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WEAK CONVERGENCE RESULTS FOR CONTROLLED BRANCHING PROCESSES: STATISTICAL APPLICATIONS

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In this communication it is proved a fluctuation limit theorem for controlled branching processes. Under the conditions that the offspring and control means tend to be *critical*, the obtained limit is a diffusion process. This result is applied to conclude that the standard parametric bootstrap weighted conditional least squares estimate for the offspring mean is asymptotically invalid in the critical case.

Keywords: Controlled branching processes, weak convergence theorem, diffusion process, conditional least squares estimation, parametric bootstrap

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1. INTRODUCTION

Branching processes are regarded as appropriate probability models for the description of the extinction/growth of populations whose developments are subject to the law of chance. In particular, controlled branching processes are useful to model some situations which require control of the population size at each generation. This consists of determining the number of individuals with reproductive capacity at each generation, mathematically through a control process.

Let us provide its formal definition: A controlled branching process (CBP) with a random control function is a stochastic process, $\{Z_n\}_{n\geq 0}$, defined recursively as follows:

$$Z_0 = N \in \mathbb{N}, \quad Z_{n+1} = \sum_{j=1}^{\phi_n(Z_n)} X_{nj}\,, \qquad n \geq 0, \tag{1}$$

where $\{X_{nj} : n = 0, 1, \ldots; j = 1, 2, \ldots\}$ and $\{\phi_n(k) : n, k = 0, 1, \ldots\}$ are two families of independent non–negative integer–valued random variables, with X_{nj}, $n = 0, 1, \ldots$; $j = 1, 2, \ldots$ being independent and identically distributed (i.i.d.) random variables having mean m and variance τ^2 (both assumed finite), and for each $n = 0, 1, \ldots$, $\{\phi_n(k)\}_{k\geq 0}$ are independent stochastic processes with equal one–dimensional probability distributions with $E[\phi_n(k)] = \varepsilon(k)$ and $Var[\phi_n(k)] = \sigma^2(k)$ (both assumed finite for each $k \geq 0$). The random variable Z_n represents the total number of individuals in generation n, starting with $Z_0 = N > 0$ progenitors. Each individual, independently of all others and all with identical probability distributions, gives rise to new individuals. The random variable X_{nj} is the number of offspring originated by the j-th individual of generation n. If in a certain generation n there are k individuals, i.e., $Z_n = k$, then, through the random variable $\phi_n(k)$, identically distributed for each n, there is produced a control in the process fixing the number of progenitors which generate Z_{n+1}. Thus the variable $\phi_n(k)$ determines the migration process in a generation of size k: for those values of the variable $\phi_n(k)$ such that $\phi_n(k) < k$, $k - \phi_n(k)$ individuals are removed from the population, and therefore do not participate in the future evolution of the process; if $\phi_n(k) > k$, $\phi_n(k) - k$ new individuals (immigrants) of the same type are added to the population participating as progenitors under the same conditions as the others. No control is applied to the population when $\phi_n(k) = k$. It is easy to see that $\{Z_n\}_{n\geq 0}$ is a homogeneous Markov chain. This model was introduced in [10] for degenerated control distributions (deterministic case) and in [11] for the random case. The probabilistic theory on this model has been developed in [1], [6], [8] and [11] (and references therein).

Let define $\tau_m(k) = k^{-1}E[Z_{n+1} \mid Z_n = k]$, $k = 1, 2, \ldots$. Intuitively $\tau_m(k)$ is interpreted as the expected growth rate per individual when, in a certain generation, there are k individuals. The process can be classified depending on the limit behaviour of the sequence $\{\tau_m(k)\}_{k\geq 1}$. In a broad sense, the cases $\limsup_{k\to\infty} \tau_m(k) < 1$, $\liminf_{k\to\infty} \tau_m(k) \leq 1 \leq \limsup_{k\to\infty} \tau_m(k)$, and $\liminf_{k\to\infty} \tau_m(k) > 1$ are referred to, respectively, as subcritical, critical, and supercritical situations for a CBP. It is easy to obtain that $\tau_m(k) = mk^{-1}\varepsilon(k)$, $k \geq 1$. Hence the classification of the process is determined essentially by the behaviour of the offspring and control means. Whenever exists the limit of the sequence $\{\tau_m(k)\}_{k\geq 1}$, as $k \to \infty$, we refer to it as the asymptotic mean growth rate.

