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METRIC CONNECTIONS ON ALMOST COMPLEX NORDEN METRIC MANIFOLDS

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Two families of metric connections on almost complex Norden metric manifolds are introduced and studied. These connections are constructed by means of the two Lie 1-forms naturally existing on the manifolds. Invariant tensors under the transformations of the Levi-Civita connection into the introduced metric connections are obtained.

Keywords: Metric connection, complex manifold, Norden metric.

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1. INTRODUCTION

Linear connections with non-vanishing torsion tensor are widely studied. In particular, two types of such connections are well known – semi-symmetric and quarter-symmetric, introduced by Friedmann and Schouten in [1] and Golab in [5], respectively. The torsion tensors of such connections are constructed by means of a 1-form and a tensor of type (1,1).

On the other hand, an object of extensive research on pseudo Riemannian manifolds are linear connections which preserve the metric tensor by covariant differentiation called metric connections. It is well known that the Levi-Civita connection of the pseudo Riemannian metric is the unique linear connection which is simultaneously metric and symmetric (i.e. torsion-free). Metric connections with non-zero torsion tensor are introduced by Hayden in [7].

By combining these both ideas, numerous authors studied semi-symmetric metric and quarter-symmetric metric connections, e.g. [8], [10], [12], [13], [17], [18]. Tripathi [16] generalized the concept of various metric and non-metric connections.

In the present work, we aim to study metric connections on almost complex Norden metric (B-metric) manifolds. These manifolds are introduced by Norden in [11] and studied for the first time in [6] under the name generalized B-manifolds. Since on such manifolds, there exist two Lie 1-forms, they can be used to generate metric connections with torsion tensors of special types, e.g. semi-symmetric, quarter-symmetric or others.

The paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we construct a 4-parametric family of metric connection on almost complex Norden metric manifolds with non-vanishing Lie 1-forms. These connections are composed by two semi-symmetric and two quarter-symmetric metric connections. We obtain necessary and sufficient conditions for the introduced connections to be invariant under the transformation of the Levi-Civita connections of the Norden metrics on a class complex Norden metric manifolds. Also, we consider tensors which under certain conditions are invariant under the transformation of the Levi-Civita connection into the constructed metric connections. In Section 4 we introduce a 2-parametric family of metric connections which are neither semi-symmetric nor quarter-symmetric and study some of their curvature properties.

2. PRELIMINARIES

A triple (M, J, g) is called an almost complex Norden metric (B-metric) manifold [6, 11] if M is a differentiable even-dimensional manifold ($\dim M = 2n$), J is an endomorphism of the tangent bundle TM, and g is a pseudo Riemannian metric on M, compatible with J, such that the following relations are satisfied:

$$J^2x = -x, \qquad g(Jx, Jy) = -g(x, y). \tag{2.1}$$

Here and further on, by x, y, z, u we denote differentiable vector fields on M, i.e. elements in the Lie algebra $\mathfrak{X}(M)$, or vectors in the tangent space T_pM at an arbitrary point $p \in M$.

Equalities (2.1) imply $g(Jx, y) = g(x, Jy)$. Hence the tensor $\widetilde{g}$ defined by

$$\widetilde{g}(x, y) = g(x, Jy) \tag{2.2}$$

is symmetric and is known as the associated (twin) metric of g. This tensor also satisfies the Norden metric property, i.e. $\widetilde{g}(Jx, Jy) = -\widetilde{g}(x, y)$. Both metrics, g and $\widetilde{g}$, are necessarily of neutral signature (n, n).

The fundamental tensor F of type (0,3) is defined by

$$F(x, y, z) = (\nabla_x\widetilde{g})(y, z) = g((\nabla_xJ)y, z), \tag{2.3}$$

where ∇ is the Levi-Civita connection of g. This tensor has the following properties $F(x,y,z) = F(x,z,y) = F(x,Jy,Jz)$.

