(n)} \right| \le \left\| \sum_{k=p_n}^{q_n} x_k e_k \right\|_\Phi \left\| y^{(n)} \right\|_\Psi^O.$$

As x is fixed and $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$ it follows that

$$\lim_{n\to\infty} \left\| \sum_{k=p_n}^{q_n} x_k e_k \right\|_\Phi = 0.$$

Thus $\{y^{(n)}\}_{n=1}^{\infty}$ is weak* null sequence. By Lemma 3.2 there is a subsequence of naturals $\{n_k\}_{k=1}^{\infty}$ so that

$$\sum_{j\neq k}\left|\sum_{i=p_{n_j}}^{q_{n_j}} y_i^{(n_j)} x_i^{(n_k)}\right| < \varepsilon_0/2.$$

We claim that

$$\lim_{k\to\infty}\sum_{j=1}^{\infty}\Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right)=\lim_{k\to\infty}\sum_{j=p_{n_k}}^{q_{n_k}}\Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right)=0, \tag{2}$$

where $\lambda > 1$ is the constant from Proposition 2.1. Indeed, by assumption h_Ψ is a stabilized asymptotic ℓ_∞ space and there exists $\lambda > 1$ such that for every $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ so that whenever $\sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j(y_j^{(n_k)}) \leq 1$ then the inequality $\sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) \leq 1/m$ holds for every $q_{n_k} \geq p_{n_k} \geq N$.

Thus $\lim_{n_k\to\infty}\sum_{j=p_{n_k}}^{q_{n_k}}\Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right)=0.$

Therefore there is subsequence $\{n_{k_m}\}_{m=1}^{\infty}$ such that

$$\sum_{m=1}^{\infty}\sum_{i=p_{n_{k_m}}}^{q_{n_{k_m}}}\Psi_i\left(\frac{y_i^{(n_{k_m})}}{\lambda}\right)\leq 1.$$

Let $y=\sum_{m=1}^{\infty} y^{(n_{k_m})}$. Obviously $y \in h_\Psi$ and since $\{x^{(n)}\}_{n=1}^{\infty}$ is weakly null we must have

$$\lim_{m\to\infty} y(x^{(n_{k_m})}) = \lim_{m\to\infty}\sum_{j=1}^{\infty}\sum_{i=p_{n_{k_j}}}^{q_{n_{k_j}}} y_i^{(n_{k_j})} x_i^{(n_{k_m})} = 0.$$

But

$$\left|\sum_{j=1}^{\infty}\sum_{i=p_{n_{k_j}}}^{q_{n_{k_j}}} y_i^{(n_{k_j})} x_i^{(n_{k_m})}\right| \geq \left|\sum_{i=p_{n_{k_m}}}^{q_{n_{k_m}}} y_i^{(n_{k_m})} x_i^{(n_{k_m})}\right| - \sum_{j\neq m}\left|\sum_{i=p_{n_{k_j}}}^{q_{n_{k_j}}} y_i^{(n_{k_j})} x_i^{(n_{k_m})}\right|$$

$$\geq \frac{3}{4}\varepsilon_0 - \frac{1}{2}\varepsilon_0 = \frac{1}{4}\varepsilon_0,$$

a contradiction. □

Let us recall that C is weakly sequentially compact if every sequence of points in C has a subsequence weakly convergent to a point of C.

For the proof of the next result we need:

Theorem 2. *(Eberlein–Smulian. see e.g. [8]) Let X be a separable Banach space and C be a weakly closed subset of X. Then C is weakly compact if and only if C is weakly sequentially compact.*

By Theorem 1 it follows immediately:

Corollary 3.1. *Let Φ be a MO function which has δ_2 condition at zero and with a complementary function Ψ such that h_Ψ is stabilized asymptotic ℓ_∞ with respect to the basis $\{e_j^*\}_{j=1}^\infty$. Then a bounded set $K \subset \ell_\Phi$ is relatively weakly compact iff K has equi–absolutely continuous norm.*

Proof. Necessity. Suppose that $K \subset h_\Phi$ is relatively weakly compact. If K fails to have equi–absolutely continuous norms then by Lemma 3.1 there are $\varepsilon_0 > 0$ and sequences $\{x^{(n)}\}_{n=1}^\infty \subset K$, $\{p_n, q_n\}_{n=1}^\infty$, $p_n, q_n \in \mathbb{N}$, $p_n \le q_n < p_{n+1}$ such that

$$\left\|\sum_{i=p_n}^{q_n} x_i^{(n)} e_i\right\| > \varepsilon_0$$

for every $n \in \mathbb{N}$.

