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NON-INTEGRABILITY OF A SYSTEM FROM THE RIGID BODY DYNAMICS

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We study the complete integrability of a system describing the motion of a rigid body with a fixed point and a particle oscillating in it in the absence of external forces. Using the Ziglin-Morales-Ramis theory, we prove rigorously that the considered system is integrable only in the case of dynamical symmetry.

Keywords: complete integrability, Ziglin-Morales-Ramis theory, system describing the motion of a rigid body with a particle oscillating in it

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1. Introduction

Consider the mechanical system which consists of a rigid body with a fixed point O and a particle, attached to a spring and oscillating in the rigid body along a line that passes through the fixed point according to the Hooke's law. Furthermore, we assume that the particle is oscillating along a principal inertia axis for the body, say, that would be the axis which inertia moment is denoted with C.

The equations of motion in the case of no external forces are [6]:

$$\begin{aligned}
&A\dot{\omega}_1 + (C-B)\omega_2\omega_3 = -2mr\dot{r}\omega_1 - mr^2\dot{\omega}_1 + mr^2\omega_2\omega_3,\\
&B\dot{\omega}_2 + (A-C)\omega_1\omega_3 = -2mr\dot{r}\omega_2 - mr^2\dot{\omega}_2 - mr^2\omega_1\omega_3,\\
&C\dot{\omega}_3 + (B-A)\omega_1\omega_2 = 0,\\
&\ddot{r} + r\left(\frac{\sigma}{m} - \omega_1^2 - \omega_1^2\right) = 0,
\end{aligned} \tag{1.1}$$

where ω_1, ω_2, ω_3 are the components of the angular velocity of the body, A, B, C are the components of the inertia tensor, r is the distance between the particle and the

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fixed point O, σ is the stiffness of the spring and m is the mass of the particle. The model can be considered as the simplest meaningful model of a flexible body [11]. Similar models are considered in [1,4], but there the oscillating particle is additionally subjected to dissipative forces.

The above system admits the integrals: the energy integral

$$H = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 + m\dot{r}^2 + mr^2(\omega_1^2 + \omega_2^2) + \sigma r^2 = 2H_0 = \text{const} \tag{1.2}$$

and

$$M^2 = (A + mr^2)^2\omega_1^2 + (B + mr^2)^2\omega_2^2 + C^2\omega_3^2 = M_0^2 = \text{const}. \tag{1.3}$$

In fact, system (1.1) can be presented in a Hamiltonian form. To see that, we first denote

$$z_1 = r, \quad z_2 = \dot{r}, \quad M_1 = (A + mz_1^2)\omega_1, \quad M_2 = (B + mz_1^2)\omega_2, \quad M_3 = C\omega_3. \tag{1.4}$$

In these variables system (1.1) becomes

$$\begin{aligned}
\dot{M}_1 &= M_2M_3\left(\frac{1}{C} - \frac{1}{B + mz_1^2}\right),\\
\dot{M}_2 &= M_1M_3\left(\frac{1}{A + mz_1^2} - \frac{1}{C}\right),\\
\dot{M}_3 &= M_1M_2\left(\frac{1}{B + mz_1^2} - \frac{1}{A + mz_1^2}\right),\\
\dot{z}_1 &= z_2,\\
\dot{z}_2 &= z_1\left(\frac{M_1^2}{(A + mz_1^2)^2} + \frac{M_2^2}{(B + mz_1^2)^2} - \frac{\sigma}{m}\right).
\end{aligned} \tag{1.5}$$

The first integrals (1.2) and (1.3) take the form

$$H = \frac{1}{2}\left(\frac{M_1^2}{(A + mz_1^2)} + \frac{M_2^2}{(B + mz_1^2)} + \frac{M_3^2}{C} + mz_2^2 + \sigma z_1^2\right) = H_0, \tag{1.6}$$

$$M^2 = M_1^2 + M_2^2 + M_3^2 = M_0^2. \tag{1.7}$$

Next, we introduce the Poisson brackets in a way similar to [2]

$$\{M_1, M_2\} = -M_3, \quad \{M_1, M_3\} = M_2, \quad \{M_2, M_3\} = -M_1, \quad \{z_1, mz_2\} = 1 \tag{1.8}$$

and the rest brackets are trivial. Then, we can write equations (1.5) as

$$\begin{aligned}
\dot{M}_i &= \{M_i, H\}, \quad i = 1, 2, 3,\\
\dot{z}_1 &= \{z_1, H\},\\
m\dot{z}_2 &= \{mz_2, H\}.
\end{aligned} \tag{1.9}$$

