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LAGRANGE OR EULER?
PART THREE: THE FUTURE

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> For every thing there is a season, and a time for every matter under heaven: a time to cast away stones, and a time to gather stones together.
>
> *Ecclesiastes*

Георги Чобанов, Иван Чобанов. ЛАГРАНЖ ИЛИ ОЙЛЕР? ЧАСТЬ ТРЕТАЯ: БУДУЩЕЕ

Эта работа является третьей частью серии статьей по вопросу о динамических традициях Эйлера и Лагранжа, первые две части [1, 2] которой были опубликованы в том же самом *Ежегоднике*. Она содержит некоторые концепции о будущем развитии исследований, направленных к стабилизации логических основ аналитической механики, отражающие взгляды авторов на эти вопросы; естественно, в этой связи нельзя игнорировать шестую проблему Гильберта об аксиоматической консолидации этих основ. После анализа настоящего состояния дела в связи с основными механическими сущностьям, как например ее геометрические предпосылки, теория стрел как подлинная математическая интерпретация понятия силы, движения в стандартных векторных пространств, понятия твердого тела и т.д., предложены соответствующие определения совместно с основными следствиями из этих определений. Статья использует на широкую ногу результаты, полученные в предыдущих публикациях авторов.

Georgi Chobanov, Ivan Chobanov. LAGRANGE OR EULER? PART THREE: THE FUTURE

This paper is the third part of a series of articles on the Eulerian and Lagrangean dynamical traditions, the first two parts [1, 2] of which have been published in this *Annual.* It contains some conceptions on the future development of the investigations on the stabilization of the logical foundations of analytical mechanics, as the authors see it; in this connection Hilbert's sixth problem on the axiomatic consolidation of these foundations is, naturally, inevitable. After an analysis of the present state of affairs concerning the basic mechanical entities, as for instance the geometrical background of analytical mechanics, the theory of arrows as the genuine mathematical interpretation of the force concept, the motions in standard vector spaces, the rigid body concept, etc., corresponding definitions are proposed together with the main corollaries they imply. The article makes on a large scale use of results obtained in previous publications of the authors.

This paper is the third and final part of a series of articles published in this *Annual* [1, 2] and dealing with the Lagrangean and Eulerian dynamical traditions. The first part gives a brief historical account of the birth and rise of these traditions, and the second treats the present state of affairs in analytical dynamics, in so far as it may be observed in the current mechanical literary sources, as well as the interplay of the Eulerian and Lagrangean dynamical traditions. This third part is dedicated to the future developments in analytical mechanics, connected with the solution of Hilbert's sixth problem, as the authors see coming events in it, relying on their own professional experience in the course of many decades and scientific researches in the domain, concerning mainly the axiomatic consolidation of the logical foundations of analytical dynamics and the realization of Hilbert's directive concerning the axiomatic construction of rational mechanics.

As it is well-known, in his famous *Vortrag* [3], reported before the Second Mathematical Congress in Paris 1900 Hilbert proposed a list of 23 mathematical problems set for this century to solve. This list includes, as problem number six, entitled *Mathematische Behandlung der Axiome der Physik*, the forming of a system of axioms for those theories of physical genesis, mechanics in the first place, for which the rôle of mathematics has already become a decisive one:

"Durch die Untersuchungen über die Grundlagen der Geometrie wird uns die Aufgabe nahe gelegt, *nach diesem Vorbilde diejenigen physikalischen Disziplinen axiomatisch zu behandeln, in denen schon heute die Mathematik eine hervorragende Rolle spielt: dies sind in erster Linie die Wahrscheinlichkeitsrechnung und die Mechanik*" [4, Bd. 3, S. 306].

After this exordium Hilbert outlined the contours of a program towards the solution of his sixth problem containing some standpoints and recommendations exposed in the typical for him optimistic tones. The ensuing development in rational mechanics, however, dashed Hilbert's hopes, and in the course of more than half a century after the announcement of his *Mathematische Probleme* the execution of his program seemed to be postponed *ad Calendas Graecas.* As Truesdell has commented apropos of Hilbert's sixth problem, "this problem, like all those he proposed concerning the relation between mathematics and physical experience, has been neglected by the mathematicians... Only two significant attempts to solve the part of Hilbert's sixth problem that concerns mechanics have been published: that of Hamel ... and that of Noll..." [5, p. 336].

As regards Hamel, Truesdell is referring to his article [6], ensued by [7]. None of these deserves the qualification "significant attempt": apropos of [6] see the critical remarks in [8]; apropos of [7], see the critical analysis in [9]. As regards Noll, Truesdell is referring to his articles [10, 11].

Noll's article [10] is the first actually serious publication in connection with the Hilbert's sixth problem. It begins with some not especially flattering constatations apropos of the *status quo* of the logical foundations of classical mechanics:

"It is a widespread belief even today that classical mechanics is a dead subject, that its foundations were made clear long ago, and that all that remained to be done is to solve special problems. This is not so. It is true that mechanics of systems of a finite number of mass points has been on a sufficiently rigorous basis since Newton. Many textbooks on theoretical mechanics dismiss continuous bodies with the remark that they can be regarded as a limiting case of a particle system with an increasing number of particles. They cannot. The erroneous belief that they can had the unfortunate effect that no serious attempt was made for a long period to put classical continuum mechanics on a rigorous axiomatic basis" (p. 266).

Noll does not mention it explicitly, but essentially his criticism is directed against the Lagrangean dynamical tradition. Indeed, this namely tradition is that regards the rigid bodies as composed by a great number of mass-points [2]. Let us remember Lagrange's own standpoint on this question cited in the second part of this series: "Je considére les corps proposés comme l'assemblage d'une infinité de corpuscules ou points massifs unis ensemble de manière qu'il gardent toujours nécéssairement entre eux les mêmes distances." Let us remember also the attitudes towards rigid bodies of the authors of the treatises [12] and [13] analysed in [2]. The first of them carries things with a high hand. He rides roughshod over any manners and customs of mathematical decency. Employing in 1965 arguments which hardly would convince Euler in 1765, he states:

"In the classical dynamics we are concerned with systems having a finite number of degrees of freedom, and it is with such systems we shall be mainly concerned in this book. Nevertheless it is natural to suppose, on physical grounds, that the fundamental equation [i.e. D'Alembert-Lagrange's principle, derived by the author of [12] for systems of finite number of mas-points only] will also hold for continuous systems, where the number of degrees is infinite — for example systems involving fluids in motion and vibrating strings. In such problems the summation occuring in the fundamental equation will be replaced by integration" (p. 37).

We shall not comment these profound thoughts recorded by an author taking infinitely many degrees of freedom with mechanics. We shall only note that such a behaviour is completely within the frames of traditional demeanour of any Lagrangeanist all over the world.

Coming back to Hilbert's sixth problem we must pass a verdict that is final and there is no appeal: the Lagrangean dynamical tradition is the only culprit for its neglecting on the part of the mathematicians. The arguments in favour of this sentence are abundantly closely discussed in [1, 2] in order to expose them here again. If we should formulate in a few words the main reasons for this state of affairs, we would adduce Truesdell's appraisal apropos of Lagrange's *Méchanique Analitique*: "... no explanation of concepts ... no attempt to justify any limit process by rigorous mathematics" [5, p. 173]. These characteristic features of the Lagrangean dynamical tradition overlived its originator in the course of two clear centuries to become nowadays its most repulsive distinguishing marks. Typical for Lagrangean mechanics are:

First, a complete lack of genuine mathematical deffinitions.

Second, a complete disregard of strict mathematical proofs.

In result, mechanical treatises are printed in Twentieth Century in which the logical spirit of the era before the French Revolution has survived.

As the case stands, which forming-up place must be chosen for the onrush against the positions of Hilbert's sixth problem? Where should one start from in order to bridge over the difficulties it conceals, to break the neck of the task?

Under the circumstances, the sole answer of these questions is: *ab ovo Ledae*. But what does *ab ovo* in analytical mechanics mean?

Let us take as a model some other mathematical theories which are already conquered territories for the axiomatic approach, such as geometry or arithmetic, for instance. What does *ab ovo* in these theories mean?

Space! The first and greatest concern of a mathematical theory in the process of an axiomatic consolidation of its logical basement is to construct the most natural for this science *medium*, in which the phenomena characteristic for the mathematical theory in question could proceed. In most cases, though not always, this medium is called space, or something of the kind, in order to evoke the mental picture of a multitude of certain objects: domain, field, line, plane, graph, tree, body; ring, group, structure, neighbourhood, etc. (One speaks, for instance, of the line of real numbers, of Gauss' plane or plane of complex numbers, of the body of quaternions, of the ring of integrals, and so on. No wonder that the term space, as well as the term point as a general denomination for the elements of certain spaces, are overburdened with usage, and they demand various adjectives with a view to their specification.) The elements of the specific medium for a given mathematical theory are, as the saying is, the bricks or the stones the edifice of this theory is constructed from. For synthetic geometry, for instance, the genuine medium is the Euclidean space; for arithmetic, depending on circumstances, it is the system of natural numbers, or the ring of integers, or the fields of rational, real, or complex numbers, etc. *Suum quique*.

For analytical mechanics the native medium — as air is the native element for birds, and water for fishes — is the real standard vector space. It is the mathematical world where the mechanical entities are borne, the place where their growth and florescence set in, the stage where the mechanical drama is running. As well as the real numbers are the building stones for the classical real analysis, and the complex numbers are the bricks for the classical complex analysis, the real standard vectors provide analytical mechanics with the most natural, the most convenient, and the most resistent building material.

Between analysis and mechanics, however, there is a difference, though technical, rather than on principle. While in analysis the terrain, the building site is cleared away and organized long ago, in analytical mechanics until very recently it was left unkempt and overgrown.

Extra special cares for the *Grundlagen der Analysis* were taken in 1930 by Edmund Landau, who, for the first time in the mathematical literature, gave a consistent, sufficiently complete, and systematical up to pedantry, exposition [14] of arithmetic of natural, whole, rational, real, and complex numbers. It is by no means absence of mind that this arithmetical treatise is entitled "Foundations of Analysis": never before Landau, in all the mathematical literature, classical analysis had at its disposal its natural medium, its genuine space — the fields of real and complex numbers respectively, constructed in a mathematically irreproachable way.

(*Entre parenthèses*: "irreproachable" is, in mathematics at least, a historical category. Landau's exposition bears the marks of his epoch and the tokens of pioneerdom. It will not be useless to discuss this question somewhat closer.

Landau skims the cream off age-old scientific toils. In [14] the roads of great mathematicians meet: it is the juncture of first-rate mathematical ideas.

As regards the natural numbers, though familiar to mankind from times memorial, they had to wait till the seventeenth century when Pascal formulated one of their most important mathematical characteristics, the mathematical induction. At first this extremely useful property of the system N of all natural numbers has been applied for the goals of arithmetical proofs only. Grassmann [15] has been the first to use it for definitional purposes: the sum $a+b$ of the natural numbers a and b he defined inductively by the two-link-chain $a+1=a'$ and $a+b'=(a+b)'$, and the product ab by $a1=a$ and $ab'=ab+a$, n' denoting the successor of the natural number n.

As regards the whole, rational, and real numbers, Landau used an approach, already classical for mathematics, according to which new mathematical object may be defined as equivalence classes in preliminarily appropriately defined sets of well-known mathematical objects. If for instance $(a, b), (c, d) \in N^2$ and by definition $(a, b) \sim (c, d)$ iff $a+d=c+b$, then $\sim$ is an equivalence relation in N^2 and any equivalence class in N^2, generated by it, is by definition a whole number. Similarly is proceeded in the definitions of rational and real numbers.

As regards the real numbers, Landau disposed of Cantor's definition from 1872, according to which a real number is an equivalence class in the set C of all *Cantorian sequences* $\{x_\nu\}_{\nu=1}^{\infty}$ of rational numbers x_ν ($\nu = 1, 2, \dots$). At that, the sequence $\{x_\nu\}_{\nu=1}^{\infty}$ is by definition Cantorian iff for any rational $\varepsilon > 0$ there exists such a natural numbers p_ε that $m > p_\varepsilon$ and $n > p_\varepsilon$ imply $|x_m - x_n| < \varepsilon$. As regards the equivalence relation $\sim$ in C, it is defined in the following manner: by definition $\{x_\nu\}_{\nu=1}^{\infty} \sim \{y_\nu\}_{\nu=1}^{\infty}$ iff for any rational $\varepsilon > 0$ there exists such a natural number q_ε that $n > q_\varepsilon$ implies $|x_n - y_n| < \varepsilon$.

As regards the complex numbers, Landau disposed of Hamilton's definition. According to Hamilton, a complex number is an element of the Cartesian square R^2 of the field R of all real numbers, supplied with operations addition and multiplication defined by

$$(x_1, y_1) + (x_2, y_2) \stackrel{d}{=} (x_1 + x_2, y_1 + y_2) \tag{1}$$

and

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \tag{2}$$

respectively.

There is much more, yet.

As regards the axiomatization principle in general, Landau disposed with Hilbert's scheme [16] from 1899, and the title of Landau's book [14] imitates the title of Hilbert's.

As regards the axiomatization principle in arithmetic of natural numbers, in patricular, Landau disposed with Peano's scheme [17] from 1889.

In such a manner, the book [14] of Landau is of a compilation character. And yet, for three at least reasons this is a remarkable work that left behind itself deep traces in modern mathematics.

First, Landau has realized for the first time in arithmetical literature that Grassmann's definition of sum and product of natural numbers requires unconditionally proofs that such things (two valued functions, as a matter of fact) really exist.

Second, as it has been already underlined, Landau's *Grundlagen der Analysis* contains the first systematical exposition of arithmetic of the mentioned number systems. For this reason it served the purposes of the popularization of modern views in the domain.

Last but not least, Landau has realized quite clearly that classical analysis, complex as well as real, is unthinkable, in up-to-date sense of the word, before a mathematically perfect arithmetic of the fields of complex and real numbers respectively is performed.

As regards some "specks" of Landau's exposition, we shall pause on one only shortcoming. The operation addition in N is defined in [14] before the order relation. This approach is, however, against the mathematical nature of these arithmetical phenomena: historically counting goes prior to computing; therefore it is rather desirable that mathematically order precedes addition. An attempt to restore the natural order of things has been undertaken in the book [18], see also [19], although the way chosen in [18] is, obviously, not the shortest possible. We close the brackets.)

We dwelt so thoroughly on these questions, since there is a complete analogy between classical analysis and analytical mechanics, on the one hand, and between the field of real numbers and the real standard vector space, on the other hand.

Vectors are discovered out and away later than numbers, no mater whether natural, whole, rational, real, or complex. In point of fact, vectors are coevals with quaternions or hypercomplex numbers. Furthermore, not only vectors and quaternions are the same age, but for a long period vector and quaternion calculus have been as inseparable as Siamese twins.

