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ESTIMATION OF THE ERROR OF RUNGE–KUTTA'S METHOD IN MULTIVARIATE CASE

Hussain Ali Al–Juboury

Хусеин Али Ал–Джубури. ОЦЕНКА ПОГРЕШНОСТЕЙ МНОГОМЕРНОГО МЕТОДА РУНГЕ–КУТТА

В работе получена оценка погрешностей многомерного метода Рунге–Кутта первого и второго порядка через усредненными модулями. Оценка получается без дополнительных ограничениях о гладкости решения. Пользуясь свойствами модулей можно вывести разные порядки сходимости.

Hussain Ali Al–Juboury. ESTIMATION OF THE ERROR OF RUNGE–KUTTA'S METHOD IN MULTIVARIATE CASE

The error estimates are obtained for Runge–Kutta's methods of first and second order in the multidimensional case by means of averaged moduli of smoothness without any additional assumptions on the solutions of the equations. The different orders of convergence can be derived from these estimates using the properties of the moduli of smoothness.

1. INTRODUCTION

In this paper we shall obtain error estimates of the numerical solution for the m-dimensional Cauchy problem of the Runge–Kutta's methods with local error of second and third degree using the averaged modulus of smoothness (which is denoted by $\tau(f;\delta)_{L_p}$, see [4]). All notations and definitions, which are used here are involved in [4].

From the properties of the averaged moduli of smoothness, which are mentioned at the end of this paragraph, together with theorems (1) and (2) below, we can obtain many consequences such as that the classical orders of the error $O(h)$ and $O(h^2)$ are obtained under weaker assumptions on the solutions.

Here we list the main properties of averaged moduli of smoothness (see [1]–[3]):

1) $\tau_k(f;\delta')_{L_p} \leq \tau_k(f;\delta'')_{L_p}$ for $\delta' \leq \delta''$;

2) $\tau_k(f+g;\delta)_{L_p} \leq \tau_k(f;\delta)_{L_p} + \tau_k(g;\delta)_{L_p}$;

3) $\tau_k(f;\delta)_{L_p} \leq 2\tau_{k-1}(f;\frac{k}{k-1}\delta)_{L_p}$;

4) $\tau_k(f;\delta)_{L_p} \leq \delta\tau_{k-1}(f';\frac{k}{k-1}\delta)_{L_p}$;

5) $\tau_k(f;n\delta)_{L_p} \leq (2n)^{k+1}\tau_k(f;\delta)_{L_p}$;

6) $\tau_k(f;\lambda\delta)_{L_p} \leq (2(\lambda+1))^{k+1}\tau_k(f;\delta)_{L_p}, \quad \lambda > 0$;

7) $\tau(f;\delta)_{L_p} \leq \delta\|f'\|_{L_p}$;

8) $\tau(f;\delta)_{L_p} \leq \delta V_a^b f$ (where $V_a^b f$ is the variation of the function f between a and b).

2. RUNGE-KUTTA'S METHODS

We shall mention briefly the result of the one dimensional case. Consider the following ordinary differential equation with the initial value:

$$y' = f(x,y), \quad x \in [0, A],\ A > 0,$$

$$y(0) = y_0,$$

and assume that the right hand side of the equation satisfies a Lipschitz condition with respect to the variable y, i.e.:

$$|f(x,y) - f(x,z)| \leq K\,|y - z|,$$

where K is an absolute constant and $x_i = ih$, $h = A/n$, $i = 0, 1, 2, \ldots, n$. If we apply Euler's method then the following estimate holds:

$$\tilde{y}_{i+1} = \tilde{y}_i + hf(x_i, \tilde{y}_i), \quad \tilde{y}_0 = y_0,$$

$$\max_{0 \leq i \leq n} |y_i - \tilde{y}_i| \leq 2e^{AK}\tau(y';h)_{L_p}$$

(see [4]).

