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EXISTENCE OF CONTINUOUS SOLUTIONS OF A PERTURBED LINEAR VOLTERRA INTEGRAL EQUATIONS

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In this paper we study the existence of continuous solutions on a compact interval of perturbed linear Volterra integral equations. The existence of such a solution is based on the well-known Leray–Schauder principle for a fixed point in Banach space. A special interest is devoted to the study of the uniqueness of continuous solution. Our numerical approach is based on a fixed point method and we apply quadrature rules to approximate the solution for the perturbed linear Volterra integral equations. The convergence of the numerical scheme is proved. Some numerical examples are given to show the applicability and accuracy of the numerical method and to validate the theoretical results.

Keywords: perturbed linear Volterra integral equation, Leray–Schauder principle, compact operator, fixed point method

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1. INTRODUCTION

Integral equations play a very important role in nonlinear analysis and have found numerous applications in engineering, mathematical physics, economics, etc. (see [2], [4], [10]). Many other applications in science are described by integral equations or integro-differential equations such as the Volterra‘s population growth model, biological species living together, propagation of stocked fish in a new lake, the heat radiation and so on [5], [6].

The existence of solutions of nonlinear integral equations has been considered in many papers and books [3], [4], [8]. In this paper, we show that under some assumptions the perturbed linear Volterra integral equation has an unique continuous solution in a bounded and closed interval.

We propose a numerical scheme to approximate the solution of this integral equation [11] and present some numerical examples to show the accuracy of our numerical method.

2. PRELIMINARIES

Let X be an arbitrary Banach space with a norm $\|\cdot\|$. By $C(X, X)$ we denote the space of all continuous operators acting in X. Set $\mathbb{R}_+ = [0, +\infty)$.

By $C([a, b]) = \{x : [a, b] \to \mathbb{R} \text{ is continuous}\}$ we denote the Banach space with the norm $\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$.

As usual, $L_p([a, b]) = \{x : [a, b] \to \mathbb{R}\,; \int_a^b |x(t)|^p\, ds < \infty\}$ stands for the Banach space with norm $\|x\|_p = \left(\int_a^b |x(t)|^p\, ds\right)^{1/p}$, $p \geq 1$.

For $r > 0$, we set $B_r = \{x \in C([a, b]); \|x\|_\infty \leq r\}$, i.e., B_r is a closed ball.

We consider the perturbed linear Volterra integral equation

$$x(t) = f(t) + \int_a^t K(t, s)x(s)ds + \int_a^t V(t, s)g(s, x(s))ds\,, \tag{2.1}$$

with given functions $f \in C([a, b])$, $g(\cdot, \cdot) : [a, b] \times \mathbb{R} \to \mathbb{R}$ and kernels $K(\cdot, \cdot)$, $V(\cdot, \cdot)$: $[a, b] \times [a, b] \to \mathbb{R}$.

We should mention that an extensive amount of work has been done on the existence and uniqueness of solutions of some special cases of Volterra integral equations, see for example [1], [3], [7], [8].

By using the following Leray–Schauder principle, we prove the existence of a solution of perturbed linear Volterra integral equation (2.1).

Theorem 1 ([7] (Leray-Schauder principle))**.** *Let X be a Banach space and the operator $T \in C(X, X)$ be compact. Suppose that any solution x of $x = \lambda T x$, $0 \leq \lambda \leq 1$ satisfies the a priori bound $\|x\| \leq M$ for some constant $M > 0$. Then T has a fixed point.*

Define the operator T on $C([a, b])$ by

$$Tx(t) = f(t) + \int_a^t K(t, s)x(s)ds + \int_a^t V(t, s)g(s, x(s))ds\,. \tag{2.2}$$