Let $\{e_i\}$ $(i = 1,2,...,2n)$ be an arbitrary basis of T_pM. The components of the inverse matrix of g with respect to this basis are denoted by g^{ij}. The Lie 1-forms θ and θ^* associated with F are defined by:

$$\theta(x) = g^{ij}F(e_i,e_j,x), \qquad \theta^*(x) = \theta(Jx). \tag{2.4}$$

We denote by Ω the Lie vector corresponding to θ, i.e. $\theta(x) = g(x,\Omega)$. Then, the vector $\Omega^* = J\Omega$ corresponds to θ^*.

A classification of the almost complex Norden metric manifolds with respect to the properties of F is introduced by Ganchev and Borisov in [2]. This classification consists of 8 classes: 3 basic classes $\mathcal{W}_i$ $(i = 1,2,3)$, their pairwise direct sums $\mathcal{W}_i \oplus \mathcal{W}_j$, the widest class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ and the class of the Kähler Norden metric manifolds $\mathcal{W}_0$ which is contained in all of the other classes and is characterized by $F = 0$ (i.e. $\nabla J = 0$). Two of the basic classes ($\mathcal{W}_1$ and $\mathcal{W}_2$) are integrable, i.e. with a vanishing Nijenhuis tensor (complex Norden metric manifolds). One of the integrable classes is said to be a main class because its characteristic condition is an explicit expression of F by means of the other structural tensors. This class is denoted by $\mathcal{W}_1$ and is defined by the condition

$$\begin{aligned} F(x,y,z) = \tfrac{1}{2n}[&g(x,y)\theta(z) + g(x,Jy)\theta(Jz) \\ &+ g(x,z)\theta(y) + g(x,Jz)\theta(Jy)]. \end{aligned} \tag{2.5}$$

A $\mathcal{W}_1$-manifold with closed Lie 1-forms θ and θ^* is called a conformal Kähler Norden metric manifold. The manifolds in this class are conformally equivalent to Kähler Norden metric manifolds by the usual conformal transformation of the metric g [3].

Since ∇ is symmetric, the 1-forms θ and θ^* are closed if and only if $(\nabla_x\theta)y = (\nabla_y\theta)x$ and $(\nabla_x\theta^*)y = (\nabla_y\theta^*)x$. By (2.3) and (2.4) it it easy to compute that $(\nabla_x\theta^*)y = (\nabla_x\theta)Jy + F(x,y,\Omega)$. Then, because of (2.5), a necessary and sufficient condition for the Lie 1-forms θ and θ^* to be closed on a $\mathcal{W}_1$-manifold is given by

$$(\nabla_x\theta)y = (\nabla_y\theta)x, \qquad (\nabla_x\theta)Jy = (\nabla_y\theta)Jx. \tag{2.6}$$

The Lie 1-forms θ and θ^* vanish on the manifolds in the classes $\mathcal{W}_0$, $\mathcal{W}_2$, $\mathcal{W}_3$ [2]. Hence, the widest class with zero Lie 1-forms is $\mathcal{W}_2 \oplus \mathcal{W}_3$.

3. SEMI-SYMMETRIC AND QUARTER-SYMMETRIC METRIC CONNECTIONS ON ALMOST COMPLEX NORDEN METRIC MANIFOLDS

Let us recall some basic definitions.

Definition 1. A linear connection ∇' on an almost complex Norden metric manifold (M,J,g) is called: (i) *almost complex* if $\nabla' J = 0$ (if the manifold is complex, a connection with this property is called *complex*); (ii) *metric* if $\nabla' g = 0$; (iii) *natural* if $\nabla' J = \nabla' g = 0$.

Because of (2.2), it is easy to prove that the defining condition in (iii) is equivalent to $\nabla' J = \nabla' \widetilde{g} = 0$ and also to $\nabla' g = \nabla' \widetilde{g} = 0$.