Notice that the defined Poisson structure is degenerated – every smooth function commutes with M^2. However, on the fixed level $M^2 = M_0^2 = \text{const}$ the Poisson structure is non-degenerated and the considered system is a Hamiltonian one with two degrees of freedom. Therefore, for complete integrability one needs one more first integral.

The only integrable case, found so far, is the case of dynamical symmetry $A = B$: the additional integral is $M_3 = \text{const}$. Later, Yehia in [16] introduced more general integrable case in which the particle moves on the axis of dynamical symmetry and it is subjected to an arbitrary conservative force that depends only on the distance from the fixed point.

Studying the integrability, the following result has been established.

Theorem 1.1 ([6]). *System (1.1) with $A > B > C$ does not possess an additional holomorphic first integral except possibly for the case*

$$\frac{C(A+B-C)}{(B-C)(A-C)} = n(n+1), \quad n \geq 1, \quad n \in \mathbb{Z}. \tag{1.10}$$

For the proof of this theorem the Ziglin's method is used (see Section 2). The exceptional cases $A = C$ and $B = C$, as well as, case (1.10) for small n were treated numerically. The Poincaré sections, obtained along the numerical experiments revealed chaotic behavior which exclude the existence of an additional first integral (see [6]). Based on these numerical experiments, it was conjectured there that system (1.1) is integrable only in the case of dynamical symmetry $A = B$.

Later, with a rather different approach, it is shown in [7] that for $B = A+\varepsilon$ and ε sufficiently small, there are horseshoes on almost every energy level of system (1.1), that is, the dynamics is chaotic. This means that the considered system does not admit additional holomorphic first integral. However, it is not clear whether such an integral exists far from $A = B$.

The goal of this paper is to give a rigorous proof that system (1.1) is non-integrable when $A \neq B$.

Theorem 1.2. *When $A \neq B$, system (1.1) is meromorphically non-integrable.*

In other words, there is no additional meromorphic first integral and the case $A = B$ is the only integrable case in sense of Liouville. The proof of this theorem is carried out with the help of the Morales-Ramis approach, based on the Differential Galois Theory.

The outline of the paper is as follows: in Section 2 we recall briefly the Ziglin-Morales-Ramis theory for integrability of Hamiltonian systems in complex domain; the proof of Theorem 2 is given in Section 3; finally, we collect some remarks in Section 4.

2. Theoretical background

In this section, in a succinct way we review basic facts about the Ziglin-Morales-Ramis theory. For more details and related differential Galois theory, we refer to [12, 13]. Some applications can be found in [9, 12].

Consider an analytic Hamiltonian function H, defined on a complex manifold M^{2n}. This Hamiltonian gives rise to a Hamiltonian system in the known way

$$\dot{\mathbf{x}} = X_H(\mathbf{x}), \quad t \in \mathbb{C}, \quad \mathbf{x} \in M^{2n}. \tag{2.1}$$

Already working in the complex domain, we accept the definition of integrability as in the real case [2]: A Hamiltonian system with n degrees of freedom is called Liouville integrable if there exist n independent (almost everywhere) first integrals $F_1 = H$, $F_2, \ldots, F_n$ in involution, i.e., $\{F_i, F_j\} = 0$ for all i, j, where $\{,\}$ is the Poisson bracket.

In most of the integrable Hamiltonian systems, the known first integrals when are considered in the complex domain are holomorphic or meromorphic functions, that is, single-valued functions. It was established that branching of solutions of Hamiltonian systems in complex time plane is an obstruction to the existence of new first integrals (see, e.g., Ziglin [17]).