3. THE EXISTENCE OF A SOLUTION

Theorem 2. *Let the following conditions be fulfilled:*
1). The function $g(s,x)$ satisfies

$$\sup_{s\in[a,b],x\in\mathbb{R}}\left(|g(s,x)|\,,\left|\frac{\partial g}{\partial x}(s,x)\right|\right)\leq G(s)\phi(|x|), \tag{3.1}$$

where $G(\cdot)$ is a positive measurable function and $\phi(\cdot)$ is positive and continuous function satisfying

$$\lim_{y\to+\infty}\frac{\phi(y)}{y}=L<\infty. \tag{3.2}$$

2). The kernels $K(t,s)$ and $V(t,s)$ are continuous with respect to t and satisfy

$$|K(t,s)|\leq K_1(t)K_2(s), \qquad |V(t,s)|\leq V_1(t)V_2(s), \tag{3.3}$$

where $K_1(\cdot),V_1(\cdot)\in C([a,b])$ and $K_2(\cdot),G(\cdot)V_2(\cdot)\in L_1([a,b])$.
Then the equation (2.1) has a solution in $C([a,b])$.

Proof. We observe that condition (3.2) implies the existence of a positive real number $A>0$ such that $\left|\frac{\phi(u)}{u}\right|\leq\frac{3}{2}L=L'$, for all $u\geq A$.

First, we shall prove that the operator $T:C([a,b])\to C([a,b])$ is continuous.

Let $x\in C([a,b])$. Hence for all $s\in[a,b]$, one conclude that $|x(s)|$ is contained in a compact set of $\mathbb{R}_+$. Moreover, $\phi(\cdot)$ is continuous over $\mathbb{R}_+$, then one concludes that there exists a positive constant N_ϕ, such that $\phi(|x(s)|\leq N_\phi$. Let $h>0$. On using assumptions 1) and 2) and applying the dominated convergence theorem, and using that $f\in C([a,b])$, we have

$$\begin{aligned}&\lim_{h\to0}|Tx(t+h)-Tx(t)|\leq\lim_{h\to0}|f(t+h)-f(t)|\\&+\|x\|_\infty\int_a^t\lim_{h\to0}|K(t+h,s)-K(t,s)|\,ds+\|K_1\|_\infty\|x\|_\infty\lim_{h\to0}\int_t^{t+h}K_2(s)ds\\&+N_\phi\int_a^t\lim_{h\to0}|V(t+h,s)-V(t,s)|\,G(s)ds+\|V_1\|_\infty N_\phi\lim_{h\to0}\int_t^{t+h}V_2(s)G(s)ds=0\,.\end{aligned}$$

Next, we shall prove that the operator T is continuous over $C([a,b])$.

Let $\{x_n\}_{n=1}^\infty\in C([a,b])$ be a sequence converging uniformly to x. Since $C([a,b])$ is complete, then $x\in C([a,b])$. Hence, for all $n\in N$, for all $s\in[a,b]$ and $\Theta_s\in[0,1]$, one concludes that $|\Theta_s x_n(s)+(1-\Theta_s)x(s)|$ is contained in a compact set of $\mathbb{R}_+$. Moreover, $\phi(\cdot)$ is continuous over $\mathbb{R}_+$, therefore there exists a positive constant M_ϕ such that $\phi(|\Theta_s x_n(s)+(1-\Theta_s)x(s)|)\leq M_\phi$.

From assumptions 1) and 2) for each $t \in [a, b]$ we have

$$
\begin{aligned}
&|Tx_n(t) - Tx(t)| \\
&\le \|x_n - x\|_\infty \left[\int_a^t |K(t,s)|\, ds + \int_a^t |V(t,s)| \left| \frac{\partial g}{\partial x}(s, \Theta_s x_n(s) + (1 - \Theta_s) x(s)) \right| ds \right] \\
&\le \|x_n - x\|_\infty \left[\|K_1\|_\infty \|K_2\|_1 + \|V_1\|_\infty \|V_2 \cdot G\|_1 M_\phi \right].
\end{aligned}
$$

Therefore, $\lim_{n\to\infty} \|Tx_n - Tx\|_\infty = 0$ or, equivalently, T is continuous over $C([a, b])$.

Next, we shall prove that the operator T is compact on $C([a, b])$. Let us set $E := \{Tx\,; x \in B_r\}$.