Definition 2. A non-symmetric linear connection ∇' with torsion tensor T is called: (i) *semi-symmetric* if $T(x,y) = \pi(y)x - \pi(x)y$; (ii) *quarter-symmetric* if $T(x,y) = \pi(y)\varphi x - \pi(x)\varphi y$, where π is a 1-form and φ is a tensor of type (1,1). In particular, if $\varphi = \mathrm{id}$, i.e. $\varphi x = x$ for all $x \in \mathfrak{X}(M)$, then a quarter-symmetric connection reduces to a semi-symmetric connection.

Further in this section, we study metric connections with torsion tensors of special types, namely semi-symmetric and quarter-symmetric ones.

Let us consider a linear connection ∇' with deformation tensor Q defined by

$$\nabla'_x y = \nabla_x y + Q(x,y). \tag{3.1}$$

Since $\nabla g = 0$, we compute $(\nabla'_x g)(y,z) = -Q(x,y,z) - Q(x,z,y)$, where $Q(x,y,z) = g(Q(x,y),z)$ is the deformation (0,3)-type tensor. Then, a necessary and sufficient condition for the connection ∇' to be metric is given by

$$Q(x,y,z) = -Q(x,z,y). \tag{3.2}$$

Because ∇ is symmetric, i.e. $\nabla_x y - \nabla_y x = [x,y]$, the torsion tensor T of ∇' is given by $T(x,y) = \nabla'_x y - \nabla'_y x - [x,y] = Q(x,y) - Q(y,x)$. Then, its corresponding (0,3)-type has the form $T(x,y,z) = g(T(x,y),z) = Q(x,y,z) - Q(y,x,z)$.

It is known that ∇' is a metric connection if and only if the following relation between its deformation tensor Q and its torsion tensor T exists

$$Q(x,y,z) = \tfrac{1}{2}\left[T(x,y,z) + T(z,y,x) - T(x,z,y)\right]. \tag{3.3}$$

Let ∇' be a semi-symmetric metric connection generated by the 1-form π, i.e. its torsion tensor is given by $T(x,y,z) = \pi(y)g(x,z) - \pi(x)g(y,z)$. Then, by applying the last equality to (3.3) we obtain the form of the deformation tensor Q of such a connection as follows [17]

$$Q(x,y,z) = \pi(y)g(x,z) - \pi(z)g(x,y). \tag{3.4}$$

We remark that the last formula is valid on an arbitrary pseudo Riemannian manifold with metric tensor g and 1-form π.

Next, we consider the case of a quarter-symmetric metric connection generated by the 1-form π and the almost complex structure J on an almost complex Norden metric manifold. Because of (2.1) and Definition 2, the torsion tensor T of such a connection is given by $T(x,y,z) = \pi(y)g(x,Jz) - \pi(x)g(y,Jz)$. Then, by (3.3) we obtain the form of the deformation tensor of a quarter-symmetric metric connection as follows

$$Q(x,y,z) = \pi(y)g(x,Jz) - \pi(z)g(x,Jy). \tag{3.5}$$

Let us remark that the last formula differs in the case of an almost Hermitian manifold because on such manifolds the tensor $\Phi(x,y)=g(Jx,y)$ is a 2-form.

Since on almost complex Norden metric manifolds which are not in the class $\mathcal{W}_2\oplus\mathcal{W}_3$, there exist two non-vanishing Lie 1-forms θ and θ^*, they can be used to generate semi-symmetric and quarter-symmetric metric connections. Four metric connections of these types can be constructed – two generated by θ (one semi-symmetric and one quarter-symmetric) and two of the same types generated by θ^*. Then, if we compose an arbitrary linear combination of the expressions in the right-hand sides of (3.4) and (3.5), and by doing so replace π with θ and θ^*, we obtain a 4-parametric family of metric connections on such manifolds. Thus, we proved the following statement.

