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MULTITIME ACTION RECURRENCES ON A MONOID

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The discrete multitime multiple recurrences are common in analysis of algorithms, computational biology, information theory, queueing theory, filters theory, statistical physics etc. We discuss in detail the cases of recurrences on a monoid, highlighting in particular algebraical aspects and original theorems on existence and uniqueness of solutions.

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1. DISCRETE MULTITIME MULTIPLE RECURRENCE

Generically, we refer to *discrete multitime multiple recurrences* of the form

$$x(t+1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \in \mathbb{Z}^m,\ t \geq t_0,\ \forall \alpha \in \{1, 2, \ldots, m\}, \tag{1.1}$$

where $F_\alpha\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_1\big\} \times M \to M$, $\alpha \in \{1, 2, \ldots, m\}$, $m \in \mathbb{N}^*$, $t_0, t_1 \in \mathbb{Z}^m$, $t_0 \geq t_1$; $1_\alpha = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^m$, i.e., 1_α has 1 on the position α and 0 otherwise; M is a nonvoid set. The unknown function is an m-sequence $x\colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to M$.

Let us start by presenting two results on the existence and uniqueness of the recurrence (1.1) solutions (see [4]).

Proposition 1. If for any $(t_0, x_0) \in \{t \in \mathbb{Z}^m \mid t \geq t_1\} \times M$, there exists at least one solution $x\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M$ which verifies the recurrence (1.1) and the condition $x(t_0) = x_0$, then

$$F_\alpha(t + 1_\beta, F_\beta(t, x)) = F_\beta(t + 1_\alpha, F_\alpha(t, x)), \tag{1.2}$$
$$\forall t \geq t_1, \ \forall x \in M, \quad \forall \alpha, \beta \in \{1, 2, \dots, m\}.$$

Theorem 1. *We consider the functions* $F_\alpha\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \times M \to M$, $\alpha \in \{1, 2, \dots, m\}$, *such that,* $\forall t \geq t_0$, $\forall x \in M$, $\forall \alpha, \beta \in \{1, 2, \dots, m\}$, *the relations* (1.2) *are fulfilled.*

Then, for any $x_0 \in M$, *there exists a unique function*

$$x\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M,$$

which verifies

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \geq t_0, \quad \forall \alpha \in \{1, 2, \dots, m\},$$

and the condition $x(t_0) = x_0$.

2. MULTITIME RECURRENCES ON A MONOID

A monoid is an algebraic structure with a single associative binary operation and an identity element. Monoids are used in computer science, both in its foundational aspects and in practical programming.

Our aim is to analyse a multitime recurrence on a monoid $(N, \cdot, e)$. We consider $\eta\colon N \times M \to M$, an action of the monoid N on the set M, i.e.

$$\eta(ab, x) = \eta\big(a, (b, x)\big), \ \eta(e, x) = x, \quad \forall a, b \in N, \forall x \in M. \tag{2.1}$$

We will use the more convenient notation

$$\eta(a, x) = ax, \quad \forall a \in N, \ \forall x \in M$$

(not to be confused with the operation of monoid N). The relations (2.1) become

$$(ab)x = a(bx), \ ex = x, \quad \forall a, b \in N, \forall x \in M.$$

The action functions $a_\alpha\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to N$, $\alpha \in \{1, 2, \dots, m\}$ (with $t_1 \in \mathbb{Z}^m$) define the action recurrence

$$x(t + 1_\alpha) = a_\alpha(t) x(t), \quad \forall \alpha \in \{1, 2, \dots, m\}, \tag{2.2}$$

with the unknown function $x\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M$, $t_0 \in \mathbb{Z}^m$, $t_0 \geq t_1$.

Introducing the set

$$\mathcal{Z} := \{t \in \mathbb{Z}^m \mid t \geq t_1\}$$

and using Proposition 1 and Theorem 1, one can prove easily the following result (see [5]):

Theorem 2. *a) If, for any $(t_0, x_0) \in \mathcal{Z} \times M$, there exists at least one function $x\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to M$, which, for any $t \geq t_0$, verifies the recurrence* (2.2) *and the condition $x(t_0) = x_0$, then*

$$a_\alpha(t + 1_\beta)a_\beta(t)x = a_\beta(t + 1_\alpha)a_\alpha(t)x, \tag{2.3}$$

$$\forall t \in \mathcal{Z},\ \forall x \in M, \quad \forall \alpha, \beta \in \{1, 2, \dots, m\}.$$

b) If the relations (2.3) *are satisfied, then, for any $(t_0, x_0) \in \mathcal{Z} \times M$, there exists a unique function $x\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to M$, which, for any $t \geq t_0$, verifies the recurrence* (2.2) *and the condition $x(t_0) = x_0$.*