On using Arzella - Ascoli theorem, the compactness of the set E will be ensured if we show that E is equicontinuous and uniformly bounded.

Let $x \in B_r$. Since $\phi(\cdot)$ is continuous over $\mathbb{R}_+$, there exists a positive constant P_ϕ such that $\phi(|x(s)|) \le P_\phi$ for each $s \in [a, b]$. From assumptions 1) and 2), for every $t \in [a, b]$ we have

$$
\begin{aligned}
&|Tx(t)| \le \|f\|_\infty + \int_a^t K_1(t) K_2(s)\, |x(s)|\, ds + \int_a^t V_1(t) V_2(s) G(s) \phi(|x(s)|) ds \\
&\le \|f\|_\infty + \|K_1\|_\infty \|K_2\|_1 \|x\|_\infty + \|V_1\|_\infty \|V_2 G\|_1 P_\phi.
\end{aligned}
$$

Consequently, E is uniformly bounded.

Let $x \in B_r$, $t', t'' \in [a, b]$ and $t' < t''$. From condition 1) and 2) we obtain

$$
\begin{aligned}
&|Tx(t'') - Tx(t')| \le |f(t'') - f(t')| \\
&+ \|x\|_\infty \int_a^{t'} |K(t'', s) - K(t', s)|\, ds + \|x\|_\infty \|K_1\|_\infty \int_{t'}^{t''} K_2(s) ds \\
&+ P_\phi \int_a^{t'} |V(t'', s) - V(t', s)|\, G(s) ds + P_\phi \|V_1\|_\infty \int_{t'}^{t''} V_2(s) G(s) ds.
\end{aligned}
$$

By applying the dominated convergence theorem to the right-hand side of the above inequality, one concludes that $\lim_{t'\to t''} |Tx(t'') - Tx(t')| = 0$.

Next, we shall prove that any solution of the equation $x = \lambda Tx$, $0 \le \lambda \le 1$ is bounded by the same constant $M > 0$. Let

$$
M_1 = |\lambda|\, \|f\|_\infty + |\lambda|\, \|V_1\|_\infty \|V_2 G\|_1 \sup_{u \in [0, A]} \phi(u)\,, \tag{3.4}
$$

$$
M_2 = \max\{\lambda \|K_1\|_\infty\,, \lambda \|V_1\|_\infty\}\,, \tag{3.5}
$$

$$
M = M_1 \exp(M_2 [\|K_2\|_1 + \|V_2 G\|_1 L'])\,, \tag{3.6}
$$

$$Q(s) = K_2(s) + V_2(s)G(s)L'. \tag{3.7}$$

Let $x \in C([a,b])$ be a solution of $x = \lambda T x$ for some $0 \le \lambda \le 1$, then we have

$$|x(t)| \le |\lambda| \, \|f\|_\infty + |\lambda| \, \|K_1\|_\infty \int_a^b K_2(s) \, |x(s)| \, ds$$

$$+ |\lambda| \, \|V_1\|_\infty \int_a^b V_2(s)G(s) \sup_{u \in [0,A]} \phi(u) ds + |\lambda| \| \, V_1\|_\infty \int_a^b V_2(s)G(s)L' \, |x(s)| \, ds$$

$$\le |\lambda| \left[\|f|_\infty + \|V_1\|_\infty \, \|V_2 G\|_1 \sup_{u \in [0,A]} \phi(u) \right] + M_2 \int_a^b [K_2(s) + V_2(s)G(s)L'] \, |x(s)| \, ds.$$

Hence, from (3.4), (3.5), (3.7) we get that $|x(t)| \le M_1 + M_2 \int_a^b Q(s) \, |x(s)| \, ds$.

