ГОДИШНИК

НА

СОФИЙСКИЯ УНИВЕРСИТЕТ
„СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ
ПО МАТЕМАТИКА И ИНФОРМАТИКА

Том 102
2015

ANNUAL

OF

SOFIA UNIVERSITY
“ST. KLIMENT OHRIDSKI”

FACULTY OF MATHEMATICS AND INFORMATICS

Volume 102
2015

СОФИЯ • 2015 • SOFIA
УНИВЕРСИТЕТСКО ИЗДАТЕЛСТВО „СВ. КЛИМЕНТ ОХРИДСКИ“
“ST. KLIMENT OHRIDSKI” UNIVERSITY PRESS

Annual of Sofia University "St. Kliment Ohridski"
Faculty of Mathematics and Informatics

Годишник на Софийския университет „Св. Климент Охридски"

Факултет по математика и информатика

Address for correspondence:

Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5, Blvd J. Bourchier, P.O. Box 48
BG-1164 Sofia, Bulgaria

Fax xx(359 2) 8687 180
Electronic mail:
annuaire@fmi.uni-sofia.bg

Aims and Scope. The *Annual* is the oldest Bulgarian journal, founded in 1904, devoted to pure and applied mathematics, mechanics and computer science. It is reviewed by *Zentralblatt für Mathematik*, *Mathematical Reviews* and the Russian *Referativnii Jurnal*. The *Annual* publishes significant and original research papers of authors both from Bulgaria and abroad in some selected areas that comply with the traditional scientific interests of the Faculty of Mathematics and Informatics at the "St. Kliment Ohridski" University of Sofia, i.e., algebra, geometry and topology, analysis, mathematical logic, theory of approximations, numerical methods, computer sciences, classical, fluid and solid mechanics, and their fundamental applications.

ISSN 0205–0808

CONTENTS

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

RESEARCH AND EDUCATION IN STOCHASTICS AT THE FACULTY OF MATHEMATICS AND INFORMATICS OF SOFIA UNIVERSITY "ST. KLIMENT OHRIDSKI"

MAROUSSIA N. SLAVTCHOVA–BOJKOVA, PLAMEN S. MATEEV

The development of the research interests and topics in Stochastic education, its place in the curriculum and the learning process are followed in parallel, from the founding of the Higher School, the predecessor of Sofia University, up to the "century of stochasticity".

Keywords:. Stochastics, mathematical statistics, education, history of Sofia University

2000 Math. Subject Classification: Primary 01A73, 01A60, Secondary 60-03, 62-03

1. INTRODUCTION

"The theory of probabilities is at bottom only common sense reduced to calculus; it make us appreciate with exactitude that which exact minds feel by a sort of instinct without being able ofttimes to give a reason for it."

Pierre-Simon, Marquis de Laplace [1]

[1]from the sixth French edition of A Philosophical Essay on Probabilities by Pierre-Simon, Marquis de Laplace translated by Frederick Wilson Truscott and Frederick Lincoln Emory, 1902, N.Y., (`http://books.google.com/laplace_A_philosophical_essay_on_probabilities.pdf`

This article is dedicated to the International year of Statistics (2013) and the 125-th anniversary of founding the Higher School, now Sofia University "St. Kliment Ohridski" [2]. Why exactly 2013, the thirteenth year of the "century of stochasticity" (see [16]) is chosen for the celebration of Statistics around the world, and is it chosen randomly? We can find explanation in some round and celebrated anniversaries. Undoubtedly, the most interesting anniversary, which can be regarded as a reason for the choice of the year 2013 is, that 300 years ago the famous book "Ars Conjectandi" [5] of Jacob Bernoulli some time after his death (August 1705) was published by his nephew Nicolas. In Chapter 2 it is said:

> "Regarding that which is certainly known and beyond doubt, we say that we *know* or *understand* [it]; concerning all the rest, — we only *conjecture* or *opine*.
>
> To make *conjectures* about something is the same as to measure its probability. Therefore, the art of *conjecturing* or *stochastics* {*ars conjectandi* sive *stochastice*} is defined as the art of measuring the probability of things as exactly as possible, to be able always to choose what will be found the best, the more satisfactory, serene and reasonable for our judgements and actions. This alone supports all the wisdom of the philosopher and the prudence of the politician." [3]

In that book a proof of the theorem with great cognitive significance, now known as the "Law of Large Numbers" is given. Its core is that the observations or data of human experience can approximate (unobservable) model of the investigated object. Particularly, it discovers that empirical distribution is a good approximation of a theoretical model under consideration, and it is shown that the probability of that increases if the number of observations increases and tends to certainty (probability equals to 1). This is the reason that the work of J. Bernoulli launches a new area of Mathematics – Probability theory and Mathematical Statistics or, with one word, "Stochastics".

It is worth noting that the year 2013 marks 250 years since the publication of another remarkable issue for Probabilities, namely the famous essay of Thomas Bayes [4], whose ideas are recently becoming more and more popular.

Let us recall that "mathematics" is a Greek word with meaning "learning, knowledge, science". The derivative word "polymath" is rarely used, but can be found in dictionaries, it means "a person who has studied and knows very much, with encyclopedic knowledge" (similar and with almost the same meaning is "polyhistor", also synonymous, but with some nuance, more widely used of the same type

[2]A variant of this text was published in Bulgarian in the on-line journal [32], `http://probablystatistics.net`

[3]Translated into English by Oscar Sheynin, Berlin 2005, ISBN 3-938417-14-5 (`http://www.sheynin.de/download/bernoulli.pdf`).

is "polyglot"). The uneducated barbarians did not have this concept and adopted the foreign word "mathematics" in their language. The only exception, according to V. I. Arnold [2], is the Dutch scientist Stevin, who managed to keep the Flemish word "wiskunde". Currently, there is no universally accepted definition, but it can be assumed that the science of abstract (imaginary or virtual) models including variables, quantities, relationships, spatial forms is Mathematics.

In Bulgarian language until recently (50-60 years ago) "smyatane" (Bulgarian "смятане"), i.e. "calculus" was a subject in elementary school (along with Arithmetics and Geometry), and the same, but further justified by adjectives such as "differential", "integral", "variational" is studied in higher education. In the meaning of this beautiful word in Bulgarian there is "thinking", and "calculating" and "guessing" and is much more suitable to the lessons in school related to quantities and quantitative variables.

Going further back to Bulgarian tradition, the education starts with "четмо" and "писмо", i.e. numeracy and literacy (although nowadays "четмо и писмо" is translating as reading and writing). In fact, the word "cheta" (Bulg."чета"), in translation "read", in Bulgarian besides its commonly used meaning, makes sense (now defined as outdated and even dialect form) of "count", where its derived words with the same root come from: "cheten" – "necheten" (Bulg. "четен" – "нечетен"), in translation — "even" – "odd", "cheta" – "chetnik" (Bulg. "чета" – "четник"), in translation – "band" and "member of the band", "razchet" (Bulg. "разчет"), in translation — "(cannon) estimate" and others.

Counting and data collection for population, soldiers and taxes are present in the Bible (the second chapter "Numbers") and in old Indian texts there are instructions on how to count the population. Recently, it was announced a discovery of a publication from IX$^{\text{th}}$ century, the Cyril and Methodius' time, as the first publication considering frequencies in the data (see [1]). The author Al - Kindi (801 – 873) discussed the possibility of breaking the cryptic messages based on analysis of the frequency distribution of the used symbols. He is an Arab philosopher to whom European civilization owes the numbers and decimal notation, brought from India [14].

The word "Statistics" is relatively new. This is the title of Gottfried Achenwall lectures (in 1749, Göttingen). The root of the word comes from the Latin "stat" (country, state, position) and the suffix for scholarly subject is adopted from "Mathematics". Until recently, the mid twentieth century, Statistics was considered as one of the social sciences, but now in all standards of international studies in classifications of occupations (ISCO) and in classifications in the areas of education (ISCED) we find the inseparable tandem "Mathematics and Statistics" (these were harmonized by the National Statistical Institute, Bulgaria, in 2011 and 2008). The word "Statistics" established itself, because it was widely used by English speakers as a synonym for "Stochastics", bearing at the same time ambiguity like in "recording data" (keep statistics), "data collection" (accumulated statistics), properties of the data distribution (averages and other descriptive statistics). In any case, the conclusions that follow are based on models of Probability theory or Stochastics.

From the very beginning of the founding of the Higher School, the predecessor of Sofia University, Stochastics has had its own place in the curriculum and the learning process. The "probabilistic revolution" observed in the beginning of the twentieth century [16] has had its role in university life as well, dated from that time up today's "century of Stochastics" as called by David Mumford [20]. We will follow in parallel the fields of research and the topics in Stochastic education as a continuation of publications [30] and [31].

1.2. FOR STOCHASTICS AND THE SOFIA UNIVERSITY

We will follow the development of education and the research at Physico-Mathematical Faculty (PhMF), now Faculty of Mathematics and Informatics at Sofia University "St. Kliment Ohridski" (FMI at SU) in the field of Stochastics structured in the following order:

- 1889 – 1945, The Beginning;
- 1945 – 1971, After the World War II;
- 1971 – 1988, Integration;
- 1988 – 2004, Disintegration;
- 2004 – 2013, Recent years.

Many available sources and articles like [10], [11] and [38], dedicated to anniversaries of the Sofia University, have been used. Of course, the task of presenting a detailed history of Stochastics in Bulgaria can not be done comprehensively in a single article. The solution is to create an accessible digital repository — a project, whose implementation requires the efforts of the whole Stochastic guild. The authors will be very grateful to everyone who can provide adequate information in this direction.

2. THE BEGINNING (1889 – 1945)

2.1. THE FIRST ONES

Soon after the founding in 1888 of the Higher School in Sofia, with a single department "History and Philology", in 1889 was formed the Department of "Physics and Mathematics" with a meaning of "natural sciences", which later becomes a faculty. The students who finished their education obtained qualification as "natural scientists", "chemists", "physicists" and "mathematicians". The last two were actually combined in one. At that time (see [23]), Mathematics, "the science of quantities" was divided into two parts: "complex" or "physical mathematics", where the variables are related to experiments and observations, and "pure", which includes Arithmetics, Geometry, Algebra, where the variables are studied "abstracted from the perceptions". This duality is perhaps the reason for the confusion of the university clerk in preparing the curriculum for the subjects Mathematics and Physics. In any case, "Probability theory and the method of least squares" is one of the "main" subjects in the curriculum (there were also "supporting" subjects).

Intuitively it is clear that "probability" is a concept of gambling and betting. However, it is also included in the so-called "Political Arithmetics", i.e. in the actuarial accounts in determining the value of the insurance, life annuities and similar quantities related to life expectancy. On the other hand, closer to the physical nature of things are the probabilities of errors in measurements and monitoring, the probabilities in processing and analysis of data from experiments and astronomical observations.

Information of the first university course in "Probability theory" can be found in the "Schedules of lectures" in 1896 [29]. Lecturer was Prof. Atanas Tinterov. We can judge about its contents by the textbook of the teacher Ch. D. Baltadziev [3] prepared for his students in seventh grade of the State High School "Alexander I" in the city of Plovdiv. In 59 pages handwritten calligraphic text 38 paragraphs are presented. We will list a few of the nine paragraphs for dividing the content:

- "Probability of simple events"
- "Principle of the complex event"
- "Probability calculated from observations"
- "On the use of Probability"
- "On mathematical hope"
- "On the games in general"
- "On life insurance"
- "Income protection insurance".

The questions in the last paragraph are solved using "The mortality tables" from the "A. V. Shourekov's logarithmic tables". It becomes clear from the title on the first page that "Probability theory" is a chapter of the subject "Algebra". The same is confirmed in first Bulgarian university textbook. It is actually volume two of N. Obreshkov's "Higher Algebra" [21] subtitled "A theory of algebraic numbers. Combinatorics. Probability theory and applications in Statistics", which lists the four parts of the book. They are distributed approximately as follows: 100 pages for "Algebraic numbers", 60 pages for "Combinatorics", 240 pages for "Probability theory" and 80 pages for "Applications of Probability theory in Statistics". The last three parts became the base of his textbook "Probability theory", which has two editions published after World War II and was the main textbook on Probability for generations of Bulgarian mathematicians.

In the library of Mathematical Institute (sometimes also called "the office") many books, thoroughly described in a special catalog by A. Shourek (1911) [37], were stored. Under the theme "Theory of Probability" he has classified 19 titles. Among them are "Analytical probability theory" of Laplace (1820, 3rd edition), the relatively modern textbook of Bertrand (1889), textbooks on Probabilities of Borel (1909) and Markov (1908) and others in French, German, Russian and one in Serbian.

Apart from them, five other titles on Mathematical Statistics, assurance and insurance are included in section "Application of Probability Theory". Statistics during that time was only "the science of the state". In Decree No. 712 from 1881 by Alexander I [25] is declared:

> The statistical division of the Ministry of Education to be raised to a separate "Bureau of Statistics". Its purpose is "to collect, process and publish annually statistics of all the branches of the state management and all the phenomena related to the physical, economical, intellectual and moral conditions of the country.

The first director of the Bureau is Mihail Sarafov [19]. He was born in Tarnovo in 1854, worked as a teacher in Tarnovo and was a participant in the April Uprising 1878. He graduated from the Faculty of Mathematics of the Polytechnic in Munich in 1880, and for the period from 1880 to 1881 he was a Minister of National Education in Karavelov's Cabinet. He discussed with Prof. K. Irechek the founding of a Higher School, but a priority became the first Census in the Principality of Bulgaria in the same 1881. Another prominent director of the statistical institution of Bulgaria (the first quarter of the twentieth century) is Kiril Georgiev Popov, who sometimes is mistaken with the famous mathematician, professor at Sofia University, Acad. Kiril Atanasov Popov. There is not much biographical information about him (K. G. P.), it is known that he was a member–founder of the Bulgarian Physico–Mathematical Society in Sofia (1898) [15]. From a brief biographical resume we learn that he was born on December 25, 1869 (old style) in the town of Varna and graduated from the Higher School in Sofia in 1895.

The Faculty of Law of the Sofia University was the third one to open in 1892 with two departments — Juristic and Economic. It functioned similarly and corresponded to the current "business schools" at university level. Statistics was set to be a "legal and public science". In the curriculum we find the first lecturer of Statistics – Assoc. Prof. Bonue Boneff. Later on he moved to the insurance business as Head of Mathematics department in the insurance company "Balkan". In 1920 he prepared and published the first issue of the "Mortality Table in Bulgaria"

[6] subsidized by seven insurance companies of that time. It is worth noting that, in addition to the table as a final result in the book, a fundamental information of the Probability theory and the algorithm used for smoothing of the Curve of Mortality were present. In modern terms, the parameters of the Gompertz – Makeham distribution were estimated and as a local approximation for over 72 years of age a cubic spline – regression was used.

The first textbook for the students of the Faculty of Juristic and Economic Department of Sofia University, for their course in Statistics, "Theory of Statistics" [8], was published in 1931 by Prof. G. Danailov. The textbook contains extensive presentation of the history of discoveries and development of the theory of probability and statistics and the author repeatedly expresses his regrets for the gaps in the explanations due to his lack of enough mathematical knowledge. The book ends with a presentation of the data with descriptive statistics and charts. According to E. Shkodrov and St. Tzvetkov [36], the first textbook in Bulgarian is "A Course of Theory of Statistics" issued in Varna in 1923 by the Russian immigrant N. V. Dolinskiy.

At that time (in the twenties) three economics institutes — Free University of Sofia, The Academy of Economics in the town of Svishtov and the Economics Institute in the town of Varna were founded. So the places to study Business science at university level become four, and Statistics took the place it deserves in each of them.

2.2. 20TH CENTURY – THE UNIVERSITY

The Higher School was raised to a rank of Sofia University with a Royal Decree in 1904. Shortly afterwards came the jeers on Ferdinand and "the university crisis" in 1907. Then the two Balkan wars and the First World War happened. A change of generations at the Mathematical Institute (the union of the four mathematical departments) occurred. New lecturers — Kiril Popov, Ivan Tzenov and Lyubomir Chakalov were recruited (in 1914). They were joined (in 1920) by Dimitar Tabakov, an assistant of Prof. Shourek before the University crisis, and the speedy graduated after the wars Nikola Obreshkov. In 1928 Obreshkov was already a full professor and a head of a department, renamed the same year to "Higher Algebra and Theory of Probabilities" [10].

In the beginning of 20-th century the challenges to Mathematics for the new century were formulated by D. Hilbert, known now as "The 24 problems of Hilbert". The sixth of them is:

> *6. Mathematical treatment of the axioms of physics*
>
> The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sci-

ences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics. [4]

Nowdays, A. N. Kolmogorov's Axiomatics, first published in 1929 in the Reports of the Academy of Sciences, is accepted as a standard. The Axiomatics earned popularity after its extended version was published in German in 1933 and translated later (1936) in Russian. Thus, the Sixth Hilbert Problem has been partially solved.

In the Jubilee book of Bulgarian Physico-Mathematical Society [15] two out of about twenty mathematical articles are dedicated to Probabilities. They give us an impression about the views and the atmosphere of the scientific research at the climax of the "Probabilistic revolution". The first one is entitled "Evolution in Probability theory" (A. Ivanov). It lists the mathematicians, whose names are "written" in "the development of the Theory of Probability", starting from Pascal and Fermat, Huygens, Jacob Bernoulli and later Laplace, Euler, Poisson, etc. But "in the beginning of this (twentieth) century Mathematics has entered into a stage in which dominates the randomness": and thanks to the hard and successful work of Poincare and others the Probability theory "was established as a separate science and emerged to become a science of the sciences". In the second article in [15] entitled "On the development of the concept of probability", the author R. Zaykov identifies three trends in understanding the concept of probability as "statisticians", "collectivists" and "experimentalists". Representatives of the first one are Karl Pearson, Ronald A. Fisher, Olaf Anderson; according to them, the probabilities are determined by "the statistical distribution". The "Collectivists" – from Richard von Mises to Abraham Wald, define a distribution on unlimited sequences (collectives). The "Experimentalists" (Jerzy Neyman) link the probability distribution to the set of experiments. With the results of the school of "Axiomatists", which replace the "phenomenological" with the "formal – mathematical concept" and the system of axioms of A. N. Kolmogrov, which "fully meets all logical requirements" and its axioms "could be proven by induction from the empirical reality" the Probability theory becomes complete.

The Probabilistic revolution was in its full swing and everybody was hopeful for a successful development of the theory and implementation of diverse and fruitful applications.

During that time the University lecturers have had strong links with the research centers in Europe. The generation of Lyubomir Chakalov, Kiril Popov, Dimitar Tabakov and Ivan Tzenov obtained specialization in European research centers such as Göttingen, Sorbonne, Nice. They maintained their contacts giving talks and presenting reports on their visits to European universities and at international congresses. Following the advice "Our most important task - said to me Einstein - is to find our successors" (from "The Autobiography of K. Popov" [26,

[4] www.ams.org/journals/bull/2000-37-04/S0273-0979-00-00881-8 /S0273-0979-00-00881 -8.pdf

p. 127]), they paved the way for the next generation led by N. Obreshkov. These are Arkady Stoyanov, Georgi Bradistilov, Blagovest Dolapchiev and Boyan Petkanchin and the younger Lyubomir Iliev, Alipi Mateev and Jaroslaw Tagamlitzki, the new generation at the Mathematical Institute of PhMF.

In 1942 the Higher Technical School in Sofia was founded. The professors of PhMF became members of its Science executive board and lecturers. In addition, G. Bradistilov and A. Stoyanov moved officially to the new academic body.

Below we list, with no claims for completeness, several publications from this period associated with Probability and Statistics.

K. Popov (according to [26]) published series of articles on the generalization of the concept of derivative in terms of Probability Theory. During his trip to Harvard University for participating in the International Organizing Committee of the Congress of Applied Mechanics, he presented a popular lecture on the principles of insurance for a Bulgarian audience in New York.

N. Obreshkov published two articles in the Proceedings of the Seminar on Mathematical Statistics at the Sorbonne. In the first one a two-dimensional distribution with Poisson marginal distributions was shown, later quoted in Alfred Renyi's book [28] as "Obreshkov Distribution".

In two consecutive issues of the Journal of Physico–Mathematical Society P. Shapkarev presented a research and modelling of time series under the heading "Decomposition of business series in time" [35]. A. Stoyanov is the author of one of publications for the role of Actuarial Mathematics in college education [33].

After breaking off his specialization in France because of the war, A. Mateev was appointed temporarily at the Central Meteorological Institute and published two articles in the collected volumes of the Institute. The first one is an overview entitled "On some methods of Mathematical Statistics for processing results from observations" [18]. In this article we see descriptive statistics, graphical representation, indicators for correlation. The second one [17] illustrates the approximation of an empirical distribution with density of the class of "Pearson curves". Data are from the minimum monthly temperatures in January for the years from 1891 to 1920.

In that period we have to mention the work of the famous Russian–German mathematician at Sofia University Oscar Anderson. He emigrated from Moscow in 1920 despite his leftist beliefs and the offer to work on the planning of Russian economy. On one hand side, he did not feel ready for such a career, and on the other side, he could not accept the attitude of the authorities towards his colleagues at the University. He firstly worked as a teacher in Hungary. From 1924 to 1933 he was a Professor at the Institute of Economics in the town of Varna. After a stay in England and Germany as a Rockefeller Fellow, he returned to Bulgaria at the end of 1934. He was appointed as Professor at the Sofia University and organized and managed the "Statistical Institute for Economics Research" at the University. As it is known, Anderson's Institute developed strong research and publication activity. In 1940 O. Anderson was seconded to Germany, where after two years of stay he accepted position Professor of Statistics at the University of Kiel, and later on in Munich, where he stayed until his death in 1960. More biographical details and information about his research interests and publications can be found in [27, 22].

3. AFTER THE WORLD WAR II (1945 – 1971)

The end of the war, the end of the bombing of Sofia and the evacuation brought hope for development of mathematical science, in which probabilities undoubtedly take prominent position with prospects for development and applications.

During 1945 the founding of a Mathematical Institute of Bulgarian Academy of Sciences (BAS) has been discussed in the Executive board of the Bulgarian Academy of Sciences. During 1946 and 1947 Academicians L. Chakalov, N. Obreshkov and K. Popov organized two committees of Statistics and Demography at BAS. In 1947 their plan succeeded. The date 27^{th} of October, 1947 was rightly named as the birthday of the Mathematical Institute (MI), now Institute of Mathematics and Informatics (IMI) [7]. On that day the Executive board of BAS approved a plan for scientific research and development for the period 1947 – 1948. This plan included work of three committees in the field of mathematical sciences:

1. Committee for demographic studies (chaired by Acad. K. Popov);
2. Committee for mathematical studies of the representative method in Statistics (chaired by Acad. N. Obreshkov);
3. Committee for financial mathematical study of state and government bonds (chaired by Acad. K. Popov),

as well as individual detailed plans of the Academicians–mathematicians I. Tzenov, L. Chakalov, N. Obreshkov and K. Popov.

At that time, with the newly approved Law on Higher Education (1947) the structure of the departments at Sofia University “St. Kliment Ohridski” was changed. The abbreviation PhMF is translated into Bulgarian language as “Faculty of Natural Sciences”. A new Department of “Mathematical Statistics and Insurance Mathematics” for applications of the Probability theory was founded. Professor Obreshkov was Head of both departments - the Department “Higher Algebra and Probability Theory” and the new one.

At the same time, Faculty of Medicine was separated from Sofia University and became Medical Academy. Moreover, Department of Economics of the Law Faculty and Institute of Statistics, together with the Free University form the Higher Institute of Economics. The number of the faculties in the Higher Technical School increased to eight and as a result HTS split into four engineering institutes — Civil-Engineering, Mechanical-Electrical Engineering, Chemical Technology and Mining – geological institutes.

Let us recall that from PhMF as a Faculty of Natural Sciences first, in 1918, Faculty of Medicine and later on, in 1921, the Agricultural and the Veterinary Faculties were separated.

Later on, one by one, Faculty of Biology, Geology and Geography (1951) and Faculty of Chemistry (1962) were separated from PhMF. It kept its original name PhMF, but not for long - in 1963 it was split into Faculty of Physics and Faculty of Mathematics.

In 1950, the "Scientific-production profile" was introduced in PhMF as a separate course of study in Mathematics. It was aimed to counteract to some opinions that the only courses at the University are pedagogically oriented, even with suggestions to be renamed to Higher Pedagogical Institute [5]. The outstanding students from the new production profile graduated by defending Diploma Theses, a significant deal of which on topics from the field of Stochastics.

Meanwhile, on June 27, 1951, the Executive Board of BAS approved the first Scientific Council of the of Mathematical Institute (MI) chaired by Acad. Obreshkov, with secretary Prof. B. Petkanchin and members Acad. L. Chakalov, Acad. K. Popov, Acad. I. Tzenov, Prof. Lyubomir Iliev, Prof. Yaroslav Tagamlitzki, Prof. Georgi Bradistilov, Prof. Arkady Stoyanov, Assoc. Prof. Alipi Mateev.

A section "Probability and Statistics" was founded at the MI in 1954. The section was headed by Prof. N. Obreshkov, Bojan Penkov was appointed as a junior researcher and Apostol Obretenov (a graduate of Acad. N. Obreshkov) was on PhD studentship. Later on, mathematicians Emanuel Simeonov and Margarita Andreeva were appointed in the section. In 1962 two more researchers were appointed – Liliana Boneva and Ivan Mirazchiyski.

In the early fifties professors D. Tabakov, K. Popov, I. Tzenov and L. Chakalov retired [6]. From the next generation the most significant steps to the prosperity of the Mathematical society are due to Prof. L. Iliev, who foresaw the emerging of Informatics as a branch of Mathematics. A specialization in "Computational Mathematics" was created on his initiative and organization in 1959–60, and the first five specialists graduated in 1961. The enrollment of students for the "production profile" also started. The first Computer center attached to the Mathematical Institute of BAS and the Department of Higher analysis of PhMF were founded in the same year (1961).

On June 6, 1960, in the Great Hall of the BAS "Extended meeting of the State Council for Science with main topic on the agenda - the development of mathematical sciences in the country" was held [12]. The main report was presented by Lyubomir Iliev. Other speakers who took part in the discussions were the academicians (K. Popov, L. Chakalov, N. Obreshkov), professors, associate professors, as well as the assistant professors Bojan Penkov and Blagovest Sendov. The State Council approved more than twenty specific proposals, including: the founding of

[5]From the speech of acad. Blagovest Sendov in the celebration of the International year of Statistics in the Great Hall of Bulgarian Academy of Sciences on November 27, 2013.

[6]D. Tabakov in 1948, I. Tzenov and K. Popov in 1951, L. Chakalov in 1952

a Computing center; the need of support of the two sections of Mathematical Institute – "Mathematical Statistics" and "Computational Mathematics and computers"; recommendations for the work of the Departments of Mathematics at Sofia University and the other universities; "creating new profiles of study – Mathematical Statistics, Mechanics and others" at Sofia University.

In the academic year 1960–61 a specialization "Mathematical Statistics" was introduced as a continuation of the tradition of education of actuaries. The lectures were provided by Prof. N. Obreshkov, Prof. Al. Mateev, Assoc. Prof. Bl. Sendov, Assoc. Prof. B. Penkov and Acad. Kiril Popov, and supported by the members of the section of MI of BAS at that time: Apostol Obretenov, Emanuel Simeonov, Margarita Andreeva, Lilyana Boneva, Ivan Mirazchiyski (see [38], [9] and [10]) in collaboration with Ivan Katzarov. The latter was Chief Actuary and, later on, Director of the National Social Assurance Institute until he had been attracted to academic career at the Economics Academy "D. Tzenov" in the town of Svishtov. He specialized in Actuarial Mathematics under the leadership of Prof. Tauber of University of Vienna [24].

In close cooperation between PhMF and MI of BAS, in 1962 the prototype of the first Bulgarian computer (on electronic lamps) "Vitosha" was invented and in 1965 the Electronic calculator "Elka" was built. In 1966 on the basis of its experience and staff the Central Institute of Computing Technology (CICT) became independent and had served as foundation of the development of computer technology production in Bulgaria until 1990.

The year 1963 is full of events for the Bulgarian Mathematical Society. After numerous splits, the Physics Faculty was separated from the PhMF. So the "nature" leaved the Faculty of Natural Sciences (Physics and Mathematics) and the "pure science" remained alone as Mathematical Faculty (MF).

The sudden death of Prof. Nikola Obreshkov beheaded two departments. Department "Algebra and Probability Theory" was renamed "Algebra" and leaded by Assoc. Prof. Ivan Duychev. "Probability" was transferred to Department of Mathematical Statistics and Insurance Mathematics" and renamed to "Probability Theory and Mathematical Statistics", with staff consisting of Assoc. Prof. Boyan Penkov and Prof. Al. Mateev (Chair). The department remained unchanged until 1965, when it was transferred to "sector" of the Department of Higher analysis with a single member – Assoc. Prof. Boyan Penkov. Specialization in Mathematical Statistics existed thanks to the teaching (lectures, seminars and practical classes) of the employees of section "Probability and Statistics" in the Mathematical Institute.

In the spring of 1964 the famous Ukrainian mathematician B. V. Gnedenko was a guest–lecturer of the MF for nearly a whole term. He read courses on Mathematical Statistics and Queueing Theory and at the same time headed an international seminar on "Reliability Theory" at the MI with participants from Bulgaria, Hungary and Germany. This visit set the beginning of a long term fruitful cooperation, which leaved significant mark in Stochastic guild in Bulgaria [13].

In 1968 the section staff was expanded with Boyan Dimitrov, Petar Petrov, Mikhail Uzunov, Maria Varbanova, Tzvetan Ignatov, Hristo Pavlov, Dimitar Vandev and Elisaveta Pancheva. Head of the section was Assoc. Prof. Boyan Penkov, who was a member of the Department of Higher Analysis. The Section was placed in one of the old buildings of the Institute of Biology at “Latinka” street (Figure 1). Next year four of the young fellows went for PhD studies - B. Dimitrov, P. Petrov and H. Pavlov to Moscow, and D. Vandev to St. Petersburg (at that time, Leningrad). Nikolay Yanev joined the section after his graduation, and Jordan Stoyanov and Miroslav Tanushev were appointed in 1970.

Figure 1: Section “Probability and Statistics” in 1969 in front of the building on “Latinka” street. Standing from left to right are: Ivan Mirazchiyski, Hristo Pavlov, Mihail Uzunov, Tzvetan Ignatov, Dimitar Vandev, second row – Maria Krasteva, Liliana Boneva, Margarita Andreeva, Elisaveta Pancheva, sitting – Apostol Obretenov and Boyan Penkov.

4. INTEGRATION (1971 – 1988)

The spreading of computers in the country was extremely fast. Computers were imported from all over the world, but also production of computers was organized urgently in the country. The need for joint efforts for training people to work with this new technology resulted in the union of the MF and the MI of BAS. The Institute of Technical Mechanics of BAS also joined them. MI became Institute of Mathematics and Mechanics (IMM), the Faculty adopted the name “Mathematics and Mechanics" (FMM) and all these institutions were united in the "United Center for Science and Education in Mathematics and Mechanics" (UCSEMM), soon after that renamed to the shorter UCMM.

This joint organization, in which the functions of the departments of the Faculty and the sections of the Institute were performed by grouping them into "sectors", lasted nearly twenty years. New sectors were founded in accordance with L. Iliev's classification of Mathematical Sciences, i.e. abstract, applied and IT oriented structures. To each area of "pure" Mathematics (abstract structure) corresponded a field of applications of classical type and an area of applications using computer technology. "Probability and Statistics", or in short "Stochastics", fell within the applied structures related to Calculus and Measure Theory. Later on, an independent unit for "Stochastic Computing" specialized in applications of Stochastics using computers was separated.

Stochastic computing encompassed development of numerical methods of Stochastics, statistical databases, intelligent statistical software and expert systems in Statistics, probabilistic and statistical simulation modelling of processes and systems with the aid of computers, computerization of Statistics education, analysis, modelling and forecasting of time series, and all other computer implementations of stochastic models and methods.

Each sector offered specialization in two–year course, the so–called block "B", which was preceded by three years of general education (block "A"). A one–year additional course after the general education, aimed for preparation for teachers, was called block "D". This organization is identical to the modern structure of our higher education with bachelor's and master's degrees. PhD degree was obtained after completion of block "C". Graduates from the basic block "A" received a Diploma of Higher education and the necessary mathematical culture and knowledge to use computers. Those completing block "B" obtained Master's degree (M.Sci.) diploma.

A separate building and premises in the building of Physics Faculty were provided to the "United center". Construction works began for a new building in the complex "IV$^{\text{th}}$ kilometer" of BAS, which was built in less than two years. The classes of block "A" took place in the current building of FMI in Lozenetz. The time–table for mathematical disciplines was scheduled from 7 a.m. to 1 p.m. Special notebooks were ordered and made for regular and compulsory homeworks and tests. Intensive language training was provided during the summer months. Staff bus service provided the transfer of lecturers between the two buildings every hour. Master's degree students (block "B") had their lectures and training in both buildings, depending on the location of the sector they specialized in.

In the summer of 1972 sector "Probability and Statistics" was situated on the fourth floor of the new building, the eastern half of the south part. In the same year the sector employed as mathematicians Plamen Mateev and Georgi Yamukov.

In the next years the sector's staff was further extended by appointing Ljuben Mutafchiev, Georgi Chobanov, Rossitca Dodunekova, Svetlozar Rachev, Valeri Stefanov, Dimitar Hadzhiev.

The first head of sector "Probability and Statistics" was Senior Research Fellow A. Obretenov. Boyan Penkov took over the management of the sector after 1979.

In 1978 a laboratory of "Computer Stochastics" was established as independent unit. The first lab researchers were Senior Research Fellow D. Vandev (head) and Junior Researcher P. Petrov. In the same year, Evgeni Dimitrov joined them. In the beginning of next year P. Mateev, after completing his PhD in Moscow State University, also became a member of the laboratory.

An unique software for statistical data analysis "Statlab" based on the platform of the first 8–bit PCs (IMKO–2 and the widely used Pravetz 8) was developed in the laboratory. Despite the memory limitations and the monochrome display with 24 lines of 40 characters, the functionality of the program includes preparation of data with up to 18 variables, including possibilities of transformations, one–dimensional descriptive statistics, single–factor dispersion analysis, original interactive step regression procedure, linear discriminant analysis also with availability of interactive selection of predictors, non–linear regression with availability for optimal choice of additional data points, factor analysis with varymax procedure, cluster analysis and multidimensional scaling. The entire system fitted on two 5–inch floppy–disks of 360 kilobytes capacity. The software system "Statlab" was developed as an educational tool for students of FMM and was part of the main set of software programs of Pravetz computers. Later on "Statlab" was adopted for the 16 bit IBM – XT and IBM – AT (and Pravetz 16 version) under the name MSTAT and its extension TSTAT designed for spectral analysis of auto-regressive models for time series. Unfortunately, the embargo on equipment with high graphical resolution (EGA, VGA, SVGA) and the economic problems in the late 80's detained the further development of this project.

Laboratory efforts were redirected towards rebuilding the knowledge in the field of Actuarial Mathematics. A joint contract of the Laboratory and FMM with a financial support from the British "Know–How Fund" set up the beginning of professional courses for actuaries to the newly established Bulgarian Actuarial Society. The leader in the implementation of this project with the most–active participation was Vladimir Kaishev until his transfer to a post at the City University, London.

The Stochastic guild of sector "Probability and Statistics" and the laboratory "Computer Stochastic" counted more than twenty people over the years. They taugh on average 10 to 12 Master's degree students in Probability and Statistics annually. In addition to the basic courses on Probability and Statistics at FMM, they also provided services to other faculties of the University, e.g., in Physics, Chemistry, Geology and Geography, Biology Departments.

The Sector was a center of the Stochastic guild for the country with many international contacts and collaborators. Since 1974 every two years the "International Summer School on Probability Theory and Mathematical Statistics" gathered more than a hundred participants. Since 1988, an annual Seminar on Statistical Data Analysis (SDA) was in operation.

Every Wednesday at 3 p.m. there were meetings of the "Common Seminar on Stochastic". It was place where reports on projects and research results were read and discussed by the members of the guild, guests–readers from home and abroad presented reports of their work, students defended their theses. In the official "Red Book" for guests of the Sector and the Seminars handwritten dedications are kept. There may be found the world famous names of statistical science – Kendall, Rao, Bolshev, Vapnik, Shiryaev, Belyaev, Solovyov, Zolotarev, Barlow, Jacod, Reves, Arato, Hawranek, Parzen, and many others.

Stochastic guild of the Unified Center initiated the founding of the "National seminar on Stochastics" (NSS) within the Union of Mathematicians in Bulgaria, aimed to focus on the problems of education in Stochastics. Director of the seminar was B. Dimitrov, D. Vandev was Scientific Secretary and E. Pancheva was Technical Secretary [9].

At the first meeting of NSS J. Stoyanov presented a report with a detailed analysis of the publications in the field of Stochastics in Bulgaria [34]. An annex to the report provided a detailed bibliography for that period awaiting its sequel. In the next ten years regular sessions of the seminar were held during the traditional Spring Conferences of the Union of Bulgarian Mathematicians (UBM). Special regular meetings of the seminar on problems of education in Stochastics were held traditionally at the Scientific center "Gyulechitza" of Sofia University situated in the Rila mountain.

An important direction in the work of the guild was the subject of statistical quality control in collaboration with the State Committees on Quality and Standards and other national institutions. An emanation of this activity was the founding of a laboratory "Statistical quality control" at IMM and "Research Laboratory on mathematical methods for quality management" at the Sofia University. Head of both laboratories was B. Dimitrov.

The two laboratories worked successfully on contracts for implementation of over ten national standards of statistical quality control, reliability, sampling control, terminological standards. They also took part in the development of three international standards and organized four national meetings of quality control professionals from the industry.

5. DISINTEGRATION (1988 – 2004)

The United Center did not fit in the standard hierarchical bureaucratic structures of both BAS and Sofia University. It survived 18 years, until 1988, when its function was officially terminated and the structure of separated Faculty and Institute with departments and sections, respectively, was restored. The Faculty enrolled students in Mathematics, Informatics, Mathematics and Informatics (teacher training) and Mechanics and adopted its current name (FMI) which suits better its educational profile. Also, the five-year course of study was restored.

The Department "Probability and Statistics" was resumed in 1988 with head Prof. B. Dimitrov and members Assoc. Prof. G. Tchobanov, Assoc. Prof. R. Dodunekova, Ass. Prof. I. Tzankova and Ass. Prof. B. Doychinov. About 40 people were transferred from IMM to FMI. Only Tz. Ignatov moved to the University. Many more were colleagues who sought for work abroad. Among those, Georgi Yanev, Mariana Beleva, Daniela Nicheva, Miroslav Tanushev, Boris Kovatchev and Rayna Robeva moved to the U.S.A.; Nikolay Kolev to Brazil; our colleagues Iva Tzankova, George Boshnakov, Yordan Stoyanov, Sahib Esa, Nikolay Trendafilov, Vladimir Kaishevin went to different European countries; Valeri Stefanov and Evgeni Dimitrov choose Australia and Maria Varbanova — South Africa.

In the mid 90's the department "Probability and Statistics" had highly reduced staff. During that period, due to the political changes, the economical problems and opening the borders of the country, a significant part of the young professionals, and not so young as well, sought for career abroad. Prof. B. Dimitrov, Assoc. Prof. R. Dodunekova and Ass. Prof. N. Ilieva started work in foreign universities. Assoc. Prof. G. Tchobanov moved to the newly established Faculty of Economics. Prof. Tz. Ignatov was the only member of the Senior staff who remained in the department.

Section "Probability and Statistics" at the Institute of BAS, which has been always supporting the teaching of Stochastic subjects at Sofia University and mainly the specialization "Probability and Statistics", also reduced significantly its staff, although it has merged with the laboratory "Computer Stochastics".

At that time there was a real danger for Stochastic guild to stop reproducing itself and for the education and training in Probability and Statistics at University to drop below the admissible minimum.

Assoc. Prof. D. Vandev was the first who tried to stop this negative trend by moving to FMI. In 1996, with his major participation, a new program – "Applied Mathematics" was opened, with specializations in Informatics, Mathematical Economics, Mechanics, Applied Statistics.

In 1997 T. Ignatov moved to Faculty of Economics. D. Vandev was doing his best to attract talented students for teaching, and also for recruiting part-time lecturers. The graduates Stiliyan Stoev and Emil Kamenov were appointed as permanent staff, but soon after St. Stoev moved to the United States.

In 2000 a new law on higher education was approved, in which the minimum number of teaching staff at one department was set to 6 persons. As a result came the merger of departments "Probability and Statistics" and "Operations Research" and a new one was formed, the department "Probability, Operations Research and Statistics", which exists at present in this form.

In 2002 the first Master's program in the Faculty, "Mathematical Modelling in Economics", which meets the new law regulations, was founded. D. Vandev actively participated in building up its curriculum. This Master's degree program was considered by Vandev as a temporary solution to the task of rebuilding of another specialization. His goal was finally realized in 2004, when an independent Master's degree program “Probability and Statistics” was founded. Meanwhile, to meet the law requirements for habilitated academic staff, Dimitar Vandev insured the opening of new positions and opportunities for habilitation and attracting of new lecturers: M. Bojkova (from IMI - BAS), L. Minkova (from Technical University of Sofia), D. Donchev (from University of Food Technologies - Plovdiv) and Vandev's collaborator for many years from the laboratory “Computer Stochastics”, P. Mateev.

Unfortunately, Assoc. Prof. D. Vandev passed away on September 25, 2004. He was not with us to see the realization of Master's program “Probability and Statistics” and his fulfilled dream for an undergraduate program “Statistics”.

We would like to express our deep appreciation for the efforts, enthusiasm and professionalism to our unforgettable colleague Assoc. Prof. Dr. Dimitar Vandev, the founder of the modern Master degree program “Probability and Statistics”. We believe that all our colleagues will join our opinion, that we remain in debt to him and his work for rebuilding and reproduction of the Stochastic guild in our country.

6. RECENT YEARS (2004 – 2013)

During that period, some of the old traditions revived and the modern trends in Stochastic education managed to make their way through the recovery of specialization “Probability and Statistics” as Master degree program. A new undergraduate program “Statistics” opened in the 2007/2008 academic year. The National Seminar on problems in Education of Stochastics (NSPES) resumed his work.

6.1. MASTER'S PROGRAM “PROBABILITY AND STATISTICS”

The Master's program is essential for the reproduction of stochastic professionals in the country. It prepares professionals able to work as independent researchers at a high level in both pure and applied science. For the period of its existence since 2004 until 2013 from the total of 70 enrolled in Master's program, 32 have completed the program and defended their theses, which is about 46% successfully completed the program. A graphical representation of the distribution by years is given in Figure 2, reflecting the data from Table 1.

Priority areas of the program are: Stochastic models and their applications, Actuarial science and Biostatistics.

A benchmark for correctness of the chosen direction of development and as an eloquent manifestation of the quality of education in “Probability and Statistics” program is the professional realization of its graduates. They pursue successful careers in the job market at home and abroad. For instance, in our country our graduates work in “Musala Soft”, Insurance Company “Unica”, State Insurance Institute, The Financial–analytical company “Finanalytica”, “Experian”, Ministry of Education, banks as Unicredit, Bulbank, Postbank, marketing firms “Alpha Research”, “Ipsos”, and others.

Figure 2. Master's degree students of the course "Probability and Statistics" in the period 2004 – 2013. A timeline of number of enrolments and number of graduates.

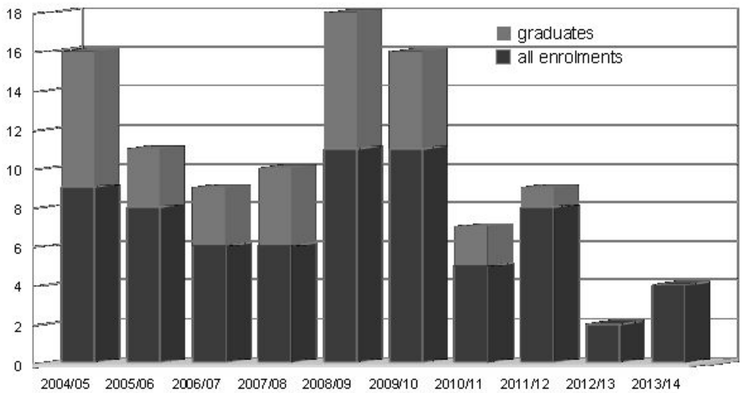


Table 1. MSci degree program "Probability and Statistics"

Year of Enrolment	Enrolled in MSci program	Graduated with MSci degree
2004/2005	9	7
2005/2006	8	3
2006/2007	6	3
2007/2008	6	4
2008/2009	11	7
2009/2010	11	5
2010/2011	5	2
2011/2012	8	1
2012/2013	2	0
2013/2014	4	0

After completing MSci [7] degree program students, seeking realization abroad continue successfully their academic development in world famous, high-profile universities such as Harvard University (USA), University of Reading and University of Bristol (UK), Humboldt University, Berlin (Germany), and others.

6.2. BACHELOR'S DEGREE PROGRAM (BSCI) "STATISTICS"

The BSci "Statistics" was established with the joint efforts of the Stochastic guild of FMI at SU and IMI - BAS on the explicit order of the Dean of the Faculty in that time, Acad. B. Boyanov. The design of the whole program and curriculum

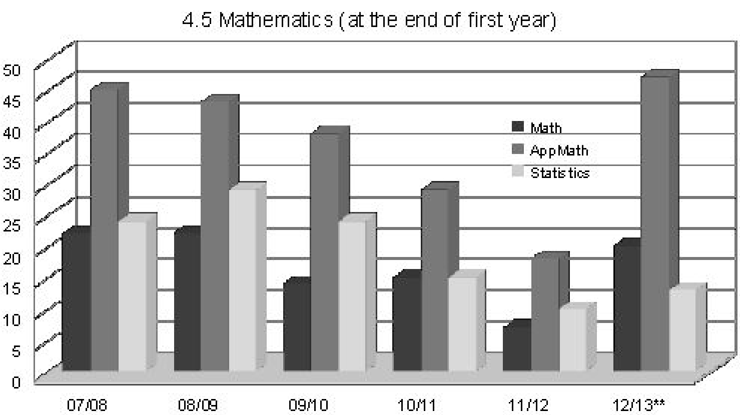
[7] Master's degree program (MSci) is three or four semester graduate education after four years (eight semesters) Bachelor's degree program (BSci). BSci diploma is delivered after successful state exam. MSci is accomplished with Diploma Thesis.

we owe to the hard work of the guild with the most active participation of P. Mateev and M. Bojkova. Programs for elective courses were also offered by Prof. Racho Dentchev, Assoc. Prof. D. Donchev, Prof. Dimitar Christozov, Assist. Prof. Vessela Stoimenova, Nina Daskalova and Dimitar Atanasov.

The program is our response to the pressing need for knowledge and expertise in this field in the community and to the world's trends. Its curriculum is based on the curriculum of the BSci program "Applied mathematics", with expanded contents for some disciplines ("Theory of Probability and Mathematical Statistics" split into two consecutive semesters, "Applied Statistics" is also divided into two semesters and renamed — "Data analysis and regression" and "Multivariate statistical models"), and adding new ones, such as "Introduction to Statistics" in the second semester of the first year and "Statistical laboratory" in the semester before the last one. Specific basic disciplines in Stochastics such as "Random processes", "Introduction to Actuarial science" are included in the curriculum with changed status from optional to mandatory.

Finally, we show some quantitative comparative data for students of BSci "Statistics" in comparison with the numbers of students of BSci "Mathematics" and "Applied Mathematics" up to 2012/13 academic year.

Figure 3. BSci degree Education filed 4.5 Mathematics at FMI for the years 2007 — 2013 (end of the first academic year)



Enrollment of the students by years is presented on Figure 3, where the distribution by years of enrolled and graduated students is given. Altogether, we enrolled 141 students and 31 of them graduated successfully, which represents 22% of the enrolled ones. All graduates are employed in their speciality and some continued their education in MSci programs in the Faculty, in other universities in Bulgaria, or abroad.

Those who choose an academic career have to demonstrate their abilities by defending a PhD Thesis. For the last five years in the Faculty there were 42 PhD students in total trained in the field "4.5 Mathematics", eight of them have chosen Stochastics (Probability and Statistics). One of those eight successfully defended a PhD Thesis, two have completed the course with a right of defence, and the others are working on their regular plans. They have published 18 research reports, participated in seven scientific forums abroad and in eleven in our country. They present their results at the annual Spring Scientific Sessions of the FMI, the European Meetings of Young Statisticians, International Conference on Probability Theory and Mathematical Statistics and the accompanying events such as seminars on Statistical Data Analysis and Branching Processes and Applications, National Seminar on Education in Stochastics and other scientific forums. Particularly strong is the participation of the PhD students – "stochasticians" in scientific schools and doctoral conferences organized within the project "Formation of a new generation of researchers in the field of Mathematics, Informatics and Computer science by supporting the creative and innovative potential of PhD students, and young post–doctorate students and researchers in FMI at SU", financed by the European Union funds.

6.4. NATIONAL SEMINAR ON PROBLEMS IN EDUCATION OF STOCHASTICS (NSPES)

In this period the work of the NSPES was renewed. The first four editions were conducted successively in 2007, 2009, 2011 and 2013 years.

The activities within NSPES will be mentioned separately below.

The first NSPES was organized in 2007. We have gathered 25 participants from across the country — Sofia University, Plovdiv University, Shoumen University, Southwestern University – Blagoevgad, Technical University – Varna , IMI – BAS, as well as participants from Macedonia and the U.S.A., and representatives of the publishing business. The seminar was dedicated to the 60-th anniversary of the Department of Mathematical Statistics at Sofia University "St. Kliment Ohridski" and 45 years of the first alumni of specialty "Mathematical Statistics".

The second NSPES was held in 2009 and is remarkable with the strong presence of students – six people from MSci program "Probability and Statistics", who shared their modest experience "on both sides of the bench" — as students–graduates and students–demonstrators. There were representatives from IMI – BAS, New Bulgarian University and foreign participants from Macedonia and the U.S.A. The focus of this seminar was on teaching probability and statistics at FMI. With the opening of new undergraduate programs at the FMI - "Computer science", "Information Systems", "Software engineering" and "Statistics" the need to include the more talented students in the Masters program of the education process extremely increased.

The third edition of NSPES, held in 2011, proved that the seminar has established itself as an useful event for discussions on mutual problems not only at national level but also internationally. The third NSPES was attended by four students from Masters program "Probability and Statistics" of FMI at SU. The participation of scientists and professors from abroad, specialists in Probability theory and Mathematical statistics, with positions in prestigious universities, is a vivid testimonial to its attractiveness and significance. It is important to note that the theme of the seminar is particularly timely and is one of the priorities for development of scientific research at Sofia University. In this format, the seminar is essential for maintaining of high level of teaching in the disciplines in the field of Stochastics, updated with accordance to the international standards and tendencies of development in recent years.

The fourth NSPES, held in 2013, was dedicated to the International Year of Statistics. Among the participants were again one student from Masters program and three PhD students and young scientists as well.

Information about the seminar and announcements of upcoming meetings is published on the Internet-site `probablystatistics.net`. More information is available at `http://probablystatistics.net`.

7. CONCLUSION

Last but not least we would like to mention that this paper is also dedicated to the 125^{th} anniversary of the FMI at SU, celebrated in 2014.

Tracing back the development of research and education in Stochastics, we may conclude that currently the FMI at SU is an established center for training of specialists in the field of Stochastics.

Near–field programs in Econometrics in higher education institutions for Economics and Business Administration and a Master's Program in Applied Statistics at New Bulgarian University are opened. Unfortunately, the resources of professionals in this field and the total load in the Faculties of Mathematics and Informatics at Plovdiv and Shoumen universities, and the Faculty of Applied mathematics and informatics at the Technical University –Sofia does not allow opening of a separate specialty.

We have been in the "Age of Stochasticity" already for fourteen years and the perspicacious business predicts that "Statistician" will be an appeal–able profession ("the sexy job") in the coming years, as claimed by Hal Varian chief economist at Google[8]:

> "I keep saying the sexy job in the next ten years will be statisticians. People think I'm joking, but who would've guessed that computer engineers would've been the sexy job of the 1990s?"

[8] An interview of Hal Varian in McKinsey Quarterly, January 2009 `.flowingdata.com/2009/02/25/googles-chief-economist-hal-varian-on-statistics-and-data`

The stochastic knowledge is a must for the modern citizen — it is a prerequisite for his informed choice. An argument supporting this statement is the positive correlation between the economic prosperity of the state and the degree of presence of Stochastic in all stages of education.

The godfathers of the discipline "Information technology" [9] define it back in 1958 as the sum of three categories:

- processing techniques,
- application of statistical and mathematical methods in decision making and
- simulation of mental activity of a higher type using computer programs.

Ahead of their time, they were under a lot of criticism, even now, in the modern definitions, the second category is neglected. Despite that, we can see an increasingly closer integration of Mathematics and Statistics on one hand and the power of computer equipment on the other, and Stochastics is taking an increasingly important place.

We strongly believe that the training of qualified specialists in the field of Stochastics, related to the scientific research and applications as well as the increase of statistical literacy of the society, is a prerequisite for its prosperity. This is a task that can be solved only by the joint efforts of the entire Mathematical guild.

In this work we tried to present a historical overview of the development of scientific and educational activities of Stochastics within the Sofia University and partially within the Bulgarian Academy of Sciences. We must confess that many important details have been left without the attention they deserve, and in many areas we remain in debt, which we hope to fix with the kind help of the reader.

ACKNOWLEDGEMENTS. Our work is partially supported by Grant 47/2015 of the state funds allocated to the Sofia University, Bulgaria. The authors thank to Rene Kamova for her assistance in translation and technical support.

8. REFERENCES

1. Al-Kindi: Wikipedia, Wikimedia Foundation `http://en.wikipedia.org/wiki/Al-Kindi` (02.11.2013)
2. Arnold, B.: What is this Mathematics? MCCME, 2002 (in Russian).
3. Baltadziev, Hr.: *Short theory of the Probability.* State Comprehensive college, "Alexander I" Plovdiv, 1890 (in Bulgarian).
4. Bayes, T., Price, R.: An Essay towards solving a Problem in the Doctrine of Chance. By the late Rev. Mr. Bayes, F.R.S. communicated by Mr. Price, in a letter to John Canton, A. M. F. R. S., *Philosophical Transactions of the Royal Society of London*, **53 (0)**, 1763, 370-–418. doi:10.1098/rstl.1763.0053

[9]Leavitt, Harold J.; Whisler, Thomas L., Management in the 1980 's. , Harvard Business Review. 11 , 1958 `hbr.org/1958/11/management-in-the-1980s`: The new technology does not yet have a single established name. We shall call it *information technology*

5. Bernoulli , Y.: Ars Conjectandi. In:*Bernulli, Laplace, Kolmogorov, Probabilities*, (B. Penkov, ed.), Science and Art, Sofia, 1982 (in Bulgarian).
6. Boneff, B.: *Table of mortality in Bulgaria*, 1920, Sofia (in Bulgarian).
7. *Bulgarian academy of sciences, Institute of Mathematics and Informatics founded 1947*, BAS, 2007, 64, ISBN 978-954-8986-27-4 (in Bulgarian).
8. Danailov, G.: *Theory of Statistics*. University Lib., No. 119, 1931 (in Bulgarian).
9. Dimitrov, B.: Ten years National Seminar of Stochastics to UBM, Years of growth. UBM, 1988, 23—32 (in Bulgarian).
10. Dimitrov, B.: *The new history of Faculty of Mathematics, 100 years UBM, section BAS-SU*, 1990, 48–57 (in Bulgarian).
11. Doychinov, D.: Ninety five years Faculty of Mathematics. *Teaching Mathematics*, **3**, 1984, 33—39 (in Bulgarian).
12. Extended meeting of the State Council for Science on the status and needs of the mathematical sciences in the country. *Physico-Mathematical Journal*, **3**, no. 3, 1960, 169–174 (in Bulgarian).
13. Geshev, A., Dimitrov, B., Pavlov, H., Obretenov, A., Penkov, B.: Acad. Boris Vladimirovich Gnedenko at 70. *Physico-Mathematical Journal*, **24(57)**, no. 4, Sofia, 1982, 307–309 (in Bulgarian).
14. Ifra, J.: *Encyclopedical history of digits*, Vol. 1 and 2. IC "Elements", 2005, 1692, ISBN: 9549414078 (transl. in Bulgarian).
15. *Jubilee volume 40 years BFMS*, 1939 , Sofia (in Bulgarian).
16. Krueger, L. et al (Ed.): *The Probabilistic Revolution*, Vol. 1: *Ideas in History*, Vol. 2: *Ideas in the Sciences*, MIT Press, 1987.
17. Matheeff, A.: On some meteorological series in relation to the Theory of Probability. *Volumes of Central Meteorological Institute*, 1943, 85—92 (in Bulgarian).
18. Matheeff, A.: On some methods of Mathematical Statistics for processing results of observation, *Volumes of Central Meteorological Institute*, 1941, 49–84 (in Bulgarian).
19. Ministry of Finance, Michael Sarafov, http://www.minfin.bg/bg/page/103 (14.05.2013) (in Bulgarian).
20. Mumford, D.: The Dawning of the Age of Stochasticity., In: *Mathematics: Frontiers and Perspectives*, (Arnold V. I., ed.), AMS, 2000, ISBN 0821826972.
21. Obreshkov, N.: *Higher Algebra*, Vol. 2, Univ. Lib. no. 163, 1935 (in Bulgarian).
22. O'Connor, J. J., Robertson, E. F.: Oskar Johann Viktor Anderson. *MacTutor History of Mathematics*, `http://www-groups.dcs.st-and.ac.uk/history/` (01.11.2013).
23. Olivie, J.: *Geometry, Trigonometry and Statics, translation and publisher Nestor Petrov*. 1871 , Ruse (transl. in Bulgarian).
24. Pavlov, N.: 100 years since the birth of Prof. Katzarov , *Journal of the National Social Assurance Institute*, IX , no. 4, 2010, 29–32 (in Bulgarian).
25. Pintev, St. and team: *120 years Bulgarian Statistics, 1880-2000*. NSI, ISBN 954-9680-03-7 (in Bulgarian).
26. Popoff, K.: *Autobiography*. Univ. Publ. "St. Kl. Ohridski", 1993, Sofia, ISBN 954-07-0179-1 (in Bulgarian).

27. Radilov , D.: Life and scientific work of Prof. Oscar Anderson. *Economic Thought*, IE -BAS , vol. XXLVII, **5**, 2002, 94–104 (in Bulgarian).

28. Renyi, A.: *Probability theory.* Budapest, Akad. Kiado Hungary, 1970.

29. Schedules of lectures in winter semester 1897/98 academic year and Schedules of lectures in summer semester of 1905 / 06, Sofia University (in Bulgarian).

30. Slavtchova - Bojkova, M., Mateev, P.: The birth and distribution of Stochastics in Bulgaria. In *Distribution and development of Physics and Mathematics in Bulgaria*, ISSP "G. Nadzakov", BAS, 2006, 85–92.

31. Slavtchova - Bojkova, M., Mateev, P.: Stochastics in Bulgaria 120 years ago. In: *120 years FMI , Sofia University "St. Kliment Ohridski"*, 2011 , ISSN 1313-9045, 50–57 (in Bulgarian).

32. Slavtchova - Bojkova, M. , P. Mateev, Research and education in Stochastics at the FMI – SU, *Stochastcs Education Problems*,**1**, 2015 , ISSN 2367-7317, 30–48 (in Bulgarian).

33. Stoyanov, A.: On teaching "Insurance" in secondary schools. *Journal of PhMD*, vol. XX, Sofia, 1935 , 237–244 (in Bulgarian).

34. Stoyanov, J.: Literature on Stochastics and its applications issued in Bulgaria in the period 1944 - 1973. In: *National Seminar on Stochastic to UBM*, 1978, 1–18 (in Bulgarian).

35. Shapkarev, P.: Decomposition of business series in time. *Journal of PhMD* vol. XXVII, Sofia, 1942, issue 7-8, 241–249 , issue 9-10, 299-325 (in Bulgarian).

36. Shkodrov, E., Tzvetkov, St.: Statistical science and education in the UNWE, *Yearbook of UNWE*, 2005, `http://yearbook.unwe.bg/2005/04.pdf` (15.05.2013) (in Bulgarian).

37. Shourek, A. B.: *Catalogue of the books of Mathematics Institute at the University of Sofia.* Sofia, 313, 1913 (in Bulgarian).

38. Vassilev, Ml.: From the history of the Faculty of Mathematics and Mechanics. *Mathematics*, **3**, 1981, 6–10 (in Bulgarian).

Received on 4 January, 2015

M. Slavtchova–Bojkova, P. Mateev

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5 blvd. J. Bourchier, BG-1164 Sofia
BULGARIA

e-mails: bojkova@fmi.uni-sofia.bg
pmat@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

ON THE INVESTIGATIONS OF IVAN PRODANOV IN THE THEORY OF ABSTRACT SPECTRA

GEORGI DIMOV, DIMITER VAKARELOV

Dedicated to the memory of Ivan Prodanov

We were invited by the Organizing Committee of the Mathematical Conference dedicated to Professor Ivan Prodanov on the occasion of the 60-th anniversary of his birth and the 10-th anniversary of his death, held on May 16, 1995 at the Faculty of Mathematics and Informatics of the Sofia University "St. Kliment Ohridski", to give a talk on his investigations in the theory of abstract spectra. All of his results in this area were announced in a short paper published in the journal "Trudy Mat. Inst. Steklova", 154, 1983, 200–208, and, as far as we know, their proofs were never written by him in the form of a manuscript, preprint or paper. The very incomplete notes which we have from Prodanov's talks on the Seminar on Spectra, organized by him in the academic year 1979/80 at the Faculty of Mathematics of the University of Sofia, seem to be the only trace of a small part of the proofs of some of the results from the cited above paper. Since the untimely death of Ivan Prodanov withheld him from preparing the full version of this paper and since, in our opinion, it contains interesting and important results, we undertook the task of writing a full version of it and thus making the results from it known to the mathematical community. So, the aim of this paper is to supply with proofs the results of Ivan Prodanov announced in the cited above paper, but we added also a small amount of new results.
The full responsibility for the correctness of the proofs of the assertions presented below in this work is taken by us; just for this reason our names appear as authors of the present paper. The talks of the participants of the conference had to be published at a special volume of the Annuaire de l'Universite de Sofia "St. Kliment Ohridski, but this never happened. That is why we have decided to publish our work separately. Since our files were lost and we had to write them once more, the paper appears only now.

Keywords: Abstract spectra, bitopological spaces, compact spaces, coherent spaces, coherent maps, distributive lattices, (pre)separative algebras, convex space, separation theorems

2010 Math. Subject Classification: Primary: 06D05; Secondary: 54H10, 54B30, 54E55, 06D50, 18A40, 18B30

1. INTRODUCTION

This paper contains an extended version of the invited talk given by the authors at the Mathematical Conference dedicated to Professor Ivan Prodanov on the occasion of the 60^{th} anniversary of his birth and the 10^{th} anniversary of his death. The conference took place on May 16, 1995 at the Faculty of Mathematics and Informatics of the Sofia University "St. Kliment Ohridski". It was planned the talks of the participants in this conference to be published in a special volume of the Annuaire de l'Universite de Sofia "St. Kliment Ohridski, Faculte de Mathematiques et Informatique, Livre 1 - Mathematiques, but this has never happened. That is why we have decided to publish our work separately. Since our files were lost and we had to write them once more, the paper appears only now.

In the academic year 1979/80 Professor Ivan Prodanov organized a seminar on spectra at the Faculty of Mathematics of Sofia University. The participants in this seminar, besides Iv. Prodanov, were G. Dimov, G. Gargov, Sv. Savchev, L. Stoyanov, V. Tchoukanov, T. Tinchev, D. Vakarelov. The talks of Iv. Prodanov on this seminar were on his own investigations in the theory of abstract spectra and the uniqueness of Pontryagin–van Kampen duality. In the reviewing talks of the other participants, Stone Duality Theorems for Boolean algebras and for distributive lattices ([41], [42]), H. A. Priestley's papers [27]-[30], M. Hochster papers [18] and [19], the topological proof of Goedel Completeness Theorem given by Rasiowa and Sikorski in [37] and many other interesting topics were discussed.

Iv. Prodanov raised a number of interesting open problems at his seminar on spectra. Two of them were solved by some of the participants of the seminar and these solutions caused, on their part, the appearance of other new papers. One of these problems was whether the category $\mathcal{L}_R$ of locally compact topological R-modules, where R is a locally compact commutative ring, admits precisely one (up to natural equivalence) functorial duality. (Using the classical Pontryagin-van Kampen duality, one easily obtains a functorial duality in $\mathcal{L}_R$, called again Pontryagin duality. Hence, there is always a functorial duality in $\mathcal{L}_R$.) L. Stoyanov [43] showed that if R is a compact commutative ring, then the Pontryagin duality is the unique functorial duality in $\mathcal{L}_R$. Later on, Gregorio [15] and Gregorio and Orsatti [16] generalized that result of Stoyanov. The second problem was whether a uniqueness theorem, like that for Pontryagin-van Kampen duality, can be proved in the cases of Stone dualities for Boolean algebras and for distributive lattices. The answers were given by G. Dimov in [8] and [9], where it was proved that the Stone duality for Boolean algebras is unique and that there are only two (up to natural equivalence) duality functors in the case of distributive lattices. Some very general results about representable dualities and the group of dualities were obtained later on by G. Dimov and W. Tholen in [11], [12]. It could be said that D. Vakarelov's paper [46] was also inspired by Prodanov's seminar on spectra. This was certainly so for the diploma thesis [39] of Sv. Savchev, written under the supervision of Professor Iv. Prodanov, and for the paper [40].

Iv. Prodanov presented his results on the uniqueness of Pontryagin-van Kampen duality in the manuscripts [32] and [33]. The more than fifty-pages-long paper [33] contains also an impressive list of open problems and conjectures. The publication of these manuscripts was postponed because Prodanov discovered that analogous results were obtained earlier by D. Roeder [38]. Prodanov's approach, however, was different and even more general than that of D. Roeder. Only his untimely death withheld him from preparing these manuscripts for publication. The task of doing that was carried out by D. Dikranjan and A. Orsatti. In their paper [7], all results from [32] and [33] were included and some of Prodanov's conjectures were answered. In such a way the manuscripts [32] and [33] became known to the mathematical community and stimulated the appearance of other papers (see [6], [14]).

The results of Iv. Prodanov on abstract spectra and separative algebras were announced in [31], but their proofs were never written by him in the form of a manuscript, preprint or paper. The very incomplete notes which we have from the Prodanov talks on the seminar on spectra seem to be the only trace of a small part of these proofs. Since, in our opinion, the results, announced in [31], are interesting and important, we decided to supply them with proofs. This is done in the present paper, where we follow, in general, the exposition of [31], but some of the announced there assertions are slightly generalized, some new statements are added and some new applications are obtained. The main of the added results is Theorem 2.39, which was formulated and proved by us as a generalization of Prodanov's assertions Corollary 2.40 and Corollary 2.41.

Section 1 of the paper is an introduction. Section 2, divided into four subsections, is devoted to the abstract spectra. In Subsection 2.1 the category $\boldsymbol{S}$ of abstract spectra and their morphisms is introduced and studied. Subsection 2.2 contains two general examples of abstract spectra (see 2.20 and 2.24). The classical spectra of rings endowed with Zariski topology appear as special cases of the first of these examples (see 2.21), while the classical spectra of distributive lattices with their Stone topology appear as special cases of both examples (see 2.22 and 2.25). In Subsection 2.3 the main theorem of Section 2 is proved (see 2.36). This theorem asserts that the category $\boldsymbol{S}$ of abstract spectra and their morphisms is isomorphic to the category $\boldsymbol{CohSp}$ of coherent spaces and coherent maps and, hence, by the Stone Duality Theorem for distributive lattices, the category $\boldsymbol{S}$ is dual to the category $\boldsymbol{DLat}$ of distributive lattices and lattice homomorphisms. It is well-known that the category $\boldsymbol{OStone}$ of ordered Stone spaces and order-preserving continuous maps is also dual to the category $\boldsymbol{DLat}$ (see [27], [28] or [20]), and that it is isomorphic to the category $\boldsymbol{CohSp}$ (see, for example, [20]). Therefore, the category $\boldsymbol{OStone}$ is isomorphic to the category $\boldsymbol{S}$. (The last fact could be also proved directly, but we do not do this.) So, each one of the categories $\boldsymbol{CohSp}$, $\boldsymbol{OStone}$ and $\boldsymbol{S}$ is dual to the category $\boldsymbol{DLat}$. In our opinion, the category $\boldsymbol{S}$ is the most natural and symmetrical one amongst all three of them. Subsection 2.4 contains two applications (see Corollary 2.40 and Corollary 2.41) of the already obtained results. The one from Corollary 2.40 is important for Section 3. These applications appear as special cases of a general theorem (see Theorem 2.39), which we formulate and prove here as a generalization of Prodanov's results Corollary 2.40 and Corollary 2.41. Theorem 2.39 was used later on by us in our paper [10].

At a first glance the advent of spectra in so general situations as in 2.20 is unexpected, since psychologically they usually are connected with separation. Actually, in general one does not know whether there are non-trivial prime ideals, but it turns out that if the operations $\times$ and $+$ from 2.20 satisfy a few not very restrictive natural conditions, then the prime ideals become as many as in the commutative rings or in distributive lattices, for example. In this way one comes to the notion of a *separative algebra* considered in Section 3.

Section 3 is divided into several subsections. In Subsection 3.1 the definition of a *preseparative algebra* as an algebra with two multivalued binary operations $\times$ and $+$ satisfying some natural axioms as commutativity and associativity is given, and some calculus with these operations is developed. Subsection 3.2 is devoted to the theory of filters and ideals in preseparative algebras. The main notion of a *separative algebra* is given in Subsection 3.3. Here a far of being complete list of examples is given: the commutative rings, the distributive lattices and also the convex spaces (= separative algebras in which the two operations coincide) are separative algebras. The main theorem for separative algebras - the Separation theorem, is proved in Subsection 3.4. In Subsection 3.5 some natural new operations in separative algebras are studied and in Subsection 3.6 a general representation theorem for separative algebras is given. Roughly speaking, every separative algebra $\underline{X} = (X, \times, +)$ can be embedded into a distributive lattice L in such a way that the operations in $\underline{X}$ are obtained easily from the operations in L. That is new even for the plane: there exists a distributive lattice $L \supseteq \boldsymbol{R}^2$ such that for each segment $ab \subset \boldsymbol{R}^2$ one has

$$ab = \{x \in \boldsymbol{R}^2 : x \leq a \vee b\} = \{x \in \boldsymbol{R}^2 : x \geq a \wedge b\}.$$

The notion of separative algebra comes from an analysis of the separation theorems connected with the convexity. The abstract study of convexity was started by Prenovitz [25] and different versions of the notion of *convex space* appeared in [34], [35], [44], [3], [4], [26]. All they are compared in [45]. The convexity was examined from other aspects in [1], [5], [17], [22] and [24], a few applications are considered in [47] and [2] contains a critique.

Y. Tagamlitzki [44] obtained a general Separation theorem for convex spaces. It was improved (again for convex spaces) and applied to analytical separation problems in [34] and [35] (cf. [1] and [4]). It seems however that the natural region for that theorem are not the convex spaces but the separative algebras: the presence of two operations makes the instrument more flexible, without additional complications (see Subsection 3.4). This permits to obtain as special cases the separation by prime ideals of an ideal and a multiplicative set in a commutative ring, or of an ideal and a filter in a distributive lattice, and also the separation of two convex sets by a convex set with convex complement.

The paper ends with Subsection 3.8 devoted to a generalization of the Separation theorem for separative algebras supplied with a topology. Thus, even restricted to convex spaces, one can find, as in [35], a few classical separation and representation theorems, but the presence of two operations enlarges the possibilities for new applications.

Let us fix the notation. If C denotes a category, we write $X \in |C|$ if X is an object of C, and $f \in C(X, Y)$ if f is a C-morphism with domain X and codomain Y. All lattices will be with top (=unit) and bottom (=zero) elements, denoted respectively by 1 and 0. We don't require the elements 0 and 1 to be distinct. As usual, the lattice homomorphisms are assumed to preserve the distinguished elements 0 and 1. ***DLat*** will stand for the category of distributive lattices and lattice homomorphisms. If X is a set then we write $Exp(X)$ for the set of all subsets of X and denote by $|X|$ the cardinality of X. If $(X, \mathcal{T})$ is a topological space and A is a subset of X then $cl_{(X,\mathcal{T})}A$ or, simply, $cl_X A$ stands for the closure of A in the space $(X, \mathcal{T})$. We denote by $\boldsymbol{D}$ the two-point discrete topological space and by ***Set*** the category of all sets and functions between them. As usual, we say that a preordered set $(X, \leq)$ (i.e. $\leq$ is a reflexive and transitive binary relation on X) is a *directed set* (resp. an *ordered set*) if for any $x, y \in X$ there exists a $z \in X$ such that $x \leq z$ and $y \leq z$ (resp. if the relation $\leq$ is also antisymmetric).

Our main references are: [20] – for category theory and Stone dualities, [13] – for general topology, and [23] – for algebra.

2. SPECTRA

2.1. THE CATEGORY OF ABSTRACT SPECTRA

Notation 2.1. Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be a non-empty bitopological space. Then we put $\mathcal{L}^+ = \{U \in \mathcal{T}^+ : S \setminus U \in \mathcal{T}^-\}$ and $\mathcal{L}^- = \{U \in \mathcal{T}^- : S \setminus U \in \mathcal{T}^+\}$.

Proposition 2.2. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be a non-empty bitopological space. Then the families $\mathcal{L}^+$ and $\mathcal{L}^-$ (see 2.1 for the notation) are closed under finite unions and finite intersections.*

Proof. It is obvious. □

Definition 2.3. *A non-empty bitopological space $(S, \mathcal{T}^+, \mathcal{T}^-)$ is called an* abstract spectrum, *if it has the following properties:*

(SP1) *$\mathcal{L}^+$ is a base for $\mathcal{T}^+$ and $\mathcal{L}^-$ is a base for $\mathcal{T}^-$;*

(SP2) *if* $F \subseteq S$ *and* $S \setminus F \in \mathcal{T}^+$ *(resp.* $S \setminus F \in \mathcal{T}^-$*), then* F *is a compact subset of the topological space* $(S, \mathcal{T}^-)$ *(resp.* $(S, \mathcal{T}^+)$*);*

(SP3) *at least one of the topological spaces* $(S, \mathcal{T}^+)$ *and* $(S, \mathcal{T}^-)$ *is a* T_0*-space.*

Proposition 2.4. *If* $(S, \mathcal{T}^+, \mathcal{T}^-)$ *is an abstract spectrum, then* $(S, \mathcal{T}^+)$ *and* $(S, \mathcal{T}^-)$ *are compact* T_0*-spaces.*

Proof. By (SP3), one of the spaces $(S, \mathcal{T}^+)$ and $(S, \mathcal{T}^-)$ is T_0-space. Let, for example, $(S, \mathcal{T}^+)$ be a T_0-space. Then we shall prove that $(S, \mathcal{T}^-)$ is also a T_0-space.

Let $x, y \in S$ and $x \neq y$. Then there exists $U \in \mathcal{T}^+$ such that $|U \cap \{x, y\}| = 1$. Let, for example, $x \in U$. Then, using (SP1), we can find a $V \in \mathcal{L}^+$ such that $x \in V \subseteq U$. Putting $W = S \setminus V$, we obtain that $W \in \mathcal{T}^-$, $y \in W$ and $x \notin W$. Therefore, $(S, \mathcal{T}^-)$ is a T_0-space.

Since S is a closed subset of $(S, \mathcal{T}^+)$, the condition (SP2) implies that S is a compact subset of $(S, \mathcal{T}^-)$.

Analogously, we obtain that $(S, \mathcal{T}^+)$ is a compact space. □

Proposition 2.5. *Let* $(S, \mathcal{T}^+, \mathcal{T}^-)$ *be an abstract spectrum. Then* $\mathcal{L}^+ = \{U \in \mathcal{T}^+ : U$ *is a compact subset of* $(S, \mathcal{T}^+)\}$ *and* $\mathcal{L}^- = \{U \in \mathcal{T}^- : U$ *is a compact subset of* $(S, \mathcal{T}^-)\}$ *(see 2.1 for the notation).*

Proof. Let us prove first that $\mathcal{L}^+ = \{U \in \mathcal{T}^+ : U$ is a compact subset of $(S, \mathcal{T}^+)\}$.

If $V \in \mathcal{L}^+$ then $S \setminus V \in \mathcal{T}^-$. Hence V is a closed subset of $(S, \mathcal{T}^-)$. This implies, by (SP2), that V is a compact subset of $(S, \mathcal{T}^+)$. Conversely, if $U \in \mathcal{T}^+$ and U is a compact subset of $(S, \mathcal{T}^+)$ then for every $x \in U$ there exists a $U_x \in \mathcal{L}^+$ such that $x \in U_x \subseteq U$. Choose a finite subcover $\{U_{x_i} : i = 1, \ldots, n\}$ of the cover $\{U_x : x \in U\}$ of the compact set U. Then $U = \bigcup\{U_{x_i} : i = 1, \ldots, n\}$ and hence, by 2.2, $U \in \mathcal{L}^+$.

The proof of the equation $\mathcal{L}^- = \{U \in \mathcal{T}^- : U$ is a compact subset of $(S, \mathcal{T}^-)\}$ is analogous. □

Proposition 2.6. *Let* $(S, \mathcal{T}^+, \mathcal{T}^-)$ *be an abstract spectrum. Then* $\mathcal{L}^+ = \{S \setminus U : U \in \mathcal{L}^-\}$ *and* $\mathcal{L}^- = \{S \setminus U : U \in \mathcal{L}^+\}$ *(see 2.1 for the notation).*

Proof. Let us prove that $\mathcal{L}^- = \{S \setminus U : U \in \mathcal{L}^+\}$.

Take $V \in \mathcal{L}^-$ and put $U = S \setminus V$. Then $U \in \mathcal{T}^+$ and $S \setminus U \in \mathcal{L}^- \subseteq \mathcal{T}^-$. Hence, $U \in \mathcal{L}^+$ and $V = S \setminus U$. Conversely, if $U \in \mathcal{L}^+$ then $V = S \setminus U \in \mathcal{T}^-$ and $S \setminus V \in \mathcal{L}^+ \subseteq \mathcal{T}^+$. Therefore, $S \setminus U \in \mathcal{L}^-$.

The proof of the equation $\mathcal{L}^+ = \{S \setminus U : U \in \mathcal{L}^-\}$ is analogous. □

Corollary 2.7. *Let* $(S, \mathcal{T}^+, \mathcal{T}_1^-)$ *and* $(S, \mathcal{T}^+, \mathcal{T}_2^-)$ *be abstract spectra. Then the topologies* $\mathcal{T}_1^-$ *and* $\mathcal{T}_2^-$ *coincide.*

Proof. It follows directly from 2.5, 2.6 and (SP1) (see 2.3). $\square$

Definition 2.8. *Let* $(S_1, \mathfrak{T}_1^+, \mathfrak{T}_1^-)$ *and* $(S_2, \mathfrak{T}_2^+, \mathfrak{T}_2^-)$ *be abstract spectra. Then a function* $f \in \boldsymbol{Set}(S_1, S_2)$ *is called an* $\boldsymbol{S}$-morphism *if* $f : (S_1, \mathfrak{T}_1^+) \longrightarrow (S_2, \mathfrak{T}_2^+)$ *and* $f : (S_1, \mathfrak{T}_1^-) \longrightarrow (S_2, \mathfrak{T}_2^-)$ *are continuous maps. The class of all abstract spectra together with the class of all* $\boldsymbol{S}$*-morphisms and the natural composition between them form, obviously, a category which will be denoted by* $\boldsymbol{S}$ *and will be called* the category of abstract spectra*.*

Definition 2.9. *An abstract spectrum* $(S, \mathfrak{T}^+, \mathfrak{T}^-)$ *is called a* Stone spectrum *if the topologies* $\mathfrak{T}^+$ *and* $\mathfrak{T}^-$ *coincide.*

Proposition 2.10. *Let* $(S, \mathfrak{T})$ *be a topological space. Then the bitopological space* $(S, \mathfrak{T}, \mathfrak{T})$ *is a Stone spectrum if and only if* $(S, \mathfrak{T})$ *is a Stone space.*

Proof. ($\Rightarrow$) Let $(S, \mathfrak{T}, \mathfrak{T})$ be a Stone spectrum. Then, by 2.4, $(S, \mathfrak{T})$ is a compact T_0-space. According to (SP1) (see 2.3), the family $\mathcal{L}^+ = \{U \in \mathfrak{T} : S \setminus U \in \mathfrak{T}\}$ is a base for $\mathfrak{T}$. Consequently $(S, \mathfrak{T})$ is a zero-dimensional space. We shall show that it is also a T_2-space. Indeed, let $x, y \in S$ and $x \neq y$. Then there exists a $U \in \mathfrak{T}$ such that $|U \cap \{x, y\}| = 1$. Let, for example, $x \in U$. Since $\mathcal{L}^+$ is a base for $\mathfrak{T}$, we can find a $V \in \mathcal{L}^+$ such that $x \in V \subseteq U$. Then $x \in V \in \mathfrak{T}$ and $y \in S \setminus V \in \mathfrak{T}$. Therefore, $(S, \mathfrak{T})$ is a T_2-space. So, we proved that $(S, \mathfrak{T})$ is a compact zero-dimensional T_2-space, i.e. a Stone space.

($\Leftarrow$) Let $(S, \mathfrak{T})$ be a Stone space. Put $\mathcal{L} = \{U \in \mathfrak{T} : S \setminus U \in \mathfrak{T}\}$ and $\mathfrak{T}^+ = \mathfrak{T}^- = \mathfrak{T}$. Then $\mathcal{L}^+ = \mathcal{L} = \mathcal{L}^-$ (see 2.1 for the notation). We shall prove that $(S, \mathfrak{T}^+, \mathfrak{T}^-)$ is an abstract spectrum. Then it will be automatically a Stone spectrum. Since $\mathcal{L}$ is a base for $(S, \mathfrak{T})$, the axiom (SP1) (see 2.3) is fulfilled. The axioms (SP2) and (SP3) are also fulfilled, since $(S, \mathfrak{T})$ is a compact T_2-space. Consequently $(S, \mathfrak{T}^+, \mathfrak{T}^-)$ is an abstract spectrum. $\square$

Proposition 2.11. *An abstract spectrum* $(S, \mathfrak{T}^+, \mathfrak{T}^-)$ *is a Stone spectrum if and only if* $(S, \mathfrak{T}^+)$ *and* $(S, \mathfrak{T}^-)$ *are* T_1*-spaces.*

Proof. ($\Rightarrow$) Since $(S, \mathfrak{T}^+, \mathfrak{T}^-)$ is a Stone spectrum, we have that $\mathfrak{T}^+ = \mathfrak{T}^-$. Then 2.10 implies that $(S, \mathfrak{T}^+)$ and $(S, \mathfrak{T}^-)$ are even T_2-spaces.

($\Leftarrow$) Let $(S, \mathfrak{T}^+)$ and $(S, \mathfrak{T}^-)$ are T_1-spaces. We shall prove that $\mathfrak{T}^+ = \mathfrak{T}^-$.

Let $U \in \mathfrak{T}^-$. Then $S \setminus U$ is closed in $(S, \mathfrak{T}^-)$ and hence, by 2.4, it is a compact subset of $(S, \mathfrak{T}^-)$. Let $x \in U$. Since $(S, \mathfrak{T}^+)$ is a T_1-space, for every $y \in S \setminus U$ there exists a $V_y \in \mathcal{L}^+$ such that $x \in V_y \subseteq S \setminus \{y\}$. Hence $y \in S \setminus V_y \subseteq S \setminus \{x\}$ and $S \setminus V_y \in \mathfrak{T}^-$. Let $\{S \setminus V_{y_i} : i = 1, \ldots, n\}$ be a finite subcover of the cover $\{S \setminus V_y : y \in S \setminus U\}$ of $S \setminus U$ and let $V_x = \bigcap\{V_{y_i} : i = 1, \ldots, n\}$. Then $x \in V_x \in \mathfrak{T}^+$ and $V_x \subseteq U$. We obtain that $U = \bigcup\{V_x : x \in U\} \in \mathfrak{T}^+$. Hence $\mathfrak{T}^- \subseteq \mathfrak{T}^+$. Analogously, using the fact that $(S, \mathfrak{T}^-)$ is a T_1-space, we prove that $\mathfrak{T}^+ \subseteq \mathfrak{T}^-$. Therefore $\mathfrak{T}^+ = \mathfrak{T}^-$, i.e. $(S, \mathfrak{T}^-, \mathfrak{T}^+)$ is a Stone spectrum. $\square$

Remark 2.12. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then, arguing as in 2.4, we obtain that $(S, \mathcal{T}^+)$ and $(S, \mathcal{T}^-)$ are T_1-spaces if and only if at least one of them is a T_1-space.*

Proposition 2.13. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum and let us put $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$. Then $(S, \mathcal{T})$ is a Stone space and hence (see 2.10) $(S, \mathcal{T}, \mathcal{T})$ is a Stone spectrum.*

Proof. The topology $\mathcal{T}$ has as a subbase the family $\mathcal{P} = \mathcal{T}^+ \cup \mathcal{T}^-$. Hence the family $\mathcal{B} = \{U^+ \cap U^- : U^+ \in \mathcal{T}^+,\ U^- \in \mathcal{T}^-\}$ is a base for $\mathcal{T}$. Then, obviously, the family $\mathcal{B}_0 = \{U^+ \cap U^- : U^+ \in \mathcal{L}^+,\ U^- \in \mathcal{L}^-\}$ is also a base for $\mathcal{T}$. For every $U \in \mathcal{L}^+$ we have that $U \in \mathcal{T}^+ \subseteq \mathcal{T}$ and $S \setminus U \in \mathcal{T}^- \subseteq \mathcal{T}$. Consequently the elements of $\mathcal{L}^+$ are clopen subsets of $(S, \mathcal{T})$. Obviously, the same is true for the elements of $\mathcal{L}^-$. Hence the elements of $\mathcal{B}_0$ are clopen in $(S, \mathcal{T})$, which implies that $(S, \mathcal{T})$ is a zero-dimensional space. This fact, together with (SP3) (see 2.3), shows that $(S, \mathcal{T})$ is a Hausdorff space.

Applying Alexander subbase theorem to the subbase $\mathcal{P}$ of $(S, \mathcal{T})$, we shall prove that $(S, \mathcal{T})$ is a compact space. Indeed, let $S = \bigcup\{U_\alpha \in \mathcal{T}^+ : \alpha \in A\} \cup \bigcup\{V_\beta \in \mathcal{T}^- : \beta \in B\}$ and $F = S \setminus \bigcup\{U_\alpha : \alpha \in A\}$. Then $F \subseteq \bigcup\{V_\beta : \beta \in B\}$ and F is closed in $(S, \mathcal{T}^+)$. Consequently, by (SP2) (see 2.3), F is a compact subset of $(S, \mathcal{T}^-)$. This implies that there exist $\beta_1, \ldots, \beta_n \in B$ such that $F \subseteq \bigcup\{V_{\beta_i} : i = 1, \ldots, n\}$. Then $G = S \setminus \bigcup\{V_{\beta_i} : i = 1, \ldots, n\} \subseteq \bigcup\{U_\alpha : \alpha \in A\}$. Since G is a closed subset of $(S, \mathcal{T}^-)$, it is a compact subset of $(S, \mathcal{T}^+)$ (by (SP2)). Hence, there exist $\alpha_1, \ldots, \alpha_m \in A$ such that $G \subseteq \bigcup\{U_{\alpha_j} : j = 1, \ldots, m\}$. Therefore, $S = \bigcup\{U_{\alpha_j} : j = 1, \ldots, m\} \cup \bigcup\{V_{\beta_i} : i = 1, \ldots, n\}$. This shows that $(S, \mathcal{T})$ is compact. Hence, $(S, \mathcal{T})$ is a Stone space. □

Remark 2.14. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum and $id : S \longrightarrow S$, $x \longrightarrow x$, be the identity function. Then, obviously, $id \in \boldsymbol{S}((S, \mathcal{T}, \mathcal{T}), (S, \mathcal{T}^+, \mathcal{T}^-))$ (see 2.13 for the notation).*

Proposition 2.15. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be a bitopological space such that $\mathcal{L}^+$ is a base for $\mathcal{T}^+$ and $\mathcal{L}^-$ is a base for $\mathcal{T}^-$ (see 2.1 for the notation). Let $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$, $(S, \mathcal{T})$ be a compact T_2-space, $S_1 \subseteq S$, $\mathcal{T}_1^+ = \{U \cap S_1 : U \in \mathcal{T}^+\}$ and $\mathcal{T}_1^- = \{U \cap S_1 : U \in \mathcal{T}^-\}$. Then the bitopological space $(S_1, \mathcal{T}_1^+, \mathcal{T}_1^-)$ is an abstract spectrum iff S_1 is a closed subset of the topological space $(S, \mathcal{T})$.*

Proof. ($\Rightarrow$) Let $\mathcal{T}_1 = \sup\{\mathcal{T}_1^+, \mathcal{T}_1^-\}$. Then, by 2.13, $(S_1, \mathcal{T}_1)$ is a Stone space. Hence it is a compact Hausdorff space. Since, obviously, $\mathcal{T}_1 = \mathcal{T}|S_1$, we obtain that S_1 is a compact subspace of the Hausdorff space $(S, \mathcal{T})$. Consequently S_1 is a closed subset of $(S, \mathcal{T})$.

($\Leftarrow$) We shall show that $(S_1, \mathcal{T}_1^+, \mathcal{T}_1^-)$ is an abstract spectrum. Let $\mathcal{L}_1^+ = \{U \cap S_1 : U \in \mathcal{L}^+\}$, $\mathcal{L}_1^- = \{U \cap S_1 : U \in \mathcal{L}^-\}$, $\mathcal{L}_{S_1}^+ = \{U \in \mathcal{T}_1^+ : S_1 \setminus U \in \mathcal{T}_1^-\}$ and $\mathcal{L}_{S_1}^- = \{U \in \mathcal{T}_1^- : S_1 \setminus U \in \mathcal{T}_1^+\}$. Then, obviously, $\mathcal{L}_1^+ \subseteq \mathcal{L}_{S_1}^+$ and $\mathcal{L}_1^- \subseteq \mathcal{L}_{S_1}^-$. Since $\mathcal{L}_1^+$ (resp. $\mathcal{L}_1^-$) is a base for $(S_1, \mathcal{T}_1^+)$ (resp. $(S_1, \mathcal{T}_1^-)$), we obtain that $\mathcal{L}_{S_1}^+$ (resp.

$\mathcal{L}^-_{S_1}$) is a base for $(S_1, \mathcal{T}^+_1)$ (resp. $(S_1, \mathcal{T}^-_1)$). Hence the condition (SP1) (see 2.3) is fulfilled.

In the part ($\Rightarrow$) of this proof, we noted that the topology $\mathcal{T}_1 = \sup\{\mathcal{T}^+_1, \mathcal{T}^-_1\}$ on S_1 coincides with the topology $\mathcal{T}|S_1$. Hence, $(S_1, \mathcal{T}_1)$ is a compact Hausdorff space (since $(S, \mathcal{T})$ is such and S_1 is a closed subset of $(S, \mathcal{T})$). Let now F be a closed subset of $(S_1, \mathcal{T}^+_1)$ (resp. $(S_1, \mathcal{T}^-_1)$). Then F is a closed subset of $(S_1, \mathcal{T}_1)$. Therefore F is a compact subset of $(S_1, \mathcal{T}_1)$. Since the identity maps $id : (S_1, \mathcal{T}_1) \longrightarrow (S_1, \mathcal{T}^+_1)$ and $id : (S_1, \mathcal{T}_1) \longrightarrow (S_1, \mathcal{T}^-_1)$ are continuous, we obtain that F is a compact subset of $(S_1, \mathcal{T}^-_1)$ (resp. $(S_1, \mathcal{T}^+_1)$). Hence, the condition (SP2) (see 2.3) is fulfilled.

For showing that the condition (SP3) (see 2.3) is fulfilled, it is enough to prove that $(S_1, \mathcal{T}^+_1)$ is a T_0-space. Let $x, y \in S_1$ and $x \neq y$. Since $(S_1, \mathcal{T}_1)$ is a T_2-space, there exist $U \in \mathcal{L}^+_1$ and $V \in \mathcal{L}^-_1$ such that $x \in U \cap V \subseteq S_1 \setminus \{y\}$. If $y \notin U$ then the element U of $\mathcal{T}^+_1$ separates x and y. If $y \in U$ then $y \notin V$. Hence $y \in S_1 \setminus V$ and $x \notin S_1 \setminus V$. Since $S_1 \setminus V \in \mathcal{T}^+_1$, we obtain that x and y are separated by an element of $\mathcal{T}^+_1$. Consequently, $(S_1, \mathcal{T}^+_1)$ is a T_0-space. □

Corollary 2.16. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum, $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$, $S_1 \subseteq S$, $\mathcal{T}^+_1 = \{U \cap S_1 : U \in \mathcal{T}^+\}$ and $\mathcal{T}^-_1 = \{U \cap S_1 : U \in \mathcal{T}^-\}$. Then the bitopological space $(S_1, \mathcal{T}^+_1, \mathcal{T}^-_1)$ is an abstract spectrum iff S_1 is a closed subset of the topological space $(S, \mathcal{T})$.*

Proof. It follows immediately from 2.15, 2.3 and 2.13. □

2.2. EXAMPLES OF ABSTRACT SPECTRA

Lemma 2.17. *Let X be a set and $Exp(X)$ be the family of all subsets of X. Let us put, for every $x \in X$, $\tilde{U}^+_x = \{A \subseteq X : x \notin A\}$ and $\tilde{U}^-_x = \{A \subseteq X : x \in A\}$. Let $\tilde{\mathcal{P}}^+ = \{\tilde{U}^+_x : x \in X\}$, $\tilde{\mathcal{P}}^- = \{\tilde{U}^-_x : x \in X\}$, $\tilde{\mathcal{T}}^+$ (resp. $\tilde{\mathcal{T}}^-$) be the topology on $Exp(X)$ having $\tilde{\mathcal{P}}^+$ (resp. $\tilde{\mathcal{P}}^-$) as a subbase and $\tilde{\mathcal{T}} = \sup\{\tilde{\mathcal{T}}^+, \tilde{\mathcal{T}}^-\}$. Let us identify the set $Exp(X)$ with the set $\boldsymbol{D}^X$ (where $\boldsymbol{D}$ is the two-point set $\{0, 1\}$) by means of the map $e : Exp(X) \longrightarrow \boldsymbol{D}^X$, $A \subseteq X \longrightarrow \chi_A$, where $\chi_A : X \longrightarrow \boldsymbol{D}$ is the characteristic function of A, i.e. $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Then the topology $\tilde{\mathcal{T}}$ on $Exp(X)$ coincides with the Tychonoff topology on $\boldsymbol{D}^X$ (where the set $\boldsymbol{D}$ is endowed with the discrete topology).*

Proof. Let $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}^+ \cup \tilde{\mathcal{P}}^-$. Then $\tilde{\mathcal{P}}$ is a subbase for the topology $\tilde{\mathcal{T}}$ on $Exp(X)$. For every $x \in X$ we have, identifying $Exp(X)$ and $\boldsymbol{D}^X$ by means of the map e, that $\tilde{U}^+_x = \{f \in \boldsymbol{D}^X : f(x) = 0\}$ and $\tilde{U}^-_x = \{f \in \boldsymbol{D}^X : f(x) = 1\}$. Now it becomes clear that the family $\tilde{\mathcal{P}}$ is also a subbase for the Tychonoff topology on $\boldsymbol{D}^X$ when $\boldsymbol{D}$ is endowed with the discrete topology. Therefore the topology $\tilde{\mathcal{T}}$ on $Exp(X)$ coincides with the Tychonoff topology on $\boldsymbol{D}^X$. □

Proposition 2.18. *Let X be a set and S be family of subsets of X (i.e. $S \subseteq Exp(X)$). Let us put, for every $x \in X$, $U_x^+ = \{p \in S : x \notin p\}$ and $U_x^- = \{p \in S : x \in p\}$. Let $\mathcal{P}^+ = \{U_x^+ : x \in X\}$, $\mathcal{P}^- = \{U_x^- : x \in X\}$, $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) be the topology on S having $\mathcal{P}^+$ (resp. $\mathcal{P}^-$) as a subbase and $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$.*

Then the following conditions are equivalent:

(a) $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum;

(b) $(S, \mathcal{T})$ is a compact T_2-space;

(c) S is a closed subset of the Cantor cube $\boldsymbol{D}^X$ (where $\boldsymbol{D}$ is the discrete two-point space and S is identified with a subset of $\boldsymbol{D}^X$ as in 2.17).

Proof. $(a) \Rightarrow (b)$. This follows from 2.13.

$(b) \Rightarrow (a)$. Let $x \in X$. Then $S \setminus U_x^+ = U_x^-$ and $S \setminus U_x^- = U_x^+$. Hence $\mathcal{P}^+ \subseteq \mathcal{L}^+$ and $\mathcal{P}^- \subseteq \mathcal{L}^-$ (see 2.1 for the notation). Consequently, using 2.2, we obtain that $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) is a base for $(S, \mathcal{T}^+)$ (resp. $(S, \mathcal{T}^-)$). This shows that putting $S_1 = S$ in 2.15, we get that $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum.

$(b) \Rightarrow (c)$. It is clear from the corresponding definitions that, using the notation of 2.17, we have $\tilde{U}_x^+ \cap S = U_x^+$ and $\tilde{U}_x^- \cap S = U_x^-$ for every $x \in X$. Hence, by 2.17, the topology $\mathcal{T}$ on S coincides with the subspace topology on S induced by the Tychonoff topology on $\boldsymbol{D}^X$. Then the condition (b) and the fact that $\boldsymbol{D}^X$ is a Hausdorff space imply that S is a closed subset of the Cantor cube $\boldsymbol{D}^X$.

$(c) \Rightarrow (b)$. In the preceding paragraph we have already noted that the topology $\mathcal{T}$ on S coincides with the subspace topology on S induced by the Tychonoff topology on $\boldsymbol{D}^X$. Therefore the condition (c) implies that $(S, \mathcal{T})$ is a compact Hausdorff space (since $\boldsymbol{D}^X$ is such). □

Definition 2.19. *Let X be a set endowed with two arbitrary multivalued binary operations $\oplus$ and $\otimes$. Let us call a subset p of X a* prime ideal *in $(X, \oplus, \otimes)$ if the following two conditions are fulfilled:*

i) if $x, y \in p$ then $x \oplus y \subseteq p$;

ii) if $(x \otimes y) \cap p \neq \emptyset$ then $x \in p$ or $y \in p$.

Let us fix two different points 0 and 1 of X. We shall say that a prime ideal $p \subseteq X$ is proper *(or, more precisely,* proper with respect to the points 0 and 1*), if $0 \in p$ and $1 \notin p$.*

A subset q of X is called a prime (proper) flter *in $(X, \oplus, \otimes)$ if the set $X \setminus q$ is a prime (proper) ideal.*

Theorem 2.20. *Let X be a set endowed with two arbitrary multivalued binary operations $\oplus$ and $\otimes$ and two fixed different points ξ_o and ξ_1. Denote by $S(X)$ (resp. $S(X)_{pr}$) the set of all (resp. all proper) prime ideals in $(X, \oplus, \otimes)$ and define the topologies $\mathcal{T}^+$ and $\mathcal{T}^-$ on $S(X)$ (resp. $\mathcal{T}_{pr}^+$ and $\mathcal{T}_{pr}^-$ on $S(X)_{pr}$) exactly as in 2.18. Then the bitopological spaces $(S(X), \mathcal{T}^+, \mathcal{T}^-)$ and $(S(X)_{pr}, \mathcal{T}_{pr}^+, \mathcal{T}_{pr}^-)$ are abstract spectra.*

Proof. We first prove that the bitopological space $(S(X), \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum. For doing this it is enough to show that $S(X)$ is a closed subset of the Cantor cube $\boldsymbol{D}^X$ (see 2.18).

Let $\{p_\sigma \in S(X) : \sigma \in \Sigma\}$ be a net in the Cantor cube $\boldsymbol{D}^X$ converging to a point $p \in \boldsymbol{D}^X$. We have to prove that $p \in S(X)$, i.e. that p is a prime ideal in $(X, \oplus, \otimes)$. Let $f_\sigma = e(p_\sigma)$ and $f = e(p)$ (see 2.17 for the notation). Then the net $\{f_\sigma, \sigma \in \Sigma\}$ in $\boldsymbol{D}^X$ converges to f, i.e., for every $x \in X$, the net $\{f_\sigma(x), \sigma \in \Sigma\}$ in the discrete space $\boldsymbol{D}$ converges to $f(x)$.

Let $a, b \in p$. Then $f(a) = f(b) = 1$. Therefore there exists a $\sigma_0 \in \Sigma$ such that $f_\sigma(a) = 1 = f_\sigma(b)$ for every $\sigma > \sigma_0$. This means that for every $\sigma > \sigma_0$ we have that $a \in p_\sigma$ and $b \in p_\sigma$. Since p_σ is a prime ideal, we obtain that $a \oplus b \subseteq p_\sigma$ for every $\sigma > \sigma_0$. Then, for every $x \in a \oplus b$ and for every $\sigma > \sigma_0$, we have that $f_\sigma(x) = 1$. This implies that $f(x) = 1$ for every $x \in a \oplus b$. Hence, if $x \in a \oplus b$ then $x \in p$, i.e. $a \oplus b \subseteq p$.

Let $a, b \in X$ and $(a \otimes b) \cap p \neq \emptyset$. Then there exists a $x \in (a \otimes b) \cap p$. Hence $f(x) = 1$. This implies that there exists a $\sigma_0 \in \Sigma$ such that $f_\sigma(x) = 1$ for every $\sigma > \sigma_0$. Consequently $x \in p_\sigma$ for every $\sigma > \sigma_0$. Then $(a \otimes b) \cap p_\sigma \neq \emptyset$ for every $\sigma > \sigma_0$. Hence, for every $\sigma > \sigma_0$, we have that $a \in p_\sigma$ or $b \in p_\sigma$, i.e. $f_\sigma(a) = 1$ or $f_\sigma(b) = 1$. Suppose that $a \notin p$ and $b \notin p$. Then $f(a) = 0 = f(b)$. Therefore, there exists a $\sigma_1 \in \Sigma$ such that $f_\sigma(a) = f_\sigma(b) = 0$ for every $\sigma > \sigma_1$. Since for every $\sigma > \sup\{\sigma_0, \sigma_1\}$ we have that $f_\sigma(a) = 1$ or $f_\sigma(b) = 1$, we get a contradiction. Hence we obtain that $a \in p$ or $b \in p$. So, we proved that p is a prime ideal in $(X, \oplus, \otimes)$. This shows that $S(X)$ is a closed subset of the Cantor cube $\boldsymbol{D}^X$. Hence, the bitopological space $(S(X), \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum.

If the prime ideals p_σ in the above proof were proper, then, obviously, p would be also proper. This shows that the set $S(X)_{pr}$ is also a closed subset of the Cantor cube $\boldsymbol{D}^X$. So, the bitopological space $(S(X)_{pr}, \mathcal{T}^+_{pr}, \mathcal{T}^-_{pr})$ is an abstract spectrum. □

Example 2.21. Let $(A, +, .)$ be a commutative ring with unit $(0 \neq 1)$, $x \oplus y$ be the ideal in the ring $(A, +, .)$ generated by $\{x, y\}$, and $x \otimes y = x.y$, for every $x, y \in A$. Then, applying the construction from 2.20 to the set A with the operations $\oplus$ and $\otimes$ and with fixed points 0 and 1, we get the topological space $(S(A)_{pr}, \mathcal{T}^+_{pr})$. We assert that it coincides with the classical spectrum of the ring $(A, +, .)$.

Proof. Recall that: a) a subgroup I of the additive group $(A, +)$ is called an *ideal* in the commutative ring $(A, +, .)$ with unit if $A.I = I$; b) an ideal $p \neq A$ in the ring A is said to be a *prime ideal* if $(x, y \in A,\ x.y \in p) \Rightarrow (x \in p \text{ or } y \in p)$; c) the set of all prime ideals in the commutative ring A is denoted by $spec(A)$; d) the family $\mathcal{Z} = \{U_I = \{p \in spec(A) : I \not\subseteq p\} : I \text{ is an ideal in } A\}$ is a topology on the set $spec(A)$, called *Zariski topology*; e) the topological space $(spec(A), \mathcal{Z})$ is the classical spectrum of the commutative ring $(A, +, .)$ with unit.

We shall denote by $I(M)$ the ideal in A generated by a subset M of A.

We first prove that the sets $spec(A)$ and $S(A)_{pr}$ coincide.

Let $p \in S(A)_{pr}$. Then $1 \notin p$ and hence $p \neq A$. If $a, b \in p$ then $a \oplus b \subseteq p$, i.e. $I(\{a, b\}) \subseteq p$. Hence, $a - b \in p$. This shows that p is an additive subgroup of A. Let $x \in A$ and $a \in p$. Since $a \oplus a = I(\{a\}) \subseteq p$, we get that $x.a \in p$. If $x, y \in A$ and $x.y \in p$, then $(x \otimes y) \cap p \neq \emptyset$ and, hence, $x \in p$ or $y \in p$. Consequently, we proved that $p \in spec(A)$.

Conversely, let $p \in spec(A)$ and $a, b \in p$. Then, obviously, $I(\{a, b\}) \subseteq p$ and, hence, $a \oplus b \subseteq p$. If $(a \otimes b) \cap p \neq \emptyset$ then $a.b \in p$. This implies that $a \in p$ or $b \in p$. Since $1 \notin p$, we get that $p \in S(A)_{pr}$. Therefore, $S(A)_{pr} = spec(A)$.

Now we prove that $\mathcal{T}^+_{pr} = \mathcal{Z}$.

Let $a \in A$. Then, obviously, $U^+_a = \{p \in S(A)_{pr} : a \notin p\} = \{p \in spec(A) : I(\{a\}) \not\subseteq p\} \in \mathcal{Z}$. Hence, $\mathcal{T}^+_{pr} \subseteq \mathcal{Z}$. Conversely, let $U \in \mathcal{Z}$. Then there exists an ideal I in A such that $U = U_I$. Let $p \in U$. Then there exists an $a = a(p) \in I \setminus p$. Hence $p \in U^+_a$. We shall prove that $U^+_a \subseteq U$. Indeed, if $q \in U^+_a$ then $a \notin q$ and, consequently, $I \not\subseteq q$. This shows that $q \in U_I = U$. So, we obtained that $p \in U^+_a \subseteq U$. Therefore, $\mathcal{Z} \subseteq \mathcal{T}^+_{pr}$. □

Example 2.22. Let $(L, \vee, \wedge)$ be a distributive lattice with 0 and 1 and let us put $x \oplus y = \{z \in L : z \leq x \vee y\}$ and $x \otimes y = \{z \in L : z \geq x \wedge y\}$, for every $x, y \in L$. Then, applying the construction from 2.20 to the set L with the operations $\oplus$ and $\otimes$ and with fixed points 0 and 1, we get the topological space $(S(L)_{pr}, \mathcal{T}^+_{pr})$. We assert that it coincides with the classical spectrum $spec(L)$ of the distributive lattice $(L, \vee, \wedge)$.

