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# NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS IV. THE EULERIAN DYNAMICAL EQUATIONS

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While the problem of oscillation of a heavy rigid body about a fixed axis had been solved correctly by Huygens, and while a more satisfactory method containing the germ of several later principles had been created by James Bernoulli in 1703, in 1750 it could not be said that the general motion of a rigid body was understood at all. Even for motion about a fixed axis, the reaction of the body upon its support could not be calculated, and no method for determining the behaviour of a spinning top was known.

Euler's "first principles" changed the scene overnight ... in the paper *Découverte d'un nouveau principe de mécanique*, written 1750 and published 1752, where these principles are published, Euler obtained the general *equations of motion of a rigid body* about its center of gravity. He applied the "first principles" to the elements of mass making up the body, at the same time replacing the acceleration of the element by its expression in terms of the *angular velocity vector*, which makes its first appearance here. Taking moments about the center of gravity then yields, after some reduction, the differential equations of motion known as "Euler's equations" for a rigid body, subject to assigned torque about its center of mass. In the process arise naturally the six components of what is now called the "tensor of inertia".

C. Truesdell: *A Program Towards Rediscovering the Rational Mechanics of the Age of Reason*

*Георги Чобанов, Иван Чобанов.* ДИНАМИЧЕСКИЕ АКСИОМЫ НЬЮТОНА И ЭЙЛЕРА. IV. ДИНАМИЧЕСКИЕ УРАВНЕНИЯ ЭЙЛЕРА

В этой четвертой части серии статьей [1–3], посвещенные динамическим аксиомам Ньютона и Эйлера, особое внимание уделено динамическим уравнениям Эйлера, управляющие движения всяких твердых тел, как свободных,так и подчиненных произвольным конечным и инфинитезимальным связям. В частности, анализированы уравнения движения твердых прутьев. Работа содержит подробный анализ динамической философии Даламбера и Лагранжа, рассматривающая "le corps proposé comme l'assemblage d'une infinité de corpuscules ou points massifs unis ensamble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances", который показывает, что подобная гипотеза приводит к противоречию с второй аксиомой Ньютона, а именно, что "mutationem motus proportionalem esse vi motrici impressae et fieri secundum lineam rectam, qua vis illa imprimitur".

*Georgi Chobanov, Ivan Chobanov.* NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS. IV. THE EULERIAN DYNAMICAL EQUATIONS

In this fourth part of the series of articles [1–3], dedicated to the Newtonian and Eulerian dynamical axioms, special stress is put on Euler's dynamical equations governing the motion of any rigid body both free and subjected to arbitrary finite and infinitesimal constraints. In particular, the equations of motion of rigid rods are discussed. The paper contains a detailed analysis of D'Alembert's and Lagrange's dynamical philosophy, regarding "le corps proposé comme l'assemblage d'une infinité de corpuscules ou points massifs unis ensemble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances", which displays clearly that such a hypothesis leads to a contradiction with Newton's second dynamical law, namely "mutationem motus proportionalem esse vi motrici impressae et fieri secundum lineam rectam, qua vis illa imprimatur".

This paper is the natural sequel of a series of articles [1–3], published in this *Annual* some time ago and concerning the logical status of the Newtonian and Eulerian dynamical axioms (the laws, or principles, or postulates, or hypotheses, etc. of momentum and of moment of momentum for mass-points and rigid bodies respectively) in the edifice of analytical mechanics and their connection with Hilbert's Sixth Problem for the axiomatic consolidation of its logical foundations. As almost any second year student has it at his finger's ends, though not every author of dynamical treatises is aware of the fact, the whole of rigid body dynamics is based on, and is developed from, the following two assumptions, or suppositions, or conjectures, or maxims, or tenets, etceteras, formulated by Euler as early as 1750 and nowadays bearing his name:

**Ax 1 E** (*First Eulerian dynamical axiom* or *principle of momentum of a rigid body*). There exists such a rigid system of reference $\Sigma$ that, all derivatives being taken with respect to $\Sigma$, for any rigid body $S$ and for any system of forces $\underset{\rightarrow}{\Phi}$, *acting* on $S$, the derivative with respect to the *time* of the momentum of $S$ equals the basis of $\underset{\rightarrow}{\Phi}$.

**Df 1 E**. Any system of reference, satisfying Ax 1 E, is called *inertial according to Euler.*

**Ax 2 E** (*Second Eulerian dynamical axiom* or *principle of moment of momentum (kinetical moment) of a rigid body*). $\Sigma$ being an inertial according to Euler system of reference and all derivatives being taken with respect to $\Sigma$, for any rigid body $S$ and for any system of forces $\underset{\rightarrow}{\Phi}$, *acting* on $S$, the derivative with respect to the *time* of the moment of momentum of $S$ equals the moment of $\underset{\rightarrow}{\Phi}$, both moments being taken with respect to the origin of $\Sigma$.

Before proceeding further, let us make a most important remark that is a matter of principle, since it concerns the logical status of Ax 1 E and Ax 2 E in the system of rigid body dynamics. Euler's dynamical axioms Ax 1 E and Ax 2 E involve a set of terms specific for analytical mechanics and proclaim certain relations between the mechanical entities these terms nominate. The terms themselves are: system of reference, rigid system of reference, origin of a system of reference, derivative (of a vector function) with respect to a system of reference, momentum of a rigid body, moment of momentum (kinetical moment) of a rigid body, system of forces, basis of a system of forces, moment of a system of forces (with respect to a given pole), acting, and time.

Now all the above terms, with the explicit exception of the last two, are capable to a strict mathematical definition — at least as strict as the term "integral" in analysis. The meaning of this statement reduces to the following mathematical fact. As it is well-known [4], a real standard vector space $V$ is defined axiomatically as a set, in which four operations (addition in $V$, multiplication of the real numbers with the elements of $V$, scalar multiplication of the elements of $V$, and vector multiplication in $V$) are defined, satisfying 15 (3, 4, 5, and 3, respectively, for any of the operations listed above) specific axioms. Now $V$ once granted, all the terms Ax 1 E and Ax 2 E include, with the exceptions of *acting* and *time*, are potentially and actually definitable by means of the algebraical and analytical apparatus in $V$. It goes without saying that the effective reproduction of the mentioned definitions is out of question here: the reader, taking an interest in this matter, may be referred to the corresponding literary sources. The cold fact remains that using the algebra and analysis in $V$ as mathematical tools and the elements of $V$ as mathematical building materials all the notions the Eulerian dynamical axioms Ax 1 E and Ax 2 E include, with the emphasized exception of *acting* and *time*, may be given specific mathematical definitions satisfying the most severe logical standards of Twentieth Century's mathematics. As regards the notions *acting* and *time*, their logical status in analytical mechanics is identical with that of the notions *point*, *line*, and *plane* in Euclidean geometry.

In other words, if all the terms the Eulerian dynamical axioms Ax 1 E and Ax 2 E include were capable of explicit mathematical definitions, then these statements would be (true of false) dynamical theorems. Now the fact that Ax 1 E and Ax 2 E involve terms incapable of such definitions reduces these statements to dynamical axioms which, in their turn, define (along with other, as yet unstated, dynamical axioms) the terms *acting* and *time* implicitly. In such a manner, Ax 1 E and Ax 2 E are mathematical predicates that are neither provable nor disprovable, just like the fifth postulate in geometry or the mathematical induction in arithmetic. Putting it in another way, one has every right to accept Ax 1 E and Ax 2 E or to reject them. In the first case one arrives at the Eulerian rigid body dynamics; as regards the second case, one is faced with one's own problems.

The Eulerian dynamical axioms are nowadays universally accepted — at least as universally as Euclidean geometry. Analytical dynamics may be then, if not defined, at least rather adequately described, as the mathematics of equilibria and motions of mass-points and rigid bodies, and of the forces, which generate these equilibria and motions and are generated by them. In the same manner, the Eulerian rigid body dynamics may be described as the set of mathematical corollaries derived from Ax 1 E and Ax 2 E. In other words, one may look upon the Eulerian dynamical axioms as questions: if Ax 1 E and Ax 2 E, then what? The answer is one and only: then modern analytical dynamics.

After these general and hence somewhat vague memoranda let us now proceed to the mathematical formalization of Ax 1 E and Ax 2 E.

First and foremost, let $Oxyz$ be an inertial according to Euler orthonormal right-hand orientated Cartesian system of reference (the existence of one at least such a system is ensured by Ax 1 E) with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of the axes $Ox$, $Oy$, $Oz$, respectively. In other words,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = 1, \quad \boldsymbol{i}\boldsymbol{j} = 0, \quad \boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}. \tag{1}$$

Second, let $\Omega\xi\eta\zeta$ be an orthonormal right-hand orientated Cartesian system of reference, invariably connected with the rigid body $S$ (the existence of one at least such a system is ensured by the very definition of the rigid body concept) with unit vectors $\bar{\xi}^0$, $\bar{\eta}^0$, $\bar{\zeta}^0$ of the axes $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$, respectively. In other words,

$$(\bar{\xi}^0)^2 = (\bar{\eta}^0)^2 = 1, \quad \bar{\xi}^0\bar{\eta}^0 = 0, \quad \bar{\zeta}^0 = \bar{\xi}^0 \times \bar{\eta}^0. \tag{2}$$

Let the *cosine-directors* $a_{\mu\nu}$ ($\mu$, $\nu = 1, 2, 3$) of $Oxyz$ and $\Omega\xi\eta\zeta$ be defined by

$$\begin{cases} \boldsymbol{i} = a_{11}\bar{\xi}^0 + a_{12}\bar{\eta}^0 + a_{13}\bar{\zeta}^0, \\ \boldsymbol{j} = a_{21}\bar{\xi}^0 + a_{22}\bar{\eta}^0 + a_{23}\bar{\zeta}^0, \\ \boldsymbol{k} = a_{31}\bar{\xi}^0 + a_{32}\bar{\eta}^0 + a_{33}\bar{\zeta}^0. \end{cases} \tag{3}$$

Then (1)–(3) imply

$$\begin{cases} \bar{\xi}^0 = a_{11}\boldsymbol{i} + a_{21}\boldsymbol{j} + a_{31}\boldsymbol{k}, \\ \bar{\eta}^0 = a_{12}\boldsymbol{i} + a_{22}\boldsymbol{j} + a_{32}\boldsymbol{k}, \\ \bar{\zeta}^0 = a_{13}\boldsymbol{i} + a_{23}\boldsymbol{j} + a_{33}\boldsymbol{k}. \end{cases} \tag{4}$$

Let $P$ be any point and

$$\boldsymbol{r} = \boldsymbol{OP}, \quad \boldsymbol{r}_\Omega = \overline{O\Omega}, \quad \bar{\rho} = \overline{\Omega P}. \tag{5}$$