Suppose system (2.1) has a non-equilibrium solution $\phi(t)$. Denote by Γ its phase curve. Along this solution we can write the variational equations (VE)

$$\dot{\boldsymbol{\xi}} = DX_H(\phi(t))\boldsymbol{\xi}, \quad \boldsymbol{\xi} \in T_\Gamma M^{2n}. \tag{2.2}$$

The first integral H gives rise to a linear integral dH of the variational equations. Using the integral dH we can reduce the variational equations. Consider the normal bundle of Γ, $F := T_\Gamma M^{2n}/TM^{2n}$ and let $\pi\colon T_\Gamma M^{2n} \to F$ be the natural projection. The system of equations (2.2) define a system of equations on F

$$\dot{\boldsymbol{\eta}} = \pi_*(DX_H(\phi(t))(\pi^{-1}\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in F \tag{2.3}$$

which is called the normal variational equations (NVE). To each meromorphic first integral of the Hamiltonian system (2.1) in the neighborhood of the curve Γ corresponds a meromorphic first integral of (NVE) [17]. In this way, the problem of complete integrability of the Hamiltonian system (2.1) reduces to the study of integrability of the linear system (2.3) (or (2.2)).

Consider such a linear non-autonomous system

$$\dot{\boldsymbol{\xi}} = A(t)\boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathbb{C}^n \tag{2.4}$$

with t defined on some Riemann surface Γ. Denote by S the set of singular points of (2.4) $S := \{t_1, t_2, \ldots, t_s\}$. Let $\Theta(t)$ be a fundamental solution of (2.4), which is analytic in a neighborhood of $t_0 \in \Gamma \setminus S$. It is said that a singular point t_i is regular if any of solutions of (2.4) has at most polynomial growth in arbitrary sector with a vertex at t_i. Otherwise the singular point is called irregular. A singular point t_i is a singularity of Fuchs type, if $A(t)$ has a simple pole at $t = t_i$. For system (2.4) singularities of Fuchs type are regular singularities. A linear system which has singularities of Fuchs type only is called Fuchsian system.

The continuation of $\Theta(t)$ along a loop (such loops usually encircle singularities t_i) on Γ defines a linear automorphism of the vector space of all solutions analytic in the vicinity of t_0, called the monodromy transformation. This linear automorphism Δ_γ, associated with a loop $\gamma \in \pi_1(\Gamma, t_0)$ corresponds to multiplication of $\Theta(t)$ from the right by a constant matrix M_γ, called monodromy matrix

$$\Delta_\gamma \Theta(t) = \Theta(t) M_\gamma.$$

The set of these matrices forms the monodromy group [13, 18].

Next, we recall tersely the essential notions and results from the differential Galois theory in order to understand the applications to the integrability of Hamiltonian systems. The detailed statements and proofs can be found in [13].

Mark the coefficient field in (2.4) by $\mathbb{K}$. A differential field $\mathbb{K}$ is a field with a derivation $\partial = '$, i.e. an additive mapping satisfying Leibnitz rule. A differential automorphism of $\mathbb{K}$ is an automorphism commuting with the derivation.

Taking ϑ_{ij} to be the elements of the fundamental matrix $\Theta(t)$, let $\mathbb{L}(\vartheta_{ij})$ be the extension of $\mathbb{K}$ generated by $\mathbb{K}$ and ϑ_{ij}. This extension is called Picard-Vessiot extension and it is a differential field. The differential Galois group $G := \mathrm{Gal}(\mathbb{L}/\mathbb{K})$ is defined to be the group of all differential automorphisms of $\mathbb{L}$ leaving the elements of $\mathbb{K}$ fixed. This group G is an algebraic group. It has a unique connected component G^0 which contains the identity and which is a normal subgroup of finite index. Furthermore, G can be represented as an algebraic linear subgroup of $\mathrm{GL}(n, \mathbb{C})$ by

$$\sigma \Theta(t) = \Theta(t) R_\sigma,$$

where $\sigma \in G$ and $R_\sigma \in \mathrm{GL}(n, \mathbb{C})$.