Proof. Recall that: a) a sub-join-semi-lattice I of the lattice L is said to be an *ideal* in L if $(a \in I,\ b \in L$ and $b \leq a) \Rightarrow (b \in I)$; b) an ideal p in L is called a *prime ideal* if $1 \notin p$ and $(a \wedge b \in p) \Rightarrow (a \in p$ or $b \in p)$; c) the set of all prime ideals in L is denoted by $spec(L)$; d) the family $\mathcal{O} = \{U_I = \{p \in spec(L) : I \not\subseteq p\} : I$ is an ideal in $L\}$ is a topology on the set $spec(L)$, called *Stone topology*; e) the topological space $(spec(L), \mathcal{O})$ is the classical spectrum of the lattice $(L, \vee, \wedge, 0, 1)$.

We first prove that the sets $spec(L)$ and $S(L)_{pr}$ coincide.

Let $p \in S(L)_{pr}$. Then $0 \in p$ and $1 \notin p$. If $a, b \in p$ then $a \oplus b \subseteq p$ and, hence, $a \vee b \in p$. Let $c \in L$, $a \in p$ and $c \leq a$. Since $a \in p$, we have that $a \oplus a \subseteq p$ and, consequently, $c \in p$. If $c, d \in L$ and $c \wedge d \in p$ then $(c \otimes d) \cap p \neq \emptyset$. Therefore $c \in p$ or $d \in p$. So, $p \in spec(L)$.

Let $p \in spec(L)$ and $a, b \in p$. Then $a \vee b \in p$ and, for all $c \in L$ such that $c \leq a \vee b$, we have that $c \in p$. Hence $a \oplus b \subseteq p$. Let $x, y \in p$ and $(x \otimes y) \cap p \neq \emptyset$. Then there exists a $z \in p$ such that $z \geq x \wedge y$. Hence $x \wedge y \in p$. This implies that $x \in p$ or $y \in p$. Since $1 \notin p$, we obtain that $p \in S(L)_{pr}$. So, $S(L)_{pr} = spec(L)$.

Now we prove that $\mathcal{T}^+_{pr} = \mathcal{O}$.

Let $a \in L$ and $I(a) = \{x \in L : x \leq a\}$. Then $I(a)$ is an ideal in L. Obviously, $U^+_a = \{p \in S(L)_{pr} : a \notin p\} = \{p \in spec(L) : I(a) \not\subseteq p\} \in \mathcal{O}$. Hence $\mathcal{T}^+_{pr} \subseteq \mathcal{O}$. Conversely, let $U \in \mathcal{O}$. Then there exists an ideal I in L such that $U = U_I$. Let $p \in U$. Then there exists an $a = a(p) \in I \setminus p$. Hence $p \in U^+_a$ and we need to prove only that $U_a^+ \subseteq U$. Let $q \in U_a^+$. Then $a \notin q$. Consequently $I \nsubseteq q$, which means that $q \in U_I = U$. So, $p \in U_a^+ \subseteq U$. We obtained that $\mathcal{O} \subseteq \mathcal{T}_{pr}^+$. □

Definition 2.23. *Let X be a set endowed with two arbitrary single-valued binary operations $+$ and $\times$. Let us call a subset p of X an* l-prime ideal in $(X, +, \times)$ *if the following two conditions are fulfilled:*

i) $x + y \in p$ *iff* $x \in p$ *and* $y \in p$*;*

ii) $x \times y \in p$ *iff* $x \in p$ *or* $y \in p$.

Let us fix two different points 0 *and* 1 *of* X. *We shall say that an l-prime ideal* $p \subseteq X$ *is* proper *(or, more precisely,* proper with respect to the points 0 and 1*), if* $0 \in p$ *and* $1 \notin p$.

Theorem 2.24. *Let X be a set endowed with two arbitrary single-valued binary operations $+$ and $\times$ and two fixed different points $\xi_o \in X$ and $\xi_1 \in X$. Denote by $S'(X)$ (resp. $S'(X)_{pr}$) the set of all (proper) l-prime ideals in $(X, +, \times)$ and define the topologies $\mathcal{T}^+$ and $\mathcal{T}^-$ on $S'(X)$ (resp. $\mathcal{T}_{pr}^+$ and $\mathcal{T}_{pr}^-$ on $S'(X)_{pr}$) exactly as in 2.18. Then the bitopological spaces $(S'(X), \mathcal{T}^+, \mathcal{T}^-)$ and $(S'(X)_{pr}, \mathcal{T}_{pr}^+, \mathcal{T}_{pr}^-)$ are abstract spectra.*

Proof. We first prove that the bitopological space $(S'(X), \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum. For doing this it is enough to show that $S'(X)$ is a closed subset of the Cantor cube $\boldsymbol{D}^X$ (see 2.18).

Let $\{p_\sigma \in S'(X) : \sigma \in \Sigma\}$ be a net in the Cantor cube $\boldsymbol{D}^X$ converging to a point $p \in \boldsymbol{D}^X$. We have to prove that $p \in S'(X)$, i.e. that p is an l-prime ideal in $(X, +, \times)$.

Exactly as in the proof of 2.20, we show that $a, b \in p$ implies that $a + b \in p$ and that if $a \times b \in p$ then $a \in p$ or $b \in p$.

Let $f_\sigma = e(p_\sigma)$ and $f = e(p)$ (see 2.17 for the notation). Then the net $\{f_\sigma, \sigma \in \Sigma\}$ in $\boldsymbol{D}^X$ converges to f, i.e., for every $x \in X$, the net $\{f_\sigma(x), \sigma \in \Sigma\}$ in the discrete space $\boldsymbol{D}$ converges to $f(x)$.

Let $a, b \in X$ and $a + b \in p$. Then $f(a + b) = 1$. Hence there exists a $\sigma_0 \in \Sigma$ such that $f_\sigma(a + b) = 1$ for every $\sigma \geq \sigma_0$. Consequently, for every $\sigma \geq \sigma_0$, we have that $a + b \in p_\sigma$. Then, for every $\sigma \geq \sigma_0$, we get that $a \in p_\sigma$ and $b \in p_\sigma$, i.e. $f_\sigma(a) = 1$ and $f_\sigma(b) = 1$. This implies that $f(a) = 1$ and $f(b) = 1$, i.e. $a \in p$ and $b \in p$.

Let $a, b \in X$ be such that $a \in p$ or $b \in p$. Suppose that $a \times b \notin p$. Then $f(a \times b) = 0$. Hence there exists a $\sigma_0 \in \Sigma$ such that $f_\sigma(a \times b) = 0$ for every $\sigma \geq \sigma_0$. This means that for every $\sigma \geq \sigma_0$, we have that $a \times b \notin p_\sigma$. Consequently, $a \notin p_\sigma$ and $b \notin p_\sigma$ for every $\sigma \geq \sigma_0$. We obtain that $f_\sigma(a) = 0$ and $f_\sigma(b) = 0$ for every $\sigma \geq \sigma_0$. This implies that $f(a) = 0$ and $f(b) = 0$, i.e. $a \notin p$ and $b \notin p$, which is a contradiction. Therefore, $a \times b \in p$. Hence, p is an l-prime ideal in $(X, +, \times)$.

This shows that $S'(X)$ is a closed subset of the Cantor cube $\boldsymbol{D}^X$. Hence, the bitopological space $(S'(X), \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum.

If the prime ideals p_σ in the above proof were proper, then, obviously, p would be also proper. This shows that the set $S'(X)_{pr}$ is also a closed subset of the Cantor cube $\boldsymbol{D}^X$. So, the bitopological space $(S'(X)_{pr}, \mathcal{T}^+_{pr}, \mathcal{T}^-_{pr})$ is an abstract spectrum.□

Example 2.25. Let $(L, \vee, \wedge)$ be a distributive lattice with 0 and 1 and let us put $x + y = x \vee y$ and $x \times y = x \wedge y$, for every $x, y \in L$. Then, applying the construction from 2.24 to the set L with the operations $+$ and $\times$ and with fixed points 0 and 1, we get the topological space $(S'(L)_{pr}, \mathcal{T}^+_{pr})$. We assert that it coincides with the classical spectrum $spec(L)$ of the distributive lattice $(L, \vee, \wedge)$.

Proof. We first prove that the sets $spec(L)$ and $S'(L)_{pr}$ coincide.

Let $p \in S'(L)_{pr}$. Then $0 \in p$ and $1 \notin p$. If $a, b \in p$ then $a + b \in p$ and, hence, $a \vee b \in p$. Let $c \in L$, $a \in p$ and $c \leq a$. Then $c \vee a = a$, i.e. $c + a \in p$. Thus $c \in p$. If $c, d \in L$ and $c \wedge d \in p$ then $c \times d \in p$. Therefore $c \in p$ or $d \in p$. So, $p \in spec(L)$.

Let $p \in spec(L)$. If $a, b \in p$ then $a \vee b \in p$, i.e. $a + b \in p$. Further, if $x, y \in L$ and $x + y \in p$, then $x \vee y \in p$ and $x \leq x \vee y$, $y \leq x \vee y$. Hence $x \in p$ and $y \in p$. So, $x + y \in p$ iff $x \in p$ and $y \in p$. Now, let $a \in p$ or $b \in p$. Then $a \wedge b \leq a$ and $a \wedge b \leq b$. Therefore $a \wedge b \in p$, i.e. $a \times b \in p$. Finally, if $x, y \in L$ and $x \times y \in p$ then $x \wedge y \in p$ and, hence, $x \in p$ or $y \in p$. So, $x \times y \in p$ iff $x \in p$ or $y \in p$. Since $0 \in p$ and $1 \notin p$, we obtain that $p \in S'(L)_{pr}$. Therefore, we proved that $S'(L)_{pr} = spec(L)$.

The proof of the equality $\mathcal{T}^+_{pr} = \mathcal{O}$ is analogous to the proof of the corresponding statement about $S(L)_{pr}$, given in the proof of 2.22. □

2.3. THE MAIN THEOREM

The main theorem of Section 2, Theorem 2.36 below, will be proved here. For doing this we need some preliminary definitions and results.

Definition 2.26. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. For every two points $a, b \in S$ we put $a \leq b$ iff $cl_{(S,\mathcal{T}^-)}\{a\} \subseteq cl_{(S,\mathcal{T}^-)}\{b\}$ (i.e., $a \leq b$ iff a is a specialization of b in the topological space $(S, \mathcal{T}^-)$).*

Remark 2.27. (a) The relation $\leq$ defined in 2.26 is a partial order on S since $(S, \mathcal{T}^-)$ is a T_0-space (see 2.4) and, as it is well known, the specialization is a partial order on every T_0-space.

(b) It is obvious that $a \leq b$ iff $a \in cl_{(S,\mathcal{T}^-)}\{b\}$ iff $b \in cl_{(S,\mathcal{T}^+)}\{a\}$ iff $cl_{(S,\mathcal{T}^+)}\{b\} \subseteq cl_{(S,\mathcal{T}^+)}\{a\}$.

(c) It is easy to see that if $a \in S$ then $cl_{(S,\mathcal{T}^+)}\{a\} = \{b \in S : b \geq a\}$ and $cl_{(S,\mathcal{T}^-)}\{a\} = \{b \in S : b \leq a\}$.

(d) If the elements of an abstract spectrum S are prime (or l-prime) ideals defined as in Section 2.2 (i.e. $S = S(X)$, where X is a set with two binary operations), then $a \leq b$ iff $a \subseteq b$, for $a, b \in S$.

Lemma 2.28. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. If the net $\{a_\sigma, \sigma \in \Sigma\}$ converges to a in $(S, \mathcal{T}^-)$, the net $\{b_\sigma, \sigma \in \Sigma\}$ converges to b in $(S, \mathcal{T}^+)$ and $a_\sigma \leq b_\sigma$ for every $\sigma \in \Sigma$, then $a \leq b$.*

Proof. Let $U \in \mathcal{L}^+$ and $b \in U$. Then there exists a $\sigma_0 \in \Sigma$ such that $b_\sigma \in U$ for every $\sigma \geq \sigma_0$. Suppose that $a \notin U$. Then $S \setminus U \in \mathcal{T}^-$ and $a \in S \setminus U$. Hence there exists a $\sigma_1 \in \Sigma$ such that $a_\sigma \in S \setminus U$ for every $\sigma \geq \sigma_1$. Putting $\sigma' = \sup\{\sigma_0, \sigma_1\}$, we obtain that $b_{\sigma'} \in U$ and $a_{\sigma'} \notin U$. Therefore $b_{\sigma'} \notin cl_{(S,T^+)}\{a_{\sigma'}\}$, i.e. $a_{\sigma'} \not\leq b_{\sigma'}$, a contradiction. Hence $a \in U$. This shows that $b \in cl_{(S,\mathcal{T}^+)}\{a\}$, i.e. $a \leq b$. □

Lemma 2.29. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. If $A \subseteq S$ and $(A, \leq)$ is a directed set (where $\leq$ is the restriction to A of the partial order defined in 2.26), then the set A has supremum in the ordered set $(S, \leq)$.*

Proof. Since $(A, \leq)$ is a directed set and $A \subseteq S$, $\{a, a \in A\}$ is a net in the compact Hausdorff space $(S, \mathcal{T})$ (where $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$) (see 2.13) and, hence, it has a cluster point $b \in S$. We shall prove that $b = \sup\{a : a \in A\}$ in $(S, \leq)$. Indeed, let $U \in \mathcal{T}^+$ and $b \in U$. Then $U \in \mathcal{T}$ and for every $a \in A$ there exists an $a' \in A$ such that $a' \geq a$ and $a' \in U$. Hence $A \subseteq U$. This shows that $b \in cl_{(S,\mathcal{T}^+)}\{a\}$ for every $a \in A$, i.e. $b \geq a$ for every $a \in A$. Let now $b' \in S$ and $b' \geq a$ for every $a \in A$. The point b is a limit in $(S, \mathcal{T})$ (and, hence, in $(S, \mathcal{T}^-)$) of a net $\{a_\sigma, \sigma \in \Sigma\}$ that is finer than the net $\{a, a \in A\}$. Put $b_\sigma = b'$ for every $\sigma \in \Sigma$. Then the net $\{b_\sigma, \sigma \in \Sigma\}$ converges to b' in $(S, \mathcal{T}^+)$. Since $a_\sigma \leq b_\sigma$ for every $\sigma \in \Sigma$, we obtain, using 2.28, that $b \leq b'$. Hence, $b = \sup A$. □

Lemma 2.30. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. If $A \subseteq S$ and $(A, \leq')$ is a directed set, where $\leq'$ is the inverse to the restriction to A of the partial order defined in 2.26 (i.e. $a' \leq' a''$ iff $a' \geq a''$, for $a', a'' \in A$), then the set A has infimum in the ordered set $(S, \leq)$.*

Proof. The proof is completely analogous to that of Lemma 2.29. □

Lemma 2.31. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then for every $s \in S$ there exists an $m \in S$ (resp. $m' \in S$) such that $s \leq m$ (resp. $m' \leq s$) and m is a maximal (resp. m' is a minimal) element of the ordered set $(S, \leq)$ (where $\leq$ is from 2.26).*

Proof. It follows from the Zorn lemma and 2.29 (resp. 2.30). □

Notation 2.32. Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. We put $Max(S) = \{m \in S : m$ is a maximal element of $(S, \leq)\}$ and $Min(S) = \{m \in S : m$ is a minimal element of $(S, \leq)\}$ (where $\leq$ is from 2.26). We shall denote by $\mathcal{T}^+_M$ (resp. $\mathcal{T}^-_M$) the induced by $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) topology on $Max(S)$, and by $\mathcal{T}^+_m$ (resp. $\mathcal{T}^-_m$) the induced by $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) topology on $Min(S)$.

Proposition 2.33. *Let $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then:*

(a) $(Max(S), \mathcal{T}^+_M)$ *and* $(Min(S), \mathcal{T}^-_m)$ *are compact* T_1*-spaces;*

(b) $(Min(S), \mathcal{T}^+_m)$ *and* $(Max(S), \mathcal{T}^-_M)$ *are* T_2*-spaces;*

(c) $Min(S)$ *is dense in* $(S, \mathcal{T}^+)$ *and* $Max(S)$ *is dense in* $(S, \mathcal{T}^-)$.

Proof. (a) We first prove that $(Max(S), \mathcal{T}^+_M)$ is a compact T_1-space. Since, for every $a \in S$, $cl_{(S,\mathcal{T}^+)}\{a\} = \{b \in S : b \geq a\}$ (see 2.27(c)), we obtain that $(Max(S), \mathcal{T}^+_M)$ is a T_1-space. Let $\{a_\sigma, \sigma \in \Sigma\}$ be a net in $(Max(S), \mathcal{T}^+_M)$. Then $\{a_\sigma, \sigma \in \Sigma\}$ is a net in the compact space $(S, \mathcal{T}^+)$ (see 2.4) and, hence, it has a cluster point $a \in S$ in $(S, \mathcal{T}^+)$. Now, we can find a net $\{a_{\sigma'}, \sigma' \in \Sigma'\}$ in $(Max(S), \mathcal{T}^+_M)$ which is finer than the net $\{a_\sigma, \sigma \in \Sigma\}$ and converges to a in $(S, \mathcal{T}^+)$. By 2.31, there exists an $a' \in Max(S)$ such that $a \leq a'$. Then $a' \in cl_{(S,\mathcal{T}^+)}\{a\}$ and, hence, the net $\{a_{\sigma'}, \sigma' \in \Sigma'\}$ converges to a' in $(Max(S), \mathcal{T}^+_M)$. This shows that the net $\{a_\sigma, \sigma \in \Sigma\}$ has a cluster point in $(Max(S), \mathcal{T}^+_M)$. Therefore, the space $(Max(S), \mathcal{T}^+_M)$ is compact.

The proof of the fact that $(Min(S), \mathcal{T}^-_m)$ is a compact T_1-space is analogous.

(b) We first prove that $(Min(S), \mathcal{T}^+_m)$ is a Hausdorff space. Indeed, let $a, b \in Min(S)$ and $a \neq b$. Suppose that for any $U, V \in \mathcal{L}^+$ such that $a \in U$ and $b \in V$, we have that $U \cap V \neq \emptyset$. Then the family $\mathcal{F} = \{W \in \mathcal{L}^+ : a \in W \text{ or } b \in W\}$ has the finite intersection property (see 2.2) and its elements are closed subsets of the compact space $(S, \mathcal{T}^-)$. Consequently there exists a $c \in \bigcap \mathcal{F}$. Since $\mathcal{L}^+$ is a base for $\mathcal{T}^+$, we obtain that $a \in cl_{(S,\mathcal{T}^+)}\{c\}$ and $b \in cl_{(S,\mathcal{T}^+)}\{c\}$. Hence $c \leq a$ and $c \leq b$. Having in mind that $a, b \in Min(S)$, we get that $c = a$ and $c = b$, i.e. $a = b$, which is a contradiction. Therefore, $(Min(S), \mathcal{T}^+_m)$ is a Hausdorff space.

Analogously, one proves that $(Max(S), \mathcal{T}^-_M)$ is a Hausdorff space.

(c) We first prove that $Min(S)$ is dense in $(S, \mathcal{T}^+)$. Indeed, let $x \in U \in \mathcal{T}^+$. By 2.31, there exists an $a \in Min(S)$ such that $a \leq x$. Then $x \in cl_{(S,\mathcal{T}^+)}\{a\}$. Hence $a \in U \cap Min(S)$. Therefore, $Min(S)$ is dense in $(S, \mathcal{T}^+)$.

The proof of the fact that $Max(S)$ is dense in $(S, \mathcal{T}^-)$ is analogous. □

Let us recall the definitions of the coherent spaces and coherent maps:

Definition 2.34. (see, for example, [20]) *Let $(X, \mathcal{T})$ be a topological space.*

(a) We shall denote by $KO(X, \mathcal{T})$ (or, simply, by $KO(X)$) the family of all compact open subsets of X.

(b) A closed subset F of X is called irreducible *if the equality $F = F_1 \cup F_2$, where F_1 and F_2 are closed subsets of X, implies that $F = F_1$ or $F = F_2$.*

(c) We say that the space $(X, \mathcal{T})$ is sober *if it is a T_0-space and for every non-void irreducible subset F of X there exists a $x \in X$ such that $F = cl_X\{x\}$.*

(d) *The space* $(X, \mathcal{T})$ *is called* coherent *if it is a compact sober space and the family* $KO(X, \mathcal{T})$ *is a closed under finite intersections base for the topology* $\mathcal{T}$.

(e) *A continuous map* $f : (X', \mathcal{T}') \longrightarrow (X'', \mathcal{T}'')$ *is called* coherent *if* $U'' \in KO(X'')$ *implies that* $f^{-1}(U'') \in KO(X')$.

Notation 2.35. We denote by $\boldsymbol{CohSp}$ the category of all coherent spaces and all coherent maps between them.

Theorem 2.36. *The categories* $\boldsymbol{S}$ *and* $\boldsymbol{CohSp}$ *are isomorphic.*

Proof. We shall construct two covariant functors $F : \boldsymbol{S} \longrightarrow \boldsymbol{CohSp}$ and $G : \boldsymbol{CohSp} \longrightarrow \boldsymbol{S}$ such that $F \circ G = Id_{\boldsymbol{CohSp}}$ and $G \circ F = Id_{\boldsymbol{S}}$.

For every $(S, \mathcal{T}^+, \mathcal{T}^-) \in |\boldsymbol{S}|$, we put $F(S, \mathcal{T}^+, \mathcal{T}^-) = (S, \mathcal{T}^+)$. We shall prove that $(S, \mathcal{T}^+) \in |\boldsymbol{CohSp}|$. Indeed, we have: a) the space $(S, \mathcal{T}^+)$ is compact (by 2.4); b) $KO(S, \mathcal{T}^+) = \mathcal{L}^+$ (by 2.5) and hence the family $KO(S, \mathcal{T}^+)$ is a closed under finite intersections base for the topology $\mathcal{T}^+$ (by 2.2 and (SP1) of 2.3). Therefore we need only to show that $(S, \mathcal{T}^+)$ is a sober space. We have that $(S, \mathcal{T}^+)$ is a T_0-space (by 2.4). Let A be a non-empty irreducible subset of $(S, \mathcal{T}^+)$. Then A is a closed subset of $(S, \mathcal{T})$, where $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$. Hence, by 2.16, $(A, \mathcal{T}^+_A, \mathcal{T}^-_A)$ is an abstract spectrum (where $\mathcal{T}^+_A$ (resp. $\mathcal{T}^-_A$) is the induced by $\mathcal{T}^+$ (resp. $\mathcal{T}^-$) topology on the subset A of S). We shall prove that $|Min(A)| = 1$. Suppose that $x, y \in Min(A)$ and $x \neq y$. Let $\mathcal{T}'$ be the induced by $\mathcal{T}^+_A$ topology on $Min(A)$. Since $(Min(A), \mathcal{T}')$ is a Hausdorff space (by 2.33(b)), there exists an $U \in \mathcal{T}'$ such that $x \in U$ and $y \notin cl_{(Min(A),\mathcal{T}')}U$. Put $B = cl_{(Min(A),\mathcal{T}')}U$ and $C = Min(A) \setminus U$. Then B and C are closed subsets of $(Min(A), \mathcal{T}')$, $Min(A) = B \cup C$, $B \neq Min(A)$ and $C \neq Min(A)$. Since $Min(A)$ is dense in $(A, \mathcal{T}^+_A)$ (by 2.33(c)), we obtain that $A = B' \cup C'$, where $B' = cl_{(A,\mathcal{T}^+_A)}B$ and $C' = cl_{(A,\mathcal{T}^+_A)}C$. The sets B' and C' are closed in $(S, \mathcal{T}^+)$ since they are closed in $(A, \mathcal{T}^+_A)$ and A is closed in $(S, \mathcal{T}^+)$. Moreover, $B' \neq A$ and $C' \neq A$, because $B' \cap Min(A) = B$ and $C' \cap Min(A) = C$. Since A is irreducible, we get a contradiction. Therefore, $|Min(A)| = 1$. Let $Min(A) = \{a\}$. Then 2.33(c) implies that $A = cl_{(S,\mathcal{T}^+)}\{a\}$. So, $(S, \mathcal{T}^+)$ is a sober space. We proved that $(S, \mathcal{T}^+)$ is a coherent space.

Let $f \in \boldsymbol{S}((S_1, \mathcal{T}^+_1, \mathcal{T}^-_1), (S_2, \mathcal{T}^+_2, \mathcal{T}^-_2))$. We denote by $F(f) : S_1 \longrightarrow S_2$ the function defined by $F(f)(x) = f(x)$ for every $x \in S_1$. We shall show that $F(f) : (S_1, \mathcal{T}^+_1) \longrightarrow (S_2, \mathcal{T}^+_2)$ is a coherent map. Indeed, since f is a $\boldsymbol{S}$-morphism, we have that $F(f) : (S_1, \mathcal{T}^+_1) \longrightarrow (S_2, \mathcal{T}^+_2)$ is a continuous map. Let $K \subseteq S_2$, $K \in \mathcal{T}^+_2$ and K be a compact subspace of $(S_2, \mathcal{T}^+_2)$. Then, by 2.5, $K \in \mathcal{L}^+_2$, i.e. $S_2 \setminus K \in \mathcal{T}^-_2$. Hence $f^{-1}(K) \in \mathcal{T}^+_1$ and $f^{-1}(S_2 \setminus K) \in \mathcal{T}^-_1$. Since $S_1 \setminus f^{-1}(K) = f^{-1}(S_2 \setminus K)$, we obtain that $f^{-1}(K) \in \mathcal{L}^+_1$. Consequently, by 2.5, $f^{-1}(K)$ is a compact subspace of $(S_1, \mathcal{T}^+_1)$. So, we proved that $F(f) \in \boldsymbol{CohSp}(F(S_1, \mathcal{T}^+_1, \mathcal{T}^-_1), F(S_2, \mathcal{T}^+_2, \mathcal{T}^-_2))$. The definition of $F(f)$ implies immediately that F preserves the identity maps and that $F(f \circ g) = F(f) \circ F(g)$. Therefore, we constructed a functor $F : \boldsymbol{S} \longrightarrow \boldsymbol{CohSp}$.

Let now $(S, \mathcal{T}^+) \in |\boldsymbol{CohSp}|$, $\mathcal{B}^+ = KO(S, \mathcal{T}^+)$ and $\mathcal{B}^- = \{S \setminus U : U \in \mathcal{B}^+\}$. Since $\mathcal{B}^+$ is closed under finite intersections and finite unions, we obtain that $\mathcal{B}^-$ has the same properties. Obviously, $\bigcup \mathcal{B}^- = S$. Hence the family $\mathcal{T}^-$ of all subsets of S that are unions of subfamilies of $\mathcal{B}^-$ is a topology on S and $\mathcal{B}^-$ is a base for the topological space $(S, \mathcal{T}^-)$. We shall show that the bitopological space $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum and we will put $G(S, \mathcal{T}^+) = (S, \mathcal{T}^+, \mathcal{T}^-)$.

It is easy to see that $\mathcal{B}^+ \subseteq \mathcal{L}^+$ and $\mathcal{B}^- \subseteq \mathcal{L}^-$ (see 2.1 for the notation). Since, by the definition of a coherent space, the family $\mathcal{B}^+$ is a base for the topological space $(S, \mathcal{T}^+)$ and since the family $\mathcal{B}^-$ is a base for the space $(X, \mathcal{T}^-)$, we obtain that $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) is a base for $(S, \mathcal{T}^+)$ (resp. $(S, \mathcal{T}^-)$). Hence the condition (SP1) of 2.3 is fulfilled. The condition (SP3) of 2.3 is also fulfilled since $(S, \mathcal{T}^+)$ is a T_0-space. Let us put $\mathcal{T} = \sup\{\mathcal{T}^+, \mathcal{T}^-\}$. We shall prove that the space $(S, \mathcal{T})$ is compact. This will imply immediately that the condition (SP2) of 2.3 is fulfilled.

Obviously, for proving that $(S, \mathcal{T})$ is compact, it is enough to show that every cover of S of the type $\Omega = \Omega^+ \cup \Omega^-$, where Ω^+ (resp. Ω^-) is a subfamily of $\mathcal{B}^+ \setminus \{S\}$ (resp. $\mathcal{B}^- \setminus \{S\}$), has a finite subcover. Let Ω^* be the family of all finite unions of the elements of Ω^-. Then $\Omega^* \subseteq \mathcal{B}^-$, $\bigcup \Omega^- = \bigcup \Omega^*$ and $(\Omega^*, \subseteq)$ is a directed set (i.e. for every $U, V \in \Omega^*$ there exists a $W \in \Omega^*$ such that $U \cup V \subseteq W$). Put $H = S \setminus \bigcup \Omega^+$. Then $H \subseteq \bigcup \Omega^*$ and H is a closed and, hence, compact subset of $(S, \mathcal{T}^+)$. If we find a $U_0 \in \Omega^*$ such that $H \subseteq U_0$ then we will have that $S \setminus U_0 \subseteq S \setminus H = \bigcup \Omega^+$. From $U_0 \in \mathcal{B}^-$ we will get that $S \setminus U_0 \in \mathcal{B}^+$ and, hence, $S \setminus U_0$ will be a compact subset of $(S, \mathcal{T}^+)$ covered by Ω^+. Consequently there will be a finite subfamily Ω^+_f of Ω^+ covering $S \setminus U_0$. Then $\Omega^+_f \cup \{U_0\}$ will cover S. Therefore, we will find a finite subcover of Ω. So, it is enough to prove that there exists an $U_0 \in \Omega^*$ such that $H \subseteq U_0$.

Put $\mathcal{H}^+ = \{V \cap H : V \in \mathcal{B}^+\}$. Then $\mathcal{H}^+$ is a base for the subspace H of $(S, \mathcal{T}^+)$, $\mathcal{H}^+$ is closed under finite unions and finite intersections, $\mathcal{H}^+$ is a distributive lattice with respect to the operations $\cup$ and $\cap$ and, since H is closed in $(S, \mathcal{T}^+)$, all elements of $\mathcal{H}^+$ are compact subsets of $(S, \mathcal{T}^+)$. Furthermore, for every $U \in \Omega^*$ we put $U^+ = S \setminus U$. Then $U^+ \in \mathcal{B}^+$ for every $U \in \Omega^*$.

Suppose that for every $U \in \Omega^*$ we have that $H \setminus U \neq \emptyset$. Then $H \cap U^+ \neq \emptyset$ for every $U \in \Omega^*$. Since for every $U, V \in \Omega^*$ there exists a $W \in \Omega^*$ such that $W^+ \subseteq U^+ \cap V^+$, the family $\{H \cap U^+ : U \in \Omega^*\}$ has the finite intersection property. Hence it generates a filter φ in $\mathcal{H}^+$. Let Φ be an ultrafilter in $\mathcal{H}^+$ containing φ and let $L = \bigcap\{cl_{(S,\mathcal{T}^+)} W : W \in \Phi\}$. Then L is a non-empty closed subset of $(S, \mathcal{T}^+)$ and $L \subseteq H$. Moreover, $L \cap W_0 \neq \emptyset$ for every $W_0 \in \Phi$. Indeed, let $W_0 \in \Phi$. Then $W_0 \in \mathcal{H}^+$ and, hence, W_0 is a compact subset of $(S, \mathcal{T}^+)$. It is easy to see that the family $\{cl_{W_0}(W_0 \cap W) : W \in \Phi\}$ has the finite intersection property. Consequently $\emptyset \neq \bigcap\{cl_{W_0}(W_0 \cap W) : W \in \Phi\} = W_0 \cap \bigcap\{cl_H(W_0 \cap W) : W \in \Phi\} \subseteq W_0 \cap \bigcap\{cl_H W : W \in \Phi\} = W_0 \cap L$. So, we proved that $L \cap W_0 \neq \emptyset$ for every $W_0 \in \Phi$. We shall prove now that L is an irreducible subset of $(S, \mathcal{T}^+)$. Indeed, suppose that $L = A \cup B$, where A and B are closed subsets of $(S, \mathcal{T}^+)$ and $A \neq L$, $B \neq L$. Then $(H \setminus A) \cap L \neq \emptyset$ and $(H \setminus B) \cap L \neq \emptyset$. Let $x \in (H \setminus A) \cap L$. Then there exists a $W' \in \mathcal{H}^+$ such that $x \in W' \subseteq H \setminus A$. Since $x \in L$, we obtain that $W' \cap W \neq \emptyset$

for every $W \in \Phi$. Consequently $W' \in \Phi$. Analogously, taking an $y \in (H \setminus B) \cap L$, we can find a $W'' \in \Phi$ such that $y \in W'' \subseteq H \setminus B$. Putting $W_0 = W' \cap W''$, we get that $W_0 \in \Phi$. Since $W_0 \subseteq (H \setminus A) \cap (H \setminus B) = H \setminus (A \cup B) = H \setminus L$, we conclude that $W_0 \cap L = \emptyset$ – a contradiction. Therefore, L is an irreducible subset of $(S, \mathcal{T}^+)$. This implies, because of the fact that $(S, \mathcal{T}^+)$ is sober, that there exists a point $l \in L$ such that $L = cl_{(S,\mathcal{T}^+)}\{l\}$. We shall show that $l \in \bigcap\{U^+ : U \in \Omega^*\}$. Indeed, let $U \in \Omega^*$. Then $H \cap U^+ \in \varphi \subseteq \Phi$. Hence $U^+ \cap L \neq \emptyset$. Let $x \in U^+ \cap L$. Then $x \in U^+ \in \mathcal{T}^+$ and $x \in L = cl_{(S,\mathcal{T}^+)}\{l\}$. Consequently $l \in U^+$. So, we proved that $l \in \bigcap\{U^+ : U \in \Omega^*\}$. On the other hand we have that $l \in L \subseteq H \subseteq \bigcup \Omega^* = \bigcup\{S \setminus U^+ : U \in \Omega^*\} = S \setminus \bigcap\{U^+ : U \in \Omega^*\}$, i.e. $l \notin \bigcap\{U^+ : U \in \Omega^*\}$ – a contradiction. It shows that there exists a $U_0 \in \Omega^*$ such that $H \subseteq U_0$. Therefore, we proved that the space $(S, \mathcal{T})$ is compact and, hence, that the condition (SP2) of 2.3 is fulfilled. So, the bitopological space $(S, \mathcal{T}^+, \mathcal{T}^-)$ is an abstract spectrum.

Let $f \in \boldsymbol{CohSp}((S_1, \mathcal{T}_1^+), (S_2, \mathcal{T}_2^+))$. We denote by $G(f) : S_1 \longrightarrow S_2$ the function defined by $G(f)(x) = f(x)$ for every $x \in S_1$. We shall show that $G(f) \in \boldsymbol{S}((S_1, \mathcal{T}_1^+, \mathcal{T}_1^-), (S_2, \mathcal{T}_2^+, \mathcal{T}_2^-))$, where $(S_i, \mathcal{T}_i^+, \mathcal{T}_i^-) = G(S_i, \mathcal{T}_i^+)$, $i = 1, 2$. Indeed, we have that $f : (S_1, \mathcal{T}_1^+) \longrightarrow (S_2, \mathcal{T}_2^+)$ is a continuous map and hence $G(f) : (S_1, \mathcal{T}_1^+) \longrightarrow (S_2, \mathcal{T}_2^+)$ is a continuous map. For proving that $G(f) : (S_1, \mathcal{T}_1^-) \longrightarrow (S_2, \mathcal{T}_2^-)$ is a continuous map it is enough to show that $U \in \mathcal{B}_2^-$ implies that $f^{-1}(U) \in \mathcal{B}_1^-$ (because $\mathcal{B}_1^-$ (resp. $\mathcal{B}_2^-$) is a base for $\mathcal{T}_1^-$ (resp. $\mathcal{T}_2^-$)) (here we use the notation introduced above in the process of the definition of G on the objects of the category $\boldsymbol{CohSp}$). So, let $U \in \mathcal{B}_2^-$. Then $S_2 \setminus U \in KO(S_2, \mathcal{T}_2^+)$. Since f is a coherent map, we obtain that $V = f^{-1}(S_2 \setminus U) \in KO(S_1, \mathcal{T}_1^+) = \mathcal{B}_1^+$. Obviously, $V = S_1 \setminus f^{-1}(U)$. Consequently $f^{-1}(U) = S_1 \setminus V \in \mathcal{B}_1^-$. So, $G(f) \in \boldsymbol{S}(G(S_1, \mathcal{T}_1^+), G(S_2, \mathcal{T}_2^+))$. The definition of $G(f)$ implies immediately that G preserves the identity maps and $G(f \circ g) = G(f) \circ G(g)$. Therefore, we constructed a functor $G : \boldsymbol{CohSp} \longrightarrow \boldsymbol{S}$.

From 2.7 and the constructions of the functors F and G we get that $F \circ G = Id_{\boldsymbol{CohSp}}$ and $G \circ F = Id_{\boldsymbol{S}}$. So, the categories $\boldsymbol{S}$ and $\boldsymbol{CohSp}$ are isomorphic. □

Corollary 2.37. *The categories* $\boldsymbol{DLat}$ *and* $\boldsymbol{S}$ *are dual.*

Proof. Since the categories $\boldsymbol{DLat}$ and $\boldsymbol{CohSp}$ are dual (see, for example, [20]), our statement follows immediately from 2.36. □

2.38. Let us recall the descriptions of the duality functors

$$F' : \boldsymbol{CohSp} \longrightarrow \boldsymbol{DLat} \quad \text{and} \quad G' : \boldsymbol{DLat} \longrightarrow \boldsymbol{CohSp}$$

(see, for example, [20]): if $(X, \mathcal{T}^+)$ is a coherent space then

$$F'(X, \mathcal{T}^+) = (KO(X, \mathcal{T}^+), \cup, \cap, \emptyset, X);$$

if $f \in \boldsymbol{CohSp}((X_1, \mathcal{T}_1^+), (X_2, \mathcal{T}_2^+))$ then $F'(f) : F'(X_2, \mathcal{T}_2^+) \longrightarrow F'(X_1, \mathcal{T}_1^+)$ is defined by the formula

$$F'(f)(U) = f^{-1}(U)$$

for every $U \in KO(X_2, \mathcal{T}_2^+)$; if $(L, \vee, \wedge, 0, 1) \in |\boldsymbol{DLat}|$ then

$$G'(L, \vee, \wedge, 0, 1) = (spec(L), \mathcal{O}),$$

where $\mathcal{O}$ is the Stone topology on $spec(L)$ (see the proof of 2.22 for the notation); if $f \in \boldsymbol{DLat}((L_1, \vee_1, \wedge_1, 0_1, 1_1), (L_2, \vee_2, \wedge_2, 0_2, 1_2))$ then

$$G'(f) : G'(L_2, \vee_2, \wedge_2, 0_2, 1_2) \longrightarrow G'(L_1, \vee_1, \wedge_1, 0_1, 1_1))$$

is defined by the formula

$$G'(f)(p) = f^{-1}(p)$$

for every $p \in spec(L_2)$. The natural equivalence $\psi : Id_{\boldsymbol{CohSp}} \longrightarrow G' \circ F'$ is given by the formula $\psi(X, \mathcal{T}^+) = \psi_{(X,\mathcal{T}^+)}$ for every $(X, \mathcal{T}^+) \in |\boldsymbol{CohSp}|$, where

$$\psi_{(X,\mathcal{T}^+)} : (X, \mathcal{T}^+) \longrightarrow (G' \circ F')(X, \mathcal{T}^+), \quad x \mapsto \{U \in F'(X, \mathcal{T}^+) : x \notin U\}.$$

In particular, $\psi_{(X,\mathcal{T}^+)}$ is a $\boldsymbol{CohSp}$-isomorphism for every coherent space $(X, \mathcal{T}^+)$. The natural equivalence $\phi : Id_{\boldsymbol{DLat}} \longrightarrow F' \circ G'$ is given by the formula $\phi(L) = \phi_L$ for every $L \in |\boldsymbol{DLat}|$, where

$$\phi_L : L \longrightarrow (F' \circ G')(L), \quad l \mapsto \{p \in G'(L) : l \notin p\}.$$

In particular, ϕ_L is a $\boldsymbol{DLat}$-isomorphism for every distributive lattice L.

2.4. SOME APPLICATIONS

Let us start with recalling that if L is a distributive lattice with 0 and 1 then its classical spectrum $spec(L)$ can be interpreted as an abstract spectrum (see 2.22, 2.6 and 2.7).

We will first prove a general theorem.

Theorem 2.39. *Let X be a set, S be a family of subsets of X (i.e. $S \subseteq Exp(X)$), $\mathcal{T}^+$ and $\mathcal{T}^-$ be the topologies on S defined in 2.18, and let the bitopological space $(S, \mathcal{T}^+, \mathcal{T}^-)$ be an abstract spectrum. Then there exist a distributive lattice L with 0 and 1, and a function $\varphi : X \longrightarrow L$ such that:*

(i) the set $\varphi(X)$ generates L;

(ii) $\varphi^{-1}(q) \in S$ for every $q \in spec(L)$ (see 2.22 for the notation);

(iii) $\Phi : spec(L) \longrightarrow S$, $q \mapsto \varphi^{-1}(q)$, is an $\boldsymbol{S}$-isomorphism;

(iv) if L' is a distributive lattice with 0 and 1, and $\theta : X \longrightarrow L'$ is a function such that:

(1) $\theta^{-1}(q) \in S$ for every $q \in spec(L')$, and

(2) $\Theta : spec(L') \longrightarrow S$, $q \mapsto \theta^{-1}(q)$, is an $\boldsymbol{S}$-morphism,

then there exists a unique lattice homomorphism $l : L \longrightarrow L'$ with $l \circ \varphi = \theta$;

(v) *if* $\varphi_1 : X \longrightarrow L_1$*, where* L_1 *is a distributive lattice with 0 and 1, is such that:*

$(1')$ $(\varphi_1)^{-1}(q) \in S$ *for every* $q \in spec(L_1)$*, and*

$(2')$ $\Phi_1 : spec(L_1) \longrightarrow S$, $q \mapsto (\varphi_1)^{-1}(q)$*, is an* $\boldsymbol{S}$*-isomorphism,*

then there exists a unique lattice isomorphism $l : L \longrightarrow L_1$ *with* $l \circ \varphi = \varphi_1$*;*

(vi) $\varphi : X \longrightarrow L$ *is an injection iff for any two different points* x *and* y *of* X *there exists a* $p \in S$ *containing exactly one of them.*

Proof. We shall use the notation of 2.18, 2.20 and 2.22.

By (the proof of) 2.36, we have that $(S, \mathcal{T}^+) \in |\boldsymbol{CohSp}|$. We put $L = F'(S, \mathcal{T}^+)$ (see 2.38), i.e. $L = \{U \in \mathcal{T}^+ : U \text{ is compact}\}$ and, hence, by 2.5, $L = \mathcal{L}^+$. Then L is a distributive lattice with 0 and 1. Define the function $\varphi : X \longrightarrow L$ by the formula $\varphi(x) = U_x^+$ for every $x \in X$ (recall that $U_x^+ = \{p \in S : x \notin p\}$ and $U_x^+ \in \mathcal{L}^+$ (see 2.18 and the part $(b) \Rightarrow (a)$ of its proof)). Hence $\varphi(X)$ $(= \{U_x^+ : x \in X\} = \mathcal{P}^+)$ is a subbase for $\mathcal{T}^+$ (see 2.18). In what follows, the topological space $(S, \mathcal{T}^+)$ will be denoted, briefly, by S.

The proof of (i): Let L^* be the set of all finite unions of the elements of the set $\mathcal{B}^+$ of all finite intersections of the elements of $\mathcal{P}^+ = \varphi(X)$. Then L^* coincides with the subset of L generated by $\varphi(X)$ and $\mathcal{B}^+$ is a base for $\mathcal{T}^+$. If $U \in L$ then U is a compact open subset of S and, hence, it is a finite union of elements of $\mathcal{B}^+$. Thus $U \in L^*$. Therefore, the set $\varphi(X)$ generates L.

The proof of (ii) and (iii): By 2.38, we have that $spec(L) = G'(L)$. Since the map $\psi_S : S \longrightarrow (G' \circ F')(S)$, $p \longrightarrow \{U \in L : p \notin U\}$ is a $\boldsymbol{CohSp}$-isomorphism (see 2.38), we get that $spec(L) = \psi_S(S)$.

Let $q \in spec(L)$. Then there exists a unique $p \in S$ such that $q = \psi_S(p)$. So, we have that $\varphi^{-1}(q) = \varphi^{-1}(\psi_S(p)) = \{x \in X : \varphi(x) \in \psi_S(p)\} = \{x \in X : U_x^+ \in \psi_S(p)\} = \{x \in X : p \notin U_x^+\} = \{x \in X : x \in p\} = p$, i.e. $\varphi^{-1}(q) = \psi_S^{-1}(q)$ for every $q \in spec(L)$. Since the function ψ_S^{-1} is a $\boldsymbol{CohSp}$-isomorphism, we conclude that the function $\Phi : spec(L) \longrightarrow S$, $q \longrightarrow \varphi^{-1}(q)$, is a $\boldsymbol{CohSp}$-isomorphism. Now, (the proof of) 2.36 implies, that Φ is an $\boldsymbol{S}$-isomorphism.

The proof of (iv): Put $\tau = \psi_S \circ \Theta$. Then, by 2.36 and 2.38,

$$\Theta : spec(L') \longrightarrow (S, \mathcal{T}^+) \text{ and } \tau : spec(L') \longrightarrow (G' \circ F')(S, \mathcal{T}^+)$$

are $\boldsymbol{CohSp}$-morphisms. Since $G'(L') = spec(L')$ and $F'(S, \mathcal{T}^+) = L$, we obtain that $F'(\tau) = F'(\Theta) \circ F'(\psi_S) : (F' \circ G')(L) \longrightarrow (F' \circ G')(L')$ (see 2.38). Put $l = \phi_{L'}^{-1} \circ F'(\tau) \circ \phi_L$ (using the notation from 2.38). Then $l : L \longrightarrow L'$ is a lattice homomorphism. We shall prove that $F'(\Theta) \circ F'(\psi_S) \circ \phi_L \circ \varphi = \phi_{L'} \circ \theta$. This will imply that $\phi_{L'}^{-1} \circ F'(\Theta) \circ F'(\psi_S) \circ \phi_L \circ \varphi = \theta$ and, hence, we wll have that $\theta = \phi_{L'}^{-1} \circ (F'(\Theta) \circ F'(\psi_S)) \circ \phi_L \circ \varphi = (\phi_{L'}^{-1} \circ F'(\tau) \circ \phi_L) \circ \varphi = l \circ \varphi$, i.e. that $\theta = l \circ \varphi$.

Let $x \in X$. Then $(\phi_{L'} \circ \theta)(x) = \phi_{L'}(\theta(x)) = \{q' \in spec(L') : \theta(x) \notin q'\}$. On the other hand, $(\phi_L \circ \varphi)(x) = \phi_L(\varphi(x)) = \{q \in spec(L) : \varphi(x) \notin q\}$. Put

$U = (F'(\psi_S) \circ \phi_L \circ \varphi)(x)$. Since $\psi_S^{-1} = \Phi$ (see the proof of (ii) and (iii) above), we get $(F'(\psi_S))^{-1} = F'(\psi_S^{-1}) = F'(\Phi)$. Hence $(F'(\Phi))(U) = (F'(\psi_S))^{-1}(U) = (\phi_L \circ \varphi)(x)$. Now, the definition of $F'(\Phi)$ (see 2.38) implies $(F'(\Phi))(U) = \Phi^{-1}(U)$. Hence $\Phi^{-1}(U) = (\phi_L \circ \varphi)(x)$. Since Φ is an isomorphism (see (iii)), we get $U = \Phi((\phi_L \circ \varphi)(x)) = \Phi(\{q \in spec(L) : \varphi(x) \notin q\}) = \{\Phi(q) : q \in spec(L), \varphi(x) \notin q\} = \{\varphi^{-1}(q) : q \in spec(L), \varphi(x) \notin q\} = \{\varphi^{-1}(q) : q \in spec(L), x \notin \varphi^{-1}(q)\} = \{p \in S : x \notin p\} = U_x^+$, i.e $U = U_x^+$. Therefore, $(F'(\psi_S) \circ \phi_L \circ \varphi)(x) = U_x^+$. Then $(F'(\Theta) \circ F'(\psi_S) \circ \phi_L \circ \varphi)(x) = (F'(\Theta))((F'(\psi_S) \circ \phi_L \circ \varphi)(x)) = (F'(\Theta))(U_x^+) = \Theta^{-1}(U_x^+) = \{q' \in spec(L') : \Theta(q') \in U_x^+\} = \{q' \in spec(L') : \theta^{-1}(q') \in U_x^+\} = \{q' \in spec(L') : x \notin \theta^{-1}(q')\} = \{q' \in spec(L') : \theta(x) \notin q'\} = (\phi_{L'} \circ \theta)(x)$. So, we proved that $\theta = l \circ \varphi$. This, combined with the fact that $\varphi(X)$ generates L (see (i)), proves the uniqueness of l.

The proof of (v): Let $\varphi_1 : X \longrightarrow L_1$ has the properties $(1')$ and $(2')$. Then, using (iv), we obtain a lattice homomorphism $l : L \longrightarrow L_1$ such that $l \circ \varphi = \varphi_1$. From the construction of l, given in (iv), we have that $l = \phi_{L_1}^{-1} \circ F'(\psi_S \circ \Phi_1) \circ \phi_L$. Since Φ_1 is an ***CohSp***-isomorphism (by $(2')$ and 2.36), we get that l is a ***DLat***-isomorphism (because all other components of the composition defining l are also ***DLat***-isomorphisms (see 2.38)).

The proof of (vi): Let $x, y \in X$ and $x \neq y$. Then $\varphi(x) = \{p \in S : x \notin p\}$ and $\varphi(y) = \{p \in S : y \notin p\}$. Hence, $\varphi(x) \neq \varphi(y)$ if and only if there exists a $p \in S$ containing exactly one of the points x and y. □

Corollary 2.40. *Let X be a set endowed with two arbitrary multivalued binary operations $\oplus$ and $\otimes$ and with two fixed different points $\xi_0 \in X$ and $\xi_1 \in X$. Then there exist a distributive lattice $(L, \vee, \wedge)$ with 0 and 1, and a function $\varphi : X \longrightarrow L$ such that:*

- *(i) the set $\varphi(X)$ generates L;*
- *(ii) $\varphi^{-1}(q) \in S(X)_{pr}$ for every $q \in spec(L)$ (resp. $\varphi^{-1}(q) \in S(X)$ for every $q \in spec(L)$) (see 2.20 and 2.22 for the notation);*
- *(iii) $\Phi : spec(L) \longrightarrow S(X)_{pr}$, $q \mapsto \varphi^{-1}(q)$ (resp. $\Phi : spec(L) \longrightarrow S(X)$, $q \longrightarrow \varphi^{-1}(q)$) is an **S**-isomorphism;*
- *(iv) if L' is a distributive lattice with 0 and 1, and $\theta : X \longrightarrow L'$ is a function such that:*

  (1) $\theta^{-1}(q) \in S(X)_{pr}$ (resp. $\theta^{-1}(q) \in S(X)$) for every $q \in spec(L')$, and

  *(2) $\Theta : spec(L') \longrightarrow S(X)_{pr}$, $q \mapsto \theta^{-1}(q)$, (resp. $\Theta : spec(L') \longrightarrow S(X)$, $q \mapsto \theta^{-1}(q)$,) is an **S**-morphism,*

  then there exists a unique lattice homomorphism $l : L \longrightarrow L'$ with $l \circ \varphi = \theta$;

(v) *if* $\varphi_1 : X \longrightarrow L_1$*, where* L_1 *is a distributive lattice with 0 and 1, is such that:*

(1′) $(\varphi_1)^{-1}(q) \in S(X)_{pr}$ *for every* $q \in spec(L_1)$ *(resp.* $(\varphi_1)^{-1}(q) \in S(X)$ *for every* $q \in spec(L_1)$*), and*

(2′) $\Phi_1 : spec(L_1) \longrightarrow S(X)_{pr}$, $q \mapsto (\varphi_1)^{-1}(q)$ *(resp.* $\Phi_1 : spec(L_1) \longrightarrow S(X)$, $q \mapsto (\varphi_1)^{-1}(q)$*) is an* ***S****-isomorphism,*

then there exists a unique lattice isomorphism $l : L \longrightarrow L_1$ *with* $l \circ \varphi = \varphi_1$*;*

(vi) $a \oplus b \subseteq \{x \in X : \varphi(x) \leq \varphi(a) \vee \varphi(b)\}$ *and* $a \otimes b \subseteq \{x \in X : \varphi(x) \geq \varphi(a) \wedge \varphi(b)\}$ *for any* $a, b \in X$*.*

Proof. Denote by S the set $S(X)_{pr}$ (resp. $S(X)$) (see 2.20 for the notation) and define the topologies $\mathcal{T}^+_{pr}$ (resp. $\mathcal{T}^+$) and $\mathcal{T}^-_{pr}$ (resp. $\mathcal{T}^-$) on S as in 2.18. Then, by 2.20, the bitopological space $(S, \mathcal{T}^+_{pr}, \mathcal{T}^-_{pr})$ (resp. $(S, \mathcal{T}^+, \mathcal{T}^-)$) is an abstract spectrum. Hence, applying Theorem 2.39, we obtain a distributive lattice

$$(L, \vee, \wedge, 0, 1)$$

and a function $\varphi : X \longrightarrow L$ satisfying conditions (i)-(v) of 2.39 and, hence, our conditions (i)-(v) as well. Consequently, we need only to check that condition (vi) is also satisfied. In what follows, the notation of the proof of 2.39 and the construction of the function φ given there are used.

Let $a, b \in X$ and $x \in a \oplus b$. Then $\varphi(a) \vee \varphi(b) = \varphi(a) \cup \varphi(b) = \{p \in S : a \notin p \text{ or } b \notin p\}$. Hence $S \setminus (\varphi(a) \cup \varphi(b)) = \{p \in S : a \in p \text{ and } b \in p\}$. Let $p' \in \varphi(x) = U^+_x = \{p \in S : x \notin p\}$ and suppose that $p' \notin \varphi(a) \cup \varphi(b)$. Then $a \in p'$ and $b \in p'$. This implies that $a \oplus b \subseteq p'$. Then $x \in p'$ and, hence, $p' \notin \varphi(x)$ – a contradiction. Therefore, $p' \in \varphi(a) \cup \varphi(b)$. This shows that $\varphi(x) \subseteq \varphi(a) \cup \varphi(b)$, i.e. $\varphi(x) \leq \varphi(a) \vee \varphi(b)$, for every $x \in a \oplus b$. Consequently, $a \oplus b \subseteq \{x \in X : \varphi(x) \leq \varphi(a) \vee \varphi(b)\}$ for any $a, b \in X$.

Let $x \in a \otimes b$. We have that $\varphi(a) \wedge \varphi(b) = \varphi(a) \cap \varphi(b) = \{p \in S : a \notin p \text{ and } b \notin p\}$. Let $p' \in \varphi(a) \cap \varphi(b)$. Then $a \notin p'$ and $b \notin p'$. Suppose that $p' \notin \varphi(x)$. Then $x \in p'$ and, hence, $(a \otimes b) \cap p' \neq \emptyset$. This implies that $a \in p'$ or $b \in p'$, i.e. we get a contradiction. Therefore, $p' \in \varphi(x)$. So, $\varphi(a) \cap \varphi(b) \subseteq \varphi(x)$, i.e. $\varphi(a) \wedge \varphi(b) \leq \varphi(x)$ for every $x \in a \otimes b$. □

Corollary 2.41. *Let* X *be a set endowed with two arbitrary single-valued binary operations* $+$ *and* $\times$ *and with two fixed different points* $\xi_o \in X$ *and* $\xi_1 \in X$*. Then there exist a distributive lattice* $(L, \vee, \wedge)$ *with 0 and 1, and a function* $\varphi : X \longrightarrow L$ *such that:*

(i) *the set* $\varphi(X)$ *generates* L*;*

(ii) $\varphi^{-1}(q) \in S'(X)$ *for every* $q \in spec(L)$ *(resp.* $\varphi^{-1}(q) \in S'(X)_{pr}$ *for every* $q \in spec(L)$*) (see 2.24, 2.22 and 2.20 for the notation);*

(iii) $\Phi : spec(L) \longrightarrow S'(X)$, $q \mapsto \varphi^{-1}(q)$, *(resp.* $\Phi : spec(L) \longrightarrow S'(X)_{pr}$, $q \mapsto \varphi^{-1}(q)$,*) is an* $\boldsymbol{S}$*-isomorphism;*

(iv) if L' *is a distributive lattice with 0 and 1, and* $\theta : X \longrightarrow L'$ *is a function such that:*

(1) $\theta^{-1}(q) \in S'(X)$ *(resp.* $\theta^{-1}(q) \in S'(X)_{pr}$*) for every* $q \in spec(L')$*, and*

(2) $\Theta : spec(L') \longrightarrow S'(X)$, $q \mapsto \theta^{-1}(q)$ *(resp.* $\Theta : spec(L') \longrightarrow S'(X)_{pr}$, $q \mapsto \theta^{-1}(q)$*) is an* $\boldsymbol{S}$*-morphism,*

then there exists a unique lattice homomorphism $l : L \longrightarrow L'$ *with* $l \circ \varphi = \theta$*;*

(v) if $\varphi_1 : X \longrightarrow L_1$*, where* L_1 *is a distributive lattice with 0 and 1, is such that:*

(1′) $(\varphi_1)^{-1}(q) \in S'(X)$ *for every* $q \in spec(L_1)$ *(resp.* $(\varphi_1)^{-1}(q) \in S'(X)_{pr}$ *for every* $q \in spec(L_1)$*), and*

(2′) $\Phi_1 : spec(L_1) \longrightarrow S'(X)$, $q \mapsto (\varphi_1)^{-1}(q)$ *(resp.* $\Phi_1 : spec(L_1) \longrightarrow S'(X)_{pr}$, $q \mapsto (\varphi_1)^{-1}(q)$*) is an* $\boldsymbol{S}$*-isomorphism,*

then there exists a unique lattice isomorphism $l : L \longrightarrow L_1$ *with* $l \circ \varphi = \varphi_1$*;*

(vi) $\varphi(a + b) = \varphi(a) \vee \varphi(b)$ *and* $\varphi(a \times b) = \varphi(a) \wedge \varphi(b)$ *for every* $a, b \in X$*.*

Proof. Denote by S the set $S'(X)$ (resp. $S'(X)_{pr}$) (see 2.24 for the notation) and introduce the topologies $\mathcal{T}^+$ (resp. $\mathcal{T}^+_{pr}$) and $\mathcal{T}^-$ (resp. $\mathcal{T}^-_{pr}$) on S as in 2.18. Then, by 2.24, the bitopological space $(S, \mathcal{T}^+, \mathcal{T}^-)$ (resp. $(S, \mathcal{T}^+_{pr}, \mathcal{T}^-_{pr})$) is an abstract spectrum. Hence, applying Theorem 2.39, we obtain a distributive lattice

$$(L, \vee, \wedge, 0, 1)$$

and a function $\varphi : X \longrightarrow L$ satisfying conditions (i)-(v) of 2.39 and, hence, our conditions (i)-(v) as well. Consequently, we need only to check that condition (vi) is also satisfied. This can be done easily (see the proof of 2.40). □

3. SEPARATIVE ALGEBRAS

The main aim of this section is to give a detailed exposition of the theory of separative algebras, introduced and announced by Prodanov in [31]. This theory is a straight generalization of the theory of convex spaces in the sense of Tagamlitzki [44], which have been also a subject of Prodanov's Ph.D. dissertation [36]. We will follow very closely the style of Prodanov's proofs from [35] and [36].

Let $X \neq \emptyset$ be a set with two binary multivalued operations denoted by " $\times$ " and " $+$ ". This means that for any $x, y \in X$, $x \times y \subseteq X$ and $x + y \subseteq X$. Later on, instead of " $\times$ " and " $+$ ", we shall use "." and " $+$ ", and following the common mathematical practice, sometimes we shall omit the sign ".".

We extend the operations "." and " $+$ " for arbitrary subsets A and B of X putting

$$A.B = \bigcup_{a \in A, b \in B} a.b \text{ and } A + B = \bigcup_{a \in A, b \in B} a + b$$

The one element subset $\{x\} \subseteq X$ will be denoted simply by x. Then for instance $x(yz)$ will mean $\{x\}.(y.z)$.

Definition 3.1. *The system* $\underline{X} = (X, ., +)$ *is called a* preseparative algebra *if* $X \neq \emptyset$*, "." and "+" are binary multivalued operations in* X *satisfying the following axioms: for arbitrary* $a, b, c, x \in X$*,*

(i) $ab = ba$; *(i′)* $a + b = b + a$;

(ii) $a(bc) = (ab)c$; *(ii′)* $a + (b + c) = (a + b) + c$;

(iii) *from* $a \in b + x$*, and* $c \in dx$*, it follows that* $(ad) \cap (b + c) \neq \emptyset$.

By means of the operations "." and " $+$ ", we introduce two new operations as follows:

division: $a/b = \{x \in X : a \in b.x\}$ and

difference: $a - b = \{x \in X : a \in b + x\}$.

We extend the operations division and difference for arbitrary subsets putting

$$A/B = \bigcup_{a \in A, b \in B} a/b, \qquad A - B = \bigcup_{a \in A, b \in B} a - b.$$

Sometimes instead of A/B we will write $A : B$ or $\frac{A}{B}$.

The following lemma follows immediately from the relevant definitions.

Lemma 3.2. *Let "*$\bullet$*" be any of the operations ".", "+", "/" and "*$-$*". Then the following conditions are true:*

(i) $A \bullet \emptyset = \emptyset \bullet A = \emptyset$;

(ii) *If* $A \subseteq A'$ *and* $B \subseteq B'$ *then* $A \bullet B \subseteq A' \bullet B'$;

(iii) $(\bigcup_{i \in I} A_i) \bullet (\bigcup_{j \in J} B_j) = \bigcup_{i \in I, j \in J} A_i \bullet B_j$ *and, in particular,*

(iii′) $A \bullet (B \cup C) = (A \bullet B) \cup (A \bullet C)$;

(iv) $(\bigcap_{i \in I} A_i) \bullet (\bigcap_{j \in J} B_j) \subseteq \bigcap_{i \in I, j \in J} A_i \bullet B_j$.

Proposition 3.3. *The following is true for arbitrary* $A, B, C \subseteq X$*:*

(i) $(A/B) \cap C \neq \emptyset$ *if and only if* $A \cap (B.C) \neq \emptyset$*;*

(ii) $(A - B) \cap C \neq \emptyset$ *if and only if* $A \cap (B + C) \neq \emptyset$*.*

Proof. (i) $(A/B) \cap C \neq \emptyset \Leftrightarrow \exists x \in X$: $x \in (A/B) \cap C \Leftrightarrow \exists x \in X$: $x \in A/B$ and $x \in C \Leftrightarrow \exists x, a, b \in X$ $a \in A$, $b \in B$ $x \in a/b$ and $x \in C \Leftrightarrow \exists x, a, b \in X$: $a \in A$, $b \in B$, $a \in b.x$ and $x \in C \Leftrightarrow \exists a \in X$: $a \in A$ and $a \in B.C \Leftrightarrow \exists a \in X$: $a \in A \cap (B.C) \Leftrightarrow A \cap (B.C) \neq \emptyset$.

The proof of (ii) is similar. □

Proposition 3.4. *The following conditions are true for arbitrary subsets* A, B *and* C *of* X*:*

(i) $AB = BA$*;* *(i′)* $A + B = B + A$*;*

(ii) $A(BC) = (AB)C$*,* *(ii′)* $A + (B + C) = (A + B) + C$*.*

Proof. As an example we shall verify (i). The proof of the remaining conditions is similar.

$x \in AB \Leftrightarrow \exists a \in A$ $\exists b \in B$: $x \in ab \Leftrightarrow$ (by commutativity of ".") $\exists a \in A$ $\exists b \in B$: $x \in ba \Leftrightarrow x \in BA$. □

Associativity enables us to write $A_1.A_2.\ldots.A_n$ and $A_1 + A_2 + \cdots + A_n$ without parentheses.

We denote $A^n = A.A\ldots A$ (n-times) and $nA = A + A + \cdots + A$ (n-times), putting $A^1 = 1A = A$.

Lemma 3.5. *The following conditions are true:*

(i) $A^i A^j = A^{i+j}$*;*

(i′) $iA + jA = (i + j)A$*;*

(ii) $(A \cup B)^2 = A^2 \cup AB \cup B^2$*;*
$(A \cup B \cup C)^2 = A^2 \cup AB \cup AC \cup BC \cup C^2$*;*

(ii′) $2(A \cup B) = 2A \cup (A + B) \cup 2B$*;*
$2(A \cup B \cup C) = 2A \cup (A + B) \cup (A + C) \cup (B + C) \cup 2C$*.*

Proof. (i) and (i′) follow immediately from the definition, and (ii) and (ii′) follow from Lemma 3.2(iii′) and commutativity. □

Proposition 3.6. *The following conditions are equivalent to the Axiom (iii) from the definition of preseparative algebras (see Definition 3.1):*

(i) $a + \frac{b}{c} \subseteq \frac{a+b}{c}$*;*

(ii) $a(b-c) \subseteq ab-c$.

Proof. As an example we show the equivalence of the Axiom (iii) with (i).

((Axiom (iii))$\longrightarrow$ (i)). Let $x \in a+\frac{b}{c}$. Then there exists $y \in X$ such that $x \in a+y$, $y \in \frac{b}{c}$ and $b \in c+y$. By Axiom (iii), $(xc) \cap (a+b) \neq \emptyset$. Then, by Proposition 3.3(i), we obtain that $x \cap \frac{a+b}{c} \neq \emptyset$ and hence $x \in \frac{a+b}{c}$. Since x is an arbitrary element of X, this shows that $a+\frac{b}{c} \subseteq \frac{a+b}{c}$.

((i)$\longrightarrow$ (Axiom (iii))). Let $a \in b+x$ and $c \in dx$. Then $x \in \frac{c}{d}$ and $c \in b+\frac{c}{d}$. Then, by (i), $c \in \frac{b+c}{d}$, so that $c \cap \frac{b+c}{d} \neq \emptyset$. Applying Proposition 3.3(i), we obtain that $(cd) \cap (b+c) \neq \emptyset$, which shows that Axiom (iii) holds.

The equivalence of Axiom (iii) with (ii) can be proved similarly by using Proposition 3.3(ii). □

Proposition 3.7. *For arbitrary subsets A, B, C, D of X, the following conditions are true:*

(i) $A+\frac{B}{C} \subseteq \frac{A+B}{C}$;

(ii) $A(B-C) \subseteq AB-C$;

(iii) $(A/B)/C = A/(BC)$;

(iv) $(A-B)-C = A-(B+C)$;

(v) $\dfrac{A}{B}+\dfrac{C}{D} \subseteq \dfrac{A+C}{B.D}$;

(vi) $(A-B)(C-D) \subseteq AC-(B+D)$.

Proof. (i) and (ii) are extensions of Proposition 3.6, (i) and (ii), for arbitrary sets and follow directly from Proposition 3.6.

(iii) Let x be an arbitrary element of X. Then, applying Proposition 3.3(i), we obtain that

$$x \in (A/B)/C \Leftrightarrow (A/B)/C \cap x \neq \emptyset (A/B) \cap Cx \neq \emptyset \Leftrightarrow A \cap (BCx) \neq \emptyset$$
$$\Leftrightarrow A \cap (BC)x \neq \emptyset \Leftrightarrow (A/(BC)) \cap x \neq \emptyset \Leftrightarrow x \in A/(BC).$$

Hence, $(A/B)/C = A/(BC)$.

(iv) The proof can be done similarly by applying Proposition 3.3(ii).

(v) $\dfrac{A}{B}+\dfrac{C}{D} \subseteq \dfrac{A/B+C}{D} \subseteq \dfrac{(A+C)/B}{D} = \dfrac{A+C}{B.D}$. We have applied two times (i) and then (iii).

(vi) The proof goes similarly by applying two times (ii) and then (iv). □

Definition 3.8. *Let $\underline{X} = (X, ., +)$ be a preseparative algebra. A subset $F \subseteq X$ is called a* filter *in X if $F.F \subseteq F$. A subset $I \subseteq X$ is called an* ideal *in X if $I + I \subseteq I$. A subset $F \subseteq X$ is called a* prime filter *in X if F is a filter and the complement $X \setminus F$ of F is an ideal in X. Dually, a subset $I \subseteq X$ is called a* prime ideal *in X if I is an ideal and $X \setminus I$ is a filter in X.*

Obviously the empty set $\emptyset$ and the whole set X are examples of a filter, ideal, prime filter and prime ideal. They are in some sense trivial examples. Nontrivial examples of filters and ideals will be given by the constructions $\mu(A)$ and $\alpha(A)$ below. Constructions of prime filters and prime ideals will be given in Section 3.4 for separative algebras.

The following lemma follows immediately from the definitions of filter and ideal.

Lemma 3.9. *The intersection of any set of filters (ideals) is a filter (ideal).*

Let $A \subseteq X$. We define $\mu(A)$ - the *multiplicative closure of* A, by putting $\mu(A)$ to be the intersection of all filters containing A. By Lemma 3.9, $\mu(A)$ is the smallest filter containing A. Analogously, the intersection of all ideals containing A, denoted by $\alpha(A)$ and called the *additive closure of* A, is the smallest ideal containing A.

Lemma 3.10. *The following claims are true:*

(i) $\mu(A) = \bigcup_{i=1}^{\infty} A^i$;

(i′) $\alpha(A) = \bigcup_{i=1}^{\infty} iA$;

(ii) *a)* *If F is a filter, then $F = \mu(F)$;*

b) *If $A \subseteq B$, then $\mu(A) \subseteq \mu(B)$;*

c) $A \subseteq \mu(A)$;

d) $\mu(\mu(A)) = \mu(A)$,

e) *$\mu(A \cup B) = \mu(A) \cup \mu(A)\mu(B) \cup \mu(B)$; if F and G are filters, then $\mu(F \cup G) = F \cup FG \cup G$; if F is a filter and $a \in X$, then $\mu(F \cup a) = F \cup F.\mu(a) \cup \mu(a)$.*

(ii′)

a) *If I is an ideal, then $I = \alpha(I)$;*

b) *If $A \subseteq B$ then $\alpha(A) \subseteq \alpha(B)$;*

c) $A \subseteq \alpha(A)$;

d) $\alpha(\alpha(A)) = \alpha(A)$;

e) $\alpha(A \cup B) = \alpha(A) \cup (\alpha(A) + \alpha(B)) \cup \alpha(B)$*; if* I *and* J *are ideals, then* $\alpha(I \cup J) = I \cup (I + J) \cup J$*; if* I *is an ideal and* $a \in X$*, then* $\alpha(I \cup a) = I \cup I.\alpha(a) \cup \alpha(a)$*.*

Proof. (i) To prove the equality (i) it suffices to show that $\bigcup_{i=1}^{\infty} A^i$ is the smallest filter containing A. By Lemma 3.2(iv), we have $(\bigcup_{i=1}^{\infty} A^i).(\bigcup_{i=1}^{\infty} A^i) \subseteq \bigcup_{i,j=1}^{\infty} A^i.A^j = \bigcup_{i,j=1}^{\infty} A^{i+j} \subseteq (\bigcup_{i=1}^{\infty} A^i)$, so $\bigcup_{i=1}^{\infty} A^i$ is a filter, which obviously contains A. To prove that $\bigcup_{i=1}^{\infty} A^i$ is the smallest filter containing A, let α be a filter and $A \subseteq \alpha$. Applying Lemma 3.2(ii), we can show by induction on i that $A^i \subseteq \alpha^i \subseteq \alpha$ and consequently $\bigcup_{i=1}^{\infty} A^i \subseteq \alpha$.

(i′) can be shown similarly.

(ii) The proof of the conditions a), b), c) and d) follows directly from the definition of μ. To prove condition e), we shall show that the set $F \cup FG \cup G$, where $F = \mu(A)$ and $G = \mu(B)$, is the smallest filter containing $A \cup B$.

By Lemma 3.5(ii), we obtain

$$(F \cup FG \cup G)^2 = F^2 \cup F^2G \cup FG \cup F^2G^2 \cup FG^2 \cup G^2 \subseteq F \cup FG \cup G.$$

This shows that $F \cup FG \cup G$ is a filter containing F and G and hence A and B. To show that $F \cup FG \cup G$ is the smallest filter containing A and B, let γ be a filter such that $A \subseteq \gamma$ and $B \subseteq \gamma$, so we have $F \subseteq \gamma$ and $G \subseteq \gamma$. Then $F \cup G \subseteq \gamma$, $FG \subseteq \gamma\gamma \subseteq \gamma$ and consequently $F \cup FG \cup G \subseteq \gamma$.

The proof of (ii′) can be obtained in a similar way. □

Proposition 3.11. *Let* F *be a filter and* I *be an ideal. Then:*

(i) $F - I$ *is a filter;*

(i′) $\frac{I}{F}$ *is an ideal;*

(ii) If $I \cap (F - I) \neq \emptyset$*, then* $F \cap I \neq \emptyset$*;*

(iii) If $F \cap \frac{I}{F} \neq \emptyset$*, then* $F \cap I \neq \emptyset$*;*

(iv) If $(F - I) \cap \frac{I}{F} \neq \emptyset$*, then* $F \cap I \neq \emptyset$*.*

Proof. We prove only (iv); the proofs of the other conditions are similar. Applying Proposition 3.3, we obtain:

$(F - I) \cap \frac{I}{F} \neq \emptyset \longleftrightarrow F \cap (I + \frac{I}{F}) \neq \emptyset$; since $I + \frac{I}{F} \subseteq \frac{I+I}{F} \subseteq \frac{I}{F}$, we get that $F \cap \frac{I}{F} \neq \emptyset$. □

Lemma 3.12. *If* $\mu(A) \cap \alpha(B) \neq \emptyset$*, then there exist finite subsets* $A' \subseteq A$ *and* $B' \subseteq B$ *such that* $\mu(A') \cap \alpha(B') \neq \emptyset$*.*

Proof. Let

$$\mu(A) \cap \alpha(B) \neq \emptyset. \tag{3.1}$$

By Lemma 3.10(i),(i′), we have that

$$\mu(A) = \bigcup_{i=1}^{\infty} A^i \qquad \text{and} \tag{3.2}$$

$$\alpha(B) = \bigcup_{j=1}^{\infty} jB. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we obtain that for some $x \in X$, $x \in \bigcup_{i=1}^{\infty} A^i$ and $x \in \bigcup_{j=1}^{\infty} jB$. Then for some i and j we have that

$$x \in A^i \qquad \text{and} \tag{3.4}$$

$$x \in jB. \tag{3.5}$$

It follows from (3.4) that there exist a set $A' = \{a_1, \dots, a_i\} \subseteq A$ such that $x \in \{a_1, \dots, a_i\}$. From here we obtain that $\{a_1 \dots a_i\} \subseteq \mu(A')$ and consequently

$$x \in \mu(A') \subseteq (A). \tag{3.6}$$

In an analogous way we obtain from (3.5) that there exists a finite subset $B' = \{b_1, \dots, b_j\} \subseteq B$ such that

$$x \in \alpha(B') \subseteq \alpha(B). \tag{3.7}$$

Then from (3.1) and (3.6) and (3.7) we obtain

$$\mu(A') \cap \alpha(B') \neq \emptyset \tag{3.8}$$

Thus, for some finite subsets $A' \subseteq A$ and $B' \subseteq B$, we have $\mu(A') \cap \alpha(B') \neq \emptyset$. □

3.3. SEPARATIVE ALGEBRAS

Let $\underline{X} = (X, ., +)$ be a preseparative algebra. For $x, y \in X$ define

$$x \leq y \ \text{ iff } \ \mu(x) \cap \alpha(y) \neq \emptyset.$$

Definition 3.13. *A preseparative algebra $\underline{X} = (X, ., +)$ is called a* separative algebra *if the following axiom is satisfied:*

(Sep_0) *The relation $\leq$ is transitive.*

A separative algebra X is called a convex space *if the operations "." and "+" coincide. In this case the filters and the ideals are called* convex sets *and the prime filters correspond to the notion of* half-space.

Convex spaces have been studied by several authors: Tagamlitzki [44], Prodanov [34] and [35], Bair [1], Bryant [3], Bryant and Webster [4].

We will now give several examples of separative algebras.

Example 3.14. Let $\underline{L} = (L, \vee, \wedge, 0, 1)$ be a distributive lattice and for $x, y \in X$ define $x \times y = \{z \in L : z \geq x \wedge y\}$ and $x + y = \{z \in L : z \leq x \vee y\}$ (see Example 2.22). Then $\underline{X}$ is a separative algebra.

Example 3.15. Let $\underline{X} = (X, 1, +, .)$ be a commutative ring and for $x, y \in X$ define $x \times y = x.y$ and $x + y = A(x, y)$, where $A(x, y)$ is the ring-ideal generated by the set $\{x, y\}$ (see Example 2.21). Then $\underline{X}$ is a separative algebra.

Example 3.16. Let $\underline{X}$ be a real linear space. For arbitrary $a, b \in X$, we set $a \times b = a + b = \{ta + (1-t)b : 0 \leq t \leq 1\}$. Then $\underline{X}$ is a convex space.

Apart from these starting examples, there is a number of other ones. It seems that whenever we have a satisfactory theory of prime ideals, then there is also a structure of separative algebra.

Example 3.17. Let X be an ordered linear topological space. Then X is a separative algebra with respect to the operations

$a \times b = \{x \in X : \exists y \in ab \text{ with } x \leq y\}$,

$a + b = \{x \in X : \exists y \in ab \text{ with } x \geq y\}$,

where $ab = \{ta + (1-t)b : 0 \leq t \leq 1\}$.

Example 3.18. Let $\underline{X} = (X, .)$ be a commutative semigroup. Then $\underline{X}$ is a convex space.

The following lemma for filters and ideals is very important.

Lemma 3.19. *Let $\underline{X}$ be a separative algebra. Then for any $A, B \subseteq X$ and $x \in X$, we have that if $\mu(A) \cap \alpha(B \cup x) \neq \emptyset$ and $\mu(x \cup A) \cap \alpha(B) \neq \emptyset$, then $\mu(A) \cap \alpha(B) \neq \emptyset$.*

Proof. Suppose that the lemma does not hold and proceed to obtain a contradiction. Then for some $A, B \subseteq X$ and $x \in X$ we have that

$$\mu(A) \cap \alpha(B \cup x) \neq \emptyset, \tag{3.9}$$

$$\mu(x \cup A) \cap \alpha(B) \neq \emptyset, \quad \text{and} \tag{3.10}$$

$$\mu(A) \cap \alpha(B) = \emptyset. \tag{3.11}$$

By Lemma 3.10((ii)e),((ii′)e), we obtain:

$$\mu(x \cup A) = \mu(A) \cup \mu(A)\mu(x) \cup \mu(x) \quad \text{and} \tag{3.12}$$

$$\alpha(B \cup x) = \alpha(B) \cup (\alpha(B) + \alpha(x)) \cup \alpha(x). \tag{3.13}$$

From (3.9), (3.11) and (3.13), we obtain that
either (a) $\mu(A) \cap (\alpha(B) + \alpha(x)) \neq \emptyset$,
or (b) $\mu(A) \cap \alpha(x) \neq \emptyset$.
From (3.10), (3.11) and (3.12), we obtain that
either (a$'$) $(\mu(A)\mu(x)) \cap \alpha(B) \neq \emptyset$,
or (b$'$) $\mu(x) \cap \alpha(B) \neq \emptyset$.

So, we have to consider and to obtain a contradiction in each of the following combinations of cases: (a, a$'$), (a, b$'$), (b, a$'$) and (b, b$'$). As an example we shall treat of only the case (a, a$'$) - the remaining cases can be treated in a similar way. For the sake of brevity, we put $F = \mu(A)$, $I = \alpha(B)$; note that F is a filter and I is an ideal. Now (a) and (a$'$) become:
(a) $F \cap (I + \alpha(x)) \neq \emptyset$ and
(a$'$) $I \cap (F.\mu(x)) \neq \emptyset$.

Applying Proposition 3.3 to (a) and (a$'$), we obtain

$$\mu(x) \cap \frac{I}{F} \neq \emptyset \text{ and} \tag{3.14}$$

$$\alpha(x) \cap (F - I) \neq \emptyset. \tag{3.15}$$

By (3.14), we conclude that there exists $y \in X$ such that

$$y \in \mu(x) \text{ and} \tag{3.16}$$

$$y \in \frac{I}{F}. \tag{3.17}$$

By (3.15), we obtain that for some $z \in X$ we have

$$z \in \alpha(x) \text{ and} \tag{3.18}$$

$$z \in F - I. \tag{3.19}$$

Conditions (3.16) and (3.18) are equivalent respectively to

$$y \cap \mu(x) \neq \emptyset \text{ and} \tag{3.20}$$

$$z \cap \alpha(x) \neq \emptyset. \tag{3.21}$$

Since $y \subseteq \alpha(y)$, using (3.20), we get

$$\mu(x) \cap \alpha(y) \neq \emptyset \tag{3.22}$$

and, consequently, $x \leq y$.

Since $z \subseteq \mu(z)$, using (3.21), we get

$$\mu(z) \cap \alpha(x) \neq \emptyset \tag{3.23}$$

and, consequently, $z \leq x$.

Now, by the axiom (Sep_0), we obtain that $z \leq y$ and, consequently,

$$\mu(z) \cap \alpha(y) \neq \emptyset. \tag{3.24}$$

By Proposition 3.11(i), $F - I$ is a filter and since, by (3.19), $z \in F - I$, we get that

$$\mu(z) \subseteq F - I. \tag{3.25}$$

By Proposition 3.11(i′), $\frac{I}{F}$ is an ideal and since, by (3.17), $y \in \frac{I}{F}$, we get that

$$\alpha(y) \subseteq \frac{I}{F}. \tag{3.26}$$

From (3.25) and (3.26), we get that

$$\mu(z) \cap \alpha(y) \subseteq (F - I) \cap \frac{I}{F}. \tag{3.27}$$

By (3.24) and (3.27), we obtain that

$$(F - I) \cap \frac{I}{F} \neq \emptyset. \tag{3.28}$$

Applying Proposition 3.11(iv), we obtain that $F \cap I \neq \emptyset$, i.e. $\mu(A) \cap \alpha(B) \neq \emptyset$, which contradicts (3.11). This completes the proof of the lemma. □

Corollary 3.20. *If F is a filter, I is an ideal and $F \cap I = \emptyset$, then, for any $x \in X$, either $\mu(F \cup x) \cap I = \emptyset$ or $F \cap \alpha(I \cup x) = \emptyset$.*

3.4. SEPARATION THEOREM

Definition 3.21. *Let $\underline{X} = (X, ., +)$ be a preseparative algebra. The following statement is called the* Separation principle for *X:*

(Sep) If F_0 is a filter, I_0 is an ideal and $F_0 \cap I_0 = \emptyset$ then there exist a prime filter F and a prime ideal I such that $F_0 \subseteq F$, $I_0 \subseteq I$ and $F \cap I = \emptyset$.

The main aim of this section is the following:

Theorem 3.22. *(Separation theorem for separative algebras) Let $\underline{X} = (X, ., +)$ be a separative algebra. Then $\underline{X}$ satisfies the Separation principle (Sep).*

Proof. Let F_0 be a filter in $\underline{X}$, I_0 be an ideal in $\underline{X}$ and $F_0 \cap I_0 = \emptyset$.

Let $M = \{F : F \text{ is a filter in } \underline{X}, F_0 \subseteq F \text{ and } F \cap I_0 = \emptyset\}$. It is easy to see that M with the set-inclusion $\subseteq$ is an inductive set and hence, by the Zorn lemma, M has a maximal element, say F.

Let $N = \{I : I \text{ is an ideal}, I_0 \subseteq I \text{ and } F \cap I = \emptyset\}$. The set N supplied with the set-inclusion is also an inductive set and hence, by the Zorn lemma, it has a maximal element, say I. We shall show that F is a prime filter and I is a prime ideal.

Since F is a filter, I is an ideal and $F \cap I = \emptyset$, it is enough to show that $F \cup I = X$. Let $x \in X$. We shall show that either $x \in F$ or $x \in I$. Since $F \cap I = \emptyset$, Corollary 3.20 implies that either $\mu(F \cup x) \cap I = \emptyset$ or $F \cap \alpha(I \cup x) = \emptyset$.

Case 1: $\mu(F \cup x) \cap I = \emptyset$. Since $I_0 \subseteq I$, we obtain that $\mu(F \cup x) \cap I_0 = \emptyset$. We also have that $F_0 \subseteq F \subseteq \mu(F \cup x)$. From here we obtain that the filter $\mu(F \cup x) \in M$. By the maximality of F in M, we obtain that $\mu(F \cup x) = F$, and hence $x \in F$.

Case 2: $F \cap \alpha(I \cup x) = \emptyset$. Since $I_0 \subseteq I \subseteq \alpha(I \cup x)$, we obtain that $\alpha(I \cup x) \in N$. Then, by the maximality of I in N, we obtain that $\alpha(I \cup x) = I$, and hence $x \in I$.

So we have found a prime filter $F \supseteq F_0$ and a prime ideal $I \supseteq I_0$ such that $F \cap I = \emptyset$, which proves the theorem. □

Let us note that Theorem 3.22 generalizes a few well known statements: the Stone separation theorem for filters and ideals in distributive lattices [42] and in Boolean algebras [41], as well as the separation theorem for convex sets in convex spaces from [44].

Theorem 3.23. *Let* $\underline{X} = (X, ., +)$ *be a preseparative algebra. Then the following conditions are equivalent:*

(i) $\underline{X}$ *is a separative algebra;*

(ii) $\underline{X}$ *satisfies the Separation principle (Sep).*

Proof. The implication (i)$\longrightarrow$(ii) is just Theorem 3.22. For the converse implication (ii)$\longrightarrow$(i), we have to show that (Sep) implies (Sep_0) (see Definition 3.13 for (Sep_0)). So, let $a, b, c \in X$,

$$a \leq b \text{ (i.e., } \mu(a) \cap \alpha(b) \neq \emptyset) \text{ and} \tag{3.29}$$

$$b \leq c \text{ (i.e., } \mu(b) \cap \alpha(c) \neq \emptyset) \tag{3.30}$$

and suppose that

$$a \not\leq c \text{ (i.e., } \mu(a) \cap \alpha(c) = \emptyset). \tag{3.31}$$

Then (3.31) and (Sep) imply that there exist a prime filter F and and a prime ideal I such that

$$F \cap I = \emptyset \text{ (i.e. } X \setminus F = I), \tag{3.32}$$

$$\mu(a) \subseteq F \text{ and} \tag{3.33}$$

$$\alpha(c) \subseteq I. \tag{3.34}$$

From (3.29) and (3.33) we obtain

$$F \cap \alpha(b) \neq \emptyset. \tag{3.35}$$

From (3.30) and (3.34) we obtain

$$\mu(b) \cap I \neq \emptyset. \tag{3.36}$$

For the element b we have, by (3.32), that either $b \in F$ or $b \in I$.

Case 1: $b \in F$. Then $\mu(b) \subseteq F$ and, by (3.36), we obtain that $F \cap I \neq \emptyset$ - a contradiction with (3.32).

Case 2: $b \in I$. Then $\alpha(b) \subseteq I$ and, by (3.35), we obtain that $F \cap I \neq \emptyset$ - again a contradiction with (3.32).

This completes the proof of the theorem. □

We shall conclude this section by showing that the Separation theorem is equivalent to the following statement, which is a generalization of the well known Wallman's lemma:

Theorem 3.24. *Let $\underline{X} = (X, ., +)$ be a preseparative algebra. Then the following conditions are equivalent:*

- *(i) $\underline{X}$ is a separative algebra;*
- *(ii) (Wallman's lemma) Let M be a filter in X and let, for any prime filter $F \supseteq M$, an element $x_F \in F$ be chosen. Then there exists a finite number of prime filters $F_i \supseteq M$, $i = 1, \ldots, n$, such that $M \cap \alpha(\{x_{F_1}, \ldots, x_{F_n}\}) \neq \emptyset$.*

Proof. (i)⟶(ii). Let $\underline{X}$ be a separative algebra and M be a filter in $\underline{X}$. Denote by N the set of all elements x_F, chosen as in the condition of the Wallman's lemma. Then $M \cap \alpha(N) \neq \emptyset$. To prove this suppose the contrary. Then there exists a prime filter $F \supseteq M$ such that $F \cap \alpha(N) = \emptyset$. But this is impossible because $x_F \in N \subseteq \alpha(N)$. So, $M \cap \alpha(N) \neq \emptyset$. Now, by Lemma 3.12, there exists a finite subset $\{x_{F_1}, \ldots, x_{F_n}\} \subseteq N$ such that $M \cap \alpha(\{x_{F_1}, \ldots, x_{F_n}\}) \neq \emptyset$.

(ii)⟶(i). Suppose the Wallman's lemma. We shall prove the Separation principle (Sep). Suppose, for the sake of contradiction, that (Sep) does not hold. Then, for some filter F_0 and some ideal I_0 such that $F_0 \cap I_0 = \emptyset$, we have that any prime filter F extending F_0 has a non-empty intersection with I_0, i.e, there exists $x_F \in F \cap I_0$. Then, by the Wallman lemma, there exists a finite set $\{x_{F_1}, \ldots, x_{F_n}\}$ such that $F_0 \cap \alpha(\{x_{F_1}, \ldots, x_{F_n}\}) \neq \emptyset$. But $\{x_{F_1}, \ldots, x_{F_n}\} \subseteq I_0$, so that $\alpha(\{x_{F_1}, \ldots, x_{F_n}\}) \subseteq I_0$, which implies $F_0 \cap I_0 \neq \emptyset$, a contradiction. □

3.5. STANDARDIZATION OF THE OPERATIONS

Here we shall consider two couples of natural operations in a given separative algebra.

Let $\underline{X} = (X, \otimes, \oplus)$ be a separative algebra and, for any $a, b \in X$, define the following two new multivalued operations, called *convex operations*:

$$a.b = \mu(\{a, b\}) \text{ and } a + b = \alpha(\{a, b\})$$

Theorem 3.25. *If $\underline{X}$ is a separative algebra then it remains separative algebra with respect to its convex operations.*

Proof. The easy proof follows from the observation that the filters and ideals with respect to convex operations remain the same. □

Let $\underline{X} = (X, \otimes, \oplus)$ be a separative algebra.For any $A \subseteq X$, let $\mu_\rho(A)$ be the intersection of all prime filters containing A, and $\alpha_\rho(A)$ be the intersections of all prime ideals containing A. A subset A of X will be called a *radical filter* (resp., a *radical ideal*) if $\mu_\rho(A) = A$ (resp., $\alpha_\rho(A) = A$).

It follows from the Separation theorem that if A is an ideal (resp. filter), then

$$\alpha_\rho(A) = \{x \in X : \ \mu(x) \cap A \neq \emptyset\}, \text{ (resp., } \mu_\rho(A) = \{x \in X : \ \alpha(x) \cap A \neq \emptyset\}).$$

The following two new operations in X are called *radical operations*:

$$a.b = \mu_\rho(\{a, b\}) \text{ and } a + b = \alpha_\rho(\{a, b\}),$$

where $a, b \in X$.

Theorem 3.26. *If $\underline{X} = (X, \otimes, \oplus)$ is a separative algebra, then it is a separative algebra with respect to its radical operations as well.*

The proof follows from the observation that the filters and ideals with respect to the radical operations are the radical filters and radical ideals with respect to the initial operations, but the order $\leq$ do not change. To show this, note that $\mu_\rho(a) = \mu_\rho(\mu(a))$ and $\alpha_\rho(b) = \alpha_\rho(\alpha(b))$. Then, by the above observation, we have that

$$\mu_\rho(a) = \mu_\rho(\mu(a)) = \{x \in X : \ \alpha(x) \cap \mu(a) \neq \emptyset\} = \{x \in X : \ a \leq x\} \text{ and}$$

$$\alpha_\rho(b) = \alpha_\rho(\alpha(b)) = \{x \in X : \ \mu(x) \cap \alpha(b) \neq \emptyset\} = \{x \in X : \ x \leq b\}.$$

Then $\mu_\rho(a) \cap \alpha_\rho(b) \neq \emptyset$ iff $\exists x$: $a \leq x$ and $x \leq b$ iff $a \leq b$. □

Let $\underline{X}$ be a separative algebra. Then $\underline{X}$ has a canonical representation $\varphi : X \to L$ into a distributive lattice with the properties from Corollary 2.40. Now φ has some additional properties.

First of all, the inequality $a \le b$ takes place if and only if $\varphi(a) \subseteq \varphi(b)$. Therefore $\varphi(a) = \varphi(b)$ if and only if the radical ideals containing a contain b. If we do not distinguish such points (which is natural, if we are interested only in radical ideals and filters), φ becomes an embedding.

Now the operations from Corollary 2.40(v) look in the following manner:

$$a.b = \{x \in X : \ \varphi(x) \le \varphi(a) \vee \varphi(b)\} \text{ and } a + b = \{x \in X : \ \varphi(x) \ge \varphi(a) \wedge \varphi(b)\},$$

where $a.b$ and $a+b$ are the radical operations. In particular, if the initial operations coincide with radical ones, as it is in Example 3.16, we can get the separative structure of X from suitable embedding of X into a distributive lattice.

Now, let $\underline{X}$ be a ring with the separative structure from Example 3.15, and let $\varphi : X \longrightarrow L$ be the canonical representation. Then L can be identified with the distributive lattice of all finitely generated radical ideals of X (the whole X is included), and, for arbitrary $a \in X$, the image $\varphi(a)$ is the radical ideal in X generated by a.

3.7. TOPOLOGICAL VERSION OF THE SEPARATION THEOREM

Definition 3.27. *We shall say that a preseparative algebra* $\underline{X} = (X, ., +)$ *is* topological, *if* X *is endowed with a topology such that the mappings* $a.x$ *and* $a + x$ *are lower semi-continuous, i.e., for every* $a \in X$, *the multi-valued maps*

$$\varphi_a : X \longrightarrow X, \quad x \mapsto a + x, \quad \textit{and} \quad \psi_a : X \longrightarrow X, \quad x \mapsto a.x,$$

are lower semi-continuous. Recall that a multi-valued map $f : X \longrightarrow Y$ *between two topological spaces* X *and* Y *is said to be* lower semi-continuous *if, for every open subset* U *of* Y, *the set* $f^{-1}(U)$ *is open in* X *(here, as usual,*

$$f^{-1}(U) = \{x \in X : \ f(x) \cap U \ne \emptyset\});$$

equivalently, f *is lower semi-continuous if, for every* $x_0 \in X$ *and every open subset* U *of* Y *with* $U \cap f(x_0) \ne \emptyset$, *there exists a neighborhood* V *of* x_0 *in* X *such that* $U \cap f(x) \ne \emptyset$, *for every* $x \in V$. *For* $a + x$, *for example, this means that if* $a, b \in X$ *and* U *is an open set with* $(a + b) \cap U \ne \emptyset$, *then there exists a neighborhood* V *of* b *such that* $(a + x) \cap U \ne \emptyset$, *for each* $x \in V$.

A topological preseparative algebra will be called a separative space *if, for each open filter* U *in* X, *the conditions* $\alpha(a) \cap U \ne \emptyset$ *and* $b \in \mu(a)$ *imply* $\alpha(b) \cap U \ne \emptyset$.

A separative space $\underline{X} = (X, ., +)$ *is called a* topological convex space *if the operations* "." *and* "+" *in* X *coincide (see [34], [35]).*

Clearly, every separative algebra X endowed with the discrete topology is a separative space, but there are also analytical examples. Now we shall only note that if $\underline{X}$ is a topological preseparative algebra such that the topology of $\underline{X}$ has a basis from open filters, then X is a separative space.

The next statement, which we include here without proof, is a topological version of the Separation theorem.

Theorem 3.28. *Let $\underline{X}$ be a separative space, I_0 be an ideal in X and F_0 be an open filter in X such that $F_0 \cap I_0 = \emptyset$. Then there exist a closed prime ideal I and an open prime filter F in X such that $F_0 \subseteq F$, $I_0 \subseteq I$ and $F \cap I = \emptyset$.*

For a proof of Theorem 3.28 for topological convex spaces see [44]. We shall notice only one application of the theorem which uses the separative (not convex) structure: Example 3.17 and Theorem 3.28 give the classical separation theorem in ordered linear spaces, and, in particular, the general representation theorem of Kadison [21].

4. REFERENCES

1. Bair, J.: Separation of two convex sets in convexity spaces and in straight line spaces. *J. Math. Anal. Appl.*, **49**, 1975, 696–704.
2. Boltjanskij, V. G., P. S. Soltan: Combinatorial geometry and convex classes. *Uspekhi Mat. Nauk*, **33**, 1978, 3–42.
3. Bryant, V. W.: Independent axioms for convexity. *J. Geometry*, **5**, 1974, 95–99.
4. Bryant, V. W., Webster, R. J.: Convexity spaces I. *J. Math. Anal. Appl.*, **37**, 1972, 206–215; Convexity spaces II. *J. Math. Anal. Appl.*, **43**, 1973, 321–327; Convexity spaces III. *J. Math. Anal. Appl.*, **57**, 1977, 382–392.
5. Cantwell, J., Kay, D. C.: Geometric convexity III. *Trans. Amer. Math. Soc.*, **246**, 1979, 211–230.
6. Dikranjan, D.: Uniqueness of dualities. In: *Abelian Groups and Modules, Proc. Padova Confer., Padova, Italy, June 23-July 1, 1994, Mathematics and its Appl.*, 343, 1995, 123–133.
7. Dikranjan, D., Orsatti, A.: On an unpublished manuscript of Ivan Prodanov concerning locally compact modules and their dualities. *Comm. Algebra*, **17**, 1989, 2739–2771.
8. Dimov, G.: On the Stone duality. In: *General Topology and its Relations to Modern Analysis and Algebra V, Proc. Fifth Prague Topol. Symp. 1981, Sigma Series in Pure Mathematics*, 3, ed. J. Novák, Heldermann Verlag Berlin, 1983, 145–146.
9. Dimov, G.: An axiomatic characterization of the Stone duality. *Serdica*, **10**, 1984, 165–173.
10. Dimov, G., Vakarelov, D.: On Scott consequence systems. *Fundamenta Informaticae*, **33**, 1998, 43–70.
11. Dimov, G., Tholen, W.: A characterization of representable dualities. In: *Proc. Internat. Conf. on Categorical Topology (Prague, 1988)*, World Scientific, Singapore, 1989, 336–357.

12. Dimov, G., Tholen, W.: Groups of dualities. *Trans. Amer. Math. Soc.*, **336**, no. 2, 1993, 901–913.
13. Engelking, R.: *General Topology.* PWN, Warszawa, 1977.
14. Gregorio, E.: Tori and continuous dualities. *Rend. Accad. Naz. Sci. dei XL, Memorie di Mat.*, **107**, 1989, 211–221.
15. Gregorio, E.: Dualities over compact rings. *Rend. Sem. Mat. Univ. Padova*, **80**, 1988, 151–174.
16. Gregorio, E., Orsatti, A.: Uniqueness and existence of dualities over compact rings. *Tsukuba J. Math.*, **18**, 1994, 39–61.
17. Guay, M. D., Naimpally, S. A.: Characterization of a convex subspace of a linear topological space. *Math. Jap.*, **20**, 1975, 37–41.
18. Hochster, M.: Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, **142**, 1969, 43–60.
19. Hochster, M.: The minimal prime spectrum of a commutative ring. *Canad. J. Math.*, **23**, 1971, 749–758.
20. Johnstone, P. T.: *Stone Spaces.* Cambridge Univ. Press, Cambridge, 1982.
21. Kadison, R. V.: A representation theory for commutative topological algebra. *Mem. Amer. Math. Soc.*, **7**, 1951.
22. Kay, D. C., Womble, E. W.: Axiomatic convexity theory and relationships between Caratheodory, Helly and Radon numbers. *Pacif. J. Math.*, **38**, 1971, 471–485.
23. Leng, S.: *Algebra.* Addison-Wesley Publ. Comp., 1965.
24. Mah, P., Naimpally, S. A., Whitfield, J. H. M.: Linearization of a convexity space. *J. London Math. Soc.*, **13**, 1976, 209–214.
25. Prenowitz, W.: Descriptive geometries as multigroups. *Trans. Amer. Math. Soc.*, **59**, 1946, 333–380.
26. Prenowitz, W., Jantosciak, J.: Geometries and join spaces. *J. Reine Angew. Math.*, **257**, 1972, 100–128.
27. Priestley, H. A.: Representation of distributive lattices by means of ordered Stone spaces. *Bull. Lond. Math. Soc.*, **2**, 1970, 186–190.
28. Priestley, H. A.: Ordered topological spaces and the representation of distributive lattices. *Proc. Lond. Math. Soc.*, **24**, 1972, 507–530.
29. Priestley, H. A.: Stone lattices, a topological approach. *Fund. Math.*, **84**, 1974, 127–143.
30. Priestley, H. A.: The construction of spaces dual to pseudocomplemented distributive lattices. *Quart. J. Math.*, **26**, 1975, 215–228.
31. Prodanov, Iv.: An abstract approach to the algebraic notion of spectrum. *Topology, Proc. Steklov Inst. Math.*, **154**, 1983, 200–208 (in Russian); English translation in: *Proc. Steklov Inst. Math.*, **154**, 1985, 215–223.
32. Prodanov, Iv.: An axiomatic characterization of the Pontryagin duality. *Unpublished manuscript.*
33. Prodanov, Iv.: Pontryagin dualities. *Unpublished manuscript.*
34. Prodanov, Iv.: A generalization of some separation theorems. *Compt. Rend. Bulg. Acad. Sci.*, **17**, 1964, 345–348.
35. Prodanov, Iv.: Double associative spaces. *Annuaire de lUniversite de Sofia "St. Kliment Ohridski, Faculte de Mathematiques et Informatique, Livre 1 - Mathematiques*, **57**, 1964, 393–422.

36. Prodanov, Iv.: *Convex spaces*. Ph.D. Dissertation, Sofia, 1963.
37. Rasiowa, H., Sikorski, R.: *The mathematics of metamathematics*. Panstwowe Wydawnictwo Naukowe, Warszawa, 1963.
38. Roeder, D. W.: Functorial characterization of Pontryagin duality. *Trans. Amer. Math. Soc.*, **154**, 1971, 151–175.
39. Savchev, S.: *Distributive semilattices and spectra*. Diploma Thesis, Sofia, 1981.
40. Savchev, S.: On the prime specrum of $C(X)$. *Unpublished manuscript*.
41. Stone, M. H.: The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, **40**, 1936, 37–111.
42. Stone, M. H.: Topological representation of distributive lattices and Brouwerian logics. *Čas. Pest. Mat. Fis.*, **67**, 1937, 1–25.
43. Stoyanov, L.: Dualities over compact commutative rings. *Rend. Accad. Naz. Sci. dei XL, Memorie di Mat.*, **101**, VII, 10, 1983, 155–176.
44. Tagamlitzki, Ja. On the separation principle in Abelian convex spaces. *Izv. Mat. Inst. Bulg. Acad. Nauk*, **7**, 1963, 402–418.
45. Teschke, L., Heidekrüger, G.: Zweifach assoziative Räume, Verbindungsräume, Konvexitätsräume und veralgemeinerte Konvexitätsräume. *Wiss. Z. Pädag. Hochschule, Halle*, **14**, 1976, 21–24.
46. Vakarelov, D.: Logical Analysis of Positive and Negative Similarity Relations in Property Systems. In: *Proc. of WOCFAI'91*, ed. Michel De Glas and Dov Gabbay, Paris, 1991, 491–499.
47. Zimmermann, K.: On the solution of some non-convex optimization problems. *Z. Angew. Math. Mech.*, **58**, 1978, 497.

Received on June 1, 2015

Georgi Dimov, Dimiter Vakarelov

Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5 J. Bourchier Blvd., BG-1164 Sofia
BULGARIA

e-mails: gdimov@fmi.uni-sofia.bg
dvak@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

DIOPHANTINE APPROXIMATION BY PRIME NUMBERS OF A SPECIAL FORM

S. I. DIMITROV, T. L. TODOROVA

We show that for $B > 1$ and for some constants λ_i, $i = 1, 2, 3$ subject to certain assumptions, there are infinitely many prime triples p_1, p_2, p_3 satisfying the inequality $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < [\log(\max p_j)]^{-B}$ and such that $p_1 + 2$, $p_2 + 2$ and $p_3 + 2$ have no more than 8 prime factors. The proof uses Davenport - Heilbronn adaption of the circle method together with a vector sieve method.

Keywords: Rosser's weights, vector sieve, circle method, almost primes, diophantine inequality

2000 Math. Subject Classification: 11D75, 11N36, 11P32

1. INTRODUCTION

The famous prime twins conjecture states that there exist infinitely many primes p such that $p + 2$ is a prime too. This hypothesis is still not proved but there are established many approximations to this result.

Throughout, P_r will stand for an integer with no more than r prime factors, counted with their multiplicities. In 1973 Chen [2] showed that there are infinitely many primes p with $p + 2 = P_2$.

Here are some examples of problems, concerning primes p with $p + 2 = P_r$ for some $r \geq 2$.

In 1937, Vinogradov [16] proved that every sufficiently large odd n can be represented as a sum

$$p_1 + p_2 + p_3 = n \tag{1.1}$$

of primes p_1, p_2, p_3. In 2000 Peneva [10] and Tolev [14] looked for representations (1.1) with primes p_i, subject to $p_i + 2 = P_{r_i}$ for some $r_i \geq 2$. It was established in [14] that if n is sufficiently large and $n \equiv 3 \pmod 6$, then (1.1) has a solution in primes p_1, p_2, p_3 with

$$p_1 + 2 = P_2, \quad p_2 + 2 = P_5, \quad p_3 + 2 = P_7.$$

In 1947 Vinogradov [17] established that if $0 < \theta < 1/5$, then there are infinitely many primes p satisfying the inequality

$$||\alpha p + \beta|| < p^{-\theta}. \tag{1.2}$$

In 2007 Todorova and Tolev [13] proved that if $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\beta \in \mathbb{R}$ and $0 < \theta \leq 1/100$, then there are infinitely many primes p with $p + 2 = P_4$, satisfying the inequality (1.2). Latter Matomäki [8] proved a Bombieri-Vinogradov type result for linear exponential sums over primes and showed that this actually holds with $p + 2 = P_2$ and $\theta = 1/1000$.