Theorem 1. *On an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^*, there exists a 4-parametric family ($\lambda_i\in\mathbb{R}$, $i=1,2,3,4$) of metric connections ∇' defined by (3.1) with deformation tensor Q given by*

$$\begin{aligned}Q(x,y)=&\lambda_1[\theta(y)x-g(x,y)\Omega]+\lambda_2[\theta(Jy)x-g(x,y)J\Omega]\\&+\lambda_3[\theta(y)Jx-g(x,Jy)\Omega]+\lambda_4[\theta(Jy)Jx-g(x,Jy)J\Omega].\end{aligned}\tag{3.6}$$

Let us have a more detailed look at the four metric connections which give rise to the family ∇' and consider if they can be natural.

If $\lambda_1\neq 0$, $\lambda_i=0$ ($i=2,3,4$), we obtain a 1-parametric family of semi-symmetric metric connections ∇^1 generated by θ. These connections cannot be complex for any values of λ_1.

If $\lambda_2\neq 0$, $\lambda_i=0$ ($i=1,3,4$), we obtain a 1-parametric family of semi-symmetric metric connections ∇^2 generated by θ^*. Because of formula (2.5), the connections ∇^2 are complex and hence natural on $\mathcal{W}_1$-manifolds if and only if $\lambda_2=\frac{1}{2n}$.

If $\lambda_3\neq 0$, $\lambda_i=0$ ($i=1,2,4$), we obtain a 1-parametric family of quarter-symmetric metric connections ∇^3 generated by θ. Because of (2.5), these connections are complex and hence natural on $\mathcal{W}_1$-manifolds if and only if $\lambda_3=-\frac{1}{2n}$.

If $\lambda_4\neq 0$, $\lambda_i=0$ ($i=1,2,3$), we obtain a 1-parametric family of quarter-symmetric metric connections ∇^4 generated by θ^*. The connections ∇^4 cannot be complex for any values of λ_4.

Let us remark that the connections ∇^2 and ∇^3 for $\lambda_2=\frac{1}{2n}$ and $\lambda_3=-\frac{1}{2n}$, respectively, i.e. $\nabla^{2n}_x y=\nabla_x y+\frac{1}{2n}[\theta(Jy)x-g(x,y)J\Omega]$ and $\nabla^{3n}_x y=\nabla_x y-\frac{1}{2n}[\theta(y)Jx-g(x,Jy)\Omega]$, are part of a 2-parametric family of natural connections ∇^n introduced and studied on $\mathcal{W}_1$-manifolds in [15]. These connections are defined by

$$\begin{aligned}\nabla^n_x y=&\nabla_x y+\tfrac{1-2p}{2n}\left[\theta(Jy)x-g(x,y)J\Omega\right]+\tfrac{1}{n}\left\{p\left[g(x,Jy)\Omega-\theta(y)Jx\right]\right.\\&\left.+q\left[g(x,y)\Omega-g(x,Jy)J\Omega-\theta(y)x+\theta(Jy)Jx\right]\right\},\quad p,q\in\mathbb{R}.\end{aligned}\tag{3.7}$$

The connections in question are obtained from (3.7) for $p=q=0$ and $p=\frac{1}{2}$, $q=0$, respectively. In the same work, we discussed that the B-connection ∇^c

[3, 4], known as Lichnerowicz first canonical connection in the Hermitian geometry, is also a member of the family (3.7) and is obtained for $p = \frac{1}{4}$, $q = 0$. Then, the B-connection is the average connection of ∇^{2n} and ∇^{3n}, i.e. $\nabla^c = \frac{1}{2}(\nabla^{2n} + \nabla^{3n})$.

3.1. INVARIANT METRIC CONNECTIONS UNDER THE TRANSFORMATION OF THE LEVI-CIVITA CONNECTIONS GENERATED BY THE NORDEN METRICS

Let ∇ and $\widetilde{\nabla}$ be the Levi-Civita connections of the metric tensors g and $\widetilde{g}$, respectively. The transformation of these connections $\nabla \to \widetilde{\nabla}$ is studied on $\mathcal{W}_1$-manifolds in [14]. In the same work, the following relations are obtained:

$$\widetilde{\nabla}_x y = \nabla_x y + \tfrac{1}{2n}[g(x, Jy)\Omega - g(x, y)J\Omega], \qquad \widetilde{\theta} = \theta, \quad \widetilde{\Omega} = -J\Omega, \tag{3.8}$$

where $\widetilde{\theta}$ and $\widetilde{\Omega}$ are the Lie 1-form and its corresponding vector defined analogously to (2.4). The general case of an almost complex Norden metric manifold is studied in [9], where it is proved that all classes of these manifolds are invariant under the considered transformation.