As a matter of fact, by its definition the monodromy group is contained in the differential Galois group of the corresponding system. Besides, for Fuchsian systems we have

Theorem 2.1 (Schlesinger). *For a system with singularities of Fuchs type, the differential Galois group coincides with the Zariski closure in* $\mathrm{GL}(n, \mathbb{C})$ *of the monodromy group.*

Further, we review some facts from the theory of the linear systems with singularities. Consider the system

$$t\dot{Y} = A(t)Y, \tag{2.5}$$

where $A(t)$ is holomorphic at $t = 0$. The point $t = 0$ is clearly a regular singular point for this system.

The change $Y = P(t)Z$, where $P(t)$ is holomorphic at $t = 0$ brings system (2.5) to the form

$$t\dot{Z} = B(t)Z, \quad \text{where} \quad B(t) = P^{-1}(t)A(t)P(t) - tP^{-1}(t)\dot{P}(t).$$

In order to find the fundamental solution and the monodromy around the singularity we can determine $P(t)$ in such a way that $B(t)$ is as simple as possible. Let

$$A(t) = \sum_{s=0}^{\infty} A_s t^s, \quad P(t) = \sum_{s=0}^{\infty} P_s t^s, \quad B(t) = \sum_{s=0}^{\infty} B_s t^s.$$

We can take $P_0 = E$ and $B_0 = A_0$. If the eigenvalues of A_0 do not differ by a positive integer a theorem from [10, 14] asserts that we can obtain all $B_s = 0$, $s \geq 1$. So, system (2.5) takes the form

$$t\dot{Z} = A(0)Z,$$

which is solved as $Z = \exp(A_0 \log t) = t^{A_0}$. Hence, $Y = P(t)t^{A_0}$. For more details, see [10, 14].

Around the 80s of the previous century Ziglin [17] obtained necessary conditions for integrability of complex-analytical Hamiltonian systems by investigating the monodromy group of normal variational equations: the monodromy group manifests the branching of solutions of (NVE). Observe that Ziglin's result does not assume that the existing n independent first integrals are in involution.

A more general and powerful approach, builded on the differential Galois theory was taken a decade later. The solutions of (2.3) define an extension $\mathbb{L}_1$ of the coefficient field $\mathbb{K}$ of (VE). This naturally defines a differential Galois group $G = \mathrm{Gal}(\mathbb{L}_1/\mathbb{K})$. Then a central result of the Morales-Ramis theory is

Theorem 2.2 (Morales-Ruiz-Ramis [12])**.** *Suppose that a Hamiltonian system has n meromorphic first integrals in involution. Then the identity component G^0 of the Galois group $G = \mathrm{Gal}(\mathbb{L}_1/\mathbb{K})$ is abelian.*

This result has been extended further with the idea to use the Galois groups of higher variational equations as obstructions to integrability. We will not use them here.

3. Proof of Theorem 1.2

When the Morales-Ramis approach is applied (Theorem 2.2) one needs to find a particular solution, to obtain the variational equations along it and then to study their Galois group.

We carry on the proof of the main Theorem in two steps. Firstly, we deal with the general case $A > B > C$. Starting with an entirely different particular solution from that found in [6], we study the monodromy group of the corresponding NVE and make use of the Schlesinger's Theorem to get information about the Galois group.

Finally, for the cases $A = C$ and $B = C$ we use a result about the solvability of double confluent Heun equation, obtained with the help of Kovacic algorithm by Duval et al. [9].

Remark 3.1. While studying the integrability we keep the parameters of the problem real and fixed (of course, due to the mechanical interpretation of these parameters we have m, $\sigma > 0$, A, B, $C > 0$ and $A+B > C$, $A+C > B$, $B+C > A$).

However, the parameters which appear in the particular solutions below are arbitrary and we chose them in an appropriate way to obtain sufficient conditions for non-integrability.

Step 1. The general case $A > B > C$.

To begin with, we assume without loss of generality that the mass of the oscillating particle is $m = 1$. Next, we have

Proposition 3.2. *Assume $A > B > C$. Then, system (1.1) possesses the following particular solution*

$$\omega_1 = \omega_2 = 0, \qquad \omega_3 = \omega_3^0 = \text{const}, \qquad r = \exp(i\sqrt{\sigma}t). \tag{3.1}$$

The proof is trivial.