Then the identity $\boldsymbol{OP} = \overline{O\Omega} + \overline{\Omega P}$ implies

$$\boldsymbol{r} = \boldsymbol{r}_\Omega + \bar{\rho}. \tag{6}$$

If

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \tag{7}$$

$$\boldsymbol{r}_\Omega = x_\Omega\boldsymbol{i} + y_\Omega\boldsymbol{j} + z_\Omega\boldsymbol{k}, \tag{8}$$

$$\bar{\rho} = \xi\bar{\xi}^0 + \eta\bar{\eta}^0 + \zeta\bar{\zeta}^0, \tag{9}$$

then (6) and (3), (4) imply

$$\begin{cases} x = x_\Omega + a_{11}\xi + a_{12}\eta + a_{13}\zeta, \\ y = y_\Omega + a_{21}\xi + a_{22}\eta + a_{23}\zeta, \\ z = z_\Omega + a_{31}\xi + a_{32}\eta + a_{33}\zeta \end{cases} \tag{10}$$

and inversely

$$(11)\quad \begin{cases} \xi = a_{11}(x - x_\Omega) + a_{21}(y - y_\Omega) + a_{31}(z - z_\Omega), \\ \eta = a_{12}(x - x_\Omega) + a_{22}(y - y_\Omega) + a_{32}(z - z_\Omega), \\ \zeta = a_{13}(x - x_\Omega) + a_{23}(y - y_\Omega) + a_{33}(z - z_\Omega). \end{cases}$$

If

$$(12)\quad \boldsymbol{k} \times \bar{\zeta}^0 \neq \boldsymbol{o},$$

then let the elementary angle $\theta$ and the orientated angles $\psi$ and $\varphi$ be defined as follows:

$$(13)\quad \cos\theta = \boldsymbol{k}\bar{\zeta}^0 \qquad (0 < \theta < \pi),$$

$$(14)\quad \sin\psi = \boldsymbol{j}\bar{\gamma}^0, \quad \cos\psi = \boldsymbol{i}\bar{\gamma}^0 \quad (0 \leqq \psi < 2\pi),$$

$$(15)\quad \sin\varphi = -\bar{\eta}^0\bar{\gamma}^0, \quad \cos\varphi = \bar{\xi}^0\bar{\gamma}^0 \quad (0 \leqq \varphi < 2\pi),$$

provided

$$(16)\quad \bar{\gamma}^0 = \frac{\boldsymbol{k} \times \bar{\zeta}^0}{\sin\theta}.$$

Then $\psi$, $\varphi$, $\theta$ are called the *Eulerian angles* of the systems of reference $Oxyz$ and $\Omega\xi\eta\zeta$, and

$$(17)\quad \begin{cases} a_{11} = \cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta, \\ a_{12} = -\cos\psi\sin\varphi - \sin\psi\cos\varphi\cos\theta, \\ a_{13} = \sin\psi\sin\theta, \\ a_{21} = \sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta, \\ a_{22} = -\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta, \\ a_{23} = -\cos\psi\sin\theta, \\ a_{31} = \sin\varphi\sin\theta, \\ a_{32} = \cos\varphi\sin\theta, \\ a_{33} = \cos\theta, \end{cases}$$

i.e.

$$(18)\quad a_{\mu\nu} = a_{\mu\nu}(\psi, \varphi, \theta) \qquad (\mu, \nu = 1, 2, 3)$$

are completely determined functions of $\psi$, $\varphi$, $\theta$.

The first and the second derivatives of the scalar functions with respect to the *time* $t$ are traditionally denoted in analytical mechanics by means of one and two dots, respectively, placed over the corresponding symbols representing those functions. As regards the vector functions, the dots and the symbols $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$ are reserved for their derivatives with respect to the system of reference $Oxyz$ only, their derivatives with respect to $\Omega\xi\eta\zeta$ being denoted by the symbols $\frac{\delta}{\delta t}$ and $\frac{\delta^2}{\delta^2 t}$.

Thus, if

$$(19)\quad \boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$

and

(20) $$\boldsymbol{a} = a_\xi \bar{\xi}^0 + a_\eta \bar{\eta}^0 + a_\zeta \bar{\zeta}^0,$$

then the derivatives of $\boldsymbol{a}$ with respect to the time and with regard to $Oxyz$ and $\Omega\xi\eta\zeta$ are

(21) $$\frac{d\boldsymbol{a}}{dt} = \dot{\boldsymbol{a}} = \dot{a}_x \boldsymbol{i} + \dot{a}_y \boldsymbol{j} + \dot{a}_z \boldsymbol{k}$$

and

(22) $$\frac{\delta \boldsymbol{a}}{\delta t} = \dot{a}_\xi \bar{\xi}^0 + \dot{a}_\eta \bar{\eta}^0 + \dot{a}_\zeta \bar{\zeta}^0,$$

respectively. The derivatives (21) and (22) are sometimes qualified by the use of the adjectives *absolute* and *relative*, respectively.

The *instantaneous angular velocity*

(23) $$\bar{\omega} = \frac{1}{2}\left(\bar{\xi}^0 \times \dot{\bar{\xi}}^0 + \bar{\eta}^0 \times \dot{\bar{\eta}}^0 + \bar{\zeta}^0 \times \dot{\bar{\zeta}}^0\right)$$

of $\Omega\xi\eta\zeta$ with respect to $Oxyz$ is defined as the only solution of the system of vector equations

(24) $$\bar{\omega} \times \bar{\xi}^0 = \dot{\bar{\xi}}^0, \quad \bar{\omega} \times \bar{\eta}^0 = \dot{\bar{\eta}}^0, \quad \bar{\omega} \times \bar{\zeta}^0 = \dot{\bar{\zeta}}^0.$$

If

(25) $$\bar{\omega} = \omega_\xi \bar{\xi}^0 + \omega_\eta \bar{\eta}^0 + \omega_\zeta \bar{\zeta}^0,$$

then the relations

(26) $$\begin{cases} \omega_\xi = \dot{\psi} \sin\theta + \dot{\theta} \cos\varphi, \\ \omega_\eta = \dot{\psi} \sin\theta - \dot{\theta} \sin\varphi, \\ \omega_\zeta = \dot{\psi} \cos\theta + \dot{\varphi} \end{cases}$$

are called the *Eulerian kinematical equations.*

If (19)–(22), then

(27) $$\frac{d\boldsymbol{a}}{dt} = \frac{\delta \boldsymbol{a}}{\delta t} + \bar{\omega} \times \boldsymbol{a},$$

whence

(28) $$\frac{d\bar{\omega}}{dt} = \frac{\delta \bar{\omega}}{\delta t}.$$

Now (28) and (25), (22) imply

(29) $$\dot{\bar{\omega}} = \dot{\omega}_\xi \bar{\xi}^0 + \dot{\omega}_\eta \bar{\eta}^0 + \dot{\omega}_\zeta \bar{\zeta}^0.$$

By definition the point $P$ belongs to the rigid body $S$ if, and only if,

(30) $$\frac{\delta \bar{\rho}}{\delta t} = \boldsymbol{o} \qquad (\forall t),$$

in other words, iff

$$\dot{\xi} = \dot{\eta} = \dot{\zeta} = 0 \qquad (\forall t) \tag{31}$$

provided (9) by virtue of (22). Since (6) and (27) imply

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}_\Omega}{dt} + \frac{\delta \bar{\rho}}{\delta t} + \bar{\omega} \times \bar{\rho}, \tag{32}$$

the definition (30) and

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}, \quad \boldsymbol{v}_\Omega = \frac{d\boldsymbol{r}_\Omega}{dt} \tag{33}$$

imply that

$$\boldsymbol{v} = \boldsymbol{v}_\Omega + \bar{\omega} \times \bar{\rho} \qquad (\forall t) \tag{34}$$

is a necessary and sufficient condition for $P$ in order to belong to the rigid body $S$.

The set $V_S$ of all points $P$ belonging to a rigid body $S$, i.e. of all (9) with (30) or, just the same, with (31), constitutes a real standard vector space. Now the very definition of the rigid body concept presupposes that a function

$$\varkappa : V_S \longrightarrow [0, \infty) \tag{35}$$

is defined, such that the integral

$$m = \int_{V_S} \varkappa(\bar{\rho})\, d\mu \tag{36}$$

exists; $\varkappa(\bar{\rho})$ is called the *density* of $S$ at the point $P$, and $m$ is called the *mass* of $S$. The density, as well as the mass of a rigid body play a fundamental role in both analytical statics and analytical dynamics. In mechanics of rigid bodies with *constant mass* it is supposed that

$$\frac{d\varkappa(\bar{\rho})}{dt} = 0 \qquad (\forall \bar{\rho} \in V_S,\ \forall t) \tag{37}$$

— a condition that will be hypothesized in the sequel.

The traditional notation

$$dm = \varkappa(\bar{\rho})\, d\mu, \tag{38}$$

as convenient as incorrect, is frequently used, $dm$ being called an *elementary mass* of $S$; besides, $V_S$ is usually omitted in the record of the integral (36), being implied by the context. Using this convention and the notation (38), the definition (36) may be written in the following popular though somewhat enigmatic form:

$$m = \int dm. \tag{39}$$

Now (36)–(39) imply

$$\frac{dm}{dt} = 0 \qquad (\forall t). \tag{40}$$

Before proceeding further, it is *nec plus ultra* necessary to say in this place some words about the integral (36) and about some other important dynamical integrals which will appear immediately below. The point is that, in the present state of affairs at least — videlicet, at such a logical level of exposition as the present one, no specification may be made as regards the mathematical nature of the process of integration in (36): in order to fix the ideas one may purely and simply suppose that the integral in (36) and elsewhere is taken *im Riemannschen Sinne*, $d\mu$ denoting infinitesimal volume (if $S$ is a 3-dimensional rigid body), or infinitesimal area (if $S$ is 2-dimensional), or infinitesimal length (if $S$ is 1-dimensional). As regards any further information, it is imbedded in the very definition of the notion of a dynamical rigid body.

Being at this stage forced into accepting those as vague as to seem void of sense explanations, let us proceed to the definition of the *mass-centre* $G$ of the rigid body $S$. It is introduced traditionally by means of the relation

$$\bar{\rho}_G = \frac{1}{m} \int \bar{\rho}\, dm \tag{41}$$

provided

$$\bar{\rho}_G = \overline{\Omega G}, \tag{42}$$

the integral being taken over $V_S$. Along with (30) and (40) the definition (41) implies

$$\frac{\delta \bar{\rho}_G}{\delta t} = 0 \qquad (\forall t), \tag{43}$$

i.e. the mass-centre of a rigid body $S$ belongs to $S$.

If by definition

$$\boldsymbol{r}_G = \boldsymbol{OG}, \tag{44}$$

then the identity $\boldsymbol{OG} = \overline{O\Omega} + \overline{\Omega G}$, together with (5) and (42), implies

$$\boldsymbol{r}_G = \boldsymbol{r}_\Omega + \bar{\rho}_G, \tag{45}$$

and (45), (41), (6), (39) imply

$$\boldsymbol{r}_G = \frac{1}{m} \int \boldsymbol{r}\, dm, \tag{46}$$

the integral being taken over $V_S$ provided (6).