Then, on a $\mathcal{W}_1$-manifold $(M, J, \widetilde{g})$ corresponding to a $\mathcal{W}_1$-manifold (M, J, g) one can construct a 4-parametric family of metric connections $\widetilde{\nabla}'$ (i.e. with the property $\widetilde{\nabla}'\widetilde{g} = 0$) defined analogously to ∇' by replacing ∇, g and θ with $\widetilde{\nabla}$, $\widetilde{g}$ and $\widetilde{\theta}$, respectively, in (3.1) and (3.6).

The following theorem is valid.

Theorem 2. *The metric connections ∇' defined by (3.1) and (3.6) are invariant under the transformation $\nabla \to \widetilde{\nabla}$ on $\mathcal{W}_1$-manifolds if and only if they coincide with the natural connections ∇^n defined by (3.7).*

Proof. Let ∇' be invariant under the transformation $\nabla \to \widetilde{\nabla}$, i.e. $\widetilde{\nabla}' = \nabla'$. Then, both metric tensors g and $\widetilde{g}$ are parallel with respect to ∇' and $\widetilde{\nabla}'$. Because of (2.2), this means that the connections ∇' and $\widetilde{\nabla}'$ are metric and complex and hence natural. By (3.1), (3.6) and (2.3) we compute

$$\begin{aligned}(\nabla'_x J)y =&(\nabla_x J)y + Q(x, Jy) - JQ(x, y)\\ =&(\lambda_1 + \lambda_4)[g(x, y)J\Omega - g(x, Jy)\Omega + \theta(Jy)x - \theta(y)Jx]\\ &+ (\lambda_3 - \lambda_2 + \frac{1}{2n})[g(x, y)\Omega + g(x, Jy)J\Omega + \theta(y)x + \theta(Jy)Jx].\end{aligned}$$

The last equality implies that $\nabla' J = 0$ if and only if $\lambda_2 = \lambda_3 + \frac{1}{2n}$ and $\lambda_1 = -\lambda_4$. In this case, by substituting $\lambda_3 = -\frac{p}{n}$ and $\lambda_4 = \frac{q}{n}$ in (3.6) we establish that ∇' coincide with ∇^n.

To prove the reverse statement, by (3.7) and (3.8) we obtain $\widetilde{\nabla}^n$ and verify that $\widetilde{\nabla}^n = \nabla^n$, i.e. ∇^n and hence ∇' are invariant under the transformation $\nabla \to \widetilde{\nabla}$. □

Let us consider the covariant derivatives of θ and θ^* with respect to the metric connections ∇'. Because of (3.1), we compute: $(\nabla'_x\theta)y = (\nabla_x\theta)y - Q(x,y,\Omega)$ and $(\nabla'_x\theta^*)y = (\nabla_x\theta^*)y - Q(x,y,J\Omega)$. Thus, we have:

$$\begin{aligned}(\nabla'_x\theta)y - (\nabla'_y\theta)x &= (\nabla_x\theta)y - (\nabla_y\theta)x + Q(y,x,\Omega) - Q(x,y,\Omega),\\ (\nabla'_x\theta^*)y - (\nabla'_y\theta^*)x &= (\nabla_x\theta^*)y - (\nabla_y\theta^*)x + Q(y,x,J\Omega) - Q(x,y,J\Omega).\end{aligned} \tag{3.9}$$

We introduce the following (0,2)-type tensors:

$$S(x,y) = (\nabla_x\theta)y - (\nabla_y\theta)x, \qquad S_*(x,y) = (\nabla_x\theta^*)y - (\nabla_y\theta^*)x. \tag{3.10}$$

By replacing ∇ with ∇' in (3.10), we obtain analogous tensors to S and S_* with respect to the connections ∇' which we denote by S' and S'_*. Then, equalities (3.9) yield that $S = S'$ and $S_* = S'_*$ if and only if $Q(x,y,\Omega) = Q(y,x,\Omega)$ and $Q(x,y,J\Omega) = Q(y,x,J\Omega)$, respectively. Having in mind (3.6), the last two conditions hold if and only if $\lambda_3 = \lambda_2$ and $\lambda_4 = -\lambda_1$, respectively. Hence, we proved the following result.