In this paper we consider an array of CBPs $\{Z_i^{(n)}\}_{i\geq 0}$, $n = 1, 2, \ldots$, defined recursively by

$$Z_0^{(n)} = N \in \mathbb{N}, \qquad Z_{i+1}^{(n)} = \sum_{j=1}^{\phi_i^{(n)}(Z_i^{(n)})} X_{ij}^{(n)}, \quad i = 0, 1, \ldots;\ n = 1, 2, \ldots \tag{2}$$

For each n, $\{X_{ij}^{(n)} : i = 0, 1, \ldots; j = 1, 2, \ldots\}$ is a sequence of i.i.d. non–negative integer–valued random variables with mean m_n and finite variance τ_n^2, and $\{\phi_i^{(n)}(k) : i = 0, 1, \ldots; k = 0, 1, \ldots\}$ are independent non–negative integer–valued random variables with means $\varepsilon_n(k)$ and finite variances $\sigma_n^2(k)$ for every $k \geq 0$. Also, for each n, we assume that $\{X_{ij}^{(n)}\}$ and $\{\phi_i^{(n)}(k)\}$ are independent.

The main aim of this paper is to provide a Feller diffusion approximation for an array of CBPs whose offspring and control means tend to be *critical*. Using operator semigroup convergence theorems, it is proved that the fluctuation limit is a diffusion process. From a practical viewpoint, the interest of developing this result stems from the usefulness of it in determining the asymptotic distributions of estimators of the main parameters of a controlled branching process. In particular, we are interested in the weighted conditional least squares (WCLS) estimator of the offspring mean. As an statistical application of the obtained fluctuation limit theorem, it is determined, in a parametric framework, the bootstrapping distribution of the WCLS estimator of the offspring mean in the critical case. From this, it is concluded that the standard parametric bootstrap WCLS estimate is asymptotically invalid in the critical case.

The communication is organized as follows. In Section 2 we prove that the functional fluctuation limit of a sequence of CBPs is a diffusion process. We present in Section 3 the WCLS estimator of the offspring mean of a CBP. We show its limit distribution from a classical viewpoint and in a parametric framework, its bootstrapping distribution by applying the obtained functional limit theorem. From the last, it is concluded that the standard parametric bootstrap WCLS estimate is asymptotically invalid in the critical case.

2. DIFFUSION APPROXIMATION THEOREM

Let consider an array of CBPs as given in (2). Let us introduce the sequence of random functions $\{W_n\}_{n\geq 1}$ as $W_n(t) = n^{-1}Z_{[nt]}^{(n)}$, $t \geq 0$, $n = 1, 2, \ldots$, with $[\cdot]$ denoting the integer part. It is clear that $\{W_n\}_{n\geq 0}$ is a $D_{[0,\infty)}[0,\infty)$–valued random variable, with $D_{[0,\infty)}[0,\infty)$ the space of non–negative functions on $[0,\infty)$ that are right continuous and have left limits. Denote by $C_c^\infty[0,\infty)$ the space of infinitely differentiable functions on $[0,\infty)$ which have compact supports. Throughout the paper "$\xrightarrow{\mathcal{D}}$" denotes the convergence of random functions in the Skorokhod topology, "$\xrightarrow{d}$" the convergence of random variables in distribution and $N(\cdot,\cdot)$ the normal distribution.

Using operator semigroup convergence theorems, we prove a weak convergence theorem for the sequence of random functions $\{W_n\}_{n\geq 0}$.

