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NONEXISTENCE OF ORBITAL MORPHISMS BETWEEN DYNAMICAL SYSTEMS ON SPHERES

Simeon T. Stefanov

Симеон Т. Стефанов. НЕСУЩЕСТВОВАНИЕ ОРБИТАЛЬНЫХ МОРФИЗМОВ МЕЖДУ ДИНАМИЧЕСКИМИ СИСТЕМАМИ НА СФЕРАХ

Рассмотрены динамические системы на нечетномерных сферах S^{2n+1}, определенные для $t \in \mathbb{R}$, $z \in S^{2n+1}$ формулой

$$tz = \left(e^{i\theta_0 t}z_0, e^{i\theta_1 t}z_1, \ldots, e^{i\theta_n t}z_n\right),$$

где θ_i — действительные числа, $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$, $\|z\| = 1$. Такие системы обозначаются через $S^{2n+1}(\theta_0, \ldots, \theta_n)$.

В работе доказано, что в случае рационально независимых θ_i не существуют орбитальных морфизмов

$$f: S^{2n+1}(\theta_0, \ldots, \theta_n) \longrightarrow S^{2n-1}(\theta'_0, \ldots, \theta'_{n-1}),$$

т. е. отображения монотонно наматывающие каждую траекторию первой динамической системы на траекторию второй. (Здесь θ'_i — произвольные.)

В случае произвольных θ_i то же самое доказательство проходит с некоторыми усложнениями.

Несуществование орбитальных морфизмов между двумя динамическими системами означает в некотором смысле, что первая система существенно сложнее второй.

Simeon T. Stefanov. NONEXISTENCE OF ORBITAL MORPHISMS BETWEEN DYNAMICAL SYSTEMS ON SPHERES

Some standard dynamical systems in the odd-dimensional spheres S^{2n+1} are considered. The nonexistence of orbital morphisms $f: S^{2n+1} \longrightarrow S^{2n-1}$, i.e. maps winding each trajectory of the first system over some trajectory of the second one, is proved.

We consider, in this note, some standard dynamical systems in the odd-dimensional spheres S^{2n+1} and prove about them a topological fact, which shows that, in some sense, the „chaos" in S^{2n+1} is essentially greater than the „chaos" in S^{2n-1} (in particular, these systems are not semiconjugated). In fact, we shall prove, that there are no orbital morphisms $f: S^{2n+1} \longrightarrow S^{2n-1}$, i.e. smooth maps winding each trajectory of S^{2n+1} over some trajectory of S^{2n-1}. This proposition has the same origin with the „nonexistence" of equivariant maps between G-spaces, namely, the Borsuk-Ulam type theorems (see [1], [2], [4]).

As usually, we shall denote by tx the $\mathbf{R}$-action defined by some dynamical system in X (here $t \in \mathbf{R}$, $x \in X$). The trajectory of an element $x \in X$ is the set

$$\operatorname{traj} x = \{tx | t \in \mathbf{R}\}.$$

Let dynamical systems in the (smooth) manifolds M and N be given.

Definition. A smooth map $f: M \to N$ is called *orbital morphism*, if for any $t \in \mathbf{R}$, $x \in M$

$$f(tx) = \varphi(t, x) f(x),$$

where $\varphi: \mathbf{R} \times M \to \mathbf{R}$ is such that $\frac{\partial \varphi}{\partial t}(t, x) \neq 0$ for any $t \in \mathbf{R}$, $x \in M$.

It is clear that f transforms any trajectory into a trajectory and that the restriction

$$f\restriction_{\operatorname{traj} x}: \operatorname{traj} x \to \operatorname{traj} f(x)$$

is a covering. (This property may be taken for definition in the continuous case.) For compact M, N one obtains also

$$f(\overline{\operatorname{traj} x}) = \overline{f(\operatorname{traj} x)}$$

(where $\overline{A}$ denotes the closure of A). It is easy to see that the composition of two orbital morphisms is also an orbital morphism. This concept is quite more general than „semiconjugacy". The last one is obtained for $\varphi(x, t) = t$.

The nonexistence of orbital morphism between M and N means, in some sense, that the dynamical system in M is essentially more complex than the system in N.