The identities (6) and (45) imply

$$\boldsymbol{r} = \boldsymbol{r}_G - \bar{\rho}_G + \bar{\rho}. \tag{47}$$

If by definition

$$\boldsymbol{v}_G = \frac{d\boldsymbol{r}_G}{dt}, \tag{48}$$

then (45), (33), (34), (43) imply

$$\boldsymbol{v}_G = \boldsymbol{v}_\Omega + \bar{\omega} \times \bar{\rho}_G, \tag{49}$$

and (49), (34) imply

$$v = v_G - \bar{\omega} \times \bar{\rho}_G + \bar{\omega} \times \bar{\rho}. \tag{50}$$

On the other hand, (46), (48), (40), (33) imply

$$v_G = \frac{1}{m} \int v \, dm. \tag{51}$$

By definition the integrals

$$K = \int v \, dm \tag{52}$$

and

$$L = \int r \times v \, dm, \tag{53}$$

taken over $V_S$ provided (6), are called the *momentum* and the *moment of momentum* (*kinetical moment*), respectively, of the rigid body $S$ with regard to $Oxyz$.

We shall now subject the quantities $K$ and $L$ to certain identical transformations. First of all, we observe that (52) and (51) imply

$$K = m v_G. \tag{54}$$

On the other hand, (53) and (47), (50) imply

$$L = \int (r_G - \bar{\rho}_G + \bar{\rho}) \times (v_G - \bar{\omega} \times \bar{\rho}_G + \bar{\omega} \times \bar{\rho}) \, dm \tag{55}$$

and (55), (39) imply

$$L = m r_G \times v_G + L_G, \tag{56}$$

where by definition

$$L_G = \int \bar{\rho} \times (\bar{\omega} \times \bar{\rho}) \, dm - m \bar{\rho}_G \times (\bar{\omega} \times \bar{\rho}_G). \tag{57}$$

If by definition

$$I_{\xi\xi} = \int (\eta^2 + \zeta^2) \, dm, \quad I_{\eta\eta} = \int (\zeta^2 + \xi^2) \, dm, \quad I_{\zeta\zeta} = \int (\xi^2 + \eta^2) \, dm, \tag{58}$$

$$I_{\eta\zeta} = \int \eta\zeta \, dm, \quad I_{\zeta\xi} = \int \zeta\xi \, dm, \quad I_{\xi\eta} = \int \xi\eta \, dm, \tag{59}$$

$$J_{\xi\xi} = m \left(\eta_G^2 + \zeta_G^2\right), \quad J_{\eta\eta} = m \left(\zeta_G^2 + \xi_G^2\right), \quad J_{\zeta\zeta} = m \left(\xi_G^2 + \eta_G^2\right), \tag{60}$$

$$J_{\eta\zeta} = m \eta_G \zeta_G, \quad J_{\zeta\xi} = m \zeta_G \xi_G, \quad J_{\xi\eta} = m \xi_G \eta_G, \tag{61}$$

$$A = I_{\xi\xi} - J_{\xi\xi}, \quad B = I_{\eta\eta} - J_{\eta\eta}, \quad C = I_{\zeta\zeta} - J_{\zeta\zeta}, \tag{62}$$

$$D = I_{\eta\zeta} - J_{\eta\zeta}, \quad E = I_{\zeta\xi} - J_{\zeta\xi}, \quad F = I_{\xi\eta} - J_{\xi\eta} \tag{63}$$

provided (9) and

$$\bar{\rho}_G = \xi_G \bar{\xi}^0 + \eta_G \bar{\eta}^0 + \zeta_G \bar{\zeta}^0, \tag{64}$$

then the quantities $A$, $B$, $C$ are called the moments of inertia and the quantities $D$, $E$, $F$ are called the moments of deviation of the rigid body $S$ with respect to $\Omega\xi\eta\zeta$.

If now

$$\boldsymbol{L}_G = L_{G\xi}\bar{\xi}^0 + L_{G\eta}\bar{\eta}^0 + L_{G\zeta}\bar{\zeta}^0, \tag{65}$$

then (57) implies

$$L_{G\xi} = \int \bar{\xi}^0 \times \bar{\rho}.\bar{\omega} \times \bar{\rho}\, dm - m\bar{\xi}^0 \times \bar{\rho}_G.\bar{\omega} \times \bar{\rho}_G, \tag{66}$$

and (66), (25), (9), (64) imply

$$L_{G\xi} = \int \left(\bar{\rho}^2\omega_\xi - (\bar{\rho}\bar{\omega})\xi\right)\, dm - m\left(\bar{\rho}_G^2\omega_\xi - (\bar{\rho}_G\bar{\omega})\xi_G\right), \tag{67}$$

whence

$$L_{G\xi} = A\omega_\xi - F\omega_\eta - E\omega_\zeta \tag{68}$$

by virtue of (58)–(63). Similarly,

$$L_{G\eta} = B\omega_\eta - D\omega_\zeta - F\omega_\xi, \tag{69}$$

$$L_{G\zeta} = C\omega_\zeta - E\omega_\xi - D\omega_\eta. \tag{70}$$

Now (68)–(70) and (65) imply

$$\boldsymbol{L}_G = (A\omega_\xi - F\omega_\eta - E\omega_\zeta)\bar{\xi}^0 + (B\omega_\eta - D\omega_\zeta - F\omega_\xi)\bar{\eta}^0 + (C\omega_\zeta - E\omega_\xi - D\omega_\eta)\bar{\zeta}^0. \tag{71}$$

On the other hand, (27) implies

$$\dot{\boldsymbol{L}}_G = \frac{\delta}{\delta t}\boldsymbol{L}_G + \bar{\omega} \times \boldsymbol{L}_G \tag{72}$$

and (72), (71), (29) imply

$$\begin{aligned} \dot{\boldsymbol{L}}_G = {} & \left(A\dot{\omega}_\xi - (B-C)\omega_\eta\omega_\zeta - D(\omega_\eta^2 - \omega_\zeta^2) - E(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) - F(\dot{\omega}_\eta - \omega_\zeta\omega_\xi)\right)\bar{\xi}^0 \\ & + \left(B\dot{\omega}_\eta - (C-A)\omega_\zeta\omega_\xi - E(\omega_\zeta^2 - \omega_\xi^2) - F(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) - D(\dot{\omega}_\zeta - \omega_\xi\omega_\eta)\right)\bar{\eta}^0 \\ & + \left(C\dot{\omega}_\zeta - (A-B)\omega_\xi\omega_\eta - F(\omega_\xi^2 - \omega_\eta^2) - D(\dot{\omega}_\eta + \omega_\zeta\omega_\xi) - E(\dot{\omega}_\xi - \omega_\eta\omega_\zeta)\right)\bar{\zeta}^0. \end{aligned} \tag{73}$$

Up to here all considerations have been purely kinematical, in the sense that no forces have ever appeared. Let us now suppose that the rigid body $S$ is subjected to the action of the system of forces

$$\overrightarrow{F}_\sigma = (\boldsymbol{F}_\sigma, \boldsymbol{M}_\sigma) \qquad (\sigma = 1, \ldots, s), \tag{74}$$

all moments $\boldsymbol{M}_\sigma$ $(\sigma = 1, \ldots, s)$ being taken with respect to $O$. Let by definition

$$\boldsymbol{F}^* = \sum_{\sigma=1}^{s} \boldsymbol{F}_\sigma \tag{75}$$

and

$$M^* = \sum_{\sigma=1}^{s} M_\sigma \tag{76}$$

be the basis and the moment, respectively, of the system (74), the latter being obviously taken with regard to $O$. If now one calls to mind that the system of reference $Oxyz$ is inertial according to Euler by hypothesis, then one may write down the Eulerian dynamical axioms Ax 1 E and Ax 2 E in the form

$$\dot{K} = F^* \tag{76'}$$

and

$$\dot{L} = M^*, \tag{77}$$

respectively.

If by definition

$$w_G = \frac{d^2 r_G}{dt^2}, \tag{78}$$

then (76′), (54), (40), (48) imply

$$m w_G = F^*. \tag{79}$$

Now (69), (78),

$$r_G = x_G i + y_G j + z_G k, \tag{80}$$

$$F^* = F_x^* i + F_y^* j + F_z^* k \tag{81}$$

imply

$$m\ddot{x}_G = F_x^*, \quad m\ddot{y}_G = F_y^*, \quad m\ddot{z}_G = F_z^*. \tag{82}$$

Let

$$M_G^* = M^* + F^* \times r_G \tag{83}$$

be the moment of the system of forces (74) with respect to the mass-centre $G$ of the rigid body $S$. Then (83) and (79) imply

$$M^* = M_G^* + m r_G \times w_G, \tag{84}$$

and (84), (77), (56) imply

$$\dot{L}_G = M_G^*. \tag{85}$$

If by definition

$$M_G^* = M_{G\xi}^* \bar{\xi}^0 + M_{G\eta}^* \bar{\eta}^0 + M_{G\zeta}^* \bar{\zeta}^0, \tag{86}$$

then (85), (73) imply

(87)

$$\begin{cases} A\dot{\omega}_\xi - (B-C)\omega_\eta\omega_\zeta - D(\omega_\eta^2 - \omega_\zeta^2) - E(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) - F(\dot{\omega}_\eta - \omega_\zeta\omega_\xi) = M_{G\xi}^*, \\ B\dot{\omega}_\eta - (C-A)\omega_\zeta\omega_\xi - E(\omega_\zeta^2 - \omega_\xi^2) - F(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) - D(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) = M_{G\eta}^*, \\ C\dot{\omega}_\zeta - (A-B)\omega_\xi\omega_\eta - F(\omega_\xi^2 - \omega_\eta^2) - D(\dot{\omega}_\eta + \omega_\zeta\omega_\xi) - E(\dot{\omega}_\xi - \omega_\eta\omega_\zeta) = M_{G\zeta}^*. \end{cases}$$

The relations (82) are called *Euler's dynamical equations for the motion of the mass-centre of a rigid body*, and the relations (87) are called *Euler's dynamical equations for the motion of a rigid body around its mass-centre*. At that, the equations (82) represent a mathematically developed equivalent of the first Eulerian dynamical axiom Ax 1 E, while the equations (87) are a mathematical reflexion, by means of the moments of inertia and of the moments of deviation of a rigid body, of the second Eulerian dynamical axiom Ax 2 E.

