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LOXODROMES ON CANAL SURFACES IN EUCLIDEAN 3-SPACE

MURAT BABAARSLAN

We obtain the differential equations of loxodromes on canal surfaces which cut all meridians or parallels at a constant angle in the Euclidean 3-space. Also we compute the arc-lengths of loxodromes and give some examples by using Mathematica to illustrate our results.

Keywords: Loxodrome, canal surface, Euclidean space
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1. INTRODUCTION

Loxodromes are curves which intersects all meridians or parallels at a constant angle on the Earth's surface. As a result of this, loxodromes do not require a change of course [4]. Therefore they are usually used in navigation.

The equations of the loxodromes on rotational surfaces were found by Noble [6].

A natural generalization of rotational surfaces is helicoidal surfaces. Also there are a lot of helicoidal objects and structures which are related to navigation in nature, science and engineering, for example; creeper plants, helicoidal staircases, moving walkways, parking garage ramps, helicoidal railways and so on [1].

The differential equations of the loxodromes on helicoidal surfaces in Euclidean 3-space $\mathbb{E}^3$ were obtained by Babaarslan and Yayli [1].

A canal surface in $\mathbb{E}^3$ can be defined as envelope of a moving sphere whose trajectory of centers is a spine curve $m(u)$ with varying radius $r(u)$. When $r(u)$ is a constant function, the canal surfaces reduce to pipe surfaces [8].

A lot of objects and structures may be represented by using canal surfaces, for example; blending surfaces and transition surfaces between pipes [7], hoses, ropes [2], 3D fonts, brass instrument, internal organs of the body in solid modeling [3], helical channel [5] and tunnels. Some particular examples of canal surfaces are cylinder, cone, torus, sphere, pipe and Dupin cyclide. Hence canal surfaces are often used in Computer Aided Geometric Design and Computer Aided Manufacturing [3].

Internal organs of the body, helical channel and tunnels are especially interesting examples of canal surfaces on which navigation is possible.

In this paper, we investigate the differential equations of loxodromes on the canal surfaces. Also we compute the arc-lengths of loxodromes and give some examples by using Mathematica.

2. LOXODROMES ON CANAL SURFACES

The parametrization of a canal surface in $\mathbb{E}^3$ is

$$C(u,v) = m(u) + r(u)\big(\sqrt{(1 - r'(u)^2)}n(u)\cos v + \sqrt{(1 - r'(u)^2)}b(u)\sin v - t(u)r'(u)\big),$$

where u is arc-length parameter, $0 \leq v < 2\pi$, t, n and b are the unit tangent, principal normal and binormal vectors of the spine curve $m(u)$, respectively.

The coefficients of first fundamental form of the canal surface C with respect to the base $\{C_u, C_v\}$ are

$$E = \langle C_u, C_u\rangle = (1 - \kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2,$$

$$F = \langle C_u, C_v\rangle = g^2\tau + gh\kappa\sin v,$$

$$G = \langle C_v, C_v\rangle = g^2.$$

Thus the first fundamental form of the canal surface C is given by the following equation

$$\begin{aligned} ds^2 &= Edu^2 + Fdudv + Gdv^2 \\ &= \big((1 - \kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)du^2 \\ &\quad +2(g^2\tau + gh\kappa\sin v)dudv + g^2dv^2, \end{aligned}$$

where $g = g(u) = r(u)\sqrt{1 - r'(u)^2}$ and $h = h(u) = r(u)r'(u)$; $\kappa = \kappa(u)$ and $\tau = \tau(u)$ are the curvature and the torsion of $m(u)$, respectively [8].

We recall that when $1 - \kappa g\cos v - h' \neq 0$, the canal surface is regular. Also, a regular canal surface is developable if and only if it is a cylinder or a cone [8].