For any point $t = (t^1, \dots, t^m) \in \mathbb{N}^m$, it is useful to denote

$$|t| := t^1 + \ldots + t^m.$$

Theorem 3. *Suppose that the monoid N is commutative. Let us consider the function (sequence) $r\colon \mathbb{N} \to N$ and the elements $q_\alpha \in N$, $\alpha \in \{1, 2, \dots, m\}$, $p_{\alpha\beta} \in N$, $\alpha, \beta \in \{1, 2, \dots, m\}$, with $p_{\alpha\beta} = p_{\beta\alpha}$, $\forall \alpha, \beta$.*

For each index $\alpha \in \{1, 2, \dots, m\}$, we define the function

$$a_\alpha\colon \mathbb{N}^m \to N,\ a_\alpha(t) = q_\alpha \cdot p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} \cdot r(|t|),\ \forall t = (t^1, \dots, t^m) \in \mathbb{N}^m.$$

We shall consider the recurrence (2.2) *defined by these functions.*

In the previous conditions, for any $x_0 \in M$, there exists a unique m-sequence $x\colon \mathbb{N}^m \to M$, which, for any $t \in \mathbb{N}^m$ verifies the recurrence (2.2), *as well as the condition $x(0) = x_0$. This m-sequence is defined by*

$$x(t) = \prod_{\alpha=1}^{m} q_\alpha^{t^\alpha} \cdot \prod_{\alpha=1}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha - 1)}{2}} \cdot \prod_{1 \leq \alpha < \beta \leq m} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|-1} r(j) \cdot x_0,\ \forall t \in \mathbb{N}^m \setminus \{0\} \tag{2.4}$$

(if $m = 1$, then the factor $\prod_{1 \leq \alpha < \beta \leq m} p_{\alpha\beta}^{t^\alpha t^\beta}$ does not appear).

Proof. For any α, β, we have

$$a_\alpha(t+1_\beta) = p_{\alpha\beta} \cdot q_\alpha \cdot p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} \cdot r(|t|+1), \quad a_\beta(t) = q_\beta \cdot p_{\beta 1}^{t^1} p_{\beta 2}^{t^2} \cdot \ldots \cdot p_{\beta m}^{t^m} \cdot r(|t|),$$

$$a_\alpha(t + 1_\beta)a_\beta(t) = p_{\alpha\beta} \cdot q_\alpha q_\beta \cdot p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} p_{\beta 1}^{t^1} p_{\beta 2}^{t^2} \cdot \ldots \cdot p_{\beta m}^{t^m} \cdot r(|t|)r(|t| + 1).$$

It follows that

$$a_\beta(t + 1_\alpha)a_\alpha(t) = p_{\beta\alpha} \cdot q_\beta q_\alpha \cdot p_{\beta 1}^{t^1} p_{\beta 2}^{t^2} \cdot \ldots \cdot p_{\beta m}^{t^m} p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} \cdot r(|t|)r(|t| + 1).$$

Since $p_{\alpha\beta} = p_{\beta\alpha}$, we can write

$$a_\alpha(t + 1_\beta)a_\beta(t) = a_\beta(t + 1_\alpha)a_\alpha(t), \quad \forall t \in \mathbb{N}^m,\ \forall \alpha, \beta \in \{1, 2, \dots, m\}.$$

We deduce that the relations (2.3) are satisfied. According to Theorem 2 (with $t_0 = t_1 = 0$), there exists a unique function $x\colon \mathbb{N}^m \to M$, which, for any $t \in \mathbb{N}^m$ verifies the recurrence (2.2), and the condition $x(0) = x_0$.

It is sufficient to show that the function defined by the formula (2.4), for $t \in \mathbb{N}^m \setminus \{0\}$, and $x(0) = x_0$, verifies the recurrence relation (2.2).

We shall verify the case $m \geq 2$; the case $m = 1$ is treated similarly.