Let us for a while come to a standstill here and let our thought dwell upon the equations (82), (87). The latter supply us with a system of 6 conditions for the motion of any rigid body concerning the quantities they involve. And which quantities do they involve? Along with the moments of inertia $A$, $B$, $C$ and the moments of deviation $D$, $E$, $F$ (which are known quantitative characteristics for any particular rigid body $S$), the equations (82), (87) include the canonic parameters

$$(88) \qquad x_\Omega,\ y_\Omega,\ z_\Omega,\ \psi,\ \varphi,\ \theta$$

of $S$ and the components

$$(89) \qquad F_x^*,\ F_y^*,\ F_z^*,\ M_{G\xi}^*,\ M_{G\eta}^*,\ M_{G\zeta}^*$$

of the basis (75) and of the moments (83) with respect to $G$ of the system of forces (74) acting on $S$. At that, according to a convention, sanctified by the centuries-old experience and tradition of analytical mechanics, it is hypothesized that

$$(90) \qquad \boldsymbol{F}^* = \boldsymbol{F}^*(x_\Omega, y_\Omega, z_\Omega, \psi, \varphi, \theta; \dot{x}_\Omega, \dot{y}_\Omega, \dot{z}_\Omega, \dot{\psi}, \dot{\varphi}, \dot{\theta}; t)$$

and

$$(91) \qquad \boldsymbol{M}_G^* = \boldsymbol{M}_G^*(x_\Omega, y_\Omega, z_\Omega, \psi, \varphi, \theta; \dot{x}_\Omega, \dot{y}_\Omega, \dot{z}_\Omega, \dot{\psi}, \dot{\varphi}, \dot{\theta}; t)$$

are certain determined functions of the canonic parameters (88), of their derivatives

$$(92) \qquad \dot{x}_\Omega,\ \dot{y}_\Omega\ \dot{z}_\Omega,\ \dot{\psi},\ \dot{\varphi},\ \dot{\theta}$$

with respect to the time $t$, and possibly of $t$ itself, whence the same supposition is valid for the components (89) of (90) and (91). In such a manner, the Eulerian dynamical equations (82), (87) represent a system of 6 differential relations of second order with respect to the time $t$ for the 6 canonic parameters (88) of the rigid body $S$.

All that would be *nuda veritas* under the assumption that all the parameters (88) of the rigid body are *mutually independent*, alias that any of them could vary, along with its derivative with respect to the time $t$, completely independently from the variations of the rest of these parameters and of their derivatives with respect to the time. Is that always the case?

If all canonic parameters (88) of a rigid body $S$ are mutually independent, then it is said that $S$ is a *free rigid body*, the term implying that no restrictions are imposed on the thinkable (or possible, or feasible, or imaginary, or potential, or virtual) positions of $S$ in space and on the velocities of its points. In the case of a free rigid body $S$ a classical for analytical mechanics hypothesis presupposes (or demands, or exacts, or requires, or insists on, or announces, or promulgates, or declares, or proclaims) that all the forces (74), acting on $S$, are *active forces*, in other words, all of them are given (or known, or familiar, or prescribed, or specified) functions of (88), of (92), and possibly of $t$. In such a way, in the case of a free rigid body the Eulerian dynamical equations (82), (87) represent a wholly determined system of 6 *genuine*, or *pure* differential equations of second order with respect to the time $t$ for the 6 unknown functions (88) of the time $t$. If now *initial values* of these functions and of their derivatives (92) with respect to $t$ are given (i.e. admissible values of (88) and (92) for any particular moment $\tau$ of $t$, say $t = 0$), then the dynamical problem concerning the motion of this free rigid body $S$ presents itself in the capacity of a perfectly correct mathematical problem with one and only solution (provided certain conditions are satisfied concerning the right-hand sides of the equations (82), (87), i.e. if some requirements affecting the analytical nature of the functions (90), (91) are fulfilled).

The situation is shifted in a trice if some of the forces (74) are unknown and, in the same time, not all of the canonic parameters (88) are mutually independent. Millenial physical experience, engineering praxis, and sound mechanical common sense display that the idyllic picture of free rigid body motions ceases to interpret adequately the dynamical realities which have surprisingly engendered nightmarish problems for all the mechanicians from the Seventeenth till Twenty-First Century. Of course, the interplay of mathematical discovery and physical experience is a dangerous game, and we by no means venture to imitate D'Alembert's unfortunate improvisations on this theme and variations, giving good reason for Truesdell's statement that "in attempting to connect physical experience with mathematics, he heaped folly on folly ... one searches for the little solid matter as a sparrow pecks out a few nutricious seeds from a dungheap — a task not altogether savory" [5]. In the same time, especially in "physikalischen Disziplinen, in denen schon heute die Mathematik eine hervorragende Rolle spielt: dies sind in erster Linie die Wahrscheinlichkeitsrechnung und die Mechanik" [6], this interaction or, should we say, heuristic symbiosis, is *un fait accompli* that no one may disregard without disturbing all sense of reality:

"... mathematics, however abstract and however precise, is a science of *experience*, for experience is not confined to the gross senses: Also the human mind can experience, and we need not be so naive as to see in an oscilloscope an instrument more precise than the brain of a man.

That rational mechanics grew out of practical mechanics and co-operated with it, if not always gracefully, to produce applied mechanics and mechanical engineering, is obvious. In writing the first treatise on rational mechanics [7] Newton established its standard of mathematical rigor as precisely that of geometry. Not always has this standard been maintained, but today as in 1687 it remains the ideal. Newton's comparison with geometry is enlightening, for geometry, too, grew from physical experience. To those who scoff at geometry for its precise calculations when all measurements are liable to error, the geometer for millenia has replied: Geometry is *mental*, not *instrumental*. The scoffers have always been with us and remain today; not only does the ultimate practical and physical value of geometry need no defense before scientists, but also no-one who has known a geometer needs reminder that practical and physical usefulness seldom has supplied or suppressed a single equation in the progress of geometrical research.

The analogy to geometry is a good one. That rational mechanics speaks not only of space and time but also of mass, force, and energy does not make it any the less precise. Since it deals with a greater number of physical concepts than does geometry, its applications to physical problems may be expected to be more frequent and more far-reaching, but physical applications are not its objective.

But does not rational mechanics deal with quantities of physical experience? Indeed it does; so does geometry, for lengths, surfaces, and volumes are equally related to physical experience. The geometer may visualize a surface in terms of a twisted strip of paper, as in mechanics one may think of a force as a push with the hand, but whatever these motivations, the symbols in the equations of geometry and mechanics are precisely defined mathematical quantities. Origin in broader experience may make mechanics more interesting, but it need not make it any less rigorous" [8, p. 335–336].

Could it be said more clearly and more simply? There are other places in [8], dedicated to the interplay of mathematics and physics, that one plainly cannot leave out not mentioned. Reminding Daniel Bernoulli's words "there is no philosophy which is not founded upon knowledge of the phenomena, but to get any profit from this knowledge it is absolutely necessary to be a mathematician" and Huygens' motto "from experience and from reason", Truesdell speculates:

"What was, then, the method? Rational mechanics was a science of *experience*, but no more than geometry was it *experimental*. While some great mechanical experiments were done in the Age of Reason, they had only occasional bearing on the growth of the theories we now regard as classical. Experiment and theory result from different kinds of reaction to experience. If, ideally, they should complement and check one another, yet even today, with all our superior knowledge not only of facts but also of scientific method, it is difficult enough to relate them, why should it has been easier 300 years ago? It was not. A factual view of the history of mechanics must concede that rational mechanics and experimental mechanics, both arising from human beings' intelligent reaction to mechanical experience, grew up separately.

Not only private, individual experimental researches were performed in the eighteenth century; there were also large, cooperative projects. As today, they cost more than real science, and they attracted administrators. But the effect of all this expense on what we now consider the achievement of the period was nil. The method used in the great researches was entirely mathematical, but the result was not what would now be called pure mathematics. *Experience* was the guide; *experience*, physical experience and the experience of accumulated previous theory. If we are to seek a word for what was done, it would not be physics and it would not be pure mathematics; least of all would it be applied mathematics. It would be *rational mechanics* ...

Without *experience*, there would be no rational mechanics, but I should mislead you if I claimed that experiment, either now or 200 years ago, had greatly influenced those who study rational mechanics. In this connection experiment, like alcohol, is a stimulant to be taken with caution. To consult the oracle of a fine vintage at decent intervals exhilirates, but excess of the common stock brings stupor" [*ibid.*, p. 135–136, 357].

One of the most primitive, most fundamental, and, together with that — no wonder *ergo propter hoc* — most complicated mathematical formalization of physical experience in rigid body analytical dynamics focalizes in the idea of — no matter accidental or intentional — restrictions imposed on the possible (virtual, potential) positions of rigid bodies in space. At first sight the underlying idea looks as simple as to seem obvious; as simple as to seem obvious are the mathematical means, too, by the aid of which, until this very day, mechanicians are trying to formalize mathematically this same idea — the height of perfection of their efforts inevitably calling to one's mind Mark Twain's observation that *for any problem there is a solution that is simple, obvious, and wrong*. In point of fact, the simplicity of the idea is spurious to such a degree that anyone who ventures to get into the swing of the work unavoidably wanders through the intricacies of a true Labyrinth with no Ariadne at its mouth.

Calling *ficus ficus*, *ligonem ligonem*, we are obliged to fathom the fact that the physical cause underlying any restriction in the positions of a rigid body in space is rooted in that attribute of matter which is described by the categorical though somewhat enigmatical term *impenetrability*. This property, characterized also by the substantives *impermeability* and *imperviousness*, is available in the very commencement of the notorious dynamical *Traité* [9] of D'Alembert — in its first sentence, to all intents and purposes, see the section *Définitions et Notions préliminaires* (p. 1) — as an inseparable part of the author's definition of the rigid body concept:

"Si deux portions d'étendüe semblables & égales entr'elles sont *impénétrables*, c'est–à–dire, si elles ne peuvent être imaginées unis & confondües l'une avec l'autre, de maniére qu'elles ne fassent qu'une même portion d'étendüe moindre que la somme des deux, chacune de ces portions d'étendüe sera ce qu'on appelle un *Corps*. L'impénétrabilité est la propriété principale par laquelle nous distinguons les Corps des parties de l'espace indéfini, où nous imaginons qu'ils sont placés."

The most natural question, coming to the mind of the reader of this *définition*, is how does its author use "la propriété principale par laquelle nous distinguons les Corps" described as *impenétrabilité* in order to achive his object so modestly proclaimed in the *Préface* of the *Traité*:

"Je me suis proposé dans cet Ouvrage de satisfaire à ce double objet, de reculer les limites de la Méchanique, & d'en applanir l'abord; & mon but principal a été de remplir en quelque sorte un de ces objets par l'autre, c'est–à–dire, nonseulement de deduire les Principes de la Méchanique des notions les plus claires, mais de les appliquer aussi à de nouveaux usages; de faire voir tout à la fois, & l'inutilité de plusieurs Principes qu'on avoit employés jusqu'ici dans la Méchanique, & l'avantage qu'on peut tirer de la combinaison des autres pour le progrès de cette Science; en un mot, d'étendre les Principes en les réduisant."