The present paper is devoted to another popular problem for primes p_i, which is studied under the additional restrictions $p_i + 2 = P_{r_i}$ for some $r_i \geq 2$. According to R. C. Vaughan's [18], there are infinitely many ordered triples of primes p_1, p_2, p_3 with

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi+\delta}$$

for $\xi = 1/10, \delta > 0$ and some constants λ_j, $j = 1, 2, 3$, η, subject to the following restrictions:

$$\lambda_i \in \mathbb{R},\ \lambda_i \neq 0, i = 1, 2, 3\,; \tag{1.3}$$

$$\lambda_1, \lambda_2, \lambda_3 \quad \text{not all of the same sign;} \tag{1.4}$$

$$\lambda_1/\lambda_2 \in \mathbb{R} \setminus \mathbb{Q}\,; \tag{1.5}$$

$$\eta \in \mathbb{R}\,. \tag{1.6}$$

Latter the upper bound for ξ was improved and the strongest published result is due to K. Matomäki with $\xi = 2/9$.

Here we prove the following result:

Theorem 1. *Let B be an arbitrary large and fixed. Then under the conditions (1.3), (1.4), (1.5), (1.6) there are infinitely many ordered triples of primes p_1, p_2, p_3 with*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < [\log(\max p_j)]^{-B} \tag{1.7}$$

and

$$p_1 + 2 = P'_8, \quad p_2 + 2 = P''_8, \quad p_3 + 2 = P'''_8.$$

2. NOTATIONS

By p and q we always denote primes. By $\varphi(n)$, $\mu(n)$, $\Lambda(n)$ we denote Euler's function, Möbius' function and Mangoldt's function, respectively. We denote by $\tau(n)$ the number of the natural divisors of n. The notations $(m_1,\, m_2)$ and $[m_1,\, m_2]$ stand for the greatest common divisor and the least common multiple of m_1, m_2, respectively. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n(k)$. As usual, $[y]$ denotes the integer part of y, $e(y) = e^{2\pi i y}$,

$$\theta(x, q, a) = \sum_{\substack{p \le x \\ p \equiv a\ (q)}} \log p\,;$$

$$E(x, q, a) = \theta(x, q, a) - \frac{x}{\varphi(q)}\,; \tag{2.1}$$

For positive A and B we write $A \asymp B$ instead of $A \ll B \ll A$.

Let q_0 be an arbitrary positive integer and X be such that

$$q_0^2 = \frac{X}{(\log X)^A}\,, \quad A \ge 5\,; \tag{2.2}$$

$$\varepsilon = \frac{1}{(\log X)^{B+1}}\ , B > 1 \text{ is arbitrary large;} \tag{2.3}$$

$$H = \frac{1000 \log X}{\varepsilon}\,; \tag{2.4}$$

$$\Delta = \frac{(\log X)^{A+1}}{X}\,; \tag{2.5}$$

$$D = \frac{X^{1/3}}{(\log X)^A}\,; \tag{2.6}$$

$$z = X^{\alpha}\,, \ \ 0 < \alpha < 1/4\,; \tag{2.7}$$

$$P(z) = \prod_{2 < p \le z} p\,;$$

$$S_k(\alpha) = \sum_{\substack{\lambda_0 X < p \le X \\ p+2 \equiv 0\ (k)}} e(\alpha p) \log p, \ \ 0 < \lambda_0 < 1\,. \tag{2.8}$$

The restrictions on A, λ_0 and the value of α will be specified latter.

3. OUTLINE OF THE PROOF

We notice that if $(p+2,\, P(z)) = 1$, then $p + 2 = P_{[1/\alpha]}$. Our aim is to prove that for a specific (as large as possible) value of α there exists a sequence $X_1,\, X_2, \ldots \to \infty$ and primes $p_i \in (\lambda_0 X_j,\, X_j]$, $i = 1, 2, 3$ with $|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \varepsilon$ and $p_i + 2 = P_{[1/\alpha]}$, $i = 1, 2, 3$. In such a way, we get an infinite sequence of triples of primes p_1, p_2, p_3 with the desired properties.

Our method goes back to Vaughan [18], but we also use the Davenport - Heilbronn adaptation of the circle method (see [19, ch. 11]) combined with a vector sieve similar to that one from [15].

We choose a function υ such that

$$\begin{aligned} &\upsilon(x) = 1 && \text{for } |x| \le \varepsilon/2; \\ &0 < \upsilon(x) < 1 && \text{for } \varepsilon/2 < |x| < \varepsilon; \\ &\upsilon(x) = 0 && \text{for } |x| \ge \varepsilon\,, \end{aligned} \tag{3.1}$$

and $\upsilon(x)$ has derivatives of sufficiently large order.

So if

$$\sum_{\substack{\lambda_0 X < p_1, p_2, p_3 \le X \\ (p_i+2, P(z))=1, i=1,2,3}} \upsilon(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \log p_1 \log p_2 \log p_3 > 0, \tag{3.2}$$

then the number of the solutions of (1.7) in primes $p_i \in (\lambda_0 X,\, X]$, $p_i + 2 = P_{[1/\alpha]}$, $i = 1,\, 2,\, 3$, is positive.

Let $\lambda^{\pm}(d)$ be the lower and upper bounds Rosser's weights of level D, hence

$$|\lambda^{\pm}(d)| \le 1, \quad \lambda^{\pm}(d) = 0 \quad \text{if} \quad d \ge D \quad \text{or} \quad \mu(d) = 0\,. \tag{3.3}$$

For further properties of Rosser's weights we refer to [5], [6].

Let $\Lambda_i = \sum\limits_{d|(p_i+2, P(z))} \mu(d)$ be the characteristic function of primes p_i, such that $(p_i + 2,\, P(z)) = 1$ for $i = 1,\, 2,\, 3$. Then from (3.2) we obtain the condition

$$\sum_{\lambda_0 X < p_1, p_2, p_3 \le X} \upsilon(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \Lambda_1 \Lambda_2 \Lambda_3 \log p_1 \log p_2 \log p_3 > 0\,. \tag{3.4}$$

To set up a vector sieve, we use the lower and the upper bounds

$$\Lambda_i^{\pm} = \sum_{d|(p_i+2, P(z))} \lambda^{\pm}(d) \quad , \quad i = 1, 2, 3\,.$$

From the linear sieve we know that $\Lambda_i^- \le \Lambda_i \le \Lambda_i^+$ (see [1, Lemma 10]). Moreover, we have the simple inequality

$$\Lambda_1 \Lambda_2 \Lambda_3 \ge \Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+\,, \tag{3.5}$$

analogous to the one in [1, Lemma 13]. Using (3.4) we get

$$\begin{aligned} &\sum_{\lambda_0 X < p_1, p_2, p_3 \le X} \upsilon(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) \\ &\times \left(\Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+\right) \log p_1 \log p_2 \log p_3 > 0\,. \end{aligned} \tag{3.6}$$

Let $\Upsilon(x) = \int\limits_{-\infty}^{\infty} \upsilon(t) e(-tx)dt$ be the Fourier transform of the function υ defined in (3.1). Then

$$|\Upsilon(x)| \leq \min\left(\frac{3\varepsilon}{2}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|}\left(\frac{k}{2\pi|x|\varepsilon/4}\right)^k\right), \tag{3.7}$$

for all $k \in \mathbb{N}$ - see [11].

We substitute the function $\upsilon(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)$ in (3.6) with its Fourier transform:

$$\sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \times \int\limits_{-\infty}^{\infty} \Upsilon(t) e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\Lambda_1\Lambda_2\Lambda_3 dt > 0\,. \tag{3.8}$$

Our next argument is based on the following consequence of (3.8).

Lemma 1. *If the following integral is positive,*

$$\begin{aligned}\Gamma(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} & e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big) \log p_1 \log p_2 \log p_3 \\ & \times (\Lambda_1^- \Lambda_2^+ \Lambda_3^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- - 2\Lambda_1^+ \Lambda_2^+ \Lambda_3^+)\, dt \\ = \Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X) - 2\Gamma_4(X) > 0\,, \end{aligned} \tag{3.9}$$

then the number of the solutions of (1.7) in primes $p_i \in (\lambda_0 X,\, X]$, $p_i + 2 = P_{[1/\alpha]}$, $i = 1,\, 2,\, 3$, *is positive. Here*

$$\Gamma_1(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \times e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\Lambda_1^- \Lambda_2^+ \Lambda_3^+\, dt\,;$$

$$\Gamma_2(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \times e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\Lambda_1^+ \Lambda_2^- \Lambda_3^+\, dt\,;$$

$$\Gamma_3(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \leq X} \log p_1 \log p_2 \log p_3 \times e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\Lambda_1^+ \Lambda_2^+ \Lambda_3^-\, dt\,;$$

$$\Gamma_4(X) = \int_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1, p_2, p_3 \le X} \log p_1 \log p_2 \log p_3$$
$$\times\, e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big) \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \, dt\,.$$

We shall estimate $\Gamma_1(X)$, the remaining integrals $\Gamma_2(X)$, $\Gamma_3(X)$, $\Gamma_4(X)$ can be treated in a similar way. Changing the order of summation we obtain

$$\Gamma_1(X) = \int_{-\infty}^{\infty} \Upsilon(t) e(\eta t) L^-(\lambda_1 t,\, X) L^+(\lambda_2 t,\, X) L^+(\lambda_3 t,\, X) dt\,, \tag{3.10}$$

where

$$L^{\pm}(t,\, X) = \sum_{d|P(z)} \lambda^{\pm}(d) \sum_{\substack{\lambda_0 X < p \le X \\ p+2 \equiv 0 (d)}} e(pt) \log p\,. \tag{3.11}$$

Let us split $\Gamma_1(X)$ into three integrals,

$$\Gamma_1(X) = \Gamma_1^{(1)}(X) + \Gamma_1^{(2)}(X) + \Gamma_1^{(3)}(X)\,, \tag{3.12}$$

where

$$\Gamma_1^{(1)}(X) = \int_{|t| \le \Delta} \Upsilon(t) e(\eta t) L^-(\lambda_1 t,\, X) L^+(\lambda_2 t,\, X) L^+(\lambda_3 t,\, X) dt, \tag{3.13}$$

$$\Gamma_1^{(2)}(X) = \int_{\Delta < |t| < H} \Upsilon(t) e(\eta t) L^-(\lambda_1 t,\, X) L^+(\lambda_2 t,\, X) L^+(\lambda_3 t,\, X) dt, \tag{3.14}$$

$$\Gamma_1^{(3)}(X) = \int_{|t| \ge H} \Upsilon(t) e(\eta t) L^-(\lambda_1 t,\, X) L^+(\lambda_2 t,\, X) L^+(\lambda_3 t,\, X) dt\,. \tag{3.15}$$

Here the functions $\Delta = \Delta(X)$ and $H = H(X)$ are defined in (2.5) and (2.4).

We estimate $\Gamma_1^{(3)}(X), \Gamma_1^{(1)}(X), \Gamma_1^{(2)}(X)$, respectively, in the sections 4, 5, 6. In section 7 we complete the proof of the theorem.

4. UPPER BOUND FOR $\Gamma_1^{(3)}(X)$.

Lemma 2. *For the integral $\Gamma_1^{(3)}(X)$, defined by (3.15), we have*

$$\Gamma_1^{(3)}(X) \ll 1.$$

Proof. From (2.8) and (3.11) it follows that

$$|L^{\pm}(t,\, X)| \leq \sum_{d|P(z)} |\lambda^{\pm}(d)|.|S_d(t)|\,.$$

For $|S_d(t)|$ we use the trivial estimate

$$|S_d(t)| \leq \sum_{\substack{n\leq X\\ n+2\equiv 0\,(d)}} \log X \leq \log X\left(\frac{X}{d}+1\right) \ll \frac{X\log X}{d} + \log X\,.$$

Combining with (3.3) we obtain

$$L^{\pm}(t,\, X) \ll \sum_{d\leq D} \log X\left(\frac{X}{d}+1\right) \ll X(\log X)^2 \tag{4.1}$$

Bearing in mind that$|\Upsilon(t)| \leq \dfrac{1}{\pi t}\left(\dfrac{k}{2\pi t\varepsilon/4}\right)^k$ (see (3.7)), from (4.1) and (3.15) one concludes that

$$\Gamma_1^{(3)}(X) \ll X^3(\log X)^6 \int\limits_H^\infty \frac{1}{t}\left(\frac{k}{2\pi t\varepsilon/4}\right)^k dt = \frac{X^3(\log X)^6}{k}\left(\frac{2k}{\pi\varepsilon H}\right)^k. \tag{4.2}$$

The choice $k = [\log X]$ provides $\log X - 1 < k \leq \log X$ and by (2.4) it follows

$$\left(\frac{2k}{\pi\varepsilon H}\right)^k \ll \left(\frac{\log X}{\varepsilon\frac{1000\log X}{\varepsilon}}\right)^{\log X} \ll \frac{1}{X^{\log 1000}}\,. \tag{4.3}$$

Finally, (4.2) and (4.3) imply

$$\Gamma_1^{(3)}(X) \ll 1. \tag{4.4}$$

5. ASYMPTOTIC FORMULA FOR $\Gamma_1^{(1)}(X)$.

We will derive the main term of the integral $\Gamma_1(X)$ from $\Gamma_1^{(1)}(X)$. Making use of (2.8), one expresses the sums (3.11) as

$$L^{\pm}(t,\, X) = \sum_{d|P(z)} \lambda^{\pm}(d)S_d(t)\,. \tag{5.1}$$

We change the order of summation and integration in (3.13) to obtain

$$\begin{aligned}\Gamma_1^{(1)}(X) = &\sum_{\substack{d_i|P(z)\\ i=1,2,3}} \lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)\\ &\times \int\limits_{|t|\leq\Delta} \Upsilon(t)e(\eta t)S_{d_1}(\lambda_1 t)S_{d_2}(\lambda_2 t)S_{d_3}(\lambda_3 t)dt\,.\end{aligned} \tag{5.2}$$

Let

$$S_i = S_{d_i}(\lambda_i t)\,, \tag{5.3}$$

$$I_i = I_{d_i}(\lambda_i t) = \frac{1}{\varphi(d_i)} \int\limits_{\lambda_0 X}^{X} e(\lambda_i t y) dy\,, \tag{5.4}$$

$$R_i = R_{d_i} = (1+\Delta X) \max_{y\in[\lambda_0 X, X]} |E(y, d_i, -2)|\,, \tag{5.5}$$

where $E(x,q,a)$ is defined by (2.1). Using (2.6), it is not difficult to prove the estimate

$$S_i \ll \frac{X \log X}{d_i}\,. \tag{5.6}$$

From the inequality $\dfrac{n}{\varphi(n)} \leq e^{\gamma} \log\log n$ (see [4, §$XVIII$, Theorem 328]) we get the following estimate for $|I_i|$:

$$|I_i| \leq \frac{X}{\varphi(d_i)} \ll \frac{X \log\log X}{d_i} \ll \frac{X \log X}{d_i}\,. \tag{5.7}$$

Our aim is to separate the main part of the sum (5.2).

As the first step, we replace the product $S_1 S_2 S_3$ by $I_1 I_2 I_3$, as far as the integral over $I_1 I_2 I_3$ is easier to be estimated. We use the identity

$$S_1 S_2 S_3 = I_1 I_2 I_3 + (S_1 - I_1) I_2 I_3 + S_1 (S_2 - I_2) I_3 + S_1 S_2 (S_3 - I_3)\,. \tag{5.8}$$

Let $2 \nmid k$. Applying Abel's transform to $S_k(\alpha)$, one gets

$$S_k(\alpha) = -\int\limits_{\lambda_0 X}^{X} \sum_{\substack{\lambda_0 X < p \leq t \\ p+2\equiv 0\ (k)}} \log p . \frac{d}{dt} e(\alpha t) dt + e(\alpha X) \sum_{\substack{\lambda_0 X < p \leq X \\ p+2\equiv 0\ (k)}} \log p\,.$$

Using (2.1), we have

$$\begin{aligned} S_k(\alpha) = &-\int\limits_{\lambda_0 X}^{X} \left[\frac{t-\lambda_0 X}{\varphi(k)} + E(t,k,-2) - E(\lambda_0 X, k, -2)\right] \frac{d}{dt} e(\alpha t) dt \\ &+ \left[\frac{X-\lambda_0 X}{\varphi(k)} + E(X,k,-2) - E(\lambda_0 X,k,-2)\right] e(\alpha X) \\ =&\frac{1}{\varphi(k)} \left[-\int\limits_{\lambda_0 X}^{X} (t-\lambda_0 X) \frac{d}{dt} e(\alpha t) dt + (X-\lambda_0 X) e(\alpha X)\right] \\ &+ \mathcal{O}\Bigg(\int\limits_{\lambda_0 X}^{X} \max_{y\in(\lambda_0 X, X]} |E(y,k,-2)| |\alpha| dt\Bigg) + \mathcal{O}\Bigg(\max_{y\in(\lambda_0 X, X]} |E(y,k,-2)|\Bigg)\,, \end{aligned}$$

whence

$$S_k(\alpha) = \frac{1}{\varphi(k)} \int\limits_{\lambda_0 X}^{X} e(\alpha t)dt + \mathcal{O}\Big(\max_{y\in(\lambda_0 X, X]} |E(y,k,-2)|(1+|\alpha|X) \Big) .$$

Let $|\alpha| \le \Delta$. Then from (5.3), (5.4) and (5.5) we obtain

$$S_i = I_i + \mathcal{O}(R_i), \quad i = 1,2,3 . \tag{5.9}$$

From (5.5) - (5.9) it follows that

$$\begin{aligned} S_1S_2S_3 - I_1I_2I_3 \ll &(X\log X)^2(1+\Delta X)\Bigg(\frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_1,-2)|}{d_2d_3} \\ &+ \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_2,-2)|}{d_1d_3} + \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_3,-2)|}{d_1d_2} \Bigg) . \end{aligned}$$

Using (5.2) and the above inequality one gets

$$\Gamma_1^{(1)}(X) = M^{(1)} + \mathcal{O}(R^{(1)}) , \tag{5.10}$$

where

$$M^{(1)} = \sum_{\substack{d_i|P(z)\\ i=1,2,3}} \lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3) \int\limits_{|t|\le\Delta} \Upsilon(t)e(\eta t)I_1(\lambda_1 t)I_2(\lambda_2 t)I_3(\lambda_3 t)dt , \tag{5.11}$$

$$\begin{aligned} R^{(1)} =&(X\log X)^2(1+\Delta X) \sum_{\substack{d_i|P(z)\\ i=1,2,3}} |\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)| \Bigg(\frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_1,-2)|}{d_2\,d_3} \\ &+ \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_2,-2)|}{d_1\,d_3} + \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_3,-2)|}{d_1\,d_2} \Bigg) \int\limits_{|t|\le\Delta} |\Upsilon(t)|\,dt. \end{aligned}$$

Let us estimate $R^{(1)}$. Since $|\Upsilon(t)| \le \frac{3\varepsilon}{2}$ (see (3.7)), we find $\int\limits_{|t|\le\Delta} |\Upsilon(t)|\,dt \ll \varepsilon\Delta$.

Then using (3.3) we obtain

$$\begin{aligned} R^{(1)} \le&\varepsilon\Delta(X\log X)^2(1+\Delta X) \sum_{\substack{d_i\le D\\ i=1,2,3\\ 2\nmid d_i}} \Bigg(\frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_1,-2)|}{d_2d_3} \\ &+ \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_2,-2)|}{d_1d_3} + \frac{\max\limits_{y\in(\lambda_0 X, X]} |E(y,d_3,-2)|}{d_1d_2} \Bigg) \\ &\ll \varepsilon\Delta(1+\Delta X)X^2(\log X)^4 \sum_{\substack{d\le D\\ 2\nmid d}} \max_{y\in(\lambda_0 X, X]} |E(y,d,-2)| . \end{aligned} \tag{5.12}$$

We shall use the following well-known result.

Theorem 2 (Bombieri - Vinogradov). *For any $A>0$ the following inequality is fulfilled (see [3, ch.28]):*

$$\sum_{q\le X^{\frac{1}{2}}/(\log X)^{C+5}} \max_{y\le X}\max_{(a,\,q)=1}\Big|E(y,\,q,\,a)\Big| \ll \frac{X}{(\log X)^C}.$$

We apply the above theorem with $C=4A+5$ to the last sum in (5.12). Using (2.6) and (2.5) we obtain

$$R^{(1)} \ll \varepsilon\Delta(1+\Delta X)X^2(\log X)^4\frac{X}{(\log X)^{4A+5}} \ll \frac{\varepsilon\Delta^2X^4}{(\log X)^{4A+1}}\,. \tag{5.13}$$

Then from (5.10) and (5.13) it follows

$$\Gamma_1^{(1)}(X) - M^{(1)} \ll \frac{\varepsilon\Delta^2X^4}{(\log X)^{4A+1}}\,. \tag{5.14}$$

As a second step we represent $M^{(1)}$ in the form

$$M^{(1)} = \sum_{\substack{d_i\,|\,P(z)\\ i=1,2,3}} \frac{\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)}B(X)+R\,, \tag{5.15}$$

where

$$B(X)=\int\limits_{-\infty}^{\infty}\Upsilon(t)e(\eta t)\Bigg(\int\limits_{\lambda_0X}^{X}\int\limits_{\lambda_0X}^{X}\int\limits_{\lambda_0X}^{X} e(t(\lambda_1y_1+\lambda_2y_2+\lambda_3y_3))dy_1dy_2dy_3\Bigg)dt\,, \tag{5.16}$$

$$R \ll \Bigg|\int\limits_{\Delta}^{\infty}\Upsilon(t)e(\eta t)\Bigg(\int\limits_{\lambda_0X}^{X} e(\lambda_1ty_1)dy_1\int\limits_{\lambda_0X}^{X} e(\lambda_2ty_2)dy_2\int\limits_{\lambda_0X}^{X} e(\lambda_3ty_3)dy_3\Bigg)dt\Bigg|$$
$$\times\sum_{\substack{d_i\,|\,P(z)\\ i=1,2,3}}\frac{|\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)|}{\varphi(d_1)\varphi(d_2)\varphi(d_3)}\,.$$

On using $\Bigg|\int\limits_{\lambda_0X}^{X} e(\lambda_ity_i)dy_i\Bigg| \ll \dfrac{1}{|\lambda_i|\,t}$ and $|\Upsilon(t)|\le\dfrac{3\varepsilon}{2}$ (see (3.7)) we obtain

$$R\ll\frac{\varepsilon}{\Delta^2}\sum_{\substack{d_i\,|\,P(z)\\ i=1,2,3}}\frac{|\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)|}{\varphi(d_1)\varphi(d_2)\varphi(d_3)}\,.$$

From (2.6), (3.3) and the equality

$$\sum_{n\le x}\frac{1}{\varphi(n)} = C\log x + C' + \mathcal{O}(x^{-1+\varepsilon})$$

(see [9, ch. 4, §4.4, ex. 4.4.14]), we find

$$R \ll \frac{\varepsilon}{\Delta^2}\left(\sum_{d\le D}\frac{1}{\varphi(d)}\right)^3 \ll \frac{\varepsilon\log^3 X}{\Delta^2}\,. \tag{5.17}$$

From (5.15) and (5.17) we obtain

$$M^{(1)} = B(X)\sum_{\substack{d_i\mid P(z)\\ i=1,2,3}}\frac{\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)} + \mathcal{O}\left(\frac{\varepsilon\log^3 X}{\Delta^2}\right)$$

and from (5.14) we have

$$\begin{aligned}\Gamma_1^{(1)}(X) =& B(X)\sum_{d_1\mid P(z)}\frac{\lambda^-(d_1)}{\varphi(d_1)}\sum_{d_2\mid P(z)}\frac{\lambda^+(d_2)}{\varphi(d_2)}\sum_{d_3\mid P(z)}\frac{\lambda^+(d_3)}{\varphi(d_3)}\\ &+\mathcal{O}\left(\frac{\varepsilon\log^3 X}{\Delta^2}\right)+\mathcal{O}\left(\frac{\varepsilon\Delta^2X^4}{(\log X)^{4A+1}}\right).\end{aligned} \tag{5.18}$$

The function Δ defined by (2.5) is such that $\dfrac{\varepsilon\log^3 X}{\Delta^2} = \dfrac{\varepsilon\Delta^2X^4}{(\log X)^{4A+1}}$. Therefore, using (2.3), (2.5) and (5.18), we find

$$\begin{aligned}\Gamma_1^{(1)}(X) = B(X)\sum_{d_1\mid P(z)}\frac{\lambda^-(d_1)}{\varphi(d_1)}\sum_{d_2\mid P(z)}\frac{\lambda^+(d_2)}{\varphi(d_2)}\sum_{d_3\mid P(z)}\frac{\lambda^+(d_3)}{\varphi(d_3)}&\\ +\mathcal{O}\left(\frac{X^2}{(\log X)^{2A+B}}\right)&.\end{aligned} \tag{5.19}$$

Let

$$G^{\pm} = \sum_{d\mid P(z)}\frac{\lambda^{\pm}(d)}{\varphi(d)}\,. \tag{5.20}$$

Then from (5.19) and (5.20) it follows

$$\Gamma_1^{(1)}(X) = B(X)G^-(G^+)^2 + \mathcal{O}\left(\frac{X^2}{(\log X)^{2A+B}}\right). \tag{5.21}$$

We conclude this section with the following lemma:

Lemma 3. *If (1.3), (1.4) hold and*

$$\lambda_0 < \min\left(\frac{\lambda_1}{4|\lambda_3|}, \frac{\lambda_2}{4|\lambda_3|}, \frac{1}{16}\right),$$

then $B(X)$ defined by (5.16) satisfies

$$B(X) \gg \varepsilon X^2\,,$$

and the constant in "$\gg$" depends only on λ_1, λ_2 and λ_3.

Proof. Let us consider $B(X)$. We change the order of integration and use that $\Upsilon(t)$ is Fourier's transform of $\upsilon(t)$ to obtain

$$B(X) = \int\limits_{\lambda_0 X}^{X} \int\limits_{\lambda_0 X}^{X} \int\limits_{\lambda_0 X}^{X} \upsilon(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta) dy_1 dy_2 dy_3 \, .$$

From the definition (3.1) of υ follows the inequality

$$B(X) \geq \iiint\limits_{V} dy_1 dy_2 dy_3 = B_1(X) \, , \tag{5.22}$$

where

$$V = \{|\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta| < \varepsilon/2, \lambda_0 X \leq y_j \leq X, j = 1, 2, 3\} \, .$$

Since $\lambda_1, \lambda_2, \lambda_3$ are not all of the same sign, we may assume that $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_3 < 0$. We substitute $\lambda_1 y_1 = z_1, \lambda_2 y_2 = z_2, \lambda_3 y_3 = -z_3$, then

$$B_1(X) = \frac{1}{\lambda_1 \lambda_2 |\lambda_3|} \iiint\limits_{V'} dz_1 dz_2 dz_3 \tag{5.23}$$

with $V' = \{(z_1, z_2, z_3) \, : \, |z_1 + z_2 - z_3 + \eta| < \varepsilon/2, \ \lambda_0 |\lambda_j| X \leq z_j \leq |\lambda_j| X, \ j = 1, 2, 3\}$. Set

$$\begin{aligned} \xi_1 &= \frac{2\lambda_0 |\lambda_3|}{\lambda_1}, \quad & \xi_2 &= \frac{2\lambda_0 |\lambda_3|}{\lambda_2} \, , \\ \xi_1' &= 2\xi_1, & \xi_2' &= 2\xi_2 \, , \\ \lambda_0 &< \min\left(\frac{\lambda_1}{4|\lambda_3|}, \frac{\lambda_2}{4|\lambda_3|}, \frac{1}{16} \right) . \end{aligned}$$

Then $\lambda_0 < \xi_1 < \xi_1' < 1, \quad \lambda_0 < \xi_2 < \xi_2' < 1,$

$$\begin{aligned} \lambda_0 \lambda_1 X < \xi_1 \lambda_1 X < & z_1 < \xi_1' \lambda_1 X < \lambda_1 X \, , \\ \lambda_0 \lambda_2 X < \xi_2 \lambda_2 X < & z_2 < \xi_2' \lambda_2 X < \lambda_2 X \, , \\ \lambda_0 |\lambda_3| X < z_1 + z_2 - \varepsilon/2 + \eta < & z_3 < z_1 + z_2 + \varepsilon/2 + \eta < |\lambda_3| X \, , \end{aligned} \tag{5.24}$$

and from (5.22), (5.23) and (5.24) there follows

$$\begin{aligned} B(X) \geq B_1(X) \gg & \int\limits_{\xi_1 \lambda_1 X}^{\xi_1' \lambda_1 X} \Bigg(\int\limits_{\xi_2 \lambda_2 X}^{\xi_2' \lambda_2 X} \Bigg(\int\limits_{z_1 + z_2 - \varepsilon/2 + \eta}^{z_1 + z_2 + \varepsilon/2 + \eta} dz_3 \Bigg) dz_2 \Bigg) dz_1 \\ & = \varepsilon (\xi_2' - \xi_2) \lambda_2 X (\xi_1' - \xi_1) \lambda_1 X = 4 \lambda_0^2 \lambda_3^2 \varepsilon X^2 \\ & \gg \varepsilon X^2 \, . \end{aligned}$$

6. UPPER BOUND FOR $\Gamma_1^{(2)}(X)$.

We shall use (2.6) and the following lemma:

Lemma 4 ([13, Lemma 1], [15, Lemma 12]). *Suppose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with a rational approximation $\dfrac{a}{q}$ satisfying $\left|\alpha - \dfrac{a}{q}\right| < \dfrac{1}{q^2}$, where $(a,q)=1, q \geq 1,\ a \neq 0$. Let D be defined by (2.6), $\xi(d)$ be complex numbers defined for $d \leq D$ and $\xi(d) \ll 1$. If*

$$\mathfrak{L}(X) = \sum_{d \leq D} \xi(d) \sum_{\substack{X/2<p\leq X \\ p+2\equiv 0\,(d)}} e(\alpha p) \log p\,, \tag{6.1}$$

then we have

$$\mathfrak{L}(X) \ll \big(\log X\big)^{37}\bigg(\frac{X}{q^{1/4}} + \frac{X}{(\log X)^{A/2}} + X^{3/4}q^{1/4}\bigg).$$

Let us consider any sum $L^{\pm}(\alpha,\, X)$ denoted by (3.11). We represent it as sum of finite number of sums of the type

$$L(\alpha,\, Y) = \sum_{d \leq D} \xi(d) \sum_{\substack{Y/2<p\leq Y \\ p+2\equiv 0(d)}} e(\alpha p) \log p\,,$$

where

$$\xi(d) = \begin{cases} \lambda^{\pm}(d), & \text{if} \quad d|\, P(z)\,, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$L^{\pm}(\alpha,\, X) \ll \max_{\lambda_0 X \leq Y \leq X} L(\alpha,\, Y)\,.$$

If

$$q \in \left[(\log X)^A,\ \frac{X}{(\log X)^A}\right], \tag{6.2}$$

then from the above lemma for the sums $L(\alpha,\, Y)$ we get

$$L(\alpha,\, Y) \ll \frac{Y}{(\log Y)^{A/4-37}}\,. \tag{6.3}$$

Therefore

$$L^{\pm}(\alpha,\, X) \ll \max_{\lambda_0 X \leq Y \leq X} \frac{Y}{(\log Y)^{A/4-37}} \ll \frac{X}{(\log X)^{A/4-37}}\,.$$

Let

$$V(t,\, X) = \min\left\{|L^{\pm}(\lambda_1 t,\, X)|, |L^{\pm}(\lambda_2 t,\, X)|\right\}. \tag{6.4}$$

We shall need the following result:

Lemma 5. *Let t, X, λ_1, $\lambda_2 \in \mathbb{R}$,*

$$|t| \in (\Delta,\, H)\,, \tag{6.5}$$

where Δ and H are defined by (2.5) and (2.4), let λ_1, λ_2 satisfy (1.5) and $V(t,\, X)$ be defined by (6.4). Then there exists a sequence of real numbers X_1, $X_2, \ldots$ with $\lim X_n = \infty$ such that

$$V(t\,, X_j) \ll \frac{X_j}{(\log X_j)^{A/4-37}}\,, \qquad j = 1, 2, \ldots\,. \tag{6.6}$$

Proof. Our goal is to prove that there exists a sequence X_1, X_2, $\ldots \to \infty$ such that for every $j \in \mathbb{N}$ at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$, with t fulfilling (6.5), can be approximated by rational numbers with denominators satisfying (6.2). Then the proof follows from (6.3) and (6.4).

Since $\dfrac{\lambda_1}{\lambda_2} \in \mathbb{R}/\mathbb{Q}$ then, by [12, Corollary 1B], there exist infinitely many fractions $\dfrac{a_0}{q_0}$ with arbitrary large denominators such that

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0}\right| < \frac{1}{q_0^2}\,, \qquad (a_0,\, q_0) = 1\,. \tag{6.7}$$

Let q_0 be sufficiently large and X be such that $q_0^2 = \dfrac{X}{(\log X)^A}$ (see (2.2)). Let us notice that there exist a_1, $q_1 \in \mathbb{Z}$ such that

$$\left|\lambda_1 t - \frac{a_1}{q_1}\right| < \frac{1}{q_1 q_0^2}\,, \qquad (a_1,\, q_1) = 1, \qquad 1 < q_1 < q_0^2, \qquad a_1 \neq 0\,. \tag{6.8}$$

The Dirichlet theorem (see [7, ch.10, §1]) implies the existence of integers a_1 and q_1 satisfying the first three conditions in (6.8). If $a_1 = 0$, then $|\lambda_1 t| < \dfrac{1}{q_1 q_0^2}$ and from (6.5) it follows

$$\lambda_1 \Delta < \lambda_1 |t| < \frac{1}{q_0^2}\,, \qquad q_0^2 < \frac{1}{\lambda_1 \Delta}\,.$$

From the last inequality, (2.2) and (2.5), one obtains

$$\frac{X}{(\log X)^A} < \frac{X}{\lambda_1 (\log X)^{A+1}}\,,$$

which is impossible for large q_0, respectively, for a large X. So $a_1 \neq 0$. By analogy there exist a_2, $q_2 \in \mathbb{Z}$, such that

$$\left|\lambda_2 t - \frac{a_2}{q_2}\right| < \frac{1}{q_2 q_0^2}\,, \qquad (a_2,\, q_2) = 1, \qquad 1 < q_2 < q_0^2, \qquad a_2 \neq 0\,. \tag{6.9}$$

If $q_i \in \left[(\log X)^A, \dfrac{X}{(\log X)^A}\right]$ for $i = 1$ or $i = 2$, then the proof is completed.

From (2.2), (6.8) and (6.9) we have

$$q_i \le \frac{X}{(\log X)^A} = q_0^2\,, \quad i = 1, 2\,.$$

Thus it remains to prove that the case

$$q_i < (\log X)^A\,, \quad i = 1, 2 \tag{6.10}$$

is impossible. Let $q_i < (\log X)^A$, $i = 1, 2$. From (6.8), (6.9) and (6.10) it follows that

$$\begin{aligned} 1 &\le |a_i| \le \frac{1}{q_0^2} + q_i\lambda_i|t| < \frac{1}{q_0^2} + q_i\lambda_i H\,, \\ 1 &\le |a_i| < \frac{1}{q_0^2} + \frac{1000(\log X)^{A+1}\lambda_i}{\varepsilon}\,, \quad i = 1,\, 2\,. \end{aligned} \tag{6.11}$$

We have

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1 t}{\lambda_2 t} = \frac{\dfrac{a_1}{q_1} + \left(\lambda_1 t - \dfrac{a_1}{q_1}\right)}{\dfrac{a_2}{q_2} + \left(\lambda_2 t - \dfrac{a_2}{q_2}\right)} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathfrak{T}_1}{1 + \mathfrak{T}_2}\,, \tag{6.12}$$

where $\mathfrak{T}_i = \dfrac{q_i}{a_i}\left(\lambda_i t - \dfrac{a_i}{q_i}\right)$, $i = 1,\, 2$. From (6.8), (6.9) and (6.12) we obtain

$$|\mathfrak{T}_i| < \frac{q_i}{|a_i|} \cdot \frac{1}{q_i q_0^2} = \frac{1}{|a_i| q_0^2} \le \frac{1}{q_0^2}\,, \quad i = 1, 2\,,$$

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathcal{O}\left(\dfrac{1}{q_0^2}\right)}{1 + \mathcal{O}\left(\dfrac{1}{q_0^2}\right)} = \frac{a_1 q_2}{a_2 q_1}\left(1 + \mathcal{O}\left(\frac{1}{q_0^2}\right)\right).$$

Thus $\dfrac{a_1 q_2}{a_2 q_1} = \mathcal{O}(1)$ and

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} + \mathcal{O}\left(\frac{1}{q_0^2}\right). \tag{6.13}$$

Therefore, both fractions $\dfrac{a_0}{q_0}$ and $\dfrac{a_1 q_2}{a_2 q_1}$ approximate $\dfrac{\lambda_1}{\lambda_2}$. Using (6.9), (6.10) and inequality (6.11) with $i = 2$ we obtain

$$|a_2| q_1 < 1 + \frac{1000(\log X)^{2A+1}\lambda_2}{\varepsilon} \ll (\log X)^{2A+B+2} < \frac{q_0}{\log X}\,, \tag{6.14}$$

so $|a_2|q_1 \neq q_0$ and the fractions $\dfrac{a_0}{q_0}$ and $\dfrac{a_1 q_2}{a_2 q_1}$ are different. On using (6.14) we obtain

$$\left|\frac{a_0}{q_0} - \frac{a_1 q_2}{a_2 q_1}\right| = \frac{|a_0 a_2 q_1 - a_1 q_2 q_0|}{|a_2| q_1 q_0} \geq \frac{1}{|a_2| q_1 q_0} \gg \frac{\log X}{q_0^2}\,. \tag{6.15}$$

On the other hand, from (6.7) and (6.13) we have

$$\left|\frac{a_0}{q_0} - \frac{a_1 q_2}{a_2 q_1}\right| \leq \left|\frac{a_0}{q_0} - \frac{\lambda_1}{\lambda_2}\right| + \left|\frac{\lambda_1}{\lambda_2} - \frac{a_1 q_2}{a_2 q_1}\right| \ll \frac{1}{q_0^2}\,,$$

which contradicts (6.15). Therefore (6.10) can not happen. Let $q_0^{(1)}$, $q_0^{(2)}$, ... be an infinite sequence of values of q_0, satisfying (6.7). Then using (2.2) one gets an infinite sequence X_1, X_2, ... of values of X, such that at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$ can be approximated by rational numbers with denominators, satisfying (6.2). The proof of Lemma 5 is completed. □

Let us estimate the integral $\Gamma_1^{(2)}(X_j)$, defined by (3.14). Using $|\Upsilon(t)| \leq \dfrac{3\varepsilon}{2}$ (see (3.7)), (6.4) and estimate (6.6), we find

$$\begin{aligned}\Gamma_1^{(2)}(X_j) &\ll \varepsilon \int\limits_{\Delta<|t|<H} V(t, X_j)\big[|L^-(\lambda_1 t, X_j)L^+(\lambda_3 t, X_j)| + |L^+(\lambda_2 t,\, X_j)L^+(\lambda_3 t,\, X_j)|\big]\,dt \\ &\ll \varepsilon \int\limits_{\Delta<|t|<H} V(t, X_j)\Big(|L^-(\lambda_1 t,\, X_j)|^2 + |L^+(\lambda_2 t,\, X_j)|^2 + |L^+(\lambda_3 t,\, X_j)|^2\Big)\,dt \\ &\ll \frac{\varepsilon X_j}{(\log X_j)^{A/4-37}} \max_{1\leq k\leq 3} \int\limits_{\Delta<|t|<H} |L^{\pm}(\lambda_k t,\, X_j)|^2\,dt\,.\end{aligned}$$

Since the above integral has the same value over the positive and the negative t, one gets

$$\Gamma_1^{(2)}(X_j) \ll \frac{\varepsilon X_j}{(\log X_j)^{A/4-37}} \max_{1\leq k\leq 3} \mathcal{I}_k\,, \tag{6.16}$$

where $\mathcal{I}_k = \int\limits_{\Delta}^{H} |L^{\pm}(\lambda_k t,\, X_j)|^2\,dt$. In order to estimate $\mathcal{I}_k$, let $y = |\lambda_k| t$, $dt = \dfrac{1}{|\lambda_k|}dy$. Using $|L^{\pm}(y,\, X_j)|^2 \geq 0$ one obtains

$$\mathcal{I}_k \leq \frac{1}{|\lambda_k|} \int\limits_0^{\big[|\lambda_k|H\big]+1} |L^{\pm}(y,\, X_j)|^2\,dy\,.$$

From (3.11) it follows

$$|L^{\pm}(y,\, X_j)|^2 = \sum_{\substack{d_i|P(z)\\ i=1,2}} \lambda^{\pm}(d_1)\lambda^{\pm}(d_2) \sum_{\substack{\lambda_0 X_j<p_1,\, p_2\leq X_j\\ p_1+2\equiv 0(d_1)\\ p_2+2\equiv 0(d_2)}} e((p_1-p_2)y)\log p_1 \log p_2\,.$$

Then

$$\mathcal{I}_k \leq \frac{1}{|\lambda_k|} \sum_{\substack{d_i | P(z) \\ i=1,2}} \lambda^{\pm}(d_1)\lambda^{\pm}(d_2)$$

$$\times \sum_{\substack{\lambda_0 X_j < p_1 ,\, p_2 \leq X_j \\ p_1+2\equiv 0(d_1) \\ p_2+2\equiv 0(d_2)}} \log p_1 \log p_2 \int\limits_0^{\left[|\lambda_k|H\right]+1} e((p_1-p_2)y)dy\,. \tag{6.17}$$

Since $e(my)$, $m \in \mathbb{Z}$ is periodical with period 1, there holds

$$\int\limits_0^{\left[|\lambda_k|H\right]+1} e((p_1-p_2)y)dy = \Big(\left[|\lambda_k|H\right]+1\Big) \int\limits_0^1 e((p_1-p_2)y)dy\,. \tag{6.18}$$

From

$$\int\limits_0^1 e((p_1-p_2)y)\,dy = \begin{cases} 1, & \text{if } p_1 = p_2\,, \\ 0, & \text{if } p_1 \neq p_2\,, \end{cases}$$

(6.18) and (6.17) one gets

$$\mathcal{I}_k \leq \frac{\left[|\lambda_k|H\right]+1}{|\lambda_k|} \sum_{\substack{d_i | P(z) \\ i=1,2}} \lambda^{\pm}(d_1)\lambda^{\pm}(d_2) \sum_{\substack{\lambda_0 X_j < p \leq X_j \\ p+2\equiv 0(d_1) \\ p+2\equiv 0(d_2)}} (\log p)^2\,.$$

From the last inequality and using (3.3) we find

$$\mathcal{I}_k \ll H(\log X_j)^2 \sum_{\substack{d_i \leq D \\ \mu(d_i)\neq 0,\, i=1,2}} \sum_{\substack{\lambda_0 X_j < p \leq X_j \\ p+2\equiv 0([d_1,\, d_2])}} 1\,. \tag{6.19}$$

Let $d = (d_1,\, d_2)$, $k_i = \dfrac{d_i}{d}$, $[d_1,\, d_2] = dk_1k_2$. Since $\mu(d_i) \neq 0$, $i = 1,\, 2$, then $(d,\, k_i) = 1$, $i = 1, 2$. Now from (2.4), (2.6) and (6.19) we obtain

$$\begin{aligned}
\mathcal{I}_k &\ll \frac{(\log X_j)^3}{\varepsilon} \sum_{d\leq D} \sum_{\substack{k_i \leq \frac{D}{d} \\ i=1,2}} \sum_{\substack{\lambda_0 X_j < n \leq X_j \\ n+2\equiv 0(dk_1k_2)}} 1 \\
&\ll \frac{(\log X_j)^3}{\varepsilon} \sum_{d\leq D} \sum_{\substack{k_i \leq \frac{D}{d} \\ i=1,2}} \frac{X_j}{dk_1k_2} \\
&= \frac{X_j(\log X_j)^3}{\varepsilon} \sum_{d\leq D} \frac{1}{d} \bigg(\sum_{l \leq \frac{D}{d}} \frac{1}{l} \bigg)^2 \ll \frac{X_j(\log X_j)^6}{\varepsilon}\,.
\end{aligned}$$

From the last inequality and using (6.16) we get

$$\Gamma_1^{(2)}(X_j) \ll \frac{\varepsilon X_j}{(\log X_j)^{A/4-37}} \cdot \frac{X_j(\log X_j)^6}{\varepsilon} \ll \frac{X_j^2}{(\log X_j)^{A/4-43}} \,. \tag{6.20}$$

Summarizing, from (3.12), (4.4), (5.21) and (6.20) we obtain

$$\Gamma_1(X_j) = B(X_j)G^-(G^+)^2 + \mathcal{O}\bigg(\frac{X_j^2}{(\log X_j)^{A/4-43}}\bigg)\,. \tag{6.21}$$

7. PROOF OF THEOREM 1.

Since the sums $\Gamma_2(X_j)$, $\Gamma_3(X_j)$ and $\Gamma_4(X_j)$ are estimated in the same fashion as $\Gamma_1(X_j)$, we obtain from (3.9) and (6.21)

$$\Gamma(X_j) \geq B(X_j)W(X_j) + \mathcal{O}\bigg(\frac{X_j^2}{(\log X_j)^{A/4-43}}\bigg)\,, \tag{7.1}$$

where

$$W(X_j) = 3(G^+)^2\bigg(G^- - \frac{2}{3}G^+\bigg)\,. \tag{7.2}$$

Let $f(s)$ and $F(s)$ are the lower and the upper functions of the linear sieve. We know that if

$$s = \frac{\log D}{\log z} = \frac{1}{3\alpha}, \qquad 2 < s < 3 \tag{7.3}$$

then

$$F(s) = 2e^\gamma s^{-1}\,, \qquad f(s) = 2e^\gamma s^{-1}\log(s-1) \tag{7.4}$$

(see [1, Lemma 10]). Using (5.20) and [1, Lemma 10], we get

$$\begin{aligned} \mathcal{F}(z)\Big(f(s) + \mathcal{O}\big((\log X)^{-1/3}\big)\Big) \leq G^- \leq \mathcal{F}(z) \leq G^+ \\ \leq \mathcal{F}(z)\Big(F(s) + \mathcal{O}\big((\log X)^{-1/3}\big)\Big)\,. \end{aligned} \tag{7.5}$$

Here,

$$\mathcal{F}(z) = \prod_{2<p\leq z}\bigg(1 - \frac{1}{p-1}\bigg) \asymp \frac{1}{\log X}\,, \tag{7.6}$$

see Mertens formula [9, ch.9, §9.1, Theorem 9.1.3] and (2.7). To estimate $W(X_j)$ from below, we shall use the inequalities (see (7.5))

$$\begin{aligned} G^- - \frac{2}{3}G^+ &\geq \mathcal{F}(z)\Big(f(s) - \frac{2}{3}F(s) + \mathcal{O}\big((\log X)^{-1/3}\big)\Big)\,, \\ G^+ &\geq \mathcal{F}(z)\,. \end{aligned} \tag{7.7}$$

Let $X = X_j$. Then from (7.2) and (7.7) it follows

$$W(X_j) \geq 3\mathcal{F}^3(z)\biggl(f(s) - \frac{2}{3}F(s) + \mathcal{O}\bigl((\log X)^{-1/3}\bigr)\biggr) . \tag{7.8}$$

We choose $s = \dfrac{\log D}{\log z} = 2.994$. Then

$$f(s) - \frac{2}{3}F(s) \geq 0,0000001 ,$$

and from (7.3) we get $\dfrac{1}{\alpha} = 8.982$. From (2.3), (7.1),(7.6), (7.8) and Lemma 3 we obtain:

$$\Gamma(X_j) \gg \frac{X_j^2}{(\log X_j)^{B+4}} + \frac{X_j^2}{(\log X_j)^{A/4-43}} . \tag{7.9}$$

We choose $A \geq 4B + 192$. Then

$$\Gamma(X_j) \gg \frac{X_j^2}{(\log X_j)^{B+4}} .$$

Finally, we note that if $\Gamma_0(X_j)$ is the number of the triples $p_i \in [\lambda_0 X_j,\, X_j]$, $p_i + 2 = P_8$, $i = 1,\, 2,\, 3$, satisfying (1.7), then there exists a positive constant c such that

$$\Gamma_0(X_j) \geq \frac{1}{(\log X_j)^3}\,\Gamma(X_j) \geq \frac{cX_j^2}{(\log X_j)^{B+7}}$$

and for every prime factor q of $p_i + 2$, $i = 1, 2, 3$ we have $q \geq X^{0.1113}$. That completes the proof of Theorem 1.

ACKNOWLEDGEMENTS. The authors are indebted to D.I. Tolev for his helpful comments and suggestions.

8. REFERENCES

1. Brüdern, J., Fouvry, E.: Lagrange's Four Squares Theorem with almost prime variables. *J. Reine Angew. Math.*, **454**, 1994, 59–96.
2. Chen, J. R.: On the representation of a large even integer as the sum of a prime and the product of at most two primes. *Sci. Sinica*, **16**, 1973, 157–176.
3. Davenport, H.: *Multiplicative number theory* (revised by H. Montgomery), Springer, 2000, Third edition.
4. Hardy, G. H., Wright, E. M.: *Higher algebra. An Introduction to the Theroy of Numbers*, Oxford Univeristy Press, 1979, Fifth edition.
5. Iwaniec, H.: Rosser's sieve. *Acta Arithmetica*, **36**, 1980, 171–202.

6. Iwaniec, H.: A new form of the error term in the linear sieve. *Acta Arithmetica*, **37**, 1980, 307–320.
7. Karatsuba, A. A.: *Basic analytic number theory*, Nauka, 1983 (in Russian).
8. Matomäki, K.: A Bombieri–Vinogradov type exponential sum result with applications. *J. Number Theory*, **129**, 2009, no. 9, 2214–2225.
9. Murty, R. M.: *Problems in Analytic Number Theory*, Springer, 2008, Second Edition.
10. Peneva, T.: On the ternary Goldbach problem with primes p such that $p+2$ are almost-prime. *Acta Math. Hungar.*, **86**, 2000, 305–318.
11. Segal, B. I.: On a theorem analogous to Waring's theorem. *Dokl. Akad. Nauk SSSR*, **2**, 1933, 47–49 (in Russian).
12. Shmidt, W. M.: *Diophantine Approximation*, 1984, (in Russian).
13. Todorova, T. L., Tolev, D. I.: On the distribution of αp modulo one for primes p of a special form. *Math. Slovaca*, **60**, 2010, 771–786.
14. Tolev, D. I.: Representations of large integers as sums of two primes of special type. *Algebraic Number Theory and Diophantine Analysis*, Walter de Gruyter, 2000, 485–495.
15. Tolev, D. I.: Arithmetic progressions of prime-almost-prime twins. *Acta Arith.*, **88**, 1999, 67–98.
16. Vinogradov, I. M.: Representation of an odd number as the sum of three primes. *Dokl. Akad. Nauk. SSSR*, **15**, 1937, 291–294, (in Russian).
17. Vinogradov, I. M.: The method of trigonometrical sums in the theory of numbers. *Trudy Math. Inst. Steklov*, **23**, 1947, 1–109, (in Russian).
18. Vaughan, R. C.: Diophantine approximation by prime numbers I. *Proc. Lond. Math. Soc.*, **28**, 1974, 373–384.
19. Vaughan, R. C.: *The Hardy–Littlewood method.* Cambridge Univ. Press, 1997, Second edition.

Received on 15 February, 2015
In revised form on 18 May, 2015

S. I. Dimitrov
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
8, St.Kliment Ohridski Blvd.
1756 Sofia, BULGARIA
e-mail: xyzstoyan@gmail.com

T. L. Todorova
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier Blvd.
1164 Sofia, BULGARIA
e-mail: tlt@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

ON VECTOR–PARAMETER FORM OF THE $\mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ MAP

VELIKO D. DONCHEV, CLEMENTINA D. MLADENOVA, IVAÏLO M. MLADENOV

By making use of the *Cayley* maps for the isomorphic Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ we have found the vector parameter form of the well-known *Wigner* group homomorphism $W : \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ and its sections. Based on it and pulling back the group multiplication in SO(3,ℝ) through the *Cayley* map $\mathfrak{su}(2) \to \mathrm{SU}(2)$ to the covering space, we present the derivation of the explicit formulas for compound rotations. It is shown that both sections are compatible with the group multiplications in SO(3,ℝ) up to a sign and this allows uniform operations with half-turns in the three-dimensional space. The vector parametrization of SU(2) is compared with that of SO(3,ℝ) generated by the *Gibbs* vectors in order to discuss their advantages and disadvantages.

Keywords: Lie groups and algebras, Cayley map, vector-parametrization of rotations
2000 Math. Subject Classification: 15A16, 15A23, 22E60, 22E70

1. INTRODUCTION

Parameterizations are used to describe Lie groups in an easier way. Let G be a finite dimensional Lie group with Lie algebra $\mathfrak{g}$. A vector parametrization of G is a map $\mathfrak{g} \to G$, which is diffeomorphic onto its image. Before studying vector parametrizations, let us compare them with the exponential map $\exp : \mathfrak{g} \to G$. It is locally bijective and need not to be such globally. For example in the case of $G = \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ for arbitrary integers $k_1,\ldots,k_n$ the diagonal matrix $\mathrm{diag}(2\pi \mathrm{i} k_1,\ldots,2\pi \mathrm{i} k_n)$ is transformed into the unit matrix $\mathcal{I}_n$. If G is connected and compact as it is in cases under consideration the exponential map is surjective, see [3]. Besides, the group multiplication $\mu : G \times G \to G$ admits a local pull-back on the Lie algebra level via the commutative diagram (see Fig. 1).

$$
\begin{array}{ccc}
\mathfrak{g} \times \mathfrak{g} & \xrightarrow{(1.1)} & \mathfrak{g} \\
\Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\
G \times G & \xrightarrow{\ \mu\ } & G
\end{array}
$$

Figure 1: Local pullback of the multiplication law μ for the Lie group G in the corresponding Lie algebra $\mathfrak{g}$.

This pull-back is given by the *Baker–Campbell–Hausdorff* formula in commutator-free form

$$BCH(X,Y) = X + Y + \sum_{n=2}^{\infty} \sum_{|\omega|=n} g_\omega \omega \,, \tag{1.1}$$

where the inner sum is over all the "words" $\omega = \omega_1 \dots \omega_n$ of length n in the alphabet $\{X, Y\}$. Here, g_ω are the Goldberg's rational coefficients [9, 15]. In general, it is difficult to compute (1.1) and there is an ongoing research in this area (see [1, 4, 17]). However, the first few terms of (1.1) in commutator form are given by the formula

$$\begin{aligned} BCH(X,Y) =& X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[Y,X]] - [Y,[X,Y]]) \\ & - \frac{1}{24}[Y,[X,[X,Y]]] + \cdots \end{aligned} \tag{1.2}$$

The image of the parametrization need not be the whole group G. For SO(3,$\mathbb{R}$), the image of the *Cayley* map consists of all rotations with angles $\theta \neq \pm\pi$, i.e., the matrices $\mathcal{R} \in$ SO(3,$\mathbb{R}$) with no eigenvalues of -1.
In Section 2 of the paper we derive a vector parametrization of SU(2) and make use of it for expressing the composition law in this group. We show that the *Cayley* map $\mathfrak{su}(2) \to$ SU(2) is bijective onto its image. Section 3 provides an explicit formula for the double cover map SU(2) $\to$ SO(3,$\mathbb{R}$) in terms of the vector parameters of the source and the target manifold.

2. VECTOR PARAMETRIZATION OF SU(2) AND THE PULL-BACK OF THE COMPOSITION LAW

2.1. THE CASE OF SO(3,$\mathbb{R}$)

The Lie algebra $\mathfrak{so}(3)$ consists of the real anti-symmetric 3×3 matrices. The *Cayley* map of $\mathfrak{so}(3) \to$ SO(3,$\mathbb{R}$) gives the so called *Gibbs* vector parametrization of SO(3,$\mathbb{R}$). The matrices

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

form a basis of $\mathfrak{so}(3)$ over the filed of the real numbers. For arbitrary $i, j, k \in \{1, 2, 3\}$ let $\varepsilon_{ijk} = 1$ if i, j, k is an even permutation of $1, 2, 3$, $\varepsilon_{ijk} = -1$ for an odd permutation of $1, 2, 3$ and $\varepsilon_{ijk} = 0$ otherwise. The following relations hold:

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad i, j, k \in \{1, 2, 3\}. \tag{2.2}$$

Any $\mathcal{C} \in \mathfrak{so}(3)$ has a unique representation

$$\boldsymbol{c} \mapsto \mathcal{C} = \mathbf{c} \cdot \mathbf{J} = c_1 J_1 + c_2 J_2 + c_3 J_3 = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix},$$

where

$$\boldsymbol{c} = (c_1, c_2, c_3), \qquad \boldsymbol{c}^2 := c_1^2 + c_2^2 + c_3^2 = \boldsymbol{c}.\boldsymbol{c} = |\boldsymbol{c}|^2 = c^2. \tag{2.3}$$

Hereafter we shall use $\boldsymbol{c}$ and c to denote respectively the vector $\boldsymbol{c}$ and its norm c. This convention applies to other vectors as well.

The *Hamilton–Cayley* theorem for $\mathcal{C}$ reads as $\mathcal{C}^3 = -c^2\mathcal{C}$. That is why the exponential map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3,\mathbb{R})$ is given explicitly by the formula

$$\exp(\mathcal{C}) = \mathcal{I} + \frac{\sin c}{c}\mathcal{C} + \frac{1 - \cos c}{c^2}\mathcal{C}^2. \tag{2.4}$$

In order to compare, let us recall that the *Cayley* map for $\mathfrak{so}(3)$ associates with $\mathbf{c} \cdot \mathbf{J} \in \mathfrak{so}(3)$ the matrix

$$\mathcal{R}(\boldsymbol{c}) = \mathrm{Cay}_{\mathfrak{so}(3)}(\boldsymbol{c}) = (\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} = (\mathcal{I} - \mathcal{C})^{-1}(\mathcal{I} + \mathcal{C}). \tag{2.5}$$

One checks immediately that

$$(\mathcal{I} - \mathcal{C})^{-1} = \mathcal{I} + \frac{1}{1 + c^2}\mathcal{C} + \frac{1}{1 + c^2}\mathcal{C}^2 \tag{2.6}$$

and (2.5) can be expressed in the form

$$\mathrm{Cay}_{\mathfrak{so}(3)}(\boldsymbol{c}) = \mathcal{I} + \frac{2}{1 + c^2}\mathcal{C} + \frac{2}{1 + c^2}\mathcal{C}^2 \tag{2.7}$$

for all $\boldsymbol{c} \in \mathbb{R}^3$. Is is well known that in $\mathrm{SO}(3,\mathbb{R})$, the half-turns are described by symmetric rotation matrices. Note that $\mathrm{Cay}_{\mathfrak{so}(3)}$ is bijective onto its image (see [14])

$$\Im\mathrm{Cay}_{\mathfrak{so}(3)} = \{\mathcal{R} \in \mathrm{SO}(3,\mathbb{R})\,;\, \mathcal{R} \neq \mathcal{R}^t\} = \mathrm{SO}(3,\mathbb{R}) \backslash \mathrm{S}(3,\mathbb{R})\,, \tag{2.8}$$

where $\mathrm{S}(3,\mathbb{R})$ is the set of all symmetric 3×3 matrices with real entries. The image $\mathcal{R}(\boldsymbol{c})$ of $\mathbf{c}$ by $\mathrm{Cay}_{\mathfrak{so}(3)}$ is

$$\mathbf{c} \to \mathcal{R}(\boldsymbol{c}) = \frac{2}{1 + c^2}\begin{pmatrix} 1 + c_1^2 & c_1c_2 - c_3 & c_1c_3 + c_2 \\ c_1c_2 + c_3 & 1 + c_2^2 & c_2c_3 - c_1 \\ c_1c_3 - c_2 & c_2c_3 + c_1 & 1 + c_3^2 \end{pmatrix} - \mathcal{I}. \tag{2.9}$$

The rotation $\mathcal{R} = \mathcal{R}(\mathbf{n}, \theta)$ at angle θ about the axis $\mathbf{n}$ is represented by *Gibbs* parameter $\mathbf{c} = \tan\frac{\theta}{2}\,\mathbf{n}$, see [2]. In order to express the group law in SO(3,$\mathbb{R}$) by the means of the *Cayley* map let us denote by $\tilde{\boldsymbol{c}}$ the vector parameter of the product $\mathcal{R}(\tilde{\boldsymbol{c}}) = \mathcal{R}(\boldsymbol{a})\mathcal{R}(\boldsymbol{c})$ of the elements of SO(3,$\mathbb{R}$), corresponding to $\boldsymbol{a}, \boldsymbol{c} \in \mathbb{R}^3$. Then, as pointed out in [7]

$$\mathcal{R}(\tilde{\boldsymbol{c}}) = \mathcal{R}(\boldsymbol{a})\mathcal{R}(\boldsymbol{c}), \qquad \tilde{\boldsymbol{c}} = \tilde{\boldsymbol{c}}(\boldsymbol{a}, \boldsymbol{c}) = \langle \boldsymbol{a}, \boldsymbol{c} \rangle = \frac{\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{a} \times \boldsymbol{c}}{1 - \boldsymbol{a}.\boldsymbol{c}} \cdot \tag{2.10}$$

In the case of $\mathfrak{so}(3)$ it is shown in [6] that the *Baker–Campbell–Hausdorff* formula takes the form

$$BCH(\mathcal{A}, \mathcal{C}) = BCH(\boldsymbol{a} \cdot \mathbf{J}, \boldsymbol{c} \cdot \mathbf{J}) = \alpha\mathcal{A} + \beta\mathcal{C} + \gamma[\mathcal{A}, \mathcal{C}]\,, \tag{2.11}$$

with

$$\alpha = \frac{\sin^{-1}(q)}{q}\frac{m}{\theta}, \qquad \beta = \frac{\sin^{-1}(q)}{q}\frac{n}{\psi}, \qquad \gamma = \frac{\sin^{-1}(q)}{q}\frac{p}{\theta\psi}\,,$$

where $\psi = |\boldsymbol{a}|$, $\theta = |\boldsymbol{c}|$, $\angle(\boldsymbol{a}, \boldsymbol{c}) = \cos^{-1}\left(\dfrac{\boldsymbol{a}.\boldsymbol{c}}{|\boldsymbol{a}||\boldsymbol{c}|}\right)$ and

$$\begin{aligned}
m &= \sin(\theta)\cos^2(\psi/2) - \sin(\psi)\sin^2(\theta/2)\cos(\angle(\boldsymbol{a}, \boldsymbol{c}))\,,\\
n &= \sin(\psi)\cos^2(\theta/2) - \sin(\theta)\sin^2(\psi/2)\cos(\angle(\boldsymbol{a}, \boldsymbol{c}))\,,\\
p &= \frac{1}{2}\sin(\theta)\sin(\psi) - 2\sin^2(\theta/2)\sin^2(\psi/2)\cos(\angle(\boldsymbol{a}, \boldsymbol{c}))\,,\\
q &= \sqrt{m^2 + n^2 + 2mn\cos(\angle(\boldsymbol{a}, \boldsymbol{c})) + p^2\sin^2(\angle(\boldsymbol{a}, \boldsymbol{c}))}\,.
\end{aligned}$$

Note that equation (2.10) is much simpler and more convenient when compared with (2.11). The vector parameter form of SO(3,$\mathbb{R}$) matrices and the corresponding composition law (2.10) are exploited in the decomposition method of the three dimensional rotations about three almost arbitrary axes, see [2]. In this vector parameter form of SO(3,$\mathbb{R}$), the half-turns, i.e., rotations at angles $\theta = \pm\pi$, can not be described. Henceforth we denote the matrix of the half-turn about the axis $\mathbf{n}$, i.e., $\mathcal{R}(\mathbf{n}, \pi)$, by $\mathcal{O}(\mathbf{n})$. The composition of the two rotations is not well defined also when $1 - \boldsymbol{a}.\boldsymbol{c} = 0$, which is exactly the condition that the compound rotation $\tilde{\boldsymbol{c}}$ is a half-turn.

2.2. DESCRIPTION OF $\mathfrak{su}(2)$

A coordinate free description [11] of $\mathfrak{su}(2)$ can be given. Let i be the imaginary unit and $\sigma_1, \sigma_2, \sigma_3$ be three elements which obey the rules

$$\begin{gathered}
\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1\\
\sigma_1\sigma_2 = -\sigma_2\sigma_1 = \mathrm{i}\sigma_3, \quad \sigma_2\sigma_3 = -\sigma_3\sigma_2 = \mathrm{i}\sigma_1, \quad \sigma_3\sigma_1 = -\sigma_1\sigma_3 = \mathrm{i}\sigma_2.
\end{gathered} \tag{2.12}$$

If we define the spin vector $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3) \tag{2.13}$$

and $\mathbf{n}$ and $\mathbf{m}$ are arbitrary unit vectors in $\mathbb{R}^3$, then the following properties hold:

$$\begin{gathered}(\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1, \qquad (\mathbf{m}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma}) = \mathbf{m}.\mathbf{n} + \mathrm{i}(\mathbf{m}\times\mathbf{n})\cdot\boldsymbol{\sigma}\,, \\ \boldsymbol{\sigma}\cdot(\mathbf{n}\cdot\boldsymbol{\sigma}) = \mathbf{n} + \mathrm{i}\mathbf{n}\times\boldsymbol{\sigma}, \qquad (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\boldsymbol{\sigma} = \mathbf{n} - \mathrm{i}\mathbf{n}\times\boldsymbol{\sigma}\,, \\ (\mathbf{m}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}(\mathbf{n}\cdot\boldsymbol{\sigma}) = (\mathbf{m}\cdot\boldsymbol{\sigma})\mathbf{n} + (\mathbf{n}\cdot\boldsymbol{\sigma})\mathbf{m} - \mathrm{i}(\mathbf{m}\times\mathbf{n}) - (\mathbf{m}.\mathbf{n})\cdot\boldsymbol{\sigma}\,.\end{gathered} \tag{2.14}$$

A concrete matrix realization of $\sigma_1, \sigma_2, \sigma_3$ in (2.12) are the Pauli's matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.15}$$

The matrices s_1, s_2 and s_3 defined by

$$s_1 = -\frac{\mathrm{i}}{2}\sigma_1, \qquad s_2 = -\frac{\mathrm{i}}{2}\sigma_2, \qquad s_3 = -\frac{\mathrm{i}}{2}\sigma_3 \tag{2.16}$$

form a $\mathbb{R}-$basis of $\mathfrak{su}(2)$. Direct calculation shows that

$$[s_i, s_j] = \epsilon_{ijk} s_k, \qquad i, j, k \in \{1, 2, 3\}. \tag{2.17}$$

Denoting $\mathbf{s} = (s_1, s_2, s_3)$ we express the $\mathfrak{su}(2)$ algebra in the following way:

$$\mathfrak{su}(2) = \left\{\mathbf{c}\cdot\mathbf{s} = \mathrm{c}_1 s_1 + \mathrm{c}_2 s_2 + \mathrm{c}_3 s_3\,;\ \mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3\right\}. \tag{2.18}$$

The corresponding matrix realization of $\mathbf{c}\cdot\mathbf{s}$ is

$$\begin{pmatrix} -\mathrm{i}\dfrac{\mathrm{c}_3}{2} & -\dfrac{\mathrm{c}_2}{2} - \mathrm{i}\dfrac{\mathrm{c}_1}{2} \\ \dfrac{\mathrm{c}_2}{2} - \mathrm{i}\dfrac{\mathrm{c}_1}{2} & \mathrm{i}\dfrac{\mathrm{c}_3}{2} \end{pmatrix}. \tag{2.19}$$

Obviously, the map

$$\mathrm{c}_1 s_1 + \mathrm{c}_2 s_2 + \mathrm{c}_3 s_3 \longrightarrow c_1 J_1 + c_2 J_2 + c_3 J_3 \tag{2.20}$$

is a linear isomorphism between $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.

2.3. CAYLEY MAP FROM $\mathfrak{su}(2)$ TO SU(2)

Till the end of this section $\mathcal{I}$ will stand for the unit matrix with dimension consistent with the context. Let

$$\mathcal{A} = \mathrm{a}_1 s_1 + \mathrm{a}_2 s_2 + \mathrm{a}_3 s_3 = -\frac{\mathrm{i}}{2}\mathbf{a}\cdot\boldsymbol{\sigma} \in \mathfrak{su}(2)\,, \tag{2.21}$$

where

$$\mathbf{a} = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3), \qquad \mathbf{a}^2 = \mathrm{a}_1^2 + \mathrm{a}_2^2 + \mathrm{a}_3^2 = \mathbf{a}.\mathbf{a} = |\mathbf{a}|^2 = \mathrm{a}^2. \tag{2.22}$$

Let us recall also that (see [8]) the exponential map for $\mathfrak{su}(2)$ is globally defined and surjective. It maps $\mathcal{A} \in \mathfrak{su}(2)$ to

$$\exp(\mathcal{A}) = \cos(\mathrm{a}/2)\mathcal{I} - \frac{\sin \mathrm{a}/2}{\mathrm{a}/2}\mathcal{A}. \tag{2.23}$$

The *Hamilton–Cayley* theorem implies the identity $\mathcal{A}^2 = -\dfrac{\mathrm{a}^2}{4}\mathcal{I}$. The image of $\mathcal{A}$ under the *Cayley* map is

$$\mathcal{U}(\mathbf{a}) = \mathrm{Cay}_{\mathfrak{su}(2)}(\mathcal{A}) = (\mathcal{I} + \mathcal{A})(\mathcal{I} - \mathcal{A})^{-1}. \tag{2.24}$$

In general, the *Cayley* map $\mathrm{Cay}_{\mathfrak{su}(n)}$ for the Lie algebra $\mathfrak{su}(n)$ of skew-hermitian matrices ($\mathcal{A}^\dagger = \overline{\mathcal{A}}^t = -\mathcal{A}$) with trace zero takes values in $\mathrm{U}(n)$. Indeed, let us take any $\mathcal{A} \in \mathfrak{su}(n)$ and its image $\mathrm{Cay}_{\mathfrak{su}(n)}(\mathcal{A}) = \mathcal{U}$. Taking into account that $(\mathcal{U}^\dagger)^{-1} = (\mathcal{U}^{-1})^\dagger$, we obtain

$$\begin{aligned} \mathcal{U}\mathcal{U}^\dagger &= (\mathcal{I} + \mathcal{A})(\mathcal{I} - \mathcal{A})^{-1}((\mathcal{I} + \mathcal{A})(\mathcal{I} - \mathcal{A})^{-1})^\dagger \\ &= (\mathcal{I} + \mathcal{A})(\mathcal{I} - \mathcal{A})^{-1}((\mathcal{I} - \mathcal{A})^{-1})^\dagger(\mathcal{I} + \mathcal{A})^\dagger \\ &= (\mathcal{I} + \mathcal{A})(\mathcal{I} - \mathcal{A})^{-1}(\mathcal{I} + \mathcal{A})^{-1}(\mathcal{I} - \mathcal{A}) \\ &= (\mathcal{I} + \mathcal{A})(\mathcal{I} - \mathcal{A})^{-1}(\mathcal{I} - \mathcal{A})(\mathcal{I} + \mathcal{A})^{-1} = \mathcal{I}. \end{aligned} \tag{2.25}$$

Lemma 1. *For each element $\mathcal{A} \in \mathfrak{su}(2)$ there holds*

$$(\mathcal{I} + \mathcal{A})(\mathcal{I} - \mathcal{A}) = (\mathcal{I} - \mathcal{A})(\mathcal{I} + \mathcal{A}) = \mathcal{I} - \mathcal{A}^2 = \big(1 + \frac{\mathrm{a}^2}{4}\big)\mathcal{I}, \tag{2.26}$$

i.e.,

$$(\mathcal{I} - \mathcal{A})^{-1} = \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}(\mathcal{I} + \mathcal{A}), \qquad (\mathcal{I} + \mathcal{A})^{-1} = \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}(\mathcal{I} - \mathcal{A}). \tag{2.27}$$

Besides (2.27), from Lemma 1 we also infer

$$\begin{aligned} \mathcal{U}(\mathbf{a}) &= (\mathcal{I} + \mathcal{A})(\mathcal{I} - \mathcal{A})^{-1} = \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}(\mathcal{I} + \mathcal{A})^2 \\ &= \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}(\mathcal{I} + 2\mathcal{A} + \mathcal{A}^2) = \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}\big(\mathcal{I} + 2\mathcal{A} - \frac{\mathrm{a}^2}{4}\mathcal{I}\big) \\ &= \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}\Big(\big(1 - \frac{\mathrm{a}^2}{4}\big)\mathcal{I} - \mathrm{i}\mathbf{a} \cdot \boldsymbol{\sigma}\Big). \end{aligned} \tag{2.28}$$

The matrix form of $\mathcal{U}(\mathbf{a})$ is

$$\mathcal{U}(\mathbf{a})=\frac{1-\dfrac{\mathrm{a}^2}{4}}{1+\dfrac{\mathrm{a}^2}{4}}\,\mathcal{I}+\frac{1}{1+\dfrac{\mathrm{a}^2}{4}}\begin{pmatrix}-\mathrm{ia}_3 & -\mathrm{a}_2-\mathrm{ia}_1\\ \mathrm{a}_2-\mathrm{ia}_1 & \mathrm{ia}_3\end{pmatrix}. \tag{2.29}$$

The matrix $\mathcal{U}(\mathbf{a})$ defined in (2.29) is unitary due to (2.25). Direct calculation shows that

$$\det\mathcal{U}(\mathbf{a})=\det\left(\left(1+\frac{\mathrm{a}^2}{4}\right)^{-1}(\mathcal{I}+\mathcal{A})^2\right)=\left(1+\frac{\mathrm{a}^2}{4}\right)^{-2}(\det(\mathcal{I}+\mathcal{A}))^2=1 \tag{2.30}$$

i.e., $\mathcal{U}(\mathbf{a})\in\mathrm{SU}(2)$. Following *Wigner* [18] we can use the explicit homomorphism map $W\colon\ \mathrm{SU}(2)\to\mathrm{SO}(3,\mathbb{R})$ given by

$$\begin{gathered}\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}=\begin{pmatrix}\alpha_1+\mathrm{i}\alpha_2 & \beta_1+\mathrm{i}\beta_2\\ -\beta_1+\mathrm{i}\beta_2 & \alpha_1-\mathrm{i}\alpha_2\end{pmatrix}\\ \xrightarrow{W}\begin{pmatrix}\alpha_1^2-\alpha_2^2-\beta_1^2+\beta_2^2 & 2(\alpha_1\alpha_2+\beta_1\beta_2) & 2(\alpha_2\beta_2-\alpha_1\beta_1)\\ 2(\beta_1\beta_2-\alpha_1\alpha_2) & \alpha_1^2-\alpha_2^2+\beta_1^2-\beta_2^2 & 2(\alpha_2\beta_1+\alpha_1\beta_2)\\ 2(\alpha_1\beta_1+\alpha_2\beta_2) & 2(\alpha_2\beta_1-\alpha_1\beta_2) & \alpha_1^2+\alpha_2^2-\beta_1^2-\beta_2^2\end{pmatrix}.\end{gathered} \tag{2.31}$$

The comparison of (2.29) and (2.31) yields

$$\alpha=\alpha_1+\mathrm{i}\alpha_2=\frac{1-\dfrac{\mathrm{a}^2}{4}}{1+\dfrac{\mathrm{a}^2}{4}}+\mathrm{i}\frac{-\mathrm{a}_3}{1+\dfrac{\mathrm{a}^2}{4}},\quad \beta=\beta_1+\mathrm{i}\beta_2=\frac{-\mathrm{a}_2}{1+\dfrac{\mathrm{a}^2}{4}}+\mathrm{i}\frac{-\mathrm{a}_1}{1+\dfrac{\mathrm{a}^2}{4}}\cdot \tag{2.32}$$

In the case of the SU(2) group manifold, which is diffeomorphic to the sphere S^3, there is a homotopy obstruction for the existence of a global diffeomorphism $\mathbb{R}^3\simeq\mathfrak{su}(2)\to\mathrm{SU}(2)\simeq S^3$, so that no vector parametrization $\mathfrak{su}(2)\to\mathrm{SU}(2)$ exists onto the entire group SU(2). Actually, the *Cayley* map provides a vector parametrization

$$\mathrm{Cay}_{\mathfrak{su}(2)}\colon\ \mathfrak{su}(2)\to\mathrm{SU}(2)\backslash\{-\mathcal{I}\}\,, \tag{2.33}$$

whose inverse is

$$\begin{gathered}\mathrm{Cay}^{-1}_{\mathfrak{su}(2)}\begin{pmatrix}\alpha_1+\mathrm{i}\alpha_2 & \beta_1+\mathrm{i}\beta_2\\ -\beta_1+\mathrm{i}\beta_2 & \alpha_1-\mathrm{i}\alpha_2\end{pmatrix}=-\frac{\mathrm{i}}{2}\mathbf{a}\cdot\boldsymbol{\sigma}\,,\\ \mathbf{a}=(\mathrm{a}_1,\mathrm{a}_2,\mathrm{a}_3)=-\frac{2}{1+\alpha_1}(\beta_2,\beta_1,\alpha_2).\end{gathered} \tag{2.34}$$

By means of (2.31) and (2.32) one calculates straightforwardly that the image $\mathcal{R}_{\mathcal{U}}(\mathbf{a})$ of $\mathcal{U}(\mathbf{a})$ under the *Wigner* map W is

$$\frac{8}{(4+\mathrm{a}^2)^2}\begin{pmatrix}\frac{(4+\mathrm{a}^2)^2}{4}-4\mathrm{a}_2^2-4\mathrm{a}_3^2 & 4\mathrm{a}_1\mathrm{a}_2-\mathrm{a}_3\left(4-\mathrm{a}^2\right) & 4\mathrm{a}_1\mathrm{a}_3+\mathrm{a}_2\left(4-\mathrm{a}^2\right)\\ 4\mathrm{a}_1\mathrm{a}_2+\mathrm{a}_3\left(4-\mathrm{a}^2\right) & \frac{(4+\mathrm{a}^2)^2}{4}-4\mathrm{a}_1^2-4\mathrm{a}_3^2 & 4\mathrm{a}_2\mathrm{a}_3-\mathrm{a}_1\left(4-\mathrm{a}^2\right)\\ 4\mathrm{a}_1\mathrm{a}_3-\mathrm{a}_2\left(4-\mathrm{a}^2\right) & 4\mathrm{a}_2\mathrm{a}_3+\mathrm{a}_1\left(4-\mathrm{a}^2\right) & \frac{(4+\mathrm{a}^2)^2}{4}-4\mathrm{a}_1^2-4\mathrm{a}_2^2\end{pmatrix}-\mathcal{I}. \tag{2.35}$$

Let $\mathcal{A} = -\frac{\mathrm{i}}{2}\mathbf{a}\cdot\boldsymbol{\sigma}$, $\mathcal{C} = -\frac{\mathrm{i}}{2}\mathbf{c}\cdot\boldsymbol{\sigma} \in \mathfrak{su}(2)$. The term of third degree in $BCH(\mathcal{A},\mathcal{C})$ (cf. (1.2)) is $\frac{1}{2}[\mathcal{A},\mathcal{C}] = -\frac{\mathrm{i}}{2}(\mathbf{a}\times\mathbf{c})\cdot\boldsymbol{\sigma}$, and that one of degree four is

$$\frac{1}{12}([\mathcal{A},[\mathcal{C},\mathcal{A}]] - [\mathcal{C},[\mathcal{A},\mathcal{C}]]) = -\frac{\mathrm{i}}{2}\tilde{\mathbf{c}}_4\cdot\boldsymbol{\sigma}, \qquad \tilde{\mathbf{c}}_4 = (u_4, v_4, w_4)\,, \tag{2.36}$$

with

$$\begin{aligned} u_4 &= \frac{1}{12}(\mathrm{a}_1\mathrm{a}_2\mathrm{c}_1 + \mathrm{a}_1\mathrm{c}_1\mathrm{c}_2 + \mathrm{a}_2\mathrm{a}_3\mathrm{c}_3 + \mathrm{a}_3\mathrm{c}_2\mathrm{c}_3 - \mathrm{a}_1^2\mathrm{c}_2 - \mathrm{a}_3^2\mathrm{c}_2 - \mathrm{a}_2\mathrm{c}_1^2 - \mathrm{a}_2\mathrm{c}_3^2)\,,\\ v_4 &= \frac{1}{12}(\mathrm{a}_1\mathrm{a}_2\mathrm{c}_2 + \mathrm{a}_1\mathrm{a}_3\mathrm{c}_3 + \mathrm{a}_2\mathrm{c}_1\mathrm{c}_2 + \mathrm{a}_3\mathrm{c}_1\mathrm{c}_3 - \mathrm{a}_2^2\mathrm{c}_1 - \mathrm{a}_3^2\mathrm{c}_1 - \mathrm{a}_1\mathrm{c}_2^2 - \mathrm{a}_1\mathrm{c}_3^2)\,,\\ w_4 &= \frac{1}{12}(\mathrm{a}_1\mathrm{c}_1\mathrm{c}_3 + \mathrm{a}_1\mathrm{a}_3\mathrm{c}_1 + \mathrm{a}_2\mathrm{c}_2\mathrm{c}_3 + \mathrm{a}_2\mathrm{a}_3\mathrm{c}_2 - \mathrm{a}_1^2\mathrm{c}_3 - \mathrm{a}_2^2\mathrm{c}_3 - \mathrm{a}_3\mathrm{c}_1^2 - \mathrm{a}_3\mathrm{c}_2^2). \end{aligned} \tag{2.37}$$

Note that the coefficients of the term of degree four are homogeneous polynomials of $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3$ of degree three. It is interesting to compare the composition rule (2.10) of SO(3,$\mathbb{R}$), expressed through the *Gibbs* vector parameter with the following formula

$$\mathcal{A} + \mathcal{C} + \frac{1}{2}[\mathcal{A},\mathcal{C}] = -\frac{\mathrm{i}}{2}\Big(\mathbf{a} + \mathbf{c} + \frac{\mathbf{a}\times\mathbf{c}}{2}\Big)\cdot\boldsymbol{\sigma}. \tag{2.38}$$

2.4. COMPOSITION LAW IN SU(2)

Proposition 1. *Let* $\mathcal{U}_1(\mathbf{c}), \mathcal{U}_2(\mathbf{a}) \in \mathrm{SU}(2)$ *are the images of* $\mathcal{A}_1 = \mathbf{c}\cdot\mathbf{s}$ *and* $\mathcal{A}_2 = \mathbf{a}\cdot\mathbf{s}$ *under the map* (2.24) *of the vectors* $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$. *Let*

$$\mathcal{U}_3(\langle\mathbf{a},\mathbf{c}\rangle_{\mathrm{SU}(2)}) = \mathcal{U}_2(\mathbf{a}).\mathcal{U}_1(\mathbf{c}) \tag{2.39}$$

denote the composition of $\mathcal{U}_2(\mathbf{a})$ *and* $\mathcal{U}_1(\mathbf{c})$ *in* SU(2). *The corresponding vector-parameter* $\widetilde{\mathbf{a}} \in \mathbb{R}^3$, *for which* $\mathrm{Cay}_{\mathfrak{su}(2)}(\mathcal{A}_3) = \mathcal{U}_3$, $\mathcal{A}_3 = \widetilde{\mathbf{a}}\cdot\mathbf{s}$ *is*

$$\widetilde{\mathbf{a}} = \frac{\left(1 - \frac{\mathrm{c}^2}{4}\right)\mathbf{a} + \left(1 - \frac{\mathrm{a}^2}{4}\right)\mathbf{c} + 4\frac{\mathbf{a}}{2}\times\frac{\mathbf{c}}{2}}{1 - 2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2} + \frac{\mathrm{a}^2}{4}\frac{\mathrm{c}^2}{4}}. \tag{2.40}$$

The vector $\widetilde{\mathbf{a}}$ *equals to* $\mathbf{0}$ *if only if* $\mathbf{c} = -\mathbf{a}$ *or* $\mathbf{c} = 2\tan\frac{\theta}{4}\,\mathbf{n}$ *and* $\mathbf{a} = 2\tan\frac{2\pi-\theta}{4}\,\mathbf{n}$, *where* $\mathbf{n}\in\mathbb{R}^3, \mathbf{n}^2 = 1$ *and* $\theta \in [0, 2\pi)$. *In both cases,* $\mathbf{c} = -\mathbf{a}$ *and* $\mathbf{c} = 2\tan\frac{\theta}{4}\,\mathbf{n}$, $\mathbf{a} = 2\tan\frac{2\pi-\theta}{4}\mathbf{n}$, *these vectors represent inverse rotations.*

Proof. From (2.28) we obtain that

$$\begin{aligned}
\mathcal{U}_3 &= \big(1+\frac{a^2}{4}\big)^{-1}\big((1-\frac{a^2}{4})\mathcal{I}-i\mathbf{a}\cdot\boldsymbol{\sigma}\big)\big(1+\frac{c^2}{4}\big)^{-1}\big((1-\frac{c^2}{4})\mathcal{I}-i\mathbf{c}\cdot\boldsymbol{\sigma}\big)\\
&\overset{(2.14)}{=} \frac{(1-\frac{a^2}{4})(1-\frac{c^2}{4})\mathcal{I}-i(1-\frac{a^2}{4})\mathbf{c}\cdot\boldsymbol{\sigma}-i(1-\frac{c^2}{4})\mathbf{a}\cdot\boldsymbol{\sigma}-\mathbf{a.c}\,\mathcal{I}-i(\mathbf{a}\times\mathbf{c})\cdot\boldsymbol{\sigma}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})} \qquad (2.41)\\
&= \frac{(1-\frac{a^2}{4})(1-\frac{c^2}{4})-\mathbf{a.c}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})}\mathcal{I}-i\frac{(1-\frac{a^2}{4})\mathbf{c}+(1-\frac{c^2}{4})\mathbf{a}+4\frac{\mathbf{a}}{2}\times\frac{\mathbf{c}}{2}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})}\cdot\boldsymbol{\sigma}.
\end{aligned}$$

The general formulas (2.29) and (2.41) will be compatible if we have simultaneously

$$\begin{aligned}
\frac{1-\frac{\widetilde{a}^2}{4}}{1+\frac{\widetilde{a}^2}{4}} &= \frac{(1-\frac{a^2}{4})(1-\frac{c^2}{4})-\mathbf{a.c}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})},\\
\frac{\widetilde{\mathbf{a}}}{1+\frac{\widetilde{a}^2}{4}} &= \frac{(1-\frac{a^2}{4})\mathbf{c}+(1-\frac{c^2}{4})\mathbf{a}+4\frac{\mathbf{a}}{2}\times\frac{\mathbf{c}}{2}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})}.
\end{aligned} \qquad (2.42)$$

From (2.42) we get

$$\frac{\widetilde{a}^2}{4} = \frac{\frac{a^2}{4}+\frac{c^2}{4}+2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}}{1-2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}+\frac{a^2}{4}\frac{c^2}{4}}, \qquad 1+\frac{\widetilde{a}^2}{4} = \frac{1+\frac{a^2}{4}+\frac{c^2}{4}+\frac{a^2}{4}\frac{c^2}{4}}{1-2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}+\frac{a^2}{4}\frac{c^2}{4}}. \qquad (2.43)$$

Taking into account that

$$1+\frac{a^2}{4}+\frac{c^2}{4}+\frac{a^2}{4}\frac{c^2}{4} = \big(1+\frac{a^2}{4}\big)\big(1+\frac{c^2}{4}\big)$$

and multiplying the numerator and denominator of the second fraction in (2.41) by $1-2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}+\frac{a^2}{4}\frac{c^2}{4}$ (when this expression is non-zero), we get the result in the second case in (2.40), i.e., the composition law in vector-parameter form for SU(2).

To rigorously see when the composition is not well defined, we investigate the case in which the denominator equals zero. According to the identity

$$1-2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}+\frac{a^2}{4}\frac{c^2}{4} = \big(1-\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}\big)^2+\big(\frac{\mathbf{a}}{2}\times\frac{\mathbf{c}}{2}\big)^2, \qquad (2.44)$$

the denominator of (2.40) vanishes if only if $\mathbf{a} = 2\tan\frac{\theta_2}{4}\mathbf{n}$, $\mathbf{c} = 2\tan\frac{\theta_1}{4}\mathbf{n}$ and

$1 = \tan\dfrac{\theta_2}{4}\tan\dfrac{\theta_1}{4}$. This implies $\cos\dfrac{\theta_1+\theta_2}{4} = 0,\ \theta_1+\theta_2 = 2\pi$ and allows to express

$$\mathbf{c} = 2\tan\frac{\theta}{4}\mathbf{n}, \qquad \mathbf{a} = 2\tan\frac{2\pi-\theta}{4}\mathbf{n}. \tag{2.45}$$

Substituting the results from (2.45) in (2.42) gives $\widetilde{\mathbf{a}}\,(\mathbf{a},\mathbf{c}) = 0$, which corresponds to the identity element $\mathcal{I}$. If $\mathbf{c} \equiv -\mathbf{a}$, then $\widetilde{\mathbf{a}} = \mathbf{0}$. □

In the particular case when one and the same rotation ($\mathbf{a} \equiv \mathbf{c}$) is applied twice the resulting vector is

$$\widetilde{\mathbf{a}} = \frac{2\big(1-\dfrac{\mathrm{a}^2}{4}\big)\mathbf{a}}{\big(1-\dfrac{\mathrm{a}^2}{4}\big)^2} = \frac{4\dfrac{\mathbf{a}}{2}}{1-\dfrac{\mathrm{a}^2}{4}}\cdot$$

It is important to investigate when the composition $\widetilde{\mathbf{a}}$ is such that $|\,\widetilde{\mathbf{a}}\,| \le 4$. Using (2.43) we obtain

$$\frac{\widetilde{\mathrm{a}}^{\,2}}{4} = \frac{\dfrac{\mathrm{a}^2}{4} + \dfrac{\mathrm{c}^2}{4} + 2\dfrac{\mathbf{a}}{2}\cdot\dfrac{\mathbf{c}}{2}}{1 - 2\dfrac{\mathbf{a}}{2}\cdot\dfrac{\mathbf{c}}{2} + \dfrac{\mathrm{a}^2}{4}\dfrac{\mathrm{c}^2}{4}} \le 1 \tag{2.46}$$

and this is equivalent to the inequality

$$\mathbf{a}.\mathbf{c} \le \big(1-\frac{\mathrm{a}^2}{4}\big)\big(1-\frac{\mathrm{c}^2}{4}\big). \tag{2.47}$$

Similar conditions for $|\,\widetilde{\mathbf{a}}\,| < 4, |\,\widetilde{\mathbf{a}}\,| = 4$ and $|\,\widetilde{\mathbf{a}}\,| > 4$ cases follow immediately.

3. THE COVERING MAP SU(2) → SO(3,ℝ) AND ITS SECTIONS IN VECTOR-PARAMETER FORM

Proposition 2. *Let* $\mathbf{a}$ *be the vector-parameter of a generic* SU(2) *element (i.e., it is not associated with some half-turn,* $\mathbf{a}^2 = 4$*). Then the Gibbs vector* $\boldsymbol{c}$*, which represents this rotation in* SO(3,ℝ)*, is given by*

$$\boldsymbol{c}(\mathbf{a}) = \frac{\mathbf{a}}{1-\dfrac{\mathrm{a}^2}{4}}\cdot \tag{3.1}$$

On the other hand, if $\boldsymbol{c}$ *is the Gibbs vector, representing a rotation from* SO(3,ℝ)*, then the preimages of this rotation in* SU(2) *correspond to the vector parameters*

$$\mathbf{a}_+(\boldsymbol{c}) = \frac{2(\sqrt{1+c^2}-1)}{c^2}\boldsymbol{c}, \qquad \mathbf{a}_-(\boldsymbol{c}) = -\frac{2(\sqrt{1+c^2}+1)}{c^2}\boldsymbol{c}. \tag{3.2}$$

Moreover, they are connected by the formulas

$$\mathbf{a}_+ = -\frac{4}{\mathrm{a}_-^2}\mathbf{a}_-, \qquad \mathbf{a}_- = -\frac{4}{\mathrm{a}_+^2}\mathbf{a}_+, \qquad \mathrm{a}_-^2\,\mathrm{a}_+^2 = 16. \tag{3.3}$$

Proof. We have to find a *Gibbs* parameter $\boldsymbol{c}$ such that

$$\mathcal{R}(\boldsymbol{c}) = \frac{2}{1+c^2}\begin{pmatrix} 1+c_1^2 & c_1c_2-c_3 & c_1c_3+c_2 \\ c_1c_2+c_3 & 1+c_2^2 & c_2c_3-c_1 \\ c_1c_3-c_2 & c_2c_3+c_1 & 1+c_3^2 \end{pmatrix} - \mathcal{I} = \mathcal{R}_{\mathcal{U}}(\mathbf{a}) \tag{3.4}$$

and where $\mathcal{R}_{\mathcal{U}}(\mathbf{a})$ is given by (2.35). Equating the corresponding matrix elements,

$$\begin{aligned} \mathcal{R}(\boldsymbol{c})_{32} - \mathcal{R}(\boldsymbol{c})_{23} &= \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{32} - \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{23} \\ \mathcal{R}(\boldsymbol{c})_{13} - \mathcal{R}(\boldsymbol{c})_{31} &= \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{13} - \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{31} \\ \mathcal{R}(\boldsymbol{c})_{21} - \mathcal{R}(\boldsymbol{c})_{12} &= \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{21} - \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{12} \\ \operatorname{tr}\mathcal{R}(\boldsymbol{c}) &= \operatorname{tr}\mathcal{R}_{\mathcal{U}}(\mathbf{a}) \end{aligned} \tag{3.5}$$

we end up with the following equalities

$$\begin{aligned} \frac{2}{1+c^2}\,c_1 &= \frac{8(4-\mathrm{a}^2)}{(4+\mathrm{a}^2)^2}\,\mathrm{a}_1\,, & \frac{2}{1+c^2}\,c_2 &= \frac{8(4-\mathrm{a}^2)}{(4+\mathrm{a}^2)^2}\,\mathrm{a}_2\,, \\ \frac{2}{1+c^2}\,c_3 &= \frac{8(4-\mathrm{a}^2)}{(4+\mathrm{a}^2)^2}\mathrm{a}_3\,, & \frac{2(3+c^2)}{1+c^2} &= \frac{8(-8\mathrm{a}^2)}{(4+\mathrm{a}^2)^2} + 6. \end{aligned} \tag{3.6}$$

From (3.6) we have

$$\frac{2}{1+c^2}\,\boldsymbol{c} = \frac{8(4-\mathrm{a}^2)}{(4+\mathrm{a}^2)^2}\,\mathbf{a} \tag{3.7}$$

and separating $1+c^2$ in (3.6) we obtain

$$\frac{2}{1+c^2} = 2\frac{(4+\mathrm{a}^2)^2 - 16\mathrm{a}^2}{(4+\mathrm{a}^2)^2} = 2\frac{(4-\mathrm{a}^2)^2}{(4+\mathrm{a}^2)^2}\cdot$$

Substituting this expression in (3.7), we obtain (3.1), which is the first statement in the proposition. To invert (3.1), we firstly calculate c^2 and get

$$c^2 = \frac{\mathrm{a}^2}{(1-\frac{\mathrm{a}^2}{4})^2}\cdot$$

If $\mathrm{a}^2 \neq 4$ (i.e., $\mathbf{a}$ does not represent a half-turn), this equality is equivalent to the following quadratic equation for a^2:

$$(\mathrm{a}^2)^2c^2 - 8(2+c^2)\mathrm{a}^2 + 16c^2 = 0. \tag{3.8}$$

The solutions of (3.8) are

$$\mathrm{a}_\pm^2 = \frac{4(2+c^2) \mp 8\sqrt{1+c^2}}{c^2}$$

and hence

$$\frac{\mathrm{a}_{\pm}^2}{4} = \frac{2 + c^2 \mp 2\sqrt{1+c^2}}{c^2} = 1 + \frac{2 \mp 2\sqrt{1+c^2}}{c^2}, \quad 1 - \frac{\mathrm{a}_{\pm}^2}{4} = -\frac{2(1 \mp \sqrt{1+c^2})}{c^2} \cdot \tag{3.9}$$

Substituting this result in (3.1) we obtain (3.2). It follows from (3.2) that

$$\begin{aligned} \mathbf{a}_+ &= \frac{2\,(\sqrt{1+c^2}-1)}{c^2}\,\boldsymbol{c} = -\frac{\sqrt{1+c^2}-1}{\sqrt{1+c^2}+1}\,\mathbf{a}_- \\ &= -\frac{2+c^2-2\sqrt{1+c^2}}{c^2}\,\mathbf{a}_- = -\frac{\mathrm{a}_+^2}{4}\,\mathbf{a}_-\,, \end{aligned} \tag{3.10}$$

therefore $\mathbf{a}_- = -\dfrac{4}{\mathrm{a}_+^2}\,\mathbf{a}_+$. From $\mathrm{a}_-^2 = \dfrac{16}{\mathrm{a}_+^4}\mathrm{a}_+^2$, $\mathrm{a}_-^2\mathrm{a}_+^2 = 16$ we find $\mathbf{a}_+ = -\dfrac{4}{\mathrm{a}_-^2}\,\mathbf{a}_-$, which completes the proof of Proposition 2. □

The relations obtained above are depicted in Fig. 2. Notice that $\mathbf{a}_\pm$ and $\boldsymbol{c}$ actually act between the algebras and also that the *Cayley* map is not surjective onto the given groups, see equations (2.8) and (2.33).

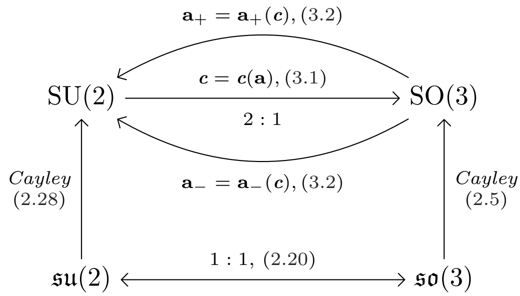


Figure 2: Informal depiction of the relations between the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ and the Lie groups SU(2) and SO(3,$\mathbb{R}$).

Viewing a_+ and a_- as functions of c (see Fig. 3) one concludes that

$$\mathrm{a}_+(c) \le 2 \le \mathrm{a}_-(c), \qquad \lim_{c\to\infty} \mathrm{a}_+(c) = \lim_{c\to\infty} \mathrm{a}_-(c) = 2. \tag{3.11}$$

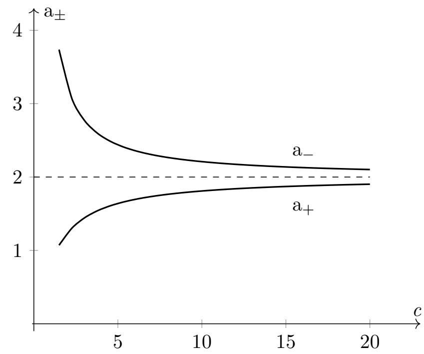


Figure 3: Graphs of a$_-$ and a$_+$ as functions of c.

In order to obtain the SU(2) elements $\mathcal{U}_{\pm}(\boldsymbol{c})$ corresponding to the SO(3,$\mathbb{R}$) rotation with vector-parameter $\boldsymbol{c}$, we substitute $\mathbf{a}_{\pm}(\boldsymbol{c})$ from (3.2) in $\mathcal{U}(\mathbf{a})$ from (2.29) and get

$$\mathcal{U}_{\pm}(\boldsymbol{c}) = \pm\frac{1}{\sqrt{1+\boldsymbol{c}^2}}\begin{pmatrix} 1-\mathrm{i}c_3 & -c_2-\mathrm{i}c_1 \\ c_2-\mathrm{i}c_1 & 1+\mathrm{i}c_3 \end{pmatrix}. \tag{3.12}$$

Let $\boldsymbol{c} = \tan\dfrac{\theta}{2}\,\mathbf{n}$ represent a SO(3,$\mathbb{R}$) rotation at angle θ about the axis $\mathbf{n}$. The corresponding SU(2) vectors $\mathbf{a}_+(\boldsymbol{c})$ and $\mathbf{a}_-(\boldsymbol{c})$ are

$$\mathbf{a}_+(\boldsymbol{c}) = 2\tan\frac{\theta}{4}\,\mathbf{n}, \qquad \mathbf{a}_-(\boldsymbol{c}) = -2\tan\frac{2\pi-\theta}{4}\,\mathbf{n}. \tag{3.13}$$

The matrix corresponding to $\mathbf{a}_+$ is the familiar axis-angle representation of rotations in SU(2), i.e.,

$$\mathcal{U}(\mathbf{a}_+) = \mathcal{U}(\mathbf{n},\theta) = \cos\frac{\theta}{2}\mathcal{I} + \sin\frac{\theta}{2}\begin{pmatrix} -\mathrm{i}n_3 & -n_2-\mathrm{i}n_1 \\ n_2-\mathrm{i}n_1 & \mathrm{i}n_3 \end{pmatrix}. \tag{3.14}$$

In SU(2) the half-turns about the axis $\mathbf{n}$ are represented by the matrices

$$\mathcal{U}(\pm\mathbf{n},\pi) = \pm\begin{pmatrix} -\mathrm{i}n_3 & -n_2-\mathrm{i}n_1 \\ n_2-\mathrm{i}n_1 & \mathrm{i}n_3 \end{pmatrix}. \tag{3.15}$$

In the derived vector-parameter form the half-turns are represented by the vectors $\pm 2\mathbf{n}$, which are well defined and are of length 2. This is an advantage, because a half-turns $\mathcal{O}(\mathbf{n})$ in the *Gibbs* vector parameter form of SO(3,$\mathbb{R}$) rotations are represented by *vectors* with infinitely large norm and direction $\pm\mathbf{n}$. Such *vectors* will be referred further on as “rays” and will be denoted by $[\mathbf{n}]$ (for more discussion, see e.g. [2] and [12]). Let $\mathcal{R} = \mathcal{O}(\mathbf{n})$ be a half-turn about the axis $\mathbf{n}$, represented by $\pm\mathbf{n}$ in SU(2). Applying the limit $\mathrm{a} \to 2$ in (3.1), we can informally write

$\lim_{\mathbf{a}\to\pm 2\mathbf{n}} \boldsymbol{c}(\mathbf{a}) = [\mathbf{n}]$. Roughly speaking, the *Gibbs* parameter, associated with $\mathcal{O}(\mathbf{n})$ is $\boldsymbol{c} = \lim_{\theta\to\pi} \tan\dfrac{\theta}{2}\,\mathbf{n} = [\mathbf{n}]$. Actually, we have

$$\lim_{\theta\to\pi} \mathcal{U}_{\pm}(\tan\frac{\theta}{2}\mathbf{n}) \overset{(3.12)}{=} \pm \lim_{c^2\to\infty} \frac{1}{\sqrt{1+\boldsymbol{c}^2}} \begin{pmatrix} 1-\mathrm{i}c_3 & -c_2-\mathrm{i}c_1 \\ c_2-\mathrm{i}c_1 & 1+\mathrm{i}c_3 \end{pmatrix} \overset{(3.15)}{=} \mathcal{U}(\pm\mathbf{n},\pi). \quad (3.16)$$

We observe that if $\boldsymbol{c} = \tan\dfrac{\theta}{2}\mathbf{n}$ represents an infinitesimal SO(3,ℝ) rotation $\mathcal{R}(\mathbf{n},\theta)$, then as SU(2) element it is represented by two vectors, one with infinitesimal norm $\mathbf{a}_+$ and the other one $\mathbf{a}_-$ with infinite norm, i.e.,

$$\lim_{c\to 0} \mathrm{a}_+^2(\boldsymbol{c}) = 0, \qquad \lim_{c\to 0} \mathrm{a}_-^2(\boldsymbol{c}) = \infty. \quad (3.17)$$

When storing infinitesimal rotations in applications, loss of information may occur because of the operations performed with very small numbers. Equation (3.17) offers an alternative way (by usage of $\mathbf{a}_-$) for computer storage of infinitesimal rotations. This is so because in many of the commercial software systems there are packages for dealing with *large* numbers.

3.1. COMPATIBILITY OF THE COMPOSITION LAWS IN SU(2) AND SO(3,ℝ)

Recall that a map $\varphi : G_1 \to G_2$ of the groups G_1, G_2 is a group homomorphism if it is compatible with the group operations in G_1 and G_2 by the rule $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G_1$. For an arbitrary subset $S_1 \subset G_1$, which is not necessarily a subgroup of G_1, we say that a map $\psi : S_1 \to G_2$ is compatible with the group operations in G_1 and G_2 if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in S_1$.

Proposition 3. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{c}$ *are some non-zero Gibbs parameters of two* SO(3,ℝ) *rotations and such that* $\boldsymbol{a}.\boldsymbol{c} \neq 1$. *Let*

$$\mathcal{U}_1(\boldsymbol{c}) = \frac{1}{\sqrt{1+c^2}} \begin{pmatrix} 1-\mathrm{i}c_3 & -c_2-\mathrm{i}c_1 \\ c_2-\mathrm{i}c_1 & 1+\mathrm{i}c_3 \end{pmatrix}, \ \mathcal{U}_2(\boldsymbol{a}) = \frac{1}{\sqrt{1+a^2}} \begin{pmatrix} 1-\mathrm{i}a_3 & -a_2-\mathrm{i}a_1 \\ a_2-\mathrm{i}a_1 & 1+\mathrm{i}a_3 \end{pmatrix}$$

be the respective images of $\boldsymbol{a}$, $\boldsymbol{c}$ *under the "+" sections of the maps* (**??**) *and* (3.12). *Then the equality*

$$\mathcal{U}_2(\boldsymbol{a})\mathcal{U}_1(\boldsymbol{c}) = \mathcal{U}(\tilde{\boldsymbol{c}}) \quad (3.18)$$

holds up to a sign, i.e., the "+" correspondences are compatible up to a sign with the group operations in SO(3,ℝ) *and* SU(2).