We fix $\gamma \in \{1, 2, \dots, m\}$. We show that for any $t \in \mathbb{N}^m$, we have

$$x(t + 1_\gamma) = a_\gamma(t)x(t).$$

Let $t \in \mathbb{N}^m \setminus \{0\}$. We need the set $\mathcal{P}_{m,2}$ of the subsets with two elements from the set $\{1, 2, \dots, m\}$, i.e., $\mathcal{P}_{m,2} = \Big\{\{\alpha, \beta\} \subseteq \{1, 2, \dots, m\} \,\Big|\, \alpha \neq \beta\Big\}$.

Since $p_{\alpha\beta}^{t^\alpha t^\beta} = p_{\beta\alpha}^{t^\beta t^\alpha}$, we observe that in the product $\prod\limits_{1\leq\alpha<\beta\leq m} p_{\alpha\beta}^{t^\alpha t^\beta}$ the factors $p_{\alpha\beta}^{t^\alpha t^\beta}$ occur, taken over all distinct elements $\{\alpha, \beta\}$ of the set $\mathcal{P}_{m,2}$.

If $m \geq 3$, we can write

$$\mathcal{P}_{m,2} = \Big\{\{\gamma, \alpha\} \,\Big|\, \alpha \in \{1, 2, \dots, m\},\, \alpha \neq \gamma\Big\} \cup$$
$$\cup \Big\{\{\alpha, \beta\} \subseteq \{1, 2, \dots, m\} \,\Big|\, \alpha \neq \gamma,\, \beta \neq \gamma\Big\},$$

hence we have

$$\prod_{1\leq\alpha<\beta\leq m} p_{\alpha\beta}^{t^\alpha t^\beta} = \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} p_{\gamma\alpha}^{t^\gamma t^\alpha} \cdot \prod_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta}. \tag{2.5}$$

For $m = 2$, one obtains the relation (2.5), but without the factor $\prod\limits_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta}$.

For $m = 2$, we denote $\prod\limits_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta} := e$. With this convention, it follows that the relation (2.5) is satisfied for any $m \geq 2$. The relation (2.4) becomes

$$\begin{aligned} x(t) = & q_\gamma^{t^\gamma} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} q_\alpha^{t^\alpha} \cdot (p_{\gamma\gamma})^{\frac{t^\gamma(t^\gamma-1)}{2}} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha-1)}{2}} \\ & \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} p_{\gamma\alpha}^{t^\gamma t^\alpha} \cdot \prod_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|-1} r(j) \cdot x_0; \end{aligned}$$

$$\begin{aligned} x(t + 1_\gamma) = & q_\gamma \cdot q_\gamma^{t^\gamma} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} q_\alpha^{t^\alpha} \cdot (p_{\gamma\gamma})^{\frac{(t^\gamma+1)t^\gamma}{2}} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha-1)}{2}} \\ & \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} (p_{\gamma\alpha}^{t^\gamma t^\alpha} \cdot p_{\gamma\alpha}^{t^\alpha}) \cdot \prod_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|} r(j) \cdot x_0. \end{aligned}$$

Since $\dfrac{(t^\gamma+1)t^\gamma}{2} = t^\gamma + \dfrac{t^\gamma(t^\gamma-1)}{2}$, it follows that

$$x(t+1_\gamma) = q_\gamma \cdot p_{\gamma\gamma}^{t^\gamma} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} p_{\gamma\alpha}^{t^\alpha} \cdot r(|t|) \cdot q_\gamma^{t^\gamma} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} q_\alpha^{t^\alpha} \cdot (p_{\gamma\gamma})^{\frac{t^\gamma(t^\gamma-1)}{2}} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha-1)}{2}}$$

$$\cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} p_{\gamma\alpha}^{t^\gamma t^\alpha} \cdot \prod_{\substack{1\le\alpha<\beta\le m\\ \alpha\neq\gamma,\,\beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|-1} r(j) \cdot x_0 = a_\gamma(t)x(t)$$

For $t = 1_\gamma$, the relation (2.4) reads as

$$x(1_\gamma) = q_\gamma \cdot r(0) \cdot x_0 = a_\gamma(0)x(0),$$

hence the equality $x(t+1_\gamma) = a_\gamma(t)x(t)$ is true also for $t=0$. □

Remark 1. If we use additive notation, i.e., the operation on N is denoted by " + " and $\eta(a,x) = a + x$ ($a \in N$, $x \in M$), then the recurrence relation (2.2) reads as

$$x(t+1_\alpha) = a_\alpha(t) + x(t), \quad \forall\alpha \in \{1,2,\ldots,m\}.$$