The arc-length of any curve on the canal surface $C(u,v)$ between u_1 and u_2 is given by

$$s = \int_{u_1}^{u_2} \sqrt{H(u,v)}du,$$

where

$$H(u,v) =(1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2 \\ + 2(g^2\tau + gh\kappa\sin v)\frac{dv}{du} + g^2\Big(\frac{dv}{du}\Big)^2.$$

A curve on the canal surface $C(u,v)$ in $\mathbb{E}^3$ which cuts all meridians (v=constant) or parallels (u=constant) at a constant angle is called *a loxodrome*.

Let us assume that a loxodrome $\alpha(t)$ is the image of a curve $(u(t),v(t))$ which lies on the (uv)-plane under H. The tangent vector $\alpha'(t)$ has coordinates (u',v') and the tangent vector C_u has coordinates $(1,0)$ with respect to the basis $\{C_u, C_v\}$. Thus, at the intersection point $C(u,v)$, we have

$$\begin{aligned} \cos\theta &= \frac{Edu + Fdv}{\sqrt{E^2du^2 + 2EFdudv + EGdv^2}} \\ &= \frac{K(u,v)}{\sqrt{L(u,v)}}, \end{aligned}$$

where

$$\begin{aligned} K(u,v) &= \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)du \\ &\quad +\big(g^2\tau + gh\kappa\sin v\big)dv, \end{aligned}$$

$$\begin{aligned} L(u,v) &= \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)^2du^2 \\ &\quad +2\big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big) \\ &\qquad \times(g^2\tau + gh\kappa\sin v)dudv \\ &\quad +\big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)g^2dv^2. \end{aligned}$$

From this equation, the differential equation of the loxodrome on the canal surface which cuts all meridians at a constant angle θ is

$$A\Big(\frac{dv}{du}\Big)^2 + B\,\frac{dv}{du} = C \tag{2.1}$$

with

$$A =\big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)g^2\cos^2\theta \\ - (g^2\tau + gh\kappa\sin v)^2,$$

$$
\begin{aligned}
B =& -2\big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big) \\
&\times (g^2\tau + gh\kappa\sin v)\sin^2\theta\,, \\
C =&\big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)^2\sin^2\theta.
\end{aligned}
$$

Also, the angle γ between the loxodrome and any parallel (u=constant) is defined by the following equation

$$
\begin{aligned}
\cos\gamma &= \frac{Fdu + Gdv}{\sqrt{EGdu^2 + 2FGdudv + G^2dv^2}} \\
&= \frac{M(u,v)}{\sqrt{N(u,v)}},
\end{aligned}
$$

where

$$
M(u,v) = \big(g^2\tau + gh\kappa\sin v\big)du + g^2dv,
$$

$$
\begin{aligned}
N(u,v) =& \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)g^2du^2 \\
&+2(g^2\tau + gh\kappa\sin v)g^2dudv + g^4dv^2.
\end{aligned}
$$

From this equation, the differential equation of the loxodrome on the canal surface which cuts all parallels at a constant angle γ is given by

$$
\widetilde{A}\,\Big(\frac{du}{dv}\Big)^2 + \widetilde{B}\,\frac{du}{dv} = \widetilde{C} \tag{2.2}
$$

with

$$
\begin{aligned}
\widetilde{A} =& \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)g^2\cos^2\gamma \\
&- (g^2\tau + gh\kappa\sin v)^2, \\
\widetilde{B} =& -2(g^2\tau + gh\kappa\sin v)g^2\sin^2\gamma, \\
\widetilde{C} =& g^4\sin^2\gamma.
\end{aligned}
$$

If we take $r = r(u)$=constant in equations (2.1) and (2.2), respectively, then we obtain the differential equation of the loxodrome on pipe surfaces which cut all meridians (resp., parallels) at a constant angle θ (resp., γ).

We have not succeeded in finding the general solutions of the differential equations (2.1) and (2.2) by using analytical methods, so it remains an open problem. To illustrate the obtained results, we give some examples produced with the help of MATHEMATICA.