In the same year, 1844, when Grassmann published the first version [20] of his *Ausdehnungslehre*, Hamilton published his first announcement concerning quaternions, shortly afterwards ensued by a series [21] of other articles on the same topic. Since the following developments are of great importance with an eye to the better understanding of the current state of affairs of vector and quaternion calculus, it will not be otiose to discuss this matter in some details.

The necessity of a geometrical calculus, alias of a mathematical method for expressing quantitatively geometrical phenomena, has been realized already by Leibniz. In a letter to Huygens dated September 8, 1679, he states:

"... je ne suit pas encor content de l'Algebre, en ce qu'elle ne donne ny les plus courtes voyes, ny les plus belles constructions de Geometrie. C'est pourquoy lorsqu'il s'agit de cela, je croy qu'il nous faut encor une autre analyse proprement geometrique ou lineaire, qui nous exprime directement situm, comme l'Algebre exprime magnitudinem."

Enclosed herewith Leibniz dispatched to Huygens an "Essay", in which he explained somewhat closser his main ideas apropos of the "geometrical symbolical language" ("nouvelle characteristique"). Huygens, whose answer dated November 20, 1679, is hardly encouraging, obviously could not realize that one of the most genial scientific foresights was concealed in Leibniz's conception. And no wonder: Leibniz's program towards the creation of a "geometrical" or "linear" calculus has been accomplished more or less satisfactory not until the end of the last century with the invention of vector algebra and vector analysis.

(Some historians of mathematics interpret Leibniz's phrase "une autre analyse proprement geometrique ou lineaire, qui nous exprime directement situm" as a foreboding of topology. We thing that such a standpoint is rather far-fetched leastwise on account of the adjective "lineaire" this phrase includes.)

In order to stimulate investigations connected with the "analyse geometrique ou lineaire" of Leibniz, the German scientific society *Fürstlich Jablonowski'sche Gesellschaft* announced a competition the aims which were "den von Leibniz erfundenen geometrischen Calcul wiederherauszustellen, oder einen ihm ähnlichen aufzustellen". Grassmann carried off the palm. His prize-winning work [22] has been published in 1847 with a commentary by Möbius. This is not by chance. Nearly twenty years before Grassmann's *Ausdehnungslehre*, Möbius published a treatise [23] on the "barycentrical calculus", invented by him, which is regarded as a predecessor of Grassmann's "geometrical analysis"; one is left with the impression that some kind of likeness in the titles of [20] and [23] is hardly a coincidence.

During the following decade in the *Journal für die reine und angewandte Mathematik* several articles of Grassmann were published where various applications of his method are made on different problems of geometry. His main work [20], however, remained for a long period obscure for the general mathematical public and did not stimulate investigations of other authors. Several reasons led to this state of affairs:

"Ein Zeitalter, welches die imaginären Größen noch als unmöglich ansah, und die "nichteuklidische" Geometrie mit Kopfschütteln betrachtete, konnte natürlich sich an den n Dimensionen der Ausdehnungslehre nur wenig Geschmack finden, und noch weniger vielleicht an den ungewohnten Operationen mit ebenso ungewohnten Objecten. Dazu kam wohl auch der Umstand, daß die mathematischen Kräfte der letzten Jahrzehnte vollauf beschäftigt waren mit der Ausbildung der Theorien eines Jacobi, Dirichlet, Steiner, Möbius, Plücker, Hesse und Anderer, die alle einen Kreis eifriger Schüler um sich sammelten, während dem Verfasser der Ausdehnungslehre das Geschmack eine gleiche Wirksamkeit versagt hatte. Endlich aber darf nicht unberücksichtigt bleiben, daß das Werk in seiner mehr philosophischen als mathematischen Form, ungewöhnlichen Inhalt in ungewöhnlicher Form bietend, dem Studium großen Schwierigkeiten entgegensetzte, deren Bewältigung nicht einmal ein sehr lockendes Ziel verhieß. Denn die Ideenkreise des Buches lagen von denen der übrigen gleichzeitigen Bestrebungen in der Mathematik weit entfernt,und die räumlichen Anwendungen kamen meist nur der Elementar-Geometrie zu Gute, welche damals noch unerschüttert auf der Eukldischen Basis thronte" [24].

The completely rehashed new edition [25] of [20] was printed in 1862. In the course of time the ideas and the technics of the geometrical analysis won it more and more followers and became generally acknowledged towards the end of the last century. About that time the complete works [26] of Grassmann have been published.

Meanwhile on the British Isles another development took place. In the course of two clear decades Hamilton published an immense series of articles [27 – 38] on various applications of quaternions, as well as the two books [39, 40]. By and by the quaternions became Hamilton's fixed-idea. The situation is described rather vividly by Felix Klein in his Lectures on the developement of mathematics in the last century [41]:

"Sehr bald wurden die Quaternionen in Dublin ein alles andere überragender Gegenstand des mathematischen Interesses, ja sogar ein officielles Examenfach, ohne dessen Kenntniss kein Absolvierung des Colleges mehr denkbar war. Hamilton selbst gestaltete sie für sich zu einer Art orthodoxaler Lehre des mathematischen Credo, in die er alle seine geometrischen und sonstigen Interessen hineinzwang, je mehr sich gegen Ende seines Lebens sein Geist vereinseitigte und unter den Folgen des Alkohols vereindüsterte.

Wie ich schon andeutete, schloss sich an Hamilton eine Schule an, die ihren Meister an Starrheit und Intoleranz noch überbot. Sie war gecignet, Gegenströmungen hervorzurufen, und so wurden denn die Quaternionen z. B. in Deutschland von der Mehrzahl der Mathematiker hartnäckig abgelehnt, bis sie auf dem Umweg über die Physik in Form der vor allem in der Dynamik unentbehrlichen Vektoranalysis dennoch eindrangen. Sollen wir heute ein Urteil über sie geben, so wäre etwa zu sagen: Die Quaternionen sind gut und brauchbar an ihrem Platze; sie reichen aber in ihrer Bedeutung an die gewöhnlichen komplexen Zahlen nicht heran."

The quaternions (hypercomplex numbers with three imaginary units) are a natural generalization of the complex number concept, whose multiplication is, in general, non-commutative. Especially remarkable is their connection with the rotations in the three-dimensional Euclidean space [42, 43]. Attaching ourselves to Klein's estimation apropos of the significance of quaternions, we should say that, while the standard vectors are the most natural means for the algebraic representation of translations in the Euclidean space, the quaternions are the genuine mathematical instrument for the algebraic expression of the rotations in the latter.

Its present aspects the vector calculus is obliged to the American theoretical physicist Gibbs. As he himself notes, his interest for vector analysis has arisen when he read the book [44] of Maxwell, in which the methdd of quaternions is applied to electrodynamical and magnetic investigations. Gibbs soon convinced himself that the quaternions are unsuitable to this end and began developing independently the modern apparatus of vector algebra and vector analysis. Meanwhile he provided himself with the books [22, 25] of Grässmann, but they exerted a slight influence on his occupations. Gibbs wrote the text-book [45] on vector calculus designed for students in physics. Later he published several articles [46 – 51] on vector calculus meant to popularize the new method. An important rôle for the confirmation of the vector calculus have played the books [52] of Peano and [53] of Hyde.

In its age of puberty the vector calculus represented a peculiar symbiosis of geometry and algebra — synthetic geometry at that. The vectors were initially introduced as "directed segments" and even "directed quantities", with a strong intuitional appeal to physical experience. Later the vectors were interpreted by means of ordered triplets (x, y, z) of real numbers x, y, and z. This atmosphere gave good grounds for the following sarcastic remark of R. von Mises:

"Wenn man in den gebräulichen Darstellungen der Vektorrechnung an ihrer Spitze die Erklärung findet, der Vektor sei eine "gerichtete Grösse", so muß jeder Vernünftige erkennen, daß damit soviel wie nichts gesagt ist. Die andere Definition, das man ebenso häufig begegnet, der Vektor sei eine "Zahlentripel", ist offenbar zu weit, denn die Zahl der Männer, Frauen und Kinder an einem Ort ist gewiss kein Vektor" [54, S. 155].

This has been written in 1924. The force of the irony in these words is so overwhelming that "jeder Vernünftige" is inclined to think that, after they were published, no author of a text-book on vector calculus would take the liberty to repeat the faults Mises is laughing at. Alas, to all appearances, such authors do not read the *Zeitschrift für angewandte Mathematik und Mechanik*. We are more than sure — we are utterly certain — that most of the text-books on vector calculus or on theoretical mechanics (including an introductory chapter on vector calculus), printed in this very day, begin with an inevitable enigmatic "definition" of the sort of "a vector is a directed quantity". The excuse that other authors are doing just the same is no exculpation — it is an accusation: *assinus assinum fricat*.

Making the long story short, we find it useful to give a first-hand account of a case in which the older of the authors of this article has been a *dramatic persona*. For the sake of brevity let us call him Ivan. When he was an assistant, the head of the chair of analytical mechanics in the Faculty of Mathematics and Physics of the University of Sofia has been Professor Blagovest Dolaptchiev, composing at this time his text-book [55] on the subject. As the foreword to the first book of [55] displays, Ivan took part in the compilation of that part of the text-book which entitled *Elements of vector calculus.* We lay emphasis on this fact, since furth down some criticism is made on the manner the vectors are introduced in [55], a it is most important to be as clear as day that this criticism is, as a matter of fa a self-criticism.

As the foreword to [55] testifies, "as a result of discussions ... an axioma exposition of vector algebra has been adapted in this book". Not entering in details, let us remark that this exposition has the following distinguishing featur

First, it has been realized that *the mathematical nature of the vector concept purely algebraic*, though the heuristic genesis of this notion is purely geometric. (that, saying *vectors*, we bear in mind here *standard* vectors; the algebraic nature other kinds of vectors, as for instance, *linear, Euclidean, Hermitean* etc. has be well-known long ago.)

Second, it has been understood that the *most natural scheme* according which the standard vectors must be introduced has to follow the specification chai *groups* ⟶ *linear spaces* ⟶ *Euclidean spaces* ⟶ *standard vector space* (thou the last term has not been used in the text-book [55]).

Third, it has been perceived that the *specification of an Euclidean space down to a standard vector space V must be accomplished by means of a fou operation* (vector multiplication) in E, introduced under the condition that it mu satisfy a certain number of specific for this operation axioms.

These specific axioms according to [55, p. 26] are:

Ax 14 D. a, $b \in V$ imply $a \times b = -(b \times a)$.

Ax 15 D. $\lambda \in R$; a, $b \in V$ imply $(\lambda a) \times b = \lambda(a \times b)$.

Ax 16 D. a, b, $c \in V$ imply $(a + b) \times c = a \times c + b \times c$.

Ax 17 D. a, b, $c \in V$ imply $a \times b \cdot c = a \cdot b \times c$.

Ax 18 D. a, b, $c \in V$ imply $(a \times b) \times c = (ac)b - (bc)a$.

Moreover, it is supposed a priori that the Euclidean space E is n-dimension n being indeterminate for the present.

Now, the greatest defect in the axiomatic exposition of the vector algebra in [5 consists in the fact that the conditions Ax 14 D — Ax 16 D are needless. In oth words, they are superfluous, redundant, surplus, otiose, unnecessary, expendab — in the capacity of axioms we mean.

Why?

Because they are theorems. They are provable properties of the vector mul plication, demonstrable on the basis of the remaining adopted axioms.

Why did the authors of the vector algebra in [55] commit this error — include theorems in the capacity of axioms?

Very simply: they were not aware of the fact. They simply did not know th Ax 14 D — Ax 16 D are theorems rather than axioms. Even today the number the mathematicians who know this fact is microscopically small. *Nemo mortali omnibus horis sapit.*

Twenty years had to pass before the truth emerged. For the first time the pro of Ax 14 D — Ax 16 D was given in the article [56] containing at the same tin the most economical system of axioms for real standard vectors hitherto know

Furthermore, it turned out that this axiomatical definition works in much more general mathematical situations than the real case represents. In other words, the field R of all real numbers may be replaced by an arbitrary ordered field F and the corresponding axioms, *mutatis mutandis* of course, still imply a consistent mathematical theory.

In order to make things crystal-clear we shall reproduce the corresponding axiomatical definition at full length.

A *standard vector space over an ordered field F* is called any set V_F, for which mappings

$$m_1 : V_F^2 \longrightarrow V_F \tag{3}$$

(*addition* in V_F),

$$m_2 : F \times V_F \longrightarrow V_F \tag{4}$$

(*multiplication* of the elements of F with the elements of V_F),

$$m_3 : V_F^2 \longrightarrow F \tag{5}$$

(*scalar multiplication* of the elements of V_F), and

$$m_4 : V_F^2 \longrightarrow V_F \tag{6}$$

(*vector multiplication* in V_F) are defined, so that, provided by definition,

$$a + b = m_1((a,\ b)) \tag{7}$$

(*sum* of a and b),

$$\alpha a = m_2((\alpha,\ a)) \tag{8}$$

(*product* of α and a),

$$ab = m_3((a,\ b)) \tag{9}$$

(*scalar product* of a and b),

$$a \times b = m_4((a,\ b)) \tag{10}$$

(*vector product* of a and b), and

$$a - b = a + (-b) \tag{11}$$

(*difference* of a and b), the following conditions are satisfied:

Ax 1 F. $a,\ b,\ c \in V_F$ imply $(a + b) + c = a + (b + c)$.

Ax 2 F. There exists $o \in V_F$ with: $a \in V_F$ implies $a + o = a$.

Ax 3 F. $a \in V_F$ implies there exists $-a \in V_F$ with $a + (-a) = o$.

Ax 4 F. $\boldsymbol{a} \in V_F$ implies: $1\boldsymbol{a} = \boldsymbol{a}$.

Ax 5 F. $\lambda,\ \mu \in F,\ \boldsymbol{a} \in V_F$ imply $(\lambda\mu)\boldsymbol{a} = \lambda(\mu\boldsymbol{a})$.

Ax 6 F. $\lambda,\ \mu \in F,\ \boldsymbol{a} \in V_F$ imply $(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \mu\boldsymbol{a}$.

Ax 7 F. $\lambda \in F;\ \boldsymbol{a},\ \boldsymbol{b} \in V_F$ imply $\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$.

Ax 8 F. $\boldsymbol{a},\ \boldsymbol{b} \in V_F$ imply $\boldsymbol{ab} = \boldsymbol{ba}$.

Ax 9 F. $\lambda \in F;\ \boldsymbol{a},\ \boldsymbol{b} \in V_F$ imply $(\lambda\boldsymbol{a})\boldsymbol{b} = \lambda(\boldsymbol{ab})$.