Suppose that we have a system of m ordinary differential equations with initial conditions as follows:

$$(y^1)' = f^1(x, y^1, \ldots, y^m), \quad y^1(0) = y_0^1,$$

$$(y^2)' = f^2(x, y^1, \ldots, y^m), \quad y^2(0) = y_0^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots\cdots\cdots$$

$$(y^m)' = f^m(x, y^1, \ldots, y^m), \quad y^m(0) = y_0^m.$$

We shall need the following generalized Lipschitz condition:

$$|f^m(x, y^1, y^2, \ldots, y^m) - f^m(x, \tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^m)|$$
$$\leq K\left\{|y^1 - \tilde{y}^1| + |y^2 - \tilde{y}^2| + \cdots + |y^m - \tilde{y}^m|\right\}.$$

By Euler's method:

$$\begin{aligned}
\widetilde{y}^1_{i+1} &= \widetilde{y}^1_i + hf^1(x_i, \widetilde{y}^1_i, \widetilde{y}^2_i, \dots, \widetilde{y}^m_i),\\
\widetilde{y}^2_{i+1} &= \widetilde{y}^2_i + hf^2(x_i, \widetilde{y}^1_i, \widetilde{y}^2_i, \dots, \widetilde{y}^m_i),\\
&\dots\dots\dots\dots\dots\dots\\
\widetilde{y}^m_{i+1} &= \widetilde{y}^m_i + hf^m(x_i, \widetilde{y}^1_i, \widetilde{y}^2_i, \dots, \widetilde{y}^m_i),
\end{aligned}$$

estimating the error in the i-th step by means of the error in the $(i-1)$-th step, it follows:

$$\begin{aligned}
&\left|y^1_{i+1} - \widetilde{y}^1_{i+1}\right|\\
&= \left|y^1_{i+1} - \widetilde{y}^1_i - hf^1(x_i, \widetilde{y}^1_i, \dots, \widetilde{y}^m_i) + hf^1(x_i, y^1_i, \dots, y^m_i) - hf^1(x_i, y^1_i, \dots, y^m_i)\right|\\
&\le \left|y^1_{i+1} - \widetilde{y}^1_i + y^1_i - y^1_i - hf^1(x_i, \widetilde{y}^1_i, \dots, \widetilde{y}^m_i) + hf^1(x_i, y^1_i, \dots, y^m_i) - hf^1(x_i, y^1_i, \dots, y^m_i)\right|\\
&\le \left|y^1_{i+1} - y^1_i - h(y^1_i)'\right| + \left|hf^1(x_i, \widetilde{y}^1_i, \dots, \widetilde{y}^m_i) - hf^1(x_i, \widetilde{y}^1_i, \dots, \widetilde{y}^m_i)\right| + \left|y^1_i - \widetilde{y}^1_i\right|,
\end{aligned}$$

and hence

$$\begin{aligned}
&\left|y^1_{i+1} - \widetilde{y}^1_{i+1}\right|\\
&\le h\omega((y^1)', x_{i+1/2}; h) + Kh\left\{\left|y^1_i - \widetilde{y}^1_i\right| + \dots + \left|y^m_i - \widetilde{y}^m_i\right|\right\} + \left|y^1_i - \widetilde{y}^1_i\right|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|y^2_{i+1} - \widetilde{y}^2_{i+1}\right|\\
&\le h\omega((y^2)', x_{i+1/2}; h) + Kh\left\{\left|y^1_i - \widetilde{y}^1_i\right| + \dots + \left|y^m_i - \widetilde{y}^m_i\right|\right\} + \left|y^2_i - \widetilde{y}^2_i\right|,\\
&\dots\dots\dots\dots\dots\dots\\
&\left|y^m_{i+1} - \widetilde{y}^m_{i+1}\right|\\
&\le h\omega((y^m)', x_{i+1/2}; h) + Kh\left\{\left|y^1_i - \widetilde{y}^1_i\right| + \dots + \left|y^m_i - \widetilde{y}^m_i\right|\right\} + \left|y^m_i - \widetilde{y}^m_i\right|.
\end{aligned}$$