Theorem 1. *Assume that*

(A1) $\quad m_n = m + \alpha n^{-1} + o(n^{-1})$ *as* $n \to \infty$, $0 < m < \infty$, $-\infty < \alpha < \infty$*;*

(A2) $\quad \tau_n^2 \to \tau^2$ *as* $n \to \infty$, $0 < \tau^2 < \infty$*;*

(A3) *for any sequence* $\{x_n\}_{n\geq 1}$ *such that* $x_n \to x$ *as* $n \to \infty$, $0 < x < \infty$, *and for all* $\epsilon > 0$,

$$\lim_{n\to\infty} \tau_n^{-2} E\left[|X_{01}^{(n)} - m_n|^2 \mathbf{1}_{\left\{|X_{01}^{(n)} - m_n| \geq \epsilon\sqrt{n x_n \tau_n^2}\right\}}\right] = 0,$$

with $\mathbf{1}_A$ *denoting the indicator function of a set* A *;*

(A4) $\varepsilon_n(k) = \varepsilon(k) + f_n(k)$, *with* $\lim_{n\to\infty} f_n(k) = 0$ *uniformly for* k *;*

(A5) $m\varepsilon(k)k^{-1} = 1 + \gamma k^{-1} + o(k^{-1})$ *as* $k \to \infty$, $-\infty < \gamma < \infty$ *;*

(A6) $\sigma_n^2(k) = \beta_n k + g_n(k)$, *with* $\lim_{k\to\infty} g_n(k)k^{-1} = 0$ *uniformly for* n, $\beta_n \to 0$ *as* $n \to \infty$.

Then $W_n \xrightarrow{\mathcal{D}} W_\alpha$ *as* $n \to \infty$, *weakly in the Skorohod space* $D_{[0,\infty)}[0,\infty)$, *where* W_α *is the diffusion process with generator*

$$A_\alpha f(x) = (\gamma + \alpha m^{-1}x)f'(x) + \frac{1}{2}\tau^2 m^{-1} x f''(x), \quad f \in C_c^\infty[0,\infty). \tag{3}$$

The proof of Theorem 1 can be found in [7].

The process W_α is the (unique) solution of the stochastic differential equation

$$dW_\alpha(t) = (\gamma + \alpha m^{-1} W_\alpha(t))dt + (\tau^2 m^{-1} W_\alpha(t))^{1/2} dB(t),\ t \geq 0,$$

where B is a standard Wiener process.

In next section it is necessary to consider a particular array version of CBPs of the general situation considered in (2). Let $\{Z_i^{(n)}\}_{i\geq 0}$, $n = 1, 2, \ldots$, be an array of CBPs with the same hypotheses about the offspring and control variables as in the definition in (2), but with the additional condition that for each $k \geq 0$, the variables $\{\phi_i^{(n)}(k)\}$, $i \geq 0$; $n \geq 1$, are identically distributed with $E[\phi_i^{(n)}(k)] = \varepsilon(k)$ and $Var[\phi_i^{(n)}(k)] = \sigma^2(k)$. In respect to the offspring law we assume conditions (A1)-(A3). Moreover, in relation to the control mean and variance we consider the following assumptions:

(B1) $m\varepsilon(k)k^{-1} = 1 + \gamma k^{-1} + o(k^{-1})$ as $k \to \infty$, $-\infty < \gamma < \infty$;

(B2) $\lim_{k\to\infty} \sigma^2(k)k^{-1} = 0$,

which are the simplified version of (A4)-(A6) in this particular case. Then, applying Theorem 1 one obtains

$$W_n \xrightarrow{\mathcal{D}} W_\alpha \text{ as } n \to \infty,$$

where W_α is the diffusion process with generator given in (3).

3. WEIGHTED CONDITIONAL LEAST SQUARES ESTIMATION AND ASYMPTOTIC RESULTS

Let consider a CBP given in (1) and let $\mathcal{F}_n$ be the $\sigma-$algebra generated by the random variables $Z_0, Z_1, \cdots, Z_n$. From the fact that $E[Z_n|\mathcal{F}_{n-1}] = m\varepsilon(Z_{n-1})$ a.s., we can represent Z_n as

$$Z_n = m\varepsilon(Z_{n-1}) + \tilde{\delta}_n, \qquad n = 1, 2, \ldots, \tag{4}$$

where the error term $\tilde{\delta}_n$ has $E\left[\tilde{\delta}_n|\mathcal{F}_{n-1}\right] = 0$. In order to obtain an efficient estimator of the offspring mean, we divide both sides of (4) by $(\varepsilon(Z_{n-1}) + 1)^{1/2}$ and rewrite the model as

$$\frac{Z_n}{(\varepsilon(Z_{n-1}) + 1)^{1/2}} = \frac{m\varepsilon(Z_{n-1})}{(\varepsilon(Z_{n-1}) + 1)^{1/2}} + \delta_n, \qquad n = 1, 2, \ldots,$$

with $\delta_n = \tilde{\delta}_n / (\varepsilon(Z_{n-1}) + 1)^{1/2}$.