Ax 10 F. $\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c} \in V_F$ imply $(\boldsymbol{a} + \boldsymbol{b})\boldsymbol{c} = \boldsymbol{ac} + \boldsymbol{bc}$.

Ax 11 F. $\boldsymbol{a} \in V_F$ implies $\boldsymbol{aa} \geqq 0$.

Ax 12 F. $\boldsymbol{a} \in V_F,\ \boldsymbol{aa} = 0$ imply $\boldsymbol{a} = \boldsymbol{o}$.

Ax 13 F. $\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c} \in V_F$ imply $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$.

Ax 14 F. $\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c} \in V_F$ imply $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{ac})\boldsymbol{b} - (\boldsymbol{bc})\boldsymbol{a}$.

Ax 15 F. There exist $\boldsymbol{a},\ \boldsymbol{b} \in V_F$ with $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{o}$.

The elements of V_F are called *F-standard vectors*. The conditions Ax 1 F — Ax 15 F are called *axioms of a standard vector space over F*. The symbols 0 (Ax 11 F, Ax 12 F) and 1 (Ax 4 F) denote the *zero* and the *unit elements* of F respectively. The elements $\boldsymbol{o}$ (Ax 2 F, Ax 3 F, Ax 12 F, Ax 15 F) and $-\boldsymbol{a}$ (Ax 3 F) of V_F are called the *zero vector* and the *opposite vector* of $\boldsymbol{a}$ respectively. The scalar product $\boldsymbol{aa}$ is denoted, for the sake of brevity, by $\boldsymbol{a}^2$ and is called *scalar square* of $\boldsymbol{a}$.

The groups of axioms Ax 1 F — Ax 3 F, Ax 1 F — Ax 7 F, and Ax 1 F — Ax 12 F display that any standard vector space over an ordered field F is a *group*, a *linear space over F*, and an *Eucledean space over F* respectively. Hence the following proposition holds good.

Pr 1 F. $\boldsymbol{a},\ \boldsymbol{b} \in V_F$ imply $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$.

The following propositions are also proved [56]:

Pr 2 F. $\boldsymbol{a},\ \boldsymbol{b} \in V_F$ imply $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$.

Pr 3 F. $\lambda \in F;\ \boldsymbol{a},\ \boldsymbol{b} \in V_F$ imply $(\lambda\boldsymbol{a}) \times \boldsymbol{b} = \lambda(\boldsymbol{a} \times \boldsymbol{b})$.

Pr 4 F. $\boldsymbol{a},\ \boldsymbol{b} \in V_F$ imply $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$.

(Let us note that Pr 2 F — Pr 4 F are analogues of Ax 14 D — Ax 16 D from [55] quoted above.)

By definition $V = V_R$ is called a *real standard vector space*, and its elements are called *real standard vectors*.

The following propositions hold:

Pr 5 F. The system of axioms Ax 1 F — Ax 15 F is consistent.

Pr 6 F. The system of axioms Ax 1 F — Ax 15 F is categorical.

Pr 7 F. Any standard vector space over an ordered field F is a 3-dimensional Euclidean space over F.

Pr 5 F is proved by the aid of a model of V_F constructed *ad hoc*. It is proposed by the Cartesian cube F^3 of F supplied with the following operations (corresponding to (3) — (6) respectively)

(12) $$(x_1,\ x_2,\ x_3) + (y_1,\ y_2,\ y_3) = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3),$$

(13) $$\lambda(x_1,\ x_2,\ x_3) = (\lambda x_1,\ \lambda x_2,\ \lambda x_3),$$

(14) $$(x_1, x_2, x_3)(y_1, y_2, y_3) = \sum_{\nu=1}^{3} x_\nu y_\nu,$$

(15) $$(x_1, x_2, x_3) \times (y_1, y_2, y_3) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

for any $\lambda \in F$ and $(x_1, x_2, x_3), (y_1, y_2, y_3) \in F^3$. Now Ax 1 F — Ax 15 F are verified for the operations (12) — (15).

In the general case of any ordered field F the F-standard vectors are deprived of modules, since the *module* of $\boldsymbol{a} \in V_F$ is, by definition, the element $\sqrt{\boldsymbol{a}^2}$ of F, and in the general case such an element does not necessarily exist. This flaw is corrigible, if one supposes that F is a *Pythagorean field.* An ordered field P is called so iff for any $\alpha \in P$ with $\alpha \geqq 0$ there exists $\beta \in P$ with $\beta \geqq 0$ and $\beta^2 = \alpha$. This β is called the *square root* of α. If P is a Pythagorean field and $\boldsymbol{a} \in V_P$, then the module of $\boldsymbol{a}$ is denoted by a or $|\boldsymbol{a}|$.

The question now quite naturally arises whether the above definition of standard vector spaces over ordered fields tolerates a complex generalization? As it is well-known, the system of axioms Ax 1 F — Ax 12 F (with $\boldsymbol{R}$ instead of F) permits such a generalization, and the result is called an *Hermitean space.* Now what about Ax 1 F — Ax 15 F?

Let us specify the above question: if H is an Hermitean space, then is it possible to define a fourth operation *vector multiplication* in H, satisfying the conditions Ax 13 F — Ax 15 F?

(Let us note that only Ax 13 F and Ax 14 F are *specific conditions* for the operation vector multiplication, whereas Ax 15 F only demands that this operation is *not a trivial one.* Indeed, if Ax 15 F is violated, then the vector multiplication would put into correspondence the zero-vector for any two vector multipliers: hardly somebody would be interested in such an operation.)

It is proved that the answer of this question is negative. In such a manner, at first sight one remains with the impression that the system of axioms Ax 1 F — Ax 15 F does not admit a complex analogue.

This conclusion is a premature one, though. Not everything is lost. A happy whim may save the situation. Indeed, in view of Ax 8 F the condition Ax 13 F may be reformulated in the following equivalent manner:

Ax 13 F bis. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_F$ imply $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{c} \cdot \boldsymbol{a}$.

Let us now reformulate the above question with Ax 13 F bis instead of Ax 13 F: If H is a Hermitean space, is it possible to define a fourth operation vector multiplication in H, satisfying the conditions Ax 13 F bis, Ax 14 F, and Ax 15 F (naturally with C instead of F)?

Strange though it may seem, this last question is answered already in the affirmative, as it is proved in the article [57]. The results are formulated immediately below.

If F is an ordered field, then the *complex extension* $C(F)$ of F is by definition the Cartesian square F^2 of F, supplied with the two operations addition (1) and multiplication (2) for any two elements (x_1, y_1) and (x_2, y_2) of F^2. After this remark, we are in a position to formulate the complex extension of Ax 1 F — Ax 15 F:

A *standard vector space over the complex extension $C(F)$ of an ordered field F* is called any set $V_{C(F)}$, for which mappings

$$m_1 : V_{C(F)}^2 \longrightarrow V_{C(F)} \tag{16}$$

(*addition* in $V_{C(F)}$),

$$m_2 : C(F) \times V_{C(F)} \longrightarrow V_{C(F)} \tag{17}$$

(*multiplication* of the elements of $C(F)$ with the elements of $V_{C(F)}$),

$$m_3 : V_{C(F)}^2 \longrightarrow C(F) \tag{18}$$

(*scalar multiplication* of the elements of $V_{C(F)}$), and

$$m_4 : V_{C(F)}^2 \longrightarrow V_{C(F)} \tag{19}$$

(*vector multiplication* in $V_{C(F)}$) are defined, so that, provided (7) — (11), the following conditions are satisfied:

Ax 1 C(F). $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_{C(F)}$ imply $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c} = \boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$.

Ax 2 C(F). There exists $\boldsymbol{o} \in V_{C(F)}$ with: $\boldsymbol{a} \in V_{C(F)}$ implies $\boldsymbol{a}+\boldsymbol{o} = \boldsymbol{a}$.

Ax 3 C(F). $\boldsymbol{a} \in V_{C(F)}$ implies: there exists $-\boldsymbol{a} \in V_{C(F)}$ with $\boldsymbol{a}+(-\boldsymbol{a}) = \boldsymbol{o}$.

Ax 4 C(F). $\boldsymbol{a} \in V_{C(F)}$ implies $1\boldsymbol{a} = \boldsymbol{a}$.

Ax 5 C(F). $\lambda, \mu \in C(F)$, $\boldsymbol{a} \in V_{C(F)}$ imply $(\lambda\mu)\boldsymbol{a} = \lambda(\mu\boldsymbol{a})$.

Ax 6 C(F). $\lambda, \mu \in C(F)$, $\boldsymbol{a} \in V_{C(F)}$ imply $(\lambda+\mu)\boldsymbol{a} = \lambda\boldsymbol{a}+\mu\boldsymbol{a}$.

Ax 7 C(F). $\lambda \in C(F)$, $\boldsymbol{a}, \boldsymbol{b} \in V_{C(F)}$ imply $\lambda(\boldsymbol{a}+\boldsymbol{b}) = \lambda\boldsymbol{a}+\lambda\boldsymbol{b}$.

Ax 8 C(F). $\boldsymbol{a}, \boldsymbol{b} \in V_{C(F)}$ imply $\boldsymbol{a}\boldsymbol{b} = \overline{\boldsymbol{b}\boldsymbol{a}}$.

Ax 9 C(F). $\lambda \in C(F)$; $\boldsymbol{a}, \boldsymbol{b} \in V_{C(F)}$ imply $(\lambda\boldsymbol{a})\boldsymbol{b} = \lambda(\boldsymbol{a}\boldsymbol{b})$.

Ax 10 C(F). $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_{C(F)}$ imply $(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{c} = \boldsymbol{a}\boldsymbol{c}+\boldsymbol{b}\boldsymbol{c}$.

Ax 11 C(F). $\boldsymbol{a} \in V_{C(F)}$ implies $\boldsymbol{a}\boldsymbol{a} \geqq 0$.

Ax 12 C(F). $\boldsymbol{a} \in V_{C(F)}$, $\boldsymbol{a}\boldsymbol{a} = 0$ imply $\boldsymbol{a} = \boldsymbol{o}$.

Ax 13 C(F). $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_{C(F)}$ imply $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{c} \cdot \boldsymbol{a}$.

Ax 14 C(F). $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_{C(F)}$ imply $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a}\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b}\boldsymbol{c})\boldsymbol{a}$.

Ax 15 C(F). There exist $\boldsymbol{a}, \boldsymbol{b} \in V_{C(F)}$ with $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{o}$.

The elements of $V_{C(F)}$ are called *$C(F)$-standard vectors.* The conditions Ax 1 C(F) — Ax 15 C(F) are called *axioms of a standard vector space over $C(F)$*. The symbols 0 (Ax 11 C(F), Ax 12 C(F)) and 1 (Ax 4 C(F)) denote the *zero* and the *unit elements* of $C(F)$ respectively. The elements $\boldsymbol{o}$ (Ax 2 C(F), Ax 3 C(F), Ax 12 C(F), Ax 15 C(F)) and $-\boldsymbol{a}$ (Ax 3 C(F)) of $V_{C(F)}$ are called the *zero vector* and the *opposite vector* of $\boldsymbol{a}$ respectively. The scalar product $\boldsymbol{a}\boldsymbol{a}$ is denoted, for the sake of brevity, by $\boldsymbol{a}^2$ and is called the *scalar square* of $\boldsymbol{a}$.

The groups of axioms Ax 1 C(F) — Ax 3 C(F), Ax 1 C(F) — Ax 7 C(F), and Ax 1 C(F) — Ax 12 C(F) display that any standard vector space over the complex extension $C(F)$ of an ordered field F is a *group*, a *linear space over $C(F)$*, and an *Hermitean space over* $C(F)$ respectively. Hence the following proposition holds good.

Pr 1 C(F). $\boldsymbol{a},\ \boldsymbol{b} \in V_{C(F)}$ imply $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$.

The following propositions are also proved [57]:

Pr 2 C(F). $\boldsymbol{a},\ \boldsymbol{b} \in V_{C(F)}$ imply $\boldsymbol{a}\times\boldsymbol{b}=-(\boldsymbol{b}\times\boldsymbol{a})$.

Pr 3 C(F). $\lambda \in C(F)$; $\boldsymbol{a},\ \boldsymbol{b} \in V_{C(F)}$ imply $(\lambda\boldsymbol{a})\times\boldsymbol{b}=\overline{\lambda}(\boldsymbol{a}\times\boldsymbol{b})$.

Pr 4 C(F). $\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c} \in V_{C(F)}$ imply $(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}$.

V_C is called a *complex standard vector space*, and its elements are called *complex standard vectors*.

The following propositions hold:

Pr 5 C(F). The system of axioms Ax 1 C(F) — Ax 15 C(F) is consistent.

Pr 6 C(F). The system of axioms Ax 1 C(F) — Ax 15 C(F) is categorical.

Pr 7 C(F). Any standard vector space over the complex extension $C(F)$ of an ordered field F is a 3-dimensional Hermitean space over $C(F)$.

Pr 5 C(F) is proved by means of a model of $V_{C(F)}$. It is proposed by the Cartesian cube $C(F)^3$ of $C(F)$, supplied with the operations (12), (13) and

(20) $$(x_1,\ x_2,\ x_3)(y_1,\ y_2,\ y_3)=\sum_{\nu=1}^{3} x_\nu \overline{y}_\nu,$$

(21) $$(x_1,\ x_2,\ x_3)\times(y_1,\ y_2,\ y_3)=(\overline{x}_2\overline{y}_3-\overline{x}_3\overline{y}_2,\ \overline{x}_3\overline{y}_1-\overline{x}_1\overline{y}_3,\ \overline{x}_1\overline{y}_2-\overline{x}_2\overline{y}_1)$$

for any $\lambda \in C(F)$ and $(x_1,\ x_2,\ x_3),\ (y_1,\ y_2,\ y_3) \in C(F)^3$, corresponding to the mappings (16) — (19) respectively. It is easily verified that these operations in $C(F)^3$ satisfy the axioms Ax 1 C(F) — Ax 15 C(F).

In the general case of any ordered field F, the $C(F)$-standard vector are deprived of modules, since the module of $\boldsymbol{a} \in V_{C(F)}$ is, by definition, the element $\sqrt{\boldsymbol{a}^2}$ of $C(F)$, and in the general case such an element does not necessarily exist. This flaw is corrigible, if one suppose that F is a Pythagorean field. If P is such and $\boldsymbol{a} \in V_{C(P)}$, then the module of $\boldsymbol{a}$ is denoted by a or $|\boldsymbol{a}|$.

A more detailed exposition of the algebras of V_F and $V_{C(F)}$ may be found in the popular booklet [58].

Finis: Et nunc, et semper, et in saecula saeculorum.