Proof. Let $\mathcal{U}_3 = \mathcal{U}_2(\boldsymbol{a})\mathcal{U}_1(\boldsymbol{c})$. We will prove that

$$\mathcal{U}_3 = \frac{\pm 1}{\sqrt{1+\tilde{c}^2}} \begin{pmatrix} 1-\mathrm{i}\tilde{c}_3 & -\tilde{c}_2-\mathrm{i}\tilde{c}_1 \\ \tilde{c}_2-\mathrm{i}\tilde{c}_1 & 1+\mathrm{i}\tilde{c}_3 \end{pmatrix}. \quad (3.19)$$

Direct multiplication shows that

$$\mathcal{U}_3 = \frac{1 - \boldsymbol{a}.\boldsymbol{c}}{\sqrt{1+a^2}\sqrt{1+c^2}} \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \tag{3.20}$$

where

$$\begin{aligned} \alpha &= 1 - \mathrm{i}\frac{a_3 + c_3 + a_1 c_2 - a_2 c_1}{1 - \boldsymbol{a}.\boldsymbol{c}} = 1 - \mathrm{i}\tilde{c}_3 \\ \beta &= -\frac{a_2 + c_2 + a_3 c_1 - a_1 c_3}{1 - \boldsymbol{a}.\boldsymbol{c}} - \mathrm{i}\frac{a_1 + c_1 + a_2 c_3 - a_3 c_2}{1 - \boldsymbol{a}.\boldsymbol{c}} = -\tilde{c}_2 - \mathrm{i}\tilde{c}_1. \end{aligned} \tag{3.21}$$

For $\tilde{\boldsymbol{c}}$ we have that

$$\tilde{c}^2 = \frac{a^2 + c^2 + (\boldsymbol{a} \times \boldsymbol{c})^2 + 2\boldsymbol{a}.\boldsymbol{c}}{(1 - \boldsymbol{a}.\boldsymbol{c})^2} = \frac{(1+c^2)(1+a^2)}{(1 - \boldsymbol{a}.\boldsymbol{c})^2} - 1. \tag{3.22}$$

Thus

$$\frac{1}{\sqrt{1+\tilde{c}^2}} = \frac{|1 - \boldsymbol{a}.\boldsymbol{c}|}{\sqrt{1+a^2}\sqrt{1+c^2}}. \tag{3.23}$$

Now from (3.19), (3.20) and (3.23) we get that $\mathcal{U}_2(\boldsymbol{a})\mathcal{U}_1(\boldsymbol{c}) = \mathcal{U}(\tilde{\boldsymbol{c}})$ up to a sign. The case $\boldsymbol{a}.\boldsymbol{c} = 1$ in Proposition 3, as well as the cases where half-turns are involved in the composition will be treated elsewhere. □

Note that Proposition 3 holds also for the negative signs of the above sections. If $\boldsymbol{c}_1, \boldsymbol{c}_2$ are represent two SO(3,ℝ) rotations and the vectors $\mathbf{a}_1, \mathbf{a}_2$ are defined by the section $\mathbf{a}_+$ in (3.2) then the SO(3,ℝ) vector parameter corresponding to $\langle \mathbf{a}_2, \mathbf{a}_1 \rangle_{\mathrm{SU}(2)}$ is exactly $\langle \boldsymbol{c}_2, \boldsymbol{c}_1 \rangle_{\mathrm{SO}(3,\mathbb{R})}$, i.e., we have the commutative diagram below. Therefore, the pull-back of the composition in SO(3,ℝ) to the covering group SU(2) allows to bypass the singularities in the vector-parameter description of the base manifold.

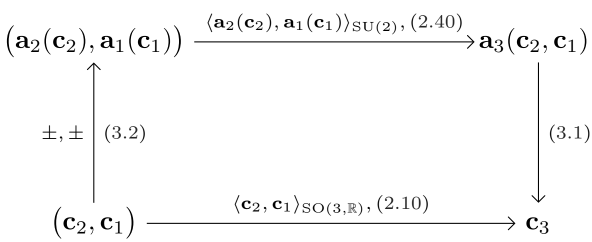


Figure 4: Composition of the three-dimensional rotations through a pull-back to the covering group SU(2).

4. CONCLUDING REMARKS

Despite of the attractive simplicity of the composition law for SO(3,$\mathbb{R}$) rotations, neither the half-turns nor the composition of rotations whose *Gibbs* vector-parameters have a scalar product equal to one are directly manageable. The derived vector-parametrization of SU(2) has the advantage to represent all rotations including the half-turns. Table 1 presents the numbers of operations needed for the composition of two rotations.

Table 1: The numbers of operations necessary to perform when composing two rotations in various representations.

Representations		Multiplications	Additions	Memory needed for the result
SO(3,$\mathbb{R}$)	matrix	27	18	9
	vector-parameter	12	12	3
SU(2)	matrix	16	16	4
	vector-parameter	28	18	3

ACKNOWLEDGEMENTS. The authors would like to thank to the anonymous referee for the helpful suggestions and for recommending various improvements in exposition. The first named author was partially supported by the Sofia University Research Fund through Grant no. 144/2015.

5. REFERENES

1. Blanes, S., Casas, F.: On the convergence and optimization of the Baker–Campbell–Hausdorff Formula. *Lin. Alg. Appl.*, **378**, 2004, 135–158.
2. Brezov, D., Mladenova, C., Mladenov, I.: Vector decompositions of rotations. *J. Geom. Symmetry Phys.*, **28**, 2012, 67–103.
3. Bröcker, T., Dieck, T.: *Representations of Compact Lie Groups*, Composition House Ltd, Salisbury, 1985.
4. Casas, F., Murua, A.: An efficient algorithm for computing the Baker–Campbell–Hausdorff series and some of its applications. *J. Math. Phys.*, **50**, 033513, 2009, 1–23.
5. Chen, C.: Application of algebra of rotations in robot kinematics. *Mech. Mach. Theory*, **22**, 1987, 77–83.
6. Engo, K.: On the Baker-Campbell-Hausdorff formula in $\mathfrak{so}(3)$. *BIT Num. Math.*, **41**, 2001, 629–632.
7. Fedorov, F.: *The Lorentz Group*, Nauka, Moscow, 1979 (in Russian).

8. Gilmore, R.: *Lie Groups, Lie Algebras and Some of Their Applications*, Wiley, New York, 1974.
9. Goldberg, K.: The formal power series for $\log e^x e^y$. *Duke Math J.*, **23**, 1956, 13–21.
10. Goldstein, H., Poole, C., Safko, J.: *Classical Mechanics*, Addison-Wesley, Boston, 2001.
11. Hill, E.: Rotations of a rigid body about a fixed point. *Am. J. Phys.*, **13**, 1945, 137–140.
12. Pina, E.: Rotations with Rodrigues' vector. *Eur. J. Phys*, **32**, 2011, 1171–1178.
13. Serre, J.: *Lie Algebras and Lie Groups*, 2^{nd} Edn, Springer, Berlin, 2006.
14. Schaub, H., Tsiotras, P., Junkins, J.: Principal rotation representation of proper $N \times N$ orthogonal matrices. *Int. J. Eng. Sci.*, **33**, 1995, 2277–2295.
15. Thompson, R.: Cyclic relations and the Goldberg coefficients. *Proc. Am. Math. Soc.*, **86**, 1982, 12–15.
16. Tsiotras, P., Junkins, J., Schaub, H.: Higher order Cayley transforms with applications to attitude representations. *J. Guidance, Control, and Dynamics*, **20**, 1997, 528–536.
17. Van-Brunt, A., Visser, M.: Special-case closed form of the Baker-Campbell-Hausdorff Formula. *J. Phys. A: Math. Theor.*, **48**, 2015, 1–10.
18. Wigner E.: *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, Academic Press, New York, 1959.

Received on February 1, 2015
In revised form on April 27, 2015

Veliko D. Donchev
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5 J. Bourchier Blvd.
1164 Sofia, BULGARIA
e-mail: vddonchev@fmi.uni-sofia.bg

Clementina D. Mladenova
Institute of Mechanics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 4
1113 Sofia, BULGARIA
e-mail: clem@imbm.bas.bg

Ivaïlo M. Mladenov
Institute of Biophysics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 21
1113 Sofia, BULGARIA
e-mail: mladenov@bio21.bas.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

A CLASSIFICATION OF CONFORMALLY FLAT RIEMANNIAN MANIFOLDS LOCALLY ISOMETRIC TO HYPERSURFACES IN EUCLIDEAN OR MINKOWSKI SPACE

GEORGI GANCHEV, VESSELKA MIHOVA

We prove that the local theory of conformally flat Riemannian manifolds, which can be locally isometrically embedded as hypersurfaces in Euclidean or Minkowski space, is equivalent to the local theory of Riemannian manifolds of quasi-constant sectional curvatures (QC-manifolds). Riemannian QC-manifolds are divided into two basic classes: with positive or negative horizontal sectional curvatures. We prove that the Riemannian QC-manifolds with positive horizontal sectional curvatures are locally equivalent to canal hypersurfaces in Euclidean space, while the Riemannian QC-manifolds with negative horizontal sectional curvatures are locally equivalent to canal space-like hypersurfaces in Minkowski space. These results give a local geometric classification of conformally flat hypersurfaces in Euclidean space and conformally flat space-like hypersurfaces in Minkowski space.

Keywords: Riemannian manifolds of quasi-constant sectional curvatures, canal space-like hypersurfaces in Minkowski space, rotational space-like hypersurfaces in Minkowski space, classification of conformally flat hypersurfaces in Euclidean or Minkowski space

2000 Math. Subject Classification: Primary 53A35, Secondary 53B20

1. INTRODUCTION

Conformally flat n-dimensional Riemannian manifolds appear as hypersurfaces in two standard models of flat spaces: Euclidean or Minkowski space. Historically, there were many attempts to describe conformally flat hypersurfaces, especially in Euclidean space. Essential steps in this direction were made by Cartan [3], Schouten [18]. Kulkarni in [15] reached to a partial description of conformally flat hypersurfaces in Euclideasn space dividing them into: hypersurfaces of constant curvature; hypersurfaces of revolution; tubes. Yano and Chen proved in [4] that canal hypersurfaces in Euclidean space, i.e. envelopes of one-parameter families of hyperspheres, are special conformally flat hypersurfaces. Compact conformally flat hypersurfaces in Euclidean space were studied in [15] and [6].

In this paper we study the close relation between the local theory of Riemannian manifolds of quasi-constant sectional curvatures and the local theory of conformally flat Riemannian hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ or in the Minkowski space $\mathbb{R}^{n+1}_1$. We give a local classification of Riemannian manifolds of quasi-constant sectional curvatures proving that they can locally be embedded as canal hypersurfaces in $\mathbb{R}^{n+1}$ or $\mathbb{R}^{n+1}_1$. Thus we obtain a geometric description of conformally flat hypersurfaces in Euclidean space and conformally flat space-like hypersurfaces in Minkowski space.

Riemannian QC-manifolds are Riemannian manifolds (M, g, ξ) endowed with a unit vector field ξ besides the metric g, satisfying the curvature condition: the sectional curvatures at any point of the manifold only depend on the point and the angle between the section and the vector ξ at that point. All tangent sections at a given point, which are perpendicular to the vector ξ at that point, have one and the same sectional curvature. We call these sectional curvatures horizontal sectional curvatures.

Everywhere in this paper we consider the case $\dim M = n \geq 4$.

The structural group of Riemannian manifolds (M, g, ξ) is $O(n-1) \times 1$ and two Riemannian manifolds (M, g, ξ) and (M', g', ξ') are equivalent if there exists a diffeomorphism $f : M \to M'$ preserving both structures: the metric g and the vector field ξ. We call such a diffeomorphism a ξ-isometry.

In [11] we proved the following statements:

Any canal hypersurface M in the Euclidean space $\mathbb{R}^{n+1}$ is a Riemannian QC-manifold with positive horizontal sectional curvatures.

Any Riemannian QC-manifold with positive horizontal sectional curvatures is locally ξ-isometric to a canal hypersurface in the Euclidean space $\mathbb{R}^{n+1}$.

The first problem we treat here is to give a local classification of Riemannian QC-manifolds with negative horizontal sectional curvatures.

In Section 3 we introduce canal space-like hypersurfaces in the Minkowski space $\mathbb{R}^{n+1}_1$ and divide them into three types. In Subsections 3.1 - 3.3 we study these three types of canal space-like hypersurfaces and show that:

Any canal space-like hypersurface in the Minkowski space $\mathbb{R}^{n+1}_1$ is a Riemannian QC-manifold with negative horizontal sectional curvatures.

The basic results are proved in Section 4. The local classification of Riemannian QC-manifolds with negative horizontal sectional curvatures is given by

Theorem 4.1:

Any Riemannian QC-manifold with negative horizontal sectional curvatures is locally ξ-isometric to a canal space-like hypersurface in the Minkowski space $\mathbb{R}_1^{n+1}$.

The second problem we deal with is to obtain a geometric description of conformally flat hypersurfaces in Euclidean space and conformally flat space-like hypersurfaces in Minkowski space. Using results of Cartan and Schouten, we are able to bring the second fundamental form of the hypersurface into consideration. This allows us to give a local geometric classification of conformally flat Riemannian hypersurfaces in Euclidean or Minkowski space:

Any conformally flat hypersurface in Euclidean space, which is free of umbilical points, locally is a part of a canal hypersurface.

Any conformally flat space-like hypersurface in Minkowski space, which is free of umbilical points, locally is a part of a canal space-like hypersurface.

The picture of the local isometric embeddings of a conformally flat Riemannian manifold into Euclidean or Minkowski space can be described briefly as follows.

Let (M, g) be a conformally flat Riemannian manifold, free of points in which all sectional curvatures are constant. The manifold (M, g) can be locally isometrically embedded into $\mathbb{R}^{n+1}$ $(\mathbb{R}_1^{n+1})$ if and only if its Ricci operator has two different from zero eigenvalues at every point: one of them of multiplicity $n-1$ and the other of multiplicity 1. The latter eigenvalue generates a unit vector field ξ, such that (M, g, ξ) is a Riemannian QC-manifold with positive (negative) horizontal sectional curvatures. Any two isometrical realizations of (M, g) are locally congruent.

Generalizing, we obtain that a conformally flat Riemannian manifold is locally isometric to a hypersurface into $\mathbb{R}^{n+1}$ $(\mathbb{R}_1^{n+1})$ if and only if its Ricci operator at any point has a root of multiplicity at least $n-1$. This fact gives an approach to further investigations of conformally flat Riemannian manifolds studying the spectrum of their Ricci operator.

It is interesting to mention Riemannian subprojective manifolds forming a subclass of Riemannian QC-manifolds characterized by the condition: the structural vector field ξ is geodesic. If (M, g, ξ) is a Riemannian subprojective manifold with scalar curvature τ $(d\tau \neq 0)$, then the structural vector field ξ is collinear with $\operatorname{grad} \tau$. Any Riemannian subprojective manifold is locally isometric (up to a motion) to a rotational hypersurface in Euclidean space or in Minkowski space.

2. PRELIMINARIES

Let (M, g, ξ) $(\dim M = n \geq 4)$ be a Riemannian manifold with metric g and a unit vec-tor field ξ. The structural group of these manifolds is $O(n-1) \times 1$. T_pM and $\mathfrak{X}M$ will stand for the tangent space to M at a point p and the algebra of smooth vector fields on M, respectively. The 1-form corresponding to the unit vector ξ is denoted by η, i.e. $\eta(X) = g(\xi, X)$, $X \in \mathfrak{X}M$. The distribution of the 1-form η is denoted by Δ, i.e.

$$\Delta(p) = \{X \in T_pM : \eta(X) = 0\}.$$

The orthogonal projection of a vector field $X \in \mathfrak{X}M$ onto the distribution Δ is denoted by the corresponding small letter x, i.e.

$$X = x + \eta(X)\,\xi. \tag{2.1}$$

Any section E in T_pM determines an angle $\angle(E, \xi)$. Then the notion analogous to the notion of a Riemannian manifold of constant sectional curvatures is described as follows [11].

Definition 2.1. A Riemannian manifold (M, g, ξ) ($\dim M \geq 3$) is said to be of quasi-constant sectional curvatures (a Riemannian QC-manifold) if for an arbitrary 2-plane E in T_pM, $p \in M$, with $\angle(E, \xi) = \varphi$, the sectional curvature of E only depends on the point p and the angle φ.

Let ∇ be the Levi-Civita connection of the metric g and $\mathcal{R}$ be its Riemannian curvature tensor. The structure (g, ξ) generates the following tensors π and Φ:

$$\begin{aligned}
\pi(X, Y, Z, U) = \quad & g(Y, Z)g(X, U) - g(X, Z)g(Y, U),\\
\Phi(X, Y, Z, U) = \quad & g(Y, Z)\eta(X)\eta(U) - g(X, Z)\eta(Y)\eta(U)\\
& + g(X, U)\eta(Y)\eta(Z) - g(Y, U)\eta(X)\eta(Z); \quad X, Y, Z, U \in \mathfrak{X}M.
\end{aligned}$$

These tensors have the symmetries of the curvature tensor $\mathcal{R}$ and are invariant under the action of the structural group of the manifold.

Riemannian manifolds of quasi-constant sectional curvatures are characterized by the following statement [11]:

Proposition 2.2. *A Riemannian manifold (M, g, ξ) is of quasi-constant sectional curvatures if and only if its curvature tensor has the form*

$$\mathcal{R} = a\,\pi + b\,\Phi, \tag{2.2}$$

where a and b are some functions on M.

Let (M, g, ξ) ($\dim M = n \geq 4$) be a Riemannian manifold of quasi-constant sectional curvatures. This means that the curvature tensor $\mathcal{R}$ of g has the form (2.2). If $b \neq 0$ everywhere, then the manifold (M, g, ξ) has the properties [11]:

- The distribution of the function a is the structural distribution Δ:

$$da = \xi(a)\,\eta. \tag{2.3}$$

- The distribution Δ is involutive, i.e.

$$d\eta(x,y) = 0, \quad x, y \in \Delta. \tag{2.4}$$

- If θ is the 1-form defined by $\theta(X) = d\eta(\xi, X),\ X \in \mathcal{X}M$, then $d\eta = \theta \wedge \eta$ and

$$\theta(x) = d\eta(\xi, x) = \frac{1}{b}\, db(x), \quad x \in \Delta. \tag{2.5}$$

- The integral submanifolds of the distribution Δ are totally umbilical in M, i.e.

$$\nabla_x \xi = k\, x, \quad k = \frac{\xi(a)}{2b}, \quad x \in \Delta. \tag{2.6}$$

- The distribution of the function k is the structural distribution Δ:

$$dk = \xi(k)\, \eta. \tag{2.7}$$

Let S_p be the maximal integral submanifold of the distribution Δ, containing a given point $p \in M$, and $\mathcal{K}$ be the curvature tensor of the Riemannian manifold (S_p, g). Then we have:

(i) All sections tangent to S_p have one and the same sectional curvature $a(p)$ with respect to the tensor $\mathcal{R}$. We say that the function $a(p)$ is the horizontal sectional curvature of the manifold.

(ii) All sections tangent to S_p have one and the same sectional curvatures $a(p) + k^2(p)$ with respect to the tensor $\mathcal{K}$.

Proposition 2.2 implies the following statement.

Proposition 2.3. *A Riemannian QC-manifold (M, g, ξ), free of points in which the sectional curvatures are constant, i.e. $b \neq 0$, is characterized by the following two conditions:*

- (M, g) is conformally flat;

- the Ricci operator ρ of (M, g) at any point has two non-zero roots, namely:
$(n-1)a + b$, *of multiplicity* $n-1$, *which generates the distribution Δ:*

$$\rho(x) = [(n-1)a + b]\, x, \quad x \in \Delta;$$

$(n-1)(a+b)$, *of multiplicity* 1, *which generates the structural vector field ξ:*

$$\rho(\xi) = (n-1)(a+b)\, \xi.$$

Proposition 2.3 implies that the notion of a QC-manifold is a notion in Riemannian geometry. The next statement is an immediate consequence from this proposition.

Theorem 2.4. *Let (M, g, ξ) and $(\bar{M}, \bar{g}, \bar{\xi})$ be two QC-manifolds free of points in which the sectional curvatures are constant. If $\varphi : M \rightarrow \bar{M}$ is an isometry, then it is a ξ-isometry, i.e. $\varphi_* \xi = \bar{\xi}$.*

The above mentioned geometric functions a and $a + k^2$ on (M, g, ξ) generate four basic classes of Riemannian manifolds of quasi-constant sectional curvatures characterized by the conditions:

1) $a > 0$;

2) $a < 0, \quad a + k^2 > 0$;

3) $a + k^2 < 0$;

4) $a + k^2 = 0$.

The class of Riemannian QC-manifolds contains the remarkable subclass of Riemannian subprojective manifolds. V. Kagan [12, 13] called an n-dimensional space A_n with symmetric linear connection ∇ a subprojective space if there exists locally a coordinate system with respect to which every geodesic of ∇ can be represented by $n - 2$ linear equations and another equation, that need not be linear (see also [19]). P. Rachevsky [17] proved necessary and sufficient conditions characterizing Riemannian subprojective spaces. T. Adati [1] studied Riemannian subprojective manifolds concerning concircular and torse-forming vector fields.

As Riemannian QC-manifolds (M, g, ξ) the Riemannian subprojective manifolds are characterized by any of the following additional properties [11]:

i) $db = \xi(b)\, \eta$;

ii) the vector field ξ is geodesic (on M);

iii) the 1-form η is closed.

Let τ be the scalar curvature of a Riemannian subprojective manifold. If $d\tau \neq 0$, then the structural distribution Δ is the distribution of the 1-form $d\tau$ and the vector field $\operatorname{grad} \tau$ is an eigenvector of the Ricci operator at every point.

3. CANAL SPACE-LIKE HYPERSURFACES IN MINKOWSKI SPACE

A hypersurface $M\, (\dim M = n)$ in the Minkowski space $\mathbb{R}^{n+1}_1$ is said to be *space-like* (or *Riemannian*) if the induced metric on M is positive definite. The normal vector field to a space-like hypersurface M in the Minkowski space $\mathbb{R}^{n+1}_1$ is necessarily time-like.

In this section we study the envelope of a one-parameter family of space-like hyperspheres $\{S^n(s)\}$, $s \in J \subset \mathbb{R}$ in $\mathbb{R}^{n+1}_1$, given as follows

$$S^n(s): \quad (Z - z(s))^2 = -R^2(s), \quad R(s) > 0,$$

where $z = z(s)$ is the center and $R(s)$ is the radius of the corresponding hypersphere $S^n(s)$.

Let the cross-section of a space-like hypersphere S^n with a hyperplane in the Minkowski space $\mathbb{R}_1^{n+1}$ be an $(n-1)$-dimensional surface. We have:

1) The cross-section of a space-like hypersphere S^n with a space-like hyperplane R^n is a Euclidean hypersphere S^{n-1} in $\mathbb{R}^n$, and S^{n-1} is of positive constant sectional curvatures.

2) The cross-section of a space-like hypersphere S^n with a time-like hyperplane R_1^n is a hyperbolic hypersphere H^{n-1} in $\mathbb{R}_1^n$, and H^{n-1} is of negative constant sectional curvatures.

3) The cross-section of a space-like hypersphere S^n with a light-like hyperplane R_0^n is a parabolic hypersphere P^{n-1} in $\mathbb{R}_0^n$, and P^{n-1} is of zero sectional curvatures.

We shall describe in more details the cross-section P^{n-1} of a space-like hypersphere $S^n(z, R)$ with a light-like hyperplane $\mathbb{R}_0^n$. It is clear that $\mathbb{R}_0^n$ can not pass through the center z of S^n. The pair $(\mathbb{R}_0^n, g)$ is an n-dimensional affine space with metric g, whose rank equals $n-1$. This means that $\mathbb{R}_0^n$ contains a light-like direction U, determined by a given light-like vector t. The light-like direction U can also be considered as a point at infinity in the infinite hyperplane of $\mathbb{R}_0^n$. Any hyperplane E^{n-1} of $\mathbb{R}_0^n$, which does not contain U, is a Euclidean hyperplane, i.e. it can be endowed with a basis $e_1, \dots, e_{n-1}$, satisfying the property $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, \dots, n-1$, δ_{ij} being the Kronecker's deltas.

Let E^{n-1} be a Euclidean hyperplane in $\mathbb{R}_0^n$ with a fixed point $T \in E^{n-1}$ and an orthonormal basis $e_1, \dots, e_{n-1}$. Adding the light-like vector t, we obtain a coordinate system $T, e_1, \dots, e_{n-1}, t$ in $\mathbb{R}_0^n$. If $Z(z_1, \dots, z_{n-1}; z_n)$ is the position vector of any point Z in $\mathbb{R}_0^n$, then we consider the quadrics $P^{n-1}(q)$ in $\mathbb{R}_0^n$, given by the equation

$$P^{n-1}(q): \; z_1^2 + \cdots + z_{n-1}^2 - 2q\, z_n = 0, \quad q = \text{const} > 0.$$

The one-parameter family of quadrics $P^{n-1}(q)$ is characterized by the properties:

(i) $P^{n-1}(q)$ is a quadric, which is tangent to the infinite hyperplane of $\mathbb{R}_0^n$ at U and to the hyperplane E^{n-1} at T;

(ii) The cross-section of $P^{n-1}(q)$ with any Euclidean hyperplane $z_n = \text{const} > 0$ (parallel to E^{n-1}) is a Euclidean hypersphere in this hyperplane.

We call these quadrics *parabolic hyperspheres* of the light-like hyperplane $(\mathbb{R}_0^n, g)$.

The parabolic hyperspheres have the following remarkable property:

Proposition 3.1. *Any parabolic hypersphere in a light-like hyperplane* $\mathbb{R}_0^n$ *is a flat* $(n-1)$*-dimensional Riemannian manifold.*

Proof: Since the only tangent hyperplane to the parabolic hypersphere $P^{n-1}(q)$, which contains U, is the infinite hyperplane of $\mathbb{R}_0^n$, then $(P^{n-1}(q), g)$ is an $(n-1)$-dimensional Riemannian manifold.

We consider the projection

$$\pi : \quad P^{n-1}(q) \ \rightarrow \ E^{n-1}$$

of the parabolic hypersphere onto the Euclidean hyperplane E^{n-1}, parallel to the direction U. It is an easy verification that the projection π is an isometry between the Riemannian manifolds $(P^{n-1}(q), g)$ and (E^{n-1}, g), excluding the common point T. This implies the assertion. □

Next, we call the $(n-1)$-dimensional cross-sections of a space-like hypersphere with a hyperplane *spheres of codimension two* and use the common denotation S^{n-1}.

Let $M = \{S^{n-1}(s)\},\ s \in J \subset \mathbb{R}$ be a space-like hypersurface in $\mathbb{R}_1^{n+1}$, which is a one-parameter family of spheres $S^{n-1}(s)$ of codimension two. Any sphere $S^{n-1}(s)$ is said to be *a spherical generator* of M.

At first canal surfaces in $\mathbb{R}^3$ have been introduced and studied in the classical works of Enneper [7, 8, 9, 10]. We use the following definition:

Definition 3.2. A space-like hypersurface $M = \{S^{n-1}(s)\},\ s \in J \subset \mathbb{R}$ in $\mathbb{R}_1^{n+1}$ is said to be *a canal space-like hypersurface* if the normals to M at the points of any fixed spherical generator pass through a fixed point.

Let now $Z = Z(s; u^1, u^2, \dots, u^{n-1}),\ s \in J,\ (u^1, u^2, \dots, u^{n-1}) \in D$ be the position vector field of a canal space-like hypersurface M. The partial derivatives of Z are denoted as follows: $Z_s = \frac{\partial Z}{\partial s},\ Z_i = \frac{\partial Z}{\partial u^i};\ i = 1, \dots, n-1$, and similar denotations are used for other vector functions.

Denoting by $z(s),\ s \in J$ the common point of the normals to M at the points of any spherical generator $S^{n-1}(s)$, we consider the space-like hypersphere $S^n(s)$ with center $z(s)$ containing $S^{n-1}(s)$. If $R(s)$ is the radius of $S^n(s)$, then the position vector Z of M satisfies the equality

$$(Z - z(s))^2 = -R^2(s), \quad R(s) > 0, \quad s \in J \subset \mathbb{R}. \tag{3.1}$$

Differentiating (3.1) with respect to the parameter s, we get

$$(Z - z(s))Z_s - (Z - z(s))z'(s) = -R(s)R'(s). \tag{3.2}$$

Under the condition that the normal to M at any point of a fixed generator $S^{n-1}(s)$ is collinear with $Z - z(s)$, the equalities (3.1) and (3.2) are equivalent to

$$\begin{aligned} &(Z - z(s))^2 = -R^2(s), \\ &(Z - z(s))z'(s) = R(s)R'(s). \end{aligned} \tag{3.3}$$

A space-like hypersurface M in $\mathbb{R}^{n+1}_1$ is said to be the *envelope* of a one-parameter family of space-like hyperspheres $\{S^n(z(s), R(s))\}$, $s \in J$ if the position vector $Z(s; u^1, \dots, u^{n-1})$ of M satisfies the equations (3.3).

Let M be a space-like hypersurface, which is the envelope of a one-parameter family of space-like hyperspheres $\{S^n(z(s), R(s))\}$, $s \in J$. It follows from (3.3) that M is a one parameter family of spheres $S^{n-1}(s)$, $s \in J$. Differentiating the first equality of (3.3), we have

$$(Z - z)Z_s = 0, \quad (Z - z)Z_i = 0, \quad i = 1, \dots, n-1,$$

which shows that the time-like vector field $Z - z$ at the points of any generator $S^{n-1}(s)$ of M is normal to both: the hypersurface M and the hypersphere $S^n(s)$.

Hence, as in the classical case [7, 8, 9, 10, 20], we have

Lemma 3.3. *A space-like hypersurface M in $\mathbb{R}^{n+1}_1$ is canal if and only if it is the envelope of a one-parameter family of space-like hyperspheres.*

Let M be a space-like canal hypersurface, given by (3.3). We denote the tangent vector to the curve of centers $z(s)$ as usual by $t(s) = z'(s)$. The unit normal vector field N to M is collinear with $Z - z$ and we always choose

$$N = -\frac{Z - z}{R}. \tag{3.4}$$

In view of (3.3), the vector field N has the properties:

$$N^2 = -1, \quad Nt = -R'.$$

Differentiating (3.4), we have

$$\begin{aligned} N_i &= -\frac{1}{R}\, Z_i, \quad i = 1, \dots, n-1, \\ Z_s + R\, N_s &= t - R'N. \end{aligned} \tag{3.5}$$

The second equality in (3.5) means that the vector field $t - R'N$ is tangent to M. Since the normals to M at the points of a spherical generator cannot be parallel to the vector t, then the vector field $t - R'N$ is space-like and $(t - R'\,N)^2 > 0$. Furthermore, the second equality in (3.3) implies that $t\,Z_i = 0$, $i = 1, ..., n-1$, and therefore $t - R'N$ is perpendicular to all Z_i.

We introduce the unit tangent vector field ξ as follows:

$$\xi := \frac{1}{\sqrt{(t - R'\,N)^2}}\,(t - R'\,N). \tag{3.6}$$

Then the distribution $\Delta := \{x \in T_pM : x \perp \xi\}$ is exactly $\Delta = \operatorname{span}\{Z_1, \dots, Z_{n-1}\}$.

For the purposes of our investigations we need to introduce three types of canal space-like hypersurfaces.

Definition 3.4. A canal space-like hypersurface M in $\mathbb{R}^{n+1}_1$, given by (3.3), is said to be *a canal space-like hypersurface of elliptic, hyperbolic* or *parabolic type* if the curve $z = z(s)$, $s \in J$ of the centers of the hyperspheres is time-like, space-like or light-like, respectively.

Rotational space-like hypersurfaces are introduced in a natural way:

Definition 3.5. A canal space-like hypersurface M in $\mathbb{R}^{n+1}_1$, given by (3.3), is said to be *a rotational space-like hypersurface* if the curve $z = z(s)$, $s \in J$ of the centers of the hyperspheres lies on a straight line.

Any of the three types of canal space-like hypersurfaces generates the corresponding subclass of rotational space-like hypersurfaces.

3.1. Canal space-like hypersurfaces of elliptic type in Minkowski space

Let M be a canal space-like hypersurface in $\mathbb{R}^{n+1}_1$ of elliptic type, given by (3.3). The curve of centers $z = z(s)$, $s \in J$, parameterized by its natural parameter, satisfies the condition $z'^2 = t^2 = -1$.

Since $(t - R'\,N)^2 = R'^2 - 1 > 0$, then the function $R(s)$ in the case of a canal space-like hypersurface of elliptic type satisfies the inequalities

$$R^2(s) > 0, \quad R'^2(s) - 1 > 0; \quad s \in J.$$

Next we find the second fundamental form of M.

Let ∇' be the standard flat Levi-Civita connection in $\mathbb{R}^{\mathbf{n+1}}_{\mathbf{1}}$ and h be the second fundamental tensor of M. The Levi-Civita connection of the induced metric on the hypersurface M is denoted by ∇. Taking into account (3.5) and (3.6), we get

$$\nabla'_{Z_i} N = N_i = -\frac{1}{R}\, Z_i, \quad \nabla'_{Z_i} \xi = \xi_i = -\frac{R'}{\sqrt{R'^2 - 1}}\, N_i = \frac{R'}{R\sqrt{R'^2 - 1}}\, Z_i.$$

These equalities can be written as follows:

$$\nabla'_x N = -\frac{1}{R}\, x, \quad \nabla'_x \xi = \nabla_x \xi = \frac{R'}{R\sqrt{R'^2 - 1}}\, x = k\, x, \quad x \in \Delta,$$

and the function k is

$$k = \frac{R'}{R\sqrt{R'^2 - 1}}\,. \tag{3.1.1}$$

Hence, the shape operator A of M satisfies

$$Ax = \frac{1}{R}\, x, \quad x \in \Delta. \tag{3.1.2}$$

Since $g(N,N)=-1$, then

$$h(x,y) = -g(Ax,y) = -\frac{1}{R}\,g(x,y), \quad x,y \in \Delta. \tag{3.1.3}$$

The equality (3.1.2) means that the tangent space Δ is invariant with respect to the shape operator A. This implies that the vector field ξ is also an eigenvector field of A, i.e.

$$A\xi = \nu\,\xi. \tag{3.1.4}$$

Assuming the standard summation convention, we can put

$$\xi = \phi^i Z_i + \phi Z_s, \quad \phi \neq 0 \tag{3.1.5}$$

for some functions $\phi^1,\dots,\phi^{n-1}$; ϕ on M. Since ξ is perpendicular to all Z_i, we have

$$\xi Z_s = \frac{1}{\phi}\,. \tag{3.1.6}$$

Taking into account (3.1.5), we compute

$$\nabla'_\xi N = \phi^i N_i + \phi N_s = -\frac{1}{R}\,\phi^i Z_i + \phi N_s. \tag{3.1.7}$$

On the other hand, because of (3.5) and (3.6), we have

$$Z_s + R\,N_s = \sqrt{R'^2 - 1}\,\xi. \tag{3.1.8}$$

In view of (3.1.5) and (3.1.8) equality (3.1.7) implies that

$$\nabla'_\xi N = -\frac{1}{R}(1 - \phi\sqrt{R'^2-1})\xi = -\nu\,\xi$$

and

$$\nu - \frac{1}{R} = -\frac{\sqrt{R'^2-1}}{R}\,\phi\,. \tag{3.1.9}$$

Using (3.1.2), (3.1.3) and (3.1.4), we obtain the shape operator of M:

$$AX = \frac{1}{R}\,X + \left(\nu - \frac{1}{R}\right)\eta(X)\,\xi, \quad X \in \mathcal{X}M.$$

The last equality and (3.1.9) imply that the second fundamental tensor of M has the form

$$h(X,Y) = -\frac{1}{R}\,g(X,Y) + \phi\,\frac{\sqrt{R'^2-1}}{R}\,\eta(X)\eta(Y), \quad X,Y \in \mathcal{X}M. \tag{3.1.10}$$

Further we replace (3.1.10) into the Gauss equation for the hypersurface M, and taking into account (3.1.6), we obtain the curvature tensor $\mathcal{R}$ of a canal space-like hypersurface M of elliptic type:

$$\mathcal{R} = -\frac{1}{R^2}\,\pi + \frac{\sqrt{R'^2-1}}{R^2(\xi Z_s)}\,\Phi = a\pi + b\Phi. \tag{3.1.11}$$

Now (3.1.11) and (3.1.1) imply that

$$a = -\frac{1}{R^2} < 0, \qquad a + k^2 = \frac{1}{R^2(R'^2 - 1)} > 0.$$

Thus we obtained the following

Proposition 3.6. *Any canal space-like hypersurface of elliptic type in $\mathbb{R}^{n+1}_1$ is a Riemannian manifold of quasi-constant sectional curvatures with functions $a < 0$ and $a + k^2 > 0$.*

Next we prove that the rotational space-like hypersurfaces of elliptic type are Riemannian subprojective manifolds satisfying the conditions in Proposition 3.6.

Using (3.1.8), we have

$$\xi Z_s + R\,(\xi N_s) = \sqrt{R'^2 - 1}.$$

In order to compute the function ξN_s, we use the equality $\xi N_s + \xi_s N = 0$. Differentiating (3.6) by s, we find

$$\xi_s N = \frac{t'N + R''}{\sqrt{R'^2 - 1}}\,.$$

Therefore

$$\xi Z_s = \frac{RR'' + R'^2 - 1 + R(t'N)}{\sqrt{R'^2 - 1}}$$

and

$$b = \frac{R'^2 - 1}{R^2\{RR'' + R'^2 - 1 + R(t'N)\}}\,.$$

According to Proposition 3.6, the hypersurface M is a Riemannian QC-manifold. Any Riemannian QC-manifold is subprojective if and only if the functions a and b generate one and the same distribution. Therefore, M is subprojective if and only if the function b does not depend on the parameters u^i; $i = 1, \dots, n-1$, i.e. $t' = 0$. Since $t' = 0$ characterizes a straight line c, we obtain the following statement.

Proposition 3.7. *A canal space-like hypersurface M of elliptic type in $\mathbb{R}^{n+1}_1$ is a Riemannian subprojective manifold if and only if M is a rotational space-like hypersurface of elliptic type.*

Combining with Proposition 3.6, we have

Proposition 3.8. *Any rotational space-like hypersurface of elliptic type in $\mathbb{R}^{n+1}_1$ is a subprojective Riemannian manifold with functions $a < 0$ and $a + k^2 > 0$.*

The curvature tensor of a rotational space-like hypersurface of elliptic type has the form

$$\mathcal{R} = -\frac{1}{R^2}\,\pi + \frac{R'^2 - 1}{R^2(RR'' + R'^2 - 1)}\,\Phi.$$

Let M be a canal space-like hypersurface of hyperbolic type, given by (3.3). The curve of centers $z = z(s)$, $s \in J$, parameterized by its natural parameter, satisfies the condition $z'^2 = t^2 = 1$.

In the case considered, the inequality $(t - R'\,N)^2 = R'^2 + 1 > 0$ is always satisfied. Hence, $R(s)$ satisfies the only condition $R(s) > 0$.

We compute

$$\nabla'_x N = -\frac{1}{R}\,x, \quad \nabla'_x \xi = \nabla_x \xi = \frac{R'}{R\sqrt{R'^2+1}}\,x = k\,x, \quad x \in \Delta,$$

where the function $k(s)$ is

$$k = \frac{R'}{R\sqrt{R'^2+1}}. \tag{3.2.1}$$

Therefore,

$$Ax = \frac{1}{R}\,x, \quad x \in \Delta \tag{3.2.2}$$

and the vector field ξ is an eigenvector for A:

$$A\xi = \nu\,\xi, \tag{3.2.3}$$

Putting $\xi = \phi^i Z_i + \phi Z_s$, we compute

$$\nabla'_\xi N = -\frac{1}{R}\phi^i Z_i + \phi N_s,$$

and taking into account that

$$Z_s + R\,N_s = \sqrt{R'^2+1}\,\xi,$$

we find

$$\nabla'_\xi N = -\frac{1}{R}(1 - \phi\sqrt{R'^2+1})\xi = -\nu\,\xi,$$

and

$$\nu = \frac{1}{R}(1 - \phi\sqrt{R'^2+1}).$$

Using (3.2.2) and (3.2.3), we obtain the second fundamental form h of the hypersurface M:

$$h = -\frac{1}{R}\,g + \phi\frac{\sqrt{R'^2+1}}{R}\,\eta\otimes\eta.$$

Applying the Gauss equation and the equality $\phi\,(\xi Z_s) = 1$, we calculate the curvature tensor of the hypersurface M.

$$\mathcal{R} = -\frac{1}{R^2}\,\pi + \frac{\sqrt{R'^2+1}}{R^2\,(\xi Z_s)}\,\Phi = a\pi + b\Phi.$$

Therefore, $a = -1/R^2$. In view of (3.2.1), we find

$$a + k^2 = \frac{-1}{R^2(R'^2+1)} < 0.$$

Thus we obtained the following statement.

Proposition 3.9. *Any canal space-like hypersurface of hyperbolic type in* $\mathbb{R}^{n+1}_1$ *is a Riemannian manifold of quasi-constant sectional curvatures with function* $a + k^2 < 0$.

Next we prove that the rotational space-like hypersurfaces of hyperbolic type are Riemannian subprojective manifolds satisfying the condition in Proposition 3.9.

Differentiating (3.6) with respect to s, we compute

$$\xi_s N = \frac{t'N + R''}{\sqrt{R'^2+1}}\,. \tag{3.2.4}$$

Using the equality $\xi_s N + \xi N_s = 0$, (3.2.4) and (3.5), we find

$$\xi Z_s = \frac{RR'' + R'^2 + 1 + R\,(t'N)}{\sqrt{R'^2+1}}$$

and

$$b = \frac{R'^2+1}{R^2\{RR'' + R'^2 + 1 + R\,(t'N)\}}.$$

Applying similar arguments as in Subsection 3.1, we conclude that M is subprojective if and only if $t' = 0$, which characterizes a straight line c.

Thus we obtained the following statement.

Proposition 3.10. *A canal space-like hypersurface* M *of hyperbolic type in* $\mathbb{R}^{n+1}_1$ *is a Riemannian subprojective manifold if and only if* M *is a rotational space-like hypersurface of hyperbolic type.*

Combining with Proposition 3.9, we have

Proposition 3.11. *Any rotational space-like hypersurface of hyperbolic type in* $\mathbb{R}^{n+1}_1$ *is a subprojective Riemannian manifold with function* $a + k^2 < 0$.

The curvature tensor of a rotational space-like hypersurface of hyperbolic type has the form

$$\mathcal{R} = -\frac{1}{R^2}\,\pi + \frac{R'^2+1}{R^2(RR'' + R'^2 + 1)}\,\Phi.$$

Let M be a canal space-like hypersurface of parabolic type, given by (3.3). The curve of centers $z = z(s),\ s \in J$, satisfies the condition $z'^2 = t^2 = 0$.

In this case $(t - R'\,N)^2 = R'^2 > 0$ and the function $R(s)$ satisfies the conditions $R(s) > 0$ and $R'(s) \neq 0$.

Next we find the second fundamental form of M.

We compute

$$\nabla'_x N = -\frac{1}{R}\,x, \quad \nabla'_x \xi = \nabla_x \xi = \frac{1}{R}\,x = kx, \quad x \in \Delta$$

and find

$$Ax = \frac{1}{R}\,x, \quad x \in \Delta, \tag{3.3.1}$$

$$k = \frac{1}{R} \tag{3.3.2}$$

and

$$A\xi = \nu\,\xi. \tag{3.3.3}$$

Further we again put

$$\xi = \phi^i Z_i + \phi Z_s$$

and compute

$$\nabla'_\xi N = \phi^i N_i + \phi N_s = -\frac{1}{R}\phi^i Z_i + \phi N_s, \quad i = 1, 2, \ldots, n-1. \tag{3.3.4}$$

Using the equality

$$Z_s = R'\,\xi - R\,N_s,$$

we obtain from (3.3.4) that

$$\nabla'_\xi N = -\frac{1}{R}(1 - \phi R')\xi = -\nu\,\xi,$$

and

$$\nu = \frac{1}{R}(1 - \phi R'). \tag{3.3.5}$$

Now equalities (3.3.1), (3.3.3) and (3.3.5) imply that

$$h = -\frac{1}{R}\,g + \phi\frac{R'}{R}\,\eta \otimes \eta.$$

Finally, replacing h into the Gauss equation and using the equality $\phi\,(\xi Z_s) = 1$, we find the curvature tensor of the hypersurface M in the form

$$\mathcal{R} = -\frac{1}{R^2}\,\pi + \frac{R'}{R^2\,(\xi Z_s)}\,\Phi = a\pi + b\Phi,$$

which shows that M is a Riemannian QC-manifold with function $a = -1/R^2$. In view of (3.3.2) we find

$$a + k^2 = 0.$$

Thus we obtained the following statement.

Proposition 3.12. *Any canal space-like hypersurface of parabolic type in* $\mathbb{R}^{n+1}_1$ *is a Riemannian manifold of quasi-constant sectional curvatures with function* $a + k^2 = 0$.

Next we prove that the rotational space-like hypersurfaces of parabolic type are Riemannian subprojective manifolds satisfying the condition in Proposition 3.12.

Differentiating (3.6) with respect to s, we get

$$\xi_s N = \frac{t' N + R''}{R'} \,. \tag{3.3.6}$$

Using the equality $\xi_s N + \xi N_s = 0$, (3.3.6) and (3.5), we find

$$\xi Z_s = \frac{RR'' + R'^2 + R' \,(t' N)}{R'}$$

and

$$b = \frac{R'^2}{R^2\{RR'' + R'^2 + R' \,(t' N)\}}.$$

Applying similar arguments as in Subsection 3.1, we conclude that M is subprojective if and only if $t' = 0$, which characterizes a straight line c.

Thus we obtained the following statement.

Proposition 3.13. *A canal space-like hypersurface* M *of parabolic type in* $\mathbb{R}^{n+1}_1$ *is a Riemannian subprojective manifold if and only if* M *is a rotational space-like hypersurface of parabolic type.*

Combining with Proposition 3.12, we have

Proposition 3.14. *Any rotational space-like hypersurface of parabolic type in* $\mathbb{R}^{n+1}_1$ *is a subprojective Riemannian manifold with function* $a + k^2 = 0$.

The curvature tensor of a rotational space-like hypersurface of parabolic type has the form

$$\mathcal{R} = -\frac{1}{R^2}\,\pi + \frac{R'^2}{R^2(RR'' + R'^2)}\,\Phi.$$

4. A LOCAL CLASSIFICATION OF RIEMANNIAN QC-MANIFOLDS

Let (M, g, ξ) $(\dim M = n \geq 4)$ be a Riemannian QC-manifold. Then the Riemannian curvature tensor $\mathcal{R}$ of M has the form

$$\mathcal{R} = a\pi + b\Phi. \tag{4.1}$$

We consider manifolds free of points in which the tensor $\mathcal{R}$ is of constant sectional curvatures, i.e. $b \neq 0$ in all points of M.

We note that the condition $a = 0$ implies that $b = 0$.

In [11] we proved that a Riemannian QC-manifold with positive horizontal sectional curvatures, i.e. $a > 0$, can be locally embedded as a canal hypersurface in Euclidean space $\mathbb{R}^{n+1}$.

In this section we study Riemannian QC-manifolds with negative horizontal sectional curvatures, i.e. $a < 0$.

The basic step in our classification of Riemannian QC-manifolds is the following theorem.

Theorem 4.1. *Let* (M, g, ξ) $(\dim M = n \geq 4)$ *be a Riemannian QC-manifold with curvature tensor* (4.1) *satisfying the conditions:*

$$b \neq 0, \qquad a < 0.$$

Then the manifold is locally ξ*-isometric to a canal space-like hypersurface in* $\mathbb{R}^{n+1}_1$.

Moreover, the manifold is locally ξ*-isometric to a canal space-like hypersurface of elliptic, hyperbolic or parabolic type, according to*

$$a + k^2 > 0, \quad a + k^2 < 0 \quad \text{or} \quad a + k^2 = 0,$$

respectively.

Proof. Under the conditions of the theorem the curvature tensor of the manifold M has the form (4.1) and all equalities (2.3) - (2.7) are valid. We put

$$\alpha = \sqrt{-a}, \qquad \beta = -\frac{b}{\sqrt{-a}}$$

and consider the symmetric tensor

$$h = \sqrt{-a}\, g - \frac{b}{\sqrt{-a}}\, \eta \otimes \eta = \alpha\, g + \beta\, \eta \otimes \eta \tag{4.2}$$

on M.

An immediate verification shows that the curvature tensor $\mathcal{R}$ of the manifold (M, g, ξ) has the following construction

$$\mathcal{R}(X, Y, Z, U) = -\{h(Y, Z)\, h(X, U) - h(X, Z)\, h(Y, U)\}, \tag{4.3}$$

i.e.

$$\mathcal{R} = -(\alpha^2\,\pi + \alpha\beta\,\Phi), \qquad a = -\alpha^2,\ b = -\alpha\beta.$$

We shall show that the tensor h satisfies the Codazzi equation

$$(\nabla'_X h)(Y,Z) - (\nabla'_Y h)(X,Z) = 0, \quad X, Y \in \mathcal{X}M. \tag{4.4}$$

Taking into account (4.2), we calculate

$$\begin{aligned}(\nabla'_X h)(Y,Z) - (\nabla'_Y h)(X,Z) = \;& d\alpha(X)\,g(Y,Z) - d\alpha(Y)\,g(X,Z)\\ &+ (d\beta(X)\,\eta(Y) - d\beta(Y)\,\eta(X))\,\eta(Z)\\ &+ \beta\,d\eta(X,Y)\,\eta(Z)\\ &+ \beta\,(\eta(Y)(\nabla'_X\eta)(Z) - \eta(X)(\nabla'_Y\eta)(Z)).\end{aligned} \tag{4.5}$$

We prove that the right hand side of (4.5) is identically zero. Since any tangent vector is decomposable as in (2.1), we divide the proof into four steps. Taking into account that $a = -\alpha^2$, $b = -\alpha\beta$, we apply equalities (2.3) - (2.7) and obtain consequently:

1) If $X = x$, $Y = y$, $Z = z$, then the right hand side of (4.5) reduces to

$$d\alpha(x)\,g(y,z) - d\alpha(y)\,g(x,z),$$

which is zero because of (2.3).

2) If $X = x$, $Y = y$, $Z = \xi$, then the right hand side of (4.5) reduces to

$$\beta\,d\eta(x,y),$$

which is zero in view of (2.4).

3) If $X = x$, $Y = \xi$, $Z = \xi$, then the right hand side of (4.5) reduces to

$$d\beta(x) + \beta\,d\eta(x,\xi),$$

which is zero as a consequence of (2.5).

4) If $X = \xi$, $Y = y$, $Z = z$, then the right hand side of (4.5) reduces to

$$\xi(\alpha)\,g(y,z) - \beta(\nabla'_y\eta)(z),$$

which is zero because of (2.6).

Combining the above cases 1) - 4), we conclude that the right hand side of (4.5) is equal to zero for all $X, Y, Z \in \mathcal{X}M$, i.e. the tensor h satisfies (4.4) identically.

Now we can apply the fundamental embedding theorem for hypersurfaces in $\mathbb{R}^{n+1}_1$ and obtain that the Riemannian QC-manifold (M, g, ξ) can be locally embedded as a hypersurface in the Minkowski space $\mathbb{R}^{n+1}_1$.

If N is the unit normal vector field to a hypersurface with second fundamental form h, then the curvature tensor $\mathcal{R}$ of this hypersurface satisfies the identity

$$\mathcal{R}(X,Y,Z,U) = g(N,N)\,\{h(Y,Z)\,h(X,U) - h(X,Z)\,h(Y,U)\}.$$

Comparing with (4.3) we obtain that the Riemannian QC-manifold (M,g,ξ) is embedded locally as a space-like hypersurface in $\mathbb{R}_1^{n+1}$. Further, we denote this hypersurface again with (M,g,ξ).

Now (M,g,ξ) is a space-like hypersurface in $\mathbb{R}_1^{n+1}$, whose second fundamental form h satisfies (4.2).

Next we prove that M is locally a part of a space-like canal hypersurface in $\mathbb{R}_1^{n+1}$.

Let Z be the position vector field of M and p be a fixed point in M. Denote by S_p the maximal integral submanifold of the distribution Δ containing p. Using the property $d\alpha = \xi(\alpha)\,\eta$, we get $\alpha = \mathrm{const}$ on S_p. Then the equality

$$\nabla'_x N = -\alpha\, x$$

implies that the vector function $Z - (1/\alpha)\,N$ is constant at the points of S_p. We set

$$z = Z - \frac{1}{\alpha}\,N,$$

and conclude that S_p lies on the time-like hypersphere S^n with center z and radius $R = (1/\alpha)$, and both hypersurfaces, M and S^n, have the same normals at the points of S_p.

Since the distribution Δ determines a one-parameter family of submanifolds $Q^{n-1}(s)$, $s \in J$ in a neighborhood U of p, then U is a part of the envelope of this family.

Finally we apply Propositions 3.6, 3.9, 3.12 and obtain the second part of the theorem. □

Applying Theorem 4.1, we obtain immediately

Theorem 4.2. *Let* (M,g,ξ) $(\dim M = n \geq 4)$ *be a subprojective Riemannian manifold with curvature tensor* (4.1) *satisfying the conditions:*

$$b \neq 0, \qquad a < 0.$$

Then the manifold is locally ξ-isometric to a rotational space-like hypersurface in $\mathbb{R}_1^{n+1}$.

Moreover, the manifold is locally ξ-isometric to a rotational space-like hypersurface of elliptic, hyperbolic or parabolic type, according to

$$a + k^2 > 0, \quad a + k^2 < 0 \quad \text{or} \quad a + k^2 = 0,$$

respectively.

5. CONFORMALLY FLAT RIEMANNIAN HYPERSURFACES IN EUCLIDEAN OR MINKOWSKI SPACE

5.1. CONFORMALLY FLAT HYPERSURFACES IN EUCLIDEAN SPACE

A hypersurface M in Euclidean space is said to be quasi-umbilical [5] if its second fundamental form h satisfies the equality

$$h = \alpha\, g + \beta\, \eta \otimes \eta \tag{5.1}$$

for some functions $\alpha \neq 0$, $\beta \neq 0$, and a unit 1-form η on M. The close relation between conformally flat hypersurfaces in Euclidean space from one hand side, and quasi-umbilical hypersurfaces in $\mathbb{R}^{n+1}$ from another hand side, is the following statement [3, 18] (see also [16]):

Lemma 5.1. (Cartan - Schouten) *Let M be a conformally flat hypersurface in $\mathbb{R}^{n+1}$. Then the shape operator of M at any point has a root of multiplicity at least $n-1$.*

As a result of Lemma 5.1 we have

Lemma 5.2. *Any conformally flat hypersurface M in Euclidean space, which is free of umbilical points, is quasi-umbilical.*

Proof. Let A be the shape operator of the hypersurface M. Since M is free of umbilical points, then according to Lemma 5.1 the operator A has at any point two different eigenvalues α and $\alpha + \beta$ of multiplicity $n-1$ and 1, respectively. Let ξ be the unit eigenvector field, corresponding to the function $\alpha + \beta$. Denoting by η the 1-form, corresponding to ξ with respect to the metric g, we obtain (5.1). □

Equality (5.1) implies that the curvature tensor $\mathcal{R}$ of M has the form

$$\mathcal{R} = \alpha^2\, \pi + \alpha\beta\, \Phi\,; \quad \alpha^2 > 0, \quad \alpha\beta \neq 0,$$

i.e. (M, g, ξ) is a Riemannian QC-manifold with positive horizontal sectional curvatures.

Applying Proposition 3 [11], we obtain:

Theorem 5.3. *Any conformally flat hypersurface M in $\mathbb{R}^{n+1}$, which is free of umbilical points, locally lies on a canal hypersurface.*

If (M, g) is a conformally flat hypersurface in $\mathbb{R}^{n+1}$, then the manifold (M, g) admits a unit vector field ξ, such that (M, g, ξ) is a Riemannian QC-manifold with positive horizontal sectional curvatures $a > 0$. Any two locally isometric conformally flat hypersurfaces are locally ξ-isometric, i.e. rigid. Taking into account

Theorem 2.4, we obtain that any isometric embedding of a conformally flat Riemannian manifold into $\mathbb{R}^{n+1}$ is locally determined up to a motion. We also recall the results of R. Beez [2] and W. Killing [14]:

A hypersurface in the Euclidean space is rigid if at least three principal curvatures are different from zero at each point of it, i.e. the hypersurface has type-number ≥ 3 at each point.

Thus, we obtained that

The local theory of conformally flat Riemannian manifolds, isometrically embedded as hypersurfaces in Euclidean space, is equivalent to the local theory of Riemannian QC-manifolds with positive horizontal sectional curvatures.

Taking into account the local classification of hypersurfaces in Euclidean space of constant sectional curvature and Theorem 5.3, we obtain a local geometric classification of conformally flat hypersurfaces in $\mathbb{R}^{n+1}$:

Theorem 5.4. *Any conformally flat hypersurface M in $\mathbb{R}^{n+1}$ is locally a part of one of the following hypersurfaces:*

(i) hyperplane $(\alpha = \beta = 0)$*;*

(ii) hypersphere $(\alpha \neq 0,\ \beta = 0)$*;*

(iii) developable hypersurface $(\alpha = 0,\ \beta \neq 0)$*;*

(iv) canal hypersurface $(\alpha \neq 0,\ \beta \neq 0)$*.*

5.2. CONFORMALLY FLAT SPACE-LIKE HYPERSURFACES IN MINKOWSKI SPACE

Let M be a space-like hypersurface in Minkowski space with second fundamental form h. Similarly to the Euclidean case, we call the hypersurface M *quasi-umbilical* if

$$h = \alpha\, g + \beta\, \eta \otimes \eta \tag{5.2}$$

for some functions $\alpha \neq 0$, $\beta \neq 0$, and a unit 1-form η on M.

The proof of Lemma 5.1 is also valid without any essential changes for conformally flat hypersurfaces in Minkowski space.

Lemma 5.5. *Let M be a conformally flat space-like hypersurface in $\mathbb{R}^{n+1}_1$. Then the shape operator of M at any point has a root of multiplicity at least $n-1$.*

Analogously to Subsection 5.1., Lemma 5.5 implies the following statement.

Lemma 5.6. *Any conformally flat space-like hypersurface M in Minkowski space, which is free of umbilical points, locally is quasi-umbilical.*

Equality (5.2) implies that the curvature tensor $\mathcal{R}$ of M has the form

$$\mathcal{R} = -\alpha^2\, \pi - \alpha\beta\, \Phi\,; \quad \alpha^2 > 0, \quad \alpha\beta \neq 0,$$

i.e. (M, g, ξ) is a Riemannian QC-manifold with negative horizontal sectional curvatures.

Applying Theorem 4.1, we obtain:

Theorem 5.7. *Any conformally flat space-like hypersurface M in Minkowski space, which is free of umbilical points, locally is a part of a canal space-like hypersurface.*

If (M, g) is a conformally flat space-like hypersurface in $\mathbb{R}^{n+1}_1$, then the manifold (M, g) admits a unit vector field ξ, such that (M, g, ξ) is a Riemannian QC-manifold with negative horizontal sectional curvatures $a < 0$. Any two locally isometric conformally flat space-like hypersurfaces are locally ξ-isometric, i.e. rigid. All isometric realizations of a conformally flat Riemannian manifold into $\mathbb{R}^{n+1}_1$ are locally congruent.

Thus we have:

The local theory of conformally flat Riemannian manifolds, isometrically immersed as space-like hypersurfaces in Minkowski space, is equivalent to the local theory of Riemannian QC-manifolds with negative horizontal sectional curvatures.

Taking into account the local classification of space-like hypersurfaces of constant sectional curvatures in Minkowski space, and Theorem 5.7, we obtain the following geometric classification of conformally flat space-like hypersurfaces in Minkowski space:

Theorem 5.8. *Any conformally flat space-like hypersurface M in Minkowski space is locally a part of one of the following hypersurfaces:*

(i) a space-like hyperplane $(\alpha = \beta = 0)$;

(ii) a space-like hypersphere $(\alpha \neq 0,\ \beta = 0)$;

(iii) a space-like developable hypersurface $(\alpha = 0,\ \beta \neq 0)$;

(iv) a space-like canal hypersurface $(\alpha \neq 0,\ \beta \neq 0)$.

ACKNOWLEDGEMENTS. The first-named author is partially supported by the National Science Fund, Ministry of Education and Science of Bulgaria under contract DFNI-I 02/14. The second-named author is partially supported by the Sofia University Research Fund through Grant 59/2014.

6. REFERENES

1. Adati, T.: On subprojective spaces I. *Tohoku Math. J.*, **3**, 1951, 159–173.
2. Beez, R.: Zur Theorie des Krümmungsmasses von Mannigfaltigkeiten höherer Ordnung. Z. *Math. Physik*, **21**, 1876, 373–401.

3. Cartan, É.: La déformation des hypersurfaces dans l'espace conforme réel à $n \geq 5$ dimensions. *Bulletin de la S.M.F.*, **45**, 1917, 57–121.
4. Chen, B.-Y., Yano, K.: Special conformally flat spaces and canal surfaces, *Tohoku Math. J.*, **25**, 1973, 177–184.
5. Chen, B.-Y., Yano, K.: Hypersurfaces of a conformally flat space. *Tensor, N.S.*, **26**, 1972, 318–322.
6. do Carmo, M., Dajczer, M., Mercuri, F.: Compact conformally flat hypersurfaces. *Trans. Amer. Math. Soc.*, **288**, 1985, 189–203.
7. Enneper, A.: Über die Cyclischen Flächen. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen*, 1866, 243–250.
8. Enneper, A.: Analytisch-geometrische Untersuchungen. III. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen*, 1867, 232–264.
9. Enneper, A.: Bemerkung über die Enveloppe einer Kugelfläche. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen*, 1873, 217–248.
10. Enneper, A.: Untersuchungen über die Flächen mit planen und sphärischen Krümmungslinien. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen*, 1880, 1–96.
11. Ganchev, G., Mihova, V.: Riemannian manifolds of quasi-constant sectional curvatures. *J. Reine Angew. Math.*, **22**, 2000, 119–141.
12. Kagan, V. F.: Über eine Erweiterung des Begriffes vom projektiven Raume und dem zugehörigen Absolut. *Abhandlungen des Seminars für Vektor und Tensoranalysis, Moskau*, **1**, 1933, 12–101.
13. Kagan, V. F.: *Subprojective spaces*, Moscow, 1961.
14. Killing, W.: *Die nicht-Euklidischen Raumformen in analytische Behandlung.* Teubner-Verlag, Leipzig, 1885.
15. Kulkarni, R.: Conformally flat manifolds. *Proc. Nat. Acad. Sci. USA*, **69**, no. 9, 1972, 2675–2676.
16. Nishikawa, S., Maeda, Y.: Conformally flat hypersurfaces in a conformally flat Riemannian manifold. *Tohoku Math. J.*, **26**, 1974, 159–168.
17. Rachevsky, P.: Caracteres tensoriels de l'espace sous-projectiv. *Abh. des Seminars für Vektor und Tensoranalysis, Moskou*, **1**, 1933, 126–140.
18. Schouten, J. A.: Über die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Maßbestimmung auf eine Mannigfaltigkeit mit Euklidischer Maßbestimmung. *Math. Z.*, **11**, 1921, 58–88.
19. Schouten, J. A.: *Ricci-Calculus.* Springer Verlag, Berlin, 1954.
20. von Lilienthal, R.: *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Band III - Geometrie, 3. Teil, D. Differentialgeometrie, 5. Besondere Flächen* (1902 - 1903), Leipzig, Verlag und Druck von B. G. Teubner, 269–354.

Received on August 7, 2014

Georgi Ganchev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., bl.8, BG-1113 Sofia
BULGARIA
e-mail: ganchev@math.bas.bg

Vesselka Mihova
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5 J. Bourchier Blvd., BG-1164 Sofia
BULGARIA
e-mail: mihova@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

ALMOST CONTACT B-METRIC STRUCTURES AND THE BIANCHI CLASSIFICATION OF THE THREE-DIMENSIONAL LIE ALGEBRAS

HRISTO M. MANEV

The object of investigation are the almost contact manifolds with B-metric in the lowest dimension three, constructed on Lie algebras. It is considered a relation between the classes in the Bianchi classification of three-dimensional real Lie algebras and the classes of a classification of the considered manifolds. There are studied some geometrical characteristics in some special classes.

Keywords: Almost contact structure, B-metric, Lie group, Lie algebra, indefinite metric

2010 Math. Subject Classification: 53C15, 53C50, 53D15

1. INTRODUCTION

The differential geometry of the manifolds equipped with an almost contact structure is well studied (see, e.g. [3]). The almost contact manifolds with B-metric are introduced and classified in [6]. These manifolds are the odd-dimensional counterpart of the almost complex manifolds with Norden metric [5, 7].

An object of special interest is the case of the lowest dimension of the considered manifolds. We investigate the almost contact B-metric manifolds in dimension three and get explicit results. Some curvature identities of the three-dimensional manifolds of this type are studied in [11, 12].

Almost contact manifolds with B-metric can be constructed on Lie algebras. It is known that all three-dimensional real Lie algebras are classified in [1, 2]. The main goal of this paper is to find a relation between the classes in the Bianchi classification and the classification of almost contact B-metric manifolds given in [6]. Moreover, the present work gives some geometrical characteristics of the considered manifolds in certain special classes.

The paper is organized as follows. In Section 2 we recall some preliminary facts about the almost contact B-metric manifolds. In Section 3 we equip each Bianchi-type Lie algebra with an almost contact B-metric structure. In Section 4 we give the relation between the Bianchi classification and the classification given in [6]. Section 5 is devoted to the curvature properties of some of the considered manifolds.

2. PRELIMINARIES

Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric or an *almost contact B-metric manifold*, where M is a $(2n+1)$-dimensional differentiable manifold, (φ,ξ,η) is an almost contact structure consisting of an endomorphism φ of the tangent bundle, a Reeb vector field ξ and its dual contact 1-form η. Moreover, M is equipped with a pseudo-Riemannian metric g, called a *B-metric*, such that the following algebraic relations are satisfied [6]:

$$\varphi\xi=0,\qquad \varphi^2=-\mathrm{Id}+\eta\otimes\xi,\qquad \eta\circ\varphi=0,\qquad \eta(\xi)=1,$$
$$g(\varphi x,\varphi y)=-g(x,y)+\eta(x)\eta(y),$$

where Id is the identity. In the latter equalities and further, x, y, z, w will stand for arbitrary elements of the algebra of the smooth vector fields on M or vectors in the tangent space T_pM of M at an arbitrary point p in M.

The associated B-metric $\tilde{g}$ of g is determined by $\tilde{g}(x,y)=g(x,\varphi y)+\eta(x)\eta(y)$. The manifold $(M,\varphi,\xi,\eta,\tilde{g})$ is also an almost contact B-metric manifold. The signature of both metrics g and $\tilde{g}$ is necessarily $(n+1,n)$. We denote the Levi-Civita connection of g and $\tilde{g}$ by ∇ and $\tilde{\nabla}$, respectively.

A classification of almost contact B-metric manifolds, consisting of eleven basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, …, $\mathcal{F}_{11}$, is given in [6]. This classification is made with respect to the tensor F of type (0,3) defined by

$$F(x,y,z)=g\big((\nabla_x\varphi)\,y,z\big) \tag{2.1}$$

and having the following properties:

$$F(x,y,z)=F(x,z,y)=F(x,\varphi y,\varphi z)+\eta(y)F(x,\xi,z)+\eta(z)F(x,y,\xi).$$

The special class determined by the condition $F(x,y,z)=0$ is denoted by $\mathcal{F}_0$. This class is the intersection of all the basic classes. Hence $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with ∇-parallel structures, i.e. $\nabla\varphi = \nabla\xi = \nabla\eta = \nabla g = \nabla\tilde{g} = 0$. Therefore $\mathcal{F}_0$ is the class of the *cosymplectic manifolds with B-metric.*

According to [10], the *square norm of* $\nabla\varphi$ is defined by:

$$\|\nabla\varphi\|^2 = g^{ij}g^{ks}g\big((\nabla_{e_i}\varphi)\,e_k, \big(\nabla_{e_j}\varphi\big)\,e_s\big). \tag{2.2}$$

It is clear that $\|\nabla\varphi\|^2 = 0$ is valid if (M,φ,ξ,η,g) is a cosymplectic manifold with B-metric, but the inverse implication is not always true. An almost contact B-metric manifold having a zero square norm of $\nabla\varphi$ is called an *isotropic-cosymplectic B-metric manifold.*

If $\{e_i;\xi\}$ $(i = 1,2,\dots,2n)$ is a basis of T_pM and $\big(g^{ij}\big)$ is the inverse matrix of (g_{ij}), then the 1-forms θ, θ^*, ω, called *Lee forms*, are associated with F and defined by:

$$\theta(z) = g^{ij}F(e_i,e_j,z), \quad \theta^*(z) = g^{ij}F(e_i,\varphi e_j,z), \quad \omega(z) = F(\xi,\xi,z).$$

Let now consider the case of the lowest dimension of the almost contact B-metric manifold M, i.e. $\dim M = 3$.

We introduce an almost contact structure (φ,ξ,η) on M defined by

$$\begin{gathered}\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad \xi = e_3,\\ \eta(e_1) = \eta(e_2) = 0, \quad \eta(e_3) = 1\end{gathered} \tag{2.3}$$

and a B-metric g such that

$$g(e_1,e_1) = -g(e_2,e_2) = g(e_3,e_3) = 1, \quad g(e_i,e_j) = 0, \;\; i \neq j \in \{1,2,3\}. \tag{2.4}$$

Let us denote the components $F_{ijk} = F(e_i,e_j,e_k)$ of F with respect to a φ*-basis* $\{e_1,e_2,e_3\}$ of T_pM.

According to [8], the components of the Lee forms are

$$\begin{array}{lll}\theta_1 = F_{111} - F_{221}, & \theta_2 = F_{112} - F_{211}, & \theta_3 = F_{113} - F_{223},\\ \theta_1^* = F_{112} + F_{211}, & \theta_2^* = F_{111} + F_{221}, & \theta_3^* = F_{123} + F_{213},\\ \omega_1 = F_{331}, & \omega_2 = F_{332}, & \omega_3 = 0.\end{array}$$

Then, if F_s $(s = 1,2,\dots,11)$ are the components of F in the corresponding basic classes $\mathcal{F}_s$ and $x = x^ie_i$, $y = y^je_j$, $z = z^ke_k$ for arbitrary vectors in T_pM, we have [8]:

$$\begin{aligned} &F_1(x,y,z) = \left(x^1\theta_1 - x^2\theta_2\right)\left(y^1z^1 + y^2z^2\right),\\ &\qquad \theta_1 = F_{111} = F_{122}, \qquad \theta_2 = -F_{211} = -F_{222};\\ &F_2(x,y,z) = F_3(x,y,z) = 0;\\ &F_4(x,y,z) = \frac{1}{2}\theta_3\Big\{x^1\left(y^3z^1 + y^1z^3\right) - x^2\left(y^3z^2 + y^2z^3\right)\Big\},\\ &\qquad \frac{1}{2}\theta_3 = F_{131} = F_{113} = -F_{232} = -F_{223};\end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&F_5(x,y,z) = \frac{1}{2}\theta_3^*\{x^1\left(y^3z^2+y^2z^3\right)+x^2\left(y^3z^1+y^1z^3\right)\},\\
&\qquad \frac{1}{2}\theta_3^* = F_{132} = F_{123} = F_{231} = F_{213};\\
&F_6(x,y,z) = F_7(x,y,z) = 0;\\
&F_8(x,y,z) = \lambda\{x^1\left(y^3z^1+y^1z^3\right)+x^2\left(y^3z^2+y^2z^3\right)\},\\
&\qquad \lambda = F_{131} = F_{113} = F_{232} = F_{223};\\
&F_9(x,y,z) = \mu\{x^1\left(y^3z^2+y^2z^3\right)-x^2\left(y^3z^1+y^1z^3\right)\},\\
&\qquad \mu = F_{132} = F_{123} = -F_{231} = -F_{213};\\
&F_{10}(x,y,z) = \nu x^3\left(y^1z^1+y^2z^2\right), \qquad \nu = F_{311} = F_{322};\\
&F_{11}(x,y,z) = x^3\{\left(y^1z^3+y^3z^1\right)\omega_1+\left(y^2z^3+y^3z^2\right)\omega_2\},\\
&\qquad \omega_1 = F_{313} = F_{331}, \qquad \omega_2 = F_{323} = F_{332}.
\end{aligned}
\tag{2.6}
$$

Obviously, the class of three-dimensional almost contact B-metric manifolds is

$$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}.$$

Let $R = [\nabla,\nabla] - \nabla_{[\ ,\ ]}$ be the curvature (1,3)-tensor of ∇. The corresponding curvature $(0,4)$-tensor is denoted by the same letter: $R(x,y,z,w) = g(R(x,y)z,w)$. The following properties are valid:

$$
\begin{aligned}
&R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z),\\
&R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0.
\end{aligned}
$$

It is known from [11] that every 3-dimensional cosymplectic B-metric manifold is flat, i.e. $R = 0$.

The Ricci tensor ρ and the scalar curvature τ for R as well as their associated quantities are defined respectively by

$$
\begin{aligned}
&\rho(y,z) = g^{ij}R(e_i,y,z,e_j), &&\tau = g^{ij}\rho(e_i,e_j),\\
&\rho^*(y,z) = g^{ij}R(e_i,y,z,\varphi e_j), &&\tau^* = g^{ij}\rho^*(e_i,e_j),
\end{aligned}
$$

where $\{e_1, e_2, \dots, e_{2n+1}\}$ is an arbitrary basis of T_pM.

Let α be a non-degenerate 2-plane (section) in T_pM. It is known that the special 2-planes with respect to (φ,ξ,η,g) are: a *totally real section* if α is orthogonal to its φ-image $\varphi\alpha$, a *φ-holomorphic section* if α coincides with $\varphi\alpha$ and a *ξ-section* if ξ lies on α.

The sectional curvature $k(\alpha;p)(R)$ of α with an arbitrary basis $\{x,y\}$ at p is

$$k(\alpha;p)(R) = \frac{R(x,y,y,x)}{g(x,x)g(y,y)-g(x,y)^2}.$$

According to [9], a manifold M whose Ricci tensor satisfies

$$\rho = \lambda g + \mu\tilde{g} + \nu\eta\otimes\eta$$

is said to be an *η-complex-Einstein manifold.*

3. EQUIPPING OF EACH BIANCHI-TYPE LIE ALGEBRA WITH ALMOST CONTACT B-METRIC STRUCTURE

It is known that L. Bianchi has categorized all three-dimensional real (and complex) Lie algebras. He proved that every three-dimensional Lie algebra is isomorphic to one, and only one, Lie algebra of his list (cf. [1, 2]). These isomorphism classes form the so-called Bianchi classification and are noted by Bia(I), Bia(II), Bia(IV), Bia(V), Bia(VI_h) ($h \leq 0$), Bia(VII_h) ($h \geq 0$), Bia(VIII) and Bia(IX). The class Bia(III) coincides with Bia(VI_{-1}). The following theorem introduces the Bianchi classification.

Theorem A. ([1, 2]) *Let $\mathfrak{l}$ be a real three-dimensional Lie algebra. Then $\mathfrak{l}$ is isomorphic to exactly one of the following Lie algebras $(\mathbb{R}^3, [\cdot,\cdot])$, where the Lie bracket is given on the canonical basis $\{e_1, e_2, e_3\}$ as follows:*

$$
\begin{array}{llll}
\mathrm{Bia(I)}: & [e_1,e_2]=o, & [e_2,e_3]=o, & [e_3,e_1]=o;\\
\mathrm{Bia(II)}: & [e_1,e_2]=o, & [e_2,e_3]=e_1, & [e_3,e_1]=o;\\
\mathrm{Bia(IV)}: & [e_1,e_2]=o, & [e_2,e_3]=e_1-e_2, & [e_3,e_1]=e_1;\\
\mathrm{Bia(V)}: & [e_1,e_2]=o, & [e_2,e_3]=e_2, & [e_3,e_1]=e_1;\\
\mathrm{Bia(VI}_h)\ (h\leq 0): & [e_1,e_2]=o, & [e_2,e_3]=e_1-he_2, & [e_3,e_1]=he_1-e_2;\\
\mathrm{Bia(VII}_h)\ (h\geq 0): & [e_1,e_2]=o, & [e_2,e_3]=e_1-he_2, & [e_3,e_1]=he_1+e_2;\\
\mathrm{Bia(VIII)}: & [e_1,e_2]=-e_3, & [e_2,e_3]=e_1, & [e_3,e_1]=e_2;\\
\mathrm{Bia(IX)}: & [e_1,e_2]=e_3, & [e_2,e_3]=e_1, & [e_3,e_1]=e_2.
\end{array}
$$

Here, o is the zero vector in $\mathfrak{l}$.

The geometrization conjecture, associated with W. Thurston, states that every closed manifold of dimension three could be decomposed in a canonical way into pieces, connected to one of the eight types of Thurston's geometric structures ([13]): Euclidean geometry E^3, Spherical geometry S^3, Hyperbolic geometry H^3, the geometry of $S^2 \times \mathbb{R}$, the geometry of $H^2 \times \mathbb{R}$, the geometry of the universal cover $\widetilde{SL}(2,\mathbb{R})$ of the special linear group $SL(2,\mathbb{R})$, the Nil geometry, the $Solv$ geometry.

Seven of the eight Thurston geometries can be associated to a class of the Bianchi classification as it is shown in the following table. The Thurston geometry on $S^2 \times \mathbb{R}$ has no such a realization (see, e.g., [4]).

TABLE 1. Relations between the Bianchi types and the Thurston geometries

Bia(I)	E^3
Bia(II)	Nil
Bia(III)	$H^2 \times \mathbb{R}$
Bia(IV)	
Bia(V)	H^3
Bia(VI_0)	$Solv$

Bia($VI_{h<0}$)	
Bia(VII_0)	E^3
Bia($VII_{h>0}$)	
Bia(VIII)	$\widetilde{SL}(2,\mathbb{R})$
Bia(IX)	S^3

Let us consider each Lie algebra from the Bianchi classification, equipped with an almost contact structure (φ, ξ, η) and a B-metric g as in (2.3) and (2.4).

The presence of the structure (φ, ξ, η, g) gives us a reason to consider the relation between the Bianchi types and the classification of almost contact B-metric manifolds in [6].

We obtain immediately the following

Proposition 3.1. *Some Bianchi types can be equipped with a structure* (φ, ξ, η, g) *in several ways. In the cases* Bia(I) *and* Bia(IX) *there is only one variant. In the remaining cases, there are three possible subtypes of each type, obtained from each other by a cyclic change of the basic vectors* e_1, e_2 *and* e_3. *All subtypes are given in Table 2:*

TABLE 2. Equipping of the Bianchi types Lie algebras with a (φ, ξ, η, g) *structure*

Bia(I)			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = o,$	$[e_3, e_1] = o$
Bia(II)			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = o$
(2)	$[e_1, e_2] = o,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2$
(3)	$[e_1, e_2] = e_3,$	$[e_2, e_3] = o,$	$[e_3, e_1] = o$
Bia(III) $\equiv$ Bia(VI$_{-1}$)			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 + e_2,$	$[e_3, e_1] = -e_1 - e_2$
(2)	$[e_1, e_2] = -e_2 - e_3,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2 + e_3$
(3)	$[e_1, e_2] = e_1 + e_3,$	$[e_2, e_3] = -e_1 - e_3,$	$[e_3, e_1] = o$
Bia(IV)			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 - e_2,$	$[e_3, e_1] = e_1$
(2)	$[e_1, e_2] = e_2,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2 - e_3$
(3)	$[e_1, e_2] = -e_1 + e_3,$	$[e_2, e_3] = e_3,$	$[e_3, e_1] = o$
Bia(V)			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_2,$	$[e_3, e_1] = e_1$
(2)	$[e_1, e_2] = e_2,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_3$
(3)	$[e_1, e_2] = e_1,$	$[e_2, e_3] = e_3,$	$[e_3, e_1] = o$
Bia(VI$_h$), $h \leq 0$			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 - he_2,$	$[e_3, e_1] = he_1 - e_2$
(2)	$[e_1, e_2] = he_2 - e_3,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2 - he_3$
(3)	$[e_1, e_2] = -he_1 + e_3,$	$[e_2, e_3] = -e_1 + he_3,$	$[e_3, e_1] = o$
Bia(VII$_h$), $h \geq 0$			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 - he_2,$	$[e_3, e_1] = he_1 + e_2$
(2)	$[e_1, e_2] = he_2 + e_3,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2 - he_3$
(3)	$[e_1, e_2] = -he_1 + e_3,$	$[e_2, e_3] = e_1 + he_3,$	$[e_3, e_1] = o$
Bia(VIII)			
(1)	$[e_1, e_2] = -e_3,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = e_2$
(2)	$[e_1, e_2] = e_3,$	$[e_2, e_3] = -e_1,$	$[e_3, e_1] = e_2$
(3)	$[e_1, e_2] = e_3,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = -e_2$
Bia(IX)			
(1)	$[e_1, e_2] = e_3,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = e_2$

4. ALMOST CONTACT B-METRIC MANIFOLDS OF EACH BIANCHI TYPE

Let us consider the Lie group L corresponding to the given Lie algebra $\mathfrak{l}$. Each definition of a Lie algebra for the different subtypes in Proposition 3.1 generates a corresponding almost contact B-metric manifold denoted by $(L, \varphi, \xi, \eta, g)$. In this section we characterize the obtained manifolds with respect to the classification in [6].

Using (2.5)–(2.6), we obtain the corresponding components of F in each subtypes (1), (2), (3) in Proposition 3.1 and determine the corresponding class of almost contact B-metric manifolds. The results are given in the following

Theorem 4.1. *The manifold $(L, \varphi, \xi, \eta, g)$, determined by each type of Lie algebra given in Proposition 3.1, belongs to a class from the classification in* [6] *as given in Table 3:*

TABLE 3. Relations between the Bianchi types and the classes in [6]

Bia(I)	
(1)	$\mathcal{F}_0$
Bia(II)	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_4 \oplus \mathcal{F}_{10}$
(3)	$\mathcal{F}_8 \oplus \mathcal{F}_{10}$
Bia(III)	
(1)	$\mathcal{F}_5 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
Bia(IV)	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
Bia(V)	
(1)	$\mathcal{F}_9$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_{11}$
Bia(VI$_0$)	
(1)	$\mathcal{F}_{10}$
(2)	$\mathcal{F}_4 \oplus \mathcal{F}_8$
(3)	$\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$

Bia(VI$_h$), $h < 0$	
(1)	$\mathcal{F}_5 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
Bia(VII$_0$)	
(1)	$\mathcal{F}_4$
(2)	$\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$
(3)	$\mathcal{F}_4 \oplus \mathcal{F}_8$
Bia(VII$_h$), $h > 0$	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_5$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{11}$
Bia(VIII)	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_8 \oplus \mathcal{F}_{10}$
(3)	$\mathcal{F}_8 \oplus \mathcal{F}_{10}$
Bia(IX)	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$

Proof. We give our arguments for the case of Bia(II), the other cases are proven in a similar way.

Using Theorem A, Eq. (2.4) and the Koszul equality

$$2g\left(\nabla_{e_i} e_j, e_k\right) = g\left([e_i, e_j], e_k\right) + g\left([e_k, e_i], e_j\right) + g\left([e_k, e_j], e_i\right),$$

we obtain the components of the Levi-Civita connection ∇ of g. Then, by them, (2.1) and (2.3), we get the following non-zero components F_{ijk} and θ_k for the different subtypes:

$$\begin{array}{ll}
(1) & F_{113}=F_{131}=-F_{223}=-F_{232}=-\frac{1}{2}, \quad F_{311}=F_{322}=-1, \quad \theta_3=-1;\\
(2) & F_{113}=F_{131}=-F_{223}=-F_{232}=-\frac{1}{2}, \quad F_{311}=F_{322}=1, \quad \theta_3=-1;\\
(3) & F_{113}=F_{131}=F_{223}=F_{232}=\frac{1}{2}, \quad F_{311}=F_{322}=1.
\end{array}$$

Bearing in mind (2.5)–(2.6), we conclude that the corresponding classes of each subtype of Bia(II) are as follows:

$$\begin{array}{ll}
(1) & (L,\varphi,\xi,\eta,g)\in\mathcal{F}_4\oplus\mathcal{F}_{10};\\
(2) & (L,\varphi,\xi,\eta,g)\in\mathcal{F}_4\oplus\mathcal{F}_{10};\\
(3) & (L,\varphi,\xi,\eta,g)\in\mathcal{F}_8\oplus\mathcal{F}_{10}.
\end{array}$$

□

5. CURVATURE PROPERTIES OF THE CONSIDERED MANIFOLDS IN SOME BIANCHI CLASSES

Now we focuss our considerations on the Bianchi classes depending on a real parameter h. They are Bia(VI$_h$) and Bia(VII$_h$). Actually, these two classes are families of manifolds whose properties are functions of h. The classes regarding F corresponding to Bia(VI$_h$), $h<0$ and Bia(VII$_h$), $h>0$, according to Theorem 4.1, can not be restricted for special values of h.

In this section our interest is in the curvature properties of these manifolds in terms of h.

In view of Proposition 3.1, it is reasonable to investigate all three subtypes of the Bianchi classes Bia(VI$_h$), $h\leq 0$ and Bia(VII$_h$), $h\geq 0$.

5.1. Bia(VI$_h$), $h\leq 0$.

Let us consider subtype (1) of this Bianchi class as given in Proposition 3.1:

$$[e_1,e_2]=o, \quad [e_2,e_3]=e_1-he_2, \quad [e_3,e_1]=he_1-e_2.$$

We calculate the non-zero components of ∇ for Bia(VI$_h$):

$$\begin{array}{lll}
\nabla_{e_1}e_1=he_3, & \nabla_{e_1}e_3=-he_1, & \nabla_{e_2}e_2=-he_3,\\
\nabla_{e_2}e_3=-he_2, & \nabla_{e_3}e_1=-e_2, & \nabla_{e_3}e_2=-e_1.
\end{array} \tag{5.1}$$

Using (2.2), (2.3), (2.4) and (5.1), we obtain for the square norm of $\nabla\varphi$

$$\|\nabla\varphi\|^2=4(2-h^2). \tag{5.2}$$

Further, we calculate the basic components $R_{ijkl}=R(e_i,e_j,e_k,e_l)$ of the curvature tensor R, $\rho_{jk}=\rho(e_j,e_k)$ of the Ricci tensor ρ, $\rho^*_{jk}=\rho^*(e_j,e_k)$ of the associated Ricci tensor ρ^*, the values of the scalar curvatures τ and τ^* and of the sectional curvatures $k_{ij} = k(e_i, e_j)$. They are as follows:

$$\begin{gathered} R_{1212} = -R_{1313} = R_{2323} = -h^2; \\ \rho_{11} = -\rho_{22} = \rho_{33} = -2h^2, \qquad \rho^*_{12} = \rho^*_{21} = -h^2; \\ \tau = -6h^2, \qquad \tau^* = 0; \\ k_{12} = k_{13} = k_{23} = -h^2. \end{gathered} \tag{5.3}$$

Using (5.3) we obtain the following

Proposition 5.1. *In the case* Bia(VI_h), *subtype (1), the following statements are valid:*

1). $(L, \varphi, \xi, \eta, g)$ *is flat if and only if* $h = 0$*;*

2). $(L, \varphi, \xi, \eta, g)$ *is an isotropic-cosymplectic B-metric manifold if and only if* $h = -\sqrt{2}$*;*

3). *The scalar curvature and the sectional curvatures are constant and non-positive;*

4). $(L, \varphi, \xi, \eta, g)$ *is* $*$*-scalar flat, i.e.* $\tau^* = 0$*;*

5). $(L, \varphi, \xi, \eta, g)$ *is an Einstein manifold.*

In the same fashion we obtain the analogues of (5.2) and (5.3) and derive the corresponding propositions in the remaining cases. For subtype (2) we have:

$$\begin{gathered} \|\nabla\varphi\|^2 = 2(1 - 5h^2); \\ R_{1212} = -R_{1313} = R_{2323} = -h^2; \\ \rho_{11} = -\rho_{22} = \rho_{33} = -2h^2, \qquad \rho^*_{12} = \rho^*_{21} = -h^2; \\ \tau = -6h^2, \qquad \tau^* = 0; \\ k_{12} = k_{13} = k_{23} = -h^2, \end{gathered}$$

whence we deduce the following

Proposition 5.2. *In the case* Bia*(VI_h), subtype (2), all the statements from Proposition 5.1 hold true, with* $h = -\sqrt{2}$ *replaced by* $h = -\frac{\sqrt{5}}{5}$ *in statement 2).*

In the case of subtype (3) we obtain:

$$\begin{gathered} \|\nabla\varphi\|^2 = 10(h^2 + 1); \\ R_{1212} = R_{2323} = h^2 + 1, \qquad R_{1313} = 1 - h^2, \qquad R_{1223} = 2h; \\ \rho_{11} = \rho_{33} = 2h^2, \qquad \rho_{13} = \rho_{31} = -2h, \qquad \rho_{22} = -2(h^2 + 1); \\ \rho^*_{12} = \rho^*_{21} = h^2 + 1, \qquad \rho^*_{23} = \rho^*_{32} = -2h; \\ \tau = 2(3h^2 + 1), \qquad \tau^* = 0; \\ k_{12} = k_{23} = h^2 + 1, \qquad k_{13} = h^2 - 1. \end{gathered}$$

The latter equalities imply

Proposition 5.3. *In the case* Bia*(VI_h), subtype (3), the following statements are valid:*

1). The square norm of $\nabla\varphi$ and the scalar curvature are positive;

2). (L,φ,ξ,η,g) is $*$-scalar flat;

3). The sectional curvatures of the φ-holomorphic sections are constant and positive.

5.2. Bia(VII_h), $h \geq 0$.

Here we focus on the three subtypes of Bia(VII_h). Firstly, let us consider the subtype (1). As in the previous subsection, we find:

$$\begin{gathered}
\|\nabla\varphi\|^2 = 4(1-h^2);\\
R_{1212} = -(h^2+1), \qquad R_{1313} = -R_{2323} = h^2-1, \qquad R_{1323} = -2h;\\
\rho_{11} = -\rho_{22} = -2h^2, \qquad \rho_{12} = \rho_{21} = 2h, \qquad \rho_{33} = 2(1-h^2);\\
\rho^*_{12} = \rho^*_{21} = -(h^2+1), \qquad \rho^*_{33} = 4h;\\
\tau = 2(1-3h^2), \qquad \tau^* = 4h;\\
k_{12} = -(h^2+1), \qquad k_{13} = k_{23} = 1-h^2.
\end{gathered}$$

Applying these results we obtain

Proposition 5.4. *In the case* Bia(VII_h)*, subtype (1), the following statements are valid:*

1). (L,φ,ξ,η,g) is an isotropic-cosymplectic B-metric manifold if and only if $h=1$;

2). (L,φ,ξ,η,g) is scalar flat if and only if $h=\frac{\sqrt{3}}{3}$;

3). (L,φ,ξ,η,g) is $*$-scalar flat if and only if $h=0$;

4). The sectional curvatures of the φ-holomorphic sections are constant and negative;

5). The sectional curvatures of the ξ-sections are constant;

6). (L,φ,ξ,η,g) is an η-complex-Einstein manifold.

Analogously, we get the corresponding results for subtype (2):

$$\begin{gathered}
\|\nabla\varphi\|^2 = -10(h^2-1);\\
R_{1212} = -R_{1313} = -(h^2-1), \qquad R_{2323} = -(h^2+1), \qquad R_{1213} = 2h;\\
\rho_{11} = -2(h^2-1), \qquad \rho_{22} = -\rho_{33} = 2h^2, \qquad \rho_{23} = \rho_{32} = -2h;\\
\rho^*_{12} = \rho^*_{21} = -(h^2-1), \qquad \rho^*_{13} = \rho^*_{31} = 2h;\\
\tau = -2(3h^2-1), \qquad \tau^* = 0;\\
k_{12} = k_{13} = -(h^2-1), \qquad k_{23} = -(h^2+1).
\end{gathered}$$

The latter equalities imply the following

Proposition 5.5. *In the case* Bia*(VII_h), subtype (2), the following statements are valid:*

1). $(L, \varphi, \xi, \eta, g)$ is an isotropic-cosymplectic B-metric manifold if and only if $h = 1$;

2). $(L, \varphi, \xi, \eta, g)$ is scalar flat if and only if $h = \frac{\sqrt{3}}{3}$;

3). $(L, \varphi, \xi, \eta, g)$ is $*$-scalar flat;

4). $(L, \varphi, \xi, \eta, g)$ is horizontal flat, i.e. $R|_H = 0$ for $H = \ker(\eta)$, if and only if $h = 1$;

5). ρ^* and $\tilde{g}$ are proportional on H as $\rho^*|_H = (h^2 - 1)\tilde{g}|_H$;

6). $(L, \varphi, \xi, \eta, g)$ is horizontal $*$-Ricci flat, i.e. $\rho^*|_H = 0$, if and only if $h = 1$.

Finally, for the case of the subtype (3) we have:

$$\begin{gathered}
\|\nabla\varphi\|^2 = 2(5h^2 + 1); \\
R_{1212} = -R_{1313} = R_{2323} = h^2; \\
\rho_{11} = -\rho_{22} = \rho_{33} = 2h^2; \\
\rho^*_{12} = \rho^*_{21} = h^2; \\
\tau = 6h^2, \qquad \tau^* = 0; \\
k_{12} = k_{13} = k_{23} = h^2,
\end{gathered}$$

whence we deduce our last proposition:

Proposition 5.6. *In the case* Bia*(VII_h), subtype (3), the following statements are valid:*

1). $(L, \varphi, \xi, \eta, g)$ is flat if and only if $h = 0$;

2). The square norm of $\nabla\varphi$ is positive;

3). $(L, \varphi, \xi, \eta, g)$ is $*$-scalar flat;

4). The scalar curvature and the sectional curvatures are constant and non-negative;

5). $(L, \varphi, \xi, \eta, g)$ is an Einstein manifold.

REFERENCES

1. Bianchi, L.: Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti. *Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, Serie Terza*, **11**, 1898, 267–352.
2. Bianchi, L.: On the three-dimensional spaces which admit a continuous group of motions. *Gen. Rel. Grav.*, **33**, 2001, 2171–2253.

3. Blair, D. E.: *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, **203**, Birkhäuser, Boston, 2002.
4. Fagundes, H.: Closed spaces in cosmology. *Gen. Rel. Grav.*, **24**, 1992, 199–217.
5. Ganchev, G., Borisov, A.: Note on the almost complex manifolds with a Norden metric. *C. R. Acad. Bulg. Sci.*, **39**, 1986, 31–34.
6. Ganchev, G., Mihova, V., Gribachev, K.: Almost contact manifolds with B-metric. *Math. Balkanica (N.S.)*, **7**, (3-4), 1993, 261–276.
7. Gribachev, K., Mekerov, D., Djelepov, G.: Generalized B-manifolds. *C. R. Acad. Bulg. Sci.*, **38**, 1985, 299–302.
8. Manev, H.: On the structure tensors of almost contact B-metric manifolds. *Filomat*, **29**, (3), 2015, 427–436.
9. Manev, H., Mekerov, D.: Lie groups as 3-dimensional almost contact B-metric manifolds. *J. Geom.*, **106**, 2015, 229–242.
10. Manev, M.: Natural connection with totally skew-symmetric torsion on almost contact manifolds with B-metric. *Int. J. Geom. Methods Mod. Phys.*, **9**, (5), 2012, 1250044 (20 pages).
11. Manev, M., Nakova, G.: Curvature properties of some three-dimensional almost contact B-metric manifolds. *Plovdiv Univ. Sci. Works – Math.*, **34**, (3), 2004, 51–60.
12. Nakova, G., Manev, M.: Curvature properties on some three-dimensional almost contact manifolds with B-metric, II. In: *Proc. 5th Internat. Conf. Geometry, Integrability & Quantization* V, (I. M. Mladenov and A. C. Hirshfeld, eds.), 2004, 169–177.
13. Thurston, W.: *Three-dimensional geometry and topology*, Vol. 1. Princeton University Press, 1997.

Received on January 6, 2015

Hristo M. Manev

Department of Pharmaceutical Sciences
Faculty of Pharmacy
Medical University of Plovdiv
15-A blvd. Vasil Aprilov, BG-4002 Plovdiv
BULGARIA

and

Department of Algebra and Geometry
Faculty of Mathematics and Informatics
Paisii Hilendarski University of Plovdiv
236 blvd. Bulgaria, BG-4027 Plovdiv
BULGARIA

e-mail: hmanev@uni-plovdiv.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

ASYMPTOTICALLY OPTIMAL QUADRATURE FORMULAE IN CERTAIN SOBOLEV CLASSES

ANA AVDZHIEVA, GENO NIKOLOV

We construct sequences of asymptotically optimal quadrature formulae in the Sobolev classes W_p^3 $(1 \le p \le \infty)$, and W_1^4. Sharp error bounds for these quadrature formulae are given.

Keywords: Quadrature formulae, Sobolev classes of functions, asymptotically optimal quadrature formulae, spline functions, Peano kernels, Euler-MacLaurin summation formulae

2000 Math. Subject Classification: 41A55, 65D30, 65D32

1. INTRODUCTION

We study quadrature formulae of the type

$$Q[f] = \sum_{i=1}^{n} a_i\, f(x_i), \qquad 0 \le x_1 < x_2 < \cdots < x_n \le 1\,, \tag{1.1}$$

that serve as an estimate for the definite integral

$$I[f] := \int_0^1 f(x)\, dx. \tag{1.2}$$

Throughout this paper π_k will stand for the set of algebraic polynomials of degree not exceeding k.

The classical approach for construction of quadrature formulae is based on the concept of *algebraic degree of precision*. The quadrature formula (1.1) is said to have algebraic degree of precision m (in short, $ADP(Q) = m$), if its remainder

$$R[Q; f] := I[f] - Q[f]$$

vanishes whenever $f \in \pi_m$, and $R[Q; f] \neq 0$ when f is a polynomial of degree $m+1$.

The ADP-concept is justified by the Weierstrass theorem about the density of algebraic polynomials in spaces of continuous functions on compacts. The pursuit of quadrature formulae (1.1) with the highest possible ADP leads to the well-known quadrature formulae of Gauss, Radau and Lobatto. The latter are uniquely determined by having ADP equal to $2n-1$, $2n-2$ and $2n-3$, respectively, where, in addition, the Radau quadrature formula has one fixed node being an end-point of the integration interval, and the Lobatto quadrature formula has two fixed nodes at the ends of the integration interval.

An alternative concept for evaluation of the quality of quadrature formulae emerged in the forties of the 20-th century, namely, the concept of optimality in a given class of functions. Its founders are A. Kolmogorov, A. N. Sard and S. M. Nikolskii. Let us briefly describe the setting of optimal quadrature formulae in a given class of functions.

Let X be a normed linear space of functions defined in $[0, 1]$, with a norm $\|\cdot\|$. For a quadrature formula Q of the form (1.1), we denote by $\mathcal{E}(Q, X)$ the largest possible error of Q for functions from the unit ball of X, i.e.

$$\mathcal{E}(Q, X) := \sup_{\|f\|_X \leq 1} |R[Q; f]|.$$

We look for the best possible choice of the coefficients $\{a_i\}_{i=1}^n$ and the nodes $\{x_i\}_{i=1}^n$ of Q, and set

$$\mathcal{E}_n(X) := \inf_{Q} \mathcal{E}(Q, X).$$

If the infimum is attained for a quadrature formula Q^{opt} of the form (1.1), then Q^{opt} is said to be an *optimal quadrature formula* of the type (1.1) in the space X. Of particular interest is the case when X is some of the Sobolev classes of functions $\widetilde{W}_p^r$ and W_p^r, defined by

$$\widetilde{W}_p^r := \{f \in C^{r-1}[0,1],\ f -\ 1\text{–periodic}\ ,\ f^{(r-1)}\ \text{ abs. cont. }, \ \|f\|_p < \infty\},$$

$$W_p^r := \{f \in C^{r-1}[0,1],\ f^{(r-1)}\ \text{ abs. cont. }, \ \|f\|_p < \infty\},$$

where

$$\|f\|_p := \Big(\int_0^1 |f(t)|^p dt\Big)^{1/p}, \ \text{ if }\ 1 \leq p < \infty, \ \text{ and }\ \|f\|_\infty = \sup_{t\in(0,1)} \text{vrai}\, |f(t)|.$$

In the periodic Sobolev classes $\widetilde{W}_p^r$ there is an universal optimal quadrature formula (i.e. optimal for all $r \in \mathbb{N}$ and $p \geq 1$) of the form (1.1), namely, the n-point rectangles quadrature formula and its translates. This is a result due to Zhensykbaev [14], special cases have been obtained earlier by Motornii [10], and Ligun [9]. The existence and uniqueness of optimal quadrature formulae in the non-periodic Sobolev spaces W_p^r is equivalent to the existence and uniqueness of specific monosplines of degree r with a minimal L_q-deviation from zero, $(1/p + 1/q = 1)$. This was proved by Zhensykbaev [15], and Bojanov extended Zhensykbaev's result to more general classes of quadrature formulae involving derivatives of the integrand. Obviously, $\mathcal{E}_n(\widetilde{W}_p^r) \leq \mathcal{E}_n(W_p^r)$, and it is known that (see Brass [6]) for $1 < p \leq \infty$,

$$\lim_{n\to\infty} \frac{\mathcal{E}_n(\widetilde{W}_p^r)}{\mathcal{E}_n(W_p^r)} = 1.$$

A drawback of the optimality concept is that, in general, the explicit form of the optimal quadrature formulae is unknown, a fact that vitiates their importance from practical point of view. In particular, except for some special cases of $r = 1$ and $r = 2$, the optimal quadrature formulae in the non-periodic Sobolev spaces W_p^r are unknown.

The way out of this situation is to step back from the requirement for optimality, and to look for quadrature formulae which are nearly optimal. A sequence $\{Q_n\}$ of quadrature formulae is said to be asymptotically optimal in the function class X, if

$$\lim_{n\to\infty} \frac{\mathcal{E}(Q_n, X)}{\mathcal{E}_n(X)} = 1$$

(here, Q_n is supposed to be a quadrature formula with n nodes).

It has been shown in [8] that the Gauss-type quadrature formulae associated with the spaces of spline functions with equidistant knots are asymptotically optimal in the non-periodic Sobolev classes W_p^r. The existence and uniqueness of such Gauss-type quadrature formulae is equivalent to the fundamental theorem of algebra for monosplines satisfying zero boundary conditions, which was proved in [7]. This fact was a motivation for investigation of such quadratures. Algorithms for the construction along with sharp error estimates of the Gauss-type quadrature formulae associated with spaces of linear and parabolic spline functions were proposed in [11] and [13] (see also [12] for the case of cubic splines with double equidistant knots). Recently, an algorithm for the construction of Gaussian quadrature formulae associated with spaces of cubic splines with equidistant knots was proposed in [1].

It should be noted that the complexity of the algorithms for the construction of Gauss-type quadrature formulae associated with spaces of spline functions with equidistant knots increases with increasing of the degree (that is, of parameter r in W_p^r). For $r \geq 3$ such quadratures are constructed only numerically. This requires high accuracy computations, especially when the number of the nodes is large. An additional difficulty causes the fact that the mutual location of the spline knots and the quadratures nodes is unknown.

In [2] we proposed an alternative approach for generation of sequences of asymptotically optimal quadrature formulae. There we constructed sequences of asymptotically optimal quadrature formulae in the Sobolev classes W_p^4, for $p = 2$ and $p = \infty$. Our approach makes use of Euler–MacLaurin–type summation formulae, in which the derivatives are replaced by suitable formulae for numerical differentiation. An advantage of our quadrature formulae, besides their asymptotical optimality, is the explicit form of their weights and nodes. In fact, most of the nodes of our quadrature formulae are either those of the compound trapezium or of the compound midpoint quadratures, to which we add a few more nodes.

Here we continue our study on this subject. The paper is organized as follows. In Section 2 we provide some well-known facts, including the Peano kernel representation of linear functionals, the Bernoulli polynomials, monosplines and numbers, the Euler–MacLaurin–type expansion formulae, and the error representation of the compound trapezium and midpoint quadrature formulae in the periodic Sobolev classes $\widetilde{W}_p^r$. In Section 3 we construct some sequences of asymptotically optimal quadrature formulae in the non-periodic Sobolev classes W_1^3, $1 \le p \le \infty$, and evaluate their sharp error constants in the cases $p = 1, 2, \infty$. In Section 4 we construct two sequences of asymptotically optimal quadrature formulae in the Sobolev classes W_1^4. Section 5 contains some concluding remarks.

2. PRELIMINARIES

2.1. SPLINE FUNCTIONS AND PEANO KERNELS OF LINEAR FUNCTIONALS

A spline function of degree $r-1$ ($r \in \mathbb{N}$) with knots $x_1 < x_2 < \cdots < x_n$ is a function $s(t)$ satisfying the requirements

$$\begin{aligned} &1) \qquad s(t)_{|t\in(x_i,x_{i+1})} \in \pi_{r-1}, \quad i = 0,\dots,n,\\ &2) \qquad s(t) \in C(\mathbb{R}), \end{aligned}$$

where $x_0 := -\infty$ and $x_{n+1} := \infty$. The set $S_{r-1}(x_1,\dots,x_n)$ of spline functions of degree $r-1$ with knots $x_1 < x_2 < \cdots < x_n$ is a linear space of dimension $n+r$, and a basis of $S_{r-1}(x_1,\dots,x_n)$ is given by the functions

$$\{1, t, \dots, t^{r-1}, (t-x_1)_+^{r-1}, \dots, (t-x_n)_+^{r-1}\},$$

where $u_+(t)$ is defined by

$$u_+(t) = \max\{t, 0\}, \quad t \in \mathbb{R}.$$

If $\mathcal{L}$ is a linear functional defined on $C[0,1]$ which vanishes on π_s, then by a classical result of Peano, for $r \in \mathbb{N}$, $1 \le r \le s+1$ and $f \in W_1^r$, $\mathcal{L}$ admits the integral representation

$$\mathcal{L}[f] = \int_0^1 K_r(t) f^{(r)}(t)\,dt, \qquad \text{where} \qquad K_r(t) = \mathcal{L}\Big[\frac{(\cdot - t)_+^{r-1}}{(r-1)!}\Big], \quad t \in [0,1].$$

In the case when $\mathcal{L}$ is the remainder $R[Q;\cdot]$ of a quadrature formula Q with algebraic degree of precision s, the function $K_r(t)=K_r(Q;t)$ is referred to as the r-th Peano kernel of Q. For Q as in (1.1), explicit representations for $K_r(Q;t)$, $t\in[0,1]$, are

$$K_r(Q;t)=\frac{(1-t)^r}{r!}-\frac{1}{(r-1)!}\sum_{i=1}^{n}a_i(x_i-t)_+^{r-1}, \tag{2.1}$$

$$K_r(Q;t)=(-1)^r\Big[\frac{t^r}{r!}-\frac{1}{(r-1)!}\sum_{i=1}^{n}a_i(t-x_i)_+^{r-1}\Big]. \tag{2.2}$$

If the integrand f belongs to the Sobolev class W_p^r, $(1\le p\le\infty)$, then from

$$R[Q;f]=\int_0^1 K_r(Q;t)f^{(r)}(t)\,dt$$

and from Hölder's inequality one obtains the sharp error estimate

$$|R[Q;f]|\le c_{r,p}(Q)\|f^{(r)}\|_p,\quad\text{where}\quad c_{r,p}(Q)=\|K_r(Q;\cdot)\|_q,\quad p^{-1}+q^{-1}=1. \tag{2.3}$$

In other words, we have $\mathcal{E}(Q,W_p^r)=c_{r,p}(Q)$. Throughout, $c_{r,p}(Q)$ will be referred to as *the error constant of* Q *in the Sobolev class* W_p^r.