The WCLS estimator of m is obtained by minimizing the expression $\sum_{i=1}^n \delta_i^2$. It is easy to check that the value of m that minimizes it is

$$\widehat{m}_n = \left(\sum_{i=1}^n \frac{Z_i \varepsilon(Z_{i-1})}{\varepsilon(Z_{i-1}) + 1}\right)\left(\sum_{i=1}^n \frac{\varepsilon^2(Z_{i-1})}{\varepsilon(Z_{i-1}) + 1}\right)^{-1}. \tag{5}$$

We are interested in the study of the limit distribution of the pivot

$$V_n = \left(\sum_{i=1}^n \frac{\varepsilon^2(Z_{i-1})}{\varepsilon(Z_{i-1}) + 1}\right)^{1/2} (\widehat{m}_n - m). \tag{6}$$

This presents different kinds of behaviour depending on the classification of the process. In [5] it was established that a CBP $\{Z_n\}_{n\geq 0}$ with $P(X_{01} = 0) > 0,\ P(X_{01} \leq 1) < 1$ and $P(\phi_0(i) > i) > 0,\ i = 0, 1, \ldots$, converges in distribution to a positive, finite and non-degenerate random variable Z.

Theorem 2. *Assume that*

i) $\limsup_{k\to\infty} \tau_m(k) < 1$ *;*

ii) $P(X_{01} = 0) > 0, \quad P(X_{01} \leq 1) < 1$ *;*

iii) $P(\phi_0(i) > i) > 0, \quad i = 0, 1, \ldots$ *;*

iv) $E[\mu_{2+\delta}(Z)] < \infty$, *with* $\mu_k(z) = E[|\phi_0(z) - \varepsilon(z)|^k], \ k \geq 1.$

Then

$$V_n \xrightarrow{d} N(0, V) \quad \textit{as} \quad n \to \infty\,,$$

where

$$V = \frac{m^2 E\left[\left(\frac{\varepsilon(Z)}{\varepsilon(Z)+1}\right)^2 \sigma^2(Z)\right] + \sigma^2 E\left[\frac{\varepsilon^3(Z)}{(\varepsilon(Z)+1)^2}\right]}{E\left[\frac{\varepsilon^2(Z)}{\varepsilon(Z)+1}\right]}.$$

The proof can be seen in [9].

In the supercritical case, we consider that

$$\lim_{n\to\infty} \tau_m(k) = m \lim_{n\to\infty} k^{-1}\varepsilon(k) = \eta m > 1.$$

Then the following result holds

$$P(Z_n \to \infty) > 0 \quad \text{and} \quad \lim_{n\to\infty} L_n = L \quad \text{a.s.,} \tag{7}$$

with $L_n = (\eta m)^{-n} Z_n$ and $P(L > 0) > 0$. Indeed, conditions that guarantee (7) can be found in the papers [3, 4].

Theorem 3. *Assume that*

i) $\limsup_{k\to\infty} \tau_m(k) > 1$ *and (7) hold;*

ii) $\lim_{k\to\infty} k^{-1}\sigma^2(k) = 0$.

Then

$$V_n \xrightarrow{d} N\left(0, \sigma^2\right), \quad \textit{as} \quad n \to \infty.$$

The details of the proof can be seen in [9].

Regarding the critical case, we obtained:

Theorem 4. *Assume that*

i) $\tau_m(k) = 1 + k^{-1}\gamma + o(k^{-1})$ *as* $k \to \infty$, *where* γ *is a real number;*

ii) $\lim_{k\to\infty} k^{-1}\sigma^2(k) = 0$.