By using the general version of Gronwall‘s inequality together with the previous inequality, one concludes that

$$|x(t)| \le M_1 \exp(M_2 \int_a^b Q(s) ds) = M_1 \exp(M_2 \left[\|K_2\|_1 + \|V_2 G\|_1 \, L' \right]) = M.$$

Since M_1 and M_2 do not depend on x, we conclude that the solutions of $x = \lambda T x$, $0 \le \lambda \le 1$ are uniformly bounded by the same constant M. Now the Leray–Schauder principle implies that T has a fixed point in $C([a,b])$. $\square$

4. NUMERICAL APPROACH AND ITS CONVERGENCE

In the proof of Theorem 2 we have shown that the continuous solutions of $x = Tx$ are uniformly bounded by the same constant M, and consequently they are contained in a closed ball B_M. We choose an initial function $x_0 \in B_M$ and construct the sequence $\{x_n(t)\}_{n=0}^\infty$ as follows

$$x_{n+1}(t) = T x_n(t), \qquad n \ge 0, \; t \in [a,b]. \tag{4.1}$$

In the next theorem we show that under certain assumptions the sequence $\{x_n(t)\}_{n=0}^\infty$ constructed by (4.1) converges to the unique fixed point $\tilde{x}$ of T.

Theorem 3. *Let the following conditions be fulfilled.*

1. *The conditions of Theorem 2 hold.*
2. *The functions $K_2(\cdot), V_2(\cdot)G(\cdot) \in L_p([a,b])$ for some $p \ge 1$;*
3. *With the constant M defined by* (3.6), *the following inequality holds:*

$$\left[\|K_1\|_\infty \, \|K_2\|_p + \|V_1\|_\infty \, \|V_2 \cdot G\|_p \max_{u \in [-M,M]} \phi(|u|) \right] (b-a)^{\frac{1}{q}} < 1,$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $p\geq 1$.

Then the operator T *is a contractive mapping in* B_M *and has exactly one fixed point, say,* $\tilde{x}(t)$. *Moreover, the generated by* (4.1) *sequence* $\{x_n(t)\}_{n=0}^{\infty}$ *convergence to this fixed point, i.e.*

$$\lim_{n\to\infty} x_n(t)=\tilde{x}(t) \quad \textit{for every} \ \ t\in[a,b], \tag{4.2}$$

and

$$\|x_n-\tilde{x}\|_{\infty}\leq \frac{L^n}{1-L}\|x_1-x_0\|_{\infty}, \tag{4.3}$$

where $0<L<1$ *is the contraction constant of* T.

Proof: Suppose that $x,y\in B_M$. For all $s\in[a,b]$ and $\Theta_s\in[0,1]$, there holds $|\Theta_s x(s)+(1-\Theta_s)y(s))\leq M$. For $t\in[a,b]$, using assumptions 1) and 2) of Theorem 2 and Holder‘s inequality, we obtain

$$\begin{aligned}
&|Tx(t)-Ty(t)|\leq \int_a^t K_1(t)K_2(s)\,|x(s)-y(s)|\,ds\\
&+\int_a^t V_1(t)V_2(s)\left|\frac{\partial g}{\partial x}(s,\Theta_s x(s)+(1-\Theta_s)y(s))\right||x(s)-y(s)|\,ds\\
&\leq \|x-y\|_{\infty}\left[\|K_1\|_{\infty}\|K_2\|_p(b-a)^{\frac{1}{q}}+\|V_1\|_{\infty}\max_{u\in[-M,M]}\phi(|u|)\,\|V_2G\|_p(b-a)^{\frac{1}{q}}\right],
\end{aligned}$$

where $1/p+1/q=1$, $p\geq 1$.

Let $L=(b-a)^{\frac{1}{q}}\left[\|K_1\|_{\infty}\|K_2\|_p+\|V_1\|_{\infty}\|V_2G\|_p\max\limits_{u\in[-M,M]}\phi(|u|)\right]$. By assumption 3), we have $L<1$, hence the operator T satisfies the Lipschitz condition

$$\|Tx-Ty\|_{\infty}\leq L\,\|x-y\|_{\infty}\,. \tag{4.4}$$

If we assume that T has two fixed point $\tilde{x},\tilde{y}\in B_M$, we would have

$$\|\tilde{x}-\tilde{y}\|_{\infty}=\|T\tilde{x}-T\tilde{y}\|_{\infty}\leq L\,\|\tilde{x}-\tilde{y}\|_{\infty}, \tag{4.5}$$

and since $0<L<1$, it follows that $\tilde{x}\equiv\tilde{y}$. Hence, the operator T has a unique fixed point in B_M.