Strange to say, the straightforward answer of this question is: not at all, not the least bit, never a whit. In spite of his promise "de déduire les Principes de la Méchanique des notions les plus claires" and "de les appliquer aussi à de nouveaux usage", D'Alembert never, nowhere, and in no wise uses "la notion" *impenétrabilité* to this end. The principles in question are formulated in the very beginning of the

*Premiere Partie. Loix générales du mouvement et de l'équilible des Corps* of [9], where one reads:

"On peut réduire tous les Principes de la Méchanique a trois, la force d'inertie, le mouvement composé, & l'équilibre. Au moins j'espere faire voir par ce Traité, que toute cette science peut être déduite de ces trois Principes. Je traiterai de chacun d'eux en particulier dans chacun des Chapitres suivans."

Well, Sir! One reads *Chapitre Premier. De la force d'inertie, et des propriétés du mouvement qui en résultent*, and one does not come across the word *impénétrabilité* at all. Afterwards one reads *Chapitre II. Du Mouvement composé*, and one does not encounter this word there too. Ultimately, one turns the pages of the book over *Chapitre III. Du Mouvement detruit ou changé par des obstacles*, and one does not run into *impénétrabilité* again. It is true that in the last chapter one reads:

"Un Corps qui se meut, peut rencontrer des obstacles qui altérent, ou même qui anéantissent tout–à–fait son Mouvement; ces derniers sont, ou invincibles par eux–mêmes, ou n'ont précisément de resistance, que ce qu'il en faut pour détruire le Mouvement imprimé au Corps.

Un obstacle invincible peut être tel, qu'il ne permette au Corps aucun Mouvement, comme quand un Corps tire une verge droit attachée a un point fixe; ou l'obstacle pourroit être de telle nature, qu'il n'empêchât pas le Corps de se mouvoir dans une autre direction que celle qu'il a, comme quand un Corps rencontre un plan inébranlable" (p. 31).

It is also true that the very idea of "obstacles" is inextricably bound up with the "propriété principale impénétrabilité". At last, it is true that a bit further down D'Alembert writes:

"Dela il s'ensuit, qu'un Corps sans ressort qui vient choquer perpendiculairement un plan immobile & *impénétrable*, doit s'arrêter après ce choc, & rester en repos. Car il est visible que si ce Corps a du Mouvement après la rencontre du plan, ce ne peut être qu'en arriére, & dans la direction de la perpendiculaire" (p. 32, our italics).

At the same time it is also true that the adjective "impénétrable" is used, in the last passage, sporadically, haphazardly, and contrastively — assigned to a subject exterior to the rigid body, the motion of which is studied rather than to this latter body itself. The same applies to other cases when the term "impénétrable" is used, for instance in 30. paragraph of the Traité, where the word "impénétrable" is used as a synonym for the word "invincible".

The reason for this state of affairs is a quite simple one: the quality "impénétrable" is an attribute to rigid bodies, whereas the *Traité de Dynamique* of D'Alembert has nothing to do with such matters: at the best it could be accepted as a writing dedicated to mass-point dynamics (if at all), as the ceaseless usage of the term "vitesse" at once displays, which becomes meaningless when assigned to rigid bodies. As regards the bodies themselves, D'Alembert is the originator of the conception, shortly afterwards adopted and developed further by his younger contemporary De la Grange — an outlook that was fated to play an extremely unenviable role in the supervening history of rational mechanics.

*Entre parenthèses*: In spite of the solemn promise of its author, promulgated in its title (namely, to "give a general principle for discovering the motions of several rigid bodies acting one upon another in an arbitrary manner"), the *Traité de Dynamique* of D'Alembert does not provide the reader with mathematical means for solving even one and only dynamical problem concerning a sole rigid body. Faced with this situation, one is at a loss for what reason has this *Traité* gained its "immortal" fame? The answer of this quite justifiable question is given by Truesdell, though his words refer to Leonardo rather than to D'Alembert:

"To learn the source, we recall the method of the Renaissance: Self-advertising ... In his skill of speech and his self-promotion he was a true son of the Renaissance. Like the humanists, with much adroitness but little solid achievement he blew himself into renown for all times" [8, p. 80–81].

In D'Alembert's case it would be appropriate to recall Vasari's words apropos of Leonardo: "Even though he talked much more about his works than he actually achieved, his name and fame will never be extinguished." We close the brackets.

In such a manner, a sound physical idea has been compromised mathematically in the *Traité de Dynamique* of D'Alembert. Without entering into details, we confine ourselves to the statement that it is discredited also in the natural logical extension [10] of D'Alembert's illogical dynamical philosophy. As a matter of fact, the collapse of the idea reaches in [10] such apocalyptic scales that evokes memories of biblical sinister omens for the original sin. *Nuda veritas* is that the reader of [10] is missing the forest for the trees. Directly contrary to Lagrange's overweening advertisements (namely that he proposes "des formules générales, dont le simple développement donne tous les équations nécessaires pour la solution de chaque problème ... la manière dont j'ai tache de remplir cet objet ne laissera rien à desirer ... Les méthodes que j'y expose ne demandent ni constructions, ni raisonnements géométriques ou méchaniques, mais seulement des opérations algébriques, assujetties à une marche régulière et uniforme"), the bulk of formulae one bumps up against in the *Méchanique Analitique* is entirely helpless when faced with the problem of the dynamical behaviour of a single rigid body subjected to any mechanical constraints: the cold fact is that the blazing upper strata of Lagrange's dynamical performances is as high as the movements (if any) of discrete systems of a finite number of masspoints. Truesdell's observations apropos of [7]: "Newton *gives no evidence of being able to set up differential equations of motion for mechanical systems* ... the cold fact is, the equations are not in Newton's book ... In Newton's *Principia* occur no equations of motion for systems of more than two free mass-points or more than one constrained mass-point; Newton's theories of fluids are largely false; and the spinning top, the bent spring, lie altogether outside Newton's range" [8, p. 92–93], may be paraphrased apropos of [10] in the following manner: Lagrange gives no evidence of being able to set up differential equations of motion for mechanical systems including rigid bodies; the cold fact is the equations are not in Lagrange's book. In Lagrange's *Méchanique Analitique* occur no equations of motion for systems of more than a finite number of mass-points; Lagrange's theories of rigid bodies are largely false; and the spinning top, the billiard ball, lie altogether outside Lagrange's range.

The long and the short of the whole span of Lagrange's mechanical philosophy, of his statical and dynamical *Weltanschauung*, may be incarnated in a sole phrase of his *Traité*:

"... considérons un système de corps, disposés les uns par rapport aux autres comme on voudra et animé par des forces accélératrices quelconques.

Soit $m$ la masse de l'un quelconque de ces corps, *regardé comme un point*" [11, p. 264; our italics].

This mechanical ideology of Lagrange's has ripened into the manhood a long time before he settled down to composing his *Méchanique Analitique* — as a matter of fact, not later than 1772 when he wrote his articles [12] wherein one reads:

"... si l'on imagine un système d'un nombre indéfini de corps considérés comme des points et liés ensemble de manière que leur distances mutuelles restent toujours les mêmes ..." (p. 579).

*Alibi*:

"En général, si l'on a un système d'autant de corps qu'on voudra, disposés de manière qu'ils soient forcés de conserver toujours les mêmes distances tant entre eux qu'à l'égard d'un point donné ..." (p. 587).

*Alibi* again:

"Je considere le corps propose comme l'assemblage d'une infinité de corpuscules ou points massifs unis ensemble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances" (p. 590).

In such a manner, there can be no mistaking Lagrange's words: his *corps* and *systèmes de corps* are special kinds of finite systems of discrete mass-points rather than rigid bodies in the genuine sense of the word. This circumstance has not been left unheeded, not mentioned, and untraversed. It did not escape Euler's attention. Apropos of [12] he wrote in [13] with undubitable while latent irony:

"But when I tried with greatest avidity to follow in detail his extremely profound thoughts, truly I could not get myself to go through all his calculations. Even the first lemma so deterred me that on account of my blindness I could not hope in any way to check through all the analytic devices he used" (quoted according to [8], p. 260).

Considerably later, in 1853 to be more precise, J. Bertrand made some critical remarks in this connection in the third edition of [10] *publiée par* himself. *Voilà* two of them, quoted after [11]:

"Le mot *corps* désigne ici un point matériel" (p. 11);

"Le mot *corps*, ici comme plus haut, désigne un point matériel" (p. 32).

In our days Noll, for instance, brought to the fore, from general considerations, the untenability of the efforts to regard "le corps proposé comme l'assemblage d'un infinité de corpuscules ou points massifs unis ensemble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances":

"Many textbooks on theoretical mechanics dismiss continuous bodies with the remark that they can be regarded as the limiting case of a particle system with an increasing number of particles. They cannot. The erroneous belief that they can had the unfortunate effect that no serious attempt was made for a long period to put classical continuum mechanics on a rigorous axiomatic basis" [14, p. 266].

Though a home truth, these statements at first sight appear to be ill-founded, since they are not substanciated by a mathematical proof. *Incredibile dictu*, as far as our knowledge goes, nobody has as yet answered mathematically the following question, fundamental for the whole of Lagrangean dynamical tradition:

**Possibility Problem**. Can rigid bodies be regarded as the limiting case of a particle system with an increasing number of particles?

Lagrange's answer is *yes*. Noll's answer is *no*. Let us cast our eyes about some other stands. *Voilà* a *Traité* [15] that out and out belongs to the mechanical classics. The feather in the author's cap, as regards the rigid body notion, consists in the following "extremely profound thoughts" in the words of Euler, *absit invidia verbo*:

"*Un corps solide est un ensemble de points matériels invariablement liés entre eux.* — Lorsqu'une force est appliquée à l'un de ces points, on dit qu'elle est appliquée au corps. Le corps solide ainsi défini est une abstraction. Tous les corps de la nature se déforment sous l'action des forces qui leur sont appliquées; mais les corps appelés communement solides subissent des déformations très petits, qui peuvent être négligees dans une première approximation" (t. I, p. 123–124).

In other words, the *Possibility Problem* is answered in the affirmative by Appell too. Skipping more than half a century, let us peek into a dynamical treatise [16] of comparatively recent time, its author promising in his *Introduction* "to give a compact, consistent, and reasonably complete account of the subject *as it now stands*" (p. VII, our italics). How does he define the rigid body concept?

*Bona venia vestra*, he does not define it at all. In the index of the book this term does not appear independently or, should we say, single-handed, unaided, off its own bat. Indeed, the text one finds there reads:

"Rigid body, motion in two dimensions, 111–113, 204; in space, 205–207. *See also* Euler's equations, spinning top, rolling sphere, rolling penny, rolling ellipsoid" (p. 640).

In order to find a description if not a definition of the notion of rigid body in [16] one must search "for the little solid matter as a sparrow pecks out a few nutricious seeds from a dungheep — a task not altogether savory", if it is permitted to use here Truesdell's words apropos of D'Alembert. While Chapter I of the book, entitled *Motion of a particle*, is dedicated to mass-point dynamics, the term "rigid body" comes into view for the first time in [16] in the beginning of Chapter II, headed *Dynamical systems*. Therein one reads:

"In the preceding chapter we considered the dynamics of a single particle. It might seem natural, following the historical order of development, to discuss next the theory of the motion of a single rigid body; this is in fact the order usually followed in a first study of rigid Dynamics. Our approach will however be somewhat different. In Analytical Dynamics we proceed directly from the single particle to the general dynamical system. The single rigid body is of course a special case of a dynamical system and indeed one that we shall frequently find useful as a special illustration" (p. 20).