This particular solution has a clear mechanical interpretation: it represents the uniform rotation of the body along the axis on which the particle oscillates.

Denoting the variations by $\xi_1 = d\omega_1$, $\xi_2 = d\omega_2$, $\xi_3 = d\omega_3$, $\xi_4 = dr$, $\xi_5 = d\dot{r}$. Then we write down just that part of variational equations along particular solution (3.1) we need

$$\begin{aligned}(A + \exp(2i\sqrt{\sigma}t))\dot{\xi}_1 &= -2i\sqrt{\sigma}\exp(2i\sqrt{\sigma}t)\xi_1 + \omega_3^0(B - C + \exp(2i\sqrt{\sigma}t))\xi_2,\\ (B + \exp(2i\sqrt{\sigma}t))\dot{\xi}_2 &= \omega_3^0(C - A - \exp(2i\sqrt{\sigma}t))\xi_1 - 2i\sqrt{\sigma}\exp(2i\sqrt{\sigma}t)\xi_2.\end{aligned} \tag{3.2}$$

The rest of variational equations are trivial, so we do not consider them.

Next, we denote $s_1 := \frac{\omega_3^0}{2\sqrt{\sigma}}$ and algebraize (3.2) by introducing a new independent variable $z := \exp(2i\sqrt{\sigma}t)$. In this way we get

$$\begin{aligned}\xi_1' &= -\frac{1}{A+z}\xi_1 - is_1\frac{B-C+z}{z(A+z)}\xi_2,\\ \xi_2' &= -is_1\frac{C-A-z}{z(B+z)}\xi_1 - \frac{1}{B+z}\xi_2, \qquad ' = d/dz.\end{aligned} \tag{3.3}$$

Clearly, the points 0, $-A$, $-B$ are regular singular points for the above system. To see the nature of the point at infinity, we put $z = \frac{1}{w}$. Then, system (3.3) obtains the form

$$\begin{aligned}\frac{d\xi_1}{dw} &= \frac{1}{w(Aw+1)}\xi_1 + is_1\frac{(B-C)w+1}{w(Aw+1)}\xi_2,\\ \frac{d\xi_2}{dw} &= is_1\frac{(C-A)w-1}{w(Bw+1)}\xi_1 + \frac{1}{w(Aw+1)}\xi_2.\end{aligned} \tag{3.4}$$

Evidently, $w = 0$ is a regular singular point for this system, and hence, ∞ is a regular singular for (3.3). Therefore, system (3.3) has four regular singular points 0, $-A$, $-B$, ∞, i.e., it is a Fuchsian one.

Now, we are going to study the monodromy group of (3.3). For this purpose, it is useful to write it in the form

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}' = \left[\frac{\Lambda_0}{z} + \frac{\Lambda_A}{z+A} + \frac{\Lambda_B}{z+B}\right]\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \tag{3.5}$$

where

$$\Lambda_0 = \begin{pmatrix}0 & -is_1\frac{B-C}{A}\\ is_1\frac{A-C}{B} & 0\end{pmatrix}, \tag{3.6}$$

with eigenvalues $\left(s_1\sqrt{\frac{(A-C)(B-C)}{AB}}, -s_1\sqrt{\frac{(A-C)(B-C)}{AB}}\right)$,

$$\Lambda_A = \begin{pmatrix} -1 & 0 \\ 0 & -is_1\frac{A+C-B}{A} \end{pmatrix}, \tag{3.7}$$

with eigenvalues $\left(-1, -is_1\frac{A+C-B}{A}\right)$, and

$$\Lambda_B = \begin{pmatrix} is_1\frac{C+B-A}{B} & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.8}$$

with eigenvalues $\left(is_1\frac{C+B-A}{B}, -1\right)$.

Notice that we can chose s_1 (or equivalently ω_3^0) in such a way that the eigenvalues of Λ_0 do not differ by a positive integer. Then, by the theory described in Section 2, in a neighborhood of $z = 0$, system (3.5) can be transformed to

$$zW' = \Lambda_0 W. \tag{3.9}$$

Hence, the local monodromy around $z = 0$ is $M_0 = \exp(2\pi i\Lambda_0)$.