By Eberlein–Smulian theorem there are $x \in \ell_\Phi$ and a subsequence $\{x^{(n_k)}\}_{n=1}^\infty$ such that $x^{(n_k)} \longrightarrow x$ weakly in ℓ_Φ. Thus by Theorem 1 $\{x^{(n_k)} - x\}_{k=1}^\infty$ has equi–absolutely continuous norms. Hence $\lim_{k\to\infty} \left\|\sum_{i=p_{n_k}}^{q_{n_k}} (x_i^{(n_k)} - x_i)e_i\right\| = 0$ and obviously $\lim_{k\to\infty} \left\|\sum_{i=p_{n_k}}^{q_{n_k}} x_i e_i\right\| = 0$. But

$$\varepsilon_0 < \left\|\sum_{i=p_{n_k}}^{q_{n_k}} x_i^{(n_k)} e_i\right\| \le \left\|\sum_{i=p_{n_k}}^{q_{n_k}} x_i e_i\right\| + \left\|\sum_{i=p_{n_k}}^{q_{n_k}} (x_i^{(n_k)} - x_i)e_i\right\| \underset{k\to\infty}{\longrightarrow} 0,$$

which is a contradiction.

Sufficiency. Let K be a bounded set with equi–absolutely continuous norms. Let $\{x^{(n)}\}_{n=1}^\infty$ be an arbitrary sequence of elements in K. Obviously there exists L such that $|x_k^{(n)}| \le L$ for every $n, k \in \mathbb{N}$. Thus there exists a subsequence $\{x^{(n_i)}\}_{i=1}^\infty$ such that $\lim_{i\to\infty} x_k^{(n_i)} = x_k$ for every $k \in \mathbb{N}$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for every $s \ge N$ and every $i \in \mathbb{N}$ the inequality holds $\left\|\sum_{k=s}^\infty x_k^{(n_i)} e_k\right\| < \varepsilon/3$. Fix $s \ge N$. There is $M \in \mathbb{N}$ such that for every $n_i, n_j \ge M$ and every $k = 1, 2, \ldots, s$ the inequality $|x_k^{(n_i)} - x_k^{(n_j)}| \le \frac{\varepsilon}{3s}$ holds. Thus we can write the inequalities:

$$\|x^{(n_i)} - x^{(n_j)}\| = \left\|\sum_{k=1}^\infty x_k^{(n_i)} e_k - \sum_{k=1}^\infty x_k^{(n_j)} e_k\right\|$$

$$\le \left\|\sum_{k=1}^s x_k^{(n_i)} e_k - \sum_{k=1}^s x_k^{(n_j)} e_k\right\| + \left\|\sum_{k=s+1}^\infty x_k^{(n_i)} e_k - \sum_{k=s+1}^\infty x_k^{(n_j)} e_k\right\|$$

$$\le \left\|\sum_{k=1}^s |x_k^{(n_i)} - x_k^{(n_j)}| e_k\right\| + \left\|\sum_{k=s+1}^\infty x_k^{(n_i)} e_k\right\| + \left\|\sum_{k=s+1}^\infty x_k^{(n_j)} e_k\right\|$$

$$< \frac{\varepsilon}{3s}s + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Consequently $\{x^{(n_i)}\}_{n=1}^{\infty}$ is a Cauchy sequence and thus it is norm convergent to $x \in \ell_\Phi$ and thus it is weakly convergent. □

Remark. Let us mention that for the proof of the sufficiency in Corollary 3.1 we do not need that ℓ_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$.

4. FIXED POINT PROPERTY FOR MO SEQUENCE SPACES

The next Lemma is similar to that in [17], where it is shown that for every normalized block basis $\{x^{(n)}\}_{n=1}^{\infty}$ of the unit vector basis $\{e_j\}_{j=1}^{\infty}$ in ℓ_Φ contains a subsequence such that $[x^{(n_i)}]_{i=1}^{\infty}$ is isomorphic to ℓ_1.

Lemma 4.1. *Let Φ be a MO function, which has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$. Then every normalized block basis $\{x^{(n)}\}_{n=1}^{\infty}$ of the unit vector basis $\{e_j\}_{j=1}^{\infty}$ in ℓ_Φ contains a subsequence $\{x^{(n_i)}\}_{i=1}^{\infty}$ such that $[x^{(n_i)}]_{i=1}^{\infty}$ is asymptotically isometric to ℓ_1.*

Proof. Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a normalized block basis of the unit vector basis $\{e_j\}_{j=1}^{\infty}$ in ℓ_Φ, where $x^{(n)} = \sum_{j=m_n+1}^{m_{n+1}} x_j^{(n)} e_j$, $\{m_n\}_{n=1}^{\infty}$ strictly increasing sequence of naturals. Let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence, such that $\lim_{n\to\infty} \lambda_n = 1$. For every $n \in \mathbb{N}$ there exists $y^{(n)} = \sum_{j=1}^{\infty} y_j^{(n)} e_j^* \in h_\Psi$ such that

$$\sum_{j=1}^{\infty} \Psi_j(y_j^{(n)}) \le 1 \quad \sum_{j=1}^{\infty} y_j^{(n)} x_j^{(n)} \ge \lambda_n.$$

WLOG we may assume that supp $y^{(n)} \equiv$ supp $x^{(n)}$.

For the sequence $\{y^{(n)}\}_{n=1}^{\infty}$ and the constant $\lambda > 1$ from Proposition 2.1 holds:

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \Psi_j\left(\frac{y_j^{(n)}}{\lambda}\right) = \lim_{n\to\infty} \sum_{j=m_n+1}^{m_{n+1}} \Psi_j\left(\frac{y_j^{(n)}}{\lambda}\right) = 0.$$

The proof is essentially the same as for (2).