Example 2.1. Let us consider the spine curve $m(u) = (0, 0, u + 1)$. Taking $r(u) = 1$, $\theta = \pi/4$, $u \in (0, 2)$, $v \in (0, 2\pi)$ and $v(0) = 0$, the arc-length of the loxodrome is computed as $2\sqrt{2}$. The loxodrome, the meridian ($v = 1$) and the canal surface are shown in Figure 1.

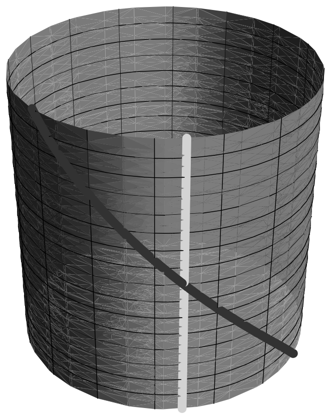

Figure. 1. The loxodrome (blue) and the meridian (green) on the canal surface (cylinder)

Example 2.2. Let us consider the spine curve $m(u) = (0, 0, u)$. Taking $r(u) = u/2$, $\theta = \pi/3$, $u \in (0, 7)$, $u_0 = 1$ and $v \in (0, 2\pi)$, the arc-length of the loxodrome is computed as $7\sqrt{3}$. Also the loxodrome, the meridian ($v = 3$) and the canal surface are shown in Figure 2.

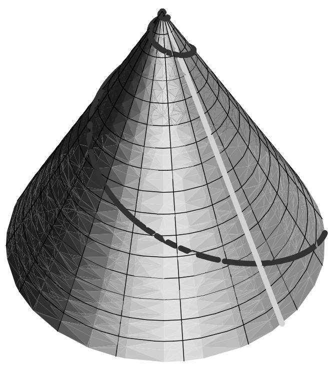

Figure. 2. The loxodrome (blue) and the meridian (green) on the canal surface (cone)

Example 2.3. Let us consider the spine curve $m(u) = (\cos u, \sin u, 0)$. Taking $r(u) = 1/2$, $\gamma = \pi/6$, $v \in (-\pi, \pi)$, $u \in (0, 2\pi)$ and $u(0) = 0$, the arc-length of the loxodrome is computed as $2\sqrt{3}\pi/3$. The loxodrome, the parallel $(u = 1/2)$ and the canal surface are depicted in Figure 3.

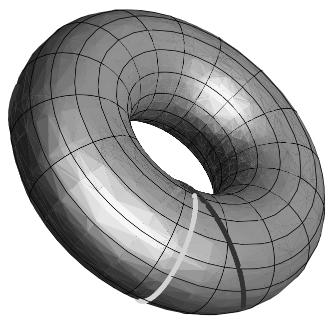

Figure. 3. The loxodrome (blue) and the parallel (green) on the canal surface (torus)

Example 2.4. Consider the spine curve $m(u) = \big(\cos(u/\sqrt{2}), \sin(u/\sqrt{2}), u/\sqrt{2}\big)$. Taking $r(u) = 1$, $\gamma = \pi/2$, $v \in (-2\pi, \pi)$, $u \in (0, 4\pi)$ and $u(0) = 0$, the arc-length of the loxodrome is computed as 14.2074. Also the loxodrome, the parallel $(u = 2\pi)$ and the canal surface are shown in Figure 4.

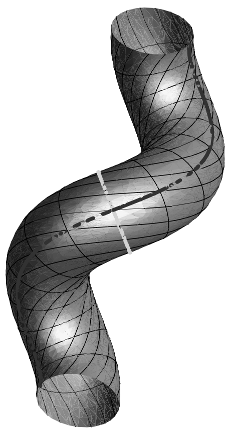

Figure. 4. The loxodrome (blue) and the parallel (green) on the canal surface (helical)

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Murat Babaarslan

Department of Mathematics
Bozok University
66100, Yozgat
TURKEY

e-mail: murat.babaarslan@bozok.edu.tr