Similarly, the local monodromies around $-A$, $-B$ are $M_A = \exp(2\pi i\Lambda_A)$ and $M_B = \exp(2\pi i\Lambda_B)$, respectively.

From (3.4) we obtain the residue matrix at $w = 0$, which is in fact the residue matrix at $z = \infty$ to be

$$\Lambda_\infty = \begin{pmatrix} 1 & is_1 \\ -is_1 & 1 \end{pmatrix}. \tag{3.10}$$

Its eigenvalues are $(1 + s_1, 1 - s_1)$ and again we can chose s in such a way that they do not differ by a positive integer. Hence, by the same arguments we get the local monodromy at infinity to be $M_\infty = \exp(2\pi i\Lambda_\infty)$. Notice that $\det M_0 = \det M_\infty = 1$, the monodromy matrices M_0, M_∞ belong to the identity component of the Galois group of (3.3).

Next, we compute the commutator $M_0 M_\infty M_0^{-1} M_\infty^{-1}$. To simplify the notations we denote

$$d := \sqrt{\frac{A(A-C)}{B(B-C)}}, \quad r := \sqrt{\frac{(A-C)(B-C)}{AB}}.$$

After some algebra we get

$$M_0 M_\infty M_0^{-1} M_\infty^{-1} = \frac{1}{8}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \tag{3.11}$$

where

$$\begin{aligned} R_{11} = {} & 2(1-i)\frac{1+d}{d} + (1+i)\frac{d-1}{d}\big[\exp(4i\pi s_1) + \exp(-4i\pi s_1)\big] \\ & + (1-i)\frac{d-1}{d}\big[\exp(4i\pi s_1 r) + \exp(-4i\pi s_1 r)\big] \\ & + (1+i)\frac{1+d}{d}\big[\exp(2i\pi s_1(r-1)) + \exp(-2i\pi s_1(r-1))\big], \end{aligned}$$

$$
\begin{aligned}
R_{12} &= (1+i)\frac{d-1}{d}\big[\exp(4i\pi s_1) - \exp(-4i\pi s_1)\big] \\
&\quad + (1-i)\frac{d-1}{d}\big[\exp(4i\pi s_1 r) + \exp(-4i\pi s_1 r)\big] \\
&\quad - (i+1)\frac{1+d}{d}\big[\exp(2i\pi s_1(r-1)) - \exp(-2i\pi s_1(r-1))\big], \\
R_{21} &= (i-1)(d-1)\big[\exp(4i\pi s_1) - \exp(-4i\pi s_1)\big] \\
&\quad + (d-1)(i+1)\big[\exp(4i\pi s_1 r) - \exp(-4i\pi s_1 r)\big] \\
&\quad + (d+1)(i-1)\big[\exp(2i\pi s_1(r-1)) - \exp(-2i\pi s_1(r-1))\big], \\
R_{22} &= 2(d+1)(i+1) + (d-1)(i-1)\big[\exp(4i\pi s_1) + \exp(-4i\pi s_1)\big] \\
&\quad - (d-1)(i+1)\big[\exp(4i\pi s_1 r) + \exp(-4i\pi s_1 r)\big] \\
&\quad + (d+1)(1-i)\big[\exp(2i\pi s_1(r-1)) + \exp(-2i\pi s_1(r-1))\big].
\end{aligned}
$$

It is clear that the above commutator is the identity only when simultaneously $d = 1$ and $r = 1$. In our case $A > B > C$ this is impossible. Therefore, the monodromy group is not abelian. It follows from Schlesinger's theorem that the identity component of the Galois group of (3.3) (or (3.5)) is also not abelian, from where non-integrability follows from the Morales-Ramis result.

Remark 3.3. Notice that the inertia moment C plays an exceptional role here, since it corresponds to the principal inertia axis along which the particle oscillates. Hence, it is worth to consider additionally the following two cases: $C > A > B$ and $A > C > B$.

The first case $C > A > B$ trivially reduces to that described above. The eigenvalues of Λ_0 can be written as $\pm s_1\sqrt{\frac{(C-A)(C-B)}{AB}}$ and the numbers d and r become $d := \sqrt{\frac{A(C-A)}{B(C-B)}}$, $r := \sqrt{\frac{(C-A)(C-B)}{AB}}$.

