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MANIFOLDS ADMITTING A STRUCTURE OF FOUR DIMENTIONAL ALGEBRA OF AFFINORS

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The purpose of this note is to describe some properties of manifolds endowed with an almost tangent structure T, $T^2 = 0$ and an almost complex structure J, $J^2 = -E$, $E = id$.

We consider a linear connection ∇ on N, which is compatible with the algebraic structure, i.e. $\nabla J = 0$, $\nabla T = 0$. The existence of ideals in corresponding algebra implies the existence of autoparallel submanifolds of N.

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1. ALGEBRAIC PRELIMINARIES

Let us consider a real associative algebra $\mathfrak{A}$ with the unit element e and two generators i, ε satisfying

$$i^2 = -e, \quad \varepsilon^2 = 0,$$

under the requirement $dim\,\mathfrak{A} = 4$ [1].

We distinguish three cases described by the relations

$$i\varepsilon = \varepsilon i\,, \tag{1.1}$$

$$i\varepsilon = -\varepsilon i\,, \tag{1.2}$$

$$i\varepsilon + \varepsilon i = e. \tag{1.3}$$

The corresponding algebras are denoted by $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, respectively.

Proposition 1. The algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ possess nontrivial ideals while $\mathfrak{A}_3$ is a simple algebra.

Proof. Let us denote $i\varepsilon = \varepsilon i = k$. Then we have the following table of multiplications of $\mathfrak{A}_1$

	e	i	ε	k
e	e	i	ε	k
i	i	$-e$	k	ε
ε	ε	k	0	0
k	k	$-\varepsilon$	0	0

Obviously, $\{e, i, \varepsilon, k\}$ is a basis of $\mathfrak{A}_1$ and $\{\varepsilon, k\}$ is an ideal with zero-multiplication.

Similarly to the previous case, $\mathfrak{A}_2$ admits an ideal, too.

Now we consider the algebra $\mathfrak{A}_3$. The mapping

$$\varphi : \mathfrak{A}_3 \to M(2)\,,$$

where $M(2)$ is the algebra of (2×2) real matrices defined by

$$\varphi(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varphi(\varepsilon) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

is an isomorphism. It is well-known that the algebra $M(2)$ is simple. That completes the proof. □

2. MANIFOLDS OVER ALGEBRAS

Let N be a manifold of class C^∞, TN - the tangent bundle of N, $\mathfrak{X}(N)$ - the $F(N)$ - module of global sections of TN, F(N)-the ring the smooth functions on N. Let $A(TN)$ be the algebra of $F(N)$ - linear operators of $\mathfrak{X}(N)$. It can be identified with the algebra of fiber–preserving automorphisms of TN.

Let us consider a real associative algebra $\mathfrak{A}$ with unit element e. A morphism of algebras $\Phi : \mathfrak{A} \to A(TN)$ such that $\Phi(e) = I$, the identity operator of $A(TN)$ will be called an $\mathfrak{A}$ - *structure* on N. A linear connection ∇ on N is said to be *compatible* with the $\mathfrak{A}$ - *structure* if $\nabla\Phi(a) = 0$, for all $a \in \mathfrak{A}$, i.e. each operator $\Phi(a)$ is parallel with respect to ∇. An $\mathfrak{A}$ - *structure* is said to be integrable if for each point p exists a neighborhood U, such that the operator $\Phi(a)$ for all $a \in \mathfrak{A}$ have constant components in corresponding coordinate chart.

If $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, we define a distribution D in TN as follows:

$$D_p = \{\Phi(b)v \in T_pN;\ for\ all\ b \in B\ and\ v \in T_pN\}.$$

In other words, at each point $p \in N$, D_p is the image of T_pN by the operators corresponding to the elements of $\mathfrak{B}$. This distribution is invariant with respect to all operators $\Phi(a)$, $a \in \mathfrak{A}$.