In Theorem 3, we have $a_\alpha(t) = q_\alpha + t^1 p_{\alpha 1} + t^2 p_{\alpha 2} + \ldots + t^m p_{\alpha m} + r(|t|)$. Formula (2.4) can be written

$$x(t) = \sum_{\alpha=1}^{m}\Big(t^\alpha q_\alpha + \frac{t^\alpha(t^\alpha-1)}{2}p_{\alpha\alpha}\Big) + \sum_{1\le\alpha<\beta\le m} t^\alpha t^\beta p_{\alpha\beta} + \sum_{j=0}^{|t|-1} r(j) + x_0.$$

Corollary 1. *Let $(M,\cdot)$ be a semigroup. We consider the function (sequence) $r\colon \mathbb{N} \to \mathbb{N}^*$ and the elements $q_\alpha \in \mathbb{N}^*$, $\alpha \in \{1,2,\ldots,m\}$, $p_{\alpha\beta} \in \mathbb{N}^*$, $\alpha,\beta \in \{1,2,\ldots,m\}$, with $p_{\alpha\beta} = p_{\beta\alpha}$, $\forall\alpha,\beta$.*

For each index $\alpha \in \{1,2,\ldots,m\}$, we define the function

$$a_\alpha\colon \mathbb{N}^m \to \mathbb{N}^*,\ a_\alpha(t) = q_\alpha \cdot p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} \cdot r(|t|),\ \forall t = (t^1,\ldots,t^m) \in \mathbb{N}^m.$$

Then, for $x_0 \in M$, there exists a unique m-sequence $x\colon \mathbb{N}^m \to M$, which, for any $t \in \mathbb{N}^m$, verifies

$$x(t+1_\alpha) = x(t)^{a_\alpha(t)}, \quad \forall\alpha \in \{1,2,\ldots,m\}, \tag{2.6}$$

and the condition $x(0) = x_0$. For any $t \in \mathbb{N}^m \setminus \{0\}$, we have

$$x(t) = x_0^{\left(\prod_{\alpha=1}^{m} q_\alpha^{t^\alpha} \cdot \prod_{\alpha=1}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha-1)}{2}} \cdot \prod_{1\le\alpha<\beta\le m} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|-1} r(j)\right)} \tag{2.7}$$

(if $m=1$, the factor $\prod_{1\le\alpha<\beta\le m} p_{\alpha\beta}^{t^\alpha t^\beta}$ does not appear).

Proof. We apply Theorem 3 to the commutative monoid $\big(N, \cdot, e\big) = \big(\mathbb{N}^*, \cdot, 1\big)$ and the action

$$\eta\colon \mathbb{N}^* \times M \to M, \quad \eta(a,x) = x^a, \quad \forall a \in \mathbb{N}^*, \forall x \in M.$$

$\square$

Remark 2. *a*) If in Corollary 1 the semigroup $(M, \cdot)$ is a monoid, then we can consider: $q_\alpha \in \mathbb{N}$, $p_{\alpha\beta} \in \mathbb{N}$ (with $p_{\alpha\beta} = p_{\beta\alpha}$) and $r\colon \mathbb{N} \to \mathbb{N}$ (i.e., q_α, $p_{\alpha\beta}$, $r(j)$ can be eventually zero). The conclusion in Corollary 1 reads similarly and the solution of the recurrence (2.6) is defined by the formula (2.7).

The proof follows by applying Theorem 3 to the commutative monoid $\big(N, \cdot, e\big) = \big(\mathbb{N}, \cdot, 1\big)$ and the action

$$\eta\colon \mathbb{N} \times M \to M, \quad \eta(a,x) = x^a, \quad \forall a \in \mathbb{N}, \forall x \in M.$$

b) If in Corollary 1 the semigroup $(M, \cdot)$ is a monoid, and the element x_0 of M is chosen invertible, then we can consider $q_\alpha \in \mathbb{Z}$, $p_{\alpha\beta} \in \mathbb{Z}$ (with $p_{\alpha\beta} = p_{\beta\alpha}$) and $r\colon \mathbb{N} \to \mathbb{Z}$ (i.e., q_α, $p_{\alpha\beta}$, $r(j)$ are integers). The conclusion in Corollary 1 writes similarly, and the solution of the recurrence (2.6) is defined also by the formula (2.7).