Theorem 3. *The tensors S and S_* given by (3.10) are invariant under the transformation of the Levi-Civita connection ∇ into the 4-parametric family of metric connections ∇' defined by (3.1) and (3.6) on an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^* if and only if the parameters satisfy the following conditions: $\lambda_3 = \lambda_2$ and $\lambda_4 = -\lambda_1$.*

Under the assumptions of the last theorem we obtain

Corollary 1. *The invariant tensors S and S_* vanish on a $\mathcal{W}_1$-manifold if and only if both Lie 1-forms θ and θ^* are closed, i.e. the manifold is conformal Kählerian.*

Further, let us consider another tensor of type (0,2) constructed by the covariant derivatives of θ and θ^* which is defined by

$$P(x,y) = (\nabla_x\theta)Jy + (\nabla_x\theta^*)y. \tag{3.11}$$

The analogous tensor P' with respect to the metric connections ∇' is given by $P'(x,y) = (\nabla'_x\theta)Jy + (\nabla'_x\theta^*)y = (\nabla_x\theta)Jy + (\nabla_x\theta^*)y - Q(x,y,J\Omega) - Q(x,Jy,\Omega)$. Hence, $P'(x,y) = P(x,y) - Q(x,Jy,\Omega) - Q(x,y,J\Omega)$, and therefore $P' = P$ if and only if

$$Q(x,Jy,\Omega) = -Q(x,y,J\Omega). \tag{3.12}$$

By applying (3.6) to the last equality we prove the following

Theorem 4. *The tensor P is invariant under the transformation of the Levi-Civita connection ∇ into the 4-parametric family of metric connections ∇' defined by (3.1) and (3.6) on an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^* if and only if the parameters satisfy the following conditions: $\lambda_3 = -\lambda_2$ and $\lambda_4 = \lambda_1$.*

If the assumptions of the last theorem are satisfied, by (2.5) we get

Corollary 2. *The invariant tensor P defined by (3.11) vanishes on a $\mathcal{W}_1$-manifold if and only if*

$$(\nabla_x\theta)y = \frac{1}{4n}\{g(x,Jy)\theta(\Omega) - g(x,y)\theta(J\Omega) + \theta(x)\theta(Jy) - \theta(Jx)\theta(y)\}. \qquad (3.13)$$

Equalities (2.6) and (3.13) yield that if P vanishes on a $\mathcal{W}_1$-manifold, the Lie 1-form θ^* is closed but θ is not closed. Also, if $P = 0$ the following relations are valid: $\operatorname{div}\Omega = -\frac{\theta(J\Omega)}{2}$ and $\operatorname{div}(J\Omega) = \frac{\theta(\Omega)}{2}$, where $\operatorname{div}\Omega = g^{ij}(\nabla_{e_i}\theta)e_j$ and $\operatorname{div}(J\Omega) = g^{ij}(\nabla_{e_i}\theta^*)e_j$.

4. OTHER TYPES OF METRIC CONNECTIONS CONSTRUCTED BY THE LIE 1-FORMS

In this section, by means of the Lie 1-forms θ and θ^* and their corresponding Lie vectors Ω and $J\Omega$, we construct a 2-parametric family of metric connection which are neither semi-symmetric, nor quarter-symmetric.