Let $\psi_i = \left|y^1_i - \widetilde{y}^1_i\right| + \dots + \left|y^m_i - \widetilde{y}^m_i\right|$, then

$$\psi_{i+1} \le h\left[\omega((y^1)', x_{i+1/2}; h) + \dots + \omega((y^m)', x_{i+1/2}; h)\right] + (1 + mKh)\psi_i,$$

and

$$\psi_i \le h\left[\omega((y^1)', x_{i-1/2}; h) + \dots + \omega((y^m)', x_{i-1/2}; h)\right] + (1 + mKh)\psi_{i-1},$$

therefore

$$\begin{aligned}
\psi_{i+1} \le\ & h(1 + mKh)\left[\omega((y^1)', x_{i-1/2}; h) + \dots + \omega((y^m)', x_{i-1/2}; h)\right]\\
&+ \left[\omega((y^1)', x_{i+1/2}; h) + \dots + \omega((y^m)', x_{i+1/2}; h)\right] + (1 + mKh)^2\psi_{i-1}.
\end{aligned}$$

If we repeat this inequality recursively on i we get

$$\psi_{i+1} \le (1 + mKh)^i \sum_{j=0}^{i} h\left[\omega((y^1)', x_{j+1/2}; h) + \dots + \omega((y^m)', x_{j+1/2}; h)\right]$$

$$\le \left(1 + \frac{mAK}{n}\right)^n \sum_{j=0}^{i}\left[\int_{x_j}^{x_{j+1}} \omega((y^1)', x_{j+1/2}; h)\,dx + \dots + \int_{x_j}^{x_{j+1}} \omega((y^m)', x_{j+1/2}; h)\,dx\right]$$

$$\leq e^{mAK}\left[\int_0^A \omega((y^1)',x;2h)\,dx+\cdots+\int_0^A \omega((y^m)',x;2h)\,dx\right]$$

$$\leq 2e^{mAK}\left[\tau((y^1)';h)+\cdots+\tau((y^m)';h)\right].$$

From the last estimations it follows that

$$\psi_{i+1}\leq 2e^{mAK}\sum_{r=1}^{m}\tau((y^r)';h)_{L_p}.$$

So we have proved the following theorem.

Theorem 1. *The following estimation is true*

$$\max\{|y_i^r-\widetilde{y}_i^r| : 1\leq r\leq m,\ 0\leq i\leq n\}\leq 2e^{mAK}\sum_{r=1}^{m}[\tau((y^r)';h)_{L_p}].$$

To estimate the error for those Runge–Kutta's methods which have local error $O(h^3)$ in multivariate case we restrict ourselves to two dependent variables as follows

$$\text{(1)}\qquad \begin{aligned} y' &= f(x,y,z), \quad y(0)=y_0,\\ z' &= g(x,y,z), \quad z(0)=z_0. \end{aligned}$$

Using the formulae

$$\text{(2)}\qquad \begin{aligned} \widetilde{y}_{i+1} &= \widetilde{y}_i+phf(x_i,\widetilde{y}_i,\widetilde{z}_i)+qhf(x_i+\alpha h,\widetilde{y}_i+\beta hf(x_i,\widetilde{y}_i,\widetilde{z}_i),\widetilde{z}_i+\beta hg(x_i,\widetilde{y}_i,\widetilde{z}_i)),\\ \widetilde{z}_{i+1} &= \widetilde{z}_i+phg(x_i,\widetilde{y}_i,\widetilde{z}_i)+qhg(x_i+\alpha h,\widetilde{y}_i+\beta hf(x_i,\widetilde{y}_i,\widetilde{z}_i),\widetilde{z}_i+\beta hg(x_i,\widetilde{y}_i,\widetilde{z}_i)) \end{aligned}$$

where the constants p, q, α, β satisfy the system

$$\text{(3)}\qquad p+q=1,\quad q\alpha=\frac{1}{2},\quad q\beta=\frac{1}{2},$$

it follows that this system has one-parameter solution of the form

$$p=1-s,\quad q=s,\quad \alpha=\beta=\frac{1}{2s}.$$

For the sake of simplicity we shall put $s=\frac{1}{2}$ (the general case can be considered in a similar way), i.e.