At the beginning of this article the following question has been put: What does ***ab ovo*** in analytical mechanics mean. Now this question may be answered quite categorically: *ab ovo* in analytical mechanics means the ***deductive development*** of this science *starting from standard vector spaces.*

Along with this affirmation, a specific formulation of Hilbert's sixth problem concerning the axiomatical consolidation of the logical foundations of analytical mechanics may be given, namely:

Starting from standard vector spaces, by the aid of specific definitions in them of the basic mechanical entities and of specific axioms concerning these entities, proceed to the construction of a mathematical theory of motion of mass-points and rigid bodies, and of the forces, which generate these motions and are generated by them.

This program may be qualified as a *program-minimum.* Indeed, after all, its realization means proceeding *in a model.* In other words, once this program settled, analytical mechanics will be shaped out of the elements of the standard vector spaces in question. Working in a model before inventing a genuine "pure" specific system of axioms is a classical mechanism, a time-honoured technique of all mathematical theories. The examples are plentiful — *nomen illis legio* — in order to adduce them here: we shall confine to the most typical. In 1872, if memory does not fail, Cantor and Dedekind independently proposed two theories of real numbers: Cantor's by means of an equivalence relation in the set of all Cantorian sequences of rational numbers, as already mentioned above; Dedekind's by the aid of Dedekindian sections in the field of all rational numbers. Both theories work in a model. Half a century had to pass, before the modern axiomatic definition of real numbers was invented. *Natura non facit saltus.* Mathematics too.

As regards a *program-maximum*, by the opinion of the authors of this paper based on the present state of affairs in analytical mechanics, to say even a word about it at all would mean to act prematurely and thoughtlessly.

Leaving programs-maximum whatever for the generations to come, let us fix our attention on the program-minimum just now formulated, and let us, first and foremost, put the following quite legitimate question: What has the problem *Lagrange or Euler?* to do with all this? Why should one connect Hilbert's sixth problem, on the one hand, with the question "Lagrange or Euler?", on the other hand.

The answer is a quite simple one.

First, because Lagrange has nothing to do with Hilbert's sixth problem.

Second, because without Euler Hilbert's sixth problem is condemned to remain a Gordian knot which will be undone by the mathematicians *cum mula peperit.*

Qui potest capere capiat. If somebody is not concordant with the first affirmation, then this is his problem: as Cicero says, *cujusvis hominis est errare, nullius, nisi insipientis, in errore perseverare.* As regards the second affirmation, its proof is a problem of the authors of the present article, as well as the main aim of the latter. As regarads the Lagrangeanists, let us remind them the following ancient words of consolation: *Temporis filia veritas.* Lagrange's times in mechanics are *tempi passati.*

Another not incurious question must be answered before proceeding further: Why should Hilbert's sixth problem be connected with such "chimeras", in a physicist's eyes at least, as complex standard vector spaces, over complex extensions of arbitrary ordered fields at that? Why not confine ourselves to the real standard vector space solely? After all, the natural phenomena of motion in this world are accomplished in such a physical space, which is interpreted mathematically most adequately namely by V, rather than by V_C or even by $V_{C(F)}$.

The answer of this question is given by Hilbert himself, in his *Mathematische Probleme*:

"Auch wird der Mathematiker, wie er es in der Geometrie getan hat, nicht bloß die der Wirklichkeit nahe kommenden, sondern überhaupt alle logisch möglichen Theorien zu berücksichtigen haben und stets darauf bedankt sein, einen vollständigen Überblick über die Gesamtheit der Folgerungen zu gewinnen, die das gerade angenommene Axiomensystem nach sich zieht."

"... wie er es in der Geometrie getan hat ..." What has the mathematician "in der Geometrie getan"?

Starting from the Euclidean geometry — a theory "der Wirklichkeit nahe kommende" — he has invented the non-Euclidean geometry of Lobatchevski — Bolyai — Gauss with its infinitely many parallels; the projective geometry in which any two lines in a plane are intersecting; the n-dimensional Euclidean geometries with an arbitrary n; the "finite" geometries the spaces of which consist of a finite number of points, lines, and planes. And so on, and so forth, etcetera.

Nicht bloß die der Wirklichkeit nahe kommenden, sondern überhaupt alle logisch möglichen Theorien zu berücksichtigen — the very essence of mathematics is incarnated in this device. Cantor gives expression of the same idea in other words: *das Wesen der Mathematik liegt eben in ihrer Freiheit.* Investigating all possible theories, the mathematician expands beyond measure the boundaries of both concepts and facts, reaching nec plus ultra borders, across which the notions become devoid of sense and the theorems untruthful.

For analytical mechanics, in particular, Hilbert's motto concerns mainly the following basic entities: the *geometry* of the mechanical space, the *forces* in statics and dynamics, the *motions*, and the most fundamental concept of *rigid bodies.* For the time being we shall confine ourselves to these remarks, postponing more circumstantial explanations till the mentioned notions are discussed elaborately.

Before we enter into details, let us bring to the fore, in connection with Hilbert's advice, one more circumstance, specific for analytical mechanics and especially important from an ideological point of view. Mechanics has always been under the action of strong centrifugal forces orientated in two different direction. The first of them is the physical tendency. There is hardly a man who has not heard sentences like "mechanics is the first chapter of physics". Not a few mathematicians sympathize with this outlook. It would be their problem, if it were not a problem of the mechanicians also. This point of view has caused damages beyond repair to rational mechanics, in general, and to analytical mechanics, in particular. The second is the engineering tendency. It is even worse than the physical one, for obvious reasons which we are in no mood to discuss here. In a few words, qualifications of the kind "mechanics is a physical science" and "mechanics is an engineering science" are by no means *rara avis in terris.*

We are in no humour for disputes about the nature of mechanics — whether it is physics or engineering — the more so that any debate of the kind is predestinated to become a dialog among stone-deaf. The problem is which mechanics? Experimental mechanics is physics, of course. Technical mechanics is engineering, O.K. But analytical mechanics? We beg your pardon.

Now an analytical mechanics in $V_{C(F)}$ is surely neither physics, nor engineering. Furthermore, such a perverted thing is beyond the mental constitution of any normal physicist and engineer. But such a thing exists — in a mathematician's head, at least. Indeed, all the basic entities of analytical mechanics, mentioned above — geometry, forces, motions, rigid bodies — may be defined and developed in $V_{C(F)}$. Moreover, the Newtonian and Eulerian dynamical axioms may be formulated in V_C at least. In such a manner, a complex analytical dynamics may be constructed, and its natural medium is the complex standard vector space V_C. Naturally, the classical real analytical mechanics (i.e. in V) is its particular case, in the sense that the real numbers are a particular case of the complex ones.

It would be ridiculous to waste time, efforts, and nerves on building-up a complex analytical mechanics only in order to stop the mouths of physicists and engineers who know better than the mechanicians what mechanics is. It would be all the same, as if Grassmann created his 1001-dimensional geometry in order to convince the geodesists that Euclidean geometry is not geodesy. The complex analytical mechanics is constructed, however, not *ad hoc* for physicists and engineers — it is a by-product in the process of realization of Hilbert's directive *nicht bloß die der Wirklichkeit nahe kommenden, sondern überhaupt alle logisch möglichen Theorien zu berücksichtigen.*

After this lyrical digression, let us return to our job.

The story begins with geometry. Above all things, let us make it pretty clear that analytical mechanics and synthetic geometry of Euclid pertain to different incompatible blood types. In order to avoid any possible *qui pro quo*, let us underline that the heuristic role of synthetic geometry for analytical mechanics is entirely out of the question: this role has been priceless in all the history of mechanics, and it will remain inestimable in the time to come. Synthetic geometry and analytical mechanics are conflicting as far as the process of logical consolidation of the latter is concerned.

Which are the reasons for these antagonistic relations?

The very nature of either of these mathematical theories. While synthetic geometry may be qualified as mathematics of *situs*, analytical mechanics is mathematics of *motus*. Although not impossible, it is, however, extremely difficult — clumsy, awkward, lubberly, unwiedly, gawky are the true words — to implant motion in synthetic-geometrical soil. As regards the mathematical description of motion, all fingers of synthetic geometry are thumbs. It is true that geometers use the term motion to describe certain geometrical operations; but in the same time it is also true that these operations are, as a matter of fact, only finite displacements. As a stimulant of mechanical intuition, as a promotive of mechanical discoveries, as visual illustrator of complicated abstract mechanical results — as all that synthetic geometry is out and out irreplaceable. As a witness of mechanical events, however, it is totally unreliable. While a great conqueror of new domains of mechanical knowledge, synthetic geometry is a poor legislator where the logical laws of analytical mechanics are concerned.

One thing must be unconditionally clear: synthetic geometry must be categorically, ruthlessly, and irretrievably expelled from analytical mechanics before the first steps towards the solution of Hilbert's sixth problem are taken. This dictum may seem *ratio ultima*; axiomatics, however, brooks no compromises. If one intends to attack Hilbert's sixth problem, he must once and for all forget not only everything he has ever learned in synthetic Euclidean geometry, but even the very word Euclid.

What then? Without geometry, analytical mechanics is flesh without skeleton. Where will the necessary geometry come from?

From the standard vector spaces. It exists there embryonically. It must be carried and born.

At that, rejecting the services of synthetic geometry as far as axiomatic processes are concerned, one is following a maxim as old as the hills and especially recommendable in such cases: *principia non sunt multiplicanda praeter necessitatem.*

Getting down to the forward delivery of the convenient geometry for analytical mechanics, let us remember something from synthetic geometry — since, as Landau [14] used to say, we have not yet forgotten it entirely. As it is well-known, in analytic geometry it is proved (by the aid of synthetic-geometrical considerations) that any line l has an equation of the kind

$$\boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{b}, \tag{22}$$

where $\boldsymbol{r}$ is a current radius-vector (the radius-vector of any point of l) and $\boldsymbol{a}$, $\boldsymbol{b}$ are given vectors with

$$\boldsymbol{a} \neq \boldsymbol{o}, \qquad \boldsymbol{ab} = 0. \tag{23}$$

At that, the equation (22) defines l unequivocally, but the inverse is not true: l has infinitely many equations of the kind (22), and any of them is obtainable from (22) by multiplying it with an appropriate real number

$$\lambda \neq 0. \tag{24}$$

In other words,

$$\boldsymbol{r} \times \boldsymbol{c} = \boldsymbol{d} \tag{25}$$

is an equation of l if, and only if, there exists (24) with

$$\boldsymbol{a} = \lambda \boldsymbol{c}, \quad \boldsymbol{b} = \lambda \boldsymbol{d}. \tag{26}$$

This fact once realized, the equation (22) becomes already needless: one may imagine a line l as the set of all ordered pairs $(\boldsymbol{a}, \boldsymbol{b})$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with (23), obtainable from one another by multiplication with arbitrary non-zero numbers λ according to (26).

Let us formalize these conclusions, regarding the set

$$\Lambda = \{(\boldsymbol{a}, \boldsymbol{b}) \in V^2 : \boldsymbol{a} \neq \boldsymbol{o}, \ \boldsymbol{ab} = 0\} \tag{27}$$

of all ordered pairs $(\boldsymbol{a}, \boldsymbol{b})$ of real standard vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with (23). Let us write

$$(\boldsymbol{a}, \boldsymbol{b}) \sim (\boldsymbol{c}, \boldsymbol{d}) \tag{28}$$

iff there exists a real λ with (24) and (26). It is easily seen that (28) is an equivalence relation in the set (27). Now any equivalence class in Λ, generated by the relation (28), may be called a *line*.

Similarly, any plane p has an equation of the kind

$$rc = \alpha, \tag{29}$$

where c is a given vector with

$$c \neq o \tag{30}$$

and α is a given real number. Now all the above considerations may be repeated with (29) instead of (22). Formalizing the corresponding conclusions, one arrives at the set

$$\Pi = \{(c,\ \alpha) \in V \times R : c \neq o\} \tag{31}$$

of all ordered pairs $(c,\ \alpha)$ of a real standard vector c with (30) and a real number α. Let us write

$$(c,\ \alpha) \sim (d,\ \beta) \tag{32}$$

iff there exists a real λ with (24) and

$$c = \lambda d, \quad \alpha = \lambda\beta. \tag{33}$$

It is easily seen again that (32) is an equivalence relation in the set (31). Now any equivalence class in Π, generated by the relation (32), may be called a *plane*.

What of it?

Neither more nor less than the fact that in such a manner the linear analytic geometry (the analytic geometry of points, lines, and planes) can be constructed axiomatically without whatever appeal to synthetic geometry — as if it does not exist at all. (It exists, of course, and moreover, it suggests what to do next in order to obtain, in analytical form, naturally, namely the synthetic Euclidean geometry and not some of its distant relatives.) The manner this end can be achieved pedantically is displayed in the article [59]; therefore we shall refrain here from any technicalities whatever. Let us note in passing that in such a way the linear analytical geometry becomes wholly emancipated from synthetic Euclidean geometry.

There are, however, some remarks to be made in this connection, which are a matter of principle.

In the first place, the definitions of lines and planes as equivalence classes in Λ and Π respectively are somewhat ungainly. This can be avoided in the following manner. (Let us note that the scheme exposed below is not specific for Λ and Π. It is applicable every time when new mathematical objects are defined as equivalence classes in appropriate set of beforehand defined objects supplied with suitable equivalence relations.)

Let the set A be supplied with an equivalence relation $\sim$ and let B and C be primary sets (i.e. unknown sets of unknown objects which only now will have to be defined implicitly), satisfying the following conditions:

Ax 1 S. $C \subset A \times B$.

Ax 2 S. $\alpha \in A$ implies: there exists $\beta \in B$ with $(\alpha, \beta) \in C$.

Ax 3 S. $(\alpha_\nu, \beta_\nu) \in C$ $(\nu = 1, 2)$, $\alpha_1 \sim \alpha_2$ imply $\beta_1 = \beta_2$.

Ax 4 S. $\beta \in B$ implies: there exists $\alpha \in A$ with $(\alpha, \beta) \in C$.

Ax 5 S. $(\alpha_\nu, \beta) \in C$ $(\nu = 1, 2)$ imply $\alpha_1 \sim \alpha_2$.

It is proved that the system axioms Ax 1 S — Ax 5 S is consistent and categorical, see [60]. This system is called the model-axiomatizing scheme and it proposes an axiomatic definition of the equivalence class concept. As a matter of fact, Ax 1 S — Ax 5 S define implicitly the elements of the set B, and the set C establishes a correspondence between the elements of A and B.

If one puts $A = \Lambda$ in Ax 1 S — Ax 5 S, then B is the set of all lines. Similarly, B is the set of all planes provided $A = \Pi$.