Then

$$V_n \xrightarrow{d} \frac{W(1) - W(0) - \gamma}{\left(\frac{1}{m}\int_0^1 W(t)dt\right)^{1/2}} \quad \textit{as} \quad n \to \infty,$$

where W *is a diffusion process with generator (3) with* $\alpha = 0$.

The reader can find the proof in [9].

This result can be generalized to the particular array version of CBPs considered in the previous section (2). We provide the behaviour of the array version of the estimator $\widehat{m}_n$ and the pivot quantity V_n, which is the interest for the study of the behaviour of the bootstrap estimator of m. Let

$$\bar{m}_n = \left(\sum_{i=1}^{n} \frac{Z_i^{(n)} \varepsilon(Z_{i-1}^{(n)})}{\varepsilon(Z_{i-1}^{(n)}) + 1} \right) \left(\sum_{i=1}^{n} \frac{\varepsilon^2(Z_{i-1}^{(n)})}{\varepsilon(Z_{i-1}^{(n)}) + 1} \right)^{-1}$$

and

$$\bar{V}_n = \left(\sum_{i=1}^{n} \frac{\varepsilon^2(Z_{i-1}^{(n)})}{\varepsilon(Z_{i-1}^{(n)}) + 1} \right)^{1/2} (\bar{m}_n - m_n).$$

Theorem 5. *Assume that assumptions (A1)–(A3) and (B1)–(B2) are satisfied. Then, as $n \to \infty$,*

$$\bar{V}_n \xrightarrow{d} \frac{W_\alpha(1) - W_\alpha(0) - \gamma}{\left(\frac{1}{m} \int_0^1 W_\alpha(t) dt \right)^{1/2}} - \alpha \left(\frac{1}{m} \int_0^1 W_\alpha(t) dt \right)^{1/2},$$

with W_α as in Theorem 1.

Briefly, three different limit distributions for V_n were obtained for three different cases, as $n \to \infty$, namely

$$V_n \xrightarrow{d} \begin{cases} N(0, V), & \text{if } \limsup_{k\to\infty} \tau_m(k) < 1 \text{ (subcritical)}, \\ \frac{W(1) - W(0) - \gamma}{\left(\frac{1}{m} \int_0^1 W(t) dt \right)^{1/2}}, & \text{if } \tau_m(k) = 1 + k^{-1}\gamma + o(k^{-1}),\ \gamma \in \mathbb{R} \text{ (critical)} \\ N(0, \sigma^2), & \text{if } \liminf_{k\to\infty} \tau_m(k) > 1 \text{ (supercritical)}, \end{cases}$$

with V and W as previously defined. Hence the classical asymptotic theory does not provide a unified estimation theory for the offspring mean. Thus it is of interest to approximate the sampling distribution of V_n by alternative methods. In particular, we are keen on the bootstrap procedure. We apply the fluctuation limit theorem previously established to determine the asymptotic distribution of the bootstrap WCLS estimator in the critical case. We consider a parametric framework and obtain as a consequence of this last limit result that the standard bootstrap version of the pivot quantity does not have the same limit distribution as V_n in such a case. Although the behaviour of the parametric bootstrap for the subcritical and supercritical cases is of interest as well, due to this fails in the critical case it will be most interesting for the future to make efforts in developing a modified bootstrap procedure to be valid in all the three cases. Let us introduce a parametric bootstrap for CBPs following analogous steps to those given in [2] for branching processes with immigration. We assume that the offspring law, p_θ, has probability mass function

$$p_\theta(k) = P_\theta(X_{01} = k),\ k = 0, 1, \ldots,$$

depending on a parameter θ where $\theta \in \Theta \subseteq \mathbb{R}$.

Consider $m = E_\theta[X_{01}] = f(\theta)$ for some function f, which we will assume to be a one-to-one mapping of Θ to $[0, \infty)$. Moreover, f is assumed to be homeomorphism between its domain and range. For instance, the power series family of distributions satisfies the conditions imposed above.