$$p=q=\frac{1}{2},\quad \alpha=\beta=1,$$

then from (1) and (2) we obtain

$$|y_{i+1}-\widetilde{y}_{i+1}|$$

$$=\left|y_{i+1}-\widetilde{y}_i-phf(x_i,\widetilde{y}_i,\widetilde{z}_i)-qhf(x_i+\alpha h,\widetilde{y}_i+\beta hf(x_i,\widetilde{y}_i,\widetilde{z}_i),\widetilde{z}_i+\beta hg(x_i,\widetilde{y}_i,\widetilde{z}_i))\right|$$

$$=\left|y_{i+1}-\widetilde{y}_i-\frac{h}{2}f(x_i,\widetilde{y}_i,\widetilde{z}_i)-\frac{h}{2}f(x_i+h,\widetilde{y}_i+hf(x_i,\widetilde{y}_i,\widetilde{z}_i),\widetilde{z}_i+hg(x_i,\widetilde{y}_i,\widetilde{z}_i))\right|$$

$$\leq\left|y_{i+1}-\widetilde{y}_i-\frac{h}{2}f(x_i,\widetilde{y}_i,\widetilde{z}_i)+\frac{h}{2}f(x_i,y_i,z_i)-\frac{h}{2}(x_i,y_i,z_i)\right.$$

$$\left.-\frac{h}{2}f(x_i+h,\widetilde{y}_i+hf(x_i,\widetilde{y}_i,\widetilde{z}_i),\widetilde{z}_i+hg(x_i,\widetilde{y}_i,\widetilde{z}_i))\right.$$

$$+\frac{h}{2}f(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i))$$

$$-\frac{h}{2}f(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i))\Big|.$$

By applying Lipschitz condition we obtain

$$
\begin{aligned}
|y_{i+1}-\tilde{y}_{i+1}| \leq\ & \frac{Kh}{2}\{|y_i-\tilde{y}_i|+|z_i-\tilde{z}_i|\} \\
&+\frac{Kh}{2}\Big\{\big|\tilde{y}_i+hf(x_i,\tilde{y}_i,\tilde{z}_i)-y_i-hf(x_i,y_i,z_i)\big| \\
&\quad+\big|\tilde{z}_i+hg(x_i,\tilde{y}_i,\tilde{z}_i)-z_i-hg(x_i,y_i,z_i)\big|\Big\} \\
&+\Big|y_{i+1}-\tilde{y}_i+y_i-y_i-\frac{h}{2}f(x_i,y_i,z_i) \\
&\quad-\frac{h}{2}f(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i))\Big| \\
\leq\ & \frac{Kh}{2}\{|y_i-\tilde{y}_i|+|z_i-\tilde{z}_i|\}+\frac{Kh}{2}\Big\{\big|\tilde{y}_i+hf(x_i,\tilde{y}_i,\tilde{z}_i)-y_i-hf(x_i,y_i,z_i)\big| \\
&\quad+\big|\tilde{z}_i+hg(x_i,\tilde{y}_i,\tilde{z}_i)-z_i-hg(x_i,y_i,z_i)\big|\Big\} \\
&\quad+\Big|y_{i+1}-\tilde{y}_i+y_i-y_i-\frac{h}{2}y_i'-\frac{h}{2}f(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i))\Big| \\
\leq\ & \frac{Kh}{2}\{|y_i-\tilde{y}_i|+|z_i-\tilde{z}_i|\}+\frac{Kh}{2}\{|y_i-\tilde{y}_i|+|z_i-\tilde{z}_i|\} \\
&\quad+\frac{K^2h^2}{2}\{|y_i-\tilde{y}_i|+|z_i-\tilde{z}_i|\}+|y_i-\tilde{y}_i| \\
&+\Big|y_{i+1}-y_i-\frac{h}{2}y_i'-\frac{h}{2}f(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i))\Big| \\
\leq\ & c_1\{|y_i-\tilde{y}_i|+|z_i-\tilde{z}_i|\}+|y_i-\tilde{y}_i|+c_2
\end{aligned}
$$