The proof can be obtained by applying Theorem 3 to the commutative monoid $\big(N, \cdot, e\big) = \big(\mathbb{Z}, \cdot, 1\big)$ and the action

$$\eta\colon \mathbb{Z} \times U(M) \to U(M), \quad \eta(a,x) = x^a, \quad \forall a \in \mathbb{Z}, \forall x \in U(M),$$

where $U(M)$ is the set of invertible elements in M; the set $U(M)$, with operation induced by that of M, is a group.

Many other original results, regarding the multitime recurrences, can be found in [3]-[8]. Some related sources are [1], [2], [9]-[12].

3. EXAMPLE OF MATRIX 3-SEQUENCE

Let us determine the matrix 3-sequence $X\colon \mathbb{N}^3 \to \mathcal{M}_2(\mathbb{R})$, which, for any $t = (t^1, t^2, t^3) \in \mathbb{N}^3$, verifies the recurrence relations

$$\begin{cases} X(t^1+1, t^2, t^3) = X(t^1, t^2, t^3)^{2^{t^1} \cdot 3^{t^2+1} \cdot 7^{t^3} \cdot (t^1+t^2+t^3+1)}, \\ X(t^1, t^2+1, t^3) = X(t^1, t^2, t^3)^{5 \cdot 3^{t^1} \cdot 2^{t^3} \cdot (t^1+t^2+t^3+1)}, \\ X(t^1, t^2, t^3+1) = X(t^1, t^2, t^3)^{11 \cdot 7^{t^1} \cdot 2^{t^2} \cdot (t^1+t^2+t^3+1)}, \end{cases}$$

and the condition $X(0,0,0) = A := \begin{pmatrix} 1 & -2 \\ 4 & 7 \end{pmatrix}$.

We are in the assumptions of Corollary 1, with $(M,\cdot) = \big(\mathcal{M}_2(\mathbb{R}),\cdot\big)$ and

$$q_1 = 3, \quad q_2 = 5, \quad q_3 = 11,$$

$$p_{11} = 2, \quad p_{22} = p_{33} = 1, \quad p_{12} = p_{21} = 3, \quad p_{13} = p_{31} = 7, \quad p_{23} = p_{32} = 2,$$
$$r\colon \mathbb{N} \to \mathbb{N}^*, \quad r(j) = j+1, \quad \forall j \in \mathbb{N}.$$

According to Corollary 1, for any $(t^1,t^2,t^3) \in \mathbb{N}^3 \setminus \{(0,0,0)\}$, we have

$$X(t^1,t^2,t^3) = A^{3^{t^1}\cdot 5^{t^2}\cdot 11^{t^3}\cdot 2^{\frac{t^1(t^1-1)}{2}}\cdot 3^{t^1t^2}\cdot 7^{t^1t^3}\cdot 2^{t^2t^3}\cdot (t^1+t^2+t^3)!},$$

$$X(t^1,t^2,t^3) = A^{2^{\frac{t^1(t^1-1)}{2}+t^2t^3}\cdot 3^{t^1(t^2+1)}\cdot 5^{t^2}\cdot 7^{t^1t^3}\cdot 11^{t^3}\cdot (t^1+t^2+t^3)!},$$

relation which is also true for $(t^1,t^2,t^3) = (0,0,0)$.

By induction one shows that

$$A^n = \begin{pmatrix} 2\cdot 3^n - 5^n & 3^n - 5^n \\ 2(5^n - 3^n) & 2\cdot 5^n - 3^n \end{pmatrix}, \quad \forall n \in \mathbb{N},$$

$$A^n = 3^n \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} + 5^n \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}, \quad \forall n \in \mathbb{N}.$$

It follows that, for any $(t^1,t^2,t^3) \in \mathbb{N}^3$, the general term is

$$X(t^1,t^2,t^3) = 3^{2^{\frac{t^1(t^1-1)}{2}+t^2t^3}\cdot 3^{t^1(t^2+1)}\cdot 5^{t^2}\cdot 7^{t^1t^3}\cdot 11^{t^3}\cdot (t^1+t^2+t^3)!}\cdot \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$$
$$+ 5^{2^{\frac{t^1(t^1-1)}{2}+t^2t^3}\cdot 3^{t^1(t^2+1)}\cdot 5^{t^2}\cdot 7^{t^1t^3}\cdot 11^{t^3}\cdot (t^1+t^2+t^3)!}\cdot \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}.$$

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