Now passing to a subsequence we get a sequence

$$\{y^{(n_k)}\}_{k\in\mathbb{N}},\ y^{(n_k)} = \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^*$$

such that

$$\sum_{k=1}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} \Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) \le 1.$$

Denote $y = \sum_{k=1}^{\infty} y^{(n_k)} = \sum_{k=1}^{\infty} \left(\sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right)$. Obviously $y \in \ell_\Psi$ and $\|y\|_\Psi \leq \lambda$. As

$$\lim_{s \to \infty} \left\| \sum_{k=s}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi = 0$$

there exists $s_0 \in \mathbb{N}$ such that

$$\left\| \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi \leq \frac{1}{2}.$$

Consequently

$$\left\| \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi^O \leq 2 \left\| \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi \leq 1.$$

Denote $\overline{y} = \sum_{k=s_0}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} e_j^*$. Then $\|\overline{y}\|_\Psi^O \leq 1$. Now using Hölder's inequality for any sequence $\{t_n\}_{n=1}^{\infty}$, such that $\sum_{k=s_0}^{\infty} t_{k-s_0+1} x^{(n_k)} \in \ell_\Phi$ we get

$$\begin{aligned} \left\| \sum_{k=s_0}^{\infty} t_{k-s_0+1} x^{(n_k)} \right\|_\Phi &\geq \frac{1}{\|\overline{y}\|_\Psi^O} \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} |t_{k-s_0+1} y_j^{(n_k)} x_j^{(n_k)}| \\ &\geq \sum_{k=s_0}^{\infty} |t_{k-s_0+1}| \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} x_j^{(n_k)} \geq \sum_{k=s_0}^{\infty} |t_{k-s_0+1}| \lambda_k. \end{aligned}$$

□

Theorem 3. *Let Φ be a MO function, which satisfies the δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$. Then ℓ_Φ is saturated with asymptotically isometric copies of ℓ_1.*

Proof. According to a well known result of Bessaga and Pelczinski [3] every infinite dimensional closed subspace Y of ℓ_Φ has a subspace Z isomorphic to a subspace of ℓ_Φ, generated by a normalized block basis of the unit vector basis of ℓ_Φ. Now to finish the proof it is enough to observe that by Lemma 4.1 the space Z contains an asymptotically isometric copy of ℓ_1. □

By using a result from [7] that states that a Banach spaces containing an asymptotically isometric copy of ℓ_1 fail the fixed point property for closed, bounded, convex sets and non–expansive (contractive) maps on them, we easily get

Corollary 4.1. *Let Φ be a MO function, which has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then ℓ_Φ fails the fixed point property (fpp) for closed, bounded, convex sets in ℓ_Φ and non-expansive (or contractive) maps on them.*

We give at the end some examples of MO sequence space, saturated with asymptotically isometric copies of ℓ_1.

Example 1.([17]) Sometimes we know only the complementary function Ψ. For example let the MO function $\Psi = \{\Psi_j\}_{j=1}^\infty$ be defined by $\Psi_j = e^{\alpha_j} e^{-\frac{\alpha_j}{|x|^{c_j}}}$, where $\lim_{j\to\infty} \alpha_j = \infty$ and $0 < c_j$. Then ℓ_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$ because

$$\lim_{j\to\infty} \inf \left\{ \frac{\Psi_j(2x)}{\Psi_j(x)} : 0 \le x \le 1 \right\}$$

$$= \lim_{j\to\infty} \inf \left\{ e^{\alpha_j \frac{2^{c_j}-1}{2^{c_j}|x|^{c_j}}} : 0 \le x \le 1 \right\} = \lim_{j\to\infty} e^{\alpha_j \frac{2^{c_j}-1}{2^{c_j}}} = \infty.$$

Thus we conclude that ℓ_Φ is saturated with asymptotically isometric copies of ℓ_1 and fails fpp for closed, bounded, convex sets in ℓ_Φ and non-expansive (or contractive) maps on them.

Example 2.([5]) Consider the Nakano sequence space $\ell_{\{p_n\}}$, where $p_n = \dfrac{\log_2(n+1)}{\log_2\left(\dfrac{n+1}{2}\right)}$. It is well known that $\ell^*_{\{p_n\}} \cong \ell_{\{q_n\}}$, where $1/p_n + 1/q_n = 1$, i.e. $q_n = \log_2(n+1)$. It is easy to see that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} \dfrac{\log_2(n+1)}{\log_2\left(\dfrac{n+1}{2}\right)} = 1$ and thus according to [4] and [12] $\ell_{\{p_n\}}$ is saturated with spaces isomorphic to ℓ_1. Moreover according to [5] $\ell_{\{q_n\}}$ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$ and thus $\ell_{\{p_n\}}$ is saturated with asymptotically isometric copies of ℓ_1 and fails fpp for closed, bounded, convex sets in ℓ_Φ and non-expansive (or contractive) maps on them.

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