The bootstrap procedure can be defined as follows: given the sample $\mathcal{M}_n = \{Z_1, ..., Z_n\}$, estimate the offspring mean by the estimator $\widehat{m}_n$ given in (5), and therefore let $\widehat{\theta}_n = f^{-1}(\widehat{m}_n)$. Conditional on $\mathcal{M}_n$, define a sequence of i.i.d. random variables X^*_{nj} having distribution given by $p_{\widehat{\theta}_n}$. The bootstrap sample $\mathcal{M}^*_n = \{Z^*_1, \ldots, Z^*_n\}$ is obtained by

$$Z^*_{n+1} = \sum_{j=1}^{\phi_n(Z^*_n)} X^*_{nj},\ n = 0, 1, \ldots,\ \text{ with } Z^*_0 = N.$$

We define the bootstrap estimator of m as $\widehat{m}^*_n$ given by

$$\widehat{m}^*_n = \left(\sum_{i=1}^{n} \frac{Z^*_i \varepsilon(Z^*_{i-1})}{\varepsilon(Z^*_{i-1}) + 1} \right) \left(\sum_{i=1}^{n} \frac{\varepsilon^2(Z^*_{i-1})}{\varepsilon(Z^*_{i-1}) + 1} \right)^{-1},$$

and the parametric bootstrap analogue, V^*_n, of the pivot quantity V_n, given in (6), as

$$V^*_n = \left(\sum_{i=1}^{n} \frac{\varepsilon^2(Z^*_{i-1})}{\varepsilon(Z^*_{i-1}) + 1} \right)^{1/2} (\widehat{m}^*_n - \widehat{m}_n).$$

Note $\varepsilon(\cdot)$ is assumed to be known and $\phi_n(\cdot)$ are observable. In this context, let denote the distribution function of V_n by $F_n(m, x) = P(V_n \le x),\ x \in \mathbb{R}$. Then, notice that

$$P(V^*_n \le x | \mathcal{M}_n) = F_n(\widehat{m}_n, x),\ x \in \mathbb{R}.$$

Our interest is to determine the limit behaviour of $F_n(\widehat{m}_n, x)$, $x \in \mathbb{R}$, assuming that the true model is a critical CBP. We check that for every $x \in \mathbb{R}$ the random variables $F_n(\widehat{m}_n, x)$ converge in distribution to a non degenerate random limit, and consequently one has that it is not verified that

$$\sup_{-\infty < x < \infty} |F_n(m, x) - F_n(\widehat{m}_n, x)| \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty, \tag{8}$$

obtaining the asymptotic invalidity of the bootstrap procedure in the critical case.

Define

$$\mathcal{W}(\alpha,m,\tau^2,\gamma)=\frac{W_\alpha(1)-W_\alpha(0)-\gamma}{\left(\frac{1}{m}\int_0^1 W_\alpha(t)dt\right)^{1/2}}-\alpha\left(\frac{1}{m}\int_0^1 W_\alpha(t)dt\right)^{1/2},$$

with W_α the diffusion process defined in Theorem 1, and

$$F(\alpha,m,\tau^2,\gamma,x)=P(\mathcal{W}(\alpha,m,\tau^2,\gamma)\le x),\ x\in\mathbb{R}.$$

As in [2], it is not hard to prove that, for each $x\in\mathbb{R}$, $F(\mathcal{V}_0,m,\tau^2,\gamma,x)$ is a random variable, with $\mathcal{V}_0=(W(1)-W(0)-\gamma)\left(\frac{1}{m}\int_0^1 W(t)dt\right)^{-1}$. Now, we are in conditions to state the result that establishes that (8) does not hold:

Theorem 6. *Assume that*

(C1) *The variance of the offspring law, τ^2, is a continuous function of θ.*

(C2) *The moment $E_\theta[|X_{01}|^{2+\delta}]$, for some $\delta>0$ is a continuous function of θ.*

Then, it is verified that for every $x\in\mathbb{R}$, as $n\to\infty$,

$$F_n(\widehat{m}_n,x)\xrightarrow{d}F(\mathcal{V}_0,m,\tau^2,\gamma,x).$$

It is not hard to check that the power distribution family verifies (C1)-(C2). The key of this proof is Theorem 5 and the details can be read in [7]. One of the reasons for the standard parametric bootstrap does not work well in such a case is the rate of convergence to the offspring mean parameter of its WCLS estimate.

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