The outcome is the same: the monodromy group of (3.5) is not commutative and by above arguments the considered system is not integrable.

Let us give a little bit more details for the case $A > C > B$. The matrix Λ_∞ remains the same. The matrix Λ_0 becomes

$$
\Lambda_0 = \begin{pmatrix} 0 & is_1\frac{C-B}{A} \\ is_1\frac{A-C}{B} & 0 \end{pmatrix}. \tag{3.12}
$$

Denote

$$
q := \sqrt{\frac{B(C-B)}{A(A-C)}}, \quad p := \sqrt{\frac{(A-C)(C-B)}{AB}}.
$$

Then, the eigenvalues of Λ_0 are $\pm is_1 p$. Additionally we denote

$$
a := \exp(2\pi s_1 p), \quad b := \exp(2\pi i s_1).
$$

Then the group commutator

$$
M_0 M_\infty M_0^{-1} M_\infty^{-1} = \frac{1}{8}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \tag{3.13}
$$

has entries

$$\begin{aligned}
R_{11} &= (1-i)\left(a+\frac{1}{a}\right)2+(1+i)\left(a+\frac{1}{a}\right)\left(\frac{b^2}{a}+\frac{a}{b^2}\right)\\
&\quad -q(1+i)\left(\frac{1}{a^2}-a^2\right)-q(1-i)\left(\frac{1}{a}-a\right)\left(\frac{a}{b^2}-\frac{b^2}{a}\right),\\
R_{12} &= (1-i)(1+iq)\left(\frac{1}{a^2}-a^2\right)\\
&\quad +(i+1)\left(\frac{b^2}{a}-\frac{a}{b^2}\right)\left(a+\frac{1}{a}+iq\left(\frac{1}{a}-a\right)\right),\\
R_{21} &= \frac{(i-1)}{q}\left(\frac{1}{a^2}-a^2\right)+(1+i)\left(\frac{1}{a^2}-a^2\right)\\
&\quad -(1-i)\left(a+\frac{1}{a}\right)\left(\frac{a}{b^2}-\frac{b^2}{a}\right)+\frac{(i-1)}{q}\left(\frac{1}{a}-a\right)\left(\frac{b^2}{a}+\frac{a}{b^2}\right),\\
R_{22} &= (i+1)\left(a+\frac{1}{a}\right)^2+(1-i)\left(a+\frac{1}{a}\right)\left(\frac{b^2}{a}+\frac{a}{b^2}\right)\\
&\quad +\frac{1}{q}\left(\frac{1}{a}-a\right)\left((1-i)\frac{b^2}{a}-(1+i)\frac{a}{b^2}\right).
\end{aligned}$$

It is clear that the above commutator is the identity only when simultaneously $a = 1$ and $b = 1$, but that contradicts to our assumption $A > C > B$. Therefore, the system is not integrable again by combination of the Schlesinger's theorem and the Morales-Ramis result.

Step 2. The cases $A = C$ and $B = C$.

It remains to deal with the exceptional cases $A = C$ and $B = C$. We will study the case $A = C$ in detail and since $B = C$ is quite similar we will give only the essentials.

Proposition 3.4. *Suppose $A = C$. Then system (1.1) admits a solution of the form*

$$\omega_2 = \omega_2^0 = \text{const}, \quad z_1 = z_2 = 0, \quad \omega_1 = \exp(is_2t), \quad \omega_3 = -i\exp(is_2t), \tag{3.14}$$

where $s_2 = \frac{A-B}{A}\omega_2^0$.

The proof is immediate.

Notice that ω_2^0 is an arbitrary constant, which we will chose appropriately later.