Finally, relations (4.2), (4.3) are proved in a standard way by using equation (4.4) and [11, p.267, Theorem 5.2.3.], with $X=C([a,b])$. □

From (3.4) and (3.6) it follows that $f\in B_M$, hence we can choose f as an initial function, $x_0\equiv f$. We apply quadrature formulae such as trapezoidal, Simpson and "3/8"-rule to evaluate numerically the integrals in the operator T.

We construct an uniform mesh on $[a,b]$ with stepsize h: $s_k = a + (k-1)h$, $k = \overline{1,n}$, where $a + nh \le b < a + (n+1)h$. We put $t = s_k$ in (2.1) and obtain the following nonlinear integral system for the unknowns $x_k = x(s_k)$, $k = \overline{1,n}$:

$$
\begin{aligned}
x_1 &= f(s_1) = f(a)\,, \\
x_k &= f(s_k) + \int_a^{s_k} K(s_k,s)\,x(s)\,ds + \int_a^{s_k} V(s_k,s)\,g(s,x(s))\,ds\,, \quad k = \overline{2,n}\,.
\end{aligned}
\tag{4.6}
$$

We apply quadrature rules for each k with nodes $s_1, s_2, \ldots, s_k$ and coefficients $h.A_{kj}$, $k = \overline{2,n}, j = \overline{1,k}$ to approximate the integrals in (4.6) for $k = \overline{2,n}$:

$$
x_k = f(s_k) + h\sum_{j=1}^{k} A_{kj}\,K(s_k,s_j)\,x_j + h\sum_{j=1}^{k} A_{kj}\,V(s_k,s_j)\,g(s_j,x_j) + R_k(x)\,,
$$

where $R_k(x) = O(h^r)$ is the error term due to the quadrature rule. We denote

$$
F_k(x_1,\ldots,x_n) = f(s_k) + h\sum_{j=1}^{k} A_{kj}\,K(s_k,s_j)\,x_j + h\sum_{j=1}^{k} A_{kj}\,V(s_k,s_j)\,g(s_j,x_j)\,.
$$

In our calculations we choose two different schemes for coefficients A_{kj}. The first scheme is constructed on the base of the trapezium quadrature formulas. The other is based on the Simpson rule and the 3/8-rule (also called Simpson 3/8) [11, Section 3.1].

The fixed point method with initial condition $x_k^0 = f(s_k)$, $k = \overline{1,n}$ is as follows:

$$
x_k^{i+1} = F_k(x_1^i,\ldots,x_n^i)\,, \quad k = \overline{2,n}\,, \quad i = 0,1,2,\ldots\,.
$$

The convergence of the numerical iterations is proved by Theorem 3.

4.2. EXPERIMENTAL RESULTS

We have tested the efficiency of the proposed numerical scheme on two Volterra integral equations given in the examples below. In our numerical scheme the iterations stop when $E^{i+1} = \|x^{i+1} - x^i\| = \max_j |x_j^{i+1} - x_j^i| \le \varepsilon$, where $\varepsilon = 10^{-7}$ is the chosen precision. All routines have been written in the software system Wolfram Mathematica 9.0.

Example 1. Consider the perturbed linear Volterra integral equation

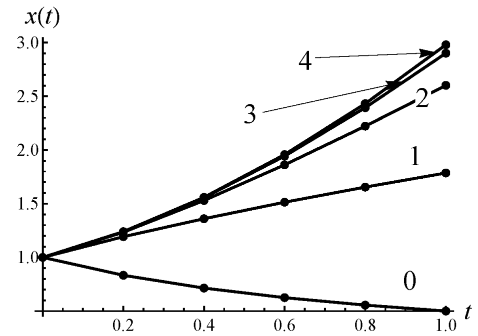
$$
x(t) = \frac{1}{t+1} + \int_0^t \frac{t+1}{s^2+s+1}\,x(s)\,ds + \int_0^t \frac{1}{t+1}\,\frac{1}{\sqrt{s^2+s+2}}\sqrt{1+x(s)}\,ds
$$

with the exact solution $x^{ex}(t) = t^2 + t + 1$ for $t \in [0, 1]$. Obviously, the assumptions of Theorem 3 are fulfilled.