The only difference between the definitions by means of equivalence classes, on the one hand, and by the aid of the model-axiomatizing scheme, on the other hand, consists in the fact that the latter is not so categorical as the first one. Indeed, in the case of lines, for instance, a line is an equivalence class in Λ generated by the equivalence relation (28) provided (26) and nothing else, whereas Ax 1 S — Ax 5 S with $A = \Lambda$ define the lines not so unambiguously.

Let us, for the sake of brevity, write $\alpha \& \beta$ iff $(\alpha, \beta) \in C$; in such a case α is called *associated* with β, and β is called *determined* by α. Under this notation Ax 2 S — Ax 5 S may be written in the equivalent form:

Ax 2*. $\alpha \in A$ implies: there exists $\beta \in B$ with $\alpha \& \beta$.

Ax 3*. $\alpha_\nu \& \beta_\nu$ $(\nu = 1, 2)$, $\alpha_1 \sim \alpha_2$ imply $\beta_1 = \beta_2$.

Ax 4*. $\beta \in B$ implies: there exists $\alpha \in A$ with $\alpha \& \beta$.

Ax 5*. $\alpha_\nu \& \beta$ $(\nu = 1, 2)$ imply $\alpha_1 \sim \alpha_2$.

After these remarks, let us say some words more about the above definitions of lines and planes.

It is immediately seen that one can substitude F and V_F for R and V respectively in the definitions (27) and (31), and these definitions still work. In such a manner one obtains not one, but infinitely many linear analytical geometries, one for every F. Since the algebras of V and V_F are identical (as far as continuity is not concerned), these analytic geometries are very similar among themselves.

Not so obvious is the fact that one can also substitute $C(F)$ and $V_{C(F)}$ for R and V respectively in the definitions (27) and (31) without turning them meaningless. We shall proceed now to the effective execution of such an exchange.

Instead of (27) and (31) let us consider the sets

$$\Lambda_{C(F)} = \{(\boldsymbol{a}, \boldsymbol{b}) \in V^2_{C(F)} : \boldsymbol{a} \neq \boldsymbol{o}, \ \boldsymbol{ab} = 0\} \tag{34}$$

and

$$\Pi_{C(F)} = \{(\boldsymbol{c}, \alpha) \in V_{C(F)} \times C(F) : \boldsymbol{c} \neq \boldsymbol{o}\} \tag{35}$$

respectively and let by definition

$$(a_1,\ b_1) \sim (a_2,\ b_2) \tag{36}$$

and

$$(c_1,\ \alpha_1) \sim (c_2,\ \alpha_2) \tag{37}$$

for any $(a_\nu,\ b_\nu) \in \Lambda_{C(F)}$, $(c_\nu,\ \alpha_\nu) \in \Pi_{C(F)}$ $(\nu = 1,\ 2)$ iff there exists $\lambda \in C(F)$ with (24) and

$$a_1 = \lambda a_2, \quad b_1 = \overline{\lambda} b_2, \tag{38}$$

$$c_1 = \lambda c_2, \quad \alpha_1 = \overline{\lambda} \alpha_2 \tag{39}$$

respectively, $\overline{\lambda}$ denoting the conjugate element of λ. It is easily seen that (36) and (37) are equivalence relations in (34) and (35) respectively.

Let us now apply the model-axiomatizing scheme to the sets (34) and (35), in other words with $A = \Lambda_{C(F)}$ and $A = \Pi_{C(F)}$ respectively, denoting B with $L_{C(F)}$ in the first case and with $P_{C(F)}$ in the second. Using the &-version of Ax 2 S — Ax 5 S, i.e. Ax 2* — Ax 5*, we obtain the following two systems of conditions:

Ax 2 L. $(a,\ b) \in \Lambda_{C(F)}$ implies: there exists $l \in L_{C(F)}$ with $(a,\ b)\&l$.

Ax 3 L. $(a_\nu,\ b_\nu)\&l_\nu$ $(\nu = 1,\ 2)$, $(a_1,\ b_1) \sim (a_2,\ b_2)$ imply $l_1 = l_2$.

Ax 4 L. $l \in L_{C(F)}$ implies: there exists $(a,\ b) \in \Lambda_{C(F)}$ with $(a,\ b)\&l$.

Ax 5 L. $(a_\nu,\ b_\nu)\&l$ $(\nu = 1,\ 2)$ imply $(a_1,\ b_1) \sim (a_2,\ b_2)$

and

Ax 2 P. $(c,\ \alpha) \in \Pi_{C(F)}$ implies: there exists $p \in P_{C(F)}$ with $(c,\ \alpha)\&p$.

Ax 3 P. $(c_\nu,\ \alpha_\nu)\&p_\nu$ $(\nu = 1,\ 2)$, $(c_1,\ \alpha_1) \sim (c_2,\ \alpha_2)$ imply $p_1 = p_2$.

Ax 4 P. $p \in P_{C(F)}$ implies: there exists $(c,\ \alpha) \in \Pi_{C(F)}$ with $(c,\ \alpha)\&p$.

Ax 5 P. $(c_\nu,\ \alpha_\nu)\&p$ $(\nu = 1,\ 2)$ imply $(c_1,\ \alpha_1) \sim (c_2,\ \alpha_2)$.

Then the elements of $L_{C(F)}$ and $P_{C(F)}$ are called $C(F)$-*lines* and $C(F)$-*planes* respectively, and the mathematical theory, based on Ax 2 L — Ax 5 L and Ax 2 P — Ax 5 P, as well on the specific properties of the equivalence relations (36) and (37) in the sets (34) and (35) respectively, is called the $C(F)$-*linear analytic geometry* and is denoted by $G_{C(F)}$. The development of $G_{C(F)}$ consists in revealing those properties of the elements of the sets (34) and (35), which are invariant with respect to the equivalence relations (36) and (37) respectively. The basic definitions and propositions of $G_{C(F)}$ are exposed pedantically in the article [61], so that we shall not dwell here on any technicalities in this connection. Two remarks are, however, expedient.

First, we shall concent to take the "real" F-case for immersed in the "complex" $C(F)$-case in the same way, as the real numbers are considered as dispersed among the complex ones, i.e. $R \subset C$. In other words, we shall agree to perceive F as a subfield of $C(F)$, i.e. $F \subset C(F)$, and similarly to graps V_F as a standard vector subspace of $V_{C(F)}$, i.e. $V_F \subset V_{C(F)}$. In other words, we shall think that V_F is obtainable from $V_{C(F)}$ by putting $\overline{\alpha} = \alpha$ for any $\alpha \in C(F)$ in the corresponding axioms of $V_{C(F)}$ (in Ax 8 C(F) in the first place). We shall not discuss the logical side of such an agreement, emphasizing here on the technical advantages: important in the moment for us is the fact that such a convention is very suitable and economizing. It saves us, for instance, the necessity to define explicitly the F-linear analytic geometry G_F, i.e. by means of Ax 2 L — Ax 5 L and Ax 2 P — Ax 5 P with F and V_F instead of $C(F)$ and $V_{C(F)}$ respectively. We shall accept that $G_F \subset G_{C(F)}$, if the notation has any sense.

The second remark concerns the relations between G_F and $G_{C(F)}$. Strange though it may seem, these geometries are rather similar, though naturally not identical. There are differences on principle, it is true. In any case, there do not exist isotropic lines and planes in $G_{C(F)}$, and this question deserves a closer discussion.

The point is that $G_{C(F)}$, or rather G_C, is not the single complex analytic geometry up to now proposed. Another version may be found, for instance, in the text-book [62] of B. Petkanchin. It is erected by means of the following constructions. Let P_C be the Cartesian cube C^3 of the field C of the complex numbers, supplied with the four operations (12) — (15) for any $\lambda \in C$ and for any (x_1, x_2, x_3), $(y_1, y_2, y_3) \in C^3$. It is proved that they satisfy Ax 1 F — Ax 10 F, Ax 13 F — Ax 15 F and Pr 1 F — Pr 4 F (with C and P_C instead of F and V_F respectively), while the statements of Ax 11 F and Ax 12 F (again with C and P_C instead of F and V_F respectively) are, in the general case, violated. Then a complex analytic geometry in the "space" P_C may be constructed that has nothing to do with G_C: the existence of isotropic lines and planes is the main if exotic characteristic features of this geometry, and the roots of this phenomenon penetrate the ruins of the said violated axioms.

(Let us note, parenthetically, that these inferences have far -reaching sequels in differential geometry. The same complex analytic geometry in P_C is used by the author of [62] in his text-book [63] on differential geometry. In the foreword of [63] one reads:

"Почти всички въпроси са разгледани в к о м п л е к с н о т о пространство. Обикновено третирането на въпросите е еднакво в реалното и в комплексното пространство, тъй като изходните положения са едни и същи. Различия се явяват понякога само в доказването на тези изходни положения. Но има и геометрични обекти, възможни с а м о в комплексното пространство, каквито са например изотропните криви. Теорията на тези криви е изложена тук с пълнота, каквато рядко се среща в книги от подобен род..."

In the introduction of the book one reads:

"В цялото следващо изложение ще разглеждаме обикновено геометрични фигури, разположени в к о м п л е к с н о т о триизмерно Евклидово пространство E_c, така че ще имаме работа с комплексни точки, прави, равнини, вектори, координатни системи и пр. Само при някои въпроси ще се ограничаваме, което ще споменаваме изрично, на фигури в реалното триизмерно Евклидово пространство E_r, което е част от E_c; тогава ще имаме пред вид само реални точки, прави, равнини, вектори, координатни системи и т.н. и ще казваме, че работим "в E_r" или "в реалния случай". Обаче от методични съображения много въпроси отначало ще бъдат третирани в E_r и след това разглежданията ще бъдат обобщавани и разширявани за E_c."

As these texts imply, the mathematical philosophy of [63] is the following one. By definition $E_r = R^3$; points and vectors are defined as elements of E_r. The analytic and differential geometries in E_r are well-known long ago. Now all the paraphernalia of these geometries are formally transplanted in E_C, provided by definition $E_c = C^3$, supplied with the operations (12) — (15) (i.e., as a matter of fact, E_c of [63] is what above has been denoted by P_C: we have good grounds to avoid the notation E_c). All these manipulations are entirely different from those by means of which G_C is constructed.

If now G_C, rather than the analytic geometry from [62], is put in the groundwork of a complex differential geometry Γ_C, the latter will be entirely different from the hybrid proposed in [63].)

A most interesting problem now arises. As it is well-known, along with the arithmetical model of V, realized by the aid of the operations (12) — (15) in R^3, there exists a geometrical model too, realized by synthetic-geometrical means. To be more specific, let P be the set of all Euclidean points (i.e. points of the synthetic Euclidean geometry) and let a relation $\sim$ be introduced in the Cartesian square P^2 of P, defined in the following manner: if $(A_\nu,\ B_\nu) \in P^2$ $(\nu = 1,\ 2)$, then by definition

$$(A_1,\ B_1) \sim (A_2,\ B_2), \tag{40}$$

iff the following conditions are satisfied:

1. $A_1 = B_1$, $A_2 = B_2$, or $A_1 \neq B_1$, $A_2 \neq B_2$ and the line A_1B_1 is *parallel* to the line A_2B_2.

2. If $A_1 \neq B_1$, $A_2 \neq B_2$, then the segments A_1B_1, A_2B_2 have the *same directions*.

3. If $A_1 \neq B_1$, $A_2 \neq B_2$, then the segments A_1B_1, A_2B_2 have the *same lengths*.

It is obvious that (40) is an equivalence relation in P^2. Now any equivalence class in P^2, generated by (40), is by definition a *real standard vector*.

In order that this denomination gets justified, definitions of operations *addition* of vectors, *multiplication* of real numbers with vectors, *scalar* and *vector multiplications* of vectors are given, and it is proved that these operations satisfy Ax 1 F — Ax 15 F (with R and V instead of F and V_F respectively). *Quod erat agendum.*

For the sake of brevity, let $G = G_R$ and let E denote the synthetic Euclidean geometry constructed axiomatically by the aid of the system of axioms S_H proposed to this end by Hilbert in his famous work [16]. The following bilateral mathematical process may now be realized.

On the one hand, one can model E in G. In other words, one can select appropriate entities in G, call them "points", "lines", "planes", etc., and prove that the axioms of S_H are satisfied under these definitions.

On the other hand, one can model G in E. Since G is constructed by means of real standard vectors, the meaning of this statement is that one can model them in E, as indicated above.

And that is all. In such a manner it is proved that there is no essential difference between E and G. All the difference consists in technicalitites, if one may say so.

So much about G. What about G_C?

The trouble is that, as far as our knowledge goes, we do not yet dispose of such a geometry — let us, hypothetically, call it E_C — that the relations between E and G described above could be imitated with E_C and G_C instead.

In other words, the geometers have not yet invented such a (hypothetical again) complex synthetic (Hermitean rather than Euclidean) geometry that should have G_C for its analytical equivalent.

Putting it another way, no complex variant of Hilbert's *Axiomensystem* S_H is up to now proposed — such at that, that the corresponding complex synthetic geometry E_C could model G_C and, inversely, could be modelled by G_C.

The solution of this *Problemata novum, ad cuius solutionem geometrici invitantur* (as Johann Bernoulli announced his problem in *Acta eruditorum*: determine that smooth curve in a vertical plane, connecting two points A and B in it, along which a heavy mass-point must move in order to arrive at B, starting from A, for the shortest time) may be facilitated in virtue of the following observation.

First, the analytic version G_C of E_C exists. Otherwise, E_C is, in a sense, familiar. The task consists, then, in discovering such complex analogues of the axioms of S_H, which could imply G_C synthetically.

Second, taking into consideration the mathematical structures of G and G_C, one could select those axioms of S_H that will remain authoritative for E_C too. As it is well-known, Hilbert divided the axioms of the Euclidean geometry into five groups:

"Die Axiomgruppe I: *Axiome der Verknüpfung.*

Die Axiomgruppe II: *Axiome der Anordnung.*

Die Axiomgruppe III: *Axiome der Kongruenz.*

Die Axiomgruppe IV: *Axiom der Parallelen.*

Die Axiomgruppe V: *Axiome der Stetigkeit*" [16, S. V; our italics].

Now everybody who is familiar with Hilbert's geometrical axiomatics [16] and with the analytic geometries in complex standard vector spaces [61] will ascertain that all Hilbert's axioms listed above must remain true for E_C, with the only exception of *die Axiomgruppe II*.

Indeed, substituting in G the field C of the complex numbers for the field R of the real numbers in order to obtain G_C, one dispenses with *Anordnung* since C, in contrast to R, is deprived of such an attribute. The situation here is very similar with that of complex analysis in comparison with the real one: the true in the real analysis theorem, according to which if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = l$ exists and if $x_n < z_n < y_n$ $(n = 1, 2, \dots)$, then $\lim_{n\to\infty} z_n$ also exists and is equal to l, becomes not only untrue, but even meaningless in the complex case, since the inequalities $x_n < z_n < y_n$ are now devoid of sense.