In such a way, the reader of [16] comes to know at the same breath the following truths as great as to seem divine revelations:

1. In Analytical Dynamics it is proceeded directly from the single particle to the general dynamical system.

2. The single rigid body is a special case of a dynamical system.

3. The latter statement needs no proof: it is "of course" true.

4. The single rigid body is frequently useful as a special illustration.

5. Ergo: the single rigid body represents no interest *in se*, that is to say as *ein Ding an sich*.

6. An indirectly implied corollary: any attributes ascribed to rigid bodies must be derived from attributes of general dynamical systems of single particles.

7. Ergo: constraints imposed on rigid bodies must be implied by constraints imposed on single particles.

We shall see now how does the author of this Treatise (having bidden fair, we recall, "to give a compact, consistent, and reasonably complete account of the subject [of analytical dynamics] as it now stands") materialize this new kind of mathematical induction — his limiting process $1 \to \infty$. *Qui habet aures audiendi, audiat*:

"The idea of a rigid body in the classical dynamics is a collection of particles set in a rigid and imponderable frame. Similarly we shall think of the general dynamical system as a collection of particles acted on by given forces and controlled by various kinds of constraints" (*ibid.*).

In such a manner, Pars answers the *Possibility Problem* also in the affirmative — in a most categorical manner at that. There is a point, however, that ought not be left unnoticed.

All those yes-answers and no-answers (or should we say can-answers and cannot-answers) are, alas, no mathematical answers at all. Quite much the reverse: those replies are sooner reflexions of inner convictions, of professional habits, of intellectual indolence, if you will, and in this respect they are not a jot more reliable than the possible responses of the question, say, which faith is more preferable — the Christian or the Mohammedan. The only way a mathematician can solve a *Possibility Problem* is to solve an *Existence Problem* — to prove that the object, the possibility of which is investigated, exists in actual fact.

It will remain an enigma of enigmas *in saecula saeculorum* why, in the course of more than two clear centuries, the idea flashed through nobody's mind that Lagrange's mental picture of "le corps ... comme l'assemblage d'un infinité de corpuscules ou points massifs unis ensemble de manière qu'ils gardent toujours nécéssairement entre eux les mêmes distances" must be unconditionally submitted to a mathematical proof or disproof, in the same manner as it must be proved, or disproved, that there exist natural numbers $x$, $y$, $z$ and $n > 2$ for which $x^n + y^n = z^n$ holds. Some mathematicians believe that such numbers exist, others disbelieve it — but, with the nasty exception of a swarm of illiterate idiots, there was a sole mathematician worthy of the name in the last four centuries, who stated he knew there exist no such numbers, and he knew it since he found a proof. However, he did not leave us such a proof, and Gauss, for instance, thought that Fermat misled himself; that is why the negation of $x^n + y^n = z^n$ is qualified by modern mathematicians as a hypothesis rather than a theorem.

If "un corps solide est un ensemble de points matériels invariablement liés entre eux", then the question quite naturally arises: what does connect them in such a manner? Since we claim to be mechanicians rather than fakirs, we accept that the only factors determining the mechanical behaviour of mass-points are forces. In such a way, the *Possibility Problem* formulated above may be re-redacted in the following manner:

**Existence Problem.** $S$ being a system of mass-points, do there exist forces acting on them and conserving invariant in the course of the time the mutual distances between these mass-points?

**Solution.** In order to accomplish a *reductio ad absurdum* let us suppose that this question is answered in the affirmative. Since the number $n$ of the points of $S$ is indeterminate, it may be supposed, without a loss of generality, that $n = 2$. Let $P_\nu$ be the points of $S$ with masses $m_\nu$, respectively, and let $\boldsymbol{r}_\nu = \boldsymbol{OP}_\nu$ $(\nu = 1, 2)$, $O$ denoting the origin of an inertial according to Newton system of reference $Oxyz$. Let $\boldsymbol{v}_\nu = \dfrac{d\boldsymbol{r}_\nu}{dt}$ $(\nu = 1, 2)$, the derivatives being taken with respect to $Oxyz$. At last, let $\boldsymbol{F}_\nu$ be the forces acting on $P_\nu$ $(\nu = 1, 2)$, respectively, in accordance with the supposition made above that such forces exist. Then, by virtue of Newton's dynamical axiom,

$$(93) \qquad \frac{d}{dt}(m_\nu \boldsymbol{v}_\nu) = \boldsymbol{F}_\nu \qquad (\nu = 1, 2),$$

the derivatives being taken with respect to $Oxyz$.

By hypothesis the forces $\boldsymbol{F}_\nu$ $(\nu = 1, 2)$ are such that

$$(94) \qquad \frac{d}{dt}(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2 = 0 \qquad (\forall t)$$

or, just the same,

$$(95) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{v}_1 - \boldsymbol{v}_2) = 0 \qquad (\forall t).$$

Let $\tau$ be a particular moment of the time $t$ and let

$$(96) \qquad \boldsymbol{r}_{\nu\tau} = \boldsymbol{r}_\nu(\tau), \quad \boldsymbol{v}_{\nu\tau} = \boldsymbol{v}_\nu(\tau) \qquad (\nu = 1, 2)$$

be the initial positions and the initial velocities, respectively — in other words, the *initial conditions* — of the dynamical problem under consideration. Since the relation (95) holds for any $t$, it is valid for $t = \tau$ too:

$$(97) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{v}_1 - \boldsymbol{v}_2) = 0 \qquad (t = \tau),$$

and (96), (97) imply

$$(98) \qquad (\boldsymbol{r}_{1\tau} - \boldsymbol{r}_{2\tau})(\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}) = 0.$$

Now the equation (98) is an *absurdity*, since it represents a restriction imposed on the initial conditions (96) of the system $S$, due to the hypothesis that there exist forces $\boldsymbol{F}_\nu$ $(\nu = 1, 2)$ for which (93) with (94) hold: it is a principle of principles in rational mechanics that the initial conditions of a mechanical system are independent of the forces acting on it, and this principle is rooted in the very essence of the theory of ordinary differential equations, according to which the initial conditions of a system of differential equations are wholly arbitrary, independent of the particular functions available there. The absurdity (98) traverses the hypothesis in question and gives a negative answer of the question posed in the *Existence Problem*. *Quod erat demonstrandum.*

**Scholium 1.** A colleague and, strange enough, a good friend of ours, when for the first time faced with the absurdity (98), ejaculated: Now the same is true for any two points of any rigid body! At first sight this is a most well-founded doubt. This is only seemingly, however.

Let $S$ be a rigid body and $P_\nu$ $(\nu = 1, 2)$ be any two of its points. Under the above notations the very definition of the rigid body concept implies the relation (95) and, following the chain of the above argumentation, leads ultimately to the conclusion (98). For a rigid body, however, the relation (98) is no restriction at all imposed on the initial position of the body in space and on its initial velocities. In other words, (98) puts no restraints on the initial values

(99) $$x_\Omega(\tau),\ y_\Omega(\tau),\ z_\Omega(\tau),\ \psi(\tau),\ \varphi(\tau),\ \theta(\tau)$$

of the canonical parameters (88) of $S$ and on the initial values

(100) $$\dot{x}_\Omega(\tau),\ \dot{y}_\Omega(\tau),\ \dot{z}_\Omega(\tau),\ \dot{\psi}(\tau),\ \dot{\varphi}(\tau),\ \dot{\theta}(\tau)$$

of their derivatives (92). As a matter of fact, in the rigid body case the relation (98) is reduced to the identity

(101) $$0 = 0,$$

as it is at once seen by a scalar multiplication with $\boldsymbol{r}_1 - \boldsymbol{r}_2$ of the necessary and sufficient condition

(102) $$\boldsymbol{v}_1 - \boldsymbol{v}_2 = \bar{\omega} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad (\forall t)$$

in order that the points $P_1$ and $P_2$ belong to $S$. *Sapienti sat.*

*Mais revenons a nos moutons!* In other words, let us return to Euler's dynamical equations (82), (87), where no specification is made as yet as regards the mechanical nature of the forces (74). As it has been underlined, the cases of a free rigid body, when all the canonic parameters (88) are mutually independent and when all the forces (74) are known beforehand as given data in the conditions of the particular dynamical problem under consideration, are as *rara avis in terris* as a honest politician; it has been emphasized also that the physical cause underlying any restriction in the position of a rigid body in space is rooted in the impenetrability of matter resulting in the phenomenon of mutual contact between bodies. The latter is a fact *homo sapiens* has been on closer acquaintance with from his very childhood in the literal as well as the metaphorical sense of the word — to such an extent as to feel it by intuition. In real fact, all the motions the same *homo* observes in nature are movements of non-free bodies, he himself being perpetually coerced to set his feet on earth.

Now that one comes to think of it, one realizes to his or her amazement that there is not a single motion in this God's earth accomplished on account of "pure" forces, that is to say without the interference of reactions due to surrounding environment. Even the free fall of ponderous bodies thrown in the air is influenced by the resistance of the medium, affecting sometimes the projectile motions to such a degree as to plunge artillerists into dispair. In reality, the only "pure" motions observable in our universe in the days of Galileo and Newton have been the planet movements; these, however, have been "polluted", first, by the Earth's own motion, and, second, by their non-observability as movements in the proper sense of the word (as, for instance, the fall of a meteor): for the naked eye, as well as for the aided by any instrument whichever, the planet motion is a series of discrete positions of the luminary rather than a continuous process in the course of time.

Let us make a parenthesis for a brief lyrical digression. Let us fancy the epoch of Galileo and Newton, at daybreak of dynamics, when no dynamical law has been as yet grasped by human mind, but hints of such one were already felt in the air. The acceleration concept has been shaped by now, the outlines of the force concept have picked out in the dark (let alone in the statical case), some kind of a mutual relation between them was already suspected, and yet nobody came to know it. If life begins *ab ovo*, then dynamics begins *ab corpusculo*: identifying, as Galileo and Newton did, bodies with mass–points, we know today that any dynamical phenomenon, observable in their days, has been governed by the law

$$mw = \boldsymbol{P} + \boldsymbol{R}, \tag{103}$$

$\boldsymbol{P}$ denoting the *innate force* of the body (in other words, the gravitational effects as established on the Earth's surface), and $\boldsymbol{R}$ — the reactions of the constraints (resistance including) imposed on the body. Now while $\boldsymbol{P}$ is a completely determined mechanical entity (at least as far as a particular geographic point is concerned), $\boldsymbol{R}$ on the contrary escapes a direct observation and measurement like a ghost. However, $\boldsymbol{R}$ being unknown and the equation (103) itself being buried in the impenetrable future, how could one hope to unearth it in broad daylight?