Now we shall define some standard dynamical systems in the odd-dimensional sphere S^{2n+1}. Consider S^{2n+1} as the unit sphere in $\mathbf{C}^{n+1}$

$$S^{2n+1} = \{z = (z_0, \ldots, z_n) \in \mathbf{C}^{n+1} \mid \|z\| = 1\}.$$

Let $\theta_0, \theta_1, \ldots, \theta_n$ be nonzero real numbers. Then set

(1) $$tz = \left(e^{i\theta_0 t} z_0, e^{i\theta_1 t} z_1, \ldots, e^{i\theta_n t} z_n\right).$$

We obtain, in this manner, some dynamical system in S^{2n+1} without stationary points that we shall denote by $S^{2n+1}(\theta_0, \ldots, \theta_n)$. Our aim is to prove that there is no orbital morphism

$$f: S^{2n+1}(\theta_0, \ldots, \theta_n) \longrightarrow S^{2n-1}(\theta'_0, \ldots, \theta'_{n-1}).$$

We shall do that in the case of rationally independent $\theta_0, \theta_1, \ldots, \theta_n$ (and arbitrary θ'_i). The same proof works with some complications in the general case of arbitrary θ_i.

The reals $\theta_0, \theta_1, \ldots, \theta_n$ are called *rationally independent*, if the equality

$$\alpha_0\theta_0 + \cdots + \alpha_n\theta_n = 0$$

for some rational $\alpha_i \in \mathbf{Q}$ implies $\alpha_0 = \cdots = \alpha_n = 0$.

Theorem. *Let $\theta_0, \ldots, \theta_n$ be rationally independent real numbers. Then there is no orbital morphism*

$$f: S^{2n+1}(\theta_0, \ldots, \theta_n) \longrightarrow S^{2n-1}(\theta_0', \ldots, \theta_n')$$

where θ_i' are arbitrary.

We shall start with some remarks concerning the system $S^{2n+1}(\theta_0, \ldots, \theta_n)$. The following one is a well-known fact that may be found, for example, in [5].

Lemma 1. *Let $T^{n+1} = (S^1)^{n+1}$ be the $(n+1)$-dimensional torus together with the $\mathbb{R}$-action in it defined by (1) with rationally independent $\theta_0, \theta_1, \ldots, \theta_n$. Then the trajectory of each point z is dense in T^{n+1}:*

$$\overline{\text{traj}\, z} = T^{n+1}.$$

Consider now the n-dimensional simplex Δ^n as

$$\Delta^n = \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \mid \sum a_i = 1,\ a_i \geqq 0\}$$

and define a map $p : S^{2n+1}(\theta_0, \ldots, \theta_n) \to \Delta^n$ as follows

$$p(z) = (\|z_0\|^2, \ldots, \|z_n\|^2).$$

Evidently, the co-image of some $(a_0, \ldots, a_n) \in \Delta^n$ is the torus $\{z \in S^{2n+1} \mid \|z_i\|^2 = a_i\}$ of dimension k, where k is the number of the nonzero a_i minus 1.

Consider also the factorspace $S^{2n+1}/_{\{\overline{\text{traj}\, z}\}}$ obtained by the identification of the points of each torus $\overline{\text{traj}\, z}$ (see lemma 1). It is clear now, that this factor is homeomorphic to Δ^n

$$S^{2n+1}/_{\{\overline{\text{traj}\, z}\}} \approx \Delta^n.$$

Really, for some $(a_0, \ldots, a_n) \in \Delta^n$ the co-image $p^{-1}(a_0, \ldots, a_n)$ is a torus T^k, but $\overline{\text{traj}\, z} = T^k$ for any $z \in T^k$ as following by lemma 1, therefore we may identify $\overline{\text{traj}\, z}$ with $(a_0, \ldots, a_n) \in \Delta^n$. The simplex Δ^n may be considered as the base of a (ramified) bundle with projection $p: S^{2n+1} \to \Delta^n$ and fiberes T^k, $k = 1, \ldots, n+1$.

Before passing to the proof of the theorem, we need two lemmas.

The homologies, we shall make use of hereafter, are Čech ones with integral coefficients. Their properties as well as some other elementary algebraic topology facts may be found, for example, in [3].

A map $f : X \to Y$ is called *trivial in dimension n*, if the induced homomorphism $f_* : H_n(X) \to H_n(Y)$ is trivial: $f_* \equiv 0$.