Proposition 2. Let $\mathfrak{A}$ is associative unitary R-algebra, N be a manifold with $\mathfrak{A}$ - *structure* and ∇ be a linear connection on N. If $\nabla\Phi(a_i) = 0$ for all basis elements a_i of $\mathfrak{A}$ then ∇ is compatible with $\mathfrak{A}$.□

Proof. The operator $\nabla : \mathfrak{D}(N) \to \mathfrak{D}(N)$ is a differentiation of the tensor algebra on N. If $\Phi(a_i) = A_i \in \mathfrak{D}_1^1(N)$, $i = 1, 2, 3, 4$, it follows that

$$\nabla_X(A_iA_j) = \nabla_X(A_i)A_j + A_i\nabla_X(A_j) = 0$$

□

The following theorem is proved in [2], p. 118.

Theorem 1. *Let (M, ∇) be an affinely connected analytical manifold equipped with an $\mathfrak{A}$ - structure compatible with ∇. Then the following properties are satisfied:*

1. The distribution D is involutive;

2. If N' is a maximal integral submanifold of D through any point of N, then it is autoparallel submanifold of N;

3. On each N' acts the quotient - algebra $\mathfrak{A}/O(\mathfrak{B})$, where $O(\mathfrak{B})$ is the annihilator of the ideal $\mathfrak{B}$ in algebra $\mathfrak{A}$.

3. ALGEBRAIC STRUCTURES $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$

The integrability conditions of these structures are given in [1].

According to the previous notations, we set $\Phi(e) = I$, $\Phi(i) = J$ and $\Phi(\varepsilon) = T$, by I we denote the unit matrix and we set $JT = K$. Moreover, we suppose that

$$Im\,T = Ker\,T = \frac{1}{2}\,dim\,N.$$

Theorem 2. *Let N be a manifold with an integrable algebraic structure of type $\mathfrak{A}_i$, $(i = 1, 2)$ and ∇ be torsion-free connection compatible with the algebraic structure, i.e. $\nabla J = 0, \nabla T = 0$. Then there exists an $\mathfrak{A}_i$ - invariant foliation N' in N, i.e. at any point $p \in N' \subset N$ the tangent space T_pN' is invariant with respect to J and T.*

Proof. *Case (1)*: $J^2 = -I, T^2 = 0, JT = TJ$ and $Ker\,T = Im\,T$. We denote by D the distribution $Ker\,T = Im\,T$. It can be easily seen that the following holds: $JD \subseteq D, TD \subseteq D$. This implies that $n \equiv 0 \pmod 4$, so we can write $n = 4m$.

For any point of N there exist an open neighborhood with a chart $(x^1, \ldots, x^{4m})$ on it such that with respect to the basis $\partial/\partial x^1, \ldots, \partial/\partial x^{4m}$ the tensors J and T have matrix expression:

$$\begin{pmatrix} 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix}.$$

We denote $(x^1, \ldots, x^{4m}) = (x^i, x^{i+m}, x^{i+2m}, x^{i+3m})$, $(i = 1, \ldots, m)$. Every integral submanifold N' of $D = Im\,T$ has coordinates $(x^i_0, x^{i+m}_0, x^{i+2m}, x^{i+3m})$.

We have $\mathfrak{A} = \{I, J, T, K\}$, $\mathfrak{B} = \{T, K\}$ - an ideal, the annihilator $O(\mathfrak{B}) = \mathfrak{B}$, $\mathfrak{A}/O(\mathfrak{B}) \approx \{I, J\}$.

The restriction of J on D is the following

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v^{i+2m} \\ v^{i+3m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -v^{i+3m} \\ v^{i+2m} \end{pmatrix}.$$

Here $v = (0, 0, v^{i+2m}, v^{i+3m}) \in D$ and by I we denote the unit $(n \times n)$- matrix.

Case (2): $JT = -TJ$.

Let M be a manifold provided with a $\mathfrak{A}_2$ - *structure*. Similarly to the previous case, one may choose an atlas, such that with respect to any chart $U_x \subset N$ the operators J and T have the form

$$\begin{pmatrix} 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix}.$$

Theorem 2 is proved. □

Now we start considering the last case.

Case (3): N is a manifold provided with a couple J, T of tensor fields of type (1,1), satisfying $J^2 = -I$, $T^2 = 0$ and $JT + TJ = I$. Here it is not necessary to require that $Ker\,T = Im\,T$, because it follows from the relation between J and T. Obviously, we can write $n = 2m$.