In Table 1 are shown the errors E^i for some iterations with different quadrature methods and different choice of the step h. It is seen that the iterations in the fixed point method are more effective for reaching the desired precision than the scheme with choice of a grid and use of quadrature rule. The left panel of Figure 1 shows the first approximations x^i, $i = \overline{0, 4}$. The exact and the approximate solution based on Simpson quadrature rule with $h = 0.2$ are shown in the right panel of Figure 1. Good agreement is demonstrated. The last approximation, the errors between the last approximate solutions E_k^{12}, $k = \overline{1, 6}$ and with the exact solution $E_k^{ex} = x_k^{12} - x_k^{ex}$, $k = \overline{1, 6}$ are listed in Table 2.

Table 1: Numerical results for Example 1.

	$h = 0.2$		$h = 0.02$		$h = 0.001$	
	Trapezoid	Simpson	Trapezoid	Simpson	Trapezoid	Simpson
$i = 1$	1.2981	1.2863	1.2862	1.2861	1.2861	1.2861
$i = 3$	0.2942	0.2997	0.3002	0.3003	0.3003	0.3003
$i = 9$	1.42E-5	8.36E-6	5.56E-6	5.49E-6	5.49E-6	5.48E-6
$i = 12$	4.02E-8	1.60E-8				



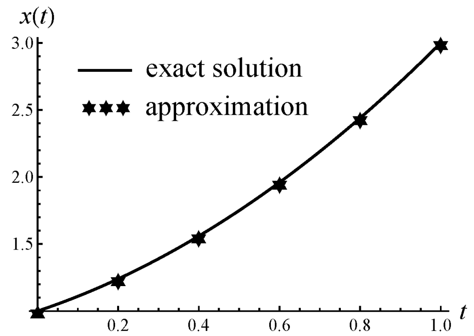


Figure. 1. Left: First 5 approximations; Right: 12-th approximation and the exact solution. Both graphics demonstrate the results using Simpson's rules with $h = 0.2$ for Example 1.

Table 2: Comparison with exact solution in Example 1.

k	1	2	3	4	5	6
x_k^{12}	1.	1.24	1.56	1.96	2.44	3.
E_k^{12}	0.	1.750E-11	1.297E-10	8.309E-10	4.031E-9	1.605E-8
E_k^{ex}	0.	-2.283E-12	-1.693E-11	-1.092E-10	-5.359E-10	-2.173E-9

Example 2. Consider the perturbed linear Volterra integral equation

$$x(t) = \frac{t^2}{4}e^t + \int\limits_0^t \frac{(t+s)^2}{4} e^{t-s} x(s)\, ds + \int\limits_0^t \frac{s\ln(1+t)}{(1+t^2)(1+s^2)} \frac{se^{-2s}}{x(s)}\, ds\,, \quad t \in [0,1]$$

It is easily seen that the functions occurring in this integral equation satisfy the assumptions of Theorem 3.

Table 3: Numerical results for Example 2.

	$h = 0.2$		$h = 0.02$		$h = 0.001$	
	Trapezoid	Simpson	Trapezoid	Simpson	Trapezoid	Simpson
$i = 1$	8.2666	8.2443	8.2444	8.2442	8.2442	8.2442
$i = 10$	0.0122	0.0008	6.50E-5	5.76E-5	5.76E-5	5.76E-5
$i = 14$	0.0003	4.19E-6	1.16E-8			
$i = 17$	2.05E-5	7.99E-8				
$i = 23$	8.40E-8					

Table 3 shows the results analogous to those in Table 1 but for Example 2. Here the role of the quadrature rule and the stepsize of the grid is significant. In Figure 2 (left panel), the approximations obtained in the first five iterations are shown. The decrease of the error with increasing the number of iterations is shown in the right panel of Figure 2, where $E^{10} = 6.06E-5$ and $E^{14} = 1.07E-8$.