$K_r(Q;t)$ is also called a monospline of degree r with knots $\{x_i\,:\,x_i\in(0,1)\}$. From $K_r(Q;x)=R[Q;(\cdot-x)_+^{r-1}/(r-1)!]$ we deduce that $K_r(Q;x)=0$ for some $x\in(0,1)$ if and only if Q evaluates to the exact value the integral of the spline function $f(t)=(t-x)_+^{r-1}$. Thus, in order that a quadrature formula Q has *maximal spline degree of precision*, i.e., Q is exact for a space of spline functions of degree $r-1$ with a maximal dimension, it is necessary and sufficient that the corresponding monospline $K_r(Q;\cdot)$ has maximal number of zeros in $(0,1)$. Quadrature formulae of the form (1.1) with maximal spline degree of precision are called, analogously to the classical algebraic case, as Gauss, Radau, and Lobatto quadrature formulae, associated with the corresponding spaces of spline functions. Similarly to the classical Gauss–type quadrature formulae, all the nodes of the Gauss-type quadratures associated with spaces of spline functions lie in the integration interval, and all their weights are positive [7, Theorem 7.1].

2.2. BERNOULLI POLYNOMIALS AND MONOSPLINES. EULER–MACLAURIN TYPE SUMMATION FORMULAE

Recall that the Bernoulli polynomials B_ν are defined recursively by

$$B_0(x)=1,\quad B_\nu'(x)=B_{\nu-1}(x),\quad\text{and}\quad\int_0^1 B_\nu(t)\,dt=0,\quad \nu\in\mathbb{N}.$$

In particular, $B_1(x)=x-\dfrac{1}{2}$, $B_2(x)=\dfrac{x^2}{2}-\dfrac{x}{2}+\dfrac{1}{12}$, $B_3(x)=\dfrac{x^3}{6}-\dfrac{x^2}{4}+\dfrac{x}{12}$,

$$B_4(x)=\frac{x^2(1-x)^2}{24}-\frac{1}{720}.$$

The Bernoulli numbers $\mathcal{B}_\nu$ are defined by $\mathcal{B}_\nu = \dfrac{B_\nu(0)}{\nu!}$.

The notation $\widetilde{B}_\nu(x)$ stands for the 1-periodic extension of the Bernoulli polynomial $B_\nu(x)$ on $\mathbb{R}$. The functions $\widetilde{B}_\nu(x)$, $\nu = 0, 1, \dots$, are called Bernoulli monosplines.

Throughout this paper, $n \in \mathbb{N}$ will be fixed, and $\{x_{k,n}\}_{k=0}^n$ and $\{y_{\ell,n}\}_{\ell=1}^n$ are given by

$$x_{k,n} = \frac{k}{n}, \quad k = 0, \dots, n; \qquad y_{\ell,n} = \frac{2\ell - 1}{2n}, \quad \ell = 1, \dots, n\,. \tag{2.4}$$

The points $\{x_{k,n}\}_{k=0}^n$ and $\{y_{\ell,n}\}_{\ell=1}^n$ are the nodes of the n-th compound trapezium and midpoint quadrature formulae Q_{n+1}^{Tr} and Q_n^{Mi}, given by

$$Q_{n+1}^{Tr}[f] = \frac{1}{2n}\big(f(x_{0,n}) + f(x_{n,n})\big) + \frac{1}{n}\sum_{k=1}^{n-1} f(x_{k,n})\,, \tag{2.5}$$

$$Q_n^{Mi}[f] = \frac{1}{n}\sum_{k=1}^{n-1} f(y_{k,n})\,. \tag{2.6}$$

Our asymptotically optimal quadrature formulae are obtained as appropriate modifications of Q_{n+1}^{Tr} and Q_n^{Mi}.

The following summation formulae of Euler–MacLaurin type (adopted for the interval $[0,1]$) are well-known, see, e.g., [6, Satz 98, 99]:

Lemma 1. *Assume that $f \in W_1^s$. Then*

$$\begin{aligned}\int_0^1 f(x)\,dx =& Q_{n+1}^{Tr}[f] - \sum_{\nu=1}^{[\frac{s}{2}]} \frac{\mathcal{B}_{2\nu}}{(2\nu)!}\,\frac{f^{(2\nu-1)}(1) - f^{(2\nu-1)}(0)}{n^{2\nu}}\\ &+ \frac{(-1)^s}{n^s}\int_0^1 \widetilde{B}_s(nx) f^{(s)}(x)\,dx\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\int_0^1 f(x)\,dx =& Q_n^{Mi}[f] + \sum_{\nu=1}^{[\frac{s}{2}]} \left(1 - 2^{1-2\nu}\right) \frac{\mathcal{B}_{2\nu}}{(2\nu)!}\,\frac{f^{(2\nu-1)}(1) - f^{(2\nu-1)}(0)}{n^{2\nu}}\\ &+ \frac{(-1)^s}{n^s}\int_0^1 \widetilde{B}_s\Big(nx - \frac{1}{2}\Big) f^{(s)}(x)\,dx\,.\end{aligned} \tag{2.8}$$

Here, $[t]$ denotes the integer part of t.

As was already mentioned, the midpoint quadrature formulae $\{Q_n^{Mi}\}_{n=1}^{\infty}$ and their translates are the unique optimal quadrature formulae in the periodic Sobolev classes $\widetilde{W}_p^r$. The trapezium quadrature formulae $\{Q_{n+1}^{Tr}\}_{n=1}^{\infty}$ also can be considered as translates of $\{Q_n^{Mi}\}_{n=1}^{\infty}$, as the values of the integrand at the endpoints are equal. For $f \in \widetilde{W}_p^s$, $1 \le p \le \infty$, the sums in the right-hand sides of (2.7) and (2.8) disappear, due to the periodicity of the integrand. Hence we obtain

$$R[Q_{n+1}^{Tr}; f] = \frac{(-1)^s}{n^s} \int_0^1 \Big[\widetilde{B}_s(nx) - d\Big] f^{(s)}(x)\, dx \tag{2.9}$$

and

$$R[Q_n^{Mi}; f] = \frac{(-1)^s}{n^s} \int_0^1 \Big[\widetilde{B}_s\Big(nx - \frac{1}{2}\Big) - d\Big] f^{(s)}(x)\, dx\,, \tag{2.10}$$

where d is an arbitrary constant. Applying Hölder's inequality to (2.9) and (2.10), and taking into account that Q_{n+1}^{Tr} and Q_n^{Mi} are optimal quadrature formulae in $\widetilde{W}_p^s$, we obtain

$$|R[Q_{n+1}^{Tr}; f[| \le \mathcal{E}_n(\widetilde{W}_p^s)\, \|f^{(s)}\|_p\,, \quad |R[Q_n^{Mi}; f[| \le \mathcal{E}_n(\widetilde{W}_p^s)\, \|f^{(s)}\|_p\,,$$

where

$$\mathcal{E}_n(\widetilde{W}_p^s) = \frac{1}{n^s} \inf_d \|B_s - d\|_q =: \|B_s - d_{s,p}\|_q\,, \qquad \frac{1}{p} + \frac{1}{q} = 1\,. \tag{2.11}$$

Some known values of the constant $d_{s,p}$ are (see, e.g., [14])

$$d_{s,p} = 0 \quad \text{for odd} \quad s \in \mathbb{N} \quad \text{and} \quad 1 \le p \le \infty, \tag{2.12}$$

$$d_{s,p} = \begin{cases} 2^{-s} B_s(0) & \text{for even} \quad s \in \mathbb{N} \quad \text{and} \quad p = 1, \\ 0 & \text{for all} \quad s \in \mathbb{N} \quad \text{and} \quad p = 2, \\ B_s\big(\frac{1}{4}\big) & \text{for even} \quad s \in \mathbb{N} \quad \text{and} \quad p = \infty\,. \end{cases} \tag{2.13}$$

We shall need constants $\mathcal{E}_n(\widetilde{W}_p^s)$ for $s = 3$, 4 and $p = 1$, 2 and ∞. In the case $s = 3$, these constants are

$$\mathcal{E}_n(\widetilde{W}_\infty^3) = \frac{1}{n^3}\, \|B_3\|_1 = \frac{1}{192\, n^3}\,, \tag{2.14}$$

$$\mathcal{E}_n(\widetilde{W}_2^3) = \frac{1}{n^3}\, \|B_3\|_2 = \frac{1}{12\sqrt{210}\, n^3}\,, \tag{2.15}$$

$$\mathcal{E}_n(\widetilde{W}_1^3) = \frac{1}{n^3}\, \|B_3\|_\infty = \frac{1}{72\sqrt{3}\, n^3}\,. \tag{2.16}$$

In the case $s = 4$, the corresponding constants are

$$\mathcal{E}_n(\widetilde{W}^4_\infty) = \frac{1}{n^4}\, \|B_4(\cdot) - B_4(1/4)\|_1 = \frac{5}{6144\, n^4}\,, \tag{2.17}$$

$$\mathcal{E}_n(\widetilde{W}^4_2) = \frac{1}{n^4}\, \|B_4\|_2 = \frac{1}{240\sqrt{21}\, n^4}\,, \tag{2.18}$$

$$\mathcal{E}_n(\widetilde{W}^4_1) = \frac{1}{n^4}\, \|B_4(\cdot) - 2^{-4}B_4(0)\|_\infty = \frac{1}{768\, n^4}\,. \tag{2.19}$$

3. ASYMPTOTICALLY OPTIMAL QUADRATURE FORMULAE IN W^3_p

Let us start with a brief outline of our method for the construction of asymptotically optimal quadrature formulae in the Sobolev classes W^3_p.

The Euler–MacLaurin summation formulae in Lemma 1 in the case $s = 3$ reduce to

$$\int_0^1 f(x)\, dx = Q^{Tr}_{n+1}[f] - \frac{1}{12\, n^2}\big[f'(1) - f'(0)\big] + \frac{1}{n^3}\int_0^1 \widetilde{B}_3(nx)\, f^{(3)}(x)\, dx \tag{3.1}$$

and

$$\int_0^1 f(x)\, dx = Q^{Mi}_{n}[f] + \frac{1}{24\, n^2}\big[f'(1) - f'(0)\big] + \frac{1}{n^3}\int_0^1 \widetilde{B}_3\Big(nx - \frac{1}{2}\Big)\, f^{(3)}(x)\, dx\,. \tag{3.2}$$

The derivatives $f'(0)$ and $f'(1)$ appearing in the right-hand side of (3.1) and (3.2) will be replaced by suitable formulae for numerical differentiation. For the sake of brevity, we give the following definition.

Definition 1. Given $0 \le t_1 < t_2 < t_3 < 1$, we denote by $D_1(t_1, t_2, t_3)[f]$ the interpolatory formula for numerical differentiation with nodes $\{t_i\}_{i=1}^3$, which approximates $f'(0)$, i.e.

$$D_1[f] = D_1(t_1, t_2, t_3)[f] = \sum_{i=1}^{3} c_i\, f(t_i) \approx f'(0)\,.$$

We shall use formulae for numerical differentiation with $t_3 = O(n^{-1})$. For instance, such a formula is

$$D_1(x_{0,n}, y_{1,n}, x_{2,n})[f] = \frac{n}{6}\big[-15 f(x_{0,n}) + 16 f(y_{1,n}) - f(x_{2,n})\big]\,.$$

For the sake of simplicity, $f'(1)$ is approximated by a numerical differentiation formula, obtained from $D_1(t_1,t_2,t_3)[f]$ by a reflection, i.e.,

$$f'(1) \approx \widetilde{D}_1[f] := D_1(t_1,t_2,t_3)[g]\,, \qquad g(t) = -f(1-t)\,.$$

The linear functionals $L[f] := f'(0) - D_1[f]$ and $\widetilde{L}[f] := f'(1) - \widetilde{D}_1[f]$ vanish on π_2, and by Peano's theorem, for $f \in W_1^3$ they are representable in the form

$$L[f] = \int_0^1 K_3(L;t) f'''(t)\,dt\,, \qquad \widetilde{L}[f] = \int_0^1 K_3(\widetilde{L};t) f'''(t)\,dt$$

with $K_3(L;t) = L\big[(\cdot - t)_+^2/2\big]$ and $K_3(\widetilde{L};t) = \widetilde{L}\big[(\cdot - t)_+^2/2\big]$. This representation also implies

$$\begin{aligned} K_3(L;t) &\equiv 0 \quad \text{for } t \in (t_3, 1]\,, \\ K_3(\widetilde{L};t) &\equiv 0 \quad \text{for } t \in [0, 1-t_3)\,. \end{aligned}$$

Replacement in (3.1) of $f'(0)$ and $f'(1)$ by $D_1[f]$ and $\widetilde{D}_1[f]$, respectively, results in a new quadrature formula Q,

$$Q[f] = Q_{n+1}^{Tr}[f] + \frac{1}{12\,n^2} \sum_{i=1}^{3} c_i \,\big[f(t_i) + f(1-t_i)\big] \tag{3.3}$$

with at most $n+7$ nodes (including $\{x_{k,n}\}_{k=0}^{n}$), and a Peano kernel $K_3(Q;t)$ given by

$$K_3(Q;t) = \frac{1}{n^3}\,\widetilde{B}_3(n\,t) + \frac{1}{12\,n^2}\big[K_3(L;t) - K_3(\widetilde{L};t)\big]\,, \quad t \in [0,1]\,.$$

Analogously, replacement in (3.1) of $f'(0)$ and $f'(1)$ by $D_1[f]$ and $\widetilde{D}_1[f]$, respectively, yields a quadrature formula Q,

$$Q[f] = Q_n^{Mi}[f] - \frac{1}{24\,n^2} \sum_{i=1}^{3} c_i \,\big[f(t_i) + f(1-t_i)\big] \tag{3.4}$$

with at most $n+6$ nodes (including $\{y_{\ell,n}\}_{\ell=1}^{n}$), and a Peano kernel $K_3(Q;t)$ given by

$$K_3(Q;t) = \frac{1}{n^3}\,\widetilde{B}_3\Big(nx - \frac{1}{2}\Big) - \frac{1}{24\,n^2}\big[K_3(L;t) - K_3(\widetilde{L};t)\big]\,, \quad t \in [0,1]\,.$$

An important observation for quadrature formulae (3.3) and (3.4) is that their third Peano kernels coincide in the interval $t \in (t_3, 1-t_3)$ with $n^{-3}\widetilde{B}_3(n\,t)$ and $n^{-3}\widetilde{B}_3(n\,t - 1/2)$, respectively. That is to say, except for some small neighborhoods of the endpoints, their third Peano kernels coincide with the third Peano kernels of Q_{n+1}^{Tr} and Q_n^{Mi} in the periodic case. Consequently, for the error constants of quadrature formulae (3.3) and (3.4) we have

$$c_{3,p}(Q) = \|K_3(Q;\cdot)\|_q = \mathcal{E}_n(\widetilde{W}_p^3)(1+o(1)) \quad \text{as} \;\; n \to \infty\,, \tag{3.5}$$

which implies their asymptotical optimality in W_p^3, $1 < p \le \infty$.

3.1. ASYMPTOTICALLY OPTIMAL QUADRATURE FORMULAE BASED ON Q_{n+1}^{Tr}

Here, we present quadrature formulae of the form (3.3) generated by some formulae for numerical differentiation.

1. A quadrature formula generated by $D_1(x_{0,n}, x_{1,n}, x_{2,n})[f]$.

Since

$$D_1(x_{0,n}, x_{1,n}, x_{2,n})[f] = \frac{n}{2}\big(-3f(x_{0,n}) + 4f(x_{1,n}) - f(x_{2,n})\big)\,,$$

the resulting quadrature formula (it is assumed that $n \ge 6$) is

$$Q_{n+1}[f] = \sum_{k=1}^{n+1} A_{k,n+1}\, f(x_{k-1,n}) \tag{3.6}$$

with

$$\begin{aligned} &A_{1,n+1} = A_{n+1,n+1} = \frac{3}{8\,n}\,, \quad A_{2,n+1} = A_{n,n+1} = \frac{7}{6\,n}\,, \\ &A_{3,n+1} = A_{n-1,n+1} = \frac{23}{24\,n}\,, \quad A_{k,n+1} = \frac{1}{n}\,, \quad 4 \le k \le n-2\,. \end{aligned} \tag{3.7}$$

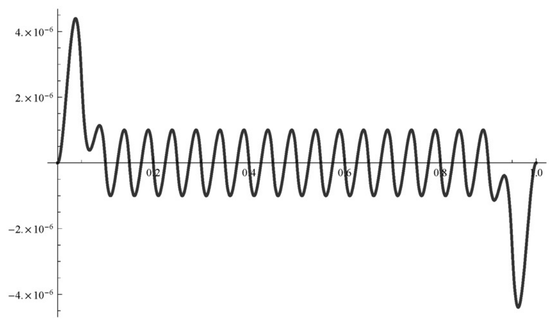


Figure. 1. The third Peano kernel $K_3(Q_{n+1};t)$ of quadrature formula (3.6), $n = 20$.

The graph of the third Peano kernel of quadrature formula (3.6) for $n = 20$ is shown on Figure 1. Since (3.6) is a symmetrical quadrature formula, $K_3(Q_{n+1};t)$

is an odd function with respect to $t = 1/2$. We proceed with evaluating the error constant $c_{3,p}(Q_{n+1})$, $p = \infty$ and $p = 2$. By symmetry, we have

$$c_{3,\infty}(Q_{n+1}) = \|K_3(Q_{n+1};\cdot)\|_1 = 2\int_0^{x_{2,n}} |K_3(Q_{n+1};t)|\,dt + \int_{x_{2,n}}^{x_{n-2,n}} |K_3(Q_{n+1};t)|\,dt\,.$$

For $t \in (x_{2,n}, x_{n-2,n})$ we have $K_3(Q_{n+1};t) = n^{-3}\widetilde{B}_3(n\,t)$, therefore for the second summand we have

$$\int_{x_{2,n}}^{x_{n-2,n}} |K_3(Q_{n+1};t)|\,dt = \frac{1}{n^3}\int_{\frac{2}{n}}^{\frac{n-2}{n}} |\widetilde{B}_3(n\,t)|\,dt = \frac{n-4}{n^3}\,\|B_3\|_1 = \frac{n-4}{192\,n^4}\,.$$

Before evaluating the first summand, we show that $K_3(Q_{n+1};t) > 0$ for $t \in (0, x_{2,n})$. Performing a change of the variable $t = u/n$, $u \in (0,2)$, we obtain, for $t \in (0, x_{2,n})$,

$$K_3(Q_{n+1};t) = -\frac{t^3}{6} + \frac{3}{16\,n}\,t^2 + \frac{7}{12\,n}(t - x_{1,n})_+^2 = \frac{1}{n^3}\Big[-\frac{u^3}{6} + \frac{3u^2}{16} + \frac{7(u-1)_+^2}{12}\Big]\,.$$

The term in the brackets is positive for $u \in (0,2)$. Indeed, if $0 < u \le 1$, then

$$-\frac{u^3}{6} + \frac{3u^2}{16} + \frac{7(u-1)_+^2}{12} = \frac{3u^2}{16}\Big(1 - \frac{8u}{9}\Big) > 0\,,$$

while, if $1 < u < 2$, then

$$-\frac{u^3}{6} + \frac{3u^2}{16} + \frac{7(u-1)_+^2}{12} = -\frac{u^3}{6} + \frac{37u^2}{48} - \frac{7u}{6} + \frac{7}{12} = (2-u)\Big(\frac{u^2}{6} - \frac{7u}{16} + \frac{7}{24}\Big) > 0\,.$$

Therefore,

$$2\int_0^{x_{2,n}} |K_3(Q_{n+1};t)|\,dt = 2\int_0^{\frac{2}{n}} \Big[-\frac{t^3}{6} + \frac{3}{16\,n}\,t^2 + \frac{7}{12\,n}\Big(t - \frac{1}{n}\Big)_+^2\Big]\,dt$$

$$= \frac{2}{n^4}\int_0^2 \Big[-\frac{u^3}{6} + \frac{3u^2}{16} + \frac{7(u-1)_+^2}{12}\Big]\,du = \frac{1}{18\,n^4}\,.$$

Hence,

$$c_{3,\infty}(Q_{n+1}) = \frac{n-4}{192\,n^4} + \frac{1}{18\,n^4} = \frac{1}{192\,n^3}\left(1 + \frac{20}{3\,n}\right).$$

In a similar manner we evaluate the error constant $c_{3,2}(Q_{n+1})$. We have

$$[c_{3,2}(Q_{n+1})]^2 = \int_0^1 [K_3(Q_{n+1};t]^2\,dt = 2\int_0^{x_{2,n}} [K_3(Q_{n+1};t)]^2\,dt + \int_{x_{2,n}}^{x_{n-2,n}} [K_3(Q_{n+1};t)]^2\,dt\,.$$

The second summand is

$$\int_{x_{2,n}}^{x_{n-2,n}} [K_3(Q_{n+1};t)]^2\,dt = \frac{1}{n^6} \int_{\frac{2}{n}}^{\frac{n-2}{n}} [\widetilde{B}_3(n\,t)]^2\,dt = \frac{n-4}{n^7}\,\|B_3\|_2^2\,,$$

and for the first one after some algebra we find

$$\begin{aligned} 2\int_0^{x_{2,n}} [K_3(Q_{n+1};t)]^2\,dt &= \frac{2}{n^7}\int_0^2 \Big[-\frac{u^3}{6}+\frac{3u^2}{16}+\frac{7(u-1)_+^2}{12}\Big]^2\,du \\ &= \frac{2}{n^7}\bigg(\int_0^1 \Big[-\frac{u^3}{6}+\frac{3u^2}{16}\Big]^2\,du + \int_1^2 \Big[-\frac{u^3}{6}+\frac{3u^2}{16}+\frac{7(u-1)^2}{12}\Big]^2\,du\bigg) \\ &= \frac{13}{10080\,n^7} = \frac{39}{n^7}\,\|B_3\|_2^2\,. \end{aligned}$$

After summing the two expressions and taking square root we obtain

$$c_{3,2}(Q_{n+1}) = \frac{1}{n^3}\,\|B_3\|_2\left(1+\frac{35}{n}\right)^{1/2} = \frac{1}{12\sqrt{210}\,n^3}\left(1+\frac{35}{n}\right)^{1/2}.$$

Comparison of the error constants $c_{3,\infty}(Q_{n+1})$ and $c_{3,2}(Q_{n+1})$ of quadrature formula (3.6) with the best possible constant (2.14) and (2.15) in the corresponding 1-periodic Sobolev classes shows the asymptotical optimality of $\{Q_{n+1}\}_{n=6}^{\infty}$ in the Sobolev classes W_∞^3 and W_2^3. Certainly, this sequence is not asymptotically optimal in W_1^3, as is seen also on Figure 1. In fact, $\|K_3(Q_{n+1};\cdot)\|_\infty$ is attained at the point $t_n^* = \frac{3}{4n}$, and

$$c_{3,1}(Q_{n+1}) = K_3(Q_{n+1};t_n^*) = \frac{9}{256\,n^3} = \frac{81\sqrt{3}}{32}\,\mathcal{E}_n(\widetilde{W}_1^3) \approx 4.384\,\mathcal{E}_n(\widetilde{W}_1^3)\,,$$

i.e., the error constant is more than four times greater than the best possible. We shall however construct sequences of quadrature formulae, which are asymptotically optimal in W_1^3, too, see quadrature formulae (3.9) and (3.13) below.

The next quadrature formulae are obtained in the same way as quadrature formula (3.6), and the evaluation of their coefficient and error constants follows the same lines as above. That is why we only give the results.

2. A quadrature formula generated by $D_1(x_{0,n}, y_{1,n}, x_{1,n})[f]$.

Here, $D_1(x_{0,n}, y_{1,n}, x_{1,n})[f] = n\big(-3f(x_{0,n}) + 4f(y_{1,n}) - f(x_{1,n})\big)$, and the resulting quadrature formula (3.3) involves $n+3$ nodes,

$$Q_{n+3}[f] = \sum_{k=1}^{n+3} A_{k,n+3}\, f(\tau_{k,n+3})\,. \tag{3.8}$$

Table 1. The coefficients, nodes and error constants of quadrature formula (3.8).

$A_{1,n+3},\ A_{n+3,n+3}$	$A_{2,n+3},\ A_{n+2,n+3}$	$A_{3,n+3},\ A_{n+1,n+3}$	$A_{k,n+3},\ 4 \le k \le n$
$\dfrac{1}{4n}$	$\dfrac{1}{3n}$	$\dfrac{11}{12n}$	$\dfrac{1}{n}$

$\tau_{1,n+3}$	$\tau_{2,n+3}$	$\tau_{k,n+3},\ 3 \le k \le n+1$	$\tau_{n+2,n+3}$	$\tau_{n+3,n+3}$
$x_{0,n}$	$y_{1,n}$	$x_{k-2,n}$	$y_{n,n}$	$x_{n,n}$

$c_{3,\infty}(Q_{n+3})$	$c_{3,2}(Q_{n+3})$
$\dfrac{1}{192\,n^3}\Big(1+\dfrac{2}{3\,n}\Big)$	$\dfrac{1}{12\sqrt{210}\,n^3}\Big(1+\dfrac{7}{4\,n}\Big)^{1/2}$

The coefficients, nodes and error constants of this quadrature formula are given in Table 1.

3. A quadrature formula generated by $\boldsymbol{D_1(x_{0,n}, x_{1,3n}, x_{2,3n})[f]}$. Here, $D_1(x_{0,n}, x_{1,3n}, x_{2,3n})[f] = \frac{3n}{2}\big(-3f(x_{0,n}) + 4f(x_{1,3n}) - f(x_{2,3n})\big)$, and by (3.3) we obtain the $(n+5)$-point quadrature formula

$$Q_{n+5}[f] = \sum_{k=1}^{n+5} A_{k,n+5} f(\tau_{k,n+5}) \tag{3.9}$$

with coefficients, nodes and error constants given in Table 2.

Table 2. The coefficients, nodes and error constants of quadrature formula (3.9).

$A_{1,n+5},\ A_{n+5,n+5}$	$A_{2,n+5},\ A_{n+4,n+5}$	$A_{3,n+5}\ ,A_{n+3,n+5}$	$A_{k,n+5},\ 5 \le k \le n+1$
$\dfrac{1}{8n}$	$\dfrac{1}{2n}$	$-\dfrac{1}{8n}$	$\dfrac{1}{n}$

$\tau_{1,n+5}$	$\tau_{2,n+5}$	$\tau_{3,n+5}$	$\tau_{k,n+5},\ 4 \le k \le n+2$	$\tau_{n+3,n+5}$	$\tau_{n+4,n+5}$	$\tau_{n+5,n+5}$
$x_{0,n}$	$x_{1,3n}$	$x_{2,3n}$	$x_{k-3,n}$	$x_{3n-2,3n}$	$x_{3n-1,3n}$	$x_{n,n}$

$c_{3,\infty}(Q_{n+5})$	$c_{3,2}(Q_{n+5})$	$c_{3,1}(Q_{n+5})$
$\dfrac{1}{192\,n^3}\Big(1-\dfrac{22}{27\,n}\Big)$	$\dfrac{1}{12\sqrt{210}\,n^3}\Big(1+\dfrac{8}{81\,n}\Big)^{1/2}$	$\dfrac{1}{72\sqrt{3}\,n^3}$

Here we would like to point out that, unlike the situation with quadrature formulae (3.6) and (3.8), here the third Peano kernel of quadrature formula (3.9) attains its $C[0,1]$-norm away from the boundary intervals affected by the numerical differentiation formulae, and therefore we have

$$c_{3,1}(Q_{n+5}) = \|K_3(Q_{n+5};\cdot)\|_\infty = n^{-3}\|B_3\|_\infty = \frac{1}{72\sqrt{3}\,n^3}\,,$$

showing that $\{Q_{n+5}\}$ is a sequence of asymptotically optimal quadrature formulae in the Sobolev class W_1^3. Figure 2 depicts $K_3(Q_{n+5};t)$ for $n = 20$.

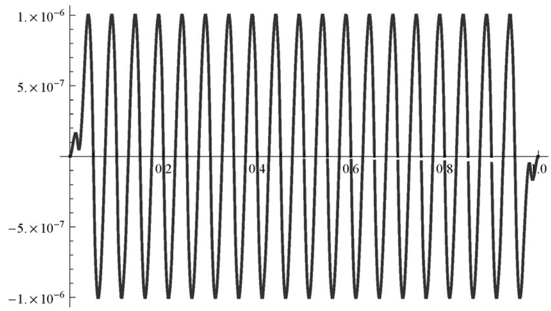


Figure. 2. The third Peano kernel $K_3(Q_{n+5};t)$ of quadrature formula (3.9), $n = 20$.

3.2. ASYMPTOTICALLY OPTIMAL QUADRATURE FORMULAE BASED ON Q_n^{Mi}

Here, we present quadrature formulae generated by various formulae for numerical differentiation through (3.4). Again, we only present in a table form the coefficients, nodes and error constants of these quadrature formulae, skipping the straightforward but sometimes tedious calculations. Occasionally, we have used Wolfram Mathematica for the evaluation of the L_1-norm of the third Peano kernels; in such cases the corresponding error constants $c_{3,\infty}$ are given with approximate numbers.

1. A quadrature formula generated by $D_1(y_{1,n}, y_{2,n}, y_{3,n})[f]$.

This choice is motivated by the aim of not introducing nodes other than $\{y_{\ell,n}\}_{\ell=1}^n$. We have

$$D_1(y_{1,n}, y_{2,n}, y_{3,n})[f] = n\big(-2f(y_{1,n}) + 3f(y_{2,n}) - f(y_{3,n})\big)\,,$$

and by (3.4) we obtain (assuming that $n > 6$) an n-point quadrature formula

$$Q_n[f] = \sum_{k=1}^{n} A_{k,n}\, f(y_{k,n}) \tag{3.10}$$

with weights $\{A_{k,n}\}$ and error constants $c_{3,\infty}(Q_n)$, $c_{3,2}(Q_n)$ as given in Table 3.

Table 3. The coefficients and error constants of quadrature formula (3.10).

$A_{1,n},\ A_{n,n}$	$A_{2,n},\ A_{n-1,n}$	$A_{3,n},\ A_{n-2,n}$	$A_{k,n},\ 3\le k\le n-3$
$\dfrac{13}{12n}$	$\dfrac{7}{8n}$	$\dfrac{25}{24n}$	$\dfrac{1}{n}$

$c_{3,\infty}(Q_n)$	$c_{3,2}(Q_n)$
$\dfrac{1}{192\,n^3}\Big(1+\dfrac{10.83836617}{n}\Big)$	$\dfrac{1}{12\sqrt{210}\,n^3}\Big(1+\dfrac{475}{4\,n}\Big)^{1/2}$

2. A quadrature formula generated by $D_1(x_{0,n},y_{1,n},x_{1,n})[f]$.

We already applied this formula for numerical differentiation in the preceding section, this time we get through (3.4) an $(n+4)$-point quadrature formula

$$Q_{n+4}[f]=\sum_{k=1}^{n+4}A_{k,n+4}\,f(\tau_{k,n+1}) \tag{3.11}$$

with coefficients, nodes and error constants given in Table 4.

Table 4. The coefficients, nodes and error constants of quadrature formula (3.11).

$A_{1,n+4},\ A_{n+4,n+4}$	$A_{2,n+4},\ A_{n+3,n+4}$	$A_{3,n+4},\ A_{n+2,n+4}$	$A_{k,n+4},\ 4\le k\le n+1$
$\dfrac{1}{8n}$	$\dfrac{5}{6n}$	$\dfrac{1}{24n}$	$\dfrac{1}{n}$

$\tau_{1,n+4}$	$\tau_{2,n+4}$	$\tau_{3,n+4}$	$\tau_{k,n+4},\ 4\le k\le n+1$	$\tau_{n+2,n+4}$	$\tau_{n+3,n+4}$	$\tau_{n+3,n+4}$
$x_{0,n}$	$y_{1,n}$	$x_{1,n}$	$y_{k-2,n}$	$x_{n-1,n}$	$y_{n-1,n}$	$x_{n,n}$

$c_{3,\infty}(Q_{n+4})$	$c_{3,2}(Q_{n+4})$
$\dfrac{1}{192\,n^3}\Big(1-\dfrac{175}{384\,n}\Big)$	$\dfrac{1}{12\sqrt{210}\,n^3}\Big(1+\dfrac{25}{16\,n}\Big)^{1/2}$

3. A quadrature formula generated by $D_1(x_{0,n},y_{1,n},y_{2,n})[f]$.

In this case, $D_1(x_{0,n},y_{1,n},y_{2,n})[f]=\frac{n}{3}\big(-8f(x_{0,n})+9f(y_{1,n})-f(y_{2,n})\big)$, and by (3.4) we obtain an $(n+2)$-point quadrature formula

$$Q_{n+2}[f]=\sum_{k=1}^{n+2}A_{k,n+2}\,f(\tau_{k,n+2}) \tag{3.12}$$

with coefficients, nodes and error constants given in Table 5.

Table 5. The coefficients, nodes and error constants of quadrature formula (3.12).

$A_{1,n+2},\ A_{n+2,n+2}$	$A_{2,n+2},\ A_{n+1,n+2}$	$A_{3,n+2},\ A_{n,n+2}$	$A_{k,n+2},\ 4\le k\le n-1$
$\dfrac{1}{9\,n}$	$\dfrac{7}{8\,n}$	$\dfrac{73}{72\,n}$	$\dfrac{1}{n}$

$\tau_{1,n+2}$	$\tau_{k,n+2},\ 2\le k\le n+1$	$\tau_{n+2,n+2}$
$x_{0,n}$	$y_{k-1,n}$	$x_{n,n}$

$c_{3,\infty}(Q_{n+2})$	$c_{3,2}(Q_{n+2})$
$\dfrac{1}{192\,n^3}\Big(1-\dfrac{0.06659022}{n}\Big)$	$\dfrac{1}{12\sqrt{210}\,n^3}\Big(1+\dfrac{19}{4\,n}\Big)^{1/2}$

4. A quadrature formula generated by $D_1(x_{0,n},x_{1,6n},x_{1,3n})[f]$.

We showed that (3.10), (3.11) and (3.12) generate sequences of asymptotically optimal quadrature formulae in the Sobolev classes W^3_∞ and W^3_2, however, the asymptotical optimality does not hold in W^3_1. With $D_1(x_{0,n},x_{1,6n},x_{1,3n})[f]$ we obtain through (3.4) an $(n+6)$-point quadrature formula

$$Q_{n+6}[f]=\sum_{k=1}^{n+6}A_{k,n+6}\,f(\tau_{k,n+6})\,, \tag{3.13}$$

which generates a sequence of asymptotically optimal quadrature formulae in all Sobolev classes W^3_p, $1\le p\le\infty$. The coefficients, nodes and error constants of (3.13) are given in Table 6.

Table 6. The coefficients, nodes and error constants of quadrature formula (3.13).

$A_{1,n+6},\ A_{n+6,n+6}$	$A_{2,n+6},\ A_{n+5,n+6}$	$A_{3,n+6}\ ,A_{n+4,n+6}$	$A_{k,n+6},\ 4\le k\le n+3$
$\dfrac{3}{8n}$	$-\dfrac{1}{2n}$	$\dfrac{1}{8n}$	$\dfrac{1}{n}$

$\tau_{1,n+6}$	$\tau_{2,n+6}$	$\tau_{3,n+6}$	$\tau_{k,n+6},\ 4\le k\le n+3$	$\tau_{n+4,n+6}$	$\tau_{n+5,n+6}$	$\tau_{n+6,n+6}$
$x_{0,n}$	$x_{1,6n}$	$x_{1,3n}$	$y_{k-3,n}$	$x_{3n-2,3n}$	$x_{6n-1,6n}$	$x_{n,n}$

$c_{3,\infty}(Q_{n+6})$	$c_{3,2}(Q_{n+6})$	$c_{3,1}(Q_{n+6})$
$\dfrac{1}{192\,n^3}\Big(1-\dfrac{4}{27\,n}\Big)$	$\dfrac{1}{12\sqrt{210}\,n^3}\Big(1+\dfrac{841}{1296\,n}\Big)^{1/2}$	$\dfrac{1}{72\sqrt{3}\,n^3}$

It is clear that quadrature formulae obtained in Sections 3.1 and 3.2 are of nearly the same quality as being asymptotically optimal in the Sobolev classes W_p^3, $1 < p \le \infty$. Nevertheless, it makes sense to compare their error constants in W_∞^3 and in W_2^3 under the assumption that they involve the *same number of nodes* n, $n \ge 7$. Interestingly, we have a clear winner in both W_∞^3 and W_2^3, namely, quadrature formula (3.12). The ranking of quadrature formulae (3.6), (3.8), (3.9), (3.10), (3.11), (3.12) and (3.13) according to the magnitude of their error constants $c_{3,\infty}(Q_n)$ and $c_{3,2}(Q_n)$ is given in Table 7 (the smaller error constant, the higher ranking).

Table 7. The ranking of quadrature formulae according to their error constants.

quadrature formula	(3.6)	(3.8)	(3.9)	(3.10)	(3.11)	(3.12)	(3.13)
position according to the size of $c_{3,\infty}(Q_n)$	2	3	6	4	5	1	7
position according to the size of $c_{3,2}(Q_n)$	6	2	4	7	3	1	5

The ranking is made assuming that n is big enough, e.g., $n \ge 59$. For small n, some small changes occur: in the ranking with respect to $c_{3,\infty}(Q_n)$, (3.10) overtakes (3.8) (if $n \le 58$) and even (3.6) (if $7 \le n \le 30$) whilst in the ranking with respect to $c_{3,2}(Q_n)$, (3.6) overtakes (3.13) if $7 \le n \le 9$.

4. ASYMPTOTICALLY OPTIMAL QUADRATURE FORMULAE IN W_1^4

In [2] the idea described in the beginning of the preceding section was exploited for the construction of asymptotically optimal quadrature formulae in the Sobolev classes W_∞^4 and W_2^4. To this, we add here two sequences of quadrature formulae, which are asymptotically optimal in the Sobolev class W_1^4.

The difference with the Sobolev classes W_p^3 is that, in the cases of W_p^4 there is a shift $d_{4,p}$ (depending on p) of the 1-periodic Bernoulli monospline $\widetilde{B}_4$ so that the shifted Bernoulli monospline has minimal L_q-deviation from zero $(1/p + 1/q = 1)$, see (2.13). In particular,

$$d_{4,1} = \frac{1}{16}\, B_4(0)\,, \tag{4.1}$$

and

$$\inf_d \|B_4 - d\|_\infty = \|B_4 - 2^{-4}\, B_4(0)\|_\infty = \frac{1}{768}\,. \tag{4.2}$$

The Euler-MacLauren formulae (2.7) and (2.8) in the case $s = 4$ reduce to

$$\int_0^1 f(x)\,dx = Q_{n+1}^{Tr}[f] - \frac{1}{12\,n^2}\big[f'(1) - f'(0)\big] + \frac{1}{720\,n^4}\big[f'''(1) - f'''(0)\big] + \frac{1}{n^4}\int_0^1 \widetilde{B}_4(n\,x)\,f^{(4)}(x)\,dx\,,$$

$$\int_0^1 f(x)\,dx = Q_{n}^{Mi}[f] + \frac{1}{24\,n^2}\big[f'(1) - f'(0)\big] - \frac{7}{5760\,n^4}\big[f'''(1) - f'''(0)\big] + \frac{1}{n^4}\int_0^1 \widetilde{B}_4\big(n\,x - 1/2\big)\,f^{(4)}(x)\,dx\,,$$

and we rewrite these formulae in the form

$$\int_0^1 f(x)\,dx = Q_{n+1}^{Tr}[f] - \frac{1}{12\,n^2}\big[f'(1) - f'(0)\big] + \frac{1}{768\,n^4}\big[f'''(1) - f'''(0)\big] + \frac{1}{n^4}\int_0^1 \big[\widetilde{B}_4(n\,x) - 2^{-4}B_4(0)\big]\,f^{(4)}(x)\,dx\,, \tag{4.3}$$

$$\int_0^1 f(x)\,dx = Q_{n}^{Mi}[f] + \frac{1}{24\,n^2}\big[f'(1) - f'(0)\big] - \frac{1}{768\,n^4}\big[f'''(1) - f'''(0)\big] + \frac{1}{n^4}\int_0^1 \big[\widetilde{B}_4\big(n\,x - 1/2\big) - 2^{-4}B_4(0)\big]\,f^{(4)}(x)\,dx\,. \tag{4.4}$$

Definition 2. Given $0 \le t_1 < t_2 < t_3 < t_4 < 1$, we denote by $D_1(t_1, t_2, t_3)[f]$ and $D_3(t_1, t_2, t_3)[f]$ the interpolatory formulae for numerical differentiation with nodes $\{t_i\}_{i=1}^4$, which approximate $f'(0)$ and $f'''(0)$, respectively, i.e.

$$D_1[f] := D_1(t_1, t_2, t_3, t_4)[f] = \sum_{i=1}^{4} c_{i,1}\,f(t_i) \approx f'(0)\,,$$

$$D_3[f] := D_3(t_1, t_2, t_3, t_4)[f] = \sum_{i=1}^{4} c_{i,3}\,f(t_i) \approx f'''(0)\,.$$

We approximate derivatives $f'(0)$ and $f'''(0)$ appearing in (4.3)–(4.4) by $D_1[f]$ and $D_3[f]$, respectively. The derivatives $f'(1)$ and $f'''(1)$ are approximated by the formulae for numerical differentiation $\widetilde{D}_1[f]$ and $\widetilde{D}_3[f]$, respectively, which are obtained from $D_1[f]$ and $D_3[f]$ by a reflection, i.e.,

$$\widetilde{D}_1[f] = D_1[g], \quad \widetilde{D}_3[f] = D_3[g], \quad g(x) := -f(1-x)\,.$$

We observe that linear functionals $L_1[f] := f'(0) - D_1[f]$, $L_3[f] := f'''(0) - D_3[f]$, $\widetilde{L}_1[f] := f'(0) - \widetilde{D}_1[f]$ and $\widetilde{L}_3[f] := f'''(0) - \widetilde{D}_3[f]$ vanish on π_3, therefore, by Peano's theorem, for $f \in W_1^4$ they possess integral representations of the form

$$L[f] = \int_0^1 K_4(L;x) f^{(4)}(x)\,dx, \qquad \text{with} \quad K_4(L;t) = L\big[(\cdot - t)_+^3/3!\big]\,.$$

Replacement of derivatives in (4.3) by the formulae for numerical differentiation yields a new quadrature formula Q,

$$\int_0^1 f(x)\,dx = Q[f] + \int_0^1 K_4(Q;x) f^{(4)}(x)\,dx\,,$$

where

$$\begin{aligned} Q[f] = Q_{n+1}^{Tr}[f] &+ \frac{1}{12\,n^2} \sum_{i=1}^4 c_{i,1}\big[f(t_i) + f(1-t_i)\big] \\ &- \frac{1}{768\,n^4} \sum_{i=1}^4 c_{i,3}\big[f(t_i) + f(1-t_i)\big]\,, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} K_4(Q;x) = \frac{1}{n^4}\big[\widetilde{B}_4(n\,x) - 2^{-4}B_4(0)\big] &+ \frac{1}{12\,n^2}\big[K_4(L_1;x) - K_4(\widetilde{L}_1;x)\big] \\ &- \frac{1}{768\,n^4}\big[K_4(L_3;x) - K_4(\widetilde{L}_3;x)\big]\,. \end{aligned} \tag{4.6}$$

Analogously, replacement of derivatives in (4.4) by the formulae for numerical differentiation yields a new quadrature formula Q,

$$\begin{aligned} Q[f] = Q_n^{Mi}[f] &- \frac{1}{24\,n^2} \sum_{i=1}^4 c_{i,1}\big[f(t_i) + f(1-t_i)\big] \\ &+ \frac{1}{768\,n^4} \sum_{i=1}^4 c_{i,3}\big[f(t_i) + f(1-t_i)\big]\,, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} K_4(Q;x) = \frac{1}{n^4}\big[\widetilde{B}_4(n\,x - 1/2) - 2^{-4}B_4(0)\big] &- \frac{1}{24\,n^2}\big[K_4(L_1;x) - K_4(\widetilde{L}_1;x)\big] \\ &+ \frac{1}{768\,n^4}\big[K_4(L_3;x) - K_4(\widetilde{L}_3;x)\big]\,. \end{aligned} \tag{4.8}$$

Here, as in the preceding section, it is assumed that $t_4 = O(n^{-1})$, and as a result, for $x \in [t_4, 1-t_4]$ the fourth Peano kernels of quadrature formulae (4.5) and (4.7) coincide with $n^{-4}\left[\widetilde{B}_4(n\,x) - 2^{-4}B_4(0)\right]$ and $n^{-4}\left[\widetilde{B}_4(n\,x - 1/2) - 2^{-4}B_4(0)\right]$, respectively. Hence, for Q being either (4.5) or (4.7) we have

$$\|K_4(Q;\cdot)\|_{C[t_4,1-t_4]} = \frac{1}{n^4}\,\|B_4 - 2^{-4}B_4(0)\|_\infty = \frac{1}{768\,n^4}\,. \tag{4.9}$$

Both (4.5) and (4.7) are symmetric quadrature formulae with at most $n+9$ nodes. In view of (2.19), (4.9) and the obvious inequality

$$\mathcal{E}_n(W_1^4) \ge \mathcal{E}_n(\widetilde{W}_1^4) = \frac{1}{768\,n^4}\,,$$

a sufficient condition for either of (4.5) and (4.7) to generate a sequence of asymptotically optimal quadrature formulae in W_1^4 is

$$\|K_4(Q;\cdot)\|_{C[0,t_4]} \le \frac{1}{768\,n^4}\,. \tag{4.10}$$

Indeed, in such a case (4.10) and (4.9) imply

$$c_{4,1}(Q) = \|K_4(Q;\cdot)\|_{C[0,1]} = \frac{1}{768\,n^4}$$

and since Q has at most $n+9$ nodes, then for Q_n, the n-point quadrature formula of the same kind, with $n > 9$, we have

$$c_{4,1}(Q_n) \le \frac{1}{768\,(n-9)^4}\,.$$

Consequently,

$$1 \le \lim_{n\to\infty} \frac{c_{4,1}(Q_n)}{\mathcal{E}_n(W_1^4)} \le \lim_{n\to\infty} \frac{\frac{1}{768\,(n-9)^4}}{\mathcal{E}_n(\widetilde{W}_1^4)} = \lim_{n\to\infty} \frac{\frac{1}{768\,(n-9)^4}}{\frac{1}{768\,n^4}} = 1\,,$$

whence the asymptotical optimality holds.

4.1. ASYMPTOTICALLY OPTIMAL QUADRATURE FORMULAE BASED ON Q_{n+1}^{Tr}

We make use of the following formulae for numerical differentiation:

$$D_1(x_{0,n}, x_{1,3n}, x_{2,3n}, x_{1,n})[f] = \frac{n}{2}\big[-11f(x_{0,n}) + 18f(x_{1,3n}) - 9f(x_{2,3n}) + 2f(x_{1,n})\big]$$

$$D_3(x_{0,n}, x_{1,3n}, x_{2,3n}, x_{1,n})[f] = 27n^3\big[-f(x_{0,n}) + 3f(x_{1,3n}) - 3f(x_{2,3n}) + f(x_{1,n})\big].$$

The resulting quadrature formula (4.5) involves $n+5$ nodes,

$$Q_{n+5} = \sum_{k=1}^{n+5} A_{k,n+5}\, f(\tau_{k,n+5})\,. \tag{4.11}$$

Table 8. The weights and the nodes of quadrature formula (4.11).

$A_{1,n+5}, A_{n+5,n+5}$	$A_{2,n+5}, A_{n+4,n+5}$	$A_{3,n+5}, A_{n+3,n+5}$	$A_{4,n+5}, A_{n+2,n+5}$	$A_{k,n+5},\ 5 \le k \le n+1$
$\dfrac{59}{768\,n}$	$\dfrac{165}{256\,n}$	$-\dfrac{69}{256\,n}$	$\dfrac{805}{768\,n}$	$\dfrac{1}{n}$

$\tau_{1,n+5}$	$\tau_{2,n+5}$	$\tau_{3,n+5}$	$\tau_{k,n+5},\ 4 \le k \le n+2$	$\tau_{n+3,n+5}$	$\tau_{n+4,n+5}$	$\tau_{n+5,n+5}$
$x_{0,n}$	$x_{1,3n}$	$x_{2,3n}$	$x_{k-3,n}$	$x_{3n-2,3n}$	$x_{3n-1,3n}$	$x_{n,n}$

The weights and the nodes of Q_{n+5} are given in Table 8.

We shall show that the fourth Peano kernel of $Q = Q_{n+5}$ satisfies condition (4.10) with $[0, t_4] = [0, x_{1,n}]$. The latter Peano kernel is given by

$$K_4(Q_{n+5}; x) = \frac{x^4}{24} - \frac{1}{6}\Big[\frac{59}{768n}\, x^3 + \frac{165}{256n}\Big(x - \frac{1}{3n}\Big)_+^3 - \frac{69}{256n}\Big(x - \frac{2}{3n}\Big)_+^3\Big], \quad x \in [0, x_{1,n}].$$

We perform change of the variable $x = u/n$, with $u \in [0, 1]$, to obtain

$$K_4(Q_{n+5}; x) = \frac{1}{24\, n^4}\Big[u^4 - \frac{59}{192}\, u^3 - \frac{165}{64}\,(u - 1/3)_+^3 + \frac{69}{64}\,(u - 2/3)_+^3\Big] =: \frac{1}{24\, n^4}\, g(u).$$

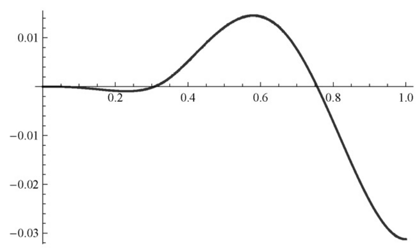


Figure. 3. The graph of $g(u)$, $u \in [0, 1]$.

A straightforward analysis shows that g attains its uniform norm in $[0, 1]$ at $u = 1$ (this is seen also from the graph of g, depicted on Figure 3). Hence,

$$\|K_4(Q_{n+5}; \cdot)\|_{C[0,x_{1,n}]} = |K_4(Q_{n+5}; x_{1,n})|\,.$$

Since

$$K_4(Q_{n+5}; x) \equiv \frac{1}{n^4}\,\big[\widetilde{B}_4(n\,x) - 2^{-4}B_4(0)\big], \quad x \in [x_{1,n}, 1 - x_{1,n}]\,,$$

we have

$$\|K_4(Q_{n+5};\cdot)\|_{C[0,x_{1,n}]} = |K_4(Q_{n+5};x_{1,n})| = \frac{1}{n^4}\big|\widetilde{B}_4(n\,x_{1,n}) - 2^{-4}B_4(0)\big|$$
$$= \frac{1-2^{-4}}{n^4}\,\big|B_4(0)\big| = \frac{1}{768\,n^4}\,.$$

Thus, condition (4.10) is verified, and the asymptotical optimality in W_1^4 of the sequence of quadrature formulae $\{Q_{n+5}\}$ given by (4.11) is proved.

4.2. ASYMPTOTICALLY OPTIMAL QUADRATURE FORMULAE BASED ON Q_n^{Mi}

Here we apply formulae for numerical differentiation with nodes $x_{0,n}$, $y_{1,3n}$, $x_{1,3n}$ and $y_{1,n}$, namely

$$D_1(x_{0,n},y_{1,3n},x_{1,3n},1_{1,n})[f] = n\,\big[-11f(x_{0,n})+18f(y_{1,3n})-9f(x_{1,3n})+2f(y_{1,n})\big]$$
$$D_3(x_{0,n},y_{1,3n},x_{1,3n},y_{1,n})[f] = 216n^3\big[-f(x_{0,n})+3f(y_{1,3n})-3f(x_{1,3n})+f(y_{1,n})\big].$$

By (4.7) we obtain a quadrature formula with $n+6$ nodes,

$$Q_{n+6} = \sum_{k=1}^{n+6} A_{k,n+6}\,f(\tau_{k,n+6})\,. \tag{4.12}$$

The weights and the nodes of Q_{n+6} are given in Table 9.

Table 9. The weights and the nodes of quadrature formula (4.12).

$A_{1,n+6},\,A_{n+6,n+6}$	$A_{2,n+6},\,A_{n+5,n+6}$	$A_{3,n+6},\,A_{n+4,n+5}$	$A_{4,n+6},\,A_{n+3,n+6}$	$A_{k,n+6},\ 5\le k\le n+2$
$\dfrac{17}{96\,n}$	$\dfrac{3}{32\,n}$	$-\dfrac{15}{32\,n}$	$\dfrac{115}{96\,n}$	$\dfrac{1}{n}$

$\tau_{1,n+6}$	$\tau_{2,n+6}$	$\tau_{3,n+6}$	$\tau_{k,n+6},\ 4\le k\le n+3$	$\tau_{n+4,n+6}$	$\tau_{n+5,n+6}$	$\tau_{n+6,n+6}$
$x_{0,n}$	$y_{1,3n}$	$x_{1,3n}$	$y_{k-3,n}$	$x_{3n-1,3n}$	$y_{3n,3n}$	$x_{n,n}$

We proceed with showing that the sequence of quadrature formulae $\{Q_{n+6}\}_{n\in\mathbb{N}}$ defined in (4.12) is asymptotically optimal in W_1^4. To this end, we need to show that the fourth Peano kernel of $Q = Q_{n+6}$ satisfies condition (3.10), with $[0,t_4]$ replaced by $[0,y_{1,n}]$. We have

$$K_4(Q_{n+6};x) = \frac{x^4}{24} - \frac{1}{6}\Big[\frac{17}{96n}\,x^3 + \frac{3}{32n}\Big(x-\frac{1}{6n}\Big)_+^3 - \frac{15}{32n}\Big(x-\frac{1}{3n}\Big)_+^3\Big],\quad x\in[0,y_{1,n}],$$

or, after change of the variable, $x = u/n$ with $u\in[0,1/2]$,

$$K_4(Q_{n+6};x) = \frac{1}{24\,n^4}\Big[u^4 - \frac{17}{24}\,u^3 - \frac{3}{8}\,(u-1/6)_+^3 + \frac{15}{8}\,(u-1/3)_+^3\Big] =: \frac{1}{24\,n^4}\,h(u).$$

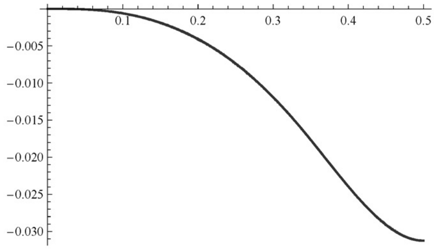


Figure. 4. The graph of $h(u)$, $u \in [0, 1/2]$.

By a straightforward analysis we see that h is monotone decreasing in the interval $[0, 1/2]$ (see Figure 4 with the graph of h), and therefore h attains its uniform norm in $[0, 1/2]$ at $u = 1/2$. Consequently,

$$\|K_4(Q_{n+6}; \cdot)\|_{C[0,y_{1,n}]} = |K_4(Q_{n+6}; y_{1,n})| \, .$$

Since

$$K_4(Q_{n+6}; x) \equiv \frac{1}{n^4} \left[\widetilde{B}_4(n\,x - 1/2) - 2^{-4} B_4(0)\right] , \quad x \in [y_{1,n}, 1 - y_{1,n}] \, ,$$

we obtain

$$\begin{aligned}\|K_4(Q_{n+6}; \cdot)\|_{C[0,y_{1,n}]} &= |K_4(Q_{n+6}; y_{1,n})| = \frac{1}{n^4}\left|\widetilde{B}_4(n\,y_{1,n} - 1/2) - 2^{-4} B_4(0)\right| \\ &= \frac{1 - 2^{-4}}{n^4}\left|B_4(0)\right| = \frac{1}{768\,n^4} \, .\end{aligned}$$

The proof that $\{Q_{n+6}\}_{n\in\mathbb{N}}$ is a sequence of asymptotically optimal quadrature formulae in W_1^4 is accomplished.

As was seen, the fourth Peano kernels of (4.11) and (4.12) have the same L_∞-norm, namely, $\frac{1}{768\,n^4}$, however, quadrature formula (4.11) can be viewed as slightly better as it involves one node less than (4.12).

5. CONCLUSIONS

We have constructed certain sequences of quadrature formulae, which are asymptotically optimal in the Sobolev classes W_p^3, $1 \le p \le \infty$ and in W_1^4. Their weights and nodes are explicitly given, and their sharp error constants for $p = 1, 2$ and ∞ and are evaluated.

For the sake of simplicity, we have considered only symmetric quadrature formulae, however, sequences of non-symmetric asymptotically optimal quadrature formulae can be generated as well by making use of different formulae for numerical differentiation for approximation of the derivatives at the end points of integration interval.

The same approach can be applied for the construction of sequences of asymptotically optimal quadrature formulae in the Sobolev classes W_p^4, $r > 4$, though the calculation of their sharp error constants $c_{r,p}$, even for $p = 1, 2, \infty$, becomes rather elaborate.

ACKNOWLEDGEMENT. The authors are partially supported by the Sofia University Research Fund through Contract 75/2015.

6. REFERENCES

[1] Avdzhieva, A., Nikolov, G.: Numerical computation of Gaussian quadrature formulae for spaces of cubic splines with equidistant knots. In: *BGSIAM'12, Proceedings of the 7th meeting of the Bulgarian Section of SIAM* (A. Slavova and Kr. Georgiev, Eds.), ISNM: 1314 - 7145, Sofia, 2012, 28–38.

[2] Avdzhieva, A., Nikolov, G.: On certain asymptotically optimal quadrature formulae. In: *Advanced Research in Mathematics and Computer Science, MIE 2014 Proceedings* (P. Sloup, Kr. Stefanov, A. Soskova, I. Koytchev, P. Boytchev, Eds.), 2014, St. Kliment Ohridski University Press, ISNM: 978–954–07–3759–1, Sofia, 3–21.

[3] Bojanov, B. D.: Uniqueness of the monosplines of least deviation. In: *Numerische Integration* (G. Hämmerlin, Ed.), ISNM 45, Birkhäuser, Basel, 1979, 67–97.

[4] Bojanov, B. D.: Existence and characterization of monosplines of least L_p deviation. In: *Constructive Function Theory '77* (Bl. Sendov and D. Vačov, Eds), Sofia, BAN, 1980, 249–268.

[5] Bojanov, B. D.: Uniqueness of the optimal nodes of quadrature formulae, *Math. Comput.*, **36**, 1981, 525–546.

[6] Braß, H.: *Quadraturverfahren.* Vandenhoeck & Ruprecht, Göttingen, 1977.

[7] Karlin, S., Micchelli, C. A.: The fundamental theorem of algebra for monosplines satisfying boundary conditions. *Israel J. Math.*, **11**, 1972, 405–451.

[8] Köhler, P., Nikolov, G. P.: Error bounds for Gauss type quadrature formulae related to spaces of splines with equidistant knots. *J. Approx. Theory*, **81**, no. 3, 1995, 368–388.

[9] Ligun, A. A.: Exact inequalities for splines and best quadrature formulas for certain classes of functions. *Mat. Zametki*, **19**, 1976, 913–926 (in Russian); English Translation in: *Math. Notes*, **19**, 1976, 533–541.

[10] Motornii, V. P.: On the best quadrature formula of the form $\sum p_k f(x_k)$ for some classes of differentiable periodic functions. *Izv. Akad. Nauk SSSR Ser. Mat.*, **38**, 1974, 583–614 (in Russian); English Translation in: *Math. USSR Izv.*, **8**, 1974, 591–620.

[11] Nikolov, G.: Gaussian quadrature formulae for splines. In: *Numerische Integration, IV* (G. Hämmerlin and H. Brass, Eds.), ISNM Vol. 112, Birkhäuser, Basel, 1993, 267–281.

[12] Nikolov, G.: On certain definite quadrature formulae. *J. Comput. Appl. Math.*, **75**, 1996, 329–343.

[13] Nikolov, G., Simian, C.: Gauss-type quadrature formulae for parabolic splines with equidistant knots. In: *Approximation and Computation - In Honor of Gradimir V. Milovanovic* (W. Gautschi, G. Mastroianni, Th. M. Rassias, eds.), Springer Optimization and its Applications, Springer Verlag, Berlin - Heidelberg - New York, 2010, 207–229.

[14] Zhensykbaev, A.: Best quadrature formulae for some classes of periodic differentiable functions. *Izv. Akad. Nauk SSSR Ser. Mat.*, **41**, 1977 (in Russian); English Translation in: *Math. USSR Izv.*, **11**, 1977, 1055–1071.

[15] Zhensykbaev, A.: Monosplines and optimal quadrature formulae for certain classes of non-periodic functions. *Anal. Math.*, **5**, 1979, 301–331 (in Russian).

Received on December 17, 2014

Ana Avdzhieva, Geno Nikolov

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., BG-1164 Sofia
BULGARIA

e-mails: aavdzhieva@fmi.uni-sofia.bg
geno@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

ON THE NOTION OF JUMP STRUCTURE

STEFAN V. VATEV

For a given countable structure $\mathfrak{A}$ and a computable ordinal α, we define its α-th jump structure $\mathfrak{A}^{(\alpha)}$. We study how the jump structure relates to the original structure. We consider a relation between structures called conservative extension and show that $\mathfrak{A}^{(\alpha)}$ conservatively extends the structure $\mathfrak{A}$. It follows that the relations definable in $\mathfrak{A}$ by computable infinitary Σ_α formulae are exactly the relations definable in $\mathfrak{A}^{(\alpha)}$ by computable infinitary Σ_1 formulae. Moreover, the Turing degree spectrum of $\mathfrak{A}^{(\alpha)}$ is equal to the α'-th jump Turing degree spectrum of $\mathfrak{A}$, where $\alpha' = \alpha + 1$, if $\alpha < \omega$, and $\alpha' = \alpha$, otherwise.

Keywords: Computability, structures, definability

2000 Math. Subject Classification: 03D45, 03D30

1. INTRODUCTION

The jump of an abstract structure is a notion that has gathered the attention of many researchers for the past decade. Various versions were suggested and studied independently. Montalbán [6] uses predicates for computable infinitary Σ_1 formulae; Baleva [3], I. Soskov and A. Soskova [10] use Moschovakis extensions; Stukachev [12] uses hereditarily finite extensions. In [7] the reader can find very good historical notes and bibliography on this topic.

Here we consider the notion of jump structure as suggested by A. Soskova and I. Soskov [10], where the first jump of a structure is defined. Later, the author [13] extended their definition to arbitrary finite jumps and studied its properties in the context of a relation between structures called *conservative extension*. In this paper, which is based on a chapter of the author's Ph.D. dissertation [14], we offer a natural continuation of this line of research. We lift the results from [13] to arbitrary computable ordinals.

We work with abstract structures of the form $\mathfrak{A} = (A; P_0, \dots, P_{s-1})$, where A is countable and infinite, the predicates $P_i \subseteq A^{n_i}$ and the equality is among $P_0, \dots, P_{s-1}$. We will use the letters $\mathfrak{A}$, $\mathfrak{B}$ to denote structures and the letters A, B to denote their domains. We call f an *enumeration* of the set A if f is a total one-to-one mapping of $\mathbb{N}$ onto A. We say that f is an enumeration of the structure $\mathfrak{A}$ if f is an enumeration of its domain A. For every $k \in \mathbb{N}$, we will implicitly use an effective encoding of $\mathbb{N}^k$ onto $\mathbb{N}$. By $\langle x_1, \dots, x_k \rangle$ we denote the natural number corresponding to the tuple $(x_1, \dots, x_k)$. If $R \subseteq A^n$, we denote the *pullback* of R as the set $f^{-1}(R) = \{\langle x_0, \dots, x_{n-1} \rangle \mid (f(x_0), \dots, f(x_{n-1})) \in R\}$.

Given a countable structure $\mathfrak{A} = (A; P_0, \dots, P_{s-1})$, we define the *copy* of $\mathfrak{A}$ via the enumeration f as the total function $f^{-1}(\mathfrak{A})$, where:

$$f^{-1}(\mathfrak{A})(u) = \begin{cases} 1, & \text{if } u = s \cdot \langle x_1, \dots, x_{n_i} \rangle + i \ \& \ i < s \ \& \ (f(x_1), \dots, f(x_{n_i})) \in P_i \\ 0, & \text{if } u = s \cdot \langle x_1, \dots, x_{n_i} \rangle + i \ \& \ i < s \ \& \ (f(x_1), \dots, f(x_{n_i})) \notin P_i. \end{cases}$$

We can also look at $f^{-1}(\mathfrak{A})$ as the structure with domain $\mathbb{N}$ obtained from $\mathfrak{A}$ via the isomorphism f. Moreover, for a structure with domain $\mathbb{N}$, let us denote by $D(\mathfrak{A})$ the set of all codes of formulae belonging to the atomic diagram of $\mathfrak{A}$, given by some Gödel numbering of all formulae in the relevant language. This means that $f^{-1}(\mathfrak{A})$ gives us the set of codes of formulae belonging to the atomic diagram of the structure obtained from $\mathfrak{A}$ via the isomorphism f. When we say that the structure $\mathfrak{A}$ is computable, or belongs to the computability-theoretic class $\mathscr{C}$, we mean that its atomic diagram $D(\mathfrak{A})$ is computable, or belongs to $\mathscr{C}$.

Definition 1 (Richter [9]). *The degree spectrum of the structure* $\mathfrak{A}$ *is the set of Turing degrees*

$$DS(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \textit{ computes a copy of } \mathfrak{A}\}.$$

For a computable ordinal α*, we define the* α*-th jump degree spectrum of* $\mathfrak{A}$ *as*

$$DS_\alpha(\mathfrak{A}) = \{\mathbf{a}^{(\alpha)} \mid \mathbf{a} \in DS(\mathfrak{A})\}.$$

A countable structure $\mathfrak{A}$ is *automorphically trivial* if there is a finite subset F of its domain A such that every permutation of A whose restriction to F is the identity, is an automorphism of $\mathfrak{A}$. A set of Turing degrees $\mathscr{A}$ is *closed upwards* if for all Turing degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \in \mathscr{A} \ \& \ \mathbf{a} \leq \mathbf{b} \rightarrow \mathbf{b} \in \mathscr{A}$.

Theorem 1 (Knight [5]). *Let* $\mathfrak{A}$ *be a countable structure in a (possibly infinite) language. Then exactly one of the following holds:*

1) *the spectrum of* $\mathfrak{A}$ *is closed upwards with respect to Turing reducibility;*

2) $\mathfrak{A}$ *is automorphically trivial.*

Henceforth, we suppose that the structures we consider are automorphically non-trivial, so their degree spectra are closed upwards. The notion of degree spectra gives us one way to compare structures. That is, for structures $\mathfrak{A}$ and $\mathfrak{B}$ and computable ordinals α, β, we ask whether $DS_\alpha(\mathfrak{A}) = DS_\beta(\mathfrak{B})$.

Now we give an informal definition of the set of the *computable infinitary* Σ_α and Π_α formulae in the language of $\mathfrak{A}$, denoted Σ^c_α and Π^c_α. The Σ^c_0 and Π^c_0 formulae are the finitary quantifier free formulae. For $\alpha > 0$, a Σ^c_α formula $\varphi(\bar{x})$ is a disjunction of a c.e. set of formulae of the form $\exists \bar{y} \psi(\bar{x}, \bar{y})$, where $\psi(\bar{x}, \bar{y})$ is a Π^c_β formula, for some $\beta < \alpha$. The Π^c_α formulae are the negations of the Σ^c_α formulae. We list a few properties of the computable infinitary formulae, which will be used throughout the paper:

- Given an index for a Σ^c_α (or Π^c_α) formula φ, we can effectively find an index for a Π^c_α (or Σ^c_α) formula $neg(\varphi)$ that is logically equivalent to $\neg\varphi$.

- Given indices for a pair of Σ^c_α (or a pair of Π^c_α) formulae φ and ψ, we can effectively find indices for two Σ^c_α (or two Π^c_α) formulae logically equivalent to $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$.

We refer the reader to the book of Ash and Knight [1, Chapter 7] for details and more background information on computable infinitary formulae.

For a set of natural numbers X and a computable ordinal α, we denote by $X^{(\alpha)}$ the α-th Turing jump of X. Moreover, we define

$$\begin{aligned}
&\Delta^0_{\alpha+1}(X) = X^{(\alpha)}, \text{ if } \alpha < \omega,\\
&\Delta^0_{\alpha+1}(X) = X^{(\alpha+1)}, \text{ if } \alpha \geq \omega,\\
&\Delta^0_{\alpha}(X) = \bigcup_p \{\langle y, p\rangle \mid y \in \Delta^0_{\alpha(p)+1}(X)\}, \text{ if } \alpha = \lim \alpha(p).
\end{aligned}$$

We write Δ^0_α for $\Delta^0_\alpha(\emptyset)$. We remark that for technical reasons, we choose at limit levels to work only with sequences of successors and if α is a computable limit ordinal such that $\alpha = \lim \alpha(p)$, then $\alpha(0) \geq 1$.

Theorem 2 (Ash [1]). *Let $\mathfrak{A}$ be an arbitrary structure with domain $\mathbb{N}$. For a formula $\varphi(\bar{x})$, let us denote $\varphi^{\mathfrak{A}} = \{\bar{a} \in A \mid \mathfrak{A} \models \varphi(\bar{a})\}$. If $\varphi(\bar{x})$ is a Σ^c_α formula, then $\varphi^{\mathfrak{A}}$ is $\Sigma^0_\alpha(D(\mathfrak{A}))$, and if $\varphi(\bar{x})$ is a Π^c_α formula, then $\varphi^{\mathfrak{A}}$ is $\Pi^0_\alpha(D(\mathfrak{A}))$. Moreover, given an index for the Σ^c_α (or Π^c_α) formula φ and a notation for the ordinal α, we can effectively find an index for $\varphi^{\mathfrak{A}}$ as a set c.e. (or co-c.e.) relative to $\Delta^0_\alpha(D(\mathfrak{A}))$. The index is independent of $\mathfrak{A}$.*

A relation $R \subseteq A^r$ is Σ^c_α (or Π^c_α) *definable* in the structure $\mathfrak{A}$ if there is a Σ^c_α (or Π^c_α) formula $\psi(\bar{x}, \overline{y})$ and a finite number of parameters $\overline{a}$ in A such that $\bar{b} \in R$ if and only if $\mathfrak{A} \models \psi(\bar{b}, \overline{a})$. We denote by $\Sigma^c_\alpha(\mathfrak{A}_A)$ (or $\Pi^c_\alpha(\mathfrak{A}_A)$) the family of all relations Σ^c_α (or Π^c_α) definable in $\mathfrak{A}$ *with parameters* in A. We will write $\Sigma^c_\alpha(\mathfrak{A})$ (or $\Pi^c_\alpha(\mathfrak{A})$) for the family of relations definable in $\mathfrak{A}$ by Σ^c_α (or Π^c_α) formulae *without parameters.*

The notion of definability gives us another way to compare structures. That is, for structures $\mathfrak{A}$, $\mathfrak{B}$ such that $A \subseteq B$ and computable ordinals α, β, we ask whether $(\forall r \in \mathbb{N})(\forall R \subseteq A^r)[R \in \Sigma^c_\alpha(\mathfrak{A}_A) \leftrightarrow R \in \Sigma^c_\beta(\mathfrak{B}_B)]$.

Definition 2. *Let $\mathfrak{A}$ be an arbitrary countable structure. We say that a relation R on A is* relatively intrinsically Σ^0_α *(or Π^0_α) on $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $f^{-1}(R)$ is c.e. (or co-c.e.) relative to $\Delta^0_\alpha(f^{-1}(\mathfrak{A}))$.*

The relation R is uniformly relatively intrinsically Σ^0_α *(or Π^0_α) on $\mathfrak{A}$ if there is an index e such that for every enumeration f of $\mathfrak{A}$, $f^{-1}(R) = W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))}$ (or $\mathbb{N} \setminus f^{-1}(R) = W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))}$). In this case we say that the number e is a Σ^0_α (or Π^0_α) index for R.*

The next theorem gives a very nice syntactical characterisation of relatively intrinsically Σ^0_α sets.

Theorem 3 (Ash-Knight-Manasse-Slaman [2], Chisholm [4])**.** *Let $\mathfrak{A}$ be a countable structure. For every relation R on A, R is relatively intrinsically Σ^0_α (or Π^0_α) on $\mathfrak{A}$ if and only if R is definable in $\mathfrak{A}$ with a Σ^c_α (or Π^c_α) formula with parameters.*

Moreover, R is uniformly *relatively intrinsically Σ^0_α on $\mathfrak{A}$ if and only if R is definable in $\mathfrak{A}$ by a Σ^c_α formula* without *parameters. Given a Σ^0_α index for R, we can effectively find an index for the Σ^c_α formula, and conversely, given an index for the Σ^c_α formula, we can effectively find a Σ^0_α index for R.*

Although the second part of Theorem 3 is not explicitly stated in [2], [4], it follows in a straightforward manner from the proof of the first part of Theorem 3.

2. CONSERVATIVE EXTENSIONS

Before turning our attention to the notion of jump structure, we need to consider how we will relate the original structure to its jump structure. I. Soskov observed that many common features are shared between the structures constructed by A. Soskova and I. Soskov [10], namely the Moschovakis' extension, the jump structure and the Marker's extension of a structure, which is a construction for obtaining jump-invert structures. It turns out that all these structures relate to the initial structure in a similar way. In the terminology that we are going to introduce, the Moschovakis' extension of $\mathfrak{A}$ is $(1, 1)$-conservative extension of $\mathfrak{A}$. One of our main results will be that the α-th jump structure of $\mathfrak{A}$ is $(\alpha', 1)$-conservative extension of $\mathfrak{A}$, where $\alpha' = \alpha + 1$, if $\alpha < \omega$, and $\alpha' = \alpha$, otherwise.

We begin by defining a relation between enumerations of structures.

Definition 3 (Soskov). *Let f and h be enumerations for the countable structures $\mathfrak{A}$ and $\mathfrak{B}$ respectively. We write $f \leq^\alpha_\beta h$ if*

1) $\Delta^0_\alpha(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B}))$ *and*

2) $E(f,h) = \{\langle x,y\rangle \mid x,y \in \mathbb{N} \;\&\; f(x) = h(y)\}$ *is* $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$.

Definition 4 (Soskov). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures, possibly in different languages.*

1) $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$ *if for every enumeration h of $\mathfrak{B}$ there exists an enumeration f of $\mathfrak{A}$ such that $f \leq^\alpha_\beta h$.*

2) $\mathfrak{A} \Leftarrow^\alpha_\beta \mathfrak{B}$ *if for every enumeration f of $\mathfrak{A}$ there exists an enumeration h of $\mathfrak{B}$ such that $h \leq^\beta_\alpha f$.*

3) $\mathfrak{A} \Leftrightarrow^\alpha_\beta \mathfrak{B}$ *if $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$ and $\mathfrak{A} \Leftarrow^\alpha_\beta \mathfrak{B}$.*

*We say that $\mathfrak{B}$ is an (α,β)-*conservative extension *of $\mathfrak{A}$ if $A \subseteq B$ and $\mathfrak{A} \Leftrightarrow^\alpha_\beta \mathfrak{B}$.*

The following theorem motivates the use of the term conservative extension, i.e. if $\mathfrak{B}$ is an (α,β)-conservative extension of $\mathfrak{A}$ then Σ^c_α definability in $\mathfrak{A}$ is equivalent to Σ^c_β definability in $\mathfrak{B}$ for the subsets of A.

Theorem 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A \subseteq B$. For all $\alpha, \beta < \omega_1^{CK}$,*

1) *if* $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$, *then* $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathfrak{A}_A) \to X \in \Sigma^c_\beta(\mathfrak{B}_B)]$;

2) *if* $\mathfrak{A} \Leftarrow^\alpha_\beta \mathfrak{B}$, *then* $(\forall X \subseteq A)[X \in \Sigma^c_\beta(\mathfrak{B}_B) \to X \in \Sigma^c_\alpha(\mathfrak{A}_A)]$;

3) *if* $\mathfrak{A} \Leftrightarrow^\alpha_\beta \mathfrak{B}$, *then* $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathfrak{A}_A) \leftrightarrow X \in \Sigma^c_\beta(\mathfrak{B}_B)]$.

Proof. 1) Let $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. Then for every enumeration h of $\mathfrak{B}$, there exists an enumeration f of $\mathfrak{A}$ such that $f \leq^\alpha_\beta h$. Let X be a subset of A such that $X \in \Sigma^c_\alpha(\mathfrak{A}_A)$. According to Theorem 3, for every enumeration f of $\mathfrak{A}$, $f^{-1}(X)$ is $\Sigma^0_\alpha(f^{-1}(\mathfrak{A}))$. We will show that for every enumeration h of $\mathfrak{B}$, $h^{-1}(X)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$.

Let us take an arbitrary enumeration h of $\mathfrak{B}$. Since $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$, there is an enumeration f of $\mathfrak{A}$ such that $\Delta^0_\alpha(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B}))$ and $E(f,h)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$. Moreover, $f^{-1}(X)$ is c.e. relative to $\Delta^0_\alpha(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B}))$. It follows from the equivalence $x \in h^{-1}(X) \leftrightarrow (\exists y \in \mathbb{N})[(y,x) \in E(f,h) \;\&\; y \in f^{-1}(X)]$ that $h^{-1}(X)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$, which is what we wanted to show.

The proof of *2)* is similar to that of *1)*. ☐

As remarked in [13], we do not always have the other directions in Theorem 4. We give a very simple counterexample. Let $\mathfrak{A} = (A;=)$ and take $\mathfrak{B} = \mathfrak{A}$. It is easy to see that for every computable ordinal α, $(\forall X \subseteq A)[X \in \Sigma^c_\alpha(\mathfrak{A}_A) \to X \in \Sigma^c_1(\mathfrak{A}_A)]$.

It we assume that we have the reverse directions in Theorem 4, then we would have $(\forall \alpha < \omega_1^{CK})[\mathfrak{A} \Rightarrow_1^\alpha \mathfrak{A}]$, which is evidently not true. To see this, it is enough to take an enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A})$ is computable. Then there is *no* enumeration h of $\mathfrak{A}$ such that $h^{-1}(\mathfrak{A})' \leq_T f^{-1}(\mathfrak{A}) \equiv_T \emptyset$.

For a computable ordinal α, we define the ordinal α' as

$$\alpha' = \begin{cases} \alpha + 1, & \text{if } \alpha < \omega \\ \alpha, & \text{if } \alpha \geq \omega. \end{cases}$$

The reason behind this notation is that a set X is Σ^0_{n+1} if and only if X is c.e. in $\emptyset^{(n)}$, when $n < \omega$, and X is Σ^0_α, $\alpha \geq \omega$ if and only if X is c.e. in $\emptyset^{(\alpha)}$. We also have that for a countable structure $\mathfrak{A}$, $DS_\alpha(\mathfrak{A}) = \{d_T(\Delta^0_{\alpha'}(f^{-1}(\mathfrak{A}))) \mid f \text{ is an enumeration of } \mathfrak{A}\}$.

Theorem 5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A \subseteq B$.*

1) *If $\mathfrak{A} \Rightarrow^{\alpha'}_{\beta'} \mathfrak{B}$ then $DS_\beta(\mathfrak{B}) \subseteq DS_\alpha(\mathfrak{A})$.*

2) *If $\mathfrak{A} \Leftarrow^{\alpha'}_{\beta'} \mathfrak{B}$ then $DS_\alpha(\mathfrak{A}) \subseteq DS_\beta(\mathfrak{B})$;*

3) *If $\mathfrak{A} \Leftrightarrow^{\alpha'}_{\beta'} \mathfrak{B}$ then $DS_\alpha(\mathfrak{A}) = DS_\beta(\mathfrak{B})$.*

Proof. We prove only 1) since the others are similar.

Let $\mathfrak{A} \Rightarrow^{\alpha'}_{\beta'} \mathfrak{B}$ and $\mathbf{b} \in DS_\beta(\mathfrak{B})$. We show that $\mathbf{b} \in DS_\alpha(\mathfrak{A})$. Since $\mathfrak{A}$ is a non-trivial structure, $DS_\alpha(\mathfrak{A})$ is closed upwards and it is enough to prove that there exists a Turing degree $\mathbf{a} \in DS_\alpha(\mathfrak{A})$ such that $\mathbf{a} \leq_T \mathbf{b}$. Let f be an enumeration of $\mathfrak{B}$ and $d_T(\Delta^0_{\beta'}(f^{-1}(\mathfrak{B}))) = \mathbf{b}$. Since $\mathfrak{A} \Rightarrow^{\alpha'}_{\beta'} \mathfrak{B}$, there is an enumeration h of $\mathfrak{A}$ such that $h \leq^{\alpha'}_{\beta'} f$. For $\mathbf{a} = d_T(\Delta^0_{\alpha'}(h^{-1}(\mathfrak{A})))$ we have $\mathbf{a} \in DS_\alpha(\mathfrak{A})$ and $\mathbf{a} \leq_T \mathbf{b}$. □

We note that we do not have the other directions in Theorem 5. For example, let us consider the structures $\mathfrak{N} = (\mathbb{N}; =)$ and $\mathfrak{M} = (\mathbb{N}; G_{Succ}, =)$, where G_{Succ} is the graph of the successor function on $\mathbb{N}$. It is easy to see that $DS(\mathfrak{N}) = DS(\mathfrak{M}) = \{\mathbf{a} \mid \mathbf{0} \leq_T \mathbf{a}\}$. If we assume that $\mathfrak{M} \Leftrightarrow^1_1 \mathfrak{N}$, then the Σ^c_1 definable sets in $\mathfrak{N}$ with parameters are also Σ^c_1 definable in $\mathfrak{M}$ with parameters. But the sets $X \in \Sigma^c_1(\mathfrak{N}_\mathbb{N})$ are just the finite and co-finite sets, whereas the sets $X \in \Sigma^c_1(\mathfrak{M}_\mathbb{N})$ are all c.e. sets. This is a contradiction.

2.1. THE NOTION OF FORCING

We define a forcing relation with conditions all finite injective mappings from $\mathbb{N}$ into the domain of the countable structure $\mathfrak{A} = (A; P_0, \dots, P_{s-1})$. We call them *finite parts* and we use the letters τ, ρ, δ to denote them. Let $\mathbb{P}_A$ be the set of all finite parts and let $\mathbb{P}_2$ be the set of all finite functions on the natural numbers taking values in $\{0,1\}$. Given a finite part τ, we define the finite function $\tau^{-1}(\mathfrak{A})$ in the following way:

$$\begin{aligned}\tau^{-1}(\mathfrak{A})(u)\downarrow= 1 \leftrightarrow (\exists i < s)(\exists x_1,\dots,x_{n_i} \in Dom(\tau))[u = s\cdot\langle x_1,\dots,x_{n_i}\rangle + i\ \&\\ (\tau(x_1),\dots,\tau(x_{n_i})) \in P_i],\\ \tau^{-1}(\mathfrak{A})(u)\downarrow= 0 \leftrightarrow (\exists i < s)(\exists x_1,\dots,x_{n_i} \in Dom(\tau))[u = s\cdot\langle x_1,\dots,x_{n_i}\rangle + i\ \&\\ (\tau(x_1),\dots,\tau(x_{n_i})) \notin P_i],\end{aligned}$$

$\tau^{-1}(\mathfrak{A})(u)\uparrow$ in all other cases. We should note that in the definition of $\tau^{-1}(\mathfrak{A})$ we make the same assumptions about the coding of tuples of natural numbers as in the definition of $f^{-1}(\mathfrak{A})$.

If φ is a partial function and $e \in \mathbb{N}$, then by W_e^φ we will denote the set of all x such that the computation $\{e\}^\varphi(x)$ halts successfully. We assume that if during a computation the oracle φ is called with an argument outside of its domain, then the computation halts unsuccessfully.

For every $e, x \in \mathbb{N}$, every finite part τ and every computable ordinal $\alpha \geq 1$, we define the forcing relations $\tau \Vdash_\alpha F_e(x)$ and $\tau \Vdash_\alpha \neg F_e(x)$ in the following way:

(i) $\tau \Vdash_1 F_e(x) \leftrightarrow x \in W_e^{\tau^{-1}(\mathfrak{A})}$.

(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned}\tau \Vdash_{\beta+1} F_e(x)\ \leftrightarrow\ &(\exists\delta \in \mathbb{P}_2)[x \in W_e^\delta\ \&\ (\forall z \in Dom(\delta))[\\ &(\delta(z) = 1\ \&\ \tau \Vdash_\beta F_z(z)) \vee\\ &(\delta(z) = 0\ \&\ \tau \Vdash_\beta \neg F_z(z))]].\end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned}\tau \Vdash_\alpha F_e(x)\ \leftrightarrow\ &(\exists\delta \in \mathbb{P}_2)[x \in W_e^\delta\ \&\ (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle\ \&\\ &((\delta(z) = 1\ \&\ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\ &(\delta(z) = 0\ \&\ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]].\end{aligned}$$

(iv) $\tau \Vdash_\alpha \neg F_e(x)\ \leftrightarrow\ (\forall\delta \in \mathbb{P}_2)[\delta \supseteq \tau \to \delta \nVdash_\alpha F_e(x)]$.

The forcing relation depends also on the structure $\mathfrak{A}$. To avoid ambiguity, we will write $\tau \Vdash_\alpha^{\mathfrak{A}} F_e(x)$, when necessary.

Lemma 1. *For every computable ordinal $\alpha \geq 1$ and every $e, x \in \mathbb{N}$, we have the following properties:*

1) *for any finite parts $\tau \subseteq \rho$, if $\tau \Vdash_\alpha F_e(x)$, then $\rho \Vdash_\alpha F_e(x)$;*

2) *for any finite parts $\tau \subseteq \rho$, if $\tau \Vdash_\alpha \neg F_e(x)$, then $\rho \Vdash_\alpha \neg F_e(x)$;*

Proof. We prove 1) and 2) simultaneously by transfinite induction on α. The case $\alpha = 1$ for 1) follows directly from the fact that $\tau \subseteq \rho \to \tau^{-1}(\mathfrak{A}) \subseteq \rho^{-1}(\mathfrak{A})$.

For 2), let $\tau \Vdash_1 \neg F_e(x)$ and assume that $\rho \in \mathbb{P}_A$ is such that $\tau \subseteq \rho$, but $\rho \nVdash_1 \neg F_e(x)$. It follows that there exists $\delta \supseteq \rho \supseteq \tau$ such that $\delta \Vdash_1 F_e(x)$. But then $(\exists \delta \supseteq \tau)[\delta \Vdash_1 F_e(x)]$ implies $\tau \nVdash_1 \neg F_e(x)$. We reach a contradiction. Therefore,

$$\tau \Vdash_1 \neg F_e(x) \to \rho \Vdash_1 \neg F_e(x).$$

Let $\alpha = \beta + 1$. By the induction hypothesis for 1) and 2),

$$\begin{aligned} \tau \Vdash_{\beta+1} F_e(x) \ &\leftrightarrow (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \& \ (\forall z \in Dom(\delta))[\\ &\quad (\delta(z) = 1 \ \& \ \tau \Vdash_\beta F_z(z)) \ \vee (\delta(z) = 0 \ \& \ \tau \Vdash_\beta \neg F_z(z))]] \\ &\to (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \& \ (\forall z \in Dom(\delta))[\\ &\quad (\delta(z) = 1 \ \& \ \rho \Vdash_\beta F_z(z)) \ \vee (\delta(z) = 0 \ \& \ \rho \Vdash_\beta \neg F_z(z))]] \\ &\leftrightarrow \rho \Vdash_{\beta+1} F_e(x). \end{aligned}$$

For 2), we apply the same argument as in the case of $\alpha = 1$. Let $\tau \Vdash_\alpha \neg F_e(x)$ and assume that $\rho \in \mathbb{P}_A$ is such that $\tau \subseteq \rho$, but $\rho \nVdash_\alpha \neg F_e(x)$. Then $(\exists \delta \supseteq \tau)[\delta \Vdash_\alpha F_e(x)]$, which implies $\tau \nVdash_\alpha \neg F_e(x)$. We reach a contradiction.

Let $\alpha = \lim \alpha(p)$. Then, again using the induction hypothesis for 1) and 2),

$$\begin{aligned} \tau \Vdash_\alpha F_e(x) \ &\leftrightarrow (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \& \ (\forall z \in Dom(\delta))[z = \langle x_z, p_z \rangle \ \& \\ &\quad ((\delta(z) = 1 \ \& \ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee (\delta(z) = 0 \ \& \ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]] \\ &\to (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \& \ (\forall z \in Dom(\delta))[z = \langle x_z, p_z \rangle \ \& \\ &\quad ((\delta(z) = 1 \ \& \ \rho \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee (\delta(z) = 0 \ \& \ \rho \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]] \\ &\leftrightarrow \rho \Vdash_\alpha F_e(x). \end{aligned}$$

For 2), we again use the same argument. □

Proposition 1. *There is a computable function h such that for any computable ordinal $\alpha > 0$, finite part τ, and natural numbers e, x,*

$$\begin{aligned} \tau \Vdash_\alpha F_e(x) \ &\leftrightarrow \ \tau \Vdash_{\alpha+1} F_{h(e)}(x); \\ \tau \Vdash_\alpha \neg F_e(x) \ &\leftrightarrow \ \tau \Vdash_{\alpha+1} \neg F_{h(e)}(x). \end{aligned}$$

Moreover, there is a computable function h' such that for any computable limit ordinal $\alpha = \lim \alpha(p)$, finite part τ, and natural numbers e, x, p,

$$\begin{aligned} \tau \Vdash_{\alpha(p)} F_e(x) \ &\leftrightarrow \ \tau \Vdash_\alpha F_{h'(p,e)}(x); \\ \tau \Vdash_{\alpha(p)} \neg F_e(x) \ &\leftrightarrow \ \tau \Vdash_\alpha \neg F_{h'(p,e)}(x). \end{aligned}$$

Proof. Firstly, it is easy to see by the relativised S^m_n theorem that there exists a computable function g such that

$$(\forall \sigma \in \mathbb{P}_2)[x \in W^\sigma_e \ \rightarrow \ W^\sigma_{g(e,x)} = \mathbb{N}],$$

$$(\forall \sigma \in \mathbb{P}_2)[x \notin W^\sigma_e \ \rightarrow \ W^\sigma_{g(e,x)} = \emptyset].$$

Then we have for any $\sigma \in \mathbb{P}_2$,

$$x \in W^\sigma_e \leftrightarrow \ W^\sigma_{g(e,x)} = \mathbb{N} \leftrightarrow \ g(e,x) \in W^\sigma_{g(e,x)},$$

and it follows that for any computable ordinal $\alpha > 0$,

$$\tau \Vdash_\alpha F_e(x) \ \leftrightarrow \tau \Vdash_\alpha F_{g(e,x)}(g(e,x)).$$

Now we take h to be a computable function such that for any e and x,

$$(\forall \sigma \in \mathbb{P}_2)[x \in W^\sigma_{h(e)} \ \leftrightarrow \ \sigma(g(e,x)) = 1]. \tag{2.1}$$

In other words, $(\forall \sigma \in \mathbb{P}_2)[x \in W^\sigma_{h(e)} \leftrightarrow \ \{\langle g(e,x), 1\rangle\} \subseteq \mathit{Graph}(\sigma)]$. Our goal is to prove that $\tau \Vdash_\alpha F_e(x)$ if and only if $\tau \Vdash_{\alpha+1} F_{h(e)}(x)$. It is enough to prove that $\tau \Vdash_\alpha F_{g(e,x)}(g(e,x))$ if and only if $\tau \Vdash_{\alpha+1} F_{h(e)}(x)$.

For the $(\rightarrow)$ part, we use that for the finite function σ with $\mathit{Graph}(\sigma) = \{\langle g(e,x), 1\rangle\}$, we have $x \in W^\sigma_{h(e)}$. Thus,

$$\begin{aligned}
\tau \Vdash_\alpha F_{g(e,x)}(g(e,x)) \ &\leftrightarrow (\exists \sigma \in \mathbb{P}_2)[\mathit{Graph}(\sigma) = \{\langle g(e,x), 1\rangle\} \& \tau \Vdash_\alpha F_{g(e,x)}(g(e,x))] \\
&\leftrightarrow (\exists \sigma \in \mathbb{P}_2)[x \in W^\sigma_{h(e)} \ \& \ \mathit{Graph}(\sigma) = \{\langle g(e,x), 1\rangle\} \ \& \\
&\qquad \tau \Vdash_\alpha F_{g(e,x)}(g(e,x))] \\
&\rightarrow (\exists \sigma \in \mathbb{P}_2)[x \in W^\sigma_{h(e)} \ \& \ (\forall z \in \mathit{Dom}(\sigma))[\\
&\qquad (\sigma(z) = 1 \ \& \ \tau \Vdash_\alpha F_z(z)) \vee (\sigma(z) = 0 \ \& \ \tau \Vdash_\alpha \neg F_z(z))]] \\
&\rightarrow \tau \Vdash_{\alpha+1} F_{h(e)}(x).
\end{aligned}$$

For the $(\leftarrow)$ part, let $\tau \Vdash_{\alpha+1} F_{h(e)}(x)$ and consider one such $\sigma \in \mathbb{P}_2$ for which we have that $x \in W^\sigma_{h(e)}$ and

$$(\forall z \in \mathit{Dom}(\sigma))[(\sigma(z) = 1 \ \& \ \tau \Vdash_\alpha F_z(z)) \ \vee (\sigma(z) = 0 \ \& \ \tau \Vdash_\alpha \neg F_z(z))]].$$

By Equivalence (2.1), since $x \in W^\sigma_{h(e)}$, it follows that the number $g(e,x)$ is among the numbers $z \in \mathit{Dom}(\sigma)$ for which $\sigma(z) = 1$. In this way, for $z = g(e,x)$, we obtain $g(e,x) \in \mathit{Dom}(\sigma)$, $\sigma(g(e,x)) = 1$ and hence $\tau \Vdash_\alpha F_{g(e,x)}(g(e,x))$. We conclude that

$$\tau \Vdash_{\alpha+1} F_{h(e)}(x) \ \rightarrow \ \tau \Vdash_\alpha F_{g(e,x)}(g(e,x)).$$

It is easy to see that we also have the following:

$$\begin{aligned}
\tau \Vdash_\alpha \neg F_e(x) \ &\leftrightarrow (\forall \rho \supseteq \tau)[\rho \nVdash_\alpha F_e(x)] \leftrightarrow (\forall \rho \supseteq \tau)[\rho \nVdash_{\alpha+1} F_{h(e)}(x)] \\
&\leftrightarrow \tau \Vdash_{\alpha+1} \neg F_{h(e)}(x).
\end{aligned}$$

For the second part, let $\alpha = \lim \alpha(p)$ and take h' to be a computable function such that for any index e and natural numbers x, p,

$$(\forall \sigma \in \mathbb{P}_2)[x \in W^{\sigma}_{h'(e,p)} \leftrightarrow \sigma(\langle g(e,x), p\rangle) = 1]. \tag{2.2}$$

In other words,

$$(\forall \sigma \in \mathbb{P}_2)[x \in W^{\sigma}_{h'(e,p)} \leftrightarrow \{\langle\langle g(e,x), p\rangle, 1\rangle\} \subseteq Graph(\sigma)].$$

It suffices to prove that $\tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))$ iff $\tau \Vdash_{\alpha} F_{h'(e,p)}(x)$. For the $(\rightarrow)$ part, we have the equivalences:

$$\begin{aligned}
\tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x)) \leftrightarrow\ & (\exists \sigma \in \mathbb{P}_2)[Graph(\sigma) = \{\langle\langle g(e,x), p\rangle, 1\rangle\} \\
& \quad \& \ \tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))] \\
\leftrightarrow\ & (\exists \sigma \in \mathbb{P}_2)[x \in W^{\sigma}_{h'(e,p)} \ \& \ Graph(\sigma) = \{\langle\langle g(e,x), p\rangle, 1\rangle\} \\
& \quad \& \ \tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))] \\
\rightarrow\ & (\exists \sigma \in \mathbb{P}_2)[x \in W^{\sigma}_{h'(e,p)} \ \& \ (\forall z \in Dom(\sigma))[z = \langle x_z, p_z\rangle \\
& \quad \& \ ((\sigma(z) = 1 \ \& \ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee \\
& \quad (\sigma(z) = 0 \ \& \ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]] \\
\rightarrow\ & \tau \Vdash_{\alpha} F_{h'(e,p)}(x).
\end{aligned}$$

Now for the $(\leftarrow)$ part, let $\tau \Vdash_{\alpha} F_{h'(e,p)}(x)$ and consider one such $\sigma \in \mathbb{P}_2$ for which we have

$$\begin{aligned}
x \in W^{\sigma}_{h'(e,p)} \ \& \ (\forall z \in Dom(\sigma))[z = \langle x_z, p_z\rangle \ \& \ ((\sigma(z) = 1 \ \& \ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee \\
(\sigma(z) = 0 \ \& \ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)))]].
\end{aligned}$$

By Equivalence (2.2), since $x \in W^{\sigma}_{h'(e,p)}$, it follows that the number $\langle g(e,x), p\rangle$ is among the numbers $\langle x_z, p_z\rangle \in Dom(\sigma)$ for which $\sigma(\langle x_z, p_z\rangle) = 1$. In this way, for $x_z = g(e,x)$ and $p_z = p$, we obtain $\langle g(e,x), p\rangle \in Dom(\sigma)$, $\sigma(\langle g(e,x), p\rangle) = 1$, and hence $\tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))$. We conclude that if $\tau \Vdash_{\alpha} F_{h'(e,p)}(x)$, then $\tau \Vdash_{\alpha(p)} F_{g(e,x)}(g(e,x))$. It is again easy to see that $\tau \Vdash_{\alpha(p)} \neg F_e(x)$ if and only if $\tau \Vdash_{\alpha} \neg F_{h'(e,p)}(x)$. □

Let f be an enumeration of $\mathfrak{A}$. For every $e, x \in \mathbb{N}$ and every computable ordinal $\alpha \geq 1$, we define the *modelling relations* $f \models_{\alpha} F_e(x)$ and $f \models_{\alpha} \neg F_e(x)$ in the following way:

(i) $f \models_1 F_e(x) \leftrightarrow x \in W_e^{f^{-1}(\mathfrak{A})}$

(ii) Let $\alpha = \beta + 1$. Then

$$\begin{aligned}
f \models_{\beta+1} F_e(x) \leftrightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^{\delta} \ \& \ (\forall z \in Dom(\delta))[\\
& (\delta(z) = 1 \ \& \ f \models_{\beta} F_z(z)) \vee \\
& (\delta(z) = 0 \ \& \ f \models_{\beta} \neg F_z(z))]].
\end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned} f \models_\alpha F_e(x) \ \leftrightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \& \ (\forall z \in Dom(\delta))[z = \langle x_z, p_z \rangle \ \& \\ & ((\delta(z) = 1 \ \& \ f \models_{\alpha(p_z)} F_{x_z}(x_z)) \vee \\ & (\delta(z) = 0 \ \& \ f \models_{\alpha(p_z)} \neg F_{x_z}(x_z)))]]. \end{aligned}$$

(iv) $f \models_\alpha \neg F_e(x) \ \leftrightarrow\ f \not\models_\alpha F_e(x)$.

Lemma 2. *For any computable ordinal $\alpha \geq 1$, and any enumeration f of $\mathfrak{A}$,*

$$\begin{aligned} x \in W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))} &\ \leftrightarrow\ f \models_\alpha F_e(x), \\ x \notin W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))} &\ \leftrightarrow\ f \models_\alpha \neg F_e(x). \end{aligned}$$

Proof. The proof is by induction on α. The case $\alpha = 1$ follows from the definition of $\models_1$. Let $\alpha = \beta + 1$. Recall that for any set of natural numbers X, $\Delta^0_\alpha(X) = (\Delta^0_\beta(X))'$. For any $\rho \in \mathbb{P}_2$, we have:

$$\begin{aligned} \rho \subseteq \Delta^0_\alpha(f^{-1}(\mathfrak{A})) \ \leftrightarrow\ & (\forall z \in Dom(\rho))[(\rho(z) = 1 \ \& \ z \in \Delta^0_\alpha(f^{-1}(\mathfrak{A}))) \\ & \vee\ (\rho(z) = 0 \ \& \ z \notin \Delta^0_\alpha(f^{-1}(\mathfrak{A})))] \\ \leftrightarrow\ & (\forall z \in Dom(\rho))[(\rho(z) = 1 \ \& \ z \in W_z^{\Delta^0_\beta(f^{-1}(\mathfrak{A}))}) \\ & \vee\ (\rho(z) = 0 \ \& \ z \notin W_z^{\Delta^0_\beta(f^{-1}(\mathfrak{A}))})] \\ \leftrightarrow\ & (\forall z \in Dom(\rho))[(\rho(z) = 1 \ \& \ f \models_\beta F_z(z)) \\ & \vee\ (\rho(z) = 0 \ \& \ f \models_\beta \neg F_z(z)),] \end{aligned}$$

where the last equivalence follows from the induction hypothesis for β. Thus, we have the equivalences:

$$\begin{aligned} x \in W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))} \ \leftrightarrow\ & (\exists \rho \in \mathbb{P}_2)[x \in W_e^\rho \ \& \ \rho \subseteq \Delta^0_\alpha(f^{-1}(\mathfrak{A}))] \\ \leftrightarrow\ & (\exists \rho \in \mathbb{P}_2)[x \in W_e^\rho \ \& \ (\forall z \in Dom(\rho))[\\ & \quad (\rho(z) = 1 \ \& \ f \models_\beta F_z(z)) \vee \\ & \quad (\rho(z) = 0 \ \& \ f \models_\beta \neg F_z(z))]] \\ \leftrightarrow\ & f \models_\alpha F_e(x). \end{aligned}$$

Let $\alpha = \lim \alpha(p)$. For any $\rho \in \mathbb{P}_2$, we have:

$$\begin{aligned} \rho \subseteq \Delta^0_\alpha(f^{-1}(\mathfrak{A})) \ \leftrightarrow\ & (\forall z \in Dom(\rho))[z = \langle x_z, p_z \rangle \ \& \\ & (\rho(z) = 1 \ \& \ x_z \in \Delta^0_{\alpha(p_z)+1}(f^{-1}(\mathfrak{A}))) \\ & \vee\ (\rho(z) = 0 \ \& \ x_z \notin \Delta^0_{\alpha(p_z)+1}(f^{-1}(\mathfrak{A})))] \\ \leftrightarrow\ & (\forall z \in Dom(\rho))[z = \langle x_z, p_z \rangle \ \& \\ & (\rho(z) = 1 \ \& \ x_z \in W_{x_z}^{\Delta^0_{\alpha(p_z)}(f^{-1}(\mathfrak{A}))}) \\ & \vee\ (\rho(z) = 0 \ \& \ x_z \notin W_{x_z}^{\Delta^0_{\alpha(p_z)}(f^{-1}(\mathfrak{A}))})] \end{aligned}$$

$$\begin{aligned}&\leftrightarrow (\forall z \in Dom(\rho))[z = \langle x_z, p_z\rangle \ \&\\ &\quad (\rho(z) = 1 \ \& \ f \models_{\alpha(p_z)} F_{x_z}(x_z))\\ &\quad \vee \ (\rho(z) = 0 \ \& \ f \models_{\alpha(p_z)} \neg F_{x_z}(x_z))],\end{aligned}$$

where we have used the induction hypothesis for ordinals $\alpha(p) < \alpha$. Let us recall that according to our definition for limit ordinals $\alpha = \lim \alpha(p)$,

$$\langle x, p\rangle \in \Delta^0_\alpha(X) \ \leftrightarrow \ x \in \Delta^0_{\alpha(p)+1}(X) \ \leftrightarrow \ x \in W_x^{\Delta^0_{\alpha(p)}(X)}.$$

Thus, we have the equivalences:

$$\begin{aligned} x \in W_e^{\Delta^0_\alpha(f^{-1}(\mathfrak{A}))} &\leftrightarrow (\exists \rho \in \mathbb{P}_2)[x \in W_e^\rho \ \& \ \rho \subseteq \Delta^0_\alpha(f^{-1}(\mathfrak{A}))]\\ &\leftrightarrow (\exists \rho \in \mathbb{P}_2)[x \in W_e^\rho \ \& \ (\forall z \in Dom(\rho))[z = \langle x_z, p_z\rangle\\ &\quad (\rho(x_z) = 1 \ \& \ f \models_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\ &\quad (\rho(x_z) = 0 \ \& \ f \models_{\alpha(p_z)} \neg F_{x_z}(x_z))]]\\ &\leftrightarrow f \models_\alpha F_e(x).\end{aligned}$$

□

Definition 5. *Let $\alpha > 1$ be a computable ordinal and $\mathfrak{A}$ a countable structure. An enumeration f of $\mathfrak{A}$ is called* α-generic *in the following two cases:*

1) $\alpha = \beta + 1$, *and for every $e, x \in \mathbb{N}$*

$$(\exists \tau \in \mathbb{P}_2)[\tau \subseteq f \ \& \ (\tau \Vdash_\beta F_e(x) \ \vee \ \tau \Vdash_\beta \neg F_e(x))].$$

2) $\alpha = \lim \alpha(p)$, *and for every $e, x, p \in \mathbb{N}$*

$$(\exists \tau \in \mathbb{P}_2)[\tau \subseteq f \ \& \ (\tau \Vdash_{\alpha(p)} F_e(x) \ \vee \ \tau \Vdash_{\alpha(p)} \neg F_e(x))].$$

Proposition 2. *For every computable ordinal $\alpha > 1$, if g is a not α-generic enumeration of $\mathfrak{A}$, then there exist numbers e, x such that*

$$(\forall \tau \subseteq g)[\tau \nVdash_\alpha F_e(x) \ \& \ \tau \nVdash_\alpha \neg F_e(x)].$$

Proof. Let $\alpha = \beta + 1$. Since g is not α-generic, there exist numbers e, x such that

$$(\forall \tau \subseteq g)[\tau \nVdash_\beta F_e(x) \ \& \ \tau \nVdash_\beta \neg F_e(x)].$$

By Proposition 1, let $e_0 = h(e)$ be such that for every finite part τ

$$\tau \Vdash_{\beta+1} F_{e_0}(x) \ \leftrightarrow \ \tau \Vdash_\beta F_e(x),$$

$$\tau \Vdash_{\beta+1} \neg F_{e_0}(x) \ \leftrightarrow \ \tau \Vdash_\beta \neg F_e(x).$$

Since $\alpha = \beta + 1$, it follows that

$$(\forall \tau \subseteq g)[\tau \nVdash_\alpha F_{e_0}(x) \ \& \ \tau \nVdash_\alpha \neg F_{e_0}(x)].$$

Let $\alpha = \lim \alpha(p)$. Since g is not α-generic, there exist numbers e, x, p for which

$$(\forall \tau \subseteq g)[\tau \nVdash_{\alpha(p)} F_e(x) \ \& \ \tau \nVdash_{\alpha(p)} \neg F_e(x)].$$

Again by Proposition 1, let $e_0 = h'(p, e)$ be such that for every finite part τ

$$\tau \Vdash_\alpha F_{e_0}(x) \ \leftrightarrow \ \tau \Vdash_{\alpha(p)} F_e(x) \ \text{ and } \ \tau \Vdash_\alpha \neg F_{e_0}(x) \ \leftrightarrow \ \tau \Vdash_{\alpha(p)} \neg F_e(x).$$

It follows that

$$(\forall \tau \subseteq g)[\tau \nVdash_\alpha F_{e_0}(x) \ \& \ \tau \nVdash_\alpha \neg F_{e_0}(x)].$$

□

Lemma 3. 1) *Let $\alpha > 1$. If g is a $(\alpha+1)$-generic enumeration of $\mathfrak{A}$, then g is also α-generic.*

2) *Let $\alpha = \lim \alpha(p)$. If g is a α-generic enumeration of $\mathfrak{A}$, then g is also $\alpha(p)$-generic for any number p.*

Proof. For the first part, suppose that g is $(\alpha+1)$-generic, but g is not α-generic. By Proposition 2, this means that there exist natural numbers e, x for which

$$(\forall \tau \subseteq g)[\tau \nVdash_\alpha F_e(x) \ \& \ \tau \nVdash_\alpha \neg F_e(x)].$$

This contradicts the fact that g is $(\alpha+1)$-generic.