Proposition 3. An $\mathfrak{A}_3$ - structure on a smooth manifold N may be given equivalently:

1. By the operators P and Q, such that

$$P^2 = I, \quad Q^2 = I \quad \text{and} \quad PQ + QP = 0,$$

2. By the operators J and P, such that

$$J^2 = -I, \quad P^2 = I \quad \text{and} \quad JP = -PJ.$$

Proof: 1. If we set $P = JT - TJ$ and $Q = J + 2T$, by using the characteristic identity in *Case (3)* we have

$$\begin{aligned} P^2 &= (JT - TJ)(JT - TJ) = JTJT - JTTJ - TJJT + TJTJ \\ &= JT(I - TJ) + TJ(I - JT) = JT + TJ = I\,, \\ Q^2 &= (J + 2T)(J + 2T) = J^2 + 2JT + 2TJ = -I + 2I = I\,. \end{aligned}$$

2. In analogy with the previous case we have

$$\begin{aligned} JP &= J(JT - TJ) = J^2T - JTJ = -T - J(I - JT) \\ &= -T - J - T = -J - 2T\,, \\ PJ &= (JT - TJ)J = JTJ + T = J(I - JT) + T = J + 2T\,. \end{aligned}$$

Remark 1. In [1] the structure $\{J, P\}$ is called a *complex product structure.*

The next theorem is a modification of the result of A. Andrada [3].

Theorem 3. *Let N be a manifold with an $\mathfrak{A}_3$ -structure, given by the operators $\{J, T\}$. Then:*

1. There exists a unique torsion-free connection ∇ with respect to which J and T are parallel;

2. The leaves of the distribution $\mathfrak{D} = Im\,T$ are flat autoparallel submanifolds of N.

Proof. The connection ∇, which preserves the tensor fields J and P is given by

$$\begin{aligned} \nabla_x Y =& \frac{1}{4}\{[X,Y] - [PX,PY] + P[X,PY] - P[PX,Y] \\ & - J[X,JY] - J[PX,Q] + Q[X,QY] + Q[PX,JY]\,, \end{aligned}$$

where $Q = -JP$.

Since $T = \frac{1}{2}(PJ - J)$, it follows that $\nabla P = \nabla J = 0$. Our assertion follows from Proposition 2.

We may choose an atlas on N, whose Jacobian matrices are local constant. Then the operators J and T have the following form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix},$$

where I is the unit $(n \times n)$ - matrix. The theorem is proved. □

Remark 2. Another proof of the existence and uniqueness of ∇ is given in [1].

Remark 3. In this case the distribution $D = ImT$ is not invariant with respect to the operator J.

Remark 4. As it is shown in [1], [3] the connection ∇ does not need to be flat.

An essential property of the tangent bundle TM is the fact that it bears a *tangent structure*. More precisely, let $\pi : TM \to M$ and $K : TTM \to TM$ be natural projection and connection maps of ∇, respectively. If X is a vector field on M, we may define vertical lift X^v and horizontal lift X^h on TM by the relations

$$(d\pi)X^v = 0, \qquad KX^v = X\,,$$
$$(d\pi)X^h = X, \qquad KX^h = 0\,.$$

From a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{X}(M)$ we get the basis of $\mathfrak{X}(TM) : \{X_k^h, X_n^v\}$, $k = 1, \ldots, n$. With respect to this basis the tangent structure has the matrix expression mentioned above. We define

$$\tilde{J} : X^h \to X^v, \quad X^v \to -X^h, \quad \tilde{J}^2 = -I.$$

By setting $J = -\tilde{J}$, this leads us to the $\mathfrak{A}_3$ algebra.

Theorem 4. *The manifold TM can be endowed with integrable operators P, Q, subject to the relations*

$$P^2 = I, \;\; Q^2 = I, \;\; PQ = QP = 0.$$

Proof. Let us set $P = JT - TJ$ and $Q = J + 2T$. By using the identity in Case (3) of Theorem 2, we can easily verify our statement.

The integrability of J and T implies the integrability of P and Q. □

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