The only chance one has at his disposal is the case

$$\boldsymbol{R} = o. \tag{104}$$

Such "pure" motions are proposed by planets, by planets only, and by nothing save planets. Newton grasped this chance — *his* chance — with both hands. The result is immortality, as far as stars are immortal, since his law governs stellar motions:

*Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.*

Now the universality of this discovery of Newton's lies in the fact that, although discovered in the special case (104) of (103), it is not only applicable — moreover, it is a *conditio sine qua non* — for the motion of any corpuscular body subjected to any constraints imposed on it, generating any reactions the Human Mind and Mother Nature may devise. This inference is one of the most daring, true though incredible, inductional hypothesis in all the history of science, with wholly nonforecastable after-effects.

Summing up, one could quite justifiably state that no rational dynamics could be created if stellar motions were un-get-at-able to observation and measurements — if, for instance, the average earth temperature was some degrees higher, so that no stars could be seen on account of clouds. *Finis* of the lyrical digression.

All those meditations are much more philosophical than mathematical by nature, and we apologize to the reader begging his pardon. And yet, the character of the mathematical phenomenon described by the enigmatic expression *geometrical constraints imposed on rigid bodies* cannot be grasped rightly without these verbal explanations. Since, *summa summarum*, all this has a bearing on one of the most fundamental concepts in rigid dymanics.

Squaring accounts as regards the heuristic origins of the notion, we must perceive that, although technically feasible by means of an infinite variety of contrivances, all restrictions on the positions of a rigid body in space, described in the mechanical literary sources by means of phrases like "the body is constrained", or "compelled", or "coerced". or "forced", or "imposed", etceteras repeatedly used, reduce, when all is said and done, to a most simple mathematical device: those are *geometrical constraints* imposed on certain points of the rigid bodies. However, since a logical *anguis in herba latet* here, and the witchcraft of the words may play a practical joke on the uninitiated, converting sound intentions into a germ of regrettable misunderstandings, it is of paramount importance to nip in the bud any chance for any misconception by taking special pains for explaining the exact meaning of those synonymous terms.

Here is a point that must become crystal clear for anybody who has made up his mind to work professionally rather than dilettantish in analytical dynamics: in spite of the fact that the combination of words *geometrical constraint* has infiltrated the whole span of mechanical language, it is by no means a mathematical term — it is a concise expression of most knotty, most catchy, and most mazy mathematical situations that badly need a formal specification in any particular case. All of those particular cases reduce to the essentiality that specific mathematical hypothesis of one kind or another must be announced in the very conditions of the dynamical problem under consideration, concerning the mechanical behaviour of one or more points of the rigid body or rigid bodies. The corresponding point or points are promulgated, or proclaimed, or declared *points of contact* between the rigid body and the geometrical constraint in question. The importance of this notion may be emphysized by the maxim *no point of contact — no dynamical problem* concerning non-free rigid bodies, in the genuine mathematical sense of the word.

There are three geometrical entities in space, and there are also three geometrical entities invariably connected with a rigid body $S$, that can be juxtaposed in such mutual relations among each other as to restrict the possible positions of $S$ in space, and these entities are points, lines, and surfaces. The relations in question reduce to one of the following combinations:

A fixed point of $S$ is constrained to coincide with a given point in space, or to describe a given curve line in space, or to lie on a given surface in space.

Or a fixed curve line in $S$ is constrained to pass through a given point in space, or to intersect a given curve line in space, or to touch a given curve line in space, or to touch a given surface in space.

Or a fixed surface in $S$ is constrained to pass through a given point in space, or to touch a given curve line in space, or to touch a given surface in space.

(In all those cases the term *in space* means *external* for the rigid body $S$; at that, the special points, lines, and surfaces may be both *scleronomic* and *rheonomic*, that is to say fixed in space, or variable in position, or in shape in the course of time, respectively.)

Whenever any of these 10 cases is at hand in a dynamical problem (separately or in combination with others), it is said that *a geometrical constraint is imposed* on the rigid body. It is immediately seen that in such a case a singular point comes out into the open, namely the particular point common for both the geometrical entity fixed in the rigid body $S$ and for the geometrical entity in space, playing the part of a geometrical constraint. This namely point is called the *point of contact* of $S$ with the geometrical constraint in question.

The cardinal significance of the notion *point of contact* for rigid mechanics is predeterminated by the following dynamical axiom, reflecting age-old practical experience.

**Ax 3 E**. Any geometrical constraint imposed on a rigid body $S$ generates a force acting on $S$, the directrix of which is passing through the point of contact of $S$ with the geometrical constraint.

**Df 2 E**. The force of Ax 3 E is called the *reaction* of the geometrical constraint.

**Scholium 2**. The term *reaction* is fabricated as an antipode, or at least in contrast, to the term *action*, by means of which the forces indicated in the conditions of the dynamical (as well as statical) problem are described. Another terminology exploits the terms *active forces* and *passive forces*, respectively. At that, *active* are by definition those forces that are completely determined in the conditions of the statical or dynamical problem for any position and any motion of the rigid body $S$, that is to say for any admissible values of the canonic parameters (88) of $S$, of their velocities (92), and possibly of the time $t$, whereas nothing else is known for the *passive* forces save what Ax 3 E sermonizes, namely that they are acting on $S$ and that their directrices are running through the corresponding points of contact with the geometrical constraints generating those reactions.

The latter statement necessitates some specification. Let $A$ be the point of contact of the rigid body $S$ with a geometrical constraint $\gamma$ and let

$$\overrightarrow{R} = (\boldsymbol{R}, \boldsymbol{N}) \tag{105}$$

be the reaction of $\gamma$, its moment $\boldsymbol{N}$ being taken with respect to $O$. As it is well-known, the equation of the directrix $d$ of (105) is

$$\boldsymbol{r} \times \boldsymbol{R} = \boldsymbol{N}, \tag{106}$$

$\boldsymbol{r} = \boldsymbol{OP}$ denoting the fluent radius-vector of any point $P$ of $d$. If by definition $\boldsymbol{r}_A = \boldsymbol{OA}$, then (106) implies

$$\boldsymbol{r}_A \times \boldsymbol{R} = \boldsymbol{N} \tag{107}$$

by virtue of Ax 3 E.

**Scholium 3**. As a matter of fact, Ax 3 E states 3 things:

1. The existence of the force $\overrightarrow{R}$.
2. $\overrightarrow{R}$ is acting on $S$.
3. $\boldsymbol{N}$ is known as far as $\boldsymbol{r}_A$ and $\boldsymbol{R}$ are known.

In other words, any constraint imposed on a rigid body $S$ introduces a new force in the right-hand sides of the equations (82), (87), governing the motion of $S$. Besides, any such constraint introduces 3 new unknown quantities in the mathematical problem to be solved, namely the components of $\boldsymbol{R}$ according to

$$\boldsymbol{R} = R_x\boldsymbol{i} + R_y\boldsymbol{j} + R_z\boldsymbol{k} \tag{108}$$

in view of (107).

Let, in a particular dynamical problem, $S$ be under the action of the active forces

$$\vec{F}_\mu = (\boldsymbol{F}_\mu, \boldsymbol{M}_\mu) \qquad (\mu = 1, \dots, m) \tag{109}$$

and let by definition

$$\boldsymbol{F} = \sum_{\mu=1}^{m} \boldsymbol{F}_\mu, \quad \boldsymbol{M} = \sum_{\mu=1}^{m} \boldsymbol{M}_\mu. \tag{110}$$

Let $n$ geometrical constraints be imposed on $S$, generating passive forces

$$\vec{R}_\nu = (\boldsymbol{R}_\nu, \boldsymbol{N}_\nu) \qquad (\nu = 1, \dots, n), \tag{111}$$

and let by definition

$$\boldsymbol{R} = \sum_{\nu=1}^{n} \boldsymbol{R}_\nu, \quad \boldsymbol{N} = \sum_{\nu=1}^{n} \boldsymbol{N}_\nu. \tag{112}$$

(Naturally, all moments $\boldsymbol{M}_\mu$ and $\boldsymbol{N}_\nu$ ($\mu = 1, \dots, m$; $\nu = 1, \dots, n$) in (109) and (111) are taken with respect to $O$.) Besides, let

$$\boldsymbol{M}_G = \boldsymbol{M} + \boldsymbol{F} \times \boldsymbol{r}_G, \quad \boldsymbol{N}_G = \boldsymbol{N} + \boldsymbol{R} \times \boldsymbol{r}_G \tag{113}$$

be the moments of the system of forces (109) and (111), respectively, with regard to the mass-centre $G$ of $S$. Under these hypothesis, the Eulerian dynamical equations (82), (87) take the form

$$m\ddot{x}_G = F_x + R_x, \quad m\ddot{y}_G = F_y + R_y, \quad m\ddot{z}_G = F_z + R_z, \tag{114}$$

$$\left\{\begin{aligned} &A\dot{\omega}_\xi - (B - C)\omega_\eta\omega_\zeta - D(\omega_\eta^2 - \omega_\zeta^2) \\ &\qquad - E(\dot{\omega}_\zeta + \omega_\xi\omega_\eta) - F(\dot{\omega}_\eta - \omega_\zeta\omega_\xi) = M_{G\xi} + N_{G\xi}, \\ &B\dot{\omega}_\eta - (C - A)\omega_\zeta\omega_\xi - E(\omega_\zeta^2 - \omega_\xi^2) \\ &\qquad - F(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) - D(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) = M_{G\eta} + N_{G\eta}, \\ &C\dot{\omega}_\zeta - (A - B)\omega_\xi\omega_\eta - F(\omega_\xi^2 - \omega_\eta^2) \\ &\qquad - D(\dot{\omega}_\eta + \omega_\zeta\omega_\xi) - E(\dot{\omega}_\xi - \omega_\eta\omega_\zeta) = M_{G\zeta} + N_{G\zeta}, \end{aligned}\right. \tag{115}$$

provided by definition

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}, \tag{116}$$

$$\boldsymbol{M}_G = M_{G\xi}\bar{\xi}^0 + M_{G\eta}\bar{\eta}^0 + M_{G\zeta}\bar{\zeta}^0, \tag{117}$$

$$\boldsymbol{N}_G = N_{G\xi}\bar{\xi}^0 + N_{G\eta}\bar{\eta}^0 + N_{G\zeta}\bar{\zeta}^0. \tag{118}$$

**Scholium 4.** Even a cursory analysis of the mathematical formalism describing a geometrical constraint of the kinds enumerated above at once displays that any such constraint imposes one, two, or at most three analytic restrictions on the canonic parameters (88) of the rigid body $S$. In the case of $n$ constraints this circumstance diminishes the number of the unknown functions

$$x_\Omega(t),\ y_\Omega(t),\ z_\Omega(t),\ \psi(t),\ \varphi(t),\ \theta(t) \tag{119}$$

of the time $t$, the determination of which as a solution of the system of differential equations (114), (115) is required, by at least $n$ and at most $3n$ units. On the other hand, the reactions (111) introduce $3n$ new unknowns. In such manner, any problem of rigid dynamics is reduced to a system of 6 ordinary differential equations (114), (115) of second order with respect to the time $t$ of a heterogenously mixed type: a part of the unknown quantities are some of the functions (119) and they are at hand in (114), (115) analytically, that is to say together with their first and second derivatives with respect to $t$; another part are the $3n$ unknown components of the reactions (111), provided

$$\boldsymbol{R}_\nu = R_{\nu x}\boldsymbol{i} + R_{\nu y}\boldsymbol{j} + R_{\nu z}\boldsymbol{k} \qquad (\nu = 1, \ldots, n), \tag{119'}$$

and they are at hand in (114), (115) algebraically, as linear unknown quantities, in point of fact.