Lemma 2. *Let X be a compact space and ω be a finite open covering of X, such that the canonical projection $\pi: X \to N_\omega$ into the nerve of ω is nontrivial in dimension n. Suppose $f: X \xrightarrow{\text{on}} Y$ is a map of X onto Y trivial in dimension n. Then for some $y_0 \in Y$ the set $f^{-1}(y_0)$ is not contained in any element of the covering ω.*

P r o o f. Suppose the contrary, i.e. that for any $y \in Y$ the set $f^{-1}(y)$ is contained in some element U_y of ω. Let V_y be an open neighbourhood of y such that $f^{-1}(V_y) \subset U_y$. The covering $\{V_y \mid y \in Y\}$ has a finite subcovering α. Clearly, the simplicial complexes N_α and $N_{f^{-1}(\alpha)}$ are isomorphic (here $f^{-1}(\alpha) = \{f^{-1}(V) \mid V \in \alpha\}$). Since the covering $f^{-1}(\alpha)$ is inscribed in ω, there exists a simplicial map $\lambda: N_{f^{-1}(\alpha)} \to N_\omega$. Consider the diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{\ f\ } & Y \\ & \swarrow^{\pi} & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ N_\omega & \xleftarrow{\ \lambda\ } & N_{f^{-1}(\alpha)} & \overset{\rho}{\approx} & N_\alpha \end{array}$$

where π_1 and π_2 are the canonical projections and ρ is the natural isomorphism. The maps π and $\lambda\pi_1$ are homotopic, since for every $x \in X$ the points $\pi(x)$ and $\lambda\pi_1(x)$ are contained in one and the same simplex. For the same reason, $\rho\pi_1$ and $\pi_2 f$ are also homotopic. Then the diagram

$$\begin{array}{ccccc} & & H_n(X) & \xrightarrow{\ f_*\ } & H_n(Y) \\ & \swarrow^{\pi_*} & \downarrow{\scriptstyle \pi_{1*}} & & \downarrow{\scriptstyle \pi_{2*}} \\ H_n(N_\omega) & \xleftarrow{\ \lambda_*\ } & H_n(N_{f^{-1}(\alpha)}) & \overset{\rho_*}{\approx} & H_n(N_\alpha) \end{array}$$

is commutative. Hence $\pi_* = \lambda_* \rho_*^{-1} \pi_{2*} f_* \equiv 0$, which contradicts the condition.

Lemma 3. *Let the n-simplex Δ^n be the union of n open subsets $\Delta^n = \bigcup_{j=1}^{n} V_j$. Then some V_j has a component K intersecting all $n-1$ dimensional faces of Δ^n.*

P r o o f. Obviously, we may suppose each V_j having a finite number of components (if not, then take their ε-neighbourhoods $O_\varepsilon V_j$ for ε sufficiently small). Consider the covering of Δ^n

$$\gamma = \{K \mid K \text{ is a component of some } V_j\}.$$

It is clear, that $\operatorname{ord}\gamma \leq n$, so the nerve N_γ is a $n-1$ dimensional polyhedron. Consider the canonical projection $\pi_\gamma : \Delta^n \to N_\gamma$, then the restriction $\pi_\gamma \restriction_{\partial\Delta^n} : \partial\Delta^n \to N_\gamma$ is homotopic to a constant and therefore is trivial in dimension $n-1$ ($\partial\Delta^n$ denotes the boundary of Δ^n).

Consider, on the other hand, the covering $\omega = \{U_0, \dots, U_n\}$ of $\partial\Delta^n$ defined by $U_i = \partial\Delta^n \setminus \Delta_i^{n-1}$ where Δ_i^{n-1} is the (closed) $n-1$ dimensional face of Δ^n opposite to the vertex a_i. It is clear, that $\bigcap U_i = \varnothing$ and the projection $\pi_\omega : \partial\Delta^n \to N_\omega$ is nontrivial in dimension $n-1$. (Really, N_ω may be identified with $\partial\Delta^n$ and π_ω with the identity.) Set $f = \pi_\gamma \restriction_{\partial\Delta^n}$. Then according to lemma 2 there exists $x \in N_\gamma$ such that $f^{-1}(x)$ is not contained in any $\partial\Delta^n \setminus \Delta_i^{n-1}$, which means that $f^{-1}(x)$ intersects all faces Δ_i^{n-1}. But $f^{-1}(x)$ lies in some element of the covering γ (since f is a restriction of the canonical projection π_γ). Hence, by the definition of γ the set $f^{-1}(x)$ lies in the component K of some V_j. Then K intersects all the $n-1$ dimensional faces of Δ^n.

Recalling the notes before lemma 2, let us now pass to the proof of the theorem.