For the second part, suppose that g is α-generic, but g is not $\alpha(p)$-generic, for some natural number p. Again by Proposition 2, there exist numbers e, x for which

$$(\forall \tau \subseteq g)[\tau \nVdash_{\alpha(p)} F_e(x) \ \& \ \tau \nVdash_{\alpha(p)} \neg F_e(x)].$$

This contradicts the fact that g is α-generic. □

Lemma 4. *For every $e, x \in \mathbb{N}$, we have the following properties:*

1) *for any enumeration f of $\mathfrak{A}$, $f \models_1 F_e(x)$ iff $(\exists \tau \subseteq f)[\tau \Vdash_1 F_e(x)]$;*

2) *for $\alpha > 1$ and every α-generic enumeration g of $\mathfrak{A}$, $g \models_\alpha F_e(x)$ iff $(\exists \tau \subseteq g)[\tau \Vdash_\alpha F_e(x)]$;*

3) *for $\alpha \geq 1$ and every $(\alpha+1)$-generic enumeration g of $\mathfrak{A}$, $g \models_\alpha \neg F_e(x)$ iff $(\exists \tau \subseteq g)[\tau \Vdash_\alpha \neg F_e(x)]$.*

Proof. Part 1) follows from the facts:

- if $\tau \subseteq f$ and $x \in W_e^{\tau^{-1}(\mathfrak{A})}$, then $x \in W_e^{f^{-1}(\mathfrak{A})}$;
- if $x \in W_e^{f^{-1}(\mathfrak{A})}$, then there is $\tau \subseteq f$ such that $x \in W_e^{\tau^{-1}(\mathfrak{A})}$.

We prove 2) and 3) by transfinite induction on α. We start with 3) for $\alpha = 1$. Let g be 2-generic. For the $(\rightarrow)$ part, let $g \models_1 \neg F_e(x)$, but assume $(\not\exists \tau \subseteq g)[\tau \Vdash_1 \neg F_e(x)]$. Since g is 2-generic, $\tau \Vdash_1 F_e(x)$, for some $\tau \subseteq g$. But by 1),

$$\tau \Vdash_1 F_e(x) \ \&\ \tau \subseteq g \ \rightarrow\ g \models_1 F_e(x).$$

We reach a contradiction.

For the direction $(\leftarrow)$, let us fix a finite part $\tau \subseteq g$ such that $\tau \Vdash_1 \neg F_e(x)$, but assume $g \not\models_1 \neg F_e(x)$, which, by definition, means $g \models_1 F_e(x)$. Then by 1), there is a finite part $\delta \subseteq g$ such that $\delta \Vdash_1 F_e(x)$. By 1) of Lemma 1, we can take δ to be such that $\tau \subseteq \delta$. But then again by Lemma 1,

$$\tau \Vdash_1 \neg F_e(x) \ \&\ \tau \subseteq \delta \ \rightarrow\ \delta \Vdash_1 \neg F_e(x).$$

It follows that $\delta \nVdash_1 F_e(x)$, which is a contradiction with our choice of δ.

Let $\alpha = \beta + 1$ and let g be α-generic. We first consider the direction $(\rightarrow)$ of 2). Suppose we have $g \models_{\beta+1} F_e(x)$. Then

$$\begin{aligned} g \models_{\beta+1} F_e(x) \ \leftrightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^{\delta} \ \&\ (\forall z \in Dom(\delta))[\\ & (\delta(z) = 1 \ \&\ g \models_\beta F_z(z)) \vee \\ & (\delta(z) = 0 \ \&\ g \models_\beta \neg F_z(z))]] \end{aligned}$$

Fix one such $\delta \in \mathbb{P}_2$. Then by the induction hypothesis for 2) and 3),

$$\begin{aligned} (\forall z \in Dom(\delta))[(\delta(z) = 1 \ \&\ (\exists \tau_z \subseteq g)[\tau_z \Vdash_\beta F_z(z)]) \vee \\ (\delta(z) = 0 \ \&\ (\exists \tau_z \subseteq g)[\tau_z \Vdash_\beta \neg F_z(z)])]]. \end{aligned}$$

Choose appropriate finite parts τ_z and let $\tau = \bigcup_{z \in Dom(\delta)} \tau_z$. Then by Lemma 1, since every $\tau_z \subseteq \tau$,

$$\tau_z \Vdash_\beta F_z(z) \ \rightarrow \tau \Vdash_\beta F_z(z),$$

$$\tau_z \Vdash_\beta \neg F_z(z) \ \rightarrow \tau \Vdash_\beta \neg F_z(z).$$

It follows that

$$\begin{aligned} g \models_{\beta+1} F_e(x) \ \rightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^{\delta} \ \&\ (\forall z \in Dom(\delta))[\\ & (\delta(z) = 1 \ \&\ \tau \Vdash_\beta F_z(z)) \vee \\ & (\delta(z) = 0 \ \&\ \tau \Vdash_\beta \neg F_z(z))]] \\ \rightarrow\ & \tau \Vdash_{\beta+1} F_e(x). \end{aligned}$$

We conclude that $g \models_{\beta+1} F_e(x) \ \rightarrow\ (\exists \tau \subseteq g)[\tau \Vdash_{\beta+1} F_e(x)]$.

Now we consider part $(\leftarrow)$ of 2). Suppose there is $\tau \subseteq g$ such that $\tau \Vdash_{\beta+1} F_e(x)$. Then, by definition and the induction hypothesis for 2) and 3),

$$\begin{aligned}
\tau \Vdash_{\beta+1} F_e(x) \leftrightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[\\
& (\delta(z) = 1 \ \&\ \tau \Vdash_\beta F_z(z)) \vee\\
& (\delta(z) = 0 \ \&\ \tau \Vdash_\beta \neg F_z(z))]]\\
\rightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[\\
& (\delta(z) = 1 \ \&\ g \models_\beta F_z(z)) \vee\\
& (\delta(z) = 0 \ \&\ g \models_\beta \neg F_z(z))]]\\
\leftrightarrow\ & g \models_{\beta+1} F_e(x).
\end{aligned}$$

We conclude that $(\exists \tau \subseteq g)[\tau \Vdash_{\beta+1} F_e(x)] \rightarrow g \models_{\beta+1} F_e(x)$.

The proof of 3) is essentially the same as in the case $\alpha = 1$.

Let $\alpha = \lim \alpha(p)$ and let g be α-generic. For the $(\rightarrow)$ part of 2), suppose $g \models_\alpha F_e(x)$.

$$\begin{aligned}
g \models_\alpha F_e(x) \ \leftrightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \ \&\\
& (\delta(z) = 1 \ \&\ g \models_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\
& (\delta(z) = 0 \ \&\ g \models_{\alpha(p_z)} \neg F_{x_z}(x_z))]].
\end{aligned}$$

Fix one such $\delta \in \mathbb{P}_2$. Then, by 1) and the induction hypothesis for 2) and 3),

$$\begin{aligned}
(\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \ \&\ (\delta(z) = 1 \ \&\ (\exists \tau_z \subseteq g)[\tau_z \Vdash_{\alpha(p_z)} F_{x_z}(x_z)]) \vee\\
(\delta(z) = 0 \ \&\ (\exists \tau_z \subseteq g)[\tau_z \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z)])]].
\end{aligned}$$

Again, choose appropriate τ_z and let $\tau = \bigcup_{z \in Dom(\delta)} \tau_z$. Then by Lemma 1, since every $\tau_z \subseteq \tau$,

$$\begin{aligned}
\tau_z \Vdash_{\alpha(p_z)} F_{x_z}(x_z) \ &\rightarrow \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z),\\
\tau_z \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z) \ &\rightarrow \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z).
\end{aligned}$$

It follows that

$$\begin{aligned}
g \models_\alpha F_e(x) \ \rightarrow\ & (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \&\ (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \ \&\\
& (\delta(z) = 1 \ \&\ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee\\
& (\delta(z) = 0 \ \&\ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z))]]\\
\rightarrow\ & \tau \Vdash_\alpha F_e(x).
\end{aligned}$$

We conclude that

$$g \models_\alpha F_e(x) \ \rightarrow\ (\exists \tau \subseteq g)[\tau \Vdash_\alpha F_e(x)].$$

For part ($\leftarrow$) of 2), suppose that there is $\tau \subseteq g$ such that $\tau \Vdash_{\beta+1} F_e(x)$. Then, by definition and the induction hypothesis for 2) and 3),

$$\begin{aligned}\tau \models_\alpha F_e(x) \ &\leftrightarrow (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \& \ (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \ \& \\ &\qquad (\delta(z) = 1 \ \& \ \tau \Vdash_{\alpha(p_z)} F_{x_z}(x_z)) \vee \\ &\qquad (\delta(z) = 0 \ \& \ \tau \Vdash_{\alpha(p_z)} \neg F_{x_z}(x_z))]] \\ &\rightarrow (\exists \delta \in \mathbb{P}_2)[x \in W_e^\delta \ \& \ (\forall z \in Dom(\delta))[z = \langle x_z, p_z\rangle \ \& \\ &\qquad (\delta(z) = 1 \ \& \ g \models_{\alpha(p_z)} F_{x_z}(x_z)) \vee \\ &\qquad (\delta(z) = 0 \ \& \ g \models_{\alpha(p_z)} \neg F_{x_z}(x_z))]] \\ &\leftrightarrow g \models_\alpha F_e(x).\end{aligned}$$

We conclude that

$$(\exists \tau \subseteq g)[\tau \Vdash_\alpha F_e(x)] \rightarrow g \models_\alpha F_e(x).$$

The proof of 3) for $\alpha = \lim \alpha(p)$ is again very similar to the proof in the case of $\alpha = 1$. ☐

Let *var* be a computable mapping of the natural numbers onto the variables. By X_i we denote the variable $var(i)$. For a finite set $D = \{d_0 < d_1 < \cdots < d_{k-1}\}$ of natural numbers and a formula Φ with free variables including $\{X_i \mid i \in D\}$, it is convenient to denote

$$(\exists_D)\Phi \ \equiv \ (\exists X_{d_0} \ldots \exists X_{d_{k-1}})\Phi.$$

Moreover, for any finite part ρ and any formula Φ, by $\Phi(\bar{\rho})$ we denote the formula obtained from Φ by replacing each occurrence of the free variable X_i in Φ by the constant $\rho(i)$, for every $i \in Dom(\rho)$.

Lemma 5 (Definability of forcing)**.** *Let $\mathfrak{A}$ be a structure in the language $\mathscr{L} = \{P_0, \ldots, P_{s-1}\}$, which include equality. Then for every non-empty finite set D of natural numbers, every natural numbers e, x and a computable ordinal $\alpha \geq 1$, we can effectively find a Σ^c_α formula $\Phi^\alpha_{D,e,x}$ and a Π^c_α formula $\Theta^\alpha_{D,e,x}$ in the language $\mathscr{L}$ with free variables in $\{X_i \mid i \in D\}$ such that for every finite part δ with $Dom(\delta) = D$, we have the following:*

$$\begin{aligned}\delta \Vdash_\alpha F_e(x) \ &\leftrightarrow \ \mathfrak{A} \models \Phi^\alpha_{D,e,x}(\bar{\delta}), \\ \delta \Vdash_\alpha \neg F_e(x) \ &\leftrightarrow \ \mathfrak{A} \models \Theta^\alpha_{D,e,x}(\bar{\delta})\end{aligned}$$

Proof. We will define the formulae $\Phi^\alpha_{D,e,x}$ by effective transfinite recursion on the computable ordinals α following the definition of the forcing relation. For every e, x, let $W_{e,x} = \{\kappa \in \mathbb{P}_2 \mid x \in W_e^\kappa\}$, which is a c.e. set.

Let $\alpha = 1$. Then, by definition,

$$\tau \Vdash_1 F_e(x) \ \leftrightarrow \ x \in W_e^{\tau^{-1}(\mathfrak{A})} \leftrightarrow (\exists \kappa \in \mathbb{P}_2)[x \in W_e^\kappa \ \& \ \kappa \subseteq \tau^{-1}(\mathfrak{A})].$$

We define the atomic formulae $\Psi^1_{D,\kappa,u}$ in the following way:

- if $u = s \cdot \langle i_1, \dots, i_{n_r} \rangle + r$ for $r < s$ and $i_1, \dots, i_{n_r} \in D$, then

$$\Psi^1_{D,\kappa,u} \equiv \begin{cases} P_r(X_{i_1}, \dots, X_{i_{n_r}}), & \text{if } \kappa(u) = 1, \\ \neg P_r(X_{i_1}, \dots, X_{i_{n_r}}), & \text{if } \kappa(u) = 0. \end{cases}$$

- otherwise, we set $\Psi^1_{D,\kappa,u} \equiv \neg(X_d = X_d)$, where d is some element of D.

We define the atomic formula $\Psi^1_{D,\kappa}$ with free variables in $\{X_i \mid i \in D\}$ as

$$\Psi^1_{D,\kappa} \equiv \bigwedge_{\substack{d \neq d', \\ d,d' \in D}} X_d \neq X_{d'} \ \& \bigwedge_{u \in Dom(\kappa)} \Psi^1_{D,\kappa,u}.$$

We have the property:

$$\kappa \subseteq \delta^{-1}(\mathfrak{A}) \ \leftrightarrow \ (\forall u \in Dom(\kappa))[\mathfrak{A} \models \Psi^1_{Dom(\delta),\kappa,u}(\bar{\delta})]$$

and hence

$$\kappa \subseteq \delta^{-1}(\mathfrak{A}) \ \leftrightarrow \ \mathfrak{A} \models \Psi^1_{Dom(\delta),\kappa}(\bar{\delta}).$$

In the end, we define

$$\Phi^1_{D,e,x} \equiv \bigvee_{\kappa \in W_{e,x}} \Psi^1_{D,\kappa},$$

which is a Σ^c_1 formula with free variables in $\{X_i \mid i \in D\}$.

Let us fix e, x and $\delta \in \mathbb{P}_A$. Let $D = Dom(\delta)$. We have the equivalences:

$$\begin{aligned} \delta \Vdash_1 F_e(x) \ &\leftrightarrow (\exists \kappa \in \mathbb{P}_2)[x \in W^\kappa_e \ \& \ \kappa \subseteq \delta^{-1}(\mathfrak{A})] \\ &\leftrightarrow \mathfrak{A} \models \bigvee_{\kappa \in W_{e,x}} \Psi^1_{D,\kappa}(\bar{\delta}) \\ &\leftrightarrow \mathfrak{A} \models \Phi^1_{D,e,x}(\bar{\delta}), \end{aligned}$$

$$\begin{aligned} \delta \Vdash_1 \neg F_e(x) \ &\leftrightarrow (\nexists \rho \in \mathbb{P}_A)[\rho \supseteq \delta \ \& \ \mathfrak{A} \models \Phi^1_{Dom(\rho),e,x}(\bar{\rho})] \\ &\leftrightarrow (\nexists D' \supseteq D)[\mathfrak{A} \models (\exists_{D' \setminus D}) \Phi^1_{D',e,x}(\bar{\delta})] \\ &\leftrightarrow \mathfrak{A} \models \neg \bigvee_{D' \supseteq D} (\exists_{D' \setminus D}) \Phi^1_{D',e,x}(\bar{\delta}). \end{aligned}$$

We set

$$\Theta^1_{D,e,x} \equiv \neg \bigvee_{D' \supseteq D} (\exists_{D' \setminus D}) \Phi^1_{D',e,x}.$$

Let $\alpha = \beta + 1$. Let us consider $\kappa \in W_{e,x}$. Then for every $u \in Dom(\kappa)$, we define

$$\Psi^\alpha_{D,\kappa,u} \equiv \begin{cases} \Phi^\beta_{D,u,u}, & \text{if } \kappa(u) = 1 \\ \\ \Theta^\beta_{D,u,u}, & \text{if } \kappa(u) = 0. \end{cases}$$

By definition, $\Psi^{\alpha}_{D,\kappa,u}$ is either a Σ^c_β or a Π^c_β formula. We let

$$\Psi^{\alpha}_{D,\kappa} \equiv \bigwedge_{\substack{d\neq d',\\ d,d'\in D}} X_d \neq X_{d'} \ \&\ \bigwedge_{u\in Dom(\kappa)} \Psi^{\alpha}_{D,\kappa,u},$$

which is a *finite* conjunction of Σ^c_β and Π^c_β formulae with free variables in $\{X_i \mid i \in D\}$. We can view $\Psi^{\alpha}_{D,\kappa}$ as a finite conjunction of $\Sigma^c_{\beta+1}$ formulae and hence it is equivalent to a $\Sigma^c_{\beta+1}$ formula. In the end, we define

$$\Phi^{\alpha}_{D,e,x} \equiv \bigvee_{\kappa\in W_{e,x}} \Psi^{\alpha}_{D,\kappa},$$

which is a Σ^c_α formula with free variables in $\{X_i \mid i \in D\}$.

Now we are ready to show that the formula $\Phi^{\alpha}_{D,e,x}$ defines the forcing relation $\delta \Vdash_\alpha F_e(x)$, where $D = Dom(\delta)$. We have the following equivalences:

$$\begin{aligned}
\delta \Vdash_\alpha F_e(x) \ &\leftrightarrow (\exists \kappa \in \mathbb{P}_2)[x \in W^{\kappa}_e \ \&\ (\forall u \in Dom(\kappa))[\\
&\qquad (\kappa(u) = 1 \ \&\ \delta \Vdash_\beta F_u(u)) \vee (\kappa(u) = 0 \ \&\ \delta \Vdash_\beta \neg F_u(u))]]\\
&\leftrightarrow \mathfrak{A} \models \bigvee_{\kappa\in W_{e,x}} \bigwedge_{u\in Dom(\kappa)} \Psi^{\alpha}_{D,\kappa,u}(\bar{\delta})\\
&\leftrightarrow \mathfrak{A} \models \Phi^{\alpha}_{D,e,x}(\bar{\delta})
\end{aligned}$$

Again, it is easy to see that the Π^c_α formula

$$\Theta^{\alpha}_{D,e,x} \equiv \neg \bigvee_{D'\supseteq D} (\exists_{D'\setminus D}) \Phi^{\alpha}_{D',e,x}$$

defines in $\mathfrak{A}$ the relation $\delta \Vdash_\alpha \neg F_e(x)$.

Let $\alpha = \lim \alpha(p)$ and consider $\kappa \in W_{e,x}$. Then for every $u \in Dom(\kappa)$ we define the formula $\Psi^{\alpha}_{D,\kappa,u}$ in the following way:

- if $u = \langle x_u, p_u\rangle$, then

$$\Psi^{\alpha}_{D,\kappa,u} \equiv \begin{cases} \Phi^{\alpha(p_u)}_{D,x_u,x_u}, & \text{if } \kappa(u) = 1\\[2ex] \Theta^{\alpha(p_u)}_{D,x_u,x_u}, & \text{if } \kappa(u) = 0 \end{cases}$$

- otherwise, we set $\Psi^{\alpha}_{D,\kappa,u} \equiv \neg(X_{d_0} = X_{d_0})$, where d_0 is some element of D.

Again we set

$$\Psi^{\alpha}_{D,\kappa} \equiv \bigwedge_{\substack{d\neq d',\\ d,d'\in D}} X_d \neq X_{d'} \ \&\ \bigwedge_{u\in Dom(\kappa)} \Psi^{\alpha}_{D,\kappa,u},$$

which is a *finite* conjunction of Σ^c_β and Π^c_β formulae, for various $\beta < \alpha$, with free variables in $\{X_i \mid i \in D\}$. Therefore, $\Psi^\alpha_{D,\kappa}$ is also a Σ^c_γ formula for some $\gamma < \alpha$.

In the end, we define the Σ^c_α formula

$$\Phi^\alpha_{D,e,x} \equiv \bigvee_{\kappa \in W_{e,x}} \Psi^\alpha_{D,\kappa}.$$

By the induction hypothesis we obtain:

$$\begin{aligned} \delta \Vdash_\alpha F_e(x) \;&\leftrightarrow (\exists \kappa \in \mathbb{P}_2)[x \in W^\kappa_e \;\&\; (\forall u \in Dom(\kappa))[u = \langle x_u, p_u\rangle \;\&\\ &\quad (\kappa(u) = 1 \;\&\; \delta \Vdash_{\alpha(p_u)} F_{x_u}(x_u)) \vee \\ &\quad (\kappa(u) = 0 \;\&\; \delta \Vdash_{\alpha(p_u)} \neg F_{x_u}(x_u))]] \\ &\leftrightarrow \mathfrak{A} \models \bigvee_{\kappa \in W_{e,x}} \bigwedge_{u \in Dom(\kappa)} \Psi^\alpha_{D,\kappa,u}(\bar{\delta}) \\ &\leftrightarrow \mathfrak{A} \models \bigvee_{\kappa \in W_{e,x}} \Psi^\alpha_{D,\kappa}(\bar{\delta}) \\ &\leftrightarrow \mathfrak{A} \models \Phi^\alpha_{D,e,x}(\bar{\delta}), \end{aligned}$$

where $D = Dom(\delta)$. Moreover, $\delta \Vdash_\alpha \neg F_e(x) \;\leftrightarrow\; \mathfrak{A} \models \Theta^\alpha_{Dom(\delta),e,x}(\bar{\delta})$, where

$$\Theta^\alpha_{D,e,x} \equiv \neg[\bigvee_{D' \supseteq D} (\exists_{D' \setminus D}) \Phi^\alpha_{D',e,x}].$$

□

2.2. MOSCHOVAKIS' EXTENSION

We proceed with the investigation of conditions under which we have the other directions in Theorem 4. For this purpose we need firstly to introduce some coding machinery and then the sets $K^{\mathfrak{A}}_\alpha$ which will serve as universal predicates for the Σ^c_α formulae.

Following Moschovakis [8], we define the least acceptable extension $\mathfrak{A}^\star$ of $\mathfrak{A}$, which we call the Moschovakis' extension of $\mathfrak{A}$. Let 0 be an object which does not belong to A and Π be a pairing operation chosen so that neither 0 nor any element of A is an ordered pair. Let $A^\star$ be the least set containing all elements of $A_0 = A \cup \{0\}$ and closed under Π.

We associate an element $n^\star$ of $A^\star$ with each $n \in \mathbb{N}$ by induction. Let

$$0^\star = 0 \text{ and } (n+1)^\star = \Pi(0, n^\star).$$

We denote by $\mathbb{N}^\star$ the set of all elements $n^\star$. Let L and R be the functions on $A^\star$ satisfying the following conditions:

$$\begin{aligned} &L(0) = R(0) = 0; \\ &(\forall t \in A)[L(t) = R(t) = 1^\star]; \\ &(\forall s, t \in A^\star)[L(\Pi(s,t)) = s \;\&\; R(\Pi(s,t)) = t]. \end{aligned}$$

The pairing function allows us to code finite sequences of elements. Let

$$\Pi_1(t_1) = t_1 \text{ and } \Pi_{n+1}(t_1, \ldots, t_{n+1}) = \Pi(t_1, \Pi_n(t_2, \ldots, t_{n+1})),$$

for every $t_1, \ldots, t_{n+1} \in A^\star$. For each predicate P_i of the structure $\mathfrak{A}$ define the respective predicate $P_i^\star$ on $A^\star$ by

$$P_i^\star(t) \leftrightarrow (\exists a_1, \ldots, a_{n_i} \in A)[t = \Pi_{n_i}(a_1, \ldots, a_{n_i}) \ \&\ P_i(a_1, \ldots, a_{n_i})].$$

For an enumeration f of $A^\star$, we denote

$$f^{-1}(\Pi_n)(x_0, \ldots, x_{n-1}) = y \ \leftrightarrow\ (\exists a_0, \ldots, a_{n-1} \in A)[\bigwedge_{i<n} f(x_i) = a_i \ \& \\ \Pi_n(a_0, \ldots, a_{n-1}) = f(y)]$$

Definition 6. *Moschovakis' extension of* $\mathfrak{A}$ *is the structure*

$$\mathfrak{A}^\star = (A^\star; A_0, P_1^\star, \ldots, P_s^\star, G_\Pi, G_L, G_R, =),$$

where G_Π, G_L *and* G_R *are the graphs of* Π, L *and* R *respectively.*

When we have two structures $\mathfrak{A}$ and $\mathfrak{B}$ with domains $A \subseteq B$, we assume that their respective Moschovakis' extensions $\mathfrak{A}^\star$ and $\mathfrak{B}^\star$ are defined so that $A^\star \subseteq B^\star$. We proceed with a few technical results which will be used often when we want to show that a property for $\mathfrak{A}$ also holds for $\mathfrak{A}^\star$ or vice-versa.

Proposition 3. *Let* f *be an enumeration of* $\mathfrak{A}$. *We define the enumeration* $f_\star$ *of* $\mathfrak{A}^\star$ *such that*

$$f_\star(0) = 0^\star,$$
$$f_\star(2n+1) = f(n),$$
$$f_\star(2^{k+1}(2n+1)) = \Pi(f_\star(k), f_\star(n)).$$

Then $f_\star \leq_1^1 f$, *and* $f \leq_1^1 f_\star$.

Proof. We follow Lemma 7 of [10] to show that $f^{-1}(\mathfrak{A}) \equiv_T f_\star^{-1}(\mathfrak{A}^\star)$.

Let $J(x,y) = 2^{x+1}(2y+1)$. Denote by induction for any $x_1, \ldots, x_n$, $J_1(x_1) = x_1$ and $J_{n+1}(x_1, \ldots, x_{n+1}) = J(x_1, J_n(x_2, \ldots, x_{n+1}))$. Let l and r be computable functions satisfying the equalities:

$$l(0) = r(0) = 0;$$

$$l(2x+1) = r(2x+1) = 2 = J(0,0);$$

$$l(J(x,y)) = x, \quad r(J(x,y)) = y.$$

It is easy to see that

$f_\star^{-1}(A_0) = \{2n+1 \mid n \in \mathbb{N}\} \cup \{0\};$

$f_\star^{-1}(G_\Pi) = \{\langle x,y,z\rangle \mid \Pi(f_\star(x), f_\star(y)) = f_\star(z)\} = \{\langle x,y,z\rangle \mid J(x,y) = z\};$

$f_\star^{-1}(G_L) = \{\langle x,y\rangle \mid L(f_\star(x)) = f_\star(y)\} = \{\langle x,y\rangle \mid l(x) = y\};$

$f_\star^{-1}(G_R) = \{\langle x,y\rangle \mid R(f_\star(x)) = f_\star(y)\} = \{\langle x,y\rangle \mid r(x) = y\}.$

Then for any relation $P \subseteq A^n$,

$$\begin{aligned}\langle x_1,\dots,x_n\rangle \in f^{-1}(P) &\leftrightarrow (f(x_1),\dots,f(x_n)) \in P\\ &\leftrightarrow (f_\star(2x_1+1),\dots,f_\star(2x_n+1)) \in P\\ &\leftrightarrow \Pi_n(f_\star(2x_1+1),\dots,f_\star(2x_n+1)) \in P^\star\\ &\leftrightarrow J_n(2x_1+1,\dots,2x_n+1) \in f_\star^{-1}(P).\end{aligned}$$

Since f and $f_\star$ are bijective, $f^{-1}(=^A) = f_\star(=^\star) = \{\langle z,z\rangle \mid z \in \mathbb{N}\}$, where $=^A$ is the equality on A and $=^\star$ is the equality on $A^\star$. We conclude that $f^{-1}(\mathfrak{A}) \equiv_T f_\star^{-1}(\mathfrak{A}^\star)$.

To prove $f_\star \leq_1^1 f$ and $f \leq_1^1 f_\star$, it is enough to check that $E(f_\star, f)$ is c.e. in $f^{-1}(\mathfrak{A})$. By the definition of $f_\star$, we have

$$E(f_\star, f) = \{\langle 2x+1, x\rangle \mid x \in \mathbb{N}\}.$$

Now it is clear that $E(f_\star, f)$ is c.e. and hence it is clearly c.e. in $f^{-1}(\mathfrak{A})$. □

Proposition 4. *Let f be an enumeration of $\mathfrak{A}^\star$. There is an enumeration $f_{\restriction A}$ of $\mathfrak{A}$ such that $f_{\restriction A} \leq_1^1 f$.*

Proof. Since A is a relation in $\mathfrak{A}^\star$, $f^{-1}(A)$ is computable in $f^{-1}(\mathfrak{A}^\star)$. Let us fix a computable in $f^{-1}(\mathfrak{A}^\star)$ enumeration $\{x_n\}_{n\in\mathbb{N}}$ of the set $f^{-1}(A)$. Define the enumeration $f_{\restriction A}$ of A as $f_{\restriction A}(n) = f(x_n)$. Then $E(f_{\restriction A}, f) = \{\langle n, x_n\rangle \mid n \in \mathbb{N}\}$ is clearly computable in $f^{-1}(\mathfrak{A}^\star)$. For any predicate P_i in $\mathfrak{A}$, the equivalences

$$\langle y_1,\dots,y_{n_i}\rangle \in f_{\restriction A}^{-1}(P_i) \leftrightarrow (\exists z)[z = f^{-1}(\Pi_{n_i})(x_{y_1},\dots,x_{y_{n_i}})\ \&\ z \in f^{-1}(P_i^\star)],$$

$$\langle y_1,\dots,y_{n_i}\rangle \notin f_{\restriction A}^{-1}(P_i) \leftrightarrow (\exists z)[z = f^{-1}(\Pi_{n_i})(x_{y_1},\dots,x_{y_{n_i}})\ \&\ z \notin f^{-1}(P_i^\star)],$$

show that $f_{\restriction A}^{-1}(P_i) \leq_T f^{-1}(\mathfrak{A}^\star)$. We conclude that $f_{\restriction A} \leq_1^1 f$. □

Proposition 5. *For any countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$, we have $\mathfrak{A} \Leftrightarrow_\alpha^\alpha \mathfrak{A}^\star$. In other words, $\mathfrak{A}^\star$ is (α,α)-conservative extension of $\mathfrak{A}$.*

Proof. Fix $\alpha > 0$. Let f be an enumeration of $\mathfrak{A}^\star$ and let $f_{\restriction A}$ be defined as in Proposition 4. Since $f_{\restriction A} \leq_1^1 f$, we have $f_{\restriction A} \leq_\alpha^\alpha f$. Thus, $\mathfrak{A} \Rightarrow_\alpha^\alpha \mathfrak{A}^\star$.

For the other direction, let f be an enumeration of $\mathfrak{A}$. Consider $f_\star$, defined as in Proposition 3. Since $f_\star \leq_1^1 f$, we have $f_\star \leq_\alpha^\alpha f$. Thus, $\mathfrak{A} \Leftarrow_\alpha^\alpha \mathfrak{A}^\star$. □

Fix an enumeration f of $\mathfrak{A}^\star$. We define a coding scheme for finite sequences of natural numbers in the following way:

$$J^f(x,y) = f^{-1}(\Pi(f(x), f(y)));$$
$$J^f_1(x) = x, \quad J^f_{n+1}(x_0,\dots,x_n) = J^f(x_0, J^f_n(x_1,\dots,x_n)).$$

We assign a measure $\|x\|^f$ for every natural number x in the following way:

$$\|x\|^f = \begin{cases} 0, & \text{if } x \in f^{-1}(A_0);\\ m+1, & \text{if } x = J^f(y,z) \ \&\ m = max\{\|y\|^f, \|z\|^f\}. \end{cases}$$

It is easy to see that J^f and $\|.\|^f$ are functions computable in $f^{-1}(\mathfrak{A}^\star)$.

Lemma 6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with domains $A \subseteq B$. Then for any computable ordinals $\alpha, \beta > 0$, $\mathfrak{A} \Leftrightarrow^\alpha_\beta \mathfrak{B}$ if and only if $\mathfrak{A}^\star \Leftrightarrow^\alpha_\beta \mathfrak{B}^\star$.*

Proof. We prove only the part $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$ if and only if $\mathfrak{A}^\star \Rightarrow^\alpha_\beta \mathfrak{B}^\star$. Then it is easy to see that we can apply a similar argument to prove that $\mathfrak{A} \Leftarrow^\alpha_\beta \mathfrak{B} \leftrightarrow \mathfrak{A}^\star \Leftarrow^\alpha_\beta \mathfrak{B}^\star$.

Let $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. We prove $\mathfrak{A}^\star \Rightarrow^\alpha_\beta \mathfrak{B}^\star$. Let h be an enumeration of $\mathfrak{B}^\star$. By Proposition 4, $h_{\restriction B}$ is an enumeration of $\mathfrak{B}$. Since $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$, there exists f of $\mathfrak{A}$ such that $f \leq^\alpha_\beta h_{\restriction B}$. We shall show that for the enumeration $f_\star$ of $\mathfrak{A}^\star$, we have $f_\star \leq^\alpha_\beta h$. Since $h_{\restriction B} \leq^1_1 h$ and $f \leq^\alpha_\beta h_{\restriction B}$, we have

$$\Delta^0_\alpha(f_\star^{-1}(\mathfrak{A}^\star)) \leq_T \Delta^0_\alpha(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(h^{-1}_{\restriction B}(\mathfrak{B})) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B}^\star)).$$

Thus, $\Delta^0_\alpha(f_\star^{-1}(\mathfrak{A}^\star)) \leq \Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))$, so we only need to prove that $E(f_\star, h)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}^\star))$. We remark that if $\langle x,y\rangle \in E(f_\star,h)$, then $\|x\|^{f_\star} = \|y\|^h$. We define the sets $E_i = \{\langle x,y\rangle \mid \|x\|^{f_\star} = \|y\|^h \leq i \ \&\ \langle x,y\rangle \in E(f_\star,h)\}$. Clearly, $E(f_\star,h) = \bigcup_{i\in\mathbb{N}} E_i$. We define by recursion on i a computable function μ such that for every i, $E_i = W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(i)}$. We will use the fact that

$$\langle x,y\rangle \in E_{i+1} \leftrightarrow \langle x,y\rangle \in E_0 \ \vee\ (\exists u,v,c,d)[x = J^{f_\star}(u,v) \ \&\ y = J^h(c,d) \ \& \\ \langle u,c\rangle \in E_i \ \&\ \langle v,d\rangle \in E_i].$$

Let $i = 0$. Fix $x_0 = f_\star^{-1}(0^\star)$ and $y_0 = h^{-1}(0^\star)$. Then

$$E_0 = \{\langle x_0,y_0\rangle\} \cup \{\langle x,y\rangle \mid x \in f_\star^{-1}(A) \ \&\ \langle x,y\rangle \in E(f_\star,h)\}$$

and by the definitions of $f_\star$ and $h_{\restriction B}$, for $u \in f_\star^{-1}(A)$,

$$\langle u,v\rangle \in E(f_\star,h) \text{ if and only if } (\exists n)[u = 2n+1 \ \&\ \langle n, x_v\rangle \in E(f, h_{\restriction B})],$$

where $\{x_n\}_{n\in\mathbb{N}}$ is a computable in $h^{-1}(\mathfrak{B}^\star)$ enumeration of $h^{-1}(B)$, which was used in the definition of $h_{\restriction B}$ in Proposition 4. We know that $E(f,h_{\restriction B})$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}^\star))$. Thus, $E_0 = W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{e_0}$ for some index e_0. Let $\mu(0) = e_0$.

Let $i = j + 1$. Since $J^{f_\star}$ and J^h are functions computable in $\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))$, define $\mu(j+1)$ to be an index such that

$$\begin{aligned}\langle x, y\rangle \in W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(j+1)} \leftrightarrow\ & \langle x, y\rangle \in W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(0)} \vee \\ & (\exists u, v, c, d)[x = J^{f_\star}(u, v)\ \&\ y = J^h(c, d)\ \& \\ & \langle u, c\rangle \in W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(j)}\ \&\ \langle u, d\rangle \in W^{\Delta^0_\beta(h^{-1}(\mathfrak{B}^\star))}_{\mu(j)}].\end{aligned}$$

Thus, $E(f_\star, h)$ is $\Sigma^0_\alpha(h^{-1}(\mathfrak{B}^\star))$ and hence $f_\star \leq^\alpha_\beta h$.

Let $\mathfrak{A}^\star \Rightarrow^\alpha_\beta \mathfrak{B}^\star$. We will prove $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. Take an enumeration h of $\mathfrak{B}$ and $h_\star$ as defined in Proposition 3. Fix the enumeration f of $\mathfrak{A}^\star$ such that $f \leq^\alpha_\beta h_\star$. We will show that $f_{\restriction A} \leq^\alpha_\beta h$. By the following chain,

$$\Delta^0_\alpha(f^{-1}_{\restriction A}(\mathfrak{A})) \leq \Delta^0_\alpha(f^{-1}(\mathfrak{A}^\star)) \leq_T \Delta^0_\beta(h^{-1}_\star(\mathfrak{B}^\star)) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B})),$$

we have $\Delta^0_\alpha(f^{-1}_{\restriction A}(\mathfrak{A})) \leq \Delta^0_\beta(h^{-1}(\mathfrak{B})$. Moreover, $\langle u, v\rangle \in E(f_{\restriction A}, h)$ if and only if $u \in f^{-1}(A)$ & $2v + 1 \in h^{-1}_\star(B)$ & $\langle x_u, 2v + 1\rangle \in E(f, h_\star)$, where $\{x_n\}_{n\in\mathbb{N}}$ is a computable in $f^{-1}(\mathfrak{A}^\star)$ enumeration of $f^{-1}(A)$, Thus, $E(f_{\restriction A}, h)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$ and $f_{\restriction A} \leq^\alpha_\beta h$. □

2.3. CODING TUPLES IN $A^\star$

For each finite part $\tau \in \mathbb{P}_A$, $\tau \neq \emptyset$ with $Dom(\tau) = \{x_1 < x_2 < \cdots < x_n\}$ and $\tau(x_i) = a_i$, we associate the element of $A^\star$, $\tau^\star = \Pi_n(\Pi(x^\star_1, a_1), \ldots, \Pi(x^\star_n, a_n))$. For $\tau = \emptyset$, let $\tau^\star = 0^\star$. We denote $\mathbb{P}^\star_A = \{\tau^\star \mid \tau \in \mathbb{P}_A\}$.

Proposition 6. *The sets $\mathbb{N}^\star$ and $\mathbb{P}^\star_A$ are uniformly relatively intrinsically computable in $\mathfrak{A}^\star$. Thus, $\mathbb{N}^\star$ and $\mathbb{P}^\star_A$ are definable in $\mathfrak{A}^\star$ by Σ^c_1 and Π^c_1 formulae without parameters.*

Proof. We briefly describe why $\mathbb{N}^\star$ is uniformly relatively intrinsically computable in $\mathfrak{A}^\star$. The proof for $\mathbb{P}^\star_A$ is similar.

For an enumeration f of $\mathfrak{A}^\star$, fix z such that $f(z) = 0^\star$. This is the unique element $z \in f^{-1}(A_0)$ such that $\langle z, z\rangle \in f^{-1}(G_R)$. Then $x \in f^{-1}(\mathbb{N}^\star)$ if and only if $x = z$ or $x = J^f_n(z, \ldots, z)$, where $n \geq 2$ is the least number such that there are numbers $y_1, \ldots, y_{n-1}$, different from z, and $\langle x, y_1\rangle \in f^{-1}(G_R), \langle y_1, y_2\rangle \in f^{-1}(G_R), \ldots, \langle y_{n-1}, z\rangle \in f^{-1}(G_R)$. □

Corollary 1. *The following relations are uniformly relatively intrinsically computable in $\mathfrak{A}^\star$:*

- *$Dm(x, y)$ if and only if $(\exists \tau \in \mathbb{P}_A)[y = \tau^\star\ \&\ x \in Dom(\tau)]$,*
- *$Rn(x, y)$ if and only if $(\exists \tau \in \mathbb{P}_A)[y = \tau^\star\ \&\ x \in Ran(\tau)]$,*

– $Sb(x,y)$ *if and only if* $(\exists\tau,\rho\in\mathbb{P}_A)[x=\tau^\star \ \&\ y=\rho^\star \ \&\ \tau\subseteq\rho]$.

Lemma 7. *For a countable structure* $\mathfrak{A}=(A;P_0,\dots,P_{s-1})$*, computable ordinal* $\alpha\geq 1$*, and natural numbers* e*,* x*,*

1) $X^\alpha_{e,x}=\{\tau^\star \mid \tau\Vdash^{\mathfrak{A}}_\alpha F_e(x)\}$ *is definable in* $\mathfrak{A}^\star$ *by a* Σ^c_α *formula* without *parameters;*

2) $Y^\alpha_{e,x}=\{\tau^\star \mid \tau\Vdash^{\mathfrak{A}}_\alpha \neg F_e(x)\}$ *is definable in* $\mathfrak{A}^\star$ *by a* Π^c_α *formula* without *parameters;*

3) $Z^\alpha_{e,x}=\{\tau^\star \mid (\exists\delta\in\mathbb{P}_A)[\delta\supseteq\tau \ \&\ \delta\Vdash^{\mathfrak{A}}_\alpha F_e(x)\}$ *is definable in* $\mathfrak{A}^\star$ *by a* Σ^c_α *formula* without *parameters.*

Given natural numbers e*,* x*, and a computable ordinal* $\alpha\geq 1$*, we can effectively find these formulae.*

Proof. Following the proof of Lemma 5 step by step, it is easy to see that for every non-empty set D of natural numbers, every e, x, and computable ordinal $\alpha\geq 1$, we can effectively find a Σ^c_α formula $\Phi^{\star,\alpha}_{D,e,x}$ and a Π^c_α formula $\Theta^{\star,\alpha}_{D,e,x}$ in the language of $\mathfrak{A}^\star$ with free variables in $\{X_i \mid i\in D\}$ such that for every $\delta\in\mathbb{P}_A$ with $Dom(\delta)=D$, we have

$$\delta\Vdash^{\mathfrak{A}}_\alpha F_e(x) \ \leftrightarrow\ \mathfrak{A}\models\Phi^\alpha_{D,e,x}(\bar\delta) \ \leftrightarrow\ \mathfrak{A}^\star\models\Phi^{\star,\alpha}_{D,e,x}(\bar\delta),$$
$$\delta\Vdash^{\mathfrak{A}}_\alpha \neg F_e(x) \ \leftrightarrow\ \mathfrak{A}\models\Theta^\alpha_{D,e,x}(\bar\delta) \ \leftrightarrow\ \mathfrak{A}^\star\models\Theta^{\star,\alpha}_{D,e,x}(\bar\delta).$$

We will just show how to produce the Σ^c_1 formulae $\Phi^{\star,1}_{D,e,x}$. We start by defining the finitary Σ_1 formulae $\Psi^{\star,1}_{D,\kappa,u}$:

- if $u=s\cdot\langle i_1,\dots,i_{n_r}\rangle+r$ for $r<s$ and $i_1,\dots,i_{n_r}\in D$, then

$$\Psi^{\star,1}_{D,\kappa,u}\equiv\begin{cases}(\exists Z)[Z=\Pi_r(X_{i_1},\dots,X_{i_{n_r}}) \ \&\ P^\star_r(Z)], & \text{if } \kappa(u)=1,\\ (\exists Z)[Z=\Pi_r(X_{i_1},\dots,X_{i_{n_r}}) \ \&\ \neg P^\star_r(Z)], & \text{if } \kappa(u)=0,\end{cases}$$

- otherwise, we set $\Psi^{\star,1}_{D,\kappa,u}\equiv\neg(X_d=X_d)$, where d is some element of D.

We define the finitary Σ_1 formula $\Psi^{\star,1}_{D,\kappa}$ with free variables in $\{X_i \mid i\in D\}$ as

$$\Psi^{\star,1}_{D,\kappa}\equiv\bigwedge_{i\in D}A(X_i) \ \&\ \bigwedge_{\substack{i\neq j\\ i,j\in D}}X_i\neq X_j \ \&\ \bigwedge_{u\in Dom(\kappa)}\Psi^{\star,1}_{D,\kappa,u},$$

where $A(X)\equiv(\exists Y,Z)[A_0(X) \ \&\ G_R(Z,Z) \ \&\ G_\Pi(Z,Z,Y) \ \&\ G_R(X,Y)]$. Here we used the fact that $A=\{x \mid x\in A_0 \ \&\ R(x)=1^\star\}$. We have the property:

$$\kappa\subseteq\delta^{-1}(\mathfrak{A}) \ \leftrightarrow\ \mathfrak{A}^\star\models\Psi^{\star,1}_{Dom(\delta),\kappa}(\bar\delta).$$

In the end, we define

$$\Phi^{\star,1}_{D,e,x} \equiv \bigvee_{\kappa\in W_{e,x}} \Psi^{\star,1}_{D,\kappa},$$

which is a Σ^c_1 formula with free variables in $\{X_i \mid i \in D\}$. Now, we have the following equivalences:

$$u \in X^\alpha_{e,x} \leftrightarrow \bigvee_{D=\{d_1<\cdots<d_n\}} (\exists a_1,\ldots,a_n)[\Pi_n(\Pi(d^\star_1,a_1),\ldots,\Pi(d^\star_n,a_n)) = u \ \& \ \mathfrak{A}^\star \models \Phi^{\star;\alpha}_{D,e,x}(a_1,\ldots,a_n)]$$

$$z \in Z^\alpha_{e,x} \leftrightarrow \bigvee_{D=\{d_1<\cdots<d_n\}} (\exists a_1,\ldots,a_n)[\Pi_n(\Pi(d^\star_1,a_1),\ldots,\Pi(d^\star_n,a_n)) = z \ \& \ \mathfrak{A}^\star \models \bigvee_{D'\supseteq D} (\exists_{D'\setminus D})\Phi^{\star;\alpha}_{D',e,x}(a_1,\ldots,a_n)]$$

Since $\Phi^{\star;\alpha}_{e,x}$ is a Σ^c_α formula, it should be clear that the right-hand sides of the equivalences can be expressed as Σ^c_α formulae. $Y^\alpha_{e,x} = \mathbb{P}^\star_A \setminus Z^\alpha_{e,x}$ and by the fact that $\mathbb{P}^\star_A \in \Pi^c_1(\mathfrak{A}^\star)$, it follows that $Y^\alpha_{e,x} \in \Pi^c_\alpha(\mathfrak{A}^\star)$ □

Since we can produce the corresponding formulae uniformly in e and x, we obtain the following corollary.

Corollary 2. *The sets* $X^\alpha = \{\Pi_3(e^\star,x^\star,\tau^\star) \mid \tau \Vdash_\alpha F_e(x)\}$ *and* $Z^\alpha = \{\Pi_3(e^\star,x^\star,\tau^\star) \mid (\exists\delta \supseteq \tau)[\delta \Vdash_\alpha F_e(x)]\}$ *are definable in* $\mathfrak{A}^\star$ *by* Σ^c_α *formulae without parameters. The set* $Y^\alpha = \{\Pi_3(e^\star,x^\star,\tau^\star) \mid \tau \Vdash_\alpha \neg F_e(x)\}$ *is definable in* $\mathfrak{A}^\star$ *by a* Π^c_α *formula without parameters. We can find indices for these formulae effectively in* α.

Proof. The sets X^α and Z^α are definable by formulae, which are essentially infinite disjunctions over e and x of all formulae Σ^c_α which define the sets $X^\alpha_{e,x}$ and $Z^\alpha_{e,x}$. Let $Y^\alpha_{e,x}$ be definable by the Π^c_α formula $\Theta^{\star;\alpha}_{e,x}$ in $\mathfrak{A}^\star$. Define the Π^c_α formula

$$\Xi^\alpha(X,Y,Z) \equiv \bigwedge_{e,x\in\mathbb{N}} [X = x^\star \ \& \ Y = e^\star \ \rightarrow \ \Theta^{\star;\alpha}_{e,x}(Z)].$$

Since $y \in Y^\alpha$ if and only if $\mathfrak{A}^\star \models \Xi^\alpha(L(y), L(R(y)), R^2(y)) \ \& \ L(y) \in \mathbb{N}^\star \ \& \ L(R(y)) \in \mathbb{N}^\star$ and $\mathbb{N}^\star \in \Pi^c_1(\mathfrak{A}^\star)$, we conclude that $Y^\alpha \in \Pi^c_\alpha(\mathfrak{A}^\star)$. □

Corollary 3. *Since we have uniformity in* e, x *and* α, *for a computable limit ordinal* $\alpha = \lim \alpha(p)$, *each of the following sets*

- $\hat{X}^\alpha = \{\Pi_4(e^\star,x^\star,p^\star,\tau^\star) \mid \tau \Vdash_{\alpha(p)} F_e(x)\}$,
- $\hat{Y}^\alpha = \{\Pi_4(e^\star,x^\star,p^\star,\tau^\star) \mid \tau \Vdash_{\alpha(p)} \neg F_e(x)\}$,
- $\hat{Z}^\alpha = \{\Pi_4(e^\star,x^\star,p^\star,\tau^\star) \mid (\exists\delta \supseteq \tau)[\delta \Vdash_{\alpha(p)} F_e(x)]\}$

is definable in $\mathfrak{A}^\star$ by a Σ^c_α formula and by a Π^c_α formula without parameters. We can find indices for these formulae effectively in the notation of α.

Proof. The fact that $\hat{X}^\alpha \in \Sigma^c_\alpha(\mathfrak{A}^\star)$ and $\hat{Z}^\alpha \in \Sigma^c_\alpha(\mathfrak{A}^\star)$ follows directly from Corollary 2, because we can find indices for the formulae defining $X^{\alpha(p)}$ and $Z^{\alpha(p)}$ uniformly in p. By the same argument $\hat{Y}^\alpha \in \Pi^c_\alpha(\mathfrak{A}^\star)$.

Since $\alpha = \lim(\alpha(p)+1)$ and $X^{\alpha(p)} \in \Pi^c_{\alpha(p)+1}(\mathfrak{A}^\star)$, $Z^{\alpha(p)} \in \Pi^c_{\alpha(p)+1}(\mathfrak{A}^\star)$, as in Corollary 2 we can show that $\hat{X}^\alpha \in \Pi^c_\alpha(\mathfrak{A}^\star)$ and $\hat{Z}^\alpha \in \Pi^c_\alpha(\mathfrak{A}^\star)$. Similarly, $\hat{Y}^\alpha \in \Sigma^c_\alpha(\mathfrak{A}^\star)$. □

2.4. CHARACTERISATION

Let us fix an enumeration f of $\mathfrak{A}^\star$. Following [10], we show how to associate a finite mapping $\tau \in \mathbb{P}_A$ with natural numbers relative to f. For every natural number n, we denote $n^f = f^{-1}(n^\star)$ and $\mathbb{N}^f = f^{-1}(\mathbb{N}^\star)$. For finite parts $\tau \in \mathbb{P}_A$, we associate with $\tau^\star$ the natural number $\tau^f = f^{-1}(\tau^\star)$. For example, if $\tau^\star = \Pi_n(\Pi(x_1^\star, a_1), \dots, \Pi(x_n^\star, a_n))$, then $\tau^f = J^f_n(J^f(x^f_1, f^{-1}(a_1)), \dots, J^f(x^f_n, f^{-1}(a_n)))$.

Sometimes we will look at τ^f as a finite mapping with $Dom(\tau^f) = \{x^f_1, \dots, x^f_n\}$ and $\tau^f(x^f_i) = f^{-1}(\tau(x_i))$. We assume that $Dom(\tau^f) = \emptyset$ if $\tau^f = 0$. Notice that $f(\tau^f(x^f)) = \tau(x)$ for all $x \in Dom(\tau)$. By Corollary 1, there exists a computable in $f^{-1}(\mathfrak{A}^\star)$ predicate P such that for $\tau, \delta \in \mathbb{P}_A$, $P(\tau^f, \delta^f) = 1$ if and only if $\tau \subseteq \delta$. We will slightly abuse our notation and write $\tau^f \subseteq \delta^f$ instead of $P(\tau^f, \delta^f) = 1$.

The next results give conditions under which we have the other directions of Theorem 4.

Theorem 6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A^\star \subseteq B$. Then for any computable ordinals $\alpha, \beta > 0$,*

$$(\forall X \subseteq A^\star)[X \in \Sigma^c_\alpha(\mathfrak{A}^\star_{A^\star}) \to X \in \Sigma^c_\beta(\mathfrak{B}_B)] \;\to\; \mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}.$$

Proof. Let us fix an enumeration f of $\mathfrak{B}$. We will show that there exists an enumeration g of $\mathfrak{A}$ such that $g \leq^\alpha_\beta f$.

Since $A \in \Sigma^c_1(\mathfrak{A}^\star_{A^\star})$, we have $A \in \Sigma^c_\beta(\mathfrak{B}_B)$ and then by Theorem 3, $f^{-1}(A)$ is $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$. Fix a bijection $\mu : \mathbb{N} \to f^{-1}(A)$, which is computable in $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$. We have two cases to consider.

Let $\alpha = 1$. We take the enumeration g of A defined as $g(n) = f(\mu(n))$. Clearly the set $E(g, f)$ is $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$, because

$$\langle x, y\rangle \in E(g, f) \;\leftrightarrow\; g(x) = f(y) \;\leftrightarrow\; y = \mu(x).$$

Let P_i be any relation in $\mathfrak{A}$. We have $P_i \in \Sigma^c_\beta(\mathfrak{B}_B)$ and $A^{n_i} \setminus P_i \in \Sigma^c_\beta(\mathfrak{B}_B)$. Thus, both $f^{-1}(P_i)$ and $f^{-1}(A^{n_i} \setminus P_i)$ are $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$. Moreover,

$$\begin{aligned} u \in g^{-1}(P_i) \ &\leftrightarrow\ (\exists x_1, \dots, x_{n_i} < u)[u = \langle x_1, \dots, x_{n_i}\rangle \ \& \\ &\qquad\qquad \langle \mu(x_1), \dots, \mu(x_{n_i})\rangle \in f^{-1}(P_i)], \\ u \in \mathbb{N} \setminus g^{-1}(P_i) \ &\leftrightarrow\ \neg(\exists x_1, \dots, x_{n_i} < u)[u = \langle x_1, \dots, x_{n_i}\rangle] \ \vee \\ &\qquad (\exists x_1, \dots, x_{n_i} < u)[u = \langle x_1, \dots, x_{n_i}\rangle \ \& \\ &\qquad\qquad \langle \mu(x_1), \dots, \mu(x_{n_i})\rangle \in f^{-1}(A^{n_i} \setminus P_i)]. \end{aligned}$$

Since $g^{-1}(P_i)$ and $\mathbb{N} \setminus g^{-1}(P_i)$ are both $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$, $g^{-1}(\mathfrak{A})$ is $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$ and hence $g \leq^1_\beta f$.

Let $\alpha > 1$. We build an α-generic enumeration g of $\mathfrak{A}$ such that $g \leq^\alpha_\beta f$. We essentially use the sets defined in Lemma 7.

- Let $\alpha = \gamma + 1$. By Corollary 2, $Y^\gamma \in \Pi^c_\gamma(\mathfrak{A}^\star)$ and hence $Y^\gamma \in \Sigma^c_\alpha(\mathfrak{A}^\star)$. It follows that the sets X^γ, Y^γ and Z^γ are all in $\Sigma^c_\beta(\mathfrak{B}_B)$. Thus, $f^{-1}(X^\gamma)$, $f^{-1}(Y^\gamma)$ and $f^{-1}(Z^\gamma)$ are all $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$.
- Let $\alpha = \lim \alpha(p)$. By Corollary 3, for the fixed enumeration f of $\mathfrak{B}$, $f^{-1}(\hat{X}^\alpha)$, $f^{-1}(\hat{Y}^\alpha)$ and $f^{-1}(\hat{Z}^\alpha)$ are all $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$.

Recall that for any natural number x, we denote by $x^f = f^{-1}(x^\star)$ and $\mathbb{N}^f$ is the set of all these x^f.

Claim 1. *There exists an α-generic enumeration g of $\mathfrak{A}$ such that g^f is $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$, where $g^f : \mathbb{N}^f \to f^{-1}(A)$ is defined as $g^f(x^f) = f^{-1}(g(x))$.*

Proof. We describe a construction in which at each stage s we define a finite part $\tau_s \subseteq \tau_{s+1}$. In the end, the α-generic enumeration of $\mathfrak{A}$ will be defined as $g = \bigcup_s \tau_s$. Let $\tau_0 = \emptyset$ and suppose we have already defined τ_s.

a) Case $s = 2r$. We make sure that g is one-to-one and onto A. Let x be the least natural number not in $Dom(\tau_s)$. Find the least p such that $\mu(p) \notin Ran(\tau^f_s)$. Set $\tau_{s+1}(x) = f(\mu(p))$ and $\tau_{s+1}(z) = \tau_s(z)$ for every $z \neq x$ and $z \in Dom(\tau_s)$. Leave $\tau_{s+1}(z)$ undefined for any other z. Since $\mathbb{N}^f$ and μ are $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$, we can find τ^f_{s+1} effectively relative to $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$.

b) Case $s = 2r + 1$. We satisfy the requirement that g is α-generic.

   Let $\alpha = \gamma + 1$ and $s = 2\langle e, x\rangle + 1$. Check whether there exists an extension δ of τ_s such that $\delta \Vdash_\gamma F_e(x)$. This is equivalent to asking which one of the following is true:
   $$J^f_3(e^f, x^f, \tau^f_s) \in f^{-1}(Y^\gamma) \text{ or } J^f_3(e^f, x^f, \tau^f_s) \in f^{-1}(Z^\gamma).$$
   We can answer this question effectively relative to the oracle $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$.

- If $J_3^f(e^f, x^f, \tau_s^f) \in f^{-1}(Y^\gamma)$, then $\tau_s \Vdash_\gamma \neg F_e(x)$ and we set $\tau_{s+1} = \tau_s$.
- If $J_3^f(e^f, x^f, \tau_s^f) \in f^{-1}(Z^\gamma)$, we search for $\delta^f \in \mathbb{P}_A^f$ such that $\tau_s^f \subseteq \delta^f$ and $J_3^f(e^f, x^f, \delta^f) \in f^{-1}(X^\gamma)$. We can find such δ^f effectively in $\Delta_\beta^0(f^{-1}(\mathfrak{B}))$. Set $\tau_{s+1} = \delta$, where δ^f is the first we find.

Let $\alpha = \lim \alpha(p)$ and $s = 2\langle e, x, p\rangle + 1$. This time we check whether there exists an extension δ of τ_s such that $\delta \Vdash_{\alpha(p)} F_e(x)$. This is equivalent to asking:

$$J_4^f(e^f, x^f, p^f, \tau_s^f) \in f^{-1}(\hat{Y}^\alpha) \text{ or } J_4^f(e^f, x^f, p^f, \tau_s^f) \in f^{-1}(\hat{Z}^\alpha).$$

Again we can answer this question effectively relative to the oracle $\Delta_\beta^0(f^{-1}(\mathfrak{B}))$. If there is no such δ, we set $\tau_{s+1} = \tau_s$. If such δ does exists, then $\tau_{s+1} = \delta$, where δ^f is the first we find. Again, we can do all this effectively relative to the oracle $\Delta_\beta^0(f^{-1}(\mathfrak{B}))$, because, as explained above, the sets $f^{-1}(\hat{X}^\alpha)$, $f^{-1}(\hat{Y}^\alpha)$, and $f^{-1}(\hat{Z}^\alpha)$ are $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$.

End of construction
It follows from the construction that the graph of g^f is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$. □

Claim 2. *For the enumeration g of $\mathfrak{A}$ we have the following:*

i) the relation $E(g, f)$ is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$;

ii) the relation $\tau^f \subseteq g^f$ is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$.

Proof. i) The equivalences $g(x) = f(y) \leftrightarrow f^{-1}(g(x)) = y \leftrightarrow g^f(x^f) = y$ and the fact that the graph of g^f is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$ imply that the set $E(g, f)$ is $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$.

ii) Since $f(g^f(x^f)) = g(x)$, $f(\tau^f(x^f)) = \tau(x)$, and equality is among the relation symbols in the language of $\mathfrak{A}^\star$, we have:

$$\begin{aligned} \tau^f \subseteq g^f &\leftrightarrow (\forall x^f \in Dom(\tau^f))[\tau^f(x^f) = g^f(x^f)] \\ &\leftrightarrow (\forall x^f \in Dom(\tau^f))[f(\tau^f(x^f)) = \tau(x) = g(x) = f(g^f(x^f)))] \\ &\leftrightarrow (\forall x^f \in Dom(\tau^f))[f(\tau^f(x^f)) = g(x)] \\ &\leftrightarrow (\forall x^f \in Dom(\tau^f))(\exists y)[g(x) = f(y) \;\&\; f(\tau^f(x^f)) = f(y)] \\ &\leftrightarrow (\forall x^f \in Dom(\tau^f))(\exists y)[\langle x, y\rangle \in E(g, f) \;\&\; \langle \tau^f(x^f), y\rangle \in f^{-1}(=^\star)]. \end{aligned}$$

Here we denote by $=^\star$ the equality on $A^\star$. Since we have all of the following:

- the sets $\{x^f \mid x \in \mathbb{N}\}$ and $\{\tau^f \mid \tau \in \mathbb{P}_A\}$ are $\Sigma_\beta^0(f^{-1}(\mathfrak{B}))$;
- given a number $x \in Dom(\tau^f)$, we can effectively relative to $\Delta_\beta^0(f^{-1}(\mathfrak{B}))$ find the value of $\tau^f(x^f)$;

- the sets $E(g, f)$ and $f^{-1}(=^\star)$ are $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$,

it follows that the relation $\tau^f \subseteq g^f$ is $\Sigma^0_\beta(f^{-1}(\mathfrak{B}))$. $\square$

We note that if $E(g, f)$ is c.e. in the set Z, then the relation $\tau^f \subseteq g^f$ is c.e. in $\Delta^0_\beta(f^{-1}(\mathfrak{B})) \oplus Z$. Since g is α-generic, we obtain the following equivalences.

Let $\alpha = \gamma + 1$. Then

$$\begin{aligned} x \in \Delta^0_\alpha(g^{-1}(\mathfrak{A})) &\leftrightarrow g \models_\gamma F_x(x) \ \leftrightarrow\ (\exists \tau \subseteq g)[\tau \Vdash_\gamma F_x(x)] \\ &\leftrightarrow (\exists \tau^f \subseteq g^f)[J^f_3(x^f, x^f, \tau^f) \in f^{-1}(X^\gamma)]. \\ x \notin \Delta^0_\alpha(g^{-1}(\mathfrak{A})) &\leftrightarrow g \models_\gamma \neg F_x(x) \ \leftrightarrow\ (\exists \tau \subseteq g)[\tau \Vdash_\gamma \neg F_x(x)] \\ &\leftrightarrow (\exists \tau^f \subseteq g^f)[J^f_3(x^f, x^f, \tau^f) \in f^{-1}(Y^\gamma)]. \end{aligned}$$

Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned} \langle x, p\rangle \in \Delta^0_\alpha(g^{-1}(\mathfrak{A})) \ &\leftrightarrow x \in \Delta^0_{\alpha(p)+1}(g^{-1}(\mathfrak{A})) \ \leftrightarrow\ g \models_{\alpha(p)} F_x(x) \\ &\leftrightarrow (\exists \tau \subseteq g)[\tau \Vdash_{\alpha(p)} F_x(x)]. \\ &\leftrightarrow (\exists \tau^f \subseteq g^f)[J^f_4(x^f, x^f, p^f, \tau^f) \in f^{-1}(\hat{X}^\alpha)]. \\ \langle x, p\rangle \notin \Delta^0_\alpha(g^{-1}(\mathfrak{A})) \ &\leftrightarrow x \notin \Delta^0_{\alpha(p)+1}(g^{-1}(\mathfrak{A})) \ \leftrightarrow\ g \models_{\alpha(p)} \neg F_x(x) \\ &\leftrightarrow (\exists \tau \subseteq g)[\tau \Vdash_{\alpha(p)} \neg F_x(x)]. \\ &\leftrightarrow (\exists \tau^f \subseteq g^f)[J^f_4(x^f, x^f, p^f, \tau^f) \in f^{-1}(\hat{Y}^\alpha)]. \end{aligned}$$

It follows that $\Delta^0_\alpha(g^{-1}(\mathfrak{A}))$ is $\Delta^0_\beta(f^{-1}(\mathfrak{B}))$. We conclude that for the enumeration g of $\mathfrak{A}$, $g \leq^\alpha_\beta f$ and hence $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. $\square$

Examining closely the proof of Theorem 6, we obtain the following corollary by isolating the requirements we need in the construction of the generic enumeration.

Corollary 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A^\star \subseteq B$, and let $\alpha > 0$, $\beta > 0$ be computable ordinals. Suppose that for every relation P_i in $\mathfrak{A}^\star$, P_i and $(A^\star)^{n_i} \setminus P_i$ are in $\Sigma^c_\beta(\mathfrak{B}_B)$, and*

- *if $\alpha \geq 2$ and $\alpha = \gamma + 1$, then $X^\gamma \in \Sigma^c_\beta(\mathfrak{B}_B)$, $Y^\gamma \in \Sigma^c_\beta(\mathfrak{B}_B)$, $Z^\gamma \in \Sigma^c_\beta(\mathfrak{B}_B)$;*

- *if α is a limit ordinal, then $\hat{X}^\alpha \in \Sigma^c_\beta(\mathfrak{B}_B)$, $\hat{Y}^\alpha \in \Sigma^c_\beta(\mathfrak{B}_B)$, $\hat{Z}^\alpha \in \Sigma^c_\beta(\mathfrak{B}_B)$.*

Then we have $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$.

Moreover, for every enumeration f of $\mathfrak{B}$ and every α-generic enumeration g of $\mathfrak{A}$, if $E(f, g)$ is c.e. in Z, then $\Delta^0_\alpha(g^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(f^{-1}(\mathfrak{B})) \oplus Z$.

Corollary 5. *For any two countable structures $\mathfrak{A}$, $\mathfrak{B}$ with domains $A \subseteq B$ and computable ordinals $\alpha, \beta > 0$,*

$$\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B} \leftrightarrow (\forall X \subseteq A^\star)[X \in \Sigma^c_\alpha(\mathfrak{A}^\star_{A^\star}) \to X \in \Sigma^c_\beta(\mathfrak{B}^\star_{B^\star})].$$

In the special case when $A = B$,

$$\mathfrak{A} \Leftrightarrow^{\alpha}_{\beta} \mathfrak{B} \leftrightarrow (\forall X \subseteq A^\star)[X \in \Sigma^c_\alpha(\mathfrak{A}^\star_{A^\star}) \leftrightarrow X \in \Sigma^c_\beta(\mathfrak{B}^\star_{B^\star})].$$

Proof. ($\rightarrow$) Let $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. By Lemma 6, we have $\mathfrak{A}^\star \Rightarrow^\alpha_\beta \mathfrak{B}^\star$. Then by Theorem 4, $(\forall X \subseteq A^\star)[X \in \Sigma^c_\alpha(\mathfrak{A}^\star_{A^\star}) \rightarrow X \in \Sigma^c_\beta(\mathfrak{B}^\star_{B^\star})]$.

($\leftarrow$) We apply Theorem 6 for the structures $\mathfrak{A}$ and $\mathfrak{B}^\star$ and obtain $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}^\star$. Take any enumeration h of $\mathfrak{B}$ and consider $h_\star$ of $\mathfrak{B}^\star$, defined as in Proposition 3. There exists f of $\mathfrak{A}$ such that $f \leq^\alpha_\beta h_\star$. Since $h_\star^{-1}(\mathfrak{B}^\star) \equiv_T h^{-1}(\mathfrak{B})$, and $E(h_\star, h)$ is computable, we obtain $E(f, h)$ is $\Sigma^0_\beta(h^{-1}(\mathfrak{B}))$ and $\Delta^0_\alpha(f^{-1}(\mathfrak{A})) \leq_T \Delta^0_\beta(h^{-1}(\mathfrak{B}))$. It follows that $f \leq^\alpha_\beta h$ and hence $\mathfrak{A} \Rightarrow^\alpha_\beta \mathfrak{B}$. □

3. JUMP STRUCTURES

For any countable structure $\mathfrak{A}$, we will define its *α-jump structure* $\mathfrak{A}^{(\alpha)}$, which $(\alpha, 1)$-conservatively extends the original structure $\mathfrak{A}$.

Definition 7. *Let $\mathfrak{A}$ be a countable structure. We define, for every computable ordinal $\alpha > 0$, the set $K^\mathfrak{A}_\alpha$ in the following way:*

- *if $\alpha < \omega$, $K^\mathfrak{A}_\alpha = \{\Pi_3(e^\star, x^\star, \tau^\star) \mid \tau \Vdash_\alpha \neg F_e(x)\ \&\ e, x \in \mathbb{N}\ \&\ \tau \in \mathbb{P}_A\}$.*
- *if $\alpha \geq \omega$ and $\alpha = \beta + 1$,*

$$K^\mathfrak{A}_\alpha = \{\Pi_3(e^\star, x^\star, \tau^\star) \mid \tau \Vdash_\beta \neg F_e(x)\ \&\ e, x \in \mathbb{N}\ \&\ \tau \in \mathbb{P}_A\}.$$

- *if $\alpha = \lim \alpha(p)$,*

$$K^\mathfrak{A}_\alpha = \{\Pi_4(e^\star, x^\star, p^\star, \tau^\star) \mid \tau \Vdash_{\alpha(p)} \neg F_e(x)\ \&\ e, x \in \mathbb{N}\ \&\ \tau \in \mathbb{P}_A\}.$$

Definition 8. *Let $\mathfrak{A}$ be a countable structure. For every computable ordinal $\alpha > 0$, we define the α-th jump of $\mathfrak{A}$ in the following way.*

$$\mathfrak{A}^{(0)} = \mathfrak{A} \text{ and } \mathfrak{A}^{(\alpha)} = (\mathfrak{A}^\star, K^\mathfrak{A}_\alpha),$$

where $\mathfrak{A}^\star$ is the Moschovakis' extension of $\mathfrak{A}$. The language of the jump structures is the language of the structure $\mathfrak{A}^\star$ plus the predicate symbol K.

We remark that A. Soskova and I. Soskov [10] define the jump structure of $\mathfrak{A}$ as $\mathfrak{A}' = (\mathfrak{A}^\star, R)$, where $R = A^\star \setminus K^\mathfrak{A}_1$. Recall that we defined $\alpha' = \alpha + 1$, if $\alpha < \omega$, and $\alpha' = \alpha$, otherwise. The next lemma explains why the definition of $K^\mathfrak{A}_\alpha$ involves so many cases for different α.

Lemma 8. *For any countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$, $K^\mathfrak{A}_\alpha$ is uniformly relatively intrinsically $\Delta^0_{\alpha'}$ on $\mathfrak{A}^\star$.*

Proof. Essentially the proof is an application of Corollary 2 and Corollary 3.

Let $\alpha < \omega$. Here $\alpha' = \alpha + 1$. In this case we have $K_\alpha^{\mathfrak{A}} = Y^\alpha$ and hence $K_\alpha^{\mathfrak{A}}$ is definable by a Π_α^c formula without parameters. Thus, $K_\alpha^{\mathfrak{A}}$ is uniformly relatively intrinsically $\Delta_{\alpha+1}^0$ on $\mathfrak{A}^\star$.

Let $\alpha \geq \omega$ and $\alpha = \beta + 1$. Here $K_\alpha^{\mathfrak{A}} = Y^\beta$ and hence $K_\alpha^{\mathfrak{A}}$ is Π_β^c definable without parameters in $\mathfrak{A}^\star$. Thus, $K_\alpha^{\mathfrak{A}}$ is uniformly relatively intrinsically Δ_α^0 on $\mathfrak{A}^\star$.

Let $\alpha = \lim \alpha(p)$. We have that $K_\alpha^{\mathfrak{A}} = \hat{Y}^\alpha$ and by the fact that $\hat{Y}^\alpha$ is definable by both Σ_α^c and Π_α^c formulae without parameters, $K_\alpha^{\mathfrak{A}}$ is uniformly relatively intrinsically Δ_α^0 on $\mathfrak{A}^\star$. ☐

Corollary 6. *For any countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$,*

$$\mathfrak{A}^{(\alpha)} \Rightarrow_{\alpha'}^1 \mathfrak{A}^\star.$$

More precisely, for any enumeration f of $\mathfrak{A}^\star$, $f^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta_{\alpha'}^0(f^{-1}(\mathfrak{A}^\star))$.

Proof. By Lemma 8, $K_\alpha^{\mathfrak{A}}$ is relatively intrinsically $\Delta_{\alpha'}^0$ on $\mathfrak{A}^\star$. Then for any enumeration f of $\mathfrak{A}^\star$, $f^{-1}(K_\alpha^{\mathfrak{A}})$ is $\Delta_{\alpha'}^0(f^{-1}(\mathfrak{A}^\star))$. Thus, $f^{-1}(\mathfrak{A}^{(\alpha)})$ is $\Delta_{\alpha'}^0(f^{-1}(\mathfrak{A}^\star))$ and hence $\mathfrak{A}^{(\alpha)} \Rightarrow_{\alpha'}^1 \mathfrak{A}^\star$. ☐

Proposition 7. *For any computable ordinal $\alpha \geq 1$, $K_\alpha^{\mathfrak{A}}$ and $A^\star \setminus K_\alpha^{\mathfrak{A}}$ are definable by Σ_1^c formulae without parameters in $\mathfrak{A}^{(\alpha+1)}$. Therefore, if a relation R is Σ_1^c definable without parameters in $\mathfrak{A}^{(\alpha)}$, given an index for this formula, we can effectively find a Σ_1^c formula without parameters which defines R in $\mathfrak{A}^{(\alpha+1)}$.*

Proof. Here h and h' are the computable functions from Proposition 1. For $\alpha = \beta + 1$, the proposition follows from the equivalence

$$u \in K_\alpha^{\mathfrak{A}} \leftrightarrow \bigvee_{(e,n) \in Graph(h)} [L(u) = e^\star \ \&\ \Pi_3(n^\star, L(R(u)), R^2(u)) \in K_{\alpha+1}^{\mathfrak{A}}].$$

For $\alpha = \lim \alpha(p)$, we can define $K_\alpha^{\mathfrak{A}}$ in a similar way, but now we use that

$$\Pi_4(e^\star, x^\star, p^\star, \tau^\star) \in K_\alpha^{\mathfrak{A}} \leftrightarrow \Pi_3((h'(e,p))^\star, x^\star, \tau^\star) \in K_{\alpha+1}^{\mathfrak{A}}.$$

☐

Proposition 7 can be extended and it can be shown that if R is relatively intrinsically c.e. on $\mathfrak{A}^{(\alpha)}$, then R is relatively intrinsically c.e. on $\mathfrak{A}^{(\gamma)}$, for any $\gamma \geq \alpha$.

Lemma 9. *Fix a countable structure $\mathfrak{A}$. For every computable ordinal $\alpha > 0$, and natural numbers e, x, we have that $X_{e,x}^\alpha \in \Sigma_1^c(\mathfrak{A}^{(\alpha)})$. Moreover, we can effectively find Σ_1^c indices for these formulae uniformly in e, x and α.*

Proof. The proof is by transfinite induction on α. The base case is for $\alpha = 1$. By Lemma 7, the sets $X^1_{e,x}$ are in $\Sigma^c_1(\mathfrak{A}^\star)$ and thus they are definable in $\mathfrak{A}'$ by the same formulae. Now consider the ordinal $\alpha + 1 < \omega$.

$$\tau \Vdash_{\alpha+1} F_e(x) \leftrightarrow (\exists \delta \in \mathbb{P}_2)[x \in W^\delta_e \ \& \ (\forall z \in Dom(\delta))[(\delta(z) = 1 \ \& \ \tau^\star \in X^\alpha_{z,z}) \\ \vee \ (\delta(z) = 0 \ \& \ \Pi_3(z^\star, z^\star, \tau^\star) \in K^{\mathfrak{A}}_\alpha)]].$$

By the induction hypothesis, $X^\alpha_{e,x}$ is definable in $\mathfrak{A}^{(\alpha)}$ by a Σ^c_1 formula, denoted $\chi^\alpha_{e,x}$, without parameters and we can effectively find an index for this formula uniformly in e, x and α. Let us define the Σ^c_1 formula without parameters:

$$\breve{\chi}^{\alpha+1}_{e,x}(X) \equiv \bigvee_{\delta \in W_{e,x}} [\bigwedge_{\delta(z)=0} \chi^\alpha_{e,x}(X) \ \wedge \bigwedge_{\delta(z)=1} K(\Pi_3(z^\star, z^\star, X))],$$

where $W_{e,x} = \{\delta \in \mathbb{P}_2 \mid x \in W^\delta_e\}$. By K we denote the relation symbol which is interpreted as $K^{\mathfrak{A}}_\alpha$ in $\mathfrak{A}^{(\alpha)}$. Therefore, $\tau \Vdash_{\alpha+1} F_e(x) \ \leftrightarrow \ \mathfrak{A}^{(\alpha)} \models \breve{\chi}^{\alpha+1}_{e,x}(\tau^\star)$. Hence $X^{\alpha+1}_{e,x} \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$ and we can find an index for $\breve{\chi}^{\alpha+1}_{e,x}$ effectively in e, x and our notation for $\alpha + 1$. By Proposition 7, we can effectively transform $\breve{\chi}^{\alpha+1}_{e,x}$ to the Σ^c_1 formula $\chi^{\alpha+1}_{e,x}$ without parameters such that $\tau \Vdash_{\alpha+1} F_e(x) \ \leftrightarrow \ \mathfrak{A}^{(\alpha+1)} \models \chi^{\alpha+1}_{e,x}(\tau^\star)$. For the case of $\alpha + 1 > \omega$, we have:

$$\tau \Vdash_{\alpha+1} F_e(x) \ \leftrightarrow (\exists \delta \in \mathbb{P}_2)[x \in W^\delta_e \ \& \ (\forall z \in Dom(\delta))[(\delta(z) = 1 \ \& \ \tau^\star \in X^\alpha_{z,z}) \\ \vee \ (\delta(z) = 0 \ \& \ \Pi_3(z^\star, z^\star, \tau^\star) \in K^{\mathfrak{A}}_{\alpha+1})]].$$

By the induction hypothesis, we effectively produce the Σ^c_1 formulae $\chi^\alpha_{e,x}$ for the sets $X^\alpha_{e,x}$ such that $t \in X^\alpha_{e,x} \leftrightarrow \mathfrak{A}^{(\alpha)} \models \chi^\alpha_{e,x}(t)$. Again by Proposition 7, we effectively transform them into the Σ^c_1 formulae $\breve{\chi}^\alpha_{e,x}$ which define the sets $X^\alpha_{e,x}$ in $\mathfrak{A}^{(\alpha+1)}$ without parameters. We define the Σ^c_1 formula

$$\chi^{\alpha+1}_{e,x}(X) \equiv \bigvee_{\delta \in W_{e,x}} [\bigwedge_{\delta(z)=0} \breve{\chi}^\alpha_{z,z}(X) \ \wedge \bigwedge_{\delta(z)=1} K(\Pi_3(z^\star, z^\star, X))],$$

for which we have $\tau \Vdash_{\alpha+1} F_e(x) \ \leftrightarrow \ \mathfrak{A}^{(\alpha+1)} \models \chi^{\alpha+1}_{e,x}(\tau^\star)$. Clearly, $\chi^{\alpha+1}_{e,x}$ defines the set $X^{\alpha+1}_{e,x}$ in $\mathfrak{A}^{(\alpha+1)}$ without parameters.

Let us consider the computable limit ordinal $\alpha = \lim \alpha(p)$. By induction hypothesis, given e, x and $\alpha(p)$, we can effectively produce the Σ^c_1 formula $\chi^{\alpha(p)}_{e,x}$ which define the set $X^{\alpha(p)}_{e,x}$ in $\mathfrak{A}^{(\alpha(p))}$ without parameters. Since $\Pi_3(e^\star, x^\star, \tau^\star) \in K^{\mathfrak{A}}_{\alpha(p)}$ if and only if $\Pi_4(e^\star, x^\star, p^\star, \tau^\star) \in K^{\mathfrak{A}}_\alpha$, we effectively transform each $\chi^{\alpha(p)}_{e,x}$ into the Σ^c_1 formula $\breve{\chi}^{\alpha(p)}_{e,x}$ which define $X^{\alpha(p)}_{e,x}$ in $\mathfrak{A}^{(\alpha)}$ without parameters. Now we define the Σ^c_1 formula for $X^\alpha_{e,x}$ as follows:

$$\chi^\alpha_{e,x}(X) \equiv \bigvee_{\delta \in W_{e,x}} [\bigwedge_{\delta(\langle z,p \rangle)=0} \breve{\chi}^{\alpha(p)}_{z,z}(X) \ \wedge \bigwedge_{\delta(\langle z,p \rangle)=1} K(\Pi_4(z^\star, z^\star, p^\star, X))].$$

Since $\tau \Vdash_\alpha F_e(x) \leftrightarrow \mathfrak{A}^{(\alpha)} \models \chi^\alpha_{e,x}(\tau^\star)$, the formula $\chi^\alpha_{e,x}$ defines the set $X^\alpha_{e,x}$ in $\mathfrak{A}^{(\alpha)}$ without parameters. □

We did all the hard work. Now we are ready to show that $\mathfrak{A}^{(\alpha)}$ is $(\alpha', 1)$-conservative extension of $\mathfrak{A}$.

Corollary 7. *For any countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$,*

$$\mathfrak{A} \Rightarrow^{\alpha'}_1 \mathfrak{A}^{(\alpha)}.$$

Moreover, for any α'-generic enumeration g of $\mathfrak{A}$,

$$\Delta^0_{\alpha'}(g^{-1}(\mathfrak{A})) \equiv_T g^{-1}(\mathfrak{A})^{(\alpha)} \equiv_T g_\star^{-1}(\mathfrak{A}^\star)^{(\alpha)} \equiv_T g_\star^{-1}(\mathfrak{A}^{(\alpha)}),$$

where $g_\star$ is defined as in Proposition 3.

Proof. First we note that, having Lemma 9, we can prove analogues to Corollary 2 and Corollary 3, that is, we can show that for any computable ordinal α, $X^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, $Z^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, and $\hat{X}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, $\hat{Z}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$. Now all we need to do is check the premises of Corollary 4 for $\beta = 1$ and $\mathfrak{B} = \mathfrak{A}^{(\alpha)}$, where we have a few cases for α to consider:

- $\alpha < \omega$, $\alpha' = \alpha + 1$. As noted above, we have that $X^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, $Z^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$. Since $Y^\alpha = K^{\mathfrak{A}}_\alpha$, we also have $Y^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$.
- $\alpha = \gamma + 1 > \omega$, $\alpha' = \alpha$. We have that $X^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\gamma)})$, $Z^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\gamma)})$. Then by Proposition 7, $X^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$ and $Z^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$. We also have $Y^\gamma = K^{\mathfrak{A}}_\alpha$ and hence $Y^\gamma \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$.
- $\alpha = \lim \alpha(p)$, $\alpha' = \alpha$. Here we have that $\hat{X}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$, $\hat{Z}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$. By definition, $\hat{Y}^\alpha = K^{\mathfrak{A}}_\alpha$. Thus, $\hat{Y}^\alpha \in \Sigma^c_1(\mathfrak{A}^{(\alpha)})$.

By Corollary 4, we conclude that $\mathfrak{A} \Rightarrow^{\alpha'}_1 \mathfrak{A}^{(\alpha)}$.

Now we will prove the second part. By Corollary 4, since g is α'-generic,

$$\Delta^0_{\alpha'}(g^{-1}(\mathfrak{A})) \leq_T g_\star^{-1}(\mathfrak{A}^{(\alpha)}) \oplus Z,$$

where Z is such that $E(g, g_\star)$ is c.e. in Z. By Proposition 3, we have that $E(g, g_\star)$ is computable. Thus, we obtain $\Delta^0_{\alpha'}(g^{-1}(\mathfrak{A})) \leq_T g_\star^{-1}(\mathfrak{A}^{(\alpha)})$. By Corollary 6, $\mathfrak{A}^{(\alpha)} \Rightarrow^1_{\alpha'} \mathfrak{A}^\star$ and hence $g_\star^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(g_\star^{-1}(\mathfrak{A}^\star))$. Again by Proposition 3, $g^{-1}(\mathfrak{A}) \equiv_T g_\star^{-1}(\mathfrak{A}^\star)$. Combining all of the above, we conclude

$$\Delta^0_{\alpha'}(g^{-1}(\mathfrak{A})) \equiv_T g^{-1}(\mathfrak{A})^{(\alpha)} \equiv_T g_\star^{-1}(\mathfrak{A}^\star)^{(\alpha)} \equiv_T g_\star^{-1}(\mathfrak{A}^{(\alpha)}).$$

□

Theorem 7. *For every countable structure $\mathfrak{A}$ and computable ordinal $\alpha > 0$,*

1) $\mathfrak{A} \Leftrightarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$, *or in other words,* $\mathfrak{A}^{(\alpha)}$ *is a* $(\alpha', 1)$*-conservative extension* $\mathfrak{A}$*;*

2) $\mathfrak{A}^\star \Leftrightarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$, *i.e.* $\mathfrak{A}^{(\alpha)}$ *is also a* $(\alpha', 1)$*-conservative extension* $\mathfrak{A}^\star$*;*

3) $\mathfrak{A}^{(\alpha)} \Rightarrow_1^1 \mathfrak{A}^{(\alpha+1)}$, *but* $\mathfrak{A}^{(\alpha)} \not\Leftarrow_1^1 \mathfrak{A}^{(\alpha+1)}$.

Proof. One direction of 1) is Corollary 7. For the other direction, let us take an enumeration f of $\mathfrak{A}$. By Proposition 3, $f_\star$ is an enumeration of $\mathfrak{A}^\star$ and hence it is an enumeration of $\mathfrak{A}^{(\alpha)}$. Moreover, by Corollary 6, $f_\star^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(f_\star^{-1}(\mathfrak{A}^\star))$. Since $f_\star^{-1}(\mathfrak{A}^\star) \equiv_T f^{-1}(\mathfrak{A})$ and $E(f_\star, f)$ is computable, we get $\mathfrak{A} \Leftarrow^1_{\alpha'} \mathfrak{A}^{(\alpha)}$.

2) We take any enumeration f of $\mathfrak{A}^{(\alpha)}$ and since by 1) $\mathfrak{A} \Rightarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$, we choose h of $\mathfrak{A}$ such that $h \leq_1^{\alpha'} f$. $h_\star$ is an enumeration of $\mathfrak{A}^\star$, $E(h_\star, h)$ is computable and $h_\star^{-1}(\mathfrak{A}^\star) \equiv_T h^{-1}(\mathfrak{A})$. Thus, $h_\star \leq_1^{\alpha'} f$ and hence $\mathfrak{A}^\star \Rightarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$. The other direction is exactly Corollary 6, because $\mathfrak{A}^\star$ and $\mathfrak{A}^{(\alpha)}$ are structures with equal domains and in this case $\mathfrak{A}^{(\alpha)} \Rightarrow^1_{\alpha'} \mathfrak{A}^\star$ is equivalent to $\mathfrak{A}^\star \Leftarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$. Therefore, $\mathfrak{A}^\star \Leftarrow_1^{\alpha'} \mathfrak{A}^{(\alpha)}$.

3) By Proposition 7, $K_\alpha^{\mathfrak{A}} \in \Sigma_1^c(\mathfrak{A}^{(\alpha+1)})$. Then by Corollary 4, we obtain $\mathfrak{A}^{(\alpha)} \Rightarrow_1^1 \mathfrak{A}^{(\alpha+1)}$. Assume $\mathfrak{A}^{(\alpha)} \Leftarrow_1^1 \mathfrak{A}^{(\alpha+1)}$ and let g be an $(\alpha'+1)$-generic enumeration of $\mathfrak{A}$. Since $g_\star$ is an enumeration of $\mathfrak{A}^{(\alpha)}$, there exists an enumeration f of $\mathfrak{A}^{(\alpha+1)}$ such that $f \leq_1^1 g_\star$ and hence $f^{-1}(\mathfrak{A}^{(\alpha+1)}) \leq_T g_\star(\mathfrak{A}^{(\alpha)})$. By Corollary 6 we have $g_\star(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(g_\star^{-1}(\mathfrak{A}^\star))$ and by Proposition 3, $g_\star^{-1}(\mathfrak{A}^\star) \equiv_T g^{-1}(\mathfrak{A})$. We conclude that $f^{-1}(\mathfrak{A}^{(\alpha+1)}) \leq_T \Delta^0_{\alpha'}(g_\star^{-1}(\mathfrak{A}^\star)) \equiv_T g(\mathfrak{A})^{(\alpha)}$.

We apply Corollary 4 for $\beta = 1$, $\mathfrak{B} = \mathfrak{A}^{(\alpha+1)}$, and obtain that for the given enumeration f of $\mathfrak{A}^{(\alpha+1)}$ and $(\alpha'+1)$-generic g enumeration of $\mathfrak{A}$, $\Delta^0_{\alpha'+1}(g^{-1}(\mathfrak{A})) \leq_T f^{-1}(\mathfrak{A}^{(\alpha+1)}) \oplus Z$, where Z is such that $E(f,g)$ is c.e. in Z. Since $(x,y) \in E(f,g)$ if and only if $(2x+1, y) \in E(f, g_\star)$ and $E(f, g_\star)$ is c.e. in $g_\star^{-1}(\mathfrak{A}^{(\alpha)})$, we can replace Z by $g_\star^{-1}(\mathfrak{A}^{(\alpha)})$. Therefore,

$$g^{-1}(\mathfrak{A})^{(\alpha+1)} \equiv_T \Delta^0_{\alpha'+1}(g^{-1}(\mathfrak{A})) \leq_T f^{-1}(\mathfrak{A}^{(\alpha+1)}) \oplus g_\star^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T g^{-1}(\mathfrak{A})^{(\alpha)}.$$

We reach a contradiction. □

Corollary 8. *For a countable structure* $\mathfrak{A}$ *and computable ordinal* $\alpha > 0$,

1) $(\forall X \subseteq A)[X \in \Sigma^c_{\alpha'}(\mathfrak{A}_A) \leftrightarrow X \in \Sigma^c_1(\mathfrak{A}^{(\alpha)}_{A^\star})]$;

2) $DS(\mathfrak{A}^{(\alpha)}) = DS_\alpha(\mathfrak{A})$.

Proof. Direct application of *1)* of Theorem 7, Theorem 4 and Theorem 5. □

Theorem 8. *For all countable structures* $\mathfrak{A}$, $\mathfrak{B}$ *with* $A \subseteq B$ *and computable ordinals* $\alpha, \beta > 0$, $\mathfrak{A} \Leftrightarrow^{\alpha'}_{\beta'} \mathfrak{B}$ *if and only if* $\mathfrak{A}^{(\alpha)} \Leftrightarrow^1_1 \mathfrak{B}^{(\beta)}$.

Proof. By Lemma 6, for any $\alpha, \beta > 0$, $\mathfrak{A} \Leftrightarrow^\alpha_\beta \mathfrak{B}$ if and only if $\mathfrak{A}^\star \Leftrightarrow^\alpha_\beta \mathfrak{B}^\star$. We explain only why $\mathfrak{A}^\star \Rightarrow^{\alpha'}_{\beta'} \mathfrak{B}^\star$ implies $\mathfrak{A}^{(\alpha)} \Rightarrow^1_1 \mathfrak{B}^{(\beta)}$. The other directions make use of similar ideas.

By 2) of Theorem 7, $\mathfrak{B}^\star \Rightarrow_1^{\beta'} \mathfrak{B}^{(\beta)}$. Take any enumeration f of $\mathfrak{B}^{(\beta)}$ and let h be an enumeration of $\mathfrak{B}^\star$ for which $h \leq_1^{\beta'} f$. Since $\mathfrak{A}^\star \Rightarrow_{\beta'}^{\alpha'} \mathfrak{B}^\star$, there exists an enumeration g of $\mathfrak{A}^\star$ such that $g \leq_{\beta'}^{\alpha'} h$. By Corollary 6, $g^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(g^{-1}(\mathfrak{A}^\star))$. We clearly have $g^{-1}(\mathfrak{A}^{(\alpha)}) \leq_T \Delta^0_{\alpha'}(g^{-1}(\mathfrak{A}^\star)) \leq_T \Delta^0_{\beta'}(h^{-1}(\mathfrak{B}^\star)) \leq_T f^{-1}(\mathfrak{B}^{(\beta)})$. Since $\langle x, y\rangle \in E(g, f)$ if and only if there is a number z such that $\langle x, z\rangle \in E(g, h)$ and $\langle z, y\rangle \in E(h, f)$, the set $E(g, f)$ is c.e. in $f^{-1}(\mathfrak{B}^{(\beta)})$. Therefore, $g \leq_1^{\alpha'} f$. We conclude that $\mathfrak{A} \Rightarrow_{\beta'}^{\alpha'} \mathfrak{B}$ implies $\mathfrak{A}^{(\alpha)} \Rightarrow_1^1 \mathfrak{B}^{(\beta)}$. □

ACKNOWLEDGEMENTS. The author would like thank the referee for numerous remarks and suggestions which improved the paper substantially.

4. REFERENCES

[1] Ash, C., Knight, J.: *Computable structures and the hyperarithmetical hierarchy.* Elsevier Science, 2000.

[2] Ash C., Knight, J., Manasse, M., Slaman, T.: Generic copies of countable structures. *Ann. Pure and Appl. Logic*, **42**, 1989, 195 – 205.

[3] Baleva, V.: The jump operation for structure degrees. *Arch. Math. Logic*, **45**, no. 3, 2006, 249 – 265.

[4] Chisholm, J.: Effective model theory vs. recursive model theory. *J. Symbol. Logic*, **55**, no. 3, 1990, 1168 – 1191.

[5] Knight, J.: Degrees coded in jumps of orderings. *J. Symbol. Logic*, **51**, no. 4, 1986, 1034 – 1042.

[6] Montalbán, A.: Notes on the jump of a structure. *Mathematical Theory and Computational Practice*, 2009, 372 – 378.

[7] Montalbán, A.: Rice sequences of relations. *Philosophical Transactions of the Royal Society A*, **370**, 2012, 3464 – 3487.

[8] Moschovakis, Y.: *Elementary induction on abstract structures.* North - Holland, Amsterdam, 1974.

[9] Richter, L.: Degrees of structures. *J. Symbol. Logic*, **46**, no. 4, 1981, 723–731.

[10] Soskova, A., Soskov, I.: A jump inversion theorem for the degree spectra. *J.f Logic and Comput.*, **19**, 2009, 199–215.

[11] Soskov, I.: Degree spectra and co-spectra of structures. *Ann. Univ. Sofia*, **96**, 2004, 45–68.

[12] Stukachev, A.: A jump inversion theorem for the semilattices of Σ-degrees. *Siberian Advances in Mathematics*, **20**, no. 1, 2010, 68 – 74.

[13] Vatev, S.: Conservative extensions of abstract structures. In: *Proceedings of CiE 2011*, eds. B. Löwe, D. Normann, I. Soskov, A. Soskova, Springer-Verlag, Vol. 6735, 2011, 300 – 309.

[14] Vatev, S.: *Ph.D. dissertation.* Sofia University, 2014.

Received on 24 March, 2014
In revised form on 7 April, 2015

Stefan V. Vatev

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5 J. Bourchier Blvd., BG-1164 Sofia
BULGARIA

e-mail: stefanv@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

DEFINABILITY OF JUMP CLASSES IN THE LOCAL THEORY OF THE ω-ENUMERATION DEGREES

HRISTO GANCHEV, ANDREY C. SARIEV

In the present paper we continue the study of the definability in the local substructure $\mathcal{G}_\omega$ of the ω-enumeration degrees, which was started in the work of Ganchev and Soskova [3]. We show that the class $\mathbf{I}$ of the intermediate degrees is definable in $\mathcal{G}_\omega$. As a consequence of our observations, we show that the first jump of the least ω-enumeration degree is also definable.

Keywords: Enumeration reducibility, ω-enumeration degrees, degree structures, local substructures, definability, jump classes

2000 Math. Subject Classification: 03D28, 03D30

1. INTRODUCTION

A major focus of research in Computability theory involves definability issues in degree structures. Considering a degree structure, natural questions arise about the definability of classes of degrees determined by the structure's jump operation. The same questions can be transferred to its local substructures as well. As an interesting special case one can ask for which natural numbers n the jump classes $\mathbf{H}_n$ and $\mathbf{L}_n$, consisting of the high$_n$ and the low$_n$ degrees respectively, are first order definable in a degree structure.

As it has been shown by Shore and Slaman in [9], the Turing jump is first order definable in the structure of the Turing degrees, $\mathcal{D}_T$, so for all natural numbers n the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are first order definable in $\mathcal{D}_T$. For the local substructure

$\mathcal{G}_T$ consisting of all Turing degrees less than or equal to the first jump of the least element in $\mathcal{D}_T$, and for the substructure $\mathcal{R}$ consisting of all computably enumerable Turing degrees, Nies, Shore and Slaman [7] showed that for each natural number n the jump classes $\mathbf{H}_n$ and $\mathbf{L}_{n+1}$ are first order definable. The question whether the class $\mathbf{L}_1$ is first order definable is still open.

In the case of the structure of the enumeration degrees, $\mathcal{D}_e$, Kalimullin [5] proved that the enumeration jump is first order definable, so all of the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are first order definable as well. In the local structure $\mathcal{G}_e$ consisting of all enumeration degrees below the first jump of the least element in $\mathcal{D}_e$, we know by a recent result of Ganchev and M. Soskova [4] that the class $\mathbf{L}_1$ is definable. The problems concerning the definability of $\mathbf{H}_1$ and of the classes of the high$_n$ and low$_n$ degrees for $n \geq 2$ still resist all attempts to be solved.

Further one can consider the questions about the first order definability of the jump classes $\mathbf{H} = \bigcup \mathbf{H}_n$ of the degrees which are high$_n$ for some $n < \omega$, $\mathbf{L} = \bigcup \mathbf{L}_n$ of the degrees which are low$_n$ for some $n < \omega$ and of the class of the intermediate degrees $\mathbf{I}$. It is known that the classes $\mathbf{H}, \mathbf{L}$ and $\mathbf{I}$ are definable in $\mathcal{D}_T$. This follows from the fact that each relation on $\mathcal{D}_T$ is definable in $\mathcal{D}_T$ *if and only if* it is invariant under the automorphisms and it is induced by a degree invariant relation on 2^ω definable in Second-Order Arithmetic, see [10]. An analogous reasoning is valid for the structure $\mathcal{D}_e$, [11].

What is the situation in the local substructures? In the case of the structure of the c.e. degrees, $\mathcal{R}$, the classes $\mathbf{H}, \mathbf{L}$ and $\mathbf{I}$ are not definable. Indeed, by Solovay (see for instance [12]), the set of the indices of the c.e. sets which are intermediate is $\Pi^0_{\omega+1}$-complete, and the sets of the indices of the c.e. sets which are in $\mathbf{H}$ and $\mathbf{L}$ respectively are both $\Sigma^0_{\omega+1}$-complete and hence are not definable in First-Order Arithmetic. On the other hand, by Nies, Shore and Slaman [7], a relation on c.e. degrees invariant under the double jump[1] is definable in $\mathcal{R}$ if and only if it is definable in First-Order Arithmetic. Therefore $\mathbf{I}, \mathbf{H}$ and $\mathbf{L}$ are not definable in $\mathcal{R}$. From this point one may conclude that $\mathbf{I}, \mathbf{H}$ and $\mathbf{L}$ are not definable in $\mathcal{G}_T$. Indeed, following Nies, Shore and Slaman [7], a relation on degrees below $\mathbf{0}'_T$ invariant under the double jump is definable in $\mathcal{G}_T$ if and only if it is definable in First-Order Arithmetic. But the classes of the indeces of the Δ^0_2-sets having Turing degrees in $\mathbf{I}, \mathbf{H}$ or $\mathbf{L}$ respectively are not definable in First-Order Arithmetic, since otherwise adding to their definitions the condition of being c.e. (which is definable in First-Order Arithmetic) would result into definitions of the indices of the c.e. sets in $\mathbf{I}, \mathbf{H}$ and $\mathbf{L}$. So again $\mathbf{I}, \mathbf{H}$ and $\mathbf{L}$ are not definable in $\mathcal{G}_T$. Finally, let us consider $\mathcal{G}_e$. Here one can argue in a manner similar to the above by noting that $\mathcal{R}$ is isomorphic to the structure of the Π^0_1 enumeration degrees [8], and that the latter are definable in First-Order Arithmetic. Now assuming that one of the classes $\mathbf{H}, \mathbf{L}$ and $\mathbf{I}$ is definable in $\mathcal{G}_e$, one can easily show the definability of the respective class of indeces of Σ^0_2-sets in First-Order Arithmetic. So a definition in First-Order

[1] A n-ary relation R on degrees is invariant under the double jump if and only if whenever $R(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{x}''_1 = \mathbf{y}''_1, \ldots, \mathbf{x}''_n = \mathbf{y}''_n$, it is also true that $R(\mathbf{y}_1, \ldots, \mathbf{y}_n)$.

Arithmetic of the corresponding class of c.e. sets is obtained, once again leading to a contradiction.

In this paper we investigate the question about the definability of the classes **I**, **H** and **L** in the local theory of the structure of the ω-enumeration degrees, $\mathcal{D}_\omega$, which is a proper extension of $\mathcal{D}_e$.

The structure of the ω-enumeration degrees was introduced by Soskov [14] and further studied in a sequence of works by Soskov, M. Soskova and Ganchev [15,17,3]. Unlike the structures of the Turing degrees and of the enumeration degrees, $\mathcal{D}_\omega$ is based on a reducibility relation between sequences of sets of natural numbers. To be more precise, a sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ is said to be *ω-enumeration reducible* to a sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ if and only if $J_\mathcal{B} \subseteq J_\mathcal{A}$, where for any sequence $\mathcal{X} = \{X_k\}_{k<\omega}$, $J_\mathcal{X}$ denotes the jump class

$$J_\mathcal{X} = \{\mathbf{d}_T(Y)|\ X_k \text{ is c.e. in } Y^{(k)} \text{ uniformly in } k\}.$$

The jump $\mathcal{A}'$ of a sequence $\mathcal{A}$ is defined [15] so that the class $J_{\mathcal{A}'}$ consists exactly of the jumps of the Turing degrees in $J_\mathcal{A}$, i.e. so that $J_{\mathcal{A}'} = J'_\mathcal{A}$. The jump operator on sequences is monotone and thus induces a jump operation $'$ in $\mathcal{D}_\omega$. Like the jump operation in $\mathcal{D}_T$, the range of the jump operation in $\mathcal{D}_\omega$ is exactly the cone above the first jump of the least element $\mathbf{0}_\omega$. In other words, a general jump inversion theorem is valid for $\mathcal{D}_\omega$. Moreover, even a stronger statement turns out to be true, namely, for every ω-enumeration degree $\mathbf{a}$ above $\mathbf{0}'_\omega$ there is a least degree with jump equals to $\mathbf{a}$. This property is neither true for $\mathcal{D}_T$ nor for $\mathcal{D}_e$.

The strong jump inversion theorem makes the structure $\mathcal{D}_\omega$ worth studying, since using it one may consider a natural copy of the structure $\mathcal{D}_e$ definable in $\mathcal{D}_\omega$ augmented by the jump operation. Moreover, the automorphism groups of $\mathcal{D}_e$ and ${\mathcal{D}_\omega}'$ (i.e. the structure of ω-enumeration degrees augmented with jump operation) are isomorphic.

The jump operation gives rise to the local substructure $\mathcal{G}_\omega$ consisting of all ω-enumeration degrees below $\mathbf{0}'_\omega$. Thanks to the strong jump inversion, $\mathcal{G}_\omega$ contains a class of remarkable degrees having no analogue in either $\mathcal{R}$, $\mathcal{G}_T$ or $\mathcal{G}_e$. These degrees are denoted by $\mathbf{o}_n$, $n < \omega$, and are defined so that $\mathbf{o}_n$ is the least degree whose n-th jump is equal to the $(n+1)$-th jump of $\mathbf{0}_\omega$. In other words, $\mathbf{o}_n$ is the least high$_n$ degree. The degrees $\mathbf{o}_n$ turn out be also connected to low$_n$ degrees. Indeed, a degree in $\mathcal{G}_\omega$ is low$_n$ *if and only if* it forms a minimal pair with $\mathbf{o}_n$.

Each one of the degrees $\mathbf{o}_n$ turns out to be definable in $\mathcal{G}_\omega$, [3], and hence so are the classes $\mathbf{H}_n$ and $\mathbf{L}_n$, for $n \in \omega$. The definition in $\mathcal{G}_\omega$ of $\mathbf{o}_n$ given by Ganchev and M. Soskova [3] is based on the notion of Kalimullin pairs, or more simply $\mathcal{K}$-pairs — a notion first introduced and studied by Kalimullin in the context of the enumeration degrees. For an arbitrary partial order $\mathcal{D} = (\mathbf{D}, \leq)$ a pair $\{\mathbf{a}, \mathbf{b}\}$ is called a $\mathcal{K}$-pair *if and only if*

$$\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b})$$

holds for every $\mathbf{x} \in \mathcal{D}$.

The $\mathcal{K}$-pairs in $\mathcal{G}_\omega$ can be separated into two disjoint classes. The first class consists of the $\mathcal{K}$-pairs formed by two almost zero degrees (a degree in $\mathcal{G}_\omega$ is called *almost zero* if and only if it is bellow each $\mathbf{o}_n$). The other class contains the $\mathcal{K}$-pairs *inherited* from $\mathcal{D}_e[\mathbf{0}_e^{(n)}, \mathbf{0}_e^{(n+1)}]$ for some natural number n. The degrees $\mathbf{o}_n$ are strongly connected with the inherited $\mathcal{K}$-pairs. In fact the degree $\mathbf{o}_n$ is the greatest degree which is the least upper bound of an inherited $\mathcal{K}$-pair, which cannot be cupped above $\mathbf{o}_{n-1}$ by a degree less then $\mathbf{o}_{n-1}$. On the other hand if a $\mathcal{K}$-pair is not inherited, then it is bounded by every $\mathbf{o}_{n-1}$ so that we can relax the condition on the $\mathcal{K}$-pairs to be inherited in the above characterisation. Since $\mathbf{o}_0$ is the top element in $\mathcal{G}_\omega$, we can define each of the degrees $\mathbf{o}_n$ inductively in $\mathcal{G}_\omega$.

In this paper we continue the study of the connections between the degrees $\mathbf{o}_n$ and the $\mathcal{K}$-pairs. Our aim is to prove the following theorem.