Let us also note that, as regards the *Axiome der Stetigkeit*, they remain true for G_C, but must be reformulated or rejected at all, if one desires to construct axiomatically $G_{C(F)}$ with an arbitrary F rather than G_C.

All these problems belong to the future. As already mentioned, the solution of the formulated above ***Problemata novum, ad cuius solutionem Geometrici invitantur*** is still *in-cunabula*. The complex analogue Σ_H of S_H once settled, G_C will propose a second, geometrical this time, model of V_C (and possibly of $V_{C(F)}$), along with the arithmetical one, proposed by C^3 together with the operations (12), (13) and (20), (21) in it.

That much about the interplay of analytical mechanics and geometry. In which manner the real and complex analytic geometries may be used in analytical mechanics in order to supply it with motions, rigid bodies, and forces — all this will be manifested later. Before we undertake the next step, let us make a brief summing-up.

Starting from the algebraic notions of ordered fields, in general, and of the fields of the real and complex numbers, in particular, we defined axiomatically V_F and $V_{C(F)}$, in general, and V and V_C, in particular. Then we proceeded to analytic geometries G_C and $G_{C(F)}$ in V_F and $V_{C(F)}$ respectively. What now lies ahead is a good algebraic theory of arrows.

Sliding vectors (*vecteurs glissants, gleitende Vektoren, скользящие векторы*) are no vectors at all, as it will be seen immediately below. Their genesis is a purely physical one. They have been destined to describe mathematically forces acting on mass-points and rigid bodies, and it is worth noting that they did their duty with merit deserving the highest praise.

Very much nonsense has been written in the philosophical, physical, and even mathematical literature in connection with the "meaning", "nature", or "essence" of forces: stupidity is unforbiddable. As a matter of fact, a mathematician is interested not in "what a mathematical object is?" (this question is void of sense), but in "how to describe it mathematically?". Making a long story short, let us pronounce the final verdict: in analytical mechanics forces are described mathematically by arrows. Now what does an "arrow" mean?

We must begin from afar.

Let us first see, in a book chosen at random, how are sliding vectors traditionally introduced in the mathematical literature:

"Дадим поэтому следующее определение:

Вектором называется величина, характеризующаяся, помимо измеряющего ее в определенных единицах меры числа, еще своим направлением в пространстве ...

Отметим, что различают векторы трех родов: с в о б о д н ы е, п е р е д в и ж н ы е [sliding vectors] и о п р е д е л е н н ы е в е к т о р ы. Введенные нами векторы относятся к типу свободных, так как точку их приложения можно выбирать по произволу. У передвижных векторов точку приложения вектора можно перемещать произвольно вдоль самого вектора, так что последный может лежать на любой части определенной прямой. Примером передвижного вектора является сила, приложенная к твердому телу, так как за точку приложения силы можно взять любую точку на линии действия силы. Наконец, у определенных векторов точка приложения вектора должна быть зафиксирована. Так, например, при рассмотрении движения жидкости за точку приложения силы, действующей на какую-либо частицу жидкости, принимается некоторая точка самой частицы.

Изучение передвижных и определенных векторов сводится к изучению свободных векторов, почему достаточно ограничиться рассмотрением только последних."

This citation is taken from the ninth edition [64, стр. 6 – 7] of a treatise especially dedicated to vector calculus and it inevitably reminds the estimation "so muß jeder Vernünftige erkennen, daß damit soviel als nichts gesagt ist" of v. Mises. The book [64] makes no unhappy exception. The same fate haunts the sliding vectors in the hands of almost any author of a book on vector calculus or theoretical mechanics. The situation reminds the legend of the destroying of the Alexandrian Library. Someone of the priesthood allegedly has said: "If these books are in concordance with the Alcoran (or the Bible, no matter), then they are redundant; if they contradict it, then they are harmful." In the same way, definitions of the kind quoted above are needless if the reader already knows what does a sliding vector mean; if he does not, then they are wholly futile.

Turning back to the already cited text-book [55] let us note, as a kind of self-criticism, that at the time when it was written, neither its author, nor the elder of the authors of this article, knew how the sliding vectors must be defined most properly. Because of that reason the exposition of the theory of sliding vectors in [55], as well as in the second edition [65] of the book, seems today unsatisfactory. In both edition about 50 pages are assigned to sliding vectors. The main reason for such a prodigality is rooted in the fact that the exposition is, in essence, synthetic-geometrical.

Let us discuss this problem in some details.

The intuitive idea, incarnated in the original notion of a sliding vectors, consists in the simultaneous perception of two different mathematical objects, genetically united in an indivisible whole. These objects are a line l and a vector $\boldsymbol{s}$, lying on l or at least parellel to it. Conceptually, this idea is connected with the image of a *force* acting on a rigid body. The line l interprets mathematically the *line of action* or the *directrix* of the force, and the vector $\boldsymbol{s}$ its *magnitude*. In older treatises on mechanics this mental picture is materialized in a pleasant if naive way by illustrations: l is portrayed in the form of a stretched string, while $\boldsymbol{s}$ in pictured as a man's hand pulling the string.

Many years ago an attempt (none too felicitous) was made to formalize mathematically these intuitive ideas. In the article [66] a sliding vector is defined, in plain words, as an ordered pair $(\boldsymbol{s}, l)$ of a vector $\boldsymbol{s}$ and a line l parallel to $\boldsymbol{s}$; this definition works, but works rather arduously.

It turns out, however, that this definition may be reformulated in a most profitable way. Let us write down an equation of the line l: since it is parellel to $\boldsymbol{s}$, there certainly exists a vector $\boldsymbol{m}$, so that one of the equations of l (there are infinitely many, as we know) reads:

$$(41) \qquad \boldsymbol{r} \times \boldsymbol{s} = \boldsymbol{m}.$$

At that, obviously

$$(42) \qquad \boldsymbol{s} \neq \boldsymbol{o}, \quad \boldsymbol{s}\boldsymbol{m} = 0.$$

This settled, let us regard the ordered pair $(\boldsymbol{s},\ \boldsymbol{m})$ with (42) instead of tl ordered pair $(\boldsymbol{s},\ l)$. Both pairs are equivalent in the following sense: if $(\boldsymbol{s},\ \boldsymbol{m})$ given, then l is given too through its equation (41), hence $(\boldsymbol{s},\ l)$ is known; inversel if $(\boldsymbol{s},\ l)$ is given, then $\boldsymbol{m}$ is given through (41), hence $(\boldsymbol{s},\ \boldsymbol{m})$ is known.

These observations lead to the following alternative definition of a sliding ve tor: so is called any ordered pair

$$(43) \qquad \vec{s} = (\boldsymbol{s},\ \boldsymbol{m}),$$

of real standard vectors $\boldsymbol{s}$, $\boldsymbol{m}$, satisfying (42). For the same reason for which tl number zero is introduced, this definition is supplemented with that of the *ze sliding vector*

$$(44) \qquad \vec{o} = (\boldsymbol{o},\ \boldsymbol{o})$$

while (43) with (42) are called *non-zero sliding vectors.*

A developed algebraic theory of sliding vectors based on the definitions (4 and (43) with (42) is proposed in the article [67].

Before proceeding further, let us underline an important act of the author [55].

As it is completely clear from the definitions (43) and (44), a sliding vecto zero or non-zero, is no vector at all: it is a pair of vectors. That is why tl term "sliding vector" is extremely inadequate, being psychologically misleading - a crying inexpedience. Obviously, a new term must be invented. After some di cussions, the author of [55] decided to substitute the term *arrow* for the tradition *sliding vector*, and nowadays, through the text-book [55], it enjoys popularity, this country at least.

(Forces being interpreted by means of arrows, do not believe an author of book on theoretical mechanics who states that "forces are vectors": he is misleadiı you.)

The definitions (44) and (43) with (42) are, however, by no means necessari connected with the real standard vector space V. Obviously, the idea to substitu V_F and even $V_{C(F)}$ for V in these definitions is quite natural and rather temptin In result one ontains not a single, but infinitely many algebraic theories of arrov — one for any F both in V_F and $V_{C(F)}$. It is natural to call the arrows of tl first kind *real*, and those of the second kind *complex* ones. We shall sketch nc the general contours of such a theory, working, for the sake of generality, in tl complex case of $V_{C(F)}$, the real one being obtainable automatically by virtue of tl agreement to accept the inclusion $V_F \subset V_{C(F)}$.

Let by definition

$$(45) \qquad A_{C(F)} = \{(s,\ m) \in V^2_{C(F)} : s \neq o,\ sm = 0 \vee s = m = o\}.$$

The elements of $A_{C(F)}$ are called $C(F)$-arrows (or simply arrows, for the sake of brevity). If

$$(46) \qquad (s,\ m) \in A_{C(F)},$$

then s and m are called the *basis* and the *moment* of $\vec{s}$ respectively, provided (43).

The definitions (34) and (45) imply the inclusion

$$(47) \qquad \Lambda_{C(F)} \subset A_{C(F)}.$$

If

$$(48) \qquad \vec{s} \in \Lambda_{C(F)},$$

then $\vec{s}$ is called a *non-zero arrow*. The arrow $\vec{o}$ defined by (44) is called the *zero-arrow*.

If (48), then there exists exactly one $l \in L_{C(F)}$ with $\vec{s}\&l$. The line l is called the *directrix* of $\vec{s}$ and is denoted by dir $\vec{s}$. In such a manner, if (48), then

$$(49) \qquad \vec{s}\ \&\ \mathrm{dir}\ \vec{s}.$$

If

$$(50) \qquad r \in V_{C(F)},$$

$$(51) \qquad \vec{s} \in A_{C(F)},$$

and (43), then the vector defined by

$$(52) \qquad \mathrm{mom}_r \vec{s} = m + s \times r$$

is called the r-*moment* of $\vec{s}$, and r is called the *pole* of $\mathrm{mom}_r\ \vec{s}$. Obviously, (51) and (43) imply

$$(53) \qquad m = \mathrm{mom}_o \vec{s}.$$

Besides

$$(54) \qquad \mathrm{mom}_r \vec{o} = o$$

for any pole r.

If (50) and

(55) $$l \in L_{C(F)},$$

(56) $$(a,\ b)\ \&\ l,$$

then r is called *incident* with l iff (22). In such a case it is written

(57) $$r\ \mathsf{Z}\ l.$$

In case (57) in violated, r is called *non-incident* with l. In such a case it is written $r\ \overline{\mathsf{Z}}\ l$.

Let

(58) $$\overline{p} \in V_{C(F)},$$

(59) $$\overline{p}\ \mathsf{Z}\ \mathrm{dir}\ \vec{s}.$$

Then (51), (43), (58), (59), and the definition of the relation (57) imply

(60) $$\overline{p} \times s = m.$$

Now (50), (60), and (52) imply

(61) $$(\overline{p} - r) \times s = \overline{p} \times s + s \times r = m + s \times r = \mathrm{mom}_r \vec{s}.$$

Most of the text-books on theoretical mechanics define $\mathrm{mom}_r \vec{s}$ by the expression in the left-hand side of (61). The definition (52), however, is preferable, since it possesses the most important for a mathematical definition characteristic of economy.

If (51), (43),

(62) $$r_\nu \in V_{C(F)} \qquad (\nu = 1,\ 2),$$

then (52) implies

(63) $$\mathrm{mom}_{r_1} \vec{s} - \mathrm{mom}_{r_2} \vec{s} = s \times (r_1 - r_2).$$

The relation (63) is called the *connection* between the moments of an arrow with respect to the poles (62). It implies

(64) $$s \cdot \mathrm{mom}_{r_1} \vec{s} = s \cdot \mathrm{mom}_{r_2} \vec{s}.$$

The relation (64) is, however, trivial, since both its sides are a priori zeroes: (52) and (42) imply

(65) $$s \cdot \mathrm{mom}_r \vec{s} = 0$$

for any pole (50). The relation

$$(66)\qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2)\cdot \mathrm{mom}_{\boldsymbol{r}_1}\vec{s} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)\cdot \mathrm{mom}_{\boldsymbol{r}_2}\vec{s}$$

implied by (63) is, however, non-trivial, except when $\boldsymbol{r}_1 = \boldsymbol{r}_2$.

The *sum* of the arrows

$$(67)\qquad \vec{s}_\nu \in A_{C(F)} \qquad (\nu = 1,\ 2),$$

defined by

$$(68)\qquad \vec{s}_1 + \vec{s}_2 = (\boldsymbol{s}_1 + \boldsymbol{s}_2,\ \boldsymbol{m}_1 + \boldsymbol{m}_2)$$

provided

$$(69)\qquad \vec{s}_\nu = (\boldsymbol{s}_\nu,\ \boldsymbol{m}_\nu) \qquad (\nu = 1,\ 2),$$

does not necessarily exist: the right-hand side of (68) belongs to $A_{C(F)}$ if, and only if,

$$(70)\qquad \boldsymbol{s}_1 + \boldsymbol{s}_2 = \boldsymbol{o}, \quad \boldsymbol{m}_1 + \boldsymbol{m}_2 = \boldsymbol{o}$$

or

$$(71)\qquad \boldsymbol{s}_1 + \boldsymbol{s}_2 \neq 0, \quad (\boldsymbol{s}_1 + \boldsymbol{s}_2)(\boldsymbol{m}_1 + \boldsymbol{m}_2) = 0.$$

The arrow

$$(72)\qquad -\vec{s} = (-\boldsymbol{s},\ -\boldsymbol{m})$$

provided (51), (43) is called the *opposite* of $\vec{s}$. On the other hand, (67), (69), and (45) imply

$$(73)\qquad \boldsymbol{s}_\nu \boldsymbol{m}_\nu = 0 \qquad (\nu = 1,\ 2).$$

Now (73) imply that (71) are equivalent to

$$(74)\qquad \boldsymbol{s}_1 + \boldsymbol{s}_2 \neq \boldsymbol{o}, \quad \boldsymbol{s}_1\boldsymbol{m}_2 + \boldsymbol{s}_2\boldsymbol{m}_1 = 0.$$

In such a manner, the sum (68) of the arrows (69) exists if, and only if, $\vec{s}_2 = -\vec{s}_1$ or (74) holds good. Hence, the set (45) of all $C(F)$-arrows is a system with partially defined addition.