The normal variational equation (NVE) is with respect to $\xi_4 = dr$ and $\xi_5 = d\dot{r}$. Writing it as a second-order equation with respect to $\xi = \xi_4$, we get

$$\ddot{\xi} + (Q_1 - \exp(2is_2t))\xi = 0, \quad \text{where} \quad Q_1 := \sigma - (\omega_2^0)^2. \tag{3.15}$$

To transform (3.15) in an algebraic form, we set $z = \exp(2is_2t)$ and obtain

$$\xi'' + \frac{1}{z}\xi' + \frac{z - Q_1}{4s_2^2z^2}\xi = 0, \quad ' = d/dz.$$

The last equation can be written in the standard form

$$\eta'' + \left[\frac{1}{4s_2^2}\frac{1}{z} - \left(\frac{Q_1}{4s_2^2} - \frac{1}{4}\right)\frac{1}{z^2}\right]\eta = 0, \tag{3.16}$$

which is a particular case of double confluent Heun family of equations with $\alpha = \beta = 0$, $\gamma = -\frac{1}{4s_2^2}$, $\delta = \frac{Q_1}{4s_2^2} - \frac{1}{4}$. Recall that for this equation, 0 is a regular singular point and ∞ is an irregular singular point. Using the Kovacic algorithm, Duval and Mitschi [9] have established that (3.16) has Liouvillian solutions, or equivalently, its Galois group is solvable if and only if

$$\delta = \frac{1}{16}(3+2l)(1-2l) \quad \text{for some} \quad l \in \mathbb{Z}.$$

In our case, the last condition reads

$$\frac{A^2}{(A-B)^2}\frac{\sigma - (\omega_2^0)^2}{(\omega_2^0)^2} = \frac{1}{4}(3+2l)(1-2l) + 1.$$

Since the maximum value of the right hand side is 2, we chose ω_2^0 in such a way that

$$\frac{A^2}{(A-B)^2}\frac{\sigma - (\omega_2^0)^2}{(\omega_2^0)^2} > 2,$$

or equivalently,

$$(\omega_2^0)^2 < \sigma\frac{1}{1 + 2\left(\frac{A-B}{A}\right)^2} < \sigma. \tag{3.17}$$

With this choice

$$\delta \neq \frac{1}{16}(3+2l)(1-2l), \quad \text{for any} \quad l \in \mathbb{Z}$$

and the Galois group of (3.16) is not solvable (and hence, not abelian). Therefore, the non-integrability in this case follows from Theorem 2.2.

The case $B = C$ is similar. We start with the particular solution

$$\omega_1 = \omega_1^0 = \text{const}, \quad z_1 = z_2 = 0, \quad \omega_2 = \exp(is_3 t), \quad \omega_3 = -i\exp(is_3 t), \tag{3.18}$$

where $s_3 = \frac{A-B}{B}\omega_1^0$.

Then, the (NVE) along this solution is

$$\ddot{\xi} + (Q_2 - \exp(2is_3 t))\xi = 0, \quad \text{where} \quad Q_2 := \sigma - (\omega_1^0)^2. \tag{3.19}$$

Similarly as in the above case, we reduce it to a particular case of the double confluent Heun equation

$$\eta'' + \left[\frac{1}{4s_3^2}\frac{1}{z} - \left(\frac{Q_2}{4s_3^2} - \frac{1}{4}\right)\frac{1}{z^2}\right]\eta = 0. \tag{3.20}$$

Similar arguments lead us to sufficient condition

$$(\omega_1^0)^2 < \sigma,$$

for which the Galois group of (3.20) is not solvable, and hence, non-integrability follows from Theorem 2.2.

This finishes the proof of Theorem 1.2. □

4. Concluding remarks

In this paper we study the integrability of a system describing the motion of a rigid body with a fixed point and a particle oscillating in it in the absence of external forces. This system is a hamiltonian one and one more additional first integral is needed for its complete integrability. We prove that such an integral exists only in the case of dynamical symmetry $A = B$. Notice that the integrable case in this model is analogue to the both integrable cases in the classical rigid body problem: the Euler's case – free motion of the rigid body, and the Lagrange case – dynamical symmetry $A = B$.

It would be interesting to study the integrability of the corresponding model subjected to certain external forces: the gravitational force [5], or some combination of gravitational, magnetic, electric and Lorenz electromagnetic forces. Probably, it is of some interest the analogues of the Kovalevskaya case or the Goryachev-Chapligin case or other completely new integrable cases to be found.

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