Theorem 1. *The classes* $\mathbf{H}, \mathbf{L}$ *and* $\mathbf{I}$ *are first order definable in the local substructure* $\mathcal{G}_\omega$ *of the* ω*-enumeration degrees.*

To obtain the above mentioned definability result it suffices to prove that the set $\mathfrak{O} = \{\mathbf{o}_n | n < \omega\}$ is definable in $\mathcal{G}_\omega$. Indeed, we obviously have that

$$\mathbf{x} \in \mathbf{H} \iff (\exists n)[\mathbf{x} \in \mathbf{H}_n] \iff (\exists n)[\mathbf{o}_n \leq_\omega \mathbf{x}] \iff (\exists \mathbf{o} \in \mathfrak{O})[\mathbf{o} \leq_\omega \mathbf{x}].$$

Similarly

$$\mathbf{x} \in \mathbf{L} \iff (\exists n)[\mathbf{x} \in \mathbf{L}_n] \iff (\exists n)[\mathbf{o}_n \wedge \mathbf{x} = \mathbf{0}_\omega] \iff (\exists \mathbf{o} \in \mathfrak{O})[\mathbf{o} \wedge \mathbf{x} = \mathbf{0}_\omega].$$

So how do we define $\mathfrak{O}$? As we stated above, each $\mathbf{o}_n$ is the least upper bound of an inherited $\mathcal{K}$-pair. Then our first goal is to define the set of the inherited $\mathcal{K}$-pairs in $\mathcal{G}_\omega$. We achieve this using a result by Kent and Sorbi [6]. Namely, we show that a $\mathcal{K}$-pair is inherited *if and only if* each of its elements bounds a non-splittable degree. So we concentrate only on least upper bounds of inherited $\mathcal{K}$-pairs. First we show that for each $\mathbf{o}_n$ and for each inherited $\mathcal{K}$-pair, the elements of the $\mathcal{K}$-pair are either bellow $\mathbf{o}_n$ or are incomparable with $\mathbf{o}_n$. Then a result by Ganchev and M. Soskova [3] allows us to show that this necessary condition is also sufficient, so that we obtain the desired definition of $\mathfrak{O}$.

Moreover, we shall extend our observations for the $\mathcal{K}$-pairs in $\mathcal{G}_\omega$ and characterise the $\mathcal{K}$-pairs in $\mathcal{D}_\omega$. We shall see that the $\mathcal{K}$-pairs in $\mathcal{D}_\omega$ either consists only of *a.z.* degrees, or are inherited just like in the case of $\mathcal{G}_\omega$. But the inherited $\mathcal{K}$-pairs are always below $\mathbf{0}'_\omega$. So, knowing how to distinguish (in $\mathcal{D}_\omega$) the inherited $\mathcal{K}$-pairs from the others and using the fact that $\mathbf{0}'_\omega$ can be represented as a least upper bound of an inherited $\mathcal{K}$-pair, we conclude that $\mathbf{0}'_\omega$ is the greatest degree which is least upper bound of an inherited $\mathcal{K}$-pair. Thus we have

Theorem 2. *The first jump of the least element* $\mathbf{0}_\omega$ *is first order definable in* $\mathcal{D}_\omega$.

We denote the set of natural numbers by ω. If not stated otherwise, a, b, $c, \ldots$ stand for natural numbers, A, B, $C, \ldots$ for sets of natural numbers, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}, \ldots$ for degrees and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}, \ldots$ for sequences of sets of natural numbers. We shall further follow the following convention: whenever a sequence is denoted by a calligraphic Latin letter, then we shall use the roman style of the same Latin letter, indexed with a natural number, say k, to denote the k-th element of the sequence (we always start counting from 0). Thus, if not stated otherwise, $\mathcal{A} = \{A_k\}_{k<\omega}$, $\mathcal{B} = \{B_k\}_{k<\omega}$, $\mathcal{C} = \{C_k\}_{k<\omega}$, etc. We shall denote the set of all sequences (*of length* ω) of sets of natural numbers by $\mathcal{S}_\omega$.

The notation $A \oplus B$ stands for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

We assume that the reader is familiar with the notion of enumeration reducibility, $\leq_e$, and with the structure of the enumeration degrees (for an introduction on the enumeration reducibilities and the respective degree structure we refer the reader to [1, 13]).

For a natural number e and a set $A \subseteq \omega$, we denote by W_e^A the domain of the partial function computed by the oracle Turing machine with index e and using A as an oracle.

Intuitively, a set A is enumeration reducible (*e-reducible*) to a set B, if there is an effective algorithm transforming each enumeration of B into an enumeration of A. More formally, $A \leq_e B$ *if and only if* there is a natural number i, such that for every enumeration f of B, the function $\{i\}^f$ is an enumeration of A. It turns out that $A \leq_e B$ *if and only if* there is a c.e. set W, such that

$$x \in A \iff (\exists u)[\langle x, u\rangle \in W \ \&\ D_u \subseteq B], \tag{2.1}$$

where $\langle x, u\rangle$ denotes the code of the pair of natural numbers (x, u) under some fixed encoding, and D_u is the finite set with canonical index u. Usually this is taken as the formal definition of the enumeration reducibility. If the set W in (2.1) has index i, we say that A is e-reducible to B via W_i, and we shall write $A = W_i(B)$.

The relation $\leq_e$ is a preorder on the powerset $\mathcal{P}(\omega)$ of the natural numbers and induces a nontrivial equivalence relation $\equiv_e$. The equivalence classes under $\equiv_e$ are called enumeration degrees. The enumeration degree which contains the set A is denoted by $\mathbf{d}_e(A)$. The set of all enumeration degrees is denoted by $\mathbf{D}_e$. The enumeration reducibility between sets induces a partial order $\leq_e$ on $\mathbf{D}_e$ by

$$\mathbf{d}_e(A) \leq_e \mathbf{d}_e(B) \iff A \leq_e B.$$

We denote by $\mathcal{D}_e$ the partially ordered set $(\mathbf{D}_e, \leq_e)$. The least element of $\mathcal{D}_e$ is the enumeration degree $\mathbf{0}_e$ of $\emptyset$. Also, the enumeration degree of $A \oplus B$ is the least upper bound of the degrees of A and B. Therefore $\mathcal{D}_e$ is an upper semilattice with least element.

By A^+ we shall denote the set $A \oplus (\omega \setminus A)$.

The enumeration jump A'_e of A is defined by $A'_e = \{x \mid x \in W_x(A)\}^+$. The jump operation preserves the enumeration reducibility, hence we can define $\mathbf{d}_e(A)' = \mathbf{d}_e(A')$. Since $A <_e A'$, then we have $\mathbf{a} <_e \mathbf{a}'$ for every enumeration degree $\mathbf{a}$. The jump operator is uniform, i.e. there exists a recursive function j such that for every sets A and B, if $A = W_e(B)$ then $A' = W_{j(e)}(B')$.

The jump operation gives rise to the local substructure $\mathcal{G}_e$, consisting of all degrees bellow $\mathbf{0}'_e$ – the jump of the least enumeration degree. Cooper [1] has proved that $\mathcal{G}_e$ is exactly the collection of all Σ^0_2 enumeration degrees.

Finally we need the following definition, which we shall use in order to characterise ω-enumeration reducibility. Given a sequence $\mathcal{A} \in S_\omega$ we define the *jump sequence* $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ as the sequence $\{P_k(\mathcal{A})\}_{k<\omega}$ such that:

1. $P_0(\mathcal{A}) = A_0$;
2. $P_{k+1}(\mathcal{A}) = P_k(\mathcal{A})' \oplus A_{k+1}$.

3. THE ω-ENUMERATION DEGREES

Soskov [14] introduced the structure of the ω-enumeration degrees $\mathcal{D}_\omega$ in the following way. For every sequence $\mathcal{A} \in \mathcal{S}_\omega$, we define its jump class $J_\mathcal{A}$ to be the set:

$$J_\mathcal{A} = \{\mathbf{d}_T(X) \mid A_k \text{ is c.e. in } X^{(k)} \text{ uniformly in } k\}. \tag{3.1}$$

We set

$$\mathcal{A} \leq_\omega \mathcal{B} \iff J_\mathcal{B} \subseteq J_\mathcal{A}.$$

Clearly $\leq_\omega$ is a reflexive and transitive relation, and the relation $\equiv_\omega$ defined by

$$\mathcal{A} \equiv_\omega \mathcal{B} \iff \mathcal{A} \leq_\omega \mathcal{B} \;\&\; \mathcal{B} \leq_\omega \mathcal{A}$$

is an equivalence relation. The equivalence classes under this relation are called ω-enumeration degrees. In particular, the equivalence class $\mathbf{d}_\omega(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$ is called the ω-enumeration degree of $\mathcal{A}$. The relation $\leq_\omega$ defined by

$$\mathbf{a} \leq_\omega \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b} (\mathcal{A} \leq_\omega \mathcal{B})$$

is a partial order on the set of all ω-enumeration degrees $\mathbf{D}_\omega$. By $\mathcal{D}_\omega$ we shall denote the structure $(\mathbf{D}_\omega, \leq_\omega)$. The ω-enumeration degree $\mathbf{0}_\omega$ of the sequence $\emptyset_\omega = \{\emptyset\}_{k<\omega}$ is the least element in $\mathcal{D}_\omega$. Further, the ω-enumeration degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \mathbf{d}_\omega(\mathcal{A})$ and $\mathbf{b} = \mathbf{d}_\omega(\mathcal{B})$. Thus $\mathcal{D}_\omega$ is an upper semi-lattice with least element.

An explicit characterisation of the ω-enumeration reducibility is derived in [16]. According to it, $\mathcal{A} \leq_\omega \mathcal{B} \iff A_n \leq_e P_n(\mathcal{B})$ uniformly in n. More formally,

$\mathcal{A} \leq_\omega \mathcal{B}$ *if and only if* there is a computable function f, such that for every natural number k, $A_k = W_{f(k)}(P_k(\mathcal{B}))$. From here, one can show that each sequence is ω-enumeration equivalent with its jump sequence, i.e. for all $\mathcal{A} \in \mathcal{S}_\omega$,

$$\mathcal{A} \equiv_\omega \mathcal{P}(\mathcal{A}). \tag{3.2}$$

Further, for the sake of convenience, for sequences $\mathcal{A}, \mathcal{B} \in S_\omega$ we shall write $\mathcal{A} \leq_e \mathcal{B}$ if and only if for each $k < \omega, A_k \leq_e B_k$ uniformly in k. So $\mathcal{A} \leq_\omega \mathcal{B} \iff \mathcal{A} \leq_e \mathcal{P}(\mathcal{B})$. Note that there exist only countably many computable functions, so that there could be only countably many sequences ω-enumeration reducible to a given sequence. In particular every ω-enumeration degree cannot contain more than countably many sequences and hence there are continuum many ω-enumeration degrees.

Given a set $A \subseteq \omega$, denote by $A \uparrow \omega$ the sequence $(A, \emptyset, \emptyset, \dots, \emptyset, \dots)$. From the definition of $\leq_\omega$ and the uniformity of the jump operation, we have that for every sets A and B,

$$A \uparrow \omega \leq_\omega B \uparrow \omega \iff A \leq_e B. \tag{3.3}$$

The last equivalence means, that the mapping $\kappa : \mathcal{D}_e \to \mathcal{D}_\omega$, defined by, $\kappa(\mathbf{x}) = \mathbf{d}_\omega(X \uparrow \omega)$, where X is an arbitrary set in $\mathbf{x}$, is an embedding of $\mathcal{D}_e$ into $\mathcal{D}_\omega$. Further, the so defined embedding κ preserves the least element and the binary least upper bound operation. We shall denote the range of κ with $\mathbf{D}_1$.

4. THE JUMP OPERATOR

Following the lines of Soskov and Ganchev [15], the ω-enumeration jump $\mathcal{A}'$ of $\mathcal{A} \in \mathcal{S}_\omega$ is defined as the sequence

$$\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \dots, A_k, \dots).$$

This operator is defined so that if $\mathcal{A}'$ is the jump of $\mathcal{A}$, then the jump class $J_{\mathcal{A}'}$ of $\mathcal{A}'$ contains exactly the jumps of the degrees in the jump class $J_\mathcal{A}$ of $\mathcal{A}$. Note also, that for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$, so $\mathcal{A}' \equiv_\omega \{P_{k+1}(\mathcal{A})\}$.

The jump operator is strictly monotone, i.e. $\mathcal{A} \lneq_\omega \mathcal{A}'$ and $\mathcal{A} \leq_\omega \mathcal{B} \Rightarrow \mathcal{A}' \leq_\omega \mathcal{B}'$. This allows to define a jump operation on the ω-enumeration degrees by setting

$$\mathbf{a}' = \mathbf{d}_\omega(\mathcal{A}'),$$

where $\mathcal{A}$ is an arbitrary sequence in $\mathbf{a}$. Clearly, $\mathbf{a} <_\omega \mathbf{a}'$ and $\mathbf{a} \leq_\omega \mathbf{b} \Rightarrow \mathbf{a}' \leq_\omega \mathbf{b}'$.

Also the jump operation on ω-enumeration degrees agrees with the jump operation on the enumeration degrees, i.e. we have

$$\kappa(\mathbf{x}') = \kappa(\mathbf{x})', \text{ for all } \mathbf{x} \in \mathcal{D}_e.$$

We shall denote by $\mathcal{A}^{(n)}$ the n-the iteration of the jump operator on $\mathcal{A}$. Let us note that

$$\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \ldots) \equiv_\omega \{P_{n+k}(\mathcal{A})\}_{k<\omega}. \tag{4.1}$$

It is clear that if $\mathcal{A} \in \mathbf{a}$, then $\mathcal{A}^{(n)} \in \mathbf{a}^{(n)}$, where $\mathbf{a}^{(n)}$ denotes the n-th iteration of the jump operation on the ω-enumeration degree $\mathbf{a}$.

The jump operator on $\mathcal{D}_\omega$ preserves the greatest lower bound, i.e. for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{D}_\omega$,

$$\mathbf{x} \wedge \mathbf{y} = \mathbf{c} \Rightarrow \mathbf{x}' \wedge \mathbf{y}' = \mathbf{c}'.[2] \tag{4.2}$$

Further, Soskov and Ganchev [15] showed that for every natural number n if $\mathbf{b}$ is above $\mathbf{a}^{(n)}$, then there is a least ω-enumeration degree $\mathbf{x}$ above $\mathbf{a}$ with $\mathbf{x}^{(n)} = \mathbf{b}$. We denote this degree by $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$. An explicit representative of $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$ can be given by setting

$$I^n_{\mathcal{A}}(\mathcal{B}) = (A_0, A_1, \ldots, A_{n-1}, B_0, B_1, \ldots, B_k, \ldots), \tag{4.3}$$

where each $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ are arbitrary.

From here it follows that for every given $\mathbf{a} \in \mathbf{D}_\omega$ and $n < \omega$, the operation $\mathbf{I}^n_{\mathbf{a}}$ is monotone. Further, its range is a downwards closed subset of the upper cone with least element $\mathbf{a}$. In fact, even a stronger property holds: if $\mathbf{x}, \mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega$ are such that $\mathbf{a} \leq_\omega \mathbf{x}$, $\mathbf{a}^{(n)} \leq_\omega \mathbf{b}$ and $\mathbf{x} \leq_\omega \mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$, then $\mathbf{x}$ is equal to $\mathbf{I}^n_{\mathbf{a}}(\mathbf{x}^{(n)})$. The above property can be easily verified by simple relativisation of claim $(I2)$ of Lemma 1 in [3].

It what follows, when $\mathbf{a} = \mathbf{0}_\omega$, we shall write $\mathbf{I}^n$ instead of $\mathbf{I}^n_{\mathbf{0}_\omega}$. Finally, we provide a property of the jump inversion operation, a proof of which can be found in [3].

$$(\mathbf{x} \vee \mathbf{I}^n(\mathbf{a}))^{(n)} = \mathbf{x}^{(n)} \vee \mathbf{a}. \tag{4.4}$$

5. THE LOCAL THEORY AND THE $\mathbf{o}_n$ DEGREES

The structure of the degrees lying beneath the first jump of the least element is usually referred to as the local structure of a degree structure. In the case of the ω-enumeration degrees we shall denote this structure by $\mathcal{G}_\omega$. When considering a local structure, one is usually concerned with questions about the definability of some classes of degrees, which have a natural definition either in the context of the global structure (for example the classes of the high and the low degrees) or in the context of the basic objects from which the degrees are built (for example the class of the Turing degrees containing a c.e. set).

Recall that a degree in the local structure is said to be *high*$_n$ for some n *if and only if* its n-th jump is as high as possible. Similarly, a degree in the local structure is said to be *low*$_n$ for some n *if and only if* its n-th jump is as low as possible. More formally, in the case of $\mathcal{G}_\omega$, a degree $\mathbf{a} \in \mathcal{G}_\omega$ is high$_n$ *if and only if* $\mathbf{a}^{(n)} = (\mathbf{0}'_\omega)^{(n)} = \mathbf{0}^{(n+1)}_\omega$, and is low$_n$ *if and only if* $\mathbf{a}^{(n)} = (\mathbf{0}'_\omega)^{(n)}$.

As usual, we denote by $\mathbf{H}_n$ the collection of all high$_n$ degrees, and by $\mathbf{L}_n$ the collection of all low$_n$ degrees. Also $\mathbf{H}$ stands for the union of all the classes $\mathbf{H}_n$ and analogously, $\mathbf{L}$ is the union of all of the classes $\mathbf{L}_n$. Finally, $\mathbf{I}$ will stay for the collection of the degrees that are neither high$_n$ nor low$_n$ for any n. The degrees in $\mathbf{I}$ shall be referred to as intermediate degrees.

Using the corresponding results for the structure of the enumeration degrees, it is easy to see that there exist intermediate degrees and for every natural number n, there are degrees in the local structure of the ω-enumeration degrees, that are high$_{(n+1)}$ (respectively low$_{(n+1)}$) but are not high$_n$ (respectively low$_n$).

Soskov and Ganchev [15] gave a characterisation of the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ that does not involve directly the jump operation. Let us set $\mathbf{o}_n$ to be the least n-th jump invert of $\mathbf{0}_\omega^{(n+1)}$, i.e., $\mathbf{o}_n = \mathbf{I}^n(\mathbf{0}^{(n+1)})$. Note that $\mathbf{o}_n$ is the least element of the class $\mathbf{H}_n$. Thus for arbitrary $\mathbf{x} \in \mathcal{G}_\omega$,

$$\mathbf{x} \in \mathbf{H}_n \iff \mathbf{o}_n \leq_\omega \mathbf{x}. \tag{5.1}$$

In particular, since every high$_n$ degree is also high$_{(n+1)}$, $\mathbf{o}_{n+1} \leq_\omega \mathbf{o}_n$. On the other hand, since $\mathbf{H}_{n+1} \setminus \mathbf{H}_n \neq \emptyset$, the equality $\mathbf{o}_{n+1} = \mathbf{o}_n$ is impossible, so that

$$\mathbf{0}'_\omega = \mathbf{o}_0 >_\omega \mathbf{o}_1 >_\omega \mathbf{o}_2 >_\omega \cdots >_\omega \mathbf{o}_n >_\omega \ldots .$$

Recall that if a degree is beneath a least n-th jump invert above $\mathbf{a}$, then it itself is a least n-th jump invert above $\mathbf{a}$. In particular, if $\mathbf{y} \leq_\omega \mathbf{o}_n$, then $\mathbf{y} = \mathbf{I}^n(\mathbf{z})$ for some degree $\mathbf{0}_\omega^{(n)} \leq_\omega \mathbf{z} \leq_\omega \mathbf{0}_\omega^{(n+1)}$ or more concretely $\mathbf{y} = \mathbf{I}^n(\mathbf{y}^{(n)})$. On the other hand if $\mathbf{y} \in \mathcal{G}_\omega$ is a least n-th jump invert, then from the monotonicity of $\mathbf{I}^n$ we have $\mathbf{y} \leq_\omega \mathbf{o}_n$. Thus

$$\{\mathbf{y} \in \mathcal{G}_\omega \mid \mathbf{y} \leq_\omega \mathbf{o}_n\} = \{\mathbf{I}^n(\mathbf{z}) \mid \mathbf{0}_\omega^{(n)} \leq_\omega \mathbf{z} \leq_\omega \mathbf{0}_\omega^{(n+1)}\}.$$

In particular, since $\mathbf{I}^n$ is injective,

$$[\mathbf{0}_\omega, \mathbf{o}_n] \simeq [\mathbf{0}_\omega^{(n)}, \mathbf{0}_\omega^{(n+1)}].$$

Ganchev and M. Soskova [3] showed that for arbitrary $\mathbf{x} \in \mathcal{G}_\omega$,

$$\mathbf{I}^n(\mathbf{x}^{(n)}) = \mathbf{x} \wedge \mathbf{o}_n. \tag{5.2}$$

Indeed, let us take an arbitrary $\mathbf{x} \in \mathcal{G}_\omega$. Clearly $\mathbf{I}^n(\mathbf{x}^{(n)}) \leq_\omega \mathbf{x}$ and $\mathbf{I}^n(\mathbf{x}^{(n)}) \leq_\omega \mathbf{o}_n$. On the other hand if $\mathbf{y}$ is such that $\mathbf{y} \leq_\omega \mathbf{x}$ and $\mathbf{y} \leq_\omega \mathbf{o}_n$, then from the second inequality we have $\mathbf{y} = \mathbf{I}^n(\mathbf{z})$ for some $\mathbf{z}$. This together with the first inequality gives us $\mathbf{z} = (\mathbf{I}^n(\mathbf{z}))^{(n)} = \mathbf{y}^{(n)} \leq_\omega \mathbf{x}^{(n)}$. Thus $\mathbf{y} = \mathbf{I}^n(\mathbf{z}) \leq_\omega \mathbf{I}^n(\mathbf{x}^{(n)})$.

This gives us a characterisation of the low$_n$ degrees in terms of the partial order $\leq_\omega$ and the degrees $\mathbf{o}_n$, namely

$$\mathbf{x} \in \mathbf{L}_n \iff \mathbf{x} \wedge \mathbf{o}_n = \mathbf{0}_\omega. \tag{5.3}$$

They also show that for arbitrary $\mathbf{a} \in \mathcal{G}_\omega$, $\mathbf{a}$ is a degree in $\mathbf{D}_1$ *iff*

$$\forall \mathbf{x} \in \mathcal{G}_\omega (\mathbf{x} \vee \mathbf{o}_1 = \mathbf{a} \vee \mathbf{o}_1 \to \mathbf{x} \geq_\omega \mathbf{a}). \tag{5.4}$$

The formula (5.4) characterises the degrees in $\mathbf{D}_1 \cap \mathcal{G}_\omega$ in terms of the ordering $\leq_\omega$ and the degree $\mathbf{o}_1$.

Soskov and Ganchev [15] introduced the *almost zero* (*a.z.*) degrees. Following their lines, the degree $\mathbf{x}$ is *a.z. if and only if* there is a representative $\mathcal{X} \in \mathbf{x}$ such that

$$(\forall k)[P_k(\mathcal{X}) \equiv_e \emptyset^{(k)}]. \tag{5.5}$$

It is clear that the class of the *a.z.* degrees is downward closed. Further, one can easily show that the only *a.z.* degree $\mathbf{a}$ for which there is a natural number n such that $\mathbf{a}^{(n)} = \mathbf{0}_\omega^{(n)}$ is the least element $\mathbf{0}_\omega$. Note also that there are continuum many *a.z.* degrees and hence not all *a.z.* degrees are in $\mathcal{G}_\omega$.

The *a.z.* degrees in $\mathcal{G}_\omega$ are exactly the degrees bounded by every degree $\mathbf{o}_n$, i.e.

$$\mathbf{x} \in \mathcal{G}_e \text{ is } a.z. \iff (\forall n < \omega)[\mathbf{x} \leq_\omega \mathbf{o}_n]. \tag{5.6}$$

Further, the classes $\mathbf{H}$ and $\mathbf{L}$ can be characterised in terms of the ordering $\leq_\omega$ and the *a.z.* degrees [15], namely

$$\mathbf{a} \in \mathbf{H} \iff (\forall \mathbf{x} - a.z.)[\mathbf{x} \leq_\omega \mathbf{a}], \tag{5.7}$$

and

$$\mathbf{a} \in \mathbf{L} \iff (\forall \mathbf{x} - a.z.)[\mathbf{x} \leq_\omega \mathbf{a} \to \mathbf{x} = \mathbf{0}_\omega], \tag{5.8}$$

where all quantifiers are restricted to degrees in $\mathcal{G}_\omega$.

From the second equivalence it follows that the only low_n *a.z.* degree is $\mathbf{0}_\omega$. Further, according to (5.1) no *a.z.* degree is high_n for any n. Thus all *a.z.* degrees are intermediate degrees.

6. DEFINABILITY IN $\mathcal{G}_\omega$

We prove in this section that the set $\mathfrak{O} = \{\mathbf{o}_n | n < \omega\}$ is first order definable in $\mathcal{G}_\omega$. Thus, by (5.1) and (5.3), we may conclude the proof of the Theorem 1. For this purpose we shall need the notion of a *Kalimullin pair* (or $\mathcal{K}$-pair).

Definition 3. *Let $\mathcal{D} = (\mathbf{D}, \leq)$ be a partial order. The pair $\{\mathbf{a}, \mathbf{b}\}$ is said to be $\mathcal{K}$-pair (strictly) over $\mathbf{u}$ for $\mathcal{D}$, if $\mathbf{a}, \mathbf{b}, \mathbf{u} \in \mathbf{D}, \mathbf{u} \leq \mathbf{a}, \mathbf{b}$ ($\mathbf{u} \lneq \mathbf{a}, \mathbf{b}$) and for all $\mathbf{x} \in \mathbf{D}$ such that $\mathbf{u} \leq \mathbf{x}$, the least upper bounds $\mathbf{x} \vee \mathbf{a}, \mathbf{x} \vee \mathbf{b}$ and greatest lower bound $(\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b})$ exist, and the following holds:*

$$\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}). \tag{6.1}$$

Further, if $\mathcal{D} = (\mathbf{D}, \leq)$ is a partially ordered set and $\mathbf{u}, \mathbf{v} \in \mathbf{D}$, we shall use the notation $\mathbf{D}[\mathbf{u}, \mathbf{v}]$ for the set $\{\mathbf{x} \in \mathbf{D} | \mathbf{u} \leq \mathbf{x} \leq \mathbf{v}\}$ together with the partial order inherited from $\mathcal{D}$.

Clearly, there exists a first order formula $\mathcal{K}$ of two free variables such that if $\mathcal{D}$ has a least element $\mathbf{0}_\mathcal{D}$, then

$$\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{b}) \iff \{\mathbf{a}, \mathbf{b}\} \text{ is a } \mathcal{K}\text{-pair strictly over } \mathbf{0}_\mathcal{D} \text{ for } \mathcal{D}.$$

Also, we shall use the fact that for each $\mathbf{a} \in \mathbf{D}$, the set

$$\mathcal{I} = \{\mathbf{b} \mid \{\mathbf{a}, \mathbf{b}\} \text{ is a } \mathcal{K}\text{-pair strictly over } \mathbf{0}_\mathcal{D} \text{ for } \mathcal{D}\}$$

is either empty or ideal, see for example [5].

The starting step of the first order definition in $\mathcal{G}_\omega$ of the set $\mathfrak{O}$ is the characterisation of the $\mathcal{K}$-pairs in $\mathcal{G}_\omega$, due to Ganchev and M. Soskova [3]. According to it, whenever $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair in $\mathcal{G}_\omega$ strictly over $\mathbf{0}_\omega$, then either $\mathbf{a}$ and $\mathbf{b}$ are both *a.z.* or the $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is inherited from the structure $\mathcal{D}_e$, i.e. there exist sets A, B and a natural number n such that:

1. $\emptyset^{(n)} <_e A, B \leq_e \emptyset^{(n+1)}$ and $A' = B' = \emptyset^{(n+1)}$;
2. $\{\mathbf{d}_e(A), \mathbf{d}_e(B)\}$ is a $\mathcal{K}$-pair in $\mathbf{D}_e[\mathbf{0}_e^{(n)}, \mathbf{0}_e^{(n+1)}]$ strictly over $\mathbf{0}_e^{(n)}$;
3. $\mathbf{a} = \mathbf{I}^n(\kappa(\mathbf{d}_e(A)))$ and $\mathbf{b} = \mathbf{I}^n(\kappa(\mathbf{d}_e(B)))$.

It is known [3] that every two degrees $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$, which are inherited from $\mathcal{D}_e$ in the above sense, form a $\mathcal{K}$-pair in $\mathcal{G}_\omega$ strictly over $\mathbf{0}_\omega$.

Note that by definitions of the embedding κ and the least jump inversion operation (4.3) the last condition of the above characterisation of the $\mathcal{K}$-pairs in the local theory is equivalent to the fact that the degrees $\mathbf{a}$ and $\mathbf{b}$ contain respectively the sequences $(\underbrace{\emptyset, \emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots)$ and $(\underbrace{\emptyset, \emptyset, \dots, \emptyset}_{n}, B, \emptyset, \dots, \emptyset, \dots)$.

Using the above characterisation, one can prove that for each $n \geq 0$, $\mathbf{o}_{n+1}$ is the greatest degree (in $\mathcal{G}_\omega$) which is the least upper bound of a $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly above $\mathbf{0}_\omega$ such that $(\forall \mathbf{x} \lneq_\omega \mathbf{o}_n)[\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n]$. Since $\mathbf{o}_0$ is the greatest degree in $\mathcal{G}_\omega$, it follows that for each natural number n, $\mathbf{o}_n$ is first order definable in $\mathcal{G}_\omega$.

Note that the *a.z.* degrees are closed under the least upper bound operation and no $\mathbf{o}_n$ is *a.z.*, thus if $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly over $\mathbf{0}_\omega$ with $\mathbf{a} \vee \mathbf{b} = \mathbf{o}_n$, then $\{\mathbf{a}, \mathbf{b}\}$ is an inherited $\mathcal{K}$-pair.

Now we shall show how to separate in $\mathcal{G}_\omega$ the inherited $\mathcal{K}$-pairs from those formed by *a.z.* degrees. Suppose that $\{\mathbf{a}, \mathbf{b}\}$ is an inherited $\mathcal{K}$-pair and let $A, B \subseteq \omega$ and $n < \omega$ be the corresponding witnesses for this. It is known by the a result of Kent and Sorbi [6], that every nonzero enumeration degree $\mathbf{x} \in \mathbf{D}_e[\mathbf{0}_e, \mathbf{0}_e']$ bounds a nonzero nonsplittable[2] degree $\mathbf{y} \in \mathbf{D}_e[\mathbf{0}_e, \mathbf{0}'_e]$. Relativising this result over $\mathbf{0}_e^{(n)}$ we conclude that there are sets $A_0, B_0 \subseteq \omega$ such that $\emptyset^{(n)} <_e A_0 \leq_e A$, $\emptyset^{(n)} <_e B_0 \leq_e B$, such that both $\mathbf{d}_e(A_0)$ and $\mathbf{d}_e(B_0)$ are nonsplittable over $\mathbf{0}_e^{(n)}$. But then the degrees $\mathbf{a}_0 = \mathbf{I}^n(\kappa(\mathbf{d}_e(A_0)))$ and $\mathbf{b}_0 = \mathbf{I}^n(\kappa(\mathbf{d}_e(B_0)))$ are nonsplittable. Indeed, assume without loss of generality that $\mathbf{a}_0$ is splittable. Then $\mathbf{a}_0 = \mathbf{c} \vee \mathbf{d}$ for some $\mathbf{0}_\omega <_\omega \mathbf{c}, \mathbf{d} <_\omega \mathbf{a}_0$ and let $\mathcal{C} = \{C_m\}_{m<\omega} \in \mathbf{c}, \mathcal{D} = \{D_m\}_{m<\omega} \in \mathbf{d}$. According to (4.3) and (3.2), $(\underbrace{\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}}_{n}, A_0, \emptyset^{(n+1)}, \ldots) \in \mathbf{a}_0$, so that

$$\mathcal{P}(\mathcal{C}) \oplus \mathcal{P}(\mathcal{D}) \leq_e (\underbrace{\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}}_{n}, A_0, \emptyset^{(n+1)}, \ldots),$$

From here $P_n(\mathcal{C}) \oplus P_n(\mathcal{D}) \leq_e A_0$,

$$\mathcal{P}(\mathcal{C}) \equiv_e (\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}, P_n(\mathcal{C}), \emptyset^{(n+1)}, \ldots)$$

and

$$\mathcal{P}(\mathcal{D}) \equiv_e (\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}, P_n(\mathcal{D}), \emptyset^{(n+1)}, \ldots).$$

Since $\mathbf{a}_0 \leq_\omega \mathbf{c} \vee \mathbf{d}$, then $(\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}, A_0, \emptyset^{(n+1)}, \ldots) \leq_e \mathcal{P}(\mathcal{P}(\mathcal{C}) \oplus \mathcal{P}(\mathcal{D}))$. Then we have that $A_0 \leq_e P_n(\mathcal{P}(\mathcal{C}) \oplus \mathcal{P}(\mathcal{D})) \equiv_e P_n(\mathcal{C}) \oplus P_n(\mathcal{D})$, [3], so finally $A_0 \equiv_e P_n(\mathcal{C}) \oplus P_n(\mathcal{D})$. Since $\mathbf{d}_e(A_0)$ is a nonsplittable degree over $\mathbf{0}_e^{(n)}$, either $P_n(\mathcal{C}) \equiv_e A_0$ or $P_n(\mathcal{D}) \equiv_e A_0$. In the first case we have that $\mathbf{a}_0 = \mathbf{c}$, and in the second – $\mathbf{a}_0 = \mathbf{d}$, i.e., we reach a contradiction.

Thus, if $\{\mathbf{a}, \mathbf{b}\}$ is an inherited $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$, then both $\mathbf{a}$ and $\mathbf{b}$ bound nonzero nonsplitting degrees. Next we shall see that if $\{\mathbf{a}, \mathbf{b}\}$ is a *a.z.* $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ then neither $\mathbf{a}$ nor $\mathbf{b}$ bounds a nonzero nonsplitting degree. Moreover, the following property holds for every *a.z.* degree in $\mathcal{D}_\omega$.

Lemma 4. *Every nonzero a.z. degree in $\mathcal{D}_\omega$ is splittable.*

Proof. Let $\mathbf{a}$ be a nonzero a.z. degree and let $\mathcal{A} \in \mathbf{a}$ satisfy (5.5). We shall construct sequences $\mathcal{B}$ and $\mathcal{C}$ such that $\emptyset_\omega \lneq_\omega \mathcal{B}, \mathcal{C} \lneq_\omega \mathcal{A}$ and $\mathcal{B} \oplus \mathcal{C} \equiv_\omega \mathcal{A}$. We shall construct $\mathcal{B} = \{B_k\}_{k<\omega}$ and $\mathcal{C} = \{C_k\}_{k<\omega}$ using induction on k. For every k we shall set either $B_k = \emptyset$ and $C_k = A_k$ or $B_k = A_k$ and $C_k = \emptyset$. This condition will ensure that $\mathcal{B} \oplus \mathcal{C} = \mathcal{A}$. So, in order to build $\mathcal{B}$ and $\mathcal{C}$ as desired, it suffices that $\mathcal{B}, \mathcal{C} \leq_\omega \mathcal{A}$ and that the following requirements are satisfied:

$$R_{2e} : \exists k \left(\varphi_e(k)\uparrow \ \vee\ A_k \neq W_{\varphi_e(k)}(P_k(\mathcal{B}))\right),$$

[2] Let $\mathcal{D} = (\mathbf{D}, \mathbf{0}, \leq, \vee)$ be an upper semilattice with a least element. Let $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ be such that $\mathbf{b} \leq \mathbf{a}$. We shall say that $\mathbf{a}$ is *splittable over* $\mathbf{b}$ if and only if there are $\mathbf{x}, \mathbf{y} \in \mathbf{D}$ such that

$$\mathbf{b} \leq \mathbf{x}, \mathbf{y} < \mathbf{a} = \mathbf{x} \vee \mathbf{y}.$$

When there are not such $\mathbf{x}$ and $\mathbf{y}$ we shall say that $\mathbf{a}$ is *nonsplittable over* $\mathbf{b}$. In the case when $\mathbf{b}$ is the least element we shall say only that $\mathbf{a}$ is splittable or nonsplittable.

[3] the last equivalence can be easily verified using induction on $n < \omega$.

$$R_{2e+1} : \exists k \left(\varphi_e(k)\uparrow \;\vee\; A_k \neq W_{\varphi_e(k)}(P_k(\mathcal{C}))\right).$$

Note that $\mathcal{B},\mathcal{C} \leq_\omega \mathcal{A}$ gives us automatically, that $\mathcal{B}$ and $\mathcal{C}$ satisfy (5.5). The requirement R_{2e} ensures that $\mathcal{A}$ can not be uniformly reduced to $\mathcal{P}(\mathcal{B})$ using the e-th computable function. Similarly, R_{2e+1} expresses that $\mathcal{A}$ can not be uniformly reduced to $\mathcal{P}(\mathcal{C})$ using the e-th computable function.

The construction: During the construction we shall use a global variable $\mathfrak{r}$ which shall show us the least requirement that is (possibly) not yet satisfied. We start by setting $\mathfrak{r} = 0$. Also we set $B_0 = B_1 = \emptyset$, $C_0 = A_0$ and $C_1 = A_1$. Let us suppose that $k \geq 2$ and that B_s and C_s are defined for $s \leq k$. Note that our assumption yields that for $s \leq k$, $P_s(\mathcal{B})$ and $P_s(\mathcal{C})$ are defined as well.

Case 1: $\mathfrak{r} = 2e$. If $\varphi_e(k-2)\uparrow$ or $A_{k-2} \neq W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{B}))$, set $B_k = A_k$, $C_k = \emptyset$ and augment $\mathfrak{r}$ by 1. Otherwise set $B_k = \emptyset$, $C_k = A_k$ and keep $\mathfrak{r}$ the same.

Case 2: $\mathfrak{r} = 2e+1$. If $\varphi_e(k-2)\uparrow$ or $A_{k-2} \neq W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{C}))$, set $B_k = \emptyset$, $C_k = A_k$ and augment $\mathfrak{r}$ by 1. Otherwise set $B_k = A_k$, $C_k = \emptyset$ and keep $\mathfrak{r}$ the same.

End of construction.

First of all let us note that, according to the definition of the jump sequence $\mathcal{P}(\mathcal{A})$, $\emptyset'' \leq_e P_k(\mathcal{A})$ for $k \geq 2$ uniformly in k. Hence for $k \geq 2$, given any enumeration of $P_k(\mathcal{A})$ we can uniformly decide if $\varphi_e(k-2)\uparrow$. Further, for $k \geq 2$, $P_{k-2}(\mathcal{A})'' \leq_e P_k(\mathcal{A})$ uniformly in k. These properties of $P_k(\mathcal{A})$ and a simple induction on $k \geq 2$ yield that given any enumeration of $P_k(\mathcal{A})$, we can uniformly answer to the questions

$$\varphi_e(k-2)\uparrow \;\vee\; A_{k-2} \neq W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{B}))$$

and

$$\varphi_e(k-2)\uparrow \;\vee\; A_{k-2} \neq W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{C})).$$

In particular, any enumeration of $P_k(\mathcal{A})$ can compute uniformly the value of $\mathfrak{r}$ at stage k and hence it can compute uniformly B_k and C_k. Therefore $\mathcal{B},\mathcal{C} \leq_\omega \mathcal{A}$.

It remains to prove that all the requirements are satisfied. Towards a contradiction assume that some requirement is not fulfilled and let n be the least index of such a requirement. Note that the construction yields that at some stage m, the global variable $\mathfrak{r}$ has been set to be equal to n, and from then on $\mathfrak{r}$ has never changed its value. First let us suppose that $n = 2e$ for some natural number e. Then for every $k > m$, $A_{k-2} = W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{B}))$, so that $B_k = \emptyset$ for $k > m$ and $A_k \leq_e P_k(\mathcal{B})$ uniformly in $k > m$. On the other hand for $0 \leq k \leq m$,

$$B_k \leq_e P_k(\mathcal{A}) \leq_e \emptyset^{(k)},$$

which together with our previous observation yields $\mathcal{B} \leq_\omega \emptyset_\omega$ and $\mathcal{A} \leq_\omega \mathcal{B}$. Thus $\mathcal{A} \leq_\omega \emptyset_\omega$, contradicting the choice of $\mathcal{A}$.

If $n = 2e + 1$, we obtain in a quite similar way $\mathcal{A} \leq_\omega \emptyset_\omega$, contradicting once again the choice of $\mathcal{A}$. Therefore our assumption that some of the requirements is not satisfied is incorrect, and hence $\emptyset_\omega \lneq_\omega \mathcal{B}, \mathcal{C} \lneq_\omega \mathcal{A}$. □

Thus, we have obtained that every inherited $\mathcal{K}$-pair bounds a nonsplitting degree, whereas every *a.z.* is splittable. Therefore we may define a first order formula $\mathcal{K}_{inh}$ separating the inherited $\mathcal{K}$-pairs from the ones formed by *a.z.* degree by setting

$$\mathcal{K}_{inh}(\mathbf{a}, \mathbf{b}) = \mathcal{K}(\mathbf{a}, \mathbf{b}) \ \& \ (\exists \mathbf{x})[x \leq_\omega \mathbf{a} \ \& \ (\forall \mathbf{u}, \mathbf{v})[\mathbf{u}, \mathbf{v} <_\omega \mathbf{x} \to \mathbf{u} \vee \mathbf{v} <_\omega \mathbf{x}]].$$

Now we have the instrument needed for the definition of the set $\mathfrak{O}$. Recall that every degree $\mathbf{o}_n$ is the least upper bound of an inherited $\mathcal{K}$-pair, so that we need just to focus on the properties of the least upper bounds of such $\mathcal{K}$-pairs.

Suppose that $\{\mathbf{a}, \mathbf{b}\}$ is an inherited $\mathcal{K}$-pair and let $A, B \subseteq \omega$ and $n < \omega$ be witnesses for this. Since

$$(\underbrace{\emptyset, \emptyset' \ldots, \emptyset^{(m-1)}}_{m}, \emptyset^{(m+1)}, \emptyset^{(m+2)}, \ldots) \in \mathbf{o}_m,$$

$$(\underbrace{\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}}_{n}, A, \emptyset^{(n+1)}, \ldots) \in \mathbf{a},$$

$$(\underbrace{\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}}_{n}, B, \emptyset^{(n+1)}, \ldots) \in \mathbf{b},$$

$\emptyset^{(n)} <_e A, B \leq_e \emptyset^{(n+1)}$ and $A' = B' = \emptyset^{(n+1)}$, we have $\mathbf{a}, \mathbf{b} <_\omega \mathbf{o}_m$ for $m \leq n$. On the other hand, $m > n$ implies that $\mathbf{a}, \mathbf{b} \not\leq_\omega \mathbf{o}_m$ and $\mathbf{o}_m \not\leq_\omega \mathbf{a}, \mathbf{b}$, for otherwise we would have $A \leq_e \emptyset^{(n)}$ and $\emptyset^{(m+1)} \leq_e \emptyset^{(m)}$, respectively.

Hence, for every $m < \omega$ and every inherited $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$, either $\mathbf{a}, \mathbf{b} <_\omega \mathbf{o}_m$ or $\mathbf{a}, \mathbf{b}|_\omega \mathbf{o}_m$.

Now we claim that whenever $\mathbf{x}$ is the least upper bound of an inherited $\mathcal{K}$-pair and $\mathbf{x}$ is not $\mathbf{o}_m$ for any natural number m, there exists an inherited $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ such that $\mathbf{a}|_\omega \mathbf{x}$ and $\mathbf{b} \leq_\omega \mathbf{x}$. Indeed, suppose that $\mathbf{x} = \mathbf{c} \vee \mathbf{d}$ for some inherited $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ and for all $m < \omega$, $\mathbf{x} \neq \mathbf{o}_m$. Let the sets C, D and the natural number n be witnessing that the $\mathcal{K}$-pair is inherited. Then the sequence $(\underbrace{\emptyset, \ldots, \emptyset}_{n}, C \oplus D, \emptyset, \ldots)$ is an element of the degree $\mathbf{x}$. Note that $C, D \leq_e \emptyset^{(n+1)}$ and $\mathbf{x} = \mathbf{c} \vee \mathbf{d} \neq \mathbf{o}_n$, so $C \oplus D \lneq_e \emptyset^{(n+1)}$. Since $\mathbf{c}$ and $\mathbf{d}$ are not *a.z.*, we have that C and D are low over $\emptyset^{(n)}$ and hence $C, D \in \Delta_2^0(\emptyset^{(n)})$. But then we have also $C \oplus D \in \Delta_2^0(\emptyset^{(n)})$. In what follows we shall need the following result due to Ganchev and M. Soskova [3].

Theorem 5. *For every total*[4] *enumeration degree* $\mathbf{g}$ *and every degree* $\mathbf{e}$*, such that* $\mathbf{g} \leq_e \mathbf{e}$ *and* $\mathbf{e}$ *contains a set* Δ_2^0 *relative to* $\mathbf{g}$*, there is a* $\mathcal{K}$*-pair* $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}$ *in* $\mathbf{D}_e[\mathbf{g}, \mathbf{g}']$ *strictly over* $\mathbf{g}$*, such that* $\widetilde{\mathbf{a}} \vee \mathbf{e} = \mathbf{g}'$*. In the case when* $\mathbf{e} \lneq_e \mathbf{g}'$ *we additionally have that* $\widetilde{\mathbf{a}}|_e\mathbf{e}$ *and* $\widetilde{\mathbf{b}} \leq_e \mathbf{e}$ *(since* $\mathbf{e} = (\widetilde{\mathbf{a}} \vee \mathbf{e}) \wedge (\widetilde{\mathbf{b}} \vee \mathbf{e})$ *and* $\widetilde{\mathbf{a}} \vee \mathbf{e} = \mathbf{g}'$*).*

Now let $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}$ be the corresponding $\mathcal{K}$-pair for $\mathbf{g} = \mathbf{0}_e^{(n)}$ and $\mathbf{e} = \mathbf{d}_e(C \oplus D)$. Let A and B be sets having enumeration degrees $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ respectively. Then the ω-enumeration degrees $\mathbf{a} = \mathbf{d}_\omega(\underbrace{\emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots)$ and $\mathbf{b} = \mathbf{d}_\omega(\underbrace{\emptyset, \dots, \emptyset}_{n}, B, \emptyset, \dots)$ form an inherited $\mathcal{K}$-pair for $\mathcal{G}_\omega$ such that $\mathbf{a}|_\omega\mathbf{x}$ and $\mathbf{b} \leq_\omega \mathbf{x}$.

Thus we have proven that a degree $\mathbf{x} \leq_\omega \mathbf{0}'_\omega$ is $\mathbf{o}_n$ for some natural number n if and only if $\mathbf{x}$ is the least upper bound of an inherited $\mathcal{K}$-pair and for each inherited $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ either $\mathbf{a}, \mathbf{b} <_\omega \mathbf{x}$ or $\mathbf{a}, \mathbf{b}|_\omega\mathbf{x}$. Namely,

$$\mathbf{x} \in \mathfrak{O} \iff (\exists \mathbf{a}, \mathbf{b})[\mathcal{K}_{inh}(\mathbf{a}, \mathbf{b}) \ \& \ \mathbf{x} = \mathbf{a} \vee \mathbf{b}] \&$$

$$(\forall \mathbf{a}, \mathbf{b})[\mathcal{K}_{inh}(\mathbf{a}, \mathbf{b}) \rightarrow \mathbf{a}, \mathbf{b} \leq_\omega \mathbf{x} \ \vee \ \mathbf{a}, \mathbf{b}|_\omega\mathbf{x}].$$

This gives us a first order definability in $\mathcal{G}_\omega$ of the set $\mathfrak{O}$ as well as of the classes $\mathbf{H}$ and $\mathbf{L}$. A direct consequence of the latter and (5.6) is the following corollary.

Corollary 6. *The set of all a.z. degrees is first order definable in* $\mathcal{G}_\omega$*.*

7. DEFINABILITY OF $\mathbf{0}'_\omega$

In this section we characterise the class of the $\mathcal{K}$-pairs strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Namely, we shall show that either such a $\mathcal{K}$-pair consists of *a.z.* degrees, or it is inherited. As a consequence of this characterisation and the fact that $\mathbf{0}'_\omega$ bounds the elements of all inherited $\mathcal{K}$-pairs we shall find a first order definition of the first jump of the least element in the structure $\mathcal{D}_\omega$.

First, let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Let $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ respectively. Using the connections between the $\mathcal{K}$-pairs in $\mathcal{D}_e$ and $\mathcal{G}_\omega$ derived in [3], we are able to conclude that for each $n < \omega$, $\{\mathbf{d}_e(P_n(\mathcal{A})), \mathbf{d}_e(P_n(\mathcal{B}))\}$ is a $\mathcal{K}$-pair over $\mathbf{0}_e^{(n)}$ for $\mathbf{D}_e[\geq \mathbf{0}_e^{(n)}]$. Hence by [5] each of $\mathbf{d}_e(P_n(\mathcal{A}))$ and $\mathbf{d}_e(P_n(\mathcal{B}))$ is quasiminimal over $\mathbf{0}_e^{(n)}$ (the enumeration degree $\mathbf{a}$ is quasiminimal over the enumeration degree $\mathbf{b} <_e \mathbf{a}$ if and only if there is no total $\mathbf{b} \lneq_e \mathbf{c} \leq_e \mathbf{a}$). Since for each n, $\mathbf{0}_e^{(n+1)} \leq_e \mathbf{d}_e(P_n(\mathcal{A}))' \leq_e \mathbf{d}_e(P_{n+1}(\mathcal{A}))$ and $\mathbf{d}_e(P_n(\mathcal{A}))'$ is total (since every jump is total), then for each n, $P_n(\mathcal{A})' \equiv_e \emptyset^{(n+1)}$. The same equivalence obviously holds also for $P_n(\mathcal{B})'$.

[4]An enumeration degree is said to be total if and only if there exists a set A such that the degree contains the set A^+. With other words a degree is total if and only if it is an image of a Turing degree under the Rogers' embedding $\iota : \mathcal{D}_T \rightarrow \mathcal{D}_e$. For example, for each n, the degree $\mathbf{0}_e^{(n)}$ is total.

Having in mind the last observation, consider a $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$ and suppose that at least one of the degrees $\mathbf{a}$ or $\mathbf{b}$ is not *a.z.*. Without loss of generality, suppose that $\mathbf{a}$ is not *a.z.* degree and let $\mathcal{A} \in \mathbf{a}$. Therefore there is $n < \omega$ such that $P_n(\mathcal{A}) \not\equiv_e \emptyset^{(n)}$. Let $\mathbf{a}^-$ be the ω-enumeration degree which contains the sequence $(\underbrace{\emptyset, \ldots, \emptyset}_{n}, P_n(\mathcal{A}), \emptyset, \ldots)$. Note that $\mathbf{a}^-$ is bellow $\mathbf{0}'_\omega$ and that $\{\mathbf{a}^-, \mathbf{b}\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Since $\mathbf{0}_e^{(n)} \leq_e \mathbf{d}_e(P_n(\mathcal{A})) \leq_e \mathbf{0}_e^{(n+1)}$, then the equality $\mathbf{d}_e(P_n(\mathcal{A}))' = \mathbf{0}_e^{(n+1)}$, together with Theorem 5, yields

$$\mathbf{d}_e(P_n(\mathcal{A})) \vee \mathbf{x} = \mathbf{x}' = \mathbf{0}_e^{(n+1)}$$

for some $\mathbf{0}_e^{(n)} \lneq_e \mathbf{x} \leq_e \mathbf{0}_e^{(n+1)}$. So for the ω-enumeration degree $\kappa(\mathbf{x})$, we have that $\mathbf{0}_\omega^{(n)} \leq_\omega \kappa(\mathbf{x}) \leq_\omega \mathbf{0}_\omega^{(n+1)}$ and since the jump inversion operation is monotone, $\mathbf{I}^n(\kappa(\mathbf{x})) \leq_\omega \mathbf{0}'_\omega$. Therefore,

$$\mathbf{I}^n(\kappa(\mathbf{x})) = (\mathbf{I}^n(\kappa(\mathbf{x})) \vee \mathbf{a}^-) \wedge (\mathbf{I}^n(\kappa(\mathbf{x})) \vee \mathbf{b}).$$

Hence, using (4.4) and (4.2), we obtain

$$\kappa(\mathbf{x}) = (\kappa(\mathbf{x}) \vee (\mathbf{a}^-)^{(n)}) \wedge (\kappa(\mathbf{x}) \vee \mathbf{b}^{(n)}).$$

By the choice of the degree $\mathbf{x}$, we have that $\kappa(\mathbf{x}) \vee (\mathbf{a}^-)^{(n)} = \mathbf{0}_\omega^{(n+1)}$. Therefore $\mathbf{b}^{(n)} \leq_\omega \kappa(\mathbf{x})$. But $\kappa(\mathbf{x})' = \mathbf{0}_\omega^{(n+1)}$, so we may conclude that $\mathbf{b}^{(n+1)} = \mathbf{0}_\omega^{(n+1)}$. From here, noting that $\mathbf{b} \neq \mathbf{0}_\omega$ and recalling that for each nonzero *a.z.* degree $\mathbf{p}$ and each $n < \omega$, $\mathbf{p}^{(n)} \not\leq_\omega \mathbf{0}_\omega^{(n)}$, we conclude that $\mathbf{b}$ is also not *a.z.* degree.

Therefore, there is $m < \omega$ such that $P_m(\mathcal{B}) \not\equiv_e \emptyset^{(m)}$. Let $\mathbf{b}^-$ be the degree containing the sequence $(\underbrace{\emptyset, \ldots, \emptyset}_{m}, P_m(\mathcal{B}), \emptyset, \ldots)$. Then $\mathbf{a}^-, \mathbf{b}^- \leq_\omega \mathbf{0}'_\omega$ and $\{\mathbf{a}^-, \mathbf{b}^-\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Note that $\{\mathbf{a}^-, \mathbf{b}^-\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ also for $\mathcal{G}_\omega$, whose elements are not *a.z.*. From the characterisation of the $\mathcal{K}$-pairs for $\mathcal{G}_\omega$ noted in the previous section, we conclude that $m = n$. Because of the choice of n and m, we have that for all $k \neq n$, $P_k(\mathcal{A}) \equiv_e P_k(\mathcal{B}) \equiv_e \emptyset^{(k)}$. Therefore $\mathbf{a} = \mathbf{a}^- \vee \mathbf{p}$ and $\mathbf{b} = \mathbf{b}^- \vee \mathbf{q}$ where $\mathbf{p}$ and $\mathbf{q}$ are both *a.z.*. But $\mathbf{p} \leq_\omega \mathbf{a}$, so if $\mathbf{p} \neq \mathbf{0}_\omega$ then $\{\mathbf{p}, \mathbf{b}\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Now since $\mathbf{b}$ is not *a.z.* we conclude that $\mathbf{p}$ is not *a.z.*. A contradiction. So $\mathbf{p}$ must be equal to $\mathbf{0}_\omega$. Analogously, $\mathbf{q} = \mathbf{0}_\omega$ and hence $\mathbf{a} = \mathbf{a}^-, \mathbf{b} = \mathbf{b}^-$. So we have the following characterisation of the $\mathcal{K}$-pairs $\{\mathbf{a}, \mathbf{b}\}$ strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$.

Theorem 7. *Let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Then exactly one of the following assertions holds:*

1. *Both $\mathbf{a}$ and $\mathbf{b}$ are a.z..*
2. *There is a natural number $n < \omega$ and sets $A, B \subseteq \omega$ such that*
   - $\emptyset^{(n)} <_e A, B \leq_e \emptyset^{(n+1)}$ *and* $A' = B' = \emptyset^{(n+1)}$*;*

- $\{\mathbf{d}_e(A), \mathbf{d}_e(B)\}$ *is* $\mathcal{K}$*-pair strictly over* $\mathbf{0}_e^{(n)}$ *for* $\mathbf{D}_e[\geq \mathbf{0}_e^{(n)}]$*;*
- $(\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, A, \emptyset, \ldots, \emptyset, \ldots) \in \mathbf{a}$ *and* $(\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, B, \emptyset, \ldots, \emptyset, \ldots) \in \mathbf{b}$.

Note that each $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$, whose elements are both not *a.z.*, is an inherited $\mathcal{K}$-pair for $\mathcal{G}_\omega$ and hence its elements are bellow $\mathbf{0}'_\omega$. So, by the observations in the previous section, each of its elements bounds a nonzero nonsplitting degree. Now, recalling Lemma 4, we have that the $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$ consists of non *a.z.* elements iff,

$$\mathcal{D}_\omega \models \mathcal{K}_{inh}(\mathbf{a}, \mathbf{b}),$$

where $\mathcal{K}_{inh}$ is the corresponding formula from the previous section. Since the elements of each inherited $\mathcal{K}$-pair are both below $\mathbf{0}'_\omega$ then their least upper bounds are also below $\mathbf{0}'_\omega$.

Now note that, by Kalimullin [5], $\mathbf{0}'_e$ can be split by a $\mathcal{K}$-pair $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}$ strictly over $\mathbf{0}_e$ for $\mathcal{D}_e$ such that $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ are low. Then $\kappa(\widetilde{\mathbf{a}})$ and $\kappa(\widetilde{\mathbf{b}})$ are not *a.z.* degrees and $\{\kappa(\widetilde{\mathbf{a}}), \kappa(\widetilde{\mathbf{b}})\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$ with $\kappa(\widetilde{\mathbf{a}}) \vee \kappa(\widetilde{\mathbf{b}}) = \mathbf{0}'_\omega$. Thus we may define $\mathbf{0}'_\omega$ as the greatest degree, which is a least upper bound of the elements of a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$, whose elements are both not *a.z.*. Thus Theorem 2 is proved.

ACKNOWLEDGEMENTS. The authors were partially supported by BNSF grant No. DMU 03-07/12.12.2011 and Sofia University Science Fund project 97, 2014.

8. REFERENCES

1. Cooper, S. B.: Enumeration reducibility, nondeterministic computations and relative computability of partial functions. In: *Recursion theory week, Oberwolfach 1989*, LNCS **1432** (K. Ambos-Spies, G. Muler, G. E. Sacks, eds.), Springer-Verlag, Heildelberg, 1990, 57–110.
2. Ganchev, H.: Exact pair theorem for the ω-enumeration degrees. In: *Computation and Logic in the Real World*, LNCS **4497** (B. Loewe, S. B. Cooper and A. Sorbi, eds.), 2007, 316–324.
3. Ganchev, H., Soskova, M. I.: The High/Low Hierarchy in the Local Structure of the ω-Enumeration Degrees. *Ann. Pure Appl. Logic*, **163**, no. 5, 2012, 547–566.
4. Ganchev, H., Soskova, M. I.: Definability via Kalimullin Pairs in the structure of the enumeration degrees. To appear in *Trans. Amer. Math. Soc.*.
5. Kalimullin, I. Sh.: Definability of the jump operator in the enumeration degrees. *J. Math. Logic*, **3**, 2003, 257–267.

6. Kent, T., Sorbi, A.: Bounding nonsplitting enumeration degrees. *J. Symb. Logic*, **72**, no. 4, 2007, 1405–1417.
7. Nies, A., Shore, R. A. , Slaman, T. A.: Interpretability and definability in the recursively enumerable degrees. *Proc. London Math. Soc.*, **77**, 1998, 241–291.
8. Rogers, H. Jr.: *Theory of recursive functions and effective computability.* McGraw-Hill Book Company, New York, 1967.
9. Shore, R. A., Slaman, T. A.: Defining the Turing jump, *Math. Res. Lett.*, **6**, no. 5–6, 1999, 711–722, 1999.
10. Slaman, T. A., Woodin, W. H.: Definability in degree structures. Preprint available at http://math.berkley.edu/slaman/talks/sw.pdf
11. Slaman, T. A., Woodin, W. H.: Definability in the enumeration degrees, *Arch. Math. Logic*, **36**, 1997, 255–267.
12. Soare, R. I.: *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets*. Springer-Verlag, Heidelberg, 1987.
13. Sorbi, A.: The enumeration degrees of the Σ_2^0 sets. In: *Complexity, Logic and Recursion Theory* (A. Sorbi, ed.), Marcel Dekker, New York, 1975, 303–330.
14. Soskov, I. N.: The ω-enumeration degrees. *J. Logic and Computation*, *17*, 2007, 1193–1214.
15. Soskov, I. N., Ganchev, H.: The jump operator on the ω-enumeration degrees. *Ann. Pure and Appl. Logic*, **160**, no. 30, 2009, 289–301.
16. Soskov, I. N., Kovachev, B.: Uniform regular enumerations. *Math. Struct. Commp. Sci.*, **16**, no. 5, 2006, 901–924.
17. Soskov, I. N., Soskova, M. I.: Kalimullin pairs of Σ_2^0 ω-enumeration degrees. *Int. J. Software Informatics* **5**, no. 4, 2011, 637–658.

Received on November 7, 2014
In revised form on January 25, 2015

Hristo Ganchev, Andrey Sariev

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5 J. Bourchier Blvd., BG-1164 Sofia
BULGARIA

e-mails: ganchev@fmi.uni-sofia.bg
andreys@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

WEIGHTED BANACH SPACES OF HOLOMORPHIC FUNCTIONS WITH LOG-CONCAVE WEIGHTS

MARTIN AT. STANEV

Some theorems on convex functions are proved and an application of these theorems in the theory of weighted Banach spaces of holomorphic functions is investigated, too. We prove that $H_v(G)$ and $H_{v_0}(G)$ are exactly the same spaces as $H_w(G)$ and $H_{w_0}(G)$ where w is the smallest log-concave majorant of v. This investigation is based on the theory of convex functions and some specific properties of the weighted banach spaces of holomorphic functions under consideration.

Keywords: Associated weights, holomorphic function, weighted banach space, convex function

2010 Math. Subject Classification: 46E15, 46B04

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane and

$$G = \{z = x + iy \mid x \in (-\infty, \infty),\ y \in (0, \infty)\} \subset \mathbb{C}$$

be the upper half plane of $\mathbb{C}$. Throughout, $v : G \to (0, \infty)$ will be a function such that $v(z) = v(x + iy) = v(iy)$ for every $z = x + iy \in G$, and

$$\inf_{y \in [\frac{1}{c}, c]} v(i\,y) > 0 \quad \text{for every} \quad c > 1. \tag{1.1}$$

We define

$$\varphi_v(y) = -\ln v(iy), \quad y \in (0, \infty)\,,$$

and property (1.1) is reformulated as the following property of $\varphi_v(y)$:

$$\sup_{y\in[\frac{1}{c},c]} \varphi_v(y) < \infty \quad \text{for every } c > 1. \tag{1.1$'$}$$

The weighted Banach spaces of holomorphic functions $H_v(G)$ and $H_{v_0}(G)$ are defined as follows

- $f \in H_v(G)$ if f is holomorphic on G and

$$\| f \|_v = \sup_{z\in G} v(z)|f(z)| < \infty\,;$$

- $f \in H_{v_0}(G)$ if $f \in H_v(G)$ and f is such that for every $\varepsilon > 0$ there exists a compact $\mathcal{K}_\varepsilon \subset G$ for which

$$\sup_{z\in G\setminus\mathcal{K}_\varepsilon} v(z)|f(z)| < \varepsilon.$$

Here, we use notations from [1, 2, 3, 4, 5].

In [1], [2] the authors find an isomorphic classification of the spaces $H_v(G)$ and $H_{v_0}(G)$ provided the weight function v satisfies some growth conditions.

In [3], [4] weighted composition operators between weighted spaces of holomorphic functions on the unit disk in the complex plane are studied and the associated weights are used in order to estimate the norm of the weighted composition operators.

The associated weights are studied in [5].

This paper is about weights that have some of the properties of the associated weights. We prove that $H_v(G)$ and $H_{v_0}(G)$ are exactly the same spaces as $H_w(G)$ and $H_{w_0}(G)$, where w is the smallest log-concave majorant of v. Here, the smallest log-concave majorant of v is exactly the associated weight, but in case of other weighted spaces this coincidence might not take place. Our work is based on the theory of convex functions and some specific properties of the weighted banach spaces of holomorphic functions under consideration.

The results of this paper are communicated at the conferences [7] and [8].

2. DEFINITIONS AND NOTATIONS

Let Φ be the set of functions φ satisfying the following conditions:

- $\varphi : (0,\infty) \to \mathbb{R}$ and
- there exists $a \in \mathbb{R}$ such that

$$\inf_{x\in(0,\infty)} \big(\varphi(x) - ax\big) > -\infty.$$

Note that if $\varphi \in \Phi$, then $-\infty < \varphi(x) < \infty$ for every $x \in (0, \infty)$.

We denote by $\widehat{a}_\varphi$ the limit inferior

$$\widehat{a}_\varphi = \liminf_{x\to\infty} \frac{\varphi(x)}{x}, \quad \varphi \in \Phi.$$

If $\varphi \in \Phi$, then

- $\widehat{a}_\varphi \in \mathbb{R} \cup \{\infty\}$, $\widehat{a}_\varphi > -\infty$;
- $\widehat{a}_\varphi = \sup\big\{a \mid a \in \mathbb{R}, \inf\limits_{x\in(0,\infty)} (\varphi(x) - ax) > -\infty\big\}$

If $\varphi \in \Phi$ is convex in $(0, \infty)$, then

$$\widehat{a}_\varphi = \lim_{x\to\infty} \frac{\varphi(x)}{x}$$

By Φ_1, Φ_2, Φ_3 we denote the following subsets of Φ:

$$\Phi_1 = \big\{\varphi \ : \ \varphi \in \Phi, \ \widehat{a}_\varphi = \infty\big\};$$
$$\Phi_2 = \big\{\varphi \ : \ \varphi \in \Phi, \ \widehat{a}_\varphi < \infty, \ \liminf_{x\to\infty}(\varphi(x) - \widehat{a}_\varphi x) = -\infty\big\};$$
$$\Phi_3 = \big\{\varphi \ : \ \varphi \in \Phi, \ \widehat{a}_\varphi < \infty, \ \liminf_{x\to\infty}(\varphi(x) - \widehat{a}_\varphi x) > -\infty\big\}.$$

Note that Φ_1, Φ_2, Φ_3 are mutually disjoint sets and $\Phi_1 \cup \Phi_2 \cup \Phi_3 = \Phi$.

If $\varphi \in \Phi_2 \cup \Phi_3$ is convex on $(0, \infty)$, then

$$\liminf_{x\to\infty}(\varphi(x) - \widehat{a}_\varphi x) = \lim_{x\to\infty}(\varphi(x) - \widehat{a}_\varphi x).$$

Note that a function $\varphi \in \Phi$ is not necessarily continuous. In fact, $\varphi \in \Phi$ is not supposed to satisfy any conditions beside those of the definition of Φ, Φ_1, Φ_2, Φ_3. There are a number of simple functions that belong to Φ, Φ_1, Φ_2, Φ_3, for instance,

- $\varphi_1(x) = x^2$ belongs to Φ_1;
- $\varphi_2(x) = x - \sqrt{x}$ belongs to Φ_2;
- $\varphi_3(x) = x^{-1}$ belongs to Φ_3,

and $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ are all convex on $(0, \infty)$.

For a $\varphi \in \Phi$ let

$$M_\varphi = \big\{(a, b) \big| \ a, \ b \in \mathbb{R}, \ \inf_{t\in(0,\infty)} (\varphi(t) - at) > b\big\}.$$

The function $\varphi^{**} : (0, \infty) \to \mathbb{R}$ is defined as

$$\varphi^{**}(x) = \sup_{(a,b)\in M_\varphi} (ax + b).$$

φ^{**} is referred to as the second Young-Fenhel conjugate of φ and it is the largest convex minorant of φ.

3. MAIN RESULTS

Here we state our main results.

Theorem 3.1. *Let* $\varphi,\ \psi \in \Phi$. *If* ψ *is convex on* $(0,\infty)$, *then*

$$\inf_{x\in(0,\infty)} \big(\varphi(x)-\psi(x)\big) = \inf_{x\in(0,\infty)} \big(\varphi^{**}(x)-\psi(x)\big)\,.$$

Theorem 3.2. *Let* $\varphi,\ \psi \in \Phi$. *If* ψ *is convex on* $(0,\infty)$ *and, in addition,* $\lim_{x\to 0^+}\big(\varphi(x)-\psi(x)\big)=\infty$, *then*

$$\lim_{x\to 0^+}\big(\varphi^{**}(x)-\psi(x)\big)=\infty\,.$$

Theorem 3.3. *Let* $\varphi\in\Phi$ *and* $\psi\in\Phi\setminus\Phi_3$. *If* ψ *is convex on* $(0,\infty)$ *and, in addition,* $\lim_{x\to\infty}\big(\varphi(x)-\psi(x)\big)=\infty$, *then*

$$\lim_{x\to\infty}\big(\varphi^{**}(x)-\psi(x)\big)=\infty\,.$$

The next examples show that the assumption for convexity of ψ in Theorems 3.1–3.3 cannot be omitted.

Example 3.1. *Theorem 3.1 does not hold with the functions*

$$\varphi(x)=\min\{x,1\}+1,\quad \psi(x)=\frac{x}{x+1},\quad x\in(0,\infty)\,.$$

Note that $\varphi\in\Phi$, *and* $\psi\in\Phi$ *is not convex on* $(0,\infty)$. *We have* $\varphi^{**}(x)=1$, $x\in(0,\infty)$, *and*

$$1=\inf_{x\in(0,\infty)} \big(\varphi(x)-\psi(x)\big) \neq \inf_{x\in(0,\infty)} \big(\varphi^{**}(x)-\psi(x)\big)=0\,.$$

Example 3.2. *Theorem 3.2 does not hold with*

$$\varphi(x)=\frac{1}{x^2}+\frac{1}{x}\sin\frac{1}{x}+\frac{2}{x},\quad \psi(x)=\varphi(x)-\frac{2}{x},\quad x\in(0,\infty)\,.$$

Note that $\varphi,\ \psi\in\Phi$, *the function* ψ *is not convex on* $(0,\infty)$ *and*

$$\infty=\lim_{x\to 0^+}\big(\varphi(x)-\psi(x)\big) > \liminf_{x\to 0^+}\big(\varphi^{**}(x)-\psi(x)\big)$$

This fact is proved in Proposition 4.1.

Example 3.3. *Theorem 3.3 does not hold with*

$$\varphi(x)=x^2+x\sin x+2x,\quad \psi(x)=\varphi(x)-2x,\quad x\in(0,\infty).$$

Note that $\varphi\in\Phi$, $\psi\in\Phi\setminus\Phi_3$, ψ *is not convex on* $(0,\infty)$ *and*

$$\infty=\lim_{x\to\infty}\big(\varphi(x)-\psi(x)\big) > \liminf_{x\to\infty}\big(\varphi^{**}(x)-\psi(x)\big)$$

This fact is proved in Proposition 4.2.

Corollary 3.1. *If φ, $\psi \in \Phi$ are such that*

$$\lim_{x\to 0^+}\big(\varphi(x)-\psi(x)\big)=\infty\,,$$

then

$$\lim_{x\to 0^+}\big(\varphi^{**}(x)-\psi^{**}(x)\big)=\infty\,. \tag{3.1}$$

Proof. Since $\psi^{**}\le\psi$, we have

$$\lim_{x\to 0^+}\big(\varphi(x)-\psi^{**}(x)\big)\ge \lim_{x\to 0^+}\big(\varphi(x)-\psi(x)\big)=\infty.$$

Now Theorem 3.2 applied to φ, ψ^{**} proves (3.1). □

Corollary 3.2. *Let $\varphi\in\Phi$ and $\psi\in\Phi\setminus\Phi_3$. If φ and ψ satisfy*

$$\lim_{x\to\infty}\big(\varphi(x)-\psi(x)\big)=\infty\,,$$

then

$$\lim_{x\to\infty}\big(\varphi^{**}(x)-\psi^{**}(x)\big)=\infty\,. \tag{3.2}$$

Proof. Note that

- $\psi^{**}\in\Phi\setminus\Phi_3$ by the Lemma 4.1;
- $\psi^{**}\le\psi$.

Theorem 3.3 applied to φ, ψ^{**} implies (3.2). □

Example 3.4. *Let $\varphi(x)=x^2+x$ and*

$$\psi(x)=\begin{cases}3x-1, & x\in(0,1]\\ 5-3x, & x\in(1,2]\\ x^2+x-7, & x\in(2,\infty)\,.\end{cases}$$

We observe that φ, $\psi\in\Phi$ and

- *φ is convex on $(0,\infty)$, and therefore $\varphi^{**}=\varphi$,*
- *ψ is not convex on $(0,\infty)$ and*

$$\psi^{**}(x)=\begin{cases}-1, & x\in(0,2]\\ x^2+x-7, & x\in(2,\infty)\,.\end{cases}$$

A direct calculation shows that

$$\inf_{x\in(0,\infty)}\big(\varphi(x)-\psi(x)\big)=0\neq 1=\inf_{x\in(0,\infty)}\big(\varphi^{**}(x)-\psi^{**}(x)\big).$$

Thus, there is no analog of Theorem 3.1 involving φ^{**} and ψ as in Corollaries 3.1 and 3.2. $\square$

4. AUXILIARY RESULTS

Proposition 4.1. *Let*

$$\varphi(x) = \frac{1}{x^2} + \frac{1}{x}\sin\frac{1}{x} + \frac{2}{x} \quad \textit{and} \quad \psi(x) = \varphi(x) - \frac{2}{x}, \quad x \in (0,\infty).$$

Then φ, $\psi \in \Phi$, *the function* ψ *is not convex on* $(0,\infty)$ *and*

$$\infty = \lim_{x\to 0^+}\big(\varphi(x) - \psi(x)\big) > \liminf_{x\to 0^+}\big(\varphi^{**}(x) - \psi(x)\big)\,.$$

Proof. The function ψ satisfies

$$\psi(x) \geq \begin{cases} \frac{1}{x^2} - \frac{1}{x}, & x \in (0,1)\,, \\ \frac{1}{x^2}\,, & x \in [1,\infty)\,, \end{cases}$$

hence, $\psi(x) \geq 0$ for every $x \in (0,\infty)$, and this implies that $\psi \in \Phi$.

Since $\varphi \geq \psi$, we have also $\varphi \in \Phi$.

Note that

$$\lim_{x\to 0^+}\big(\varphi(x) - \psi(x)\big) = \lim_{x\to 0^+}\frac{2}{x} = \infty.$$

Let

$$x_k = \frac{1}{\frac{3\pi}{2} + 2k\pi}, \quad \widetilde{x}_k = \frac{1}{\frac{5\pi}{2} + 2k\pi}, \quad k = 0,1,2,\dots\,.$$

We observe that $x_k > \widetilde{x}_k > x_{k+1} > 0$, $\lim\limits_{k\to\infty} x_k = 0$, and the harmonic mean of x_k, x_{k+1} is equal to $\widetilde{x}_k$. A direct computation shows that $\psi''(\widetilde{x}_0) < 0$, therefore ψ is not convex on $(0,\infty)$.

Let

$$f(x) = \frac{1}{x^2} + \frac{1}{x}, \quad x \in (0,\infty).$$

The function f is convex on $(0,\infty)$ and $f(x) \leq \varphi(x)$, $x \in (0,\infty)$. So, f is a convex minorant of φ and thus $f \leq \varphi^{**}$.

Therefore, $f(x_k) \leq \varphi^{**}(x_k) \leq \varphi(x_k) = f(x_k)$ and this implies that

$$f(x_k) = \varphi^{**}(x_k), \quad k = 1,2,3,\dots\,.$$

Furthermore,

$$\psi(\widetilde{x}_k) = f(\widetilde{x}_k) \leq \varphi^{**}(\widetilde{x}_k) \leq \frac{x_k - \widetilde{x}_k}{x_k - x_{k+1}}\,\varphi^{**}(x_{k+1}) + \frac{\widetilde{x}_k - x_{k+1}}{x_k - x_{k+1}}\,\varphi^{**}(x_k)\,,$$

because of the convexity of φ^{**}. Thus,

$$0 \le \varphi^{**}(\widetilde{x}_k) - \psi(\widetilde{x}_k) \le \frac{x_k - \widetilde{x}_k}{x_k - x_{k+1}} f(x_{k+1}) + \frac{\widetilde{x}_k - x_{k+1}}{x_k - x_{k+1}} f(x_k) - f(\widetilde{x}_k)\,.$$

After some simple calculations we obtain

$$\frac{x_k - \widetilde{x}_k}{x_k - x_{k+1}} f(x_{k+1}) + \frac{\widetilde{x}_k - x_{k+1}}{x_k - x_{k+1}} f(x_k) - f(\widetilde{x}_k) = (3 + \widetilde{x}_k)\pi^2\,.$$

Consequently, $0 \le \varphi^{**}(\widetilde{x}_k) - \psi(\widetilde{x}_k) \le (3 + \widetilde{x}_k)\pi^2$, $k = 1, 2, 3, \ldots$, and

$$\liminf_{x\to 0^+} \big(\varphi^{**}(x) - \psi(x)\big) < \infty. \qquad \square$$

Proposition 4.2. *Let*

$$\varphi(x) = x^2 + x\sin x + 2x \quad \textit{and} \quad \psi(x) = \varphi(x) - 2x, \qquad x \in (0, \infty)\,.$$

Then $\varphi \in \Phi$, $\psi \in \Phi \setminus \Phi_3$, *the function* ψ *is not convex on* $(0, \infty)$ *and*

$$\infty = \lim_{x\to\infty} \big(\varphi(x) - \psi(x)\big) > \liminf_{x\to\infty} \big(\varphi^{**}(x) - \psi(x)\big)\,.$$

Proof. The function ψ satisfies the inequalities

$$\psi(x) \ge \begin{cases} x^2, & x \in (0, \pi)\,, \\ x^2 - x, & x \in [\pi\,, \infty)\,, \end{cases}$$

therefore $\psi(x) \ge 0$, $x \in (0, \infty)$, and thus $\psi \in \Phi$. Moreover,

$$\widehat{a}_\psi = \liminf_{x\to\infty} \frac{\psi(x)}{x} \ge \liminf_{x\to\infty} \frac{x^2 - x}{x} = \infty\,,$$

therefore $\widehat{a}_\psi = \infty$ and thus $\psi \in \Phi_1 \subset \Phi \setminus \Phi_3$.

Now $\varphi \in \Phi$ since $\varphi \ge \psi$ and $\psi \in \Phi$. Moreover,

$$\lim_{x\to\infty} \big(\varphi(x) - \psi(x)\big) = \lim_{x\to\infty} 2x = \infty\,.$$

Let

$$x_k = \frac{3\pi}{2} + 2k\pi, \quad \widetilde{x}_k = \frac{5\pi}{2} + 2k\pi\,, \quad k = 0, 1, 2, \ldots$$

Note that, for $k \in \mathbb{N}$, $0 < x_k < \widetilde{x}_k < x_{k+1}$, $x_k + x_{k+1} = 2\widetilde{x}_k$ and $\lim\limits_{k\to\infty} x_k = \infty$.

A direct computation shows that $\psi''(\widetilde{x}_0) < 0$, hence ψ is not convex on $(0, \infty)$.

Let

$$f(x) = x^2 + x, \quad x \in (0, \infty).$$

The function f is convex on $(0,\infty)$ and $f \le \varphi$ therein. So, f is a convex minorant of φ and thus $f \le \varphi^{**}$. Therefore, $f(x_k) \le \varphi^{**}(x_k) \le \varphi(x_k) = f(x_k)$, and this implies

$$f(x_k) = \varphi^{**}(x_k), \quad k \in \mathbb{N}\,.$$

Furthermore, by the convexity of φ^{**}, we have

$$\psi(\widetilde{x}_k) = f(\widetilde{x}_k) \le \varphi^{**}(\widetilde{x}_k) \le \frac{x_{k+1}-\widetilde{x}_k}{x_{k+1}-x_k}\,\varphi^{**}(x_k) + \frac{\widetilde{x}_k - x_k}{x_{k+1}-x_k}\,\varphi^{**}(x_{k+1})\,.$$

Thus,

$$0 \le \varphi^{**}(\widetilde{x}_k) - \psi(\widetilde{x}_k) \le \frac{x_{k+1}-\widetilde{x}_k}{x_{k+1}-x_k}\,f(x_k) + \frac{\widetilde{x}_k - x_k}{x_{k+1}-x_k}\,f(x_{k+1}) - f(\widetilde{x}_k)\,.$$

After some simple calculations we obtain

$$\frac{x_{k+1}-\widetilde{x}_k}{x_{k+1}-x_k}\,f(x_k) + \frac{\widetilde{x}_k - x_k}{x_{k+1}-x_k}\,f(x_{k+1}) - f(\widetilde{x}_k) = \pi^2, \quad k \in \mathbb{N}\,.$$

Consequently, $0 \le \varphi^{**}(\widetilde{x}_k) - \psi(\widetilde{x}_k) \le \pi^2$, $k \in \mathbb{N}$, and

$$\liminf_{x\to\infty}\big(\varphi^{**}(x) - \psi(x)\big) < \infty. \qquad \square$$

Lemma 4.1. *If* $\varphi \in \Phi$, *then*

(1) $\liminf\limits_{x\to 0^+} \varphi(x) = \lim\limits_{x\to 0^+} \varphi^{**}(x)$ *;*

(2) $\liminf\limits_{x\to \infty} \dfrac{\varphi(x)}{x} = \lim\limits_{x\to \infty} \dfrac{\varphi^{**}(x)}{x}$.

Proof. Let $\varphi \in \Phi$. Then

$$\liminf_{x\to 0^+} \varphi(x) \ge \liminf_{x\to 0^+} \varphi^{**}(x) = \lim_{x\to 0^+} \varphi^{**}(x)\,,$$
$$\liminf_{x\to\infty} \frac{\varphi(x)}{x} \ge \liminf_{x\to\infty} \frac{\varphi^{**}(x)}{x} = \lim_{x\to\infty} \frac{\varphi^{**}(x)}{x}\,.$$

Let $a_0,\ b_0 \in \mathbb{R}$ be such that $a_0 x + b_0 \le \varphi(x)$, $x \in (0,\infty)$. Then

$$\liminf_{x\to 0^+} \varphi(x) \ge b_0 > -\infty\,, \tag{4.1}$$
$$\liminf_{x\to\infty} \frac{\varphi(x)}{x} \ge a_0 > -\infty\,. \tag{4.2}$$

Let b be such that $\liminf\limits_{x\to 0^+} \varphi(x) > b > -\infty$. We choose $\delta > 0$ so that

$$\inf_{0<x<\delta} \varphi(x) > b.$$

Then

$$\inf_{x>0} \frac{\varphi(x)-b}{x} \geq \min\Big\{\inf_{0<x<\delta} \frac{\varphi(x)-b}{x}\,,\ \inf_{\delta\leq x} \frac{\varphi(x)-b}{x}\Big\}$$
$$\geq \min\Big\{0,\ \inf_{\delta\leq x}\big(a_0 + \frac{b_0-b}{x}\big)\Big\} > -\infty\,.$$

Set $a = \min\big\{0,\ \inf_{\delta\leq x}\big(a_0+\frac{b_0-b}{x}\big)\big\}$, then $(a,b)\in M_\varphi$ and consequently $\varphi^{**}(x)\geq ax+b$, $x\in(0,\infty)$. Thus,

$$\lim_{x\to 0^+}\varphi^{**}(x)\geq b$$

and, by our choice of b,

$$\lim_{x\to 0^+}\varphi^{**}(x)\geq \liminf_{x\to 0^+}\varphi(x)\,.$$

Hence, assertion (1) of Lemma 4.1 is proved.

Let α be such that $\liminf_{x\to\infty}\frac{\varphi(x)}{x} > \alpha > -\infty$. We choose $\Delta>0$ such that

$$\inf_{x>\Delta}\frac{\varphi(x)}{x} > \alpha.$$

Then

$$\inf_{x>0}\big(\varphi(x)-\alpha x\big) \geq \min\big\{\inf_{0<x\leq\Delta}(\varphi(x)-\alpha x),\ \inf_{x>\Delta}(\varphi(x)-\alpha x)\big\}$$
$$\geq \min\big\{\inf_{0<x\leq\Delta}(a_0x+b_0-\alpha x),\ 0\big\} > -\infty$$

Let $\beta = \min\big\{\inf_{0<x\leq\Delta}(a_0x+b_0-\alpha x),\ 0\big\}$. Then $(\alpha,\beta)\in M_\varphi$ and consequently, $\varphi^{**}(x)\geq\alpha x+\beta$ for every $x\in(0,\infty)$. Therefore,

$$\lim_{x\to\infty}\frac{\varphi^{**}(x)}{x}\geq\alpha$$

and, by our choice of α,

$$\lim_{x\to\infty}\frac{\varphi^{**}(x)}{x}\geq\liminf_{x\to\infty}\frac{\varphi(x)}{x}$$

Thus, assertion (2) of Lemma 4.1 is proved. □

Lemma 4.2. $\varphi\in\Phi_i \iff \varphi^{**}\in\Phi_i,\ \ i=1,\,2,\,3.$

Proof. The assertion $\varphi\in\Phi_1 \iff \varphi^{**}\in\Phi_1$ is proved as (1) of Lemma 4.1. The proof of Lemma 4.2 will be completed once we prove that

$$\varphi\in\Phi_3 \iff \varphi^{**}\in\Phi_3.$$

Let $\varphi \in \Phi_2 \cup \Phi_3$ and

$$\widehat{a}_\varphi = \liminf_{x\to\infty} \frac{\varphi(x)}{x} = \lim_{x\to\infty} \frac{\varphi^{**}(x)}{x}.$$

If $\varphi^{**} \in \Phi_3$, then $\varphi \geq \varphi^{**}$ implies $\varphi \in \Phi_3$.

Now let us suppose that $\varphi \in \Phi_3$. Let $a_0, b_0 \in \mathbb{R}$ be such that $a_0 x + b_0 \leq \varphi(x)$ for every $x \in (0,\infty)$. Let $b \in \mathbb{R}$ be such that $\liminf_{x\to\infty} \big(\varphi(x) - \widehat{a}_\varphi x\big) > b > -\infty$.

Let $\Delta > 0$ satisfy

$$\inf_{x>\Delta} \big(\varphi(x) - \widehat{a}_\varphi x\big) > b.$$

Then,

$$\begin{aligned}\inf_{x>0} \big(\varphi(x) - \widehat{a}_\varphi x\big) &\geq \min\big\{\inf_{0<x\leq\Delta} (\varphi(x) - \widehat{a}_\varphi x),\ \inf_{x>\Delta} (\varphi(x) - \widehat{a}_\varphi x)\big\} \\ &\geq \min\big\{\inf_{0<x\leq\Delta} (a_0 x + b_0 - \widehat{a}_\varphi x),\ b\big\} > -\infty\,.\end{aligned}$$

Let $\widehat{b} = \min\big\{\inf_{0<x\leq\Delta} (a_0 x + b_0 - \widehat{a}_\varphi x),\ b\big\}$. Then $(\widehat{a}_\varphi, \widehat{b}) \in M_\varphi$ and consequently $\varphi^{**}(x) \geq \widehat{a}_\varphi x + \widehat{b}$ for every $x \in (0,\infty)$. Thus,

$$\liminf_{x\to\infty}\big(\varphi^{**}(x) - \widehat{a}_\varphi x\big) \geq \widehat{b} > -\infty\,,$$

and $\varphi^{**} \in \Phi_3$. □

Lemma 4.3. *Let $\varphi \in \Phi$. If a is such that $a < \widehat{a}_\varphi$, then*

$$\inf_{x>0} \big(\varphi(x) - ax\big) > -\infty$$

and

$$\lim_{x\to\infty} \big(\varphi(x) - ax\big) = \infty\,.$$

Proof. Let $a_0, b_0 \in \mathbb{R}$ be such that $(a_0, b_0) \in M_\varphi$, let a and a_1 satisfy the inequalities $-\infty < a < a_1 < \widehat{a}_\varphi$, and $\Delta > 0$ be such that

$$\inf_{x>\Delta} \frac{\varphi(x)}{x} > a_1.$$

So, $\varphi(x) - ax > (a_1 - a)x$ for $x > \Delta$ and $\lim_{x\to\infty} \big(\varphi(x) - ax\big) = \infty$. Therefore,

$$\begin{aligned}\inf_{x>0} \big(\varphi(x) - ax\big) &= \min\Big\{\inf_{0<x\leq\Delta} \big(\varphi(x) - ax\big);\ \inf_{x>\Delta} \big(\varphi(x) - ax\big)\Big\} \\ &\geq \min\Big\{\inf_{0<x\leq\Delta} \big(a_0 x + b_0 - ax\big);\ \inf_{x>\Delta} (a_1 - a)x\Big\} > -\infty.\end{aligned}$$

Lemma 4.3 is proved. □

Lemma 4.4. *Let $\psi : (0,\infty) \to \mathbb{R}$ be convex on $(0,\infty)$ and*

$$\widehat{\psi}(x) = \psi(x) - \psi'(x^-)x, \quad x \in (0,\infty)\,,$$

where $\psi'(x^-) = \lim\limits_{t\to x^-} \frac{\psi(t)-\psi(x)}{t-x}$. If $0 < x_1 < x_2$, then

$$\widehat{\psi}(x_1) \geq \widehat{\psi}(x_2).$$

Proof. Let $x_3 = \frac{x_1+x_2}{2}$. Note that

- $2\psi(x_3) \leq \psi(x_1) + \psi(x_2)$,
- $f(u,v) = \frac{\psi(u)-\psi(v)}{u-v}$ is a monotone non-decreasing function of each variable u, $v > 0$, $u \neq v$, and

$$-\infty < \lim_{t\to x^-} \frac{\psi(t)-\psi(x)}{t-x} = \psi'(x^-) \leq \psi'(x^+) = \lim_{v\to x^+} \frac{\psi(v)-\psi(x)}{v-x} < \infty\,, \quad x > 0\,.$$

Now, $\widehat{\psi}(x_2) \leq \widehat{\psi}(x_1)$ follows from the inequalities

$$\begin{aligned}
\widehat{\psi}(x_2) &= \psi(x_2) - \psi'(x_2^-)\,x_2 \leq \psi(x_2) - \frac{\psi(x_2)-\psi(x_3)}{x_2-x_3}\,x_2 \\
&= \big(\psi(x_3)-\psi(x_2)\big)\,\frac{2x_2}{x_2-x_1} - \psi(x_2) = \psi(x_3)\,\frac{2x_2}{x_2-x_1} - \psi(x_2)\,\frac{x_2+x_1}{x_2-x_1} \\
&\leq \big(\psi(x_2)+\psi(x_1)\big)\,\frac{x_2}{x_2-x_1} - \psi(x_2)\,\frac{x_2+x_1}{x_2-x_1} \\
&= \psi(x_1)\,\frac{x_2+x_1}{x_2-x_1} - \big(\psi(x_1)+\psi(x_2)\big)\,\frac{x_1}{x_2-x_1} \\
&\leq \psi(x_1)\,\frac{x_2+x_1}{x_2-x_1} - \psi(x_3)\,\frac{2x_1}{x_2-x_1} = \psi(x_1) - \big(\psi(x_3)-\psi(x_1)\big)\,\frac{2x_1}{x_2-x_1} \\
&= \psi(x_1) - \frac{\psi(x_3)-\psi(x_1)}{x_3-x_1}\,x_1 \leq \psi(x_1) - \psi'(x_1^+)\,x_1 \leq \psi(x_1) - \psi'(x_1^-)\,x_1 \\
&= \widehat{\psi}(x_1).
\end{aligned}$$

□

Lemma 4.5. *Let $\psi \in \Phi_2 \cup \Phi_3$. If ψ is convex on $(0,\infty)$ and*

$$\lim_{x\to\infty} \big(\psi(x) - \psi'(x^-)x\big) > -\infty\,,$$

then $\psi \in \Phi_3$.

Proof. Note that the limit value exists due to Lemma 4.4.

Let $\alpha \in \mathbb{R}$ satisfy

$$\lim_{x\to\infty} \big(\psi(x) - \psi'(x^-)x\big) > \alpha > -\infty.$$

Let $\Delta > 0$ be such that $\inf\limits_{x>\Delta} \big(\psi(x) - \psi'(x^-)x\big) > \alpha$. Then,

$$\psi(x) - \frac{\psi(t) - \psi(x)}{t - x} x > \alpha\,, \qquad \Delta < t < x$$

and therefore

$$\frac{\psi(t) - \alpha}{t} \geq \frac{\psi(x) - \alpha}{x}\,, \qquad \Delta < t < x\,.$$

Consequently,

$$\frac{\psi(x) - \alpha}{x} \geq \lim_{x\to\infty} \frac{\psi(x) - \alpha}{x} = \lim_{x\to\infty} \frac{\psi(x)}{x} = \widehat{a}_\psi\,, \qquad x > \Delta\,.$$

Thus, $\psi(x) - \widehat{a}_\psi x \geq \alpha$ for $x > \Delta$, and

$$\lim_{x\to\infty} \big(\psi(x) - \widehat{a}_\psi x\big) \geq \alpha > -\infty\,,$$

i.e. $\psi \in \Phi_3$. □

As a direct consequence from Lemma 4.5 we obtain

Corollary 4.3. *Let $\psi \in \Phi_2$. If ψ is convex on $(0,\infty)$, then*

$$\lim_{x\to\infty} \big(\psi(x) - \psi'(x^-)x\big) = -\infty\,.$$

5. PROOFS OF THE MAIN RESULTS

Proof of Theorem 3.1. Let $\varphi \in \Phi$, $\psi \in \Phi$ and ψ be convex on $(0,\infty)$. Since $\varphi \geq \varphi^{**}$, we have

$$\inf_{x\in(0,\infty)} \big(\varphi(x) - \psi(x)\big) \geq \inf_{x\in(0,\infty)} \big(\varphi^{**}(x) - \psi(x)\big)\,. \tag{5.1}$$

We consider separately two cases:

Case 1. $\inf\limits_{x\in(0,\infty)} \big(\varphi(x) - \psi(x)\big) = -\infty$. We have

$$\inf_{x\in(0,\infty)} \big(\varphi(x) - \psi(x)\big) = \inf_{x\in(0,\infty)} \big(\varphi^{**}(x) - \psi(x)\big) = -\infty\,.$$

Case 2. $c := \inf\limits_{x\in(0,\infty)} \big(\varphi(x) - \psi(x)\big) > -\infty$. In this case,

$$\varphi(x) \geq \psi(x) + c, \qquad x \in (0,\infty)$$

and $\psi + c$ is a convex minorant of φ. Therefore, $\varphi^{**}(x) \geq \psi(x) + c, \quad x \in (0,\infty)$, i.e. $\inf\limits_{x>0} (\varphi^{**}(x) - \psi(x)) \geq c$ and

$$\inf_{x>0} (\varphi^{**}(x) - \psi(x)) \geq \inf_{x\in(0,\infty)} \big(\varphi(x) - \psi(x)\big)\,.$$

It follows from here and inequality (5.1) that

$$\inf_{x\in(0,\infty)}\big(\varphi(x)-\psi(x)\big)=\inf_{x\in(0,\infty)}\big(\varphi^{**}(x)-\psi(x)\big). \qquad \square$$

Proof of Theorem 3.2. Recall that φ, $\psi\in\Phi$, ψ is convex on $(0,\infty)$ and

$$\lim_{x\to 0^+}\big(\varphi(x)-\psi(x)\big)=\infty\,.$$

Note that $\lim_{x\to 0^+}\psi(x)=:\psi(0^+)\in\mathbb{R}\cup\{\infty\}$ and $\psi(0^+)>-\infty$, because $\psi\in\Phi$ and ψ is convex on $(0,\infty)$. Therefore, $\varphi(0^+)=\infty$ and from Lemma 4.1 we obtain that $\varphi^{**}(0^+)=\infty$.

If $\psi(0^+)<\infty$, then

$$\lim_{x\to 0^+}(\varphi^{**}(x)-\psi(x))=\varphi^{**}(0^+)-\psi(0^+)=\infty.$$

In order to complete the proof, we have to examine the alternative when ψ satisfies $\psi(0^+)=\infty$. We shall define a new function $\widetilde{\psi}$ that is a convex minorant of φ.

Let a_0, $b_0\in\mathbb{R}$ be such that $(a_0,b_0)\in M_\varphi$ and $c\in\mathbb{R}$. We choose $\Delta_1>0$ such that

$$\inf_{0<x<\Delta_1}\big(\varphi(x)-\psi(x)\big)>c\,.$$

Next, we choose Δ_2 such that $\Delta_1>\Delta_2>0$ and

$$\inf_{0<x<\Delta_2}\big(\psi(x)+c-(a_0x+b_0)\big)>0.$$

Now, we choose Δ_3, $\Delta_2>\Delta_3>0$, so that $\psi(x)$ is monotone non-increasing on $(0,\Delta_3)$. For $x\in(0,\Delta_3)$ we have the following inequalities for the convex function ψ:

$$0\ge\frac{\psi(\Delta_3)-\psi(x)}{\Delta_3}\ge\frac{\psi(\Delta_3)-\psi(x)}{\Delta_3-x}\ge\limsup_{t\to x^+}\;\frac{\psi(t)-\psi(x)}{t-x}=:\psi'(x^+)$$

and from $\psi(0^+)=\infty$ it follows that

$$\lim_{x\to 0^+}\psi'(x^+)=-\infty.$$

Further, we choose Δ_4, $\Delta_3>\Delta_4>0$, such that

$$\sup_{0<x<\Delta_4}\psi'(x^+)<a_0.$$

If $x\in(0,\Delta_4)$, then

$$\begin{aligned}\limsup_{x\to 0^+}&\big(\psi'(x^+)(\Delta_1-x)+\psi(x)+c\big)\\&=\limsup_{x\to 0^+}\big(\psi'(x^+)(\Delta_1/2-x)+\psi(x)+c+\psi'(x^+)\Delta_1/2\big)\\&\le\limsup_{x\to 0^+}\big(\psi(\Delta_1/2)+c+\psi'(x^+)\Delta_1/2\big)=-\infty\,.\end{aligned}$$

Finally, we choose Δ_5 so that $\Delta_4 > \Delta_5 > 0$ and

$$\sup_{0<x<\Delta_5} \big(\psi'(x^+)(\Delta_1 - x) + \psi(x) + c\big) < a_0\Delta_1 + b_0 \,.$$

Let $x_1 \in (0, \Delta_5)$. We set

$$a_1 := \psi'(x_1^+), \qquad b_1 := -\psi'(x_1^+)x_1 + \psi(x_1) + c\,,$$

hence,

$$\psi'(x_1^+)(x - x_1) + \psi(x_1) + c = a_1 x + b_1, \quad x \in (0, \infty).$$

From

$$\begin{aligned} a_1x_1 + b_1 &= \psi(x_1) + c \geq a_0x_1 + b_0, \\ a_1\Delta_1 + b_1 &< a_0\Delta_1 + b_0 \end{aligned}$$

we conclude that there exists $x_2 \in [x_1, \Delta_1]$ such that $a_1x_2 + b_1 = a_0x_2 + b_0$.

We define a function $\widetilde{\psi}$ as follows:

$$\widetilde{\psi}(x) := \begin{cases} \psi(x) + c, & x \in (0, x_1) \\ a_1 x + b_1, & x \in [x_1, x_2] \\ a_0 x + b_0, & x \in (x_2, \infty). \end{cases}$$

The function $\widetilde{\psi}$ is convex on $(0, \infty)$ because it is continuous, $\psi'(x_1^-) \leq a_1 \leq a_0$ and $\psi + c$ is convex on $(0, x_1)$.

Furthermore,

$$\begin{aligned} &\widetilde{\psi}(x) = \psi(x) + c \leq \varphi(x), \quad x \in (0, x_1)\,, \\ &\widetilde{\psi}(x) = a_1x + b_1 \leq \psi(x) + c \leq \varphi(x), \quad x \in [x_1, x_2]\,, \\ &\widetilde{\psi}(x) = a_0x + b_0 \leq \varphi(x), \quad x \in (x_2, \infty)\,. \end{aligned}$$

Hence, $\widetilde{\psi}$ is a convex minorant of φ, and $\varphi^{**}(x) \geq \widetilde{\psi}(x)$, $x \in (0, \infty)$.

Thus $\varphi^{**}(x) \geq \psi(x) + c$, $x \in (0, x_1)$, and

$$\liminf_{x\to 0^+} \big(\varphi^{**}(x) - \psi(x)\big) \geq c\,,$$

which, according to the choice of c, implies that

$$\lim_{x\to 0^+} \big(\varphi^{**}(x) - \psi(x)\big) = \infty. \qquad \square$$

Proof of Theorem 3.3. Recall that $\varphi \in \Phi$, $\psi \in \Phi \setminus \Phi_3$, ψ is convex on $(0, \infty)$ and $\lim\limits_{x\to\infty} \big(\varphi(x) - \psi(x)\big) = \infty$.

By Lemma 4.1 we have

$$\widehat{a}_\varphi = \liminf_{x\to\infty} \frac{\varphi(x)}{x} = \lim_{x\to\infty} \frac{\varphi^{**}(x)}{x},$$
$$\widehat{a}_\psi = \liminf_{x\to\infty} \frac{\psi(x)}{x} = \lim_{x\to\infty} \frac{\psi^{**}(x)}{x}$$

Let $\Delta > 0$ be such that

$$\inf_{x>\Delta} \big(\varphi(x) - \psi(x)\big) > 0,$$

then $\varphi(x) \geq \psi(x)$ for every $x \in (\Delta, \infty)$ and

$$\widehat{a}_\varphi = \liminf_{x\to\infty} \frac{\varphi(x)}{x} \geq \liminf_{x\to\infty} \frac{\psi(x)}{x} = \widehat{a}_\psi.$$

The proof proceeds with separate consideration of several cases:

Case 1: $\widehat{a}_\varphi > \widehat{a}_\psi$.

Let a_1, $a_2 \in \mathbb{R}$ be such that

$$\widehat{a}_\varphi > a_1 > a_2 > \widehat{a}_\psi.$$

We choose $\Delta < \Delta_1$ such that

$$\frac{\varphi^{**}(x)}{x} > a_1 > a_2 > \frac{\psi(x)}{x}, \quad x \in (\Delta_1, \infty).$$

Hence, $\varphi^{**}(x) - \psi(x) > (a_1 - a_2)x,\ x \in (\Delta_1, \infty)$, and

$$\lim_{x\to\infty} \big(\varphi^{**}(x) - \psi(x)\big) = \infty\,.$$

Thus Case 1 is settled.

Case 2: $\widehat{a}_\varphi = \widehat{a}_\psi$. This case is split into three subcases.

Case 2.1: $\varphi \in \Phi_3$. We make the following observations:

- Lemma 3.2 implies that $\varphi^{**} \in \Phi_3$.
- $\psi \in \Phi_2$ and since ψ is convex, we have

$$\liminf_{x\to\infty} \big(\psi(x) - \widehat{a}_\psi x\big) = \lim_{x\to\infty} \big(\psi(x) - \widehat{a}_\psi x\big) = -\infty\,.$$

We claim that

$$\inf_{x>0} \left(\varphi^{**}(x) - \widehat{a}_\varphi x\right) > -\infty. \tag{5.2}$$

Indeed, let us choose the real numbers b, Δ_2, a_0 and b_0 in the following way:

– b is such that

$$\liminf_{x\to\infty}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big) > b\,;$$

– $\Delta_2 > 0$ is such that

$$\inf_{x>\Delta_2}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big) > b\,;$$

– a_0 and b_0 are such that $(a_0, b_0) \in M_\varphi$ and therefore

$$a_0x + b_0 \le \varphi^{**}(x)\,, \qquad x \in (0,\infty).$$

We have

$$\begin{aligned}\inf_{x>0}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big) &= \min\Big\{\inf_{0<x\le\Delta_2}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big);\ \inf_{x>\Delta_2}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big)\Big\}\\ &\ge \min\Big\{\inf_{0<x\le\Delta_2}\big(a_0x+b_0-\widehat{a}_\varphi x\big);\ b\Big\} > -\infty\end{aligned}$$

and claim (5.2) is proved. Now,

$$\begin{aligned}\liminf_{x\to\infty}\big(\varphi^{**}(x)-\psi(x)\big) &= \liminf_{x\to\infty}\Big((\varphi^{**}(x)-\widehat{a}_\varphi x)+(\widehat{a}_\psi x-\psi(x))\Big)\\ &\ge \inf_{x>0}(\varphi^{**}(x)-\widehat{a}_\varphi x)+\liminf_{x\to\infty}(\widehat{a}_\psi x-\psi(x))\\ &= \inf_{x>0}(\varphi^{**}(x)-\widehat{a}_\varphi x)+\lim_{x\to\infty}(\widehat{a}_\psi x-\psi(x)) = \infty\end{aligned}$$

Case 2.1 is settled.

Case 2.2: $\varphi \in \Phi_2$ and *Case 2.3:* $\varphi \in \Phi_1$.

Let $c \in \mathbb{R}$, $\Delta > 0$ satisfy $\inf\limits_{x>\Delta}(\varphi(x)-\psi(x)) > c$, and a_0, b_0 be such that $(a_0, b_0) \in M_\varphi$. In the present cases, the assumptions imply that $\varphi \in \Phi_2$ ($\varphi \in \Phi_1$) and $\widehat{a}_\varphi > a_0$. So, $\widehat{a}_\psi = \widehat{a}_\varphi > a_0$, $\psi \in \Phi_2$ ($\psi \in \Phi_1$), and by Lemma 4.3,

$$\lim_{x\to\infty}(\psi(x)-a_0x) = \infty.$$

Let $\Delta_1 > \Delta$ be such that

$$\inf_{x>\Delta_1}\big(\psi(x)+c-(a_0x+b_0)\big) > 0.$$

Since $\psi \in \Phi_2$ ($\psi \in \Phi_1$) is a convex function, we have

$$\psi'(x^-) \le \psi'(x^+) < \widehat{a}_\psi, \qquad x > 0.$$

For a fixed x' and $\infty > x > x' > 0$ we have

$$\begin{aligned}&\frac{\psi(x')-\psi(x)}{x'-x} \le \lim_{t\to x^-}\frac{\psi(t)-\psi(x)}{t-x} = \psi'(x^-) < \widehat{a}_\psi\\ \Longrightarrow\ &\lim_{x\to\infty}\frac{\frac{\psi(x)}{x}-\frac{\psi(x')}{x}}{1-\frac{x'}{x}} = \lim_{x\to\infty}\frac{\psi(x')-\psi(x)}{x'-x} \le \lim_{x\to\infty}\psi'(x^-) \le \widehat{a}_\psi\,,\end{aligned}$$

therefore

$$\lim_{x\to\infty} \psi'(x^-) = \widehat{a}_\psi\,. \tag{5.3}$$

Let $\Delta_2 > \Delta_1$ be such that

$$\inf_{x>\Delta_2} \psi'(x^-) > a_0.$$

We claim that there exists $\Delta_3 > \Delta_2$ such that

$$\psi'(x^-)(\Delta - x) + \psi(x) + c < a_0\Delta + b_0, \qquad x > \Delta_3. \tag{5.4}$$

The arguments for the proof of this claim in *Case 2.2* and *Case 2.3* are different.