Let us pass over to *systems* of arrows. By this term in analytical mechanics finite sets

$$\underset{\rightarrow}{s} = \{\vec{s}_\nu\}_{\nu=1}^n \tag{75}$$

of arrows

$$\vec{s}_\nu \in A_{C(F)} \qquad (\nu = 1, \ldots, n) \tag{76}$$

are understood. Let

$$\vec{s}_\nu = (\boldsymbol{s}_\nu, \boldsymbol{m}_\nu) \qquad (\nu = 1, \ldots, n). \tag{77}$$

Then the sums

$$\boldsymbol{s} = \sum_{\nu=1}^n \boldsymbol{s}_\nu, \quad \boldsymbol{m} = \sum_{\nu=1}^n \boldsymbol{m}_\nu \tag{78}$$

are called the *basis* and the *moment* of $\underset{\rightarrow}{s}$ respectively.

If (50), then the vector defined by

$$\operatorname{mom}_{\boldsymbol{r}} \underset{\rightarrow}{s} = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r} \tag{79}$$

is called the *moment* of the system of arrows $\underset{\rightarrow}{s}$ with respect to $\boldsymbol{r}$, and $\boldsymbol{r}$ is called the *pole* of $\operatorname{mom}_{\boldsymbol{r}} \underset{\rightarrow}{s}$. The analogy between the definitions (52) and (79) is obvious. Evidently,

$$\boldsymbol{m} = \operatorname{mom}_{\boldsymbol{o}} \underset{\rightarrow}{s}, \tag{80}$$

similar to (53). The relations (78), (79), and (52) imply

$$\begin{aligned} \operatorname{mom}_{\boldsymbol{r}} \underset{\rightarrow}{s} &= \sum_{\nu=1}^n \boldsymbol{m}_\nu + \sum_{\nu=1}^n \boldsymbol{s}_\nu \times \boldsymbol{r} \\ &= \sum_{\nu=1}^n (\boldsymbol{m}_\nu + \boldsymbol{s}_\nu \times \boldsymbol{r}) = \sum_{\nu=1}^n \operatorname{mom}_{\boldsymbol{r}} \vec{s}_\nu. \end{aligned} \tag{81}$$

If (75) — (78), (62), then (79) implies

$$\operatorname{mom}_{\boldsymbol{r}_1} \underset{\rightarrow}{s} - \operatorname{mom}_{\boldsymbol{r}_2} \underset{\rightarrow}{s} = \boldsymbol{s} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2) \tag{82}$$

similar to (63). The relation (82) is called the *connection* between the moments of a system of arrows with respect to the poles (62). It implies

$$\boldsymbol{s} \cdot \operatorname{mom}_{\boldsymbol{r}_1} \underset{\rightarrow}{s} = \boldsymbol{s} \cdot \operatorname{mom}_{\boldsymbol{r}_2} \underset{\rightarrow}{s} \tag{83}$$

and

$$(r_1 - r_2) \cdot \mathrm{mom}_{r_1} \underline{s} = (r_1 - r_2) \cdot \mathrm{mom}_{r_2} \underline{s} \tag{84}$$

similar to (64) and (66) respectively. Contrary to (64), however, the relation (83) is by no means trivial, since, in general,

$$s \cdot \mathrm{mom}_r \underline{s} \neq 0. \tag{85}$$

Since

$$s \cdot \mathrm{mom}_r \underline{s} = sm \tag{86}$$

for any pole r by virtue of (79), the quantity

$$I = sm \tag{87}$$

is called the *first scalar invariant* of $\underline{s}$. If

$$s \neq o, \tag{88}$$

then the quantity

$$II = \frac{sm}{s^2} \tag{89}$$

is called the *second scalar invariant* of $\underline{s}$.

The analogue for systems of arrows of the notion *directrix* of an arrow is the so-called *axis* of the system in question. As well as the notion directrix becomes meaningless in the case of the zero-arrow, the axis of a system of arrows does not exist save in the case (88).

Let (88) hold and let the following problem be put: determine all poles (50) for which

$$s \times \mathrm{mom}_r \underline{s} = o. \tag{90}$$

The condition (90) is a vector-algebraic equation with respect to r. In view of (79), it is equaivalent to

$$(m + s \times r) \times s = o, \tag{91}$$

i.e. to

$$(r \times s) \times s = m \times s. \tag{92}$$

The equation (92), together with (88), implies that there exists $\alpha \in C(F)$ with

$$\text{(93)} \qquad \boldsymbol{r} \times \boldsymbol{s} = \alpha \boldsymbol{s} + \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2}.$$

A scalar multiplication of (93) with $\boldsymbol{s}$ implies

$$\text{(94)} \qquad \alpha \boldsymbol{s}^2 = 0$$

and (94), (88) imply

$$\text{(95)} \qquad \alpha = 0.$$

Now (93), (95) imply

$$\text{(96)} \qquad \boldsymbol{r} \times \boldsymbol{s} = \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2}.$$

Now, by virtue of (88),

$$\text{(97)} \qquad \left(\boldsymbol{s}, \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2}\right) \in \Lambda_{C(F)}.$$

Hence there exists exactly one $l \in L_{C(F)}$ with

$$\text{(98)} \qquad \left(\boldsymbol{s}, \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2}\right) \& l.$$

It is called the *axis* of $\underset{\rightarrow}{s}$ and is denoted by ax $\underset{\rightarrow}{s}$. In such a manner,

$$\text{(99)} \qquad \left(\boldsymbol{s}, \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2}\right) \& \operatorname{ax} \underset{\rightarrow}{s}.$$

In such a way, the geometrical place of all points $\boldsymbol{r} \in V_{C(F)}$, for which (90) holds provided (88), is the line ax $\underset{\rightarrow}{s}$, defined by (99).

If (51), (43), (88) hold and

$$\text{(100)} \qquad \underset{\rightarrow}{s} = \{\vec{s}\},$$

i.e. the system $\underset{\rightarrow}{s}$ consists of the single non-zero arrow $\vec{s}$, then the relation (99) changes into

$$\text{(101)} \qquad (\boldsymbol{s}, \boldsymbol{m}) \& \operatorname{ax}\{\vec{s}\}.$$

Since by definition

$$\text{(102)} \qquad (\boldsymbol{s}, \boldsymbol{m}) \& \operatorname{dir}\vec{s},$$

it is immediately seen that the notion *axis* of a system of arrows indeed is a generalization of the notion *directrix* of a single arrow, provided (88) in both cases.

It is proved that if (50), (58),

(103) $$\boldsymbol{r} \mathsf{Z} \text{ ax } \underset{\rightarrow}{s}, \quad \overline{\boldsymbol{p}} \overline{\mathsf{Z}} \text{ ax } \underset{\rightarrow}{s},$$

then

(104) $$(\text{mom}_{\boldsymbol{r}} \underset{\rightarrow}{s})^2 < (\text{mom}_{\overline{\boldsymbol{p}}} \underset{\rightarrow}{s})^2.$$

In such a manner, the moments of a system of arrows have minimal modules with respect to the poles incident with its axis (if it exists, and if modules exist in $V_{C(F)}$).

A most important characteristic of a system of arrows $\underset{\rightarrow}{s}$ is its *rank*, denoted by rank $\underset{\rightarrow}{s}$. It is defined in the following manner.

If $G \subset V_{C(F)}$ and a mapping

(105) $$F: G \longrightarrow V_{C(F)}$$

is defined, then it is called a *vector field* over G.

A system $\underset{\rightarrow}{s}$ of arrows being given, the mapping

(106) $$\mu: V_{C(F)} \longrightarrow V_{C(F)}$$

defined by

(107) $$\mu(\boldsymbol{r}) = \text{mom}_{\boldsymbol{r}} \underset{\rightarrow}{s} \qquad (\boldsymbol{r} \in V_{C(F)})$$

is called the moment field of $\underset{\rightarrow}{s}$; (107) and (79) imply

(108) $$\mu(\boldsymbol{r}) = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r} \qquad (\boldsymbol{r} \in V_{C(F)})$$

provided (75) — (78).

The *maximal number* of linearly independent elements of the moment field of a system $\underset{\rightarrow}{s}$ of arrows, i.e. *of the set*

(109) $$\mu(V_{C(F)}),$$

representing the image of $V_{C(F)}$ through μ, is called the *rank* of $\underset{\rightarrow}{s}$. According to this definition, rank $\underset{\rightarrow}{s}$ is one of the numbers 0, 1, 2, and 3, any four elements of $V_{C(F)}$ being linearly dependent.

A system $\underset{\rightarrow}{s}$ of arrows being given, the direct determination of rank $\underset{\rightarrow}{s}$, i.e. by an immediate application of the above definition, is an unpleasing task. Indeed, to prove that rank $\underset{\rightarrow}{s} = 3$ means to find such poles

(110) $$\boldsymbol{r}_\nu \in V_{C(F)} \qquad (\nu = 1, 2, 3),$$

that $\mathrm{mom}_{r_\nu} \underset{\rightarrow}{s}$ ($\nu = 1, 2, 3$) are linearly independent, i.e.

(111) $$\mathrm{mom}_{r_1} \underset{\rightarrow}{s} \times \mathrm{mom}_{r_2} \underset{\rightarrow}{s} \cdot \mathrm{mom}_{r_3} \underset{\rightarrow}{s} \neq 0.$$

Similarly, to prove that rank $\underset{\rightarrow}{s} = 2$ means, to demonstrate that (110) imply

(112) $$\mathrm{mom}_{r_1} \underset{\rightarrow}{s} \times \mathrm{mom}_{r_2} \underset{\rightarrow}{s} \cdot \mathrm{mom}_{r_3} \underset{\rightarrow}{s} = 0,$$

and, second, to find such poles (62), that $\mathrm{mom}_{r_\nu} \underset{\rightarrow}{s}$ ($\nu = 1, 2$) are linearly independent, i.e.

(113) $$\mathrm{mom}_{r_1} \underset{\rightarrow}{s} \times \mathrm{mom}_{r_2} \underset{\rightarrow}{s} \neq o.$$

Further, to prove that rank $\underset{\rightarrow}{s} = 1$ means, first, to demonstrate that (62) imply

(114) $$\mathrm{mom}_{r_1} \underset{\rightarrow}{s} \times \mathrm{mom}_{r_2} \underset{\rightarrow}{s} = o$$

and, second, to find such a pole (50) that

(115) $$\mathrm{mom}_{r} \underset{\rightarrow}{s} \neq o.$$

At last, to prove that rank $\underset{\rightarrow}{s} = 0$ means to demonstrate

(116) $$\mathrm{mom}_{r} \underset{\rightarrow}{s} = o \qquad (r \in V_{C(F)}).$$

All this is, as already said, an unwellcome procedure. Fortunately, it becomes unnecessary in view of the following statement, usually called the rank-theorem for systems of arrows:

The conditions (75) — (78) imply

(117) $$\mathrm{rank}\, \underset{\rightarrow}{s} = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \quad \text{iff} \quad \begin{cases} s = o, \; m = o, \\ s = o, \; m \neq o, \\ s \neq o, \; sm = 0, \\ sm \neq 0 \end{cases}$$

respectively.

Two diferent proofs of the rank-theorem may be found in the article [68].

It is obvious that the application of the rank-theorem reduces the determination of rank $\underset{\rightarrow}{s}$ to a *Schablon*: a system of arrows $\underset{\rightarrow}{s}$ being defined by (75) — (77), one has to form s and m according to (78) and to see, which of the mutually exclusive cases in the right-hand side of (117) is at hand.

Let us note, by the way, that the definition (79) implies: $\mathrm{mom}_{r} \underset{\rightarrow}{s}$ is invariant with respect to the pole r iff $s = o$.

The rank of a system of arrows plays an important role in the theory of such systems. For the time being we shall pause upon one only of its applications.

A key role in the theory of arrows and in its applications to statics and dynamics of mass-points and especially of rigid bodies plays the notion of *equivalent systems of arrows.*

While the basis $\boldsymbol{s}$ and the moment $\boldsymbol{m}$ of a single arrow wholly determine it, the same is not true as regards systems of arrows. In other words, there exist two at least (infinitely many, as a matter of fact) diferent systems of arrows with the same basis and the same moment. Yet, we shall use the notation $\underset{\rightarrow}{s}(\boldsymbol{s}, \boldsymbol{m})$ in order to mark that the system $\underset{\rightarrow}{s}$ has the basis $\boldsymbol{s}$ and the moment $\boldsymbol{m}$: all the same, $\boldsymbol{s}$ and $\boldsymbol{m}$ are certain characteristics of $\underset{\rightarrow}{s}$, as for instance the rank-theorem displays.

Two systems of arrows $\underset{\rightarrow}{s_\nu}(\boldsymbol{s}_\nu, \boldsymbol{m}_\nu)$ $(\nu = 1, 2)$ are called *equivalent* iff

$$\boldsymbol{s}_1 = \boldsymbol{s}_2, \quad \boldsymbol{m}_1 = \boldsymbol{m}_2. \tag{118}$$

In such a case, it is written

$$\underset{\rightarrow}{s_1} \sim \underset{\rightarrow}{s_2}. \tag{119}$$

It is trivially seen that (119) is an *equivalence relation* in the set $S_{C(F)}$ of all systems of $C(F)$-arrows. Any equivalence class in $S_{C(F)}$ generated by it is called a $C(F)$*-action* or simply an *action.*

The time-honoured experience of mechanicians, both experimental and theoretical, displays that the mechanical behaviour, statical as well as dynamical, of mass-points and rigid bodies is predestinated by the actions of forces, rather than by the special systems of forces determining these actions, inasmuch as forces are represented mathematically by arrows. In other words, if a mass-point or a rigid body is in equilibrium or in a motion under the action of a given system of arrows $\underset{\rightarrow}{s_1}(\boldsymbol{s}_1, \boldsymbol{m}_1)$, and if the system $\underset{\rightarrow}{s_2}(\boldsymbol{s}_2, \boldsymbol{m}_2)$ is substituted for $\underset{\rightarrow}{s_1}$, then the mass-point or the rigid body in question will remain in equilibrium or will accomplish the same motion as before if, and only if, $\underset{\rightarrow}{s_1}$ and $\underset{\rightarrow}{s_2}$ are equivalent, i.e. iff (119) or, just the same, (118) holds.

As the case stands, it is easy to explain why did mechanicians, especially at the early age of mechanics, when the analytical methods were not yet developed, and the mechanicians were compelled to work synthetically, strive to substitute a system of forces $\underset{\rightarrow}{s_1}$, acting on a mass-point or on a rigid body, with another one $\underset{\rightarrow}{s_2}$, equivalent to $\underset{\rightarrow}{s_1}$, but "simpler" than $\underset{\rightarrow}{s_1}$, especially in cases, when $\underset{\rightarrow}{s_1}$ was a "complicated" one. In spite of the fact that the characteristics "simple" and "complicated" are not unambiguous ones, it is intuitively clear what is meant by them: for instance, consisting of a smaller number of elements. This process has been called the *reducation* of the system of forces in question, and it has been accomplished by the succesive application of a certain amount of synthetic-geometrical operations called *elementar-statical operations.* They are four in number:

1. If a given system of forces $\underset{\rightarrow}{s}$ contains two forces $\vec{s}_1$ and $\vec{s}_2$ which possess a sum $\vec{s}_1 + \vec{s}_2$, then substitute $\vec{s}_1 + \vec{s}_2$ in $\underset{\rightarrow}{s}$ for $\vec{s}_1$ and $\vec{s}_2$.