P r o o f o f t h e t h e o r e m. Suppose the contrary, i.e. that there is an orbital morphism

$$f : S^{2n+1}(\theta_0, \dots, \theta_n) \longrightarrow S^{2n-1}(\theta'_0, \dots, \theta'_{n-1}).$$

Consider the projection $q : S^{2n-1} \to \Delta^{n-1}$ defined with

$$q(z) = (\|z_0\|^2, \dots, \|z_{n-1}\|^2).$$

We have

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{f} & S^{2n-1} \\ \downarrow p & & \downarrow q \\ \Delta^n & & \Delta^{n-1} \end{array}$$

Let $\Delta^{n-1} = \bigcup_{i=0}^{n-1} W_i$ be the union of n open W_i such that W_i does not intersect the $(n-1)$-face opposite to the vertex a_i. Consider the open sets $f^{-1}q^{-1}(W_i)$ which cover S^{2n+1}. We shall prove first that for every i there exists an orbital morphism

$$\varphi_i : f^{-1}q^{-1}(W_i) \longrightarrow S^1.$$

Really, if $z \in f^{-1}q^{-1}(W_i)$, then the i-th coordinate of $f(z)$ is nonzero, and we put $\varphi_i = \pi_i f$ where $\pi_i : q^{-1}(W_i) \to S^1$ is the projection $\pi_i(z) = z_i/\|z_i\|$. Then evidently φ_i is an orbital morphism (as a composition of orbital morphisms)

$$\varphi_i : f^{-1}q^{-1}(W_i) \longrightarrow S^1(\theta_i').$$

Consider furthermore the open sets $V_i = pf^{-1}q^{-1}(W_i)$, which form an open covering of Δ^n. According to lemma 3, some V_i has a component K intersecting all the $(n-1)$-faces of Δ^n. Let $a \in K$ be an arbitrary point of K, then $p^{-1}(a)$ is a torus in $f^{-1}q^{-1}(W_i)$; we shall denote it by T. Our goal is to show, that the homomorphism induced by the embedding

(2) $$j_* : H_1(T) \longrightarrow H_1(f^{-1}q^{-1}(W_i))$$

is trivial. The group $H_1(T)$ is generated by α_i represented by the circle

$$A_i = \{z \in T \mid \|z_i\|^2 = a_i;\ z_j = z_j^0,\ j \neq i\}$$

where z_j^0 are arbitrary constants with $\|z_j^0\|^2 = a_j$. But the component K intersects the face Δ_i^{n-1}, where $z_i = 0$. Since K is arcwise, we can connect a and Δ_i^{n-1} with a continuous curve lying in K. It enables us to obtain a continuous family of circles A_i^t such that $A_i^0 = A_i$, A_i^1 degenerates to a point and all A_i^t lie in $p^{-1}(K) \subset f^{-1}q^{-1}(W_i)$. Therefore, we found in $f^{-1}q^{-1}(W_i)$ a deformation of A_i into a point, which means that $j_*(\alpha_i) = 0$, hence the homomorphism (2) is trivial.

As it was pointed out before, there exists an orbital morphism $\varphi : f^{-1}q^{-1}(W_i) \to S^1$. Then $\varphi|_T : T \to S^1$ is an orbital morphism, as a composition of j and φ. On the other hand, the homomorphism $(\varphi|_T)_*$ is trivial in dimension 1 (as well as in each other dimension), since $(\varphi|_T)_* = \varphi_* j_* \equiv 0$. But then $\varphi|_T$ induces a trivial homomorphism between the fundamental groups of T and S^1 and therefore $\varphi|_T$ can be lifted to $\tilde{\varphi} : T \to \mathbb{R}$ so that

$$\exp \tilde{\varphi} = \varphi|_T$$

where $\exp(x) = e^{ix}$ is the exponential map (see [3] for details). Since $\varphi|_T$ is an orbital morphism, $\tilde{\varphi}$ is monotone over each trajectory traj z. Fix some $z \in T$. The set $\tilde{\varphi}(T)$ is compact in $\mathbb{R}$, so the following supremum exists:

$$m_0 = \sup \tilde{\varphi}(\text{traj}\, z).$$

But then the map $\varphi|_{\text{traj}\, z} = \exp \tilde{\varphi}|_{\text{traj}\, z}$ is not a covering over the point $\exp(m_0)$, which contradicts the definition of orbital morphism.

REFERENCES

1. Fadell, E. R., P. H. Rabinovitz. Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. *Invent. Math.*, 45, 1978, 139–174.
2. Munkholm, H. J. Borsuk-Ulam type theorems for proper $\mathbb{Z}_p$-actions on (mod p homology) n-spheres. *Math. Scand.*, 24, 1969, 167–185.
3. Spanier, E. H. Algebraic topology. New York – San Francisco – St. Louis – Toronto – London – Sydney, 1966.
4. Yang, C. T. On theorems of Borsuk–Ulam, Kakutani–Yamabe–Yujobo and Dyson. I. *Ann. Math.*, 60, 1954, 262–282.
5. Корнфельд, И. П., Я. Г. Синай, С. В. Фомин. Эргодическая теория. М., „Наука", 1980.

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