**Scholium 5.** The first query arising when a problem of rigid dynamics is put for discussion is the question, whether the system (114), (115) of differential equations is consistent, i.e. whether it does or does not possess a solution. In other words, this is the *Existence Problem* for the dynamical problem under consideration or, in view of the physical interpretation of the mathematical circumstances, the *Possibility Problem* for the motion of the rigid body under the conditions this dynamical problem announces.

On account of the mathematical complications the existence problem gives rise to, it is an object of a particular investigation we shall soon turn back to. For the time being we shall confine us to the remark that most authors of mechanical writings leave the existence problem out in the cold in the most flagrant manner: not only they do not proceed to its solution, but even do not make mention of the existence of the existence problem.

**Scholium 6.** We shall bring our exposition to an end with a note concerning the application of the Eulerian dynamical equations (114), (115) to that special kind of rigid bodies, which are known under the name of *rigid rods*.

A rigid rod $L$ is a rigid body the density (35) of which has the eccentricity to be zero everywhere save along a straight line $l$, called the *directrix* of $L$. Let us connect with $L$ invariably an orthonormal right-hand orientated Cartesian system of reference $\Omega\xi\eta\zeta$ in the following manner: the axis $\Omega\xi$ coincides with the directrix $l$; the axis $\Omega\zeta$ is parallel to the line of intersection of the plane $Oxy$ with the plane through $\Omega$ perpendicular to $\Omega\xi$ (supposing those two planes non-parallel); the unit vectors $\bar{\xi}^0$ and $\bar{\zeta}^0$ of the axes $\Omega\xi$ and $\Omega\zeta$, respectively, once defined, the axis $\Omega\eta$ is determined by its unit vector $\bar{\eta}^0 = \bar{\zeta}^0 \times \bar{\xi}^0$. The axis $Oz$ being obviously perpendicular to the axis $\Omega\zeta$, the definition (13) implies

$$\theta = \frac{\pi}{2}, \tag{120}$$

and (120), (17) imply

$$(121)\qquad \begin{cases} a_{11} = \cos\psi\cos\varphi, & a_{12} = -\cos\psi\sin\varphi, & a_{13} = \sin\psi, \\ a_{21} = \sin\psi\cos\varphi, & a_{22} = -\sin\psi\sin\varphi, & a_{23} = -\cos\psi, \\ a_{31} = \sin\varphi, & a_{32} = \cos\varphi, & a_{33} = 0. \end{cases}$$

Besides, (120) and (26) imply

$$(122)\qquad \omega_\xi = \dot\psi\sin\varphi, \quad \omega_\eta = \dot\psi\cos\varphi, \quad \omega_\zeta = \dot\varphi.$$

By virtue of the condition (120) a dynamical problem concerning a rigid rod is presumably overdetermined. Indeed, the canonic parameters of $L$ are now 5 in number, namely

$$(123)\qquad x_\Omega,\ y_\Omega,\ z_\Omega,\ \psi,\ \varphi,$$

whereas there are 6 equations (114), (115) for their determination: in the case of a free rigid rod they outnumber the unknown quantities (123). This contradiction is, however, only an ostensible one.

Let us take a closer view of the situation. The definition of a rigid rod implies

$$(124)\qquad \varkappa(\bar\rho) = 0$$

for

$$(125)\qquad \eta \neq 0$$

or

$$(126)\qquad \zeta \neq 0,$$

whence, formally at least,

$$(127)\qquad dm = \varkappa(\bar\rho)\, d\xi d\eta d\zeta$$

implies

$$(128)\qquad dm = \varkappa(\xi)\, d\xi$$

provided (9). Now (128), (124)–(126), and (41) imply

$$(129)\qquad \bar\rho_G = \frac{1}{m}\int \xi\varkappa(\xi)\, d\xi\bar\xi^0,$$

i.e.

$$(130)\qquad \eta_G = \zeta_G = 0$$

provided (64).

On the other hand, (128), (124)–(126) and (58), (59) imply

$$(131)\qquad I_{\xi\xi} = 0, \quad I_{\eta\eta} = I_{\zeta\zeta} = \int \xi^2\varkappa(\xi)\, d\xi, \quad I_{\eta\zeta} = I_{\zeta\xi} = I_{\xi\eta} = 0$$

and (130), (60), (61) imply

$$(132)\qquad J_{\xi\xi} = 0, \quad J_{\eta\eta} = J_{\zeta\zeta} = m\xi_G^2, \quad J_{\eta\zeta} = J_{\zeta\xi} = J_{\xi\eta} = 0.$$

Now (131), (132), (62), (63) imply

$$(133)\qquad A = 0, \quad B = C = I, \quad D = E = F = 0,$$

provided by definition

$$I = \int \xi^2 \varkappa(\xi)\, d\xi - m\xi_G^2, \tag{134}$$

and (133), (115) imply

$$\begin{cases} 0 = M_{G\xi} + N_{G\xi}, \\ I(\dot\omega_\eta - \omega_\zeta\omega_\xi) = M_{G\eta} + N_{G\eta}, \\ I(\dot\omega_\zeta + \omega_\xi\omega_\eta) = M_{G\zeta} + N_{G\zeta}. \end{cases} \tag{135}$$

We are faced now with a most interesting and instructive phenomenon — a danger hanging like the sword of Damocles over the head of everyone working in rational mechanics. Let us first suppose that the rigid rod is free; then

$$N_{G\xi} = 0 \tag{136}$$

and the first equation (135) implies

$$M_{G\xi} = 0. \tag{137}$$

In other words, (137) is a necessary condition for a free rigid rod dynamical problem to be consistent, videlicet to possess a solution or, using a mechanical language, in order that the rigid body could move. Now is this *conditio sine qua non* satisfied indeed?

This is a question God Almighty cannot answer.

A Mister Someone with a more physical than mathematical mental constitution would at once exclaim: Nonsense! You bet (137) is true!

What are his motives?

His mental picture of a rigid rod is suggested by his everyday experience. He cannot imagine a spade, or a mattock, or an ax working save when hands are holding its shank, in other words, save when the forces acting on the instrument are applied on its handle. And the meaning of the term "applied" in this context is: when the directrices of the forces intersect the directrix of the rod.

Since the latter in our case is the axis $\Omega\xi$, Mister Someone presupposes that the directices

$$\boldsymbol{r} \times \boldsymbol{F}_\mu = \boldsymbol{M}_\mu \qquad (\mu = 1, \ldots, m) \tag{138}$$

of the forces (109) intersect $\Omega\xi$, the equation of which is

$$\boldsymbol{r} \times \bar\xi^0 = \boldsymbol{r}_\Omega \times \bar\xi^0 \tag{139}$$

or

$$\bar\rho \times \bar\xi^0 = \boldsymbol{o} \tag{140}$$

in view of (6). On the other hand, (64) and (130) imply

$$\bar\rho_G = \xi_G \bar\xi^0, \tag{141}$$

and (138), (6), (45) imply

$$(\bar\rho + \boldsymbol{r}_G - \bar\rho_G) \times \boldsymbol{F}_\mu = \boldsymbol{M}_\mu \qquad (\mu = 1, \ldots, m), \tag{142}$$

whence

$$(\bar\rho + \boldsymbol{r}_G - \bar\rho_G) \times \boldsymbol{F} = \boldsymbol{M} \tag{143}$$

by virtue of (110) or, just the same,

$$\bar{\rho} \times \boldsymbol{F} = \boldsymbol{M}_G + \xi_G \bar{\xi}^0 \times \boldsymbol{F} \tag{144}$$

in view of (141) and

$$\boldsymbol{M}_G = \boldsymbol{M} + \boldsymbol{F} \times \boldsymbol{r}_G. \tag{145}$$

Since the equations (140) and (144) are, by a physical hypothesis, consistent, the relation

$$\bar{\rho} = \lambda \bar{\xi}^0 \tag{146}$$

with an appropriate $\lambda$ according to (140) and (144) imply

$$\lambda \bar{\xi}^0 \times \boldsymbol{F} = \boldsymbol{M}_G + \xi_G \bar{\xi}^0 \times \boldsymbol{F}, \tag{147}$$

whence

$$\bar{\xi}^0 \boldsymbol{M}_G = 0, \tag{148}$$

i.e. (137) by virtue of (117).

In such a manner, the necessary condition (137) for a free rigid rod dynamical problem to be consistent is a corollary from the hypothesis that the directrices of all active forces applied on the rod intersect the directrix of the rod, i.e. the line along which its density is different from zero. But this hypothesis does not follow from the hitherto formulated definition of the rigid rod concept consisting in the only requirement (124) for (125) or (126): it is a new aspect of this notion that has been just now substantiated physically and mathematically and that must necessarily take part in the definition of this concept.

In such a manner we arrive at the following newly improved formulation:

A rigid rod is a rigid body the density of which is zero anywhere save along a straight line (its directrix), where its density is such that the integral (39) is non-zero; moreover, if an active force is acting on a rigid rod, its directrix intersects (or coincides with) the directrix of the rod.

This definition accepted, (137) implies that the first equation (135) becomes (136). The relation (136), however, is by no means an obligatory one. The meaning of this statement is that the condition (136) is both beyond proof and beyond disproof. Now we are faced with the same logical perplexities as in the case of the necessary condition (137). This dilemma is settled in the same way as in the preceding case. In other words, it is supposed that the only points of contact of a rigid rod with any geometrical constraint, imposed on it, must be lying on its directrix.

Summing up, we may now state that (137) and (136) are presumptive necessary conditions for any problem of rigid rod dynamics. *Praemonitus et praemunitus* with this new clause, one has now every right to state that in the case of a rigid rod the first equation (115) turns out to become an identity of the kind (101) (provided the system of reference $\Omega\xi\eta\zeta$, invariably connected with the rigid body, is chosen in such a manner that (120) and (124) provided (125) or (126) hold). As a result, in the case of rigid rod dynamics one has at his disposal exactly 5 equations of motion, namely (114) and

$$I(\dot{\omega}_\eta - \omega_\zeta \omega_\xi) = M_{G\eta} + N_{G\eta}, \quad I(\dot{\omega}_\zeta + \omega_\xi \omega_\eta) = M_{G\zeta} + N_{G\zeta}, \tag{149}$$

while the number of the unknown quantities in the dynamical problem is not lesser than 5.

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