On an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^*, let us consider a family of metric connections $\widehat{\nabla}$ defined by (3.1) with torsion tensors T given by

$$T(x,y) = [\theta(x)\theta(Jy) - \theta(Jx)\theta(y)][s\Omega + tJ\Omega], \quad s,t \in \mathbb{R}. \qquad (4.1)$$

Then, by (3.3) and (4.1) we obtain the form of the deformation tensor Q of the metric connections $\widehat{\nabla}$. Thus, we establish the truthfulness of the following theorem.

Theorem 5. *On an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^*, there exists a 2-parametric family ($s,t \in \mathbb{R}$) of metric connections $\widehat{\nabla}$ defined by*

$$\widehat{\nabla}_x y = \nabla_x y + Q(x,y), \quad Q(x,y) = [s\theta(x) + t\theta(Jx)][\theta(Jy)\Omega - \theta(y)J\Omega]. \qquad (4.2)$$

Let R be the curvature tensor of the Levi-Civita connection ∇, i.e. $R(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z$. The corresponding tensor of type (0,4) is denoted by the same letter and is defined by $R(x,y,z,u) = g(R(x,y)z,u)$. Analogously, let us denote by $\widehat{R}$ the curvature tensor of $\widehat{\nabla}$ and define its corresponding (0,4)-tensor with respect to g by $\widehat{R}(x,y,z,u) = g(\widehat{R}(x,y)z,u)$. Then, the following relation between R and $\widehat{R}$ is known to be valid: $\widehat{R}(x,y,z,u) = R(x,y,z,u) + (\nabla_x Q)(y,z,u) - (\nabla_y Q)(x,z,u) + Q(x,Q(y,z),u) - Q(y,Q(x,z),u)$. If we apply (4.2) to the last formula and take into account (2.6), we obtain the following

Theorem 6. *On a conformal Kähler Norden metric manifold, the tensors R $\widehat{R}$ of ∇ and $\widehat{\nabla}$, respectively, are related as follows*

$$\widehat{R}(x,y,z,u) = R(x,y,z,u) + [s\theta(y) + t\theta(Jy)][\theta(u)(\nabla_x\theta^*)z - \theta(z)(\nabla_x\theta^*)u$$
$$+\theta^*(z)(\nabla_x\theta)u - \theta^*(u)(\nabla_x\theta)z] - [s\theta(x) + t\theta(Jx)][\theta(u)(\nabla_y\theta^*)z - \theta(z)(\nabla_y\theta^*)u$$
$$+\theta^*(z)(\nabla_y\theta)u - \theta^*(u)(\nabla_y\theta)z].$$

Let us remark that $\widehat{R}(x,y,z,u)$ is anti-symmetric by its first and last pair of arguments, but since it does not satisfy Bianchi's first identity it is not a curvature-like tensor of type (0,4).

Having in mind the definitions of the scalar curvatures τ and $\widehat{\tau}$ of R and $\widehat{R}$, respectively, i.e. $\tau = g^{is}g^{jk}R(e_i,e_j,e_k,e_s)$ and $\widehat{\tau} = g^{is}g^{jk}\widehat{R}(e_i,e_j,e_k,e_s)$, from Theorem 6 we obtain $\widehat{\tau} = \tau + \frac{2s}{n}[\theta(\Omega)^2 + \theta(J\Omega)^2] + 2\mathrm{div}\,\Omega\,[s\theta(J\Omega) - t\theta(\Omega)] - 2\mathrm{div}\,(J\Omega)\,[s\theta(\Omega) + t\theta(J\Omega)]$.

Let us consider the transformation of the Levi-Civita connection ∇ into the metric connections $\widehat{\nabla}$ and the tensor P defined in the previous section by (3.11). Having in mind (4.2), we establish that the deformation tensor Q of $\widehat{\nabla}$ satisfies property (3.12) for all $s,t \in \mathbb{R}$. Thus, we proved

Theorem 7. *The tensor P given by (3.11) is invariant under the transformation of the Levi-Civita connection ∇ into the 2-parametric family of metric connections $\widehat{\nabla}$ defined by (4.2) on an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^*.*

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