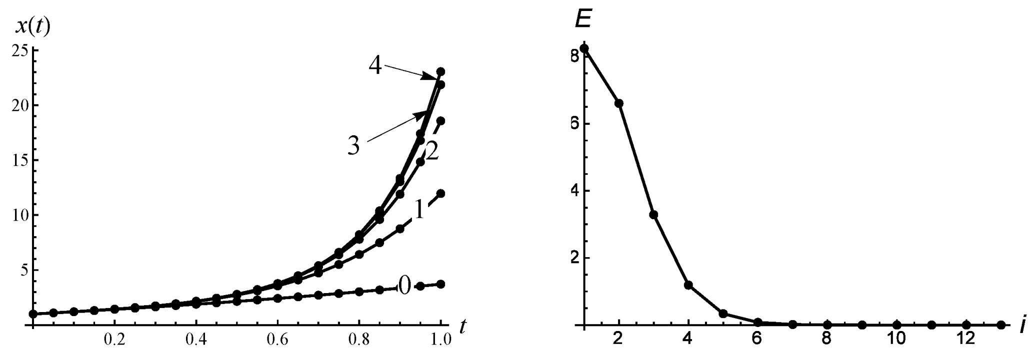


Figure. 2. Left: First 5 approximations; Right: The maximum error related to each iteration. Both graphics demonstrate the results using Simpson's rules with $h = 0.05$ for Example 2.

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5. REFERENCES

[1] Afif Ben, A.: Some fixed points theorems and existence of weak solutions of Volterra integral equations under Henstock-Kurzweil-Petttis integrability. *Comentationes Mathematicae Universitatis Carolinae*, **52**, no. 2, 2011, 177-190.

[2] Agarwal, R. P., O'Regan, D., Wong P. J. Y.: *Possitive solutions of differential, difference and integral equations*, Kluwer, Dordrecht, 1999.

[3] Banas, J., Darwish, M. A., Olszowy, L., Sadarangani, K.: Existence and Asymtotic Behaviour of Solutions of Differential and Integral Equations in Some Functional Spaces. *J. Function Spaces*, art. ID 235043, 2015.

[4] Burton, T. A.: *Volterra Integral and Differential Equations, Mathematics in science and engineering*, **202**, Amsterdam, Boston, Elsevier, 2005.

[5] Corduneanu, C.: *Abstract Volterra equations and weak topologies, Delay Differential Equations and Dynamical Systems (Claremont, Clif., 1990)*, Lecture notes in Math., **1475**, Springer, Berlin, 1991, 110–115.

[6] Gripenberg, S., Londen, O., Staffans, O.: *Volterra Integral and Functional Equations.* Encyclopedia of Mathematics and Its Applications, **34**, Cambridge University Press, 1990.

[7] Karoui, A.: Existence and approximate solutions of nonlinear integral equations. *J. Inequal. Appl.*, **5**, 2005, 569–581.

[8] Mateknejad, K., Nouri, K., Mollapourasi, R.: Existence of solutions for some nonlinear integral equations. *Communications in nonlinear science and numerical simulations*, **14**, no. 6, 2009, 2559–2564.

[9] Nemati, S.: Numerical Solution of Volterra-Fredholm integral equations using Legendre collocation method. *J. Comp. Appl. Math.s*, **278**, 2015, 29–36.

[10] O'Regan, D., Meehan, M.: *Existence Theory for Nonlinear Integral and Integro-Differential Equations*. Kluwer, Dordrecht, **445**, 1998.

[11] Stoer, J., Bulirsch R.: *Introduction to Numerical Analysis*, third ed., Springer Verlag, New York, 2002.

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