In *Case 2.2* we have $\widehat{a}_\psi < \infty$, and Corollary 4.3 applied to ψ imply

$$\lim_{x\to\infty} (\psi(x) - \psi'(x^-)x) = -\infty\,.$$

Therefore by (5.3) we obtain

$$\lim_{x\to\infty} \big(\psi'(x^-)(\Delta - x) + \psi(x) + c\big) = -\infty$$

On the other hand, in *Case 2.3* we have $\lim\limits_{x\to\infty} \psi'(x^-) = \widehat{a}_\psi = \infty$ and

$$\begin{aligned}\psi'(x^-)(\Delta - x) + \psi(x) + c &\le \psi'(x^-)(2\Delta - x) + \psi(x) + c - \psi'(x^-)\Delta\\ &\le \psi(2\Delta) + c - \psi'(x^-)\Delta\,,\end{aligned}$$

since $\psi'(x^-)(t - x) + \psi(x) \le \psi(t)$ for x, $t > 0$. Hence,

$$\lim_{x\to\infty} (\psi'(x^-)(\Delta - x) + \psi(x) + c) = -\infty\,.$$

Thus (5.4) is proved and let $\Delta_3 > \Delta_2$ be such that (5.4) is fulfilled. For $x_1 > \Delta_3$ we set

$$a_1 = \psi'(x_1^-), \quad b_1 = -\psi'(x_1^-)x_1 + \psi(x_1) + c.$$

Note that $a_1 > a_0$. Then

$$\begin{aligned}&a_1x + b_1 \le \psi(x) + c, \quad \forall x \in (0,\infty)\,,\\ &a_1x_1 + b_1 = \psi(x_1) + c \ge a_0x_1 + b_0\,,\\ &a_1\Delta + b_1 < a_0\Delta + b_0\,.\end{aligned}$$

We choose $x_2 \in (\Delta, x_1]$ so that

$$a_1x_2 + b_1 = a_0x_2 + b_0,$$

and define a function $\widetilde{\psi} : (0,\infty) \to \mathbb{R}$ as follows:

$$\widetilde{\psi}(x) = \begin{cases} a_0x + b_0, & x \in (0, x_2]\,,\\ a_1x + b_1, & x \in (x_2, x_1]\,,\\ \psi(x) + c, & x \in (x_1, \infty)\,. \end{cases}$$

Notice that $\widetilde{\psi}$ is convex on $(0,\infty)$, because it is continuous, $a_0 \le a_1 \le \psi'(x_1^-)$ and $\psi + c$ is convex on (x_1,∞). Moreover, $\widetilde{\psi}(x) \le \varphi(x)$ for every $x \in (0,\infty)$. Therefore, $\widetilde{\psi}(x) \le \varphi^{**}(x)$, $x \in (0,\infty)$.

Thus for $x > x_1$ we have $\psi(x) + c \le \varphi^{**}(x)$ and

$$\liminf_{x\to\infty}\bigl(\varphi^{**}(x) - \psi(x)\bigr) \ge c\,.$$

It follows from our choice of c that

$$\lim_{x\to\infty}\bigl(\varphi^{**}(x) - \psi(x)\bigr) = \infty. \qquad \square$$

6. APPLICATION

In this section we apply Theorems 3.1, 3.2 and 3.3 to the theory of spaces $H_v(G)$ and $H_{v_0}(G)$.

We make use of the following notation:

$$Mf(y) = \sup_{x\in(-\infty,\infty)} |f(x+iy)|,$$
$$\psi_f(y) = \ln Mf(y), \qquad \forall y > 0, f \in \Lambda(p)\,,$$

where f is a holomorphic function defined on the upper half plane G.

Note that

$$-\ln \| f \|_v = \inf_{y>0}\bigl(\varphi_v(y) - \psi_f(y)\bigr)$$

Here we reformulate our results from [6].

Theorem A. [6, Th. 1.2] *If φ satisfies condition (1.1$'$), then*

$$H_v(G) \ne \{0\} \iff \varphi \in \Phi\,,$$

where $v = e^{-\varphi}$.

Theorem B. [6, Th. 1.3] *If φ satisfies condition (1.1$'$), then*

$$H_{v_0}(G) \ne \{0\} \iff \left|\begin{array}{l}\varphi \in \Phi,\\ \varphi(0^+) = \infty\,,\end{array}\right.$$

where $v = e^{-\varphi}$.

Theorem C [6, Th. 1.4] *If φ satisfies condition (1.1$'$) and $H_{v_0}(G) \ne \{0\}$, then*

$$\psi_f \in \Phi \setminus \Phi_3 \quad \textit{for every} \quad f \in H_{v_0}(G) \setminus \{0\},$$

where $v = e^{-\varphi}$.

Note that ψ_f is convex on $(0,\infty)$ and $\psi_f \in \Phi$, $\forall f \in H_v(G) \setminus \{0\}$.

In this section we prove two new theorems.

Theorem 6.1. *If φ satisfies condition (1.1′) and $\varphi \in \Phi$, then*

$$\big(H_v(G), \|\cdot\|_v\big) \equiv \big(H_w(G), \|\cdot\|_w\big),$$

where $v = e^{-\varphi}$ and $w = e^{-\varphi^{**}}$.

Proof. Let $v = e^{-\varphi}$ and $w = e^{-\varphi^{**}}$. The following implications hold:

- $\varphi > \varphi^{**} \implies \varphi^{**}$ satisfies condition (1.1′);
- $\varphi \in \Phi \implies \varphi^{**} \in \Phi$, because $M_{\varphi^{**}} = M_\varphi \neq \emptyset$.

Thus, $H_v(G) \neq \{0\}$ and $H_w(G) \neq \{0\}$, by Theorem A. Moreover, $H_v(G) \supset H_w(G)$, because $\| f \|_v \leq \| f \|_w < \infty$, $\forall f \in H_w(G)$.

Note that for every $f \in H_v(G) \neq \{0\}$ the function $\psi_f = \ln Mf$ is convex on $(0, \infty)$ and $\psi_f \in \Phi$. Therefore, by Theorem 3.1,

$$\inf_{x\in(0,\infty)} \big(\varphi(x) - \psi_f(x)\big) = \inf_{x\in(0,\infty)} \big(\varphi^{**}(x) - \psi_f(x)\big)$$

Thus $f \in H_w(G) \neq \{0\}$ and $\| f \|_v = \| f \|_w$. □

Theorem 6.2. *If φ satisfies condition (1.1′), $\varphi \in \Phi$ and $\varphi(0^+) = \infty$, then*

$$\big(H_{v_0}(G), \|\cdot\|_v\big) \equiv \big(H_{w_0}(G), \|\cdot\|_w\big),$$

where $v = e^{-\varphi}$ and $w = e^{-\varphi^{**}}$.

Proof. Let $v = e^{-\varphi}$ and $w = e^{-\varphi^{**}}$. The following implications hold:

- $\varphi > \varphi^{**} \implies \varphi^{**}$ satisfies condition (1.1′);
- $\varphi \in \Phi \implies \varphi^{**} \in \Phi$, since $M_{\varphi^{**}} = M_\varphi \neq \emptyset$;
- $\varphi^{**}(0^+) = \varphi(0^+) = \infty$, by Lemma 4.1 (1).

By Theorem B, $H_{v_0}(G) \neq \{0\}$ and $H_{w_0}(G) \neq \{0\}$. Moreover, $H_{v_0}(G) \supset H_{w_0}(G)$, because of

$$0 \leq v(iy)|f(x+iy)| \leq w(iy)|f(x+iy)|$$

for every $f \in H_{w_0}(G)$ and $x \in (-\infty, \infty)$, $y \in (0; \infty)$.

By Theorem 6.1, $\| f \|_v = \| f \|_w$ for every $f \in H_{v_0}(G) \neq \{0\}$.

We have to prove that $f \in H_{w_0}(G) \neq \{0\}$ for every $f \in H_{v_0}(G) \neq \{0\}$. Let $f \in H_{v_0}(G) \neq \{0\}$. In view of the definition of $H_{v_0}(G)$,

$$\lim_{\mathcal{K}\uparrow G} \sup_{z\in G\setminus\mathcal{K}} v(z)|f(z)| = 0,$$

where $\mathcal{K} \subset G$ and $\mathcal{K}$ is compact. So,

$$\lim_{y\to 0^+} v(iy)Mf(y) = 0, \quad \lim_{y\to\infty} v(iy)Mf(y) = 0\,,$$

and after reformulation,

$$\lim_{y\to 0^+} \big(\varphi(y) - \psi_f(y)\big) = \infty, \qquad \lim_{y\to\infty} \big(\varphi(y) - \psi_f(y)\big) = \infty\,.$$

By Theorem C, $\psi_f \in \Phi \setminus \Phi_3$. By Theorem 3.2 and Theorem 3.3 we have

$$\lim_{y\to 0^+} \big(\varphi^{**}(y) - \psi_f(y)\big) = \infty, \qquad \lim_{y\to\infty} \big(\varphi^{**}(y) - \psi_f(y)\big) = \infty\,,$$

i.e.

$$\lim_{y\to 0^+} w(iy)Mf(y) = 0, \quad \lim_{y\to\infty} w(iy)Mf(y) = 0\,.$$

For an arbitrary $\varepsilon > 0$ we choose $c > 1$ such that

$$\sup_{y<\frac{1}{c}} w(iy)Mf(y) < \varepsilon, \quad \sup_{y>c} w(iy)Mf(y) < \varepsilon\,.$$

The quantity

$$m = \frac{\sup\limits_{\frac{1}{c}\le y\le c} w(iy)}{\inf\limits_{\frac{1}{c}\le y\le c} v(iy)}$$

satisfies $m < \infty$, since $\varphi^{**} \in \Phi$ and therefore $\inf_{\frac{1}{c}\le x\le c} \varphi^{**}(x) > -\infty$ for every $c > 1$.

In view of the definition of $H_{v_0}(G)$ there exist $x_1 > 0$, $c_1 > c$ and a compact

$$\mathcal{K}_1 = \{x + iy \mid -x_1 \le x \le x_1, \frac{1}{c_1} \le y \le c_1\}$$

satisfying

$$\sup_{x+iy\in G\setminus\mathcal{K}_1} v(iy)|f(x+iy)| \le \frac{\varepsilon}{m}\,.$$

Let $\mathcal{K} = \{x + iy \mid -x_1 \le x \le x_1, \frac{1}{c} \le y \le c\}$, then

$$\begin{aligned}
&\sup_{x+iy\in G\setminus\mathcal{K}} w(iy)|f(x+iy)| \\
&\qquad = \max\Big\{ \sup_{y<\frac{1}{c}} w(iy)Mf(y),\ \sup_{\substack{|x|>x_1,\\ \frac{1}{c}\le y\le c}} w(iy)Mf(y),\ \sup_{y>c} w(iy)Mf(y) \Big\} \\
&\qquad \le \max\Big\{ \varepsilon,\ \sup_{\substack{|x|>x_1,\\ \frac{1}{c}\le y\le c}} v(iy)m|f(x+iy)|,\ \varepsilon \Big\} \le \varepsilon
\end{aligned}$$

and therefore $f \in H_{w_0}(G)$. □

7. REFERENCES

1. Ardalani, M., Lusky, W.: Weighted spaces of holomorphic functions on the upper halfplane. *Math. Scand.*, **111**, 2012, 244-260, Zbl 1267.30111.
2. Harutyunyan, A., Lusky, W.: A remark on the isomorphic classification of weighted spaces of holomorphic functions on the upper half plane. *Ann. Univ. Sci. Budap. Sect. Comp.*, **39**, 2013, 125–135, Zbl 1289.46045.
3. Bonet, J., Domanski, P., Lindstrom, M., Taskinen, J.: Composition operators between weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. (Series A)*, **64**, 1998, 101–118, Zbl 0912.47014.
4. Contreras, M., Hernandez-Diaz, A.: Weighted composition operators in weighted Banach spaces of analytic functions. *J. Austral. Math. Soc. (Series A)*, **69**, 2000, 41–60, Zbl 0990.47018.
5. Bierstedt, K., Bonet, J., Taskinen, J.: Associated weights and spaces of holomorphic functions. *Stud. Math.*, **127**, no. 2, 1998, 137–168, Zbl 0934.46027.
6. Stanev, M. At.: Weighted Banach spaces of holomorphic functions in the upper half plane. `http://arxiv.org/abs/math/9911082`
7. Stanev, M. At.: Log-convexity of the weight of a weighted function space. *"Complex Analysis and Applications '13"*, International Memorial Conference for the 100-th Anniversary of Acad. Ljubomir Iliev), IMI - Sofia, 31 Oct. - 2 Nov. 2013. `http://www.math.bas.bg/complan/caa13/`
8. Stanev, M. At.: Weighted Banach spaces of holomorphic functions with log-concave weight function. *Jubilee Conference 125 years of Mathematics and Natural Sciences at Sofia University St. Kliment Ohridski, December 6-7, 2014.* `http://125years.fmi.uni-sofia.bg`

Received on 26 February, 2015

Martin At. Stanev

Dept. of Mathematics and Physics
University of Forestry
10 Kl. Ohridski Blvd., BG-1756 Sofia
BULGARIA

e-mail: martin_stanev@yahoo.com

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

WEAK CONVERGENCE RESULTS FOR CONTROLLED BRANCHING PROCESSES: STATISTICAL APPLICATIONS

MIGUEL GONZÁLEZ, CARMEN MINUESA, INÉS DEL PUERTO

In this communication it is proved a fluctuation limit theorem for controlled branching processes. Under the conditions that the offspring and control means tend to be *critical*, the obtained limit is a diffusion process. This result is applied to conclude that the standard parametric bootstrap weighted conditional least squares estimate for the offspring mean is asymptotically invalid in the critical case.

Keywords: Controlled branching processes, weak convergence theorem, diffusion process, conditional least squares estimation, parametric bootstrap

2000 Math. Subject Classification: 60J80, 62M05

1. INTRODUCTION

Branching processes are regarded as appropriate probability models for the description of the extinction/growth of populations whose developments are subject to the law of chance. In particular, controlled branching processes are useful to model some situations which require control of the population size at each generation. This consists of determining the number of individuals with reproductive capacity at each generation, mathematically through a control process.

Let us provide its formal definition: A controlled branching process (CBP) with a random control function is a stochastic process, $\{Z_n\}_{n\geq 0}$, defined recursively as follows:

$$Z_0 = N \in \mathbb{N}, \quad Z_{n+1} = \sum_{j=1}^{\phi_n(Z_n)} X_{nj}\,, \qquad n \geq 0, \tag{1}$$

where $\{X_{nj} : n = 0,1,\dots; j = 1,2,\dots\}$ and $\{\phi_n(k) : n,k = 0,1,\dots\}$ are two families of independent non–negative integer–valued random variables, with X_{nj}, $n = 0,1,\dots;\ j = 1,2,\dots$ being independent and identically distributed (i.i.d.) random variables having mean m and variance τ^2 (both assumed finite), and for each $n = 0,1,\dots$, $\{\phi_n(k)\}_{k\geq 0}$ are independent stochastic processes with equal one–dimensional probability distributions with $E[\phi_n(k)] = \varepsilon(k)$ and $Var[\phi_n(k)] = \sigma^2(k)$ (both assumed finite for each $k \geq 0$). The random variable Z_n represents the total number of individuals in generation n, starting with $Z_0 = N > 0$ progenitors. Each individual, independently of all others and all with identical probability distributions, gives rise to new individuals. The random variable X_{nj} is the number of offspring originated by the j-th individual of generation n. If in a certain generation n there are k individuals, i.e., $Z_n = k$, then, through the random variable $\phi_n(k)$, identically distributed for each n, there is produced a control in the process fixing the number of progenitors which generate Z_{n+1}. Thus the variable $\phi_n(k)$ determines the migration process in a generation of size k: for those values of the variable $\phi_n(k)$ such that $\phi_n(k) < k$, $k - \phi_n(k)$ individuals are removed from the population, and therefore do not participate in the future evolution of the process; if $\phi_n(k) > k$, $\phi_n(k) - k$ new individuals (immigrants) of the same type are added to the population participating as progenitors under the same conditions as the others. No control is applied to the population when $\phi_n(k) = k$. It is easy to see that $\{Z_n\}_{n\geq 0}$ is a homogeneous Markov chain. This model was introduced in [10] for degenerated control distributions (deterministic case) and in [11] for the random case. The probabilistic theory on this model has been developed in [1], [6], [8] and [11] (and references therein).

Let define $\tau_m(k) = k^{-1}E[Z_{n+1} \mid Z_n = k],\ k = 1,2,\dots$. Intuitively $\tau_m(k)$ is interpreted as the expected growth rate per individual when, in a certain generation, there are k individuals. The process can be classified depending on the limit behaviour of the sequence $\{\tau_m(k)\}_{k\geq 1}$. In a broad sense, the cases $\limsup_{k\to\infty}\tau_m(k) < 1$, $\liminf_{k\to\infty}\tau_m(k) \leq 1 \leq \limsup_{k\to\infty}\tau_m(k)$, and $\liminf_{k\to\infty}\tau_m(k) > 1$ are referred to, respectively, as subcritical, critical, and supercritical situations for a CBP. It is easy to obtain that $\tau_m(k) = mk^{-1}\varepsilon(k),\ k \geq 1$. Hence the classification of the process is determined essentially by the behaviour of the offspring and control means. Whenever exists the limit of the sequence $\{\tau_m(k)\}_{k\geq 1}$, as $k \to \infty$, we refer to it as the asymptotic mean growth rate.

In this paper we consider an array of CBPs $\{Z_i^{(n)}\}_{i\geq 0}$, $n = 1,2,\dots$, defined recursively by

$$Z_0^{(n)} = N \in \mathbb{N}, \qquad Z_{i+1}^{(n)} = \sum_{j=1}^{\phi_i^{(n)}(Z_i^{(n)})} X_{ij}^{(n)}, \quad i = 0,1,\dots;\ n = 1,2,\dots \tag{2}$$

For each n, $\{X_{ij}^{(n)} : i = 0,1,\dots; j = 1,2,\dots\}$ is a sequence of i.i.d. non–negative integer–valued random variables with mean m_n and finite variance τ_n^2, and $\{\phi_i^{(n)}(k) : i = 0,1,\dots; k = 0,1,\dots\}$ are independent non–negative integer–valued random variables with means $\varepsilon_n(k)$ and finite variances $\sigma_n^2(k)$ for every $k \geq 0$. Also, for each n, we assume that $\{X_{ij}^{(n)}\}$ and $\{\phi_i^{(n)}(k)\}$ are independent.

The main aim of this paper is to provide a Feller diffusion approximation for an array of CBPs whose offspring and control means tend to be *critical*. Using operator semigroup convergence theorems, it is proved that the fluctuation limit is a diffusion process. From a practical viewpoint, the interest of developing this result stems from the usefulness of it in determining the asymptotic distributions of estimators of the main parameters of a controlled branching process. In particular, we are interested in the weighted conditional least squares (WCLS) estimator of the offspring mean. As an statistical application of the obtained fluctuation limit theorem, it is determined, in a parametric framework, the bootstrapping distribution of the WCLS estimator of the offspring mean in the critical case. From this, it is concluded that the standard parametric bootstrap WCLS estimate is asymptotically invalid in the critical case.

The communication is organized as follows. In Section 2 we prove that the functional fluctuation limit of a sequence of CBPs is a diffusion process. We present in Section 3 the WCLS estimator of the offspring mean of a CBP. We show its limit distribution from a classical viewpoint and in a parametric framework, its bootstrapping distribution by applying the obtained functional limit theorem. From the last, it is concluded that the standard parametric bootstrap WCLS estimate is asymptotically invalid in the critical case.

2. DIFFUSION APPROXIMATION THEOREM

Let consider an array of CBPs as given in (2). Let us introduce the sequence of random functions $\{W_n\}_{n\geq 1}$ as $W_n(t) = n^{-1} Z_{[nt]}^{(n)}$, $t \geq 0$, $n = 1, 2, \ldots$, with $[\cdot]$ denoting the integer part. It is clear that $\{W_n\}_{n\geq 0}$ is a $D_{[0,\infty)}[0,\infty)$–valued random variable, with $D_{[0,\infty)}[0,\infty)$ the space of non–negative functions on $[0,\infty)$ that are right continuous and have left limits. Denote by $C_c^\infty[0,\infty)$ the space of infinitely differentiable functions on $[0,\infty)$ which have compact supports. Throughout the paper "$\xrightarrow{\mathcal{D}}$" denotes the convergence of random functions in the Skorokhod topology, "$\xrightarrow{d}$" the convergence of random variables in distribution and $N(\cdot,\cdot)$ the normal distribution.

Using operator semigroup convergence theorems, we prove a weak convergence theorem for the sequence of random functions $\{W_n\}_{n\geq 0}$.

Theorem 1. *Assume that*

(A1) $\quad m_n = m + \alpha n^{-1} + o(n^{-1})$ *as* $n \to \infty$, $0 < m < \infty$, $-\infty < \alpha < \infty$ *;*

(A2) $\quad \tau_n^2 \to \tau^2$ *as* $n \to \infty$, $0 < \tau^2 < \infty$*;*

(A3) *for any sequence* $\{x_n\}_{n\geq 1}$ *such that* $x_n \to x$ *as* $n\to\infty$, $0<x<\infty$, *and for all* $\epsilon>0$,

$$\lim_{n\to\infty} \tau_n^{-2} E\left[|X_{01}^{(n)} - m_n|^2 \mathbf{1}_{\left\{|X_{01}^{(n)}-m_n|\geq \epsilon\sqrt{n x_n \tau_n^2}\right\}}\right] = 0,$$

with $\mathbf{1}_A$ *denoting the indicator function of a set* A *;*

(A4) $\varepsilon_n(k) = \varepsilon(k) + f_n(k)$, *with* $\lim_{n\to\infty} f_n(k) = 0$ *uniformly for* k *;*

(A5) $m\varepsilon(k)k^{-1} = 1 + \gamma k^{-1} + o(k^{-1})$ *as* $k\to\infty$, $-\infty<\gamma<\infty$ *;*

(A6) $\sigma_n^2(k) = \beta_n k + g_n(k)$, *with* $\lim_{k\to\infty} g_n(k)k^{-1} = 0$ *uniformly for* n, $\beta_n \to 0$ *as* $n\to\infty$.

Then $W_n \xrightarrow{\mathcal{D}} W_\alpha$ *as* $n\to\infty$, *weakly in the Skorohod space* $D_{[0,\infty)}[0,\infty)$, *where* W_α *is the diffusion process with generator*

$$A_\alpha f(x) = (\gamma + \alpha m^{-1}x)f'(x) + \frac{1}{2}\tau^2 m^{-1} x f''(x), \quad f\in C_c^\infty[0,\infty). \tag{3}$$

The proof of Theorem 1 can be found in [7].

The process W_α is the (unique) solution of the stochastic differential equation

$$dW_\alpha(t) = (\gamma + \alpha m^{-1} W_\alpha(t))dt + (\tau^2 m^{-1} W_\alpha(t))^{1/2} dB(t),\ t\geq 0,$$

where B is a standard Wiener process.

In next section it is necessary to consider a particular array version of CBPs of the general situation considered in (2). Let $\{Z_i^{(n)}\}_{i\geq 0}$, $n=1,2,\ldots$, be an array of CBPs with the same hypotheses about the offspring and control variables as in the definition in (2), but with the additional condition that for each $k\geq 0$, the variables $\{\phi_i^{(n)}(k)\}$, $i\geq 0$; $n\geq 1$, are identically distributed with $E[\phi_i^{(n)}(k)] = \varepsilon(k)$ and $Var[\phi_i^{(n)}(k)] = \sigma^2(k)$. In respect to the offspring law we assume conditions (A1)-(A3). Moreover, in relation to the control mean and variance we consider the following assumptions:

(B1) $m\varepsilon(k)k^{-1} = 1 + \gamma k^{-1} + o(k^{-1})$ as $k\to\infty$, $-\infty<\gamma<\infty$;

(B2) $\lim_{k\to\infty} \sigma^2(k)k^{-1} = 0$,

which are the simplified version of (A4)-(A6) in this particular case. Then, applying Theorem 1 one obtains

$$W_n \xrightarrow{\mathcal{D}} W_\alpha \text{ as } n\to\infty,$$

where W_α is the diffusion process with generator given in (3).

3. WEIGHTED CONDITIONAL LEAST SQUARES ESTIMATION AND ASYMPTOTIC RESULTS

Let consider a CBP given in (1) and let $\mathcal{F}_n$ be the $\sigma-$algebra generated by the random variables $Z_0, Z_1, \cdots, Z_n$. From the fact that $E[Z_n|\mathcal{F}_{n-1}] = m\varepsilon(Z_{n-1})$ a.s., we can represent Z_n as

$$Z_n = m\varepsilon(Z_{n-1}) + \tilde{\delta}_n, \qquad n = 1, 2, \ldots, \tag{4}$$

where the error term $\tilde{\delta}_n$ has $E\left[\tilde{\delta}_n|\mathcal{F}_{n-1}\right] = 0$. In order to obtain an efficient estimator of the offspring mean, we divide both sides of (4) by $(\varepsilon(Z_{n-1}) + 1)^{1/2}$ and rewrite the model as

$$\frac{Z_n}{(\varepsilon(Z_{n-1}) + 1)^{1/2}} = \frac{m\varepsilon(Z_{n-1})}{(\varepsilon(Z_{n-1}) + 1)^{1/2}} + \delta_n, \qquad n = 1, 2, \ldots,$$

with $\delta_n = \tilde{\delta}_n / (\varepsilon(Z_{n-1}) + 1)^{1/2}$.

The WCLS estimator of m is obtained by minimizing the expression $\sum_{i=1}^n \delta_i^2$. It is easy to check that the value of m that minimizes it is

$$\widehat{m}_n = \left(\sum_{i=1}^n \frac{Z_i \varepsilon(Z_{i-1})}{\varepsilon(Z_{i-1}) + 1}\right) \left(\sum_{i=1}^n \frac{\varepsilon^2(Z_{i-1})}{\varepsilon(Z_{i-1}) + 1}\right)^{-1}. \tag{5}$$

We are interested in the study of the limit distribution of the pivot

$$V_n = \left(\sum_{i=1}^n \frac{\varepsilon^2(Z_{i-1})}{\varepsilon(Z_{i-1}) + 1}\right)^{1/2} (\widehat{m}_n - m). \tag{6}$$

This presents different kinds of behaviour depending on the classification of the process. In [5] it was established that a CBP $\{Z_n\}_{n\geq 0}$ with $P(X_{01} = 0) > 0$, $P(X_{01} \leq 1) < 1$ and $P(\phi_0(i) > i) > 0$, $i = 0, 1, \ldots$, converges in distribution to a positive, finite and non-degenerate random variable Z.

Theorem 2. *Assume that*

i) $\limsup_{k\to\infty} \tau_m(k) < 1$;

ii) $P(X_{01} = 0) > 0, \quad P(X_{01} \leq 1) < 1$;

iii) $P(\phi_0(i) > i) > 0, \quad i = 0, 1, \ldots$;

iv) $E[\mu_{2+\delta}(Z)] < \infty$, *with* $\mu_k(z) = E[|\phi_0(z) - \varepsilon(z)|^k], \ k \geq 1.$

Then

$$V_n \xrightarrow{d} N(0,V) \quad as \quad n\to\infty\,,$$

where

$$V = \frac{m^2 E\left[\left(\frac{\varepsilon(Z)}{\varepsilon(Z)+1}\right)^2 \sigma^2(Z)\right] + \sigma^2 E\left[\dfrac{\varepsilon^3(Z)}{(\varepsilon(Z)+1)^2}\right]}{E\left[\dfrac{\varepsilon^2(Z)}{\varepsilon(Z)+1}\right]}.$$

The proof can be seen in [9].

In the supercritical case, we consider that

$$\lim_{n\to\infty} \tau_m(k) = m \lim_{n\to\infty} k^{-1}\varepsilon(k) = \eta m > 1.$$

Then the following result holds

$$P(Z_n\to\infty) > 0 \quad \text{and} \quad \lim_{n\to\infty} L_n = L \quad \text{a.s.}, \tag{7}$$

with $L_n = (\eta m)^{-n} Z_n$ and $P(L>0)>0$. Indeed, conditions that guarantee (7) can be found in the papers [3, 4].

Theorem 3. *Assume that*

i) $\limsup_{k\to\infty} \tau_m(k) > 1$ *and (7) hold;*

ii) $\lim_{k\to\infty} k^{-1}\sigma^2(k) = 0$.

Then

$$V_n \xrightarrow{d} N\left(0,\sigma^2\right), \quad as \quad n\to\infty.$$

The details of the proof can be seen in [9].

Regarding the critical case, we obtained:

Theorem 4. *Assume that*

i) $\tau_m(k) = 1 + k^{-1}\gamma + o(k^{-1})$ *as* $k\to\infty$, *where* γ *is a real number;*

ii) $\lim_{k\to\infty} k^{-1}\sigma^2(k) = 0$.

Then

$$V_n \xrightarrow{d} \frac{W(1)-W(0)-\gamma}{\left(\frac{1}{m}\int_0^1 W(t)dt\right)^{1/2}} \quad as \quad n\to\infty,$$

where W *is a diffusion process with generator (3) with* $\alpha = 0$.

The reader can find the proof in [9].

This result can be generalized to the particular array version of CBPs considered in the previous section (2). We provide the behaviour of the array version of the estimator $\widehat{m}_n$ and the pivot quantity V_n, which is the interest for the study of the behaviour of the bootstrap estimator of m. Let

$$\bar{m}_n = \left(\sum_{i=1}^{n} \frac{Z_i^{(n)} \varepsilon(Z_{i-1}^{(n)})}{\varepsilon(Z_{i-1}^{(n)}) + 1} \right) \left(\sum_{i=1}^{n} \frac{\varepsilon^2(Z_{i-1}^{(n)})}{\varepsilon(Z_{i-1}^{(n)}) + 1} \right)^{-1}$$

and

$$\bar{V}_n = \left(\sum_{i=1}^{n} \frac{\varepsilon^2(Z_{i-1}^{(n)})}{\varepsilon(Z_{i-1}^{(n)}) + 1} \right)^{1/2} (\bar{m}_n - m_n).$$

Theorem 5. *Assume that assumptions (A1)–(A3) and (B1)–(B2) are satisfied. Then, as $n \to \infty$,*

$$\bar{V}_n \xrightarrow{d} \frac{W_\alpha(1) - W_\alpha(0) - \gamma}{\left(\frac{1}{m} \int_0^1 W_\alpha(t) dt \right)^{1/2}} - \alpha \left(\frac{1}{m} \int_0^1 W_\alpha(t) dt \right)^{1/2},$$

with W_α as in Theorem 1.

Briefly, three different limit distributions for V_n were obtained for three different cases, as $n \to \infty$, namely

$$V_n \xrightarrow{d} \begin{cases} N(0, V), & \text{if } \limsup_{k\to\infty} \tau_m(k) < 1 \text{ (subcritical)}, \\ \frac{W(1) - W(0) - \gamma}{\left(\frac{1}{m} \int_0^1 W(t) dt \right)^{1/2}}, & \text{if } \tau_m(k) = 1 + k^{-1}\gamma + o(k^{-1}),\ \gamma \in \mathbb{R} \text{ (critical)} \\ N(0, \sigma^2), & \text{if } \liminf_{k\to\infty} \tau_m(k) > 1 \text{ (supercritical)}, \end{cases}$$

with V and W as previously defined. Hence the classical asymptotic theory does not provide a unified estimation theory for the offspring mean. Thus it is of interest to approximate the sampling distribution of V_n by alternative methods. In particular, we are keen on the bootstrap procedure. We apply the fluctuation limit theorem previously established to determine the asymptotic distribution of the bootstrap WCLS estimator in the critical case. We consider a parametric framework and obtain as a consequence of this last limit result that the standard bootstrap version of the pivot quantity does not have the same limit distribution as V_n in such a case. Although the behaviour of the parametric bootstrap for the subcritical and supercritical cases is of interest as well, due to this fails in the critical case it will be most interesting for the future to make efforts in developing a modified bootstrap procedure to be valid in all the three cases. Let us introduce a parametric bootstrap for CBPs following analogous steps to those given in [2] for branching processes with immigration. We assume that the offspring law, p_θ, has probability mass function

$$p_\theta(k) = P_\theta(X_{01} = k),\ k = 0, 1, \ldots,$$

depending on a parameter θ where $\theta \in \Theta \subseteq \mathbb{R}$.

Consider $m = E_\theta[X_{01}] = f(\theta)$ for some function f, which we will assume to be a one-to-one mapping of Θ to $[0, \infty)$. Moreover, f is assumed to be homeomorphism between its domain and range. For instance, the power series family of distributions satisfies the conditions imposed above.

The bootstrap procedure can be defined as follows: given the sample $\mathcal{M}_n = \{Z_1, ..., Z_n\}$, estimate the offspring mean by the estimator $\widehat{m}_n$ given in (5), and therefore let $\widehat{\theta}_n = f^{-1}(\widehat{m}_n)$. Conditional on $\mathcal{M}_n$, define a sequence of i.i.d. random variables X^*_{nj} having distribution given by $p_{\widehat{\theta}_n}$. The bootstrap sample $\mathcal{M}^*_n = \{Z^*_1, \ldots, Z^*_n\}$ is obtained by

$$Z^*_{n+1} = \sum_{j=1}^{\phi_n(Z^*_n)} X^*_{nj},\ n = 0, 1, \ldots,\ \text{with } Z^*_0 = N.$$

We define the bootstrap estimator of m as $\widehat{m}^*_n$ given by

$$\widehat{m}^*_n = \left(\sum_{i=1}^{n} \frac{Z^*_i \varepsilon(Z^*_{i-1})}{\varepsilon(Z^*_{i-1}) + 1} \right) \left(\sum_{i=1}^{n} \frac{\varepsilon^2(Z^*_{i-1})}{\varepsilon(Z^*_{i-1}) + 1} \right)^{-1},$$

and the parametric bootstrap analogue, V^*_n, of the pivot quantity V_n, given in (6), as

$$V^*_n = \left(\sum_{i=1}^{n} \frac{\varepsilon^2(Z^*_{i-1})}{\varepsilon(Z^*_{i-1}) + 1} \right)^{1/2} (\widehat{m}^*_n - \widehat{m}_n).$$

Note $\varepsilon(\cdot)$ is assumed to be known and $\phi_n(\cdot)$ are observable. In this context, let denote the distribution function of V_n by $F_n(m, x) = P(V_n \leq x),\ x \in \mathbb{R}$. Then, notice that

$$P(V^*_n \leq x | \mathcal{M}_n) = F_n(\widehat{m}_n, x),\ x \in \mathbb{R}.$$

Our interest is to determine the limit behaviour of $F_n(\widehat{m}_n, x)$, $x \in \mathbb{R}$, assuming that the true model is a critical CBP. We check that for every $x \in \mathbb{R}$ the random variables $F_n(\widehat{m}_n, x)$ converge in distribution to a non degenerate random limit, and consequently one has that it is not verified that

$$\sup_{-\infty < x < \infty} |F_n(m, x) - F_n(\widehat{m}_n, x)| \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty, \tag{8}$$

obtaining the asymptotic invalidity of the bootstrap procedure in the critical case.

Define

$$\mathcal{W}(\alpha, m, \tau^2, \gamma) = \frac{W_\alpha(1) - W_\alpha(0) - \gamma}{\left(\frac{1}{m}\int_0^1 W_\alpha(t)dt\right)^{1/2}} - \alpha\left(\frac{1}{m}\int_0^1 W_\alpha(t)dt\right)^{1/2},$$

with W_α the diffusion process defined in Theorem 1, and

$$F(\alpha, m, \tau^2, \gamma, x) = P(\mathcal{W}(\alpha, m, \tau^2, \gamma) \leq x), \ x \in \mathbb{R}.$$

As in [2], it is not hard to prove that, for each $x \in \mathbb{R}$, $F(\mathcal{V}_0, m, \tau^2, \gamma, x)$ is a random variable, with $\mathcal{V}_0 = (W(1) - W(0) - \gamma)\left(\frac{1}{m}\int_0^1 W(t)dt\right)^{-1}$. Now, we are in conditions to state the result that establishes that (8) does not hold:

Theorem 6. *Assume that*

(C1) *The variance of the offspring law, τ^2, is a continuous function of θ.*

(C2) *The moment $E_\theta[|X_{01}|^{2+\delta}]$, for some $\delta > 0$ is a continuous function of θ.*

Then, it is verified that for every $x \in \mathbb{R}$, as $n \to \infty$,

$$F_n(\widehat{m}_n, x) \xrightarrow{d} F(\mathcal{V}_0, m, \tau^2, \gamma, x).$$

It is not hard to check that the power distribution family verifies (C1)-(C2). The key of this proof is Theorem 5 and the details can be read in [7]. One of the reasons for the standard parametric bootstrap does not work well in such a case is the rate of convergence to the offspring mean parameter of its WCLS estimate.

ACKNOWLEDGEMENTS. Research supported by the Ministerio de Educación, Cultura y Deporte (grant FPU13/03213), the Ministerio de Economía y Competitividad (grant MTM2012-31235), the Junta de Extremadura (grant GR15105) and the FEDER through the Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnolólgica, and by the National Fund for Scientific Research at the Ministry of Education and Science of Bulgaria (grant NFSI I02/17).

4. REFERENCES

1. Bruss, F. T.: A counterpart of the Borel-Cantelli lemma. *Journal of Applied Probability*, **17**, 1980, 1094–1101.
2. Datta, S., Sriram, T. N.: A modified bootstrap for branching processes with immigration. *Stochastic Processes and their Applications*, **56**, 1995, 275–294.

3. González, M., Martínez, R., Mota, M.: On the geometric growth in a class of homogeneous multitype Markov chain, *Journal of Applied Probability*, **42**, 2005, 1015–1030.
4. González, M., Martínez, R., Mota, M.: On the unlimited growth of a class of homogeneous multitype Markov chains. *Bernoulli*, **11**, 2005, 559–570.
5. González, M., Molina, M., del Puerto, I.: Limiting distribution for subcritical controlled branching processes with random control function. *Statistics and Probability Letters*, **67**, 2004, 277–284
6. González, M., Molina, M., del Puerto, I.: Geometric growth for stochastic difference equations with application to branching populations. *Bernoulli*, **12**, 2006, 931–942.
7. González, M., del Puerto, I.: Diffusion approximation of an array of controlled branching processes. *Methodology and Computing in Applied Probability*, **14**, 2012, 843–861.
8. Nakagawa, T.: The $L^\alpha(1 < \alpha \leq 2)$ convergence of a controlled branching process in a random environment. *Bull Gen Ed Dokkyo Univ School Medicine*, **17**, 1994, 17–24.
9. Sriram, T.N., Bhattacharya, A., González, M., Martínez, R., del Puerto, I.: Estimation of the offspring mean in a controlled branching process with a random control function. *Stochastic Processes and their Applications*, **117**, 2007, 928–946.
10. Sevast'yanov, B. A., Zubkov, A. N.: Controlled branching processes. *Theory of Probability and its Applications*, **19**, 1974, 14–24.
11. Yanev, N. M.: Conditions for degeneracy of ϕ-branching processes with random ϕ. *Theory of Probability and its Applications*, **20**, 1975, 421–428.

Received on 15 April, 2015

Miguel González, Carmen Minuesa, Inés del Puerto

Department of Mathematics and
Instituto de Computación Científica Avanzada (ICCAEx)
University of Extremadura, Avda Elvas s/n
06006 - Badajoz
SPAIN

e-mails: mvelasco@unex.es
cminuesaa@unex.es
idelpuerto@unex.es

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

ON A GENERALIZATION OF CRITERIA *A* AND *D* FOR CONGRUENCE OF TRIANGLES

VESSELKA MIHOVA, JULIA NINOVA

The conditions determining that two triangles are congruent play a basic role in planimetry. By comparing not congruent triangles with respect to given sets of corresponding elements it is important to discover if they have any common geometric properties characterizing them. The present paper is devoted to an answer of this question. We give a generalization of criteria A and D for congruence of triangles and apply it to prove some selected geometric problems.

Keywords: Congruence of triangles, comparison of triangles

2000 Math. Subject Classification: Primary 51F20, Secondary 51M15

1. INTRODUCTION

There are six essential elements of every triangle - three angles and three sides. The method of constructing a triangle varies according to the facts which are known about its sides and angles.

It is important to know what is the minimum knowledge about the sides and angles which is necessary to construct a particular triangle.

Clearly all triangles constructed in the same way with the same data must be identically equal, i. e. they must be of exactly the same size and shape and their areas must be the same.

Triangles which are equal in all respects are called *congruent triangles*.

The four sets of minimal conditions for two triangles to be congruent are set out in the following geometric criteria.

Criterion A. *Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other.*

Criterion B. *Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and a side of the other.*

Criterion C. *Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other.*

Criterion D. *Two triangles are congruent if two sides and the angle opposite the greater side of one triangle are respectively equal to two sides and the angle opposite the greater side of the other.*

We notice that in *criteria* A and D the sets of corresponding equal elements are two sides and an angle. The given angle may be any one of the three angles of the triangle. The problem "*Construct a triangle with two of its sides* a *and* b, $a < b$, *and angle* α *opposite the smaller side*" has not a unique solution. There are two triangles each of which satisfies the given conditions.

In the present paper we compare not congruent triangles with respect to given sets of corresponding elements and answer the question what are the geometric properties characterizing such couples of triangles.

2. THEORETICAL BASIS OF THE PROPOSED METHOD FOR COMPARING TRIANGLES

Throughout, for the elements of two triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ we shall use the notations $AB = c$, $BC = a$, $CA = b$; $A_1B_1 = c_1$, $B_1C_1 = a_1$, $C_1A_1 = b_1$. Moreover, θ and θ_1 will stand for two corresponding angles of $\triangle ABC$ and $\triangle A_1B_1C_1$, respectively.

Suppose that in $\triangle ABC$ and $\triangle A_1B_1C_1$ the relations $a = a_1$, $b = b_1$ and $\theta = \theta_1$ hold. We consider four possible cases.

- The angle θ is included between the sides a and b, i.e., $\theta = \sphericalangle ACB$ and $\theta_1 = \sphericalangle A_1C_1B_1$. The triangles are congruent by *Criterion A*.
- Let $a = b$, i.e., $\triangle ABC$ and $\triangle A_1B_1C_1$ are isosceles. Since $\theta = \theta_1$, the triangles are congruent as a consequence of *Criterion A*.
- Let $a > b$ and the angle θ be opposite the greater side a. In this case the triangles are congruent in view of *Criterion D*.
- Let $a > b$ and the angle θ is opposite the smaller side b. In this case the triangles are either congruent or not.

- If the triangles are congruent, then the angles opposite the greater sides are necessarily equal. It could happen that the sum of the equal angles opposite the greater sides equals 180^0, then obviously the triangles are right-angled.
- If the triangles are not congruent, then we show that the sum of the angles opposite the greater sides is always equal to 180^0.

Lemma 2.1. *Let $\triangle ABC$ and $\triangle ABD$ be not congruent triangles, and let $AC = AD$. If $\sphericalangle ABC = \sphericalangle ABD$, then $\sphericalangle ACB + \sphericalangle ADB = 180^0$.*

Proof. Since $\triangle ABC$ and $\triangle ABD$ are not congruent, then $AC < AB$ (and hence $AD < AB$). Let us denote $\sphericalangle ACB = \alpha$ and $\sphericalangle ADB = \beta$.

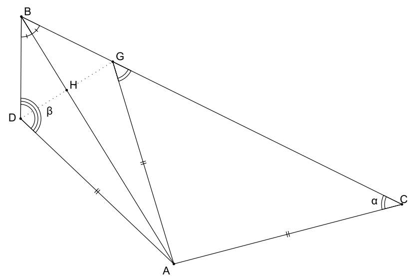


Fig. 1.

There are two possible locations of the points C and D with respect to the straight line AB.

(i) *The points C and D lie on opposite sides of AB.*

The symmetry with respect to the straight line AB transforms $\triangle ABD$ into a congruent $\triangle ABG$ which lies on the same side of AB as $\triangle ABC$ (see Fig. 1). Since $\triangle ABC \not\cong \triangle ABD$, then $\triangle ABC \not\cong \triangle ABG$. The condition $\sphericalangle ABC = \sphericalangle ABD$ implies that the straight line AB is the bisector of $\sphericalangle DBC$. From the symmetry with respect to AB it follows that $G \in BC$ and $BG \neq BC$. Let, e.g., G/BC (the case C/BG is analogous). Clearly, if the assumptions of Lemma 2.1 are fulfilled for $\triangle ABC$ and $\triangle ABD$, then they are also valid for $\triangle ABC$ and $\triangle ABG$ and vice versa.

Let us consider $\triangle ABC$ and $\triangle ABG$. The side AB and $\sphericalangle ABC$ are common for both triangles. In view of the symmetry with respect to AB and $AC = AD$, we get $AD = AG = AC$. Hence, $\triangle ACG$ is isosceles and $\sphericalangle ACG = \alpha = \sphericalangle AGC$. The angles $\sphericalangle AGC$ and $\sphericalangle AGB = \sphericalangle ADB = \beta$ are adjacent and hence $\sphericalangle AGC + \sphericalangle AGB = \sphericalangle ACB + \sphericalangle ADB = \alpha + \beta = 180^0$.

Remark 2.2. The quadrilateral $ACBD$ can be inscribed in a circle.

(*ii*) *The points C and D lie on one and the same side of AB.*

This case was already considered in (i), with $D \equiv G$. □

Remark 2.3. In the case when $\triangle ABC$ and $\triangle A_1B_1C_1$ are not congruent, the relations $AB = A_1B_1$, $AC = A_1C_1$ and $\sphericalangle ABC = \sphericalangle A_1B_1C_1$ are fulfilled and the triangles have no common side, we can choose a suitable congruence and transform $\triangle A_1B_1C_1$ into a congruent $\triangle ABD$ so that $\triangle ABC$ and $\triangle ABD$ satisfy the assumptions of Lemma 2.1.

Based on the above arguments we formulate a theorem, which is a generalization of *criteria A* and *D* for congruence of triangles (see also [6], p. 12).

Theorem 2.4. *Assume that $\triangle ABC$ and $\triangle A_1B_1C_1$ have two pairs of equal sides, $a = a_1$, $b = b_1$, and equal corresponding angles, $\theta = \theta_1$. Then $\triangle ABC$ and $\triangle A_1B_1C_1$ are either congruent, or not congruent, in which case the sum of the other two angles, not included between the given sides, is equal to 180^0.*

Lemma 2.1 and Theorem 2.4 can be used as alternative methods of comparing different triangles.

3. APPLICATION OF THEOREM 2.4 TO TWO GEOMETRIC PROBLEMS

The solutions of next selected problems are based on Theorem 2.4.

Problem 3.1 ([4, Problems 4.20 and 4.23]; [5])**.** *Let the middle points of the sides BC, CA and AB of $\triangle ABC$ be F, D, and E, respectively. If the center G of the circumscribed circle k of $\triangle FDE$ lies on the bisector of $\sphericalangle ACB$, prove that $\triangle ABC$ is either isosceles ($CA{=}CB$), or not isosceles, in which case $\sphericalangle ACB{=}60^0$.*

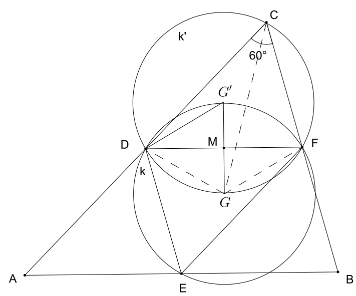


Fig. 2.

Proof. Let the center G of the circumscribed circle k of $\triangle FDE$ lie on the bisector of $\sphericalangle ACB$ (Fig. 2). Since $\triangle CGD$ and $\triangle CGF$ have a common side

CG, equal corresponding angles $\sphericalangle DCG = \sphericalangle FCG$ and equal corresponding sides $DG = FG$ (as radii of k), the assumptions of Theorem 2.4 are satisfied.

(i) If $\triangle CGD$ and $\triangle CGF$ are congruent, then $CD = CF$ and hence $CA = CB$, i.e., $\triangle ABC$ is isosceles.

Remark 3.2. There are two possibilities for $\sphericalangle ACB$: either $\sphericalangle ACB = 60^0$, in which case $\triangle ABC$ is equilateral, or $\sphericalangle ACB \neq 60^0$, and then $\triangle ABC$ is isosceles.

(ii) If $\triangle CGD$ and $\triangle CGF$ are not congruent, then in view of Lemma 2.1 $\sphericalangle CDG + \sphericalangle CFG = 180^0$ and the quadrilateral $CDGF$ can be inscribed in a circle k' (Fig. 2).

It is easily seen that $\triangle EFD \cong \triangle CDF$ and their circumscribed circles k and k' have equal radii. The circles k and k' are symmetrically located with respect to their common chord FD. Since the center G of k lies on k', then the center G' of k' lies on k. Hence, $\triangle DGG' \cong \triangle FGG'$, both triangles are equilateral, $\sphericalangle DGF = 120^0$ and $\sphericalangle ACB = 60^0$. □

Problem 3.3 ([3, Problem 8]; [4, Problem 4.12]). *Let in $\triangle ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, be the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$, respectively. Let also $AA_1 \cap BB_1 = J$. If $JA_1 = JB_1$, prove that $\triangle ABC$ is either isosceles ($CA = CB$), or not isosceles, in which case $\sphericalangle ACB = 60^0$.*

Proof. Let $\sphericalangle BAC = 2\alpha$, $\sphericalangle ABC = 2\beta$, $\sphericalangle ACB = 2\gamma$. Since J is the cut point of the angle bisectors AA_1 and BB_1 of $\triangle ABC$, then the straight line CJ is the bisector of $\sphericalangle ACB$ and $\alpha + \beta + \gamma = 90^0$ (Fig. 3).

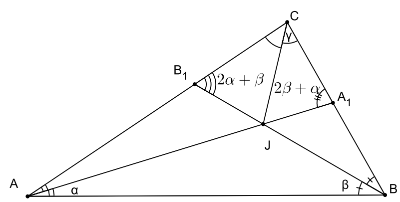


Fig. 3.

Since $\sphericalangle CB_1J$ is an exterior angle of $\triangle ABB_1$, then $\sphericalangle CB_1J = 2\alpha + \beta$. Since $\sphericalangle CA_1J$ is an exterior angle of $\triangle ABA_1$, then $\sphericalangle CA_1J = 2\beta + \alpha$.

Let us compare $\triangle CA_1J$ and $\triangle CB_1J$. They have a common side CJ, corresponding equal sides $JA_1 = JB_1$ and angles $\sphericalangle A_1CJ = \sphericalangle B_1CJ$. We observe that $\triangle CA_1J$ and $\triangle CB_1J$ satisfy the assumptions of Theorem 2.4.

(i) If $\triangle CA_1J$ and $\triangle CB_1J$ are congruent, then their corresponding elements are equal, in particular,

$$\sphericalangle CB_1J = \sphericalangle CA_1J \ \Leftrightarrow \ 2\alpha + \beta = 2\beta + \alpha \ \Leftrightarrow \ \alpha = \beta.$$

Hence, $\triangle ABC$ is isosceles with $CA = CB$.

Remark 3.4. There are two possibilities for $\sphericalangle ACB$: either $\sphericalangle ACB = 60^0$, and then $\triangle ABC$ is equilateral, or $\sphericalangle ACB \neq 60^0$, in which case $\triangle ABC$ is isosceles.

(*ii*) If $\triangle CA_1J$ and $\triangle CB_1J$ are not congruent, then by Theorem 2.4,

$$\sphericalangle CB_1J + \sphericalangle CA_1J = 180^0 \;\Leftrightarrow\; (2\alpha + \beta) + (2\beta + \alpha) = 180^0 \;\Leftrightarrow\; \alpha + \beta = 60^0.$$

Hence, $\sphericalangle ACB = 180^0 - 2(\alpha + \beta) = 60^0$. □

4. GROUPS OF PROBLEMS

In this section we illustrate the composing technology of new problems as an interpretation of specific logical models.

Our aim is the *basic problem* in each of the groups under consideration to be with (exclusive or not exclusive) disjunction as a logical structure in the conclusion and its proof to be based on Lemma 2.1 or Theorem 2.4.

4.1. PROBLEMS OF GROUP I

Suitable logical models for formulation of *equivalent* problems and *generating* problems from a given problem are described in detail in [3, 4]. The basic statements we need in this group of problems are:

$t :=$ { *A square with center O is inscribed in ΔABC so that the vertices of the square lie on the sides of ΔABC and two of them are on the side AB.*}

$p := \{\sphericalangle ACB = 90^0\}$

$q := \{CA = CB\}$

$r := \{\sphericalangle ACO = \sphericalangle BCO\}$

We describe the logical scheme for the composition of Basic problem 4.4, which has not exclusive disjunction as a logical structure in the conclusion:

- First we formulate (and prove) the *generating* problems - Problem 4.1 with a logical structure $t \wedge p \to r$ and Problem 4.3 with a logical structure $t \wedge q \to r$.
- To generate problems with logical structure $(*)\quad t \wedge (p \vee q) \to r$ we use the logical equivalence

$$(t \wedge p \to r) \wedge (t \wedge q \to r) \;\Leftrightarrow\; t \wedge (p \vee q) \to r.$$

- Finally, the formulated *inverse* problem - Basic problem 4.4 - to the problem with structure $(*)$ has the logical structure $t \wedge r \to p \vee q$.

Problem 4.1. *A square with center O is inscribed in $\triangle ABC$ so that the vertices of the square lie on the sides of $\triangle ABC$ and two of them are on the side AB. If $\sphericalangle ACB = 90^0$, prove that $\sphericalangle ACO = \sphericalangle BCO$.*

Proof. Let the quadrilateral $MNPQ$, $M \in AB$, $N \in AB$, $P \in BC$, $Q \in AC$, be the inscribed in $\triangle ABC$ square (Fig. 4). Since the diagonals of a square are

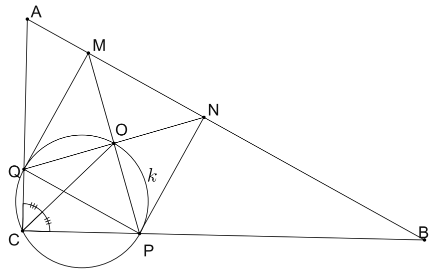


Fig. 4.

equal, intersect at right angles, bisect each other and bisect the opposite angles, then $OP = OQ$ and $\sphericalangle POQ = 90^0$. The quadrilateral $OPCQ$ can be inscribed in a circle k with diameter PQ. To the equal chords OQ and OP of k correspond equal angles, hence $\sphericalangle ACO = \sphericalangle BCO$. □

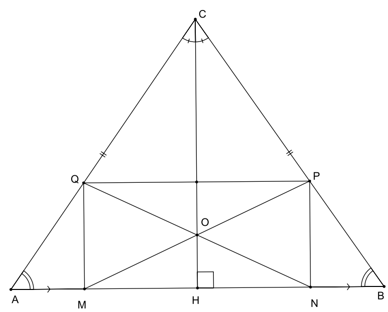


Fig. 5.

Problem 4.2. *A rectangle with center O is inscribed in $\triangle ABC$ so that the vertices of the rectangle lie on the sides of $\triangle ABC$ and two of them are on the side AB. If $CA = CB$, prove that $\sphericalangle ACO = \sphericalangle BCO$.*

Proof. Let the quadrilateral $MNPQ$, $M \in AB$, $N \in AB$, $P \in BC$, $Q \in AC$, be the inscribed in $\triangle ABC$ rectangle (Fig. 5). Since the diagonals of a rectangle are equal and bisect each other, then $OM = ON = OP = OQ$.

Let $CH \perp AB$, $H \in AB$. Since $\triangle ABC$ is isosceles with $CA = CB$, H is the middle point of AB and CH is the bisector of $\sphericalangle ACB$.

Since $MQ \parallel NP$, $NP \parallel CH$ and $MQ = NP$, it follows that $\triangle AMQ \cong \triangle BNP$ (by *Criterion B*) and $AM = BN$. Hence, H is also the middle point of MN. Since $\triangle MON$ is isosceles, then its median OH is also an altitude, i.e., $OH \perp MN$. This means that $O \in CH$ and $\sphericalangle ACO = \sphericalangle BCO$. □

A special case of Problem 4.2 is Problem 4.3 with a logical structure $t \wedge q \rightarrow r$.

Problem 4.3. *A square with center O is inscribed in $\triangle ABC$ so that the vertices of the square lie on the sides of the triangle and two of them are on the side AB. If $CA = CB$, prove that $\sphericalangle ACO = \sphericalangle BCO$.*

Now we formulate and prove the *Basic problem* in this group.

Basic problem 4.4. *A square with center O is inscribed in $\triangle ABC$ so that the vertices of the square lie on the sides of the triangle and two of them are on the side AB. If $\sphericalangle ACO = \sphericalangle BCO$, prove that $CA = CB$ or $\sphericalangle ACB = 90^0$.*

Proof. Let the quadrilateral $MNPQ$, $M \in AB$, $N \in AB$, $P \in BC$, $Q \in AC$, be the inscribed in $\triangle ABC$ square (Fig. 6). Since the diagonals of any square are

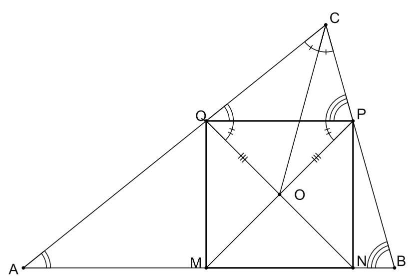


Fig. 6.

equal, intersect at right angles, bisect each other and bisect the opposite angles, then $OP = OQ$ and $\sphericalangle OPQ = \sphericalangle OQP = 45^0$.

We compare $\triangle CQO$ and $\triangle CPO$. They have a common side CO, respectively equal sides $OQ = OP$ and angles $\sphericalangle QCO = \sphericalangle PCO$. We find $\sphericalangle CQO = \sphericalangle CAB + 45^0$ and $\sphericalangle CPO = \sphericalangle CBA + 45^0$ as exterior angles of $\triangle QAN$ and $\triangle PBM$ respectively. Therefore, $\triangle CQO$ and $\triangle CPO$ satisfy the assumptions of Theorem 2.4. We consider separately the two possibilities.

(i) If $\triangle CQO$ and $\triangle CPO$ are congruent, then $\sphericalangle CQO = \sphericalangle CPO$ and hence $\sphericalangle CAB = \sphericalangle CBA$, i.e., $CA = CB$ and $\triangle ABC$ is isosceles.

In this case $\sphericalangle ACB$ is either a right angle and $\triangle ABC$ is isosceles right-angled, or not a right angle and $\triangle ABC$ is only isosceles.

(ii) If $\triangle CQO$ and $\triangle CPO$ are not congruent, then, according to Lemma 2.1, $\sphericalangle CQO + \sphericalangle CPO = 180^0$ and hence $\sphericalangle CAB + \sphericalangle CBA = 90^0$, i.e., $\triangle ABC$ is right-angled with $\sphericalangle ACB = 90^0$. □

Remark 4.5. A logically incorrect version of Basic problem 4.4 is Problem 1.54 in [1].

We reformulate Problem 4.4 by keeping the condition of homogeneity of the conclusion.

Problem 4.6. *A square with center O is inscribed in $\triangle ABC$ so that the vertices of the square lie on the sides of the triangle and two of them are on the side AB. If $\sphericalangle ACO = \sphericalangle BCO$, then $\triangle ABC$ is either isosceles with $CA = CB$ or not isosceles but right-angled with $\sphericalangle ACB = 90^0$.*

4.2. PROBLEMS OF GROUP II

By formulating appropriate statements and giving suitable logical models we get two *generating* problems that are needed for the construction of Basic problem 4.9. The basic statements we use are:

$t :=$ {*In $\triangle ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, are the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$, respectively.*}

$p := \{\sphericalangle ACB = 60^0\}$

$q := \{\sphericalangle CAB = 120^0\}$

$r := \{\sphericalangle BB_1A_1 = 30^0\}$

Since the sum of the angles of any triangle is equal to 180^0, statements p and q are mutually exclusive. Hence, if p is true, so is $\neg q$ and vice versa.

We describe the logical scheme for the composition of Basic problem 4.9, which has exclusive disjunction as a logical structure in the conclusion:

- First we formulate (and prove) two *generating* problems - Problem 4.7 with a logical structure $t \wedge p \to r$ and Problem 4.8 with a logical structure $t \wedge q \to r$.

- Since statements p and q are mutually exclusive, the equivalences $p \wedge \neg q \Leftrightarrow p$ and $\neg p \wedge q \Leftrightarrow q$ are true. As a consequence of these facts problems with logical structures $t \wedge p \rightarrow r$ and $t \wedge (p \wedge \neg q) \rightarrow r$ are equivalent. So are the problems with logical structures $t \wedge q \rightarrow r$ and $t \wedge (q \wedge \neg p) \rightarrow r$.

  To generate problems with a logical structure $(**)$ $t \wedge (p \veebar q) \rightarrow r$ we use the logical equivalence

$$(t \wedge (p \wedge \neg q) \rightarrow r) \wedge (t \wedge (\neg p \wedge q) \rightarrow r) \quad \Leftrightarrow \quad t \wedge (p \veebar q) \rightarrow r.$$

- Finally, the formulated *inverse* problem - the Basic problem 4.9 - to the problem with structure $(**)$ has the logical structure $t \wedge r \rightarrow p \veebar q$.

Problem 4.7. *In $\triangle ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, are the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$, respectively. If $\sphericalangle ACB = 60^0$, prove that $\sphericalangle BB_1A_1 = 30^0$.*

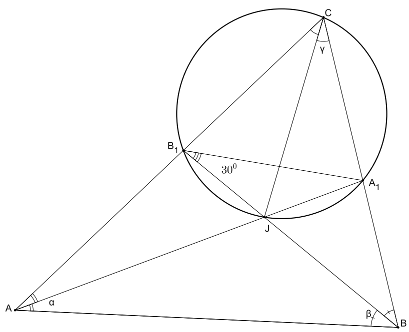


Fig. 7.

Proof. Let $\sphericalangle BAA_1 = \sphericalangle CAA_1 = \alpha$, $\sphericalangle ABB_1 = \sphericalangle CBB_1 = \beta$, $J = AA_1 \cap BB_1$. Since J is the intersection point of the angle bisectors of $\triangle ABC$, we have that $\sphericalangle JCA = \sphericalangle JCB = \gamma = 30^0$ (Fig. 7).

From $\alpha + \beta + \gamma = 90^0$ we find that $\sphericalangle AJB = 120^0$. Hence, the quadrilateral CA_1JB_1 can be inscribed in a circle. Then $\sphericalangle JA_1B_1 = \sphericalangle JCB_1 = 30^0$ and $\sphericalangle JB_1A_1 = \sphericalangle JCA_1 = 30^0$ as angles corresponding to the same segment of this circle. □

Problem 4.8. *In $\triangle ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, are the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$ respectively. If $\sphericalangle BAC = 120^0$, prove that $\sphericalangle BB_1A_1 = 30^0$.*

Proof. Let $J = AA_1 \cap BB_1$, $E = A_1B_1 \cap CJ$, $C_1 = CJ \cap AB$. Since $\sphericalangle BAC = 120^0$, its adjacent angles have a measure of 60^0. It is easily seen that the

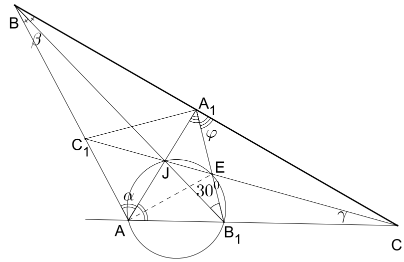


Fig. 8.

point B_1 is equidistant from the straight lines BA, BC, AA_1 and that the straight line A_1B_1 is the bisector of $\sphericalangle CA_1A$ (Fig. 8). The proof that the straight line A_1C_1 is the bisector of $\sphericalangle BA_1A$ is analogous. It follows that $\sphericalangle B_1A_1C_1$ is a right angle (the bisectors of any two adjacent angles are perpendicular to each other) (see also [2], p. 194, Problem 156).

As a consequence we get that E is the intersection point of the angle bisectors CJ and A_1B_1 of $\triangle AA_1C$ and hence $\sphericalangle JAE = \sphericalangle EAB_1 = 30^0$.

Let $\varphi = \sphericalangle CA_1B_1 = \sphericalangle B_1A_1A$ and $\gamma = \sphericalangle C_1CA = \sphericalangle C_1CB$. Then $\sphericalangle A_1B_1C = 60^0 + \varphi$ as an exterior angle of $\triangle A_1B_1A$, the sum of the angles of $\triangle AA_1C$ is $60^0 + 2\varphi + 2\gamma = 180^0$, i. e. $\varphi + \gamma = 60^0$ and hence $\sphericalangle JEB_1 = 120^0$.

Thus, the quadrilateral $AJEB_1$ can be inscribed in a circle. We conclude that $\sphericalangle JAE = \sphericalangle JB_1E = 30^0$ as angles in the same segment of this circle. Hence, $\sphericalangle BB_1A_1 = 30^0$. □

Now we formulate and prove the *Basic problem* in this group.

Basic problem 4.9. *In $\triangle ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, are the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$ respectively. If $\sphericalangle BB_1A_1 = 30^0$, prove that either $\sphericalangle ACB = 60^0$ or $\sphericalangle BAC = 120^0$.*

Proof. Let us denote $\sphericalangle BAA_1 = \sphericalangle CAA_1 = \alpha$, $\sphericalangle ABB_1 = \sphericalangle CBB_1 = \beta$, $AA_1 \cap BB_1 = J$. Since J is the intersection point of the angle bisectors of $\triangle ABC$, then the straight line CJ is the bisector of $\sphericalangle ACB$. Denoting $\gamma = \sphericalangle JCA = \sphericalangle JCB$ we get $\alpha + \beta + \gamma = 90^0$ (Fig. 9). Let the point A' be orthogonally symmetric to the point A_1 with respect to the axis BB_1. It follows that $A' \neq A$. (If $A' \equiv A$ then $\triangle ABC$ does not exist.) The straight line BB_1 is the bisector of $\sphericalangle ABC$ and consequently $A' \in AB$ and $B_1A_1 = B_1A'$. On the other hand, $\sphericalangle BB_1A_1 = 30^0$ and hence $\triangle A_1B_1A'$ is equilateral.

We find $\sphericalangle AA'B_1 = 30^0 + \beta$ (as an exterior angle of $\triangle A'BB_1$), $\sphericalangle AA'A_1 = 90^0 + \beta$ (as an exterior angle of $\triangle A'BE$), $\sphericalangle AB_1A' = 60^0 + \gamma - \alpha$ and $\sphericalangle AB_1A_1 = 120^0 + \gamma - \alpha$.

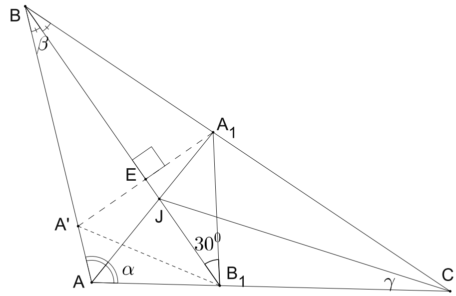


Fig. 9.

Let us compare $\triangle AA_1B_1$ and $\triangle AA_1A'$. They have a common side AA_1, equal corresponding sides $A_1B_1 = A_1A'$ and angles $\sphericalangle B_1AA_1 = \sphericalangle A'AA_1 = \alpha$. Hence Theorem 2.4 is applicable to $\triangle AA_1B_1$ and $\triangle AA_1A'$. We have two possibilities:

(i) $\triangle AA_1B_1$ and $\triangle AA_1A'$ are congruent. Then $\sphericalangle AB_1A_1 = \sphericalangle AA'A_1$, i. e. $120^0 + \gamma - \alpha = 90^0 + \beta$. Hence, $2\gamma = \sphericalangle ACB = 60^0$.

(ii) $\triangle AA_1B_1$ and $\triangle AA_1A'$ are not congruent. By Theorem 2.4 it follows that $\sphericalangle AB_1A_1 + \sphericalangle AA'A_1 = 180^0$, i. e. $(120^0 + \gamma - \alpha) + (90^0 + \beta) = 180^0$. Hence, $2\alpha = \sphericalangle BAC = 120^0$. □

Remark 4.10. An alternative version of Problem 4.9 is Problem 6 in [6].

To formulate a special type equivalent problem (see also [4]) to this Basic problem we need

Proposition 4.11. *If the statements p and q are mutually exclusive, then the following equivalences are true:*

$$(\neg(p \veebar q)) \;\Leftrightarrow\; (p \vee \neg q) \wedge (\neg p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q.$$

Proof. We have

$$(\neg(p \veebar q)) \;\Leftrightarrow\; \neg((p \wedge \neg q) \vee (\neg p \wedge q))$$

$$\Leftrightarrow\; (p \vee \neg q) \wedge (\neg p \vee q) \;\Leftrightarrow\; p \wedge (\neg p \vee q) \vee \neg q \wedge (\neg p \vee q)$$

$$\Leftrightarrow\; (p \wedge \neg p) \vee (p \wedge q) \vee (\neg q \wedge \neg p) \vee (q \wedge \neg q) \;\Leftrightarrow\; \neg p \wedge \neg q.$$

□

By Proposition 4.11, problems with logical structures $t \wedge (\neg(p \veebar q)) \;\to\; \neg r$ and $t \wedge (\neg p \wedge \neg q) \;\to\; \neg r$ are equivalent.

The following problem is equivalent to Basic problem 4.9.

Problem 4.12. *In* $\triangle ABC$ *the straight lines* AA_1, $A_1 \in BC$, *and* BB_1, $B_1 \in AC$, *are the bisectors of* $\sphericalangle CAB$ *and* $\sphericalangle CBA$, *respectively. If* $\sphericalangle ACB \neq 60^0$ *and* $\sphericalangle CAB \neq 120^0$, *prove that* $\sphericalangle BB_1A_1 \neq 30^0$.

Proof. Assuming that the opposite statement is true, i.e., $\sphericalangle BB_1A_1 = 30^0$, we would get a contradiction to Basic problem 4.9. □

ACKNOWLEDGEMENTS. The first-named author is partially supported by the Sofia University Research Fund through Grant 99/2013. The second-named author is partially supported by the Sofia University Research Fund through Grant 159/2013.

5. REFERENES

1. Kolarov, K., Lesov H.: *Sbornik ot zadachi po geometria, VIII-XII klass*, Integral, Dobritsh, 2000 (in Bulgarian).
2. Modenov, P. S.: *Sbornik zadach special'nomu kursu elementarnoj matematiki*, Sovetskaja nauka, Moscow, 1957 (in Russian).
3. Ninova, J., Mihova, V.: Equivalence problems. In: *Mathematics and Education in Mathematics, Proceedings of the Forty Second Spring Conference of the Union of Bulgarian Mathematicians, Borovetz, April 2-6, 2013*, 424–429.
4. Ninova, J., Mihova, V.: Composition of inverse problems with a given logical structure. *Ann. Sofia Univ., Fac. Math. Inf.*, **101**, 167–181.
5. Tvorcheskij konkurs uchitelej, Zadacha 12, 2004, `http://www.mccme.ru/oluch/`
6. Sharigin, I. F.: Tursete variantite. O*buchenie po matematika i informatika*, **1**, 5–12, **2**, 21–30, 1988 (in Bulgarian).

Received on March 15, 2014

Vesselka Mihova, Julia Ninova

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., BG-1164 Sofia
BULGARIA

e-mails: mihova@fmi.uni-sofia.bg
julianinova@hotmail.com

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

ACCURACY IMPROVEMENT BY NEW SENSOR SYSTEM FOR AUTOMATIC BONE DRILLING IN THE ORTHOPEDIC SURGERY

VL. KOTEV, G. BOIADJIEV, K. DELCHEV, T. BOIADJIEV, K. ZAGURSKI,
H. KAWASAKI, T. MOURI

Many orthopaedic operations involve drilling before the insertion of implants into the bones. Usually drilling is executed manually, which may cause some problems. In free hand performance of drilling some errors such as an inaccurate penetration and dilate of bone hole, overheating, harm soft tissues could be occurring. Automatic drilling is recommended to avoid such problems and reduce the subjective factor. The aim of this paper is to select, develop and test a new sensor system for a bone drilling robotized system. More in particular we utilize a sensor to measure thrust force during the bone drilling manipulation execution. Therefore a force sensor is fixed to the drilling robotized system. Moreover, an experimental identification of the drilling technical parameters such as bone resistant force and feed rates are done. The resistant forces are measured and plotted. The control algorithms and programs for drilling have done based on the experiments.

Keywords: Automatic bone drilling, sensor system, experiments, orthopedic surgery

1. INTRODUCTION

In the orthopaedic surgery many interventions involved freehand bone drilling procedures. Total knee (TKR) and hip (THR) replacement are ones of the most frequent performed orthopaedic operations [1-5]. In the both operations surgeons have to perform drilling manipulations in order to insert implant components into bones. Late detection of bone/soft tissue breakthroughs can cause unnecessary damage to the patient [1-12]. In manual operations, breakthrough detection is based on surgery's skills and visual inspections of the drill tip using imagining devices like x-rays [1-4, 6, 8-10, 12]. However, frequently exposes of x-rays is not useful for both surgeon and patient [1-4, 6]. The breakthrough detection based on thrust force measurement on the drill bit could be reduced or eliminated the need of x-ray imaging [1, 3, 6, 8, 9]. The successful execution of bone drilling requires a high level of precision, dexterity and experience [1-10, 12-15] because the drilling resistance is large and sometimes vibrates violently to difficultly grasp the hand-piece or even break the slender drill. Relatively large forces experienced during bone drilling pose significant challenges to effective application of bone drilling [7-15]. Drill bit breakage occurs frequently, and since the broken drill could obstruct placement of other devices and cause adverse histological effects due to corrosive reactions with the surrounding soft tissue, commonly necessitates follow-on procedures for removal of the broken drill bit [6, 8-11]. Generally, the increased torque during drilling induces shear stresses that exceed the strength of the drill bit, causing it to fracture [8, 11]. Similarly, uncontrolled or unpredictable bone drilling forces may result in drill breakthrough, causing considerable damage to surrounding tissue [4, 6-13]. Furthermore, drilling forces are the main source of heat generation during bone drilling [3, 4, 11-13]. Increased temperatures on the bone could induce thermo necrosis, and therefore, significant trauma to the bone tissue [2-4, 10-13].

The results show the automatized bone drilling manipulators or robots improve the quality of the drilling procedures [1-6, 10, 12, 13]. Moreover, the utilizing of the mechatronic drilling tools and robots will reduce/eliminate the need for X-rays imaging used in traditional bone fixation [4-8, 10, 12, 13]. In addition, there are several studies which refer to measurement of thrust force, feed rate, and detected breakthrough [1, 4, 6-8, 11-15]. It is will know that computer assisted surgery (CAS) and robots extremely decrease errors and time for orthopaedic surgery operations [1-5]. Usually, orthopaedic robot-assisted drilling systems consist of two modules first-one is executive drilling module and second one is assistant robot (manipulator) [1-3]. These days CAS robotized systems like Da Vinci and The RIO Robotic Arm of MAKO have been installed in many hospitals and performed many operations successfully [1, 2]. Unlike of big and expensive robots with high degree of freedom (DOF) and master slave systems [1-3], a small sizes, cost effective with special purpose robots and intelligent tools have been developing most recently [1, 2, 5, 15]. A miniature orthopaedic robot MARS with parallel structure is developed [1, 2, 5]. Praxiteles is a bone mounted guide positioning robot for TKA operation [4]. In order to remove the subjective factor and avoid the problems in hand bone drilling manipulations, the robot DORO (Drilling Orthopaedic RObot) has been created [11- 13]. Orthopaedic Drilling Robot (ODRO) has been developed latter [14-16]. This robot is intended to increase the patients safety in view point of it is accuracy, performance and sterilization. At the same time it has to be affordable for hospitals (low cost) and user friendly. ODRO can monitor time, linear velocity, angular velocity, resistant force, depth of penetration and temperature during the drilling process as well as bone breakthrough [11-13]. ODRO has own control/power block meets medical requirements. The aim of the present study is to select, develop and test a new sensor system for a bone drilling robotized system in order to increase accuracy and develop the control algorithms of it. First a small-sized compression load cell is selected to measure the thrust force in bone drilling procedure. Second we have designed a box to attach the load cell into a bone drilling robotized module. Third experiments on a pork bones are made to measure thrust force.

2. AN AUTOMATIC BONE DRILLING SYSTEM

During orthopedic surgery, a primary concern is to penetrate the bone tissue without causing mechanical and thermal damage. Therefore, without careful attention to the thermal and mechanical issues, bone drilling could impart considerable damage to the musculoskeletal system, reducing effectiveness of the surgical operation and increasing the post-operation recovery time. We are working on development of a hand-held robotized system for bone drilling procedures (Fig.1) to avoid the mentioned above problems. It is intended to perform drilling with preliminary setting of depth and stop automatically after the cutting process is completed. Drilling conditions would be changed automatically in accordance with bone density.

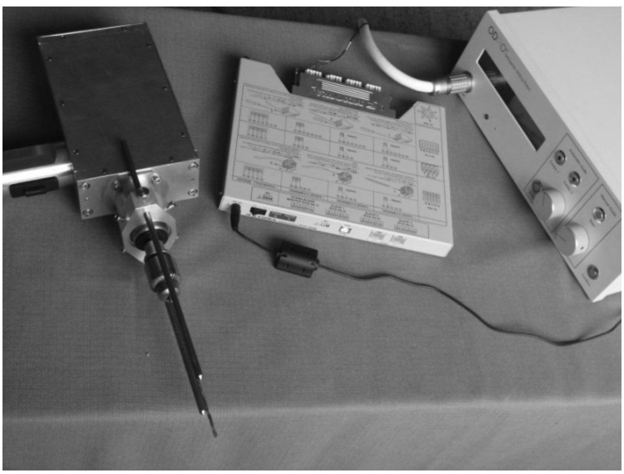


Fig. 1. A bone drilling experimental set up.

On the Fig. 2a are shown the executive drilling module and the control system of the experimental set up. In order to decrease length and increase of working zone of the executive drilling module we suggest the axis of motors to be parallel [17], unlike of these of DORO [12, 13] and ODRO [14-16]. Regards to the parallel structure the working zone becomes 121 mm, the length becomes 220 mm and height becomes 110 mm. The developed bone drilling experimental set up has the following basic components:

- Brushless DC motor MAXON [18]. The motor is equipped by servo controller/driver 1-Q-CE and amplifier DEC 50-5 [19].
- Linear motor 43000-17 [20]. It is a stepper motor with embedded screw for linear motion.
- Small and lightweight LMB-A force sensor (Fig.2.b) to measure thrust force [21].
- PCD-300 Series Sensor Interfaces [22].

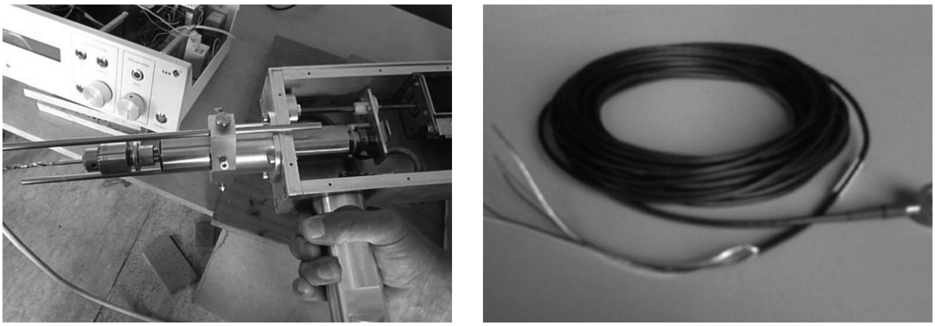


Fig. 2. The experimental bone drilling set up (a). The Kyowa's force sensor LMB-A (b).

In order to increase the accuracy of the bone drilling set up based on the experiments, date of literature, and companies' catalogues we have selected a force sensor LMB-A made by Kyowa (Fig.2b) to measure thrust force. It is a compact, lightweight, and low price load cell [21]. Moreover, to measure the thrust force more precisely during the drilling procedure execution the force sensor LMB-A is connected to the Kyowa's sensor interface PCD-300 series [22]. It is shown on the Fig.1. on the middle. The sensor interface PCD-300 series is a measuring instrument that can easily carry out measurements simply by connecting to a PC using a USB interface. We have designed and manufactured a box (Fig.2.a red arrow) for the LMB-A cell load in order to attach it to the moving part of the experimental drilling set up.

Control system of the experimental set up gives information about the drilling process execution in real time, for successful end of the task. The control block has terminals for connection with PC. They give a possibility to re-program the software, which is recorded in the controllers. Controllers can change and update the programs and to transfer the information between the sensors and PC while the drilling is executed in real time.

3. EXPERIMENTS

Bone is an in homogenous and anisotropic material, consisting two different types of bones: cortical and cancellous bone respectively. These two types of bone tissue differ in density, or tightness of the packed tissue. In a long bone such as tibia or femur, the outer shell is the cortical bone, and the inner layer is cancellous bone. Cortical and cancellous bone comprise the diaphysis of long bones and the thin shell that surrounds the metaphyses. In addition, cancellous bone is the metaphyses and epiphyses. The outside of the bones consist of a layer of connective tissue called the periosteum. The interior part of the long bone is the medullary cavity with the inner core of the bone cavity being composed of yellow marrow in adults [1, 4, 11]. The inhomogeneous structure of human bone, including a cortical (dense) portion at the outer part, followed by a cancellous (highly porous) portion and bone marrow, brings considerable complexities to application of bone drilling. The structure of bone varies between different bones (e g., femur vs. vertebra), between person to person, and between different age groups [6, 8, 11].

3.1. BONE DRILLING EXPERIMENTS. DETERMINATION OF THRUST FORCE IN DRILLING

The experiments were carried out under the following conditions: object of drilling - a pork bone; diameter of the orthopaedic drill - 4 mm; depth in bone drilling of tubular bones - 10 mm; depth of bone drilling in sponge-like bones - 20 mm; data reading - every 100 ms; velocity of drilling - 6 mm/s. Some of the obtained results are illustrated on the charts in Fig. 3 and Fig. 4.

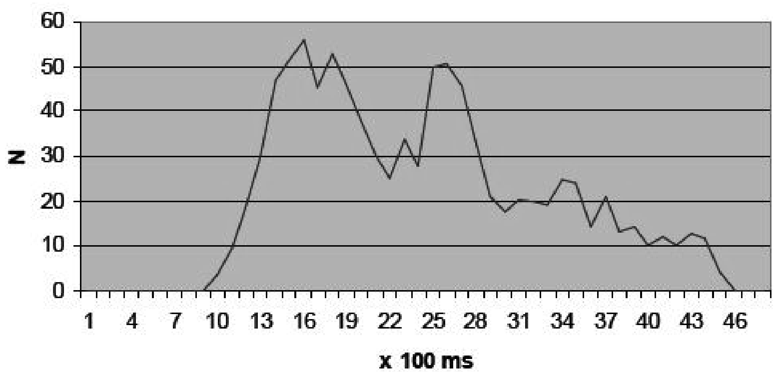


Fig. 3. Thrust force of drilling of cancellous bone.

It can be seen from the given results that in sponge-like bone drilling the resistance force varies within 30-55 N, while in tubular bone drilling the resistance force reaches up to 90-100 N, i.e. for one and the same bone depending on its structure the resistance force varies from 30 to 100 N. This means that during the performance of the operation the force of pressure should be consistent with the specific object and must be controlled accordingly.

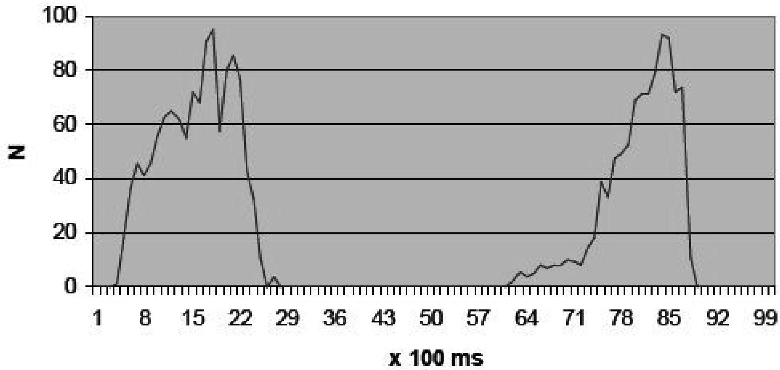


Fig. 4. Thrust force of drilling of cortical bone.

3.2. RESULTS

Specific drilling effects are revealed during the experiments. The thrust force is achieved by controlled automatic bone drilling regime in comparison with hand-drilling one. Comparison of new sensor system, implemented in the robot, with the old one is done and its better functional abilities are shown. Algorithms are created and their software realization is made. Curves of resistant force with respect of the time are presented.

4. CONCLUSIONS

Automatic bone drilling can solve the problems which arise during manual drilling. An experimental setup is designed to identify some parameters of bone drilling such as the resistant force due to variable bone density, the appropriate mechanical torque of drilling, the linear speed of the drill, and the electromechanical characteristics of motors, drives and corresponding controllers. The last leads to main conclusion that the automatic drilling guarantees higher safety for the patient. This will reduce/eliminate the need for X-rays imaging used in traditional bone fixation. The result has shown that, the bone drilling operation can be handled by a robot manipulator to improve the quality of the drilling operation. With this system, the bone breakthrough can be easily detected and further damage of the healthy patient tissue would be avoided.

ACKNOWLEDGEMENTS. Authors express their gratitude to Japan Society for the Promotion of Science (JSPS) for the support.

5. REFERENCES

1. Rosen, J., Hannaford, B., Satava, R.: *Surgical Robotics*. Springer Science+Business Media, LLC 2011.
2. Gomes, P.: Surgical robotics: Reviewing the past, analysing the present, imagining the future. *Robotics and Computer-Integrated Manufacturing*, **27**, 2011, 261–266.
3. Sugano, N.: Computer-assisted orthopedic surgery, *Journal of Orthopaedic Science*, **8**, 2003, 442–448.
4. Plaskos, C., Cinquin, P., Lavallé, S., Hodgson, A.: Praxiteles: a miniature bone-mounted robot for minimal access total knee arthroplasty. *Int. J. Medical Robotics and Computer Assisted Surgery*, **1**, no. 4, 2005, 67–79.
5. Shoham, M., Burman, M., Zehavi, E., Joskowicz, L., Batkilin, E., Kunicher, Y.: Bone-Mounted Miniature Robot for Surgical Procedures: Concept and Clinical Applications, *IEEE Transactions on Robotics and Automation*, **19**, no. 5, 2003, 893–901.
6. Allotta B., Giacalone, G., Rinaldi, L.: A hand-held drilling tool for orthopaedic surgery. *IEEE Trans. on Mechatronics*, **2**, no 4, 1997, 218–229.
7. Lee, W., Shih, C.–L.: Control and breakthrough detection of a three-axis robotic bone drilling system. *Mechatronics*, **16**, 2005, 73–84.
8. Taha, Z., Salah, A., Lee, J.: Bone Breakthrough Detection for Orthopaedic Robot-Assisted Surgery. In: *APIEMS 2008 Proc. of the 9th Asia Pasific Industrial Engineering & Management Systems Conf.*, 2008, 2742–2746.
9. Tsai, M., Hsieh, M., Tsai, C.: Bone drilling haptic interaction for orthopaedic surgical simulator. *Computers in Biology and Medicine*, **37**, 2007, 1709–1718.
10. Kasahara, Y., Kawana, H., Usuda, S., Ohnishi, K.: Telerobotic - assisted bone-drilling system using bilateral control with feed operation scaling and cutting force scaling. *Int J Med Robotics Comput Assist Surgery*, **8**, 2012, 221–229.
11. Lee, J., Gozen, B., Ozdoganlar, O.: Modelling and experimentation of bone drilling forces. *Journal of Biomechanics*, **45**, no. 6, 2012, 1076–1083.
12. Boiadjiev, G., Boiadjiev, T., Vitkov, Vl., Delchev, K., Kastelov, R., Zagurski, K.: Robotized System for Automation of the Drilling in the Orthopedic Surgery. Control Algorithms and Experimental Results. In: *Proceedings of the 9th IFAC Symp. on Robot Control SYROCO'09, Gifu, Japan, 2009*, 633–638.
13. Boiadjiev, T., Zagurski, K., Boiadjiev, G., Delchev, K., Vitkov, Vl., Veneva, I., Kastelov, R.: Identification of the Bone Structure during the Automatic Drilling in the Orthopaedic surgery. *J. Mechanics Based Design of Structures and Machines*, **39**, 2011, 285–302.

14. Boiadjiev, G., Zagurski, K., Boiadjiev, T., Delchev, K., Kastelov, R., Kotev, Vl.: Robot application in orthopedic surgery: drilling control. *GSTF Journal of Engineering Technology*, **1**, no. 1, 2012, 125–130.
15. Boiadjiev G., Kastelov, R., Boiadjiev, T., Kotev, Vl., Delchev, K., Zagurski, K., Vitkov, Vl.: Design and performance study of an orthopedic surgery robotized module for automatic bone drilling. *International Journal of Medical Robotics and Computer Assisted Surgery*, **9**, no 4, 2013, 455–463.
16. Kotev, Vl., Boiadjiev, G., Kawasaki, H., Mouri, T., Delchev, K., Boiadjiev, T.: Design of a hand-held robotized module for bone drilling and cutting in orthopedic surgery. *2012 IEEE/SICE International Symposium on System Integration (SII), Kyushu University, Fukuoka, Japan*, 2012, 504–509.
17. Kotev, Vl., Boiadjiev, G., Mouri, T., Delchev, K., Kawasaki, H., Boiadjiev, T.: A Design Concept of an Orthopaedic Bone Drilling Mechatronics System. *Applied Mechanics and Materials*, vol.: *Advanced Engineering and Materials*, 2013, 248–252.
18. Maxon, motor catalogue 2012/13, p. 177.
19. Maxon, motor catalogue 2012/13, p. 239.
20. `http://www.haydonkerk.com/LinearMotionProducts/StepperMotorLinearActuators/tabid/66/Default.aspx`
21. `http://www.kyowa-ei.co.jp/eng/product/sensors/loadcell/lmb_a.html`
22. `http://www.kyowa-ei.com/eng/product/category/acquisition/s_pcd-300_series/index.html`

Received on 23 April, 2015

Vl. Kotev, K. Delchev
Institute of Mechanics
Bulgarian Academy of Sciences
Acad. G. Bonchev St., bl. 4
1113 Sofia
BULGARIA
e-mails: vladokotev@abv.bg
kamen@imbm.bas.bg

G. Boiadjiev
Faculty of Mathematics and Informatics
Information Technology and Analytical Mechanics Dept.
"St. Kl. Ohridski" University of Sofia
5 J. Bourchier Blvd., BG-1164 Sofia
BULGARIA
e-mail: george@fmi.uni-sofia.bg

T. Boiadjiev

Institute of Information and Communication Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev St., bl. 2
1113 Sofia
BULGARIA

e-mail: tonyboiadjiev@mail.bg

K. Zagurski

Institute of System Engineering and Robotics
Bulgarian Academy of Sciences
Acad. G. Bonchev St., bl. 2
1113 Sofia
BULGARIA

e-mail: kazz@gmail.com

H. Kawasaki, T. Mouri

Gifu University
Faculty of Engineering
Department of Human and Information Systems
JAPAN

e-mails: h_kawasa@gifu-u.ac.jp
mouri@gifu-u.ac.jp

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

CLASSIFICATION OF DIFFERENT TYPES BEE HONEY ACCORDING TO PHYSICO-CHEMICAL CHARACTERISTICS

KRASTENA T. NIKOLOVA, DOYCHIN T. BOYADZHIEV

The physicochemical parameters (refractive index, water content, β-carotene, color parameters and content of glucose, fructose, sucrose and oligosaccharides for 14 types of bee honey have been investigated. They are grouping according to the following parameters:

1. Geographic region 1 – valley-mountain
2. Geographic region2 – North or South Bulgaria
3. Year of producing – 2008 or 2009
4. Botanical origin – honeydew, multiflorous, sunflower, lime.

Analysis of the data gives the opportunity for characterizing the samples of bee honey by using discriminant analysis. The models correctly present geographic region, year of producing, botanical origin and it can be used for determining the type of unidentified samples.

Keywords: Bee honey, physicochemical properties, discriminant analysis, mathematical modeling

2000 Math. Subject Classification: 62P30

1. INTRODUCTION

Bee honey contains a variety of different sugars, more than 180 ingredients such as enzymes, organic acids, vitamins, minerals, polyphenols, carotenoids, antioxidants, flavonoids, etc [1, 2, 3]. As is well known, one of the parameters to estimate the quality of honey is the contents of sugars. The most common glucose, fructose and saccharose are contained in honey in proportions as follows: 31.3%, 38% and 8% [4]. The variety of components of bee honey are an important criterion for the quality and mark some particular features of the corresponding sample. A number of authors have sought to identify the most significant parameters in order to classify bee honey. Models to classify citrus and eucalyptus honey by studying the water content, electric conductivity, pH factor, contents of glucose, saccharose and fructose have been proposed [5]. Color coordinates x, y and lumineance L are of essential significance for the classification of 15 types of Spanish honey – forest, lavender, eucalyptus, rosemary, citrus etc. [6]. Data about step-by-step discriminatory analysis, principal factor analysis for Spanish, Italian, Iranian and African honeys have been reported [7, 8]. There is comparatively little data on Bulgarian honeys such as multi-flower, acacia, lime, sunflower, forest honeydew.

The objective of this work is to test discriminatory models using the analyzed indicators to discern the geographic origin (field, mountain or Northern–Southern Bulgaria), year of production (2008–2009) and botanic origin (multiflorous or sunflower).

The objective defined requires the solution of the following problems:

- Creation of database, including types of bee honey of different botanic origin and region of cultivation;
- Determination of physical-chemical parameters (color coordinates a^* and b^*, x and y in two colorimetric systems SIE Lab and XYZ, correspondingly, luminance L^*, content of pigments such as β-carotene and chlorophyll, water content, index of refraction, sugar content).
- Establishing of significant differences in the parameters under study.
- Modeling and analysis of the groups by types of honey, yield, and regional origin.
- Test of the obtained model by using independent samples.

2. MATERIALS AND METHODS

2.1. SAMPLES

The basic data includes 14 types of bee honey of field and mountain regions in Northern and Southern Bulgaria. The samples were purchased from producers and suppliers, from two years – 2008 and 2009. Four samples of multi-floral honey with commercially available sweeteners were used to test the models.

2.2. METHODS

The color parameters of two different colorimetric systems – XYZ (aimed at large color differences) and CIE Lab (aimed at small color differences) [9] are measured. A colorimeter Lovibond PFX 880 (UK) and a cuvette with a 10 mm thickness are used. To determine the water content the refractive index is measured using an Abbe refractometer (Carl Zeiss, Germany) at $20 \pm 0.5\,°C$ The equivalent water content is determined from a table, given in Official Methods of Analysis [10]. After the honey solution is filters, the sugar content is determined using liquid chromatography with an IR detector (Waters). The parameters of the methods are: column Aminex HPX-87H; detector Differential refractometer R401, (Waters); temperature of the column and the detector is 30°C, the volume of the sample injected was $10\,l$, speed – $0.5\,ml/min$. The software "Statistica" to process the data was used. Their distribution is normal according to the Kolmogorov–Smirnov criterion [11, 12]. To establish the statistically significant differences between the indicators for the different sorts Tukey criterion for multiple comparisons was applied [13].

Discriminatory analysis is used to model the group with a priori equal probabilities to fall into the groups [14].

3. RESULTS AND DISCUSSION

The database includes 14 types of bee honey from different regions (valley-mountain or southern-northern Bulgaria). For each of the samples studied, four independent measurements have been performed. The Scheffe criterion shows significant statistical differences in the studied types of honey. The presence of considerable difference in the physicochemical characteristics of honey provides the reason for a subsequent modeling of its origin. To model the honeys by region valley-mountain a step-by-step linear discriminatory analysis was used.

A model with grouping parameter "extraction area" was obtained and it includes the following parameters by the order of introduction into the model: x, oligosaccharides and refractive index. The classification of the different sorts according to the extraction area is 100% (Table 1).

TABLE 1. Classification of the samples by the model Valley–Mountain

Group	Percent correct	Mountain $p = 0.357$	Valley $p = 0.643$
Mountain	100	20	0
Valley	100	0	36
Total	100	20	36

It has been attempted to discern the samples by the geographical region with a grouping variable Geographic area 2: Northern-Southern Bulgaria. With a classifying parameter "Geographical area 2" (Northern or Southern Bulgaria) we observe a

76.8% correct classification, with all samples from two types from southern Bulgaria and one from Northern Bulgaria were incorrectly classified.

Except by region, the samples are subdivided by the year of extraction. More parameters are included in the new model – the color coordinates, luminance and beta carotene. The presence of more variables in the model is easy to explain because the content of sugars is decisive for the crystallization of honey, while the refractive index (water content) – for the development of microorganisms in the product. The indicated parameters are related to the kinetics of the process in the bee honey during storage. The classifying parameter (Year of production) (2008 and 2009) 94.64% of the samples are recognizable, of them only three fall into a wrong group (Table 2).

TABLE 2. Classification of the samples by the year of production

Group	Percent correct	2008 $p = 0.286$	2009 $p = 0.714$
2008	87.50	14	2
2009	97.50	1	39
Total	94.64	15	41

With a classifying parameter "Botanical origin" two models are possible: with color parameters included and physicochemical indicators arranged as luminance, parameter a, saccharose, or only with color parameters: y, L, a, b and refractive index.

TABLE 3. Modeling of botanical origin by color and physicochemical indicators

Group	Percent correct	Honeydew p=0.273	Multi-floral p=0.455	Sunflower p=0.273
Honeydew	100.00	12	0	0
Multi-floral	80.00	0	16	4
Sunflower	100.00	0	0	12
Total	90.91	12	16	16

For the classifying parameter "Botanical origin" both model have the same sample identification capability of 90.91%, with four samples in the first model move from multi-floral group into the sunflower, while in the second model three multi-floral samples (wrongly identified in the first model as well) go into the sunflower group, while one sunflower sample was identified as multi-floral. The results are presented in Tables 3 and 4.

TABLE 4. Modeling the botanical origin by color indicators

Group	Percent correct	Honeydew p=0.273	Multi-floral p=0.455	Sunflower p=0.273
Honeydew	100.00	12	0	0
Multi-floral	85.00	0	17	3
Sunflower	91.67	0	1	11
Total	90.91	12	18	14

For a better visualization of the results a subsequent canonical analysis was performed. On the basis of the first two canonical variables, the position of the separate samples for the model with included color parameters and physicochemical indicators is presented in Fig. 1, the four wrongly identified samples being marked.

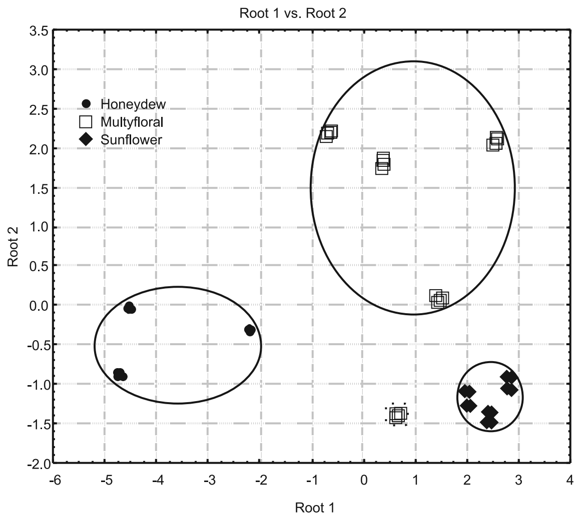


Fig. 1. Disposition of the three sorts of honey in the plane of the first two canonical variables

The figure confirms the stated hypothesis for the presence if significant differences between the separate types of honey. The analysis of the Mahalanobis distances between the three basic groups shows that the sunflower and the multi-floral honeys are close to each other and are relatively far from the honeydew. This is clearly seen from the figure shown if we trace the projections of the clouds of the various sorts upon the first canonical variable which plays an important role in discrimination of the groups – the sunflower and multi-floral are projected on the positive, while honeydew is projected on the negative direction.

The samples used are for the control of the adequacy of the created model for the description of the botanical origin of honey. From the remaining three types of honey with a known botanical origin, the samples which are insufficient to form separate groups two-lime and acacia can be classified as multi-floral, while that of thistle – as sunflower. This can be explained with the different seasons during which they are collected – the former two in the spring while the third in the summer. Honeydew is classified correctly, lime honey and acacia honey are in the group of multi-floral honey. The results from the classification according to the obtained models are presented in Table 5.

TABLE 5. Verification of the model for botanical origin with independent samples

Sort	Classified as
Lime	Polyfloral
Acacia	Polyfloral
Honeydew	Honeydew
Thistle	Sunflower
With sweetener	Polyfloral, Honeydew

4. CONCLUSIONS

The analysis of the data base give the opportunity to characterize different types of bee honey by using the discriminant analysis. It provides an efficient tool for the qualitative distinction of natural bee honey and adulterated honey containing admixtures from sugar or glucose. The models and the associated Mahalanobis distances enable the classification of unknown samples or samples with admixtures.

5. REFERENCES

1. Gheldorf, N., Wang, X. H., Engeseth, N. J.: Identification and quantification of antioxidant components of honey from various floral courses. *Journal of Agricultural and Food Chemistry*, **50**, 2002, 5870–5877.
2. Aljadi, A. M., Kamaruddin, M. Y.: Evaluation of the phenolic content and antioxidant capacities of two Malaysian floral honey. *Food Chemistry*, **85**, no. 4, 2004, 513–518.
3. Al-Mamary, Al-Meeri, Al-Habori: Antioxidant activities and total phenolics of different types of honey. *Nutrition research*, **22**, no. 9, 2002, 1041–1047.
4. Al-Khalifa, Al-Arify: Physicochemical characteristics and pollen spectrum of some Saudi honeys. *Food Chemistry*, **67**, 1999, 21–25.
5. Serrano, S., Villarejo, M., Espejo, R., Jodral, M.: Chemical and physical parameters of Andalusian honey: Classification of Citrus and Eucalyptus honeys by discriminant analysis. *Food Chemistry*, **87**, 2004, 619–625.
6. Mateo, R., Bosch-Reig, F.: Classification of Spanish Unifloral Honeys by Discriminant Analysis of Electrical Conductivity, Color, Water Content, Sugars, and pH. *J. Agric. Food Chem.*, **46**, no. 2, 1998, 393–400.
7. Baroni, M. V., Chiabrando, G. A., Costa, C., Wunderlin, D. A.: Assessment of the Floral Origin of Honey by SDS-Page Immunoblot Techniques. *J. Agric. Food Chem.*, **50**, no. 6, 2002, 1362–1367.
8. Iglesias, T., Martín-Álvarez, P. J., Carmen, P. M., de Lorenzo, C., Pueyo, E.: *Journal of Agricultural and Food Chemistry*, **54**, no. 21, 2006, 8322–8327.

9. Commission Internationale de I Elairage Recommendations on uniform color spaces, color difference equations, psychometric color terms. CIE publication no. 15 (F.1.3.1.), 1971, supplement 2. Burean central. de la Commission Internationale de I Eclairage. Vienna, 1978.
10. AOAC, Official methods of Analysis, Association of Official Analytical Chemists, Arlington, VA, 1995.
11. McLachlan, G. J.: *Discriminant Analysis and Statistical Pattern Recognition.* John Wiley & Sons. Inc, 1992.
12. Vandeginste, B. G. M., Massart, D. L., Buydens, L. M. C., De Jong, S., Lewi, P. J., Smeyers-Verbeke, J. (ed.): *Handbook of Chemometrics and Qualimetrics, Part A, Part B.* Elsevier, 1998.
13. Kalinov, K.: *Statistical methods in behavioral and social sciences.* New Bulgarian University, Sofia, 2001 (in Bulgarian).
14. Lakin, G.: *Biometrics.* Vyshaya shkola, Moscow, 1990 (in Russian).

Received on 3 March, 2015

Krastena T. Nikolova
University of Food Technologies
26 Maritsa Blvd, 4000 Plovdiv
BULGARIA
e-mail: kr.nikolova@abv.bg

Doychin T. Boyadzhiev
Plovdiv University "Paisii Hilendarski"
236 Bulgaria Blvd, 4003 Plovdiv
BULGARIA
e-mail: dtb@uni-plovdiv.bg

Submission of manuscripts. The *Annual* is published once a year. No deadline exists. Once received by the editors, the manuscript will be subjected to rapid, but thorough review process. If accepted, it is immediately scheduled for the nearest forthcoming issue. No page charge is made. The author(s) will be provided with a free of charge printable pdf file of their published paper.

The submission of a paper implies that it has not been published, or is not under consideration for publication elsewhere. In case it is accepted, it implies as well that the author(s) transfers the copyright to the Faculty of Mathematics and Informatics at the "St. Kliment Ohridski" University of Sofia, including the right to adapt the article for use in conjunction with computer systems and programs and also reproduction or publication in machine-readable form and incorporation in retrieval systems.

Instructions to Contributors. Preferences will be given to papers, not longer than 25 pages, written in English and typeset by means of a TeX system. A simple specimen file, exposing in detail the instruction for preparation of the manuscripts, is available upon request from the electronic address of the Editorial Board.

Manuscripts should be submitted for editorial consideration in pdf-format by e-mail to `annuaire@fmi.uni-sofia.bg`. Upon acceptance of the paper, the authors will be asked to send the text of the papers in `.tex` format and the appropriate graphic files (preferably in `.eps` format).

The manuscripts should be prepared for publication in accordance with the instructions, given below.

The first page of manuscripts must contain a title, name(s) of the author(s), a short abstract, a list of keywords and the appropriate 2010 MSC codes (primary and secondary, if necessary). The affiliation(s), including the electronic address, should be given at the end of the manuscripts.

Figures have to be inserted in the text near their first reference. If the author cannot supply and/or incorporate the graphic files, drawings (in black ink and on a good quality paper) should be enclosed separately. If photographs are to be used, only black and white ones are acceptable.

Tables should be inserted in the text as close to the point of reference as possible. Some space should be left above and below the table.

Footnotes, which should be kept to a minimum and should be brief, must be numbered consecutively.

References must be cited in the text in square brackets, like [3], or [5, 7], or [11, p. 123], or [16, Ch. 2.12]. They have to be numbered either in the order they appear in the text or alphabetically. Examples (please note order, style and punctuation):

For books: Obreshkoff, N.: Higher algebra. Nauka i Izkustvo, 2nd edition, Sofia, 1963 (in Bulgarian).

For journal articles: Frisch, H. L.: Statistics of random media. *Trans. Soc. Rheology*, **9**, 1965, 293–312.

For articles in edited volumes or proceedings: Friedman, H. Axiomatic recursive function theory. In: *Logic Colloquim 95*, eds. R. Gandy and F. Yates, North-Holland, 1971, 188–195.