2. If $\vec{s} \in \underset{\rightarrow}{s}$ and $\vec{s}_1, \vec{s}_2$ are such arrows that $\vec{s}_1 + \vec{s}_2 = \vec{s}$, then substitute $\vec{s}_1$ and $\vec{s}_2$ in $\underset{\rightarrow}{s}$ for $\vec{s}$.

3. If $\vec{o} \in \underset{\rightarrow}{s}$, then eliminate it from $\underset{\rightarrow}{s}$.

4. Add $\vec{o}$ to $\underset{\rightarrow}{s}$.

It should be noted that the elementar-statical operations, accomplished on a system of arrows $\underset{\rightarrow}{s}$, do not alter the basis and the moment of $\underset{\rightarrow}{s}$, so that, no matter how many elementar-statical operations are performed on $\underset{\rightarrow}{s}$, the result is always a system of arrows equivalent to $\underset{\rightarrow}{s}$.

In order to realize the importance that formerly was attached to the process of reducation of systems of forces, let us note that 30 clear pages are set aside in the text-book [55] for various particularities connected with equivalence and reduction of systems of arrows.

In the course of time, the mechanicians arrived at four "simplest" or *elementar* kinds of systems of arrows, to one of which any system of arrows can be reduced by means of elementar-statical operations. These four types of systems are as follows:

1. The zero-system $\underset{\rightarrow}{o} = \{\vec{o}\}$, consisting of the zero-arrow $\vec{o}$ only.

2. A dipol $\underset{\rightarrow}{\delta}$, consisting of two non-zero arrows $\vec{s}_\nu$ ($\nu = 1, 2$) with $\boldsymbol{s}_1 + \boldsymbol{s}_2 = \boldsymbol{o}$, $\boldsymbol{m}_1 + \boldsymbol{m}_2 = \boldsymbol{o}$.

3. The mono-system $\underset{\rightarrow}{\mu} = \{\vec{s}\}$, consisting of a single non-zero arrow $\vec{s}$.

4. A bi-system $\underset{\rightarrow}{\beta}$, consisting of two non-zero arrows $\vec{s}_\nu$ ($\nu = 1, 2$) with non-parallel non-intersecting directrices.

The following remarks are not useless.

First, while the elementar systems $\underset{\rightarrow}{o}$ and $\underset{\rightarrow}{\mu}$ (cases 1 and 3 respectively) are wholly determined, the elementar systems $\underset{\rightarrow}{\delta}$ and $\underset{\rightarrow}{\beta}$ (cases 2 and 4 respectively) are not. The meaning of this statement is that there exist two at least (infinitely many, as a matter of fact) different dipols $\underset{\rightarrow}{\delta_1}$ and $\underset{\rightarrow}{\delta_2}$ with $\underset{\rightarrow}{\delta_1} \sim \underset{\rightarrow}{\delta_2}$, as well as two at least (also infinitely many, in point of fact) different bi-system $\underset{\rightarrow}{\beta_1}$ and $\underset{\rightarrow}{\beta_2}$ with $\underset{\rightarrow}{\beta_1} \sim \underset{\rightarrow}{\beta_2}$. This is the meaning of the use of the definite atricle "the" in the cases 1 and 3, and of the indefinite article "a" in cases 2 and 4.

Second, if $\vec{s}_\nu = (\boldsymbol{s}_\nu, \boldsymbol{m}_\nu)$ ($\nu = 1, 2$), then by definition dir $\vec{s}_\nu$ ($\nu = 1, 2$) are intersecting if, and only if,

$$\boldsymbol{s}_1 \times \boldsymbol{s}_2 \neq \boldsymbol{o}, \qquad \boldsymbol{s}_1 \boldsymbol{m}_2 + \boldsymbol{s}_2 \boldsymbol{m}_1 \neq 0. \tag{120}$$

The rank-theorem proposes a criterion for a *natural classification* in the set $S_{C(F)}$ of all systems of $C(F)$-arrows, possessing a completely determined mechanical significance. It is based on the following *reduction-lemma*:

$$\underset{\rightarrow}{s}(\boldsymbol{s},\ \boldsymbol{m}) \sim \begin{cases} \underset{\rightarrow}{o} \\ \underset{\rightarrow}{\delta} \\ \underset{\rightarrow}{\mu} \\ \underset{\rightarrow}{\beta} \end{cases} \quad \text{iff} \quad \begin{cases} \boldsymbol{s} = \boldsymbol{o}, \quad \boldsymbol{m} = \boldsymbol{o}, \\ \boldsymbol{s} = \boldsymbol{o}, \quad \boldsymbol{m} \neq \boldsymbol{o}, \\ \boldsymbol{s} \neq \boldsymbol{o}, \quad \boldsymbol{sm} = 0, \\ \qquad \boldsymbol{sm} \neq 0. \end{cases} \tag{121}$$

Now the rank-theorem and the reduction-lemme imply the so-called *reduction-theorem*:

If $\underset{\rightarrow}{s} \in S_{C(F)}$, then:

$$\underset{\rightarrow}{s} \sim \begin{cases} \underset{\rightarrow}{o} \\ \underset{\rightarrow}{\delta} \\ \underset{\rightarrow}{\mu} \\ \underset{\rightarrow}{\beta} \end{cases} \quad \text{iff} \quad \text{rank } \underset{\rightarrow}{s} = \begin{cases} 0, \\ 1, \\ 2, \\ 3. \end{cases} \tag{122}$$

The rank-theorem plays an important role in kinematics of rigid bodies, more exactly in the so-called *statical-kinematical analogy,* but this is a problem we shall discuss in due time.

This is almost all one must know about arrows at all costs. The theory of arrows is a vast one, but all that remains are, from a mechanical point of view at least, particularities.

Up to now all considerations have been of a purely algebraic nature. The following step toward the axiomatic consolidation of the logical foundations of analytical mechanics consists in development of theories of motions and rigid bodies. Now, on the very threshold of these theories one is faced with the necessity of vector analysis.

Those are topics that will be discussed in the continuation of this paper. For the time being we shall append the last pieces of information from vector algebra that will be necessary in the following exposition. They concern mainly the so-called *vectors of Gibbs*.

Let $H_{C(F)}$ be an *Hermitean space over* $C(F)$, i.e. a set, in which three operations (16) — (18) are defined, satisfying the conditions Ax 1 C(F) — Ax 12 C(F) (with H instead of V), and let

$$\boldsymbol{a}_\nu \in H_{C(F)} \qquad (\nu = 1, \dots, n), \tag{123}$$

$$G(\boldsymbol{a}_\nu)_{\nu=1}^n \neq 0 \tag{124}$$

provided by definition

$$G(\boldsymbol{a}_\nu)_{\nu=1}^n = \begin{vmatrix} \boldsymbol{a}_1^2 & \dots & \boldsymbol{a}_1\boldsymbol{a}_n \\ \dots & \dots & \dots \\ \boldsymbol{a}_n\boldsymbol{a}_1 & \dots & \boldsymbol{a}_n^2 \end{vmatrix}. \tag{125}$$

Then the vectors defined by

$$(126) \qquad a_\mu^{-1} = \frac{(-1)^{\mu+1}}{G(a_\nu)_{\nu=1}^n} \begin{vmatrix} a_1 & \dots & a_n \\ a_1^2 & \dots & a_n a_1 \\ \dots & \dots & \dots \\ a_1 a_{\mu-1} & \dots & a_n a_{\mu-1} \\ a_1 a_{\mu+1} & \dots & a_n a_{\mu+1} \\ \dots & \dots & \dots \\ a_1 a_n & \dots & a_n^2 \end{vmatrix} \qquad (\mu = 1, \dots, n)$$

are called *Gibbs' vectors* or *reciprocal vectors* of the vectors (123).

(As it is well-known, the condition (124) is necessary and sufficient for the *linear independence* of the vectors (123); for n linearly independent elements of $H_{C(F)}$ it is said that they constitute a *reper*.)

The following propositions describe the basic properties of reciprocal vectors.

Pr 8 C(F). (123), (124) imply

$$(127) \qquad a_\mu^{-1} a_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu,\ \nu = 1, \dots, n).$$

Pr 9 C(F). (123), (124) imply

$$(128) \qquad a_\mu a_\nu^{-1} = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu,\ \nu = 1, \dots, n).$$

Pr 10 C(F). (123), (124) imply

$$(129) \qquad G(a_\nu^{-1})_{\nu=1}^n \neq 0.$$

Pr 11 C(F). (123), (124) imply

$$(130) \qquad G(a_\nu)_{\nu=1}^n G(a_\nu^{-1})_{\nu=1}^n = 1.$$

Pr 12 C(F). (123), (124) imply

$$(131) \qquad (a_\nu^{-1})^{-1} = a_\nu \qquad (\nu = 1, \dots, n).$$

Pr 13 C(F). (123), (124) imply

$$(132) \qquad a_\nu^{-1} = a_\nu \qquad (\nu = 1, \dots, n)$$

iff

$$(133) \qquad a_\mu a_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu,\ \nu = 1, \dots, n).$$

Let $H(\boldsymbol{a}_\nu)_{\nu=1}^n$ denote the linear span of (123) provided (124), i.e. the set

$$H(\boldsymbol{a}_\nu)_{\nu=1}^n = \left\{ \sum_{\nu=1}^n \alpha_\nu \boldsymbol{a}_\nu \,:\, \alpha_\nu \in C(F) \quad (\nu = 1, \dots, n) \right\}. \tag{134}$$

Then:

Pr 14 C(F). (123), (124),

$$\boldsymbol{r} \in H(\boldsymbol{a}_\nu)_{\nu=1}^n \tag{135}$$

imply

$$\boldsymbol{r} = \sum_{\nu=1}^n (\boldsymbol{r}\boldsymbol{a}_\nu^{-1})\boldsymbol{a}_\nu. \tag{136}$$

Pr 15 C(F). (123), (124), (135) imply

$$\boldsymbol{r} = \sum_{\nu=1}^n (\boldsymbol{r}\boldsymbol{a}_\nu)\boldsymbol{a}_\nu^{-1}. \tag{137}$$

Pr 16 C(F). (123), (124),

$$\alpha_\nu \in C(F) \qquad (\nu = 1, \dots, n) \tag{138}$$

imply: there exist exactly one (135) with

$$\boldsymbol{r}\boldsymbol{a}_\nu = \alpha_\nu \qquad (\nu = 1, \dots, n), \tag{139}$$

namely

$$\boldsymbol{r} = \sum_{\nu=1}^n \alpha_\nu \boldsymbol{a}_\nu^{-1}. \tag{140}$$

Pr 17 C(F). (123), (124) imply: there exists exactly one (135) with

$$\boldsymbol{r}\boldsymbol{a}_\nu = 0 \qquad (\nu = 1, \dots, n), \tag{141}$$

namely

$$\boldsymbol{r} = \boldsymbol{o}. \tag{142}$$

Pr 18 C(F). If

$$\boldsymbol{a}_\nu \in V_{C(F)} \qquad (\nu = 1, 2, 3), \tag{143}$$

$$a_1 \times a_2 \cdot a_3 \neq 0 \tag{144}$$

then

$$a_\nu^{-1} = \frac{a_{\nu+1} \times a_{\nu+2}}{a_1 \times a_2 \cdot a_3} \qquad (\nu = 1,\ 2,\ 3) \tag{145}$$

provided

$$a_{\nu+3} = a_\nu \qquad (\nu = 1,\ 2). \tag{146}$$

Pr 19 C(F). If (50),

$$a_\nu \in V_{C(F)} \qquad (\nu = 1,\ 2), \tag{147}$$

$$b_\nu \in V_{C(F)} \qquad (\nu = 1,\ 2) \tag{148}$$

$$r \times a_\nu = b_\nu \qquad (\nu = 1,\ 2) \tag{149}$$

then

$$a_\mu b_\nu + a_\nu b_\mu = 0 \qquad (\mu,\ \nu = 1,\ 2). \tag{150}$$

Pr 20 C(F). (50), (147),

$$a_1 \times a_2 \neq o, \tag{151}$$

$$r \times a_\nu = o \qquad (\nu = 1,\ 2) \tag{152}$$

imply (142).

Pr 21 C(F). (143), (144),

$$b_\nu \in V_{C(F)} \qquad (\nu = 1,\ 2,\ 3), \tag{153}$$

$$a_\mu b_\nu + a_\nu b_\mu = 0 \qquad (\mu,\ \nu = 1,\ 2,\ 3) \tag{154}$$

imply: there exists exactly one (50) with

$$r \times a_\nu = b_\nu \qquad (\nu = 1,\ 2,\ 3), \tag{155}$$

namely

$$r = \frac{1}{2} \sum_{\nu=1}^{3} a_\nu^{-1} \times b_\nu. \tag{156}$$

Pr 22 C(F). (147), 148), (150), (151) imply: there exists exactly one (50) with (149), namely (156) provided

$$a_3 = a_1 \times a_2, \tag{157}$$

$$b_3 = (b_1 \cdot a_2 \times a_1)a_1^{-1} + (b_2 \cdot a_2 \times a_1)a_2^{-1}. \tag{158}$$

The mechnical praxis necessitates the solving of four only systems of algebraic vector equations in the main. Three of them are the systems (139), (149), and (155), considered in Pr 16 C(F), Pr 22 C(F), and Pr 21 C(F) respectively. The fourth is discussed in the following proposition.

Pr 23 C(F). $\alpha \in C(F)$; $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c} \in V_{C(F)}$, $\boldsymbol{a} \neq \boldsymbol{o}$, $\boldsymbol{ab} = 0$ imply:

1. If $\boldsymbol{ac} \neq 0$, then there exists exactly one (50) with

$$\boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{b}, \tag{159}$$

$$\boldsymbol{rc} = \alpha, \tag{160}$$

namely

$$\boldsymbol{r} = \frac{\alpha \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b}}{\boldsymbol{ac}}. \tag{161}$$

2. If $\boldsymbol{ac} = 0$, $\alpha\boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b} = \boldsymbol{o}$, then any solution (50) of (159) is a solution of (160) too, but the inverse is not true.
3. If $\boldsymbol{ac} = 0$, $\alpha\boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b} \neq \boldsymbol{o}$, then there exists no (50) with (159), (160).

(To be continued.)

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