$$(4)\quad \leq (1+c_1)|y_i-\tilde{y}_i|+c_1|z_i-\tilde{z}_i|+c_2,$$

where $c_1=\dfrac{2Kh+K^2h^2}{2}$, and

$$c_2=\Big|y_{i+1}-y_i-\frac{h}{2}y_i'-\frac{h}{2}f(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i))\Big|.$$

Let us estimate c_2

$$
\begin{aligned}
c_2=\ &\Big|y_{i+1}-y_i-\frac{h}{2}y_i'-\frac{h}{2}f(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i)) \\
(5)\quad &-hy'_{i+1/2}+hy'_{i+1/2}-\frac{h}{2}f(x_i+h,y_{i+1},z_{i+1})+\frac{h}{2}f(x_i+h,y_{i+1},z_{i+1})\Big| \\
\leq\ &\big|y_{i+1}-y_i-hy'_{i+1/2}\big|+h\Big|y'_{i+1/2}-\frac{1}{2}y_i'-\frac{1}{2}y'_{i+1}\Big|
\end{aligned}
$$

$$+\frac{h}{2}\left|f(x_{i+1},y_{i+1},z_{i+1})-f\big(x_{i+1},y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i)\big)\right|,$$

where $x_{i+1}=x_i+h, y_{i+1}=y(x_{i+1}), y'_{i+1/2}=y'(x_{i+1/2})$. Now estimating the first term in the right hand side of inequality (5) we get

$$\left|y_{i+1}-y_i-hy'_{i+1/2}\right|=h\left|\frac{y_{i+1}-y_i}{h}-y'_{i+1/2}\right|$$

$$=\left|\int\limits_{x_i}^{x_{i+1}}\left[y'(t)-y'_{i+1/2}\right]dt\right|=h\left|\int\limits_{-1/2}^{1/2}\left[y'(x_{i+1/2}+th)-y'_{i+1/2}\right]dt\right|$$

$$=h\left|\int\limits_{0}^{1/2}\left[y'(x_{i+1/2}+th)-2y'_{i+1/2}+y'(x_{i+1/2}-th)\right]dt\right|$$

$$\text{(6)}\qquad \le h\int\limits_{0}^{1/2}\omega_2\left(y',x_{i+1/2};\frac{h}{2}\right)dt=\frac{h}{2}\omega_2\left(y',x_{i+1/2};\frac{h}{2}\right).$$

In order to estimate the following term $\left|y'_{i+1/2}-\frac{1}{2}(y'_i+y'_{i+1})\right|$, let p be the algebraic polynomial of first degree, which interpolates the function y' at the points x_i and x_{i+1}. We have (see [4], lemma 2.3, p. 30)

$$\text{(7)}\qquad \|y'-p\|_{C[x_i,x_{i+1}]}\le\omega_2(y',x_{i+1/2};h/2),$$

where $p(x_i)=p_i,\ p(x_{i+1})=p_{i+1}$.

From (7) we get

$$\left|y'_{i+1/2}-\frac{1}{2}(y'_i+y'_{i+1})\right|$$

$$\le\left|y'_{i+1/2}-p_{i+1/2}-\frac{1}{2}(y'_i-p_i)-\frac{1}{2}(y'_{i+1}-p_{i+1/2})\right|+\left|p_{i+1/2}-\frac{1}{2}(p_i+p_{i+1/2})\right|$$

$$\le\left|y'_{i+1/2}-p_{i+1/2}\right|+\frac{1}{2}\left|y'_i-p_i\right|+\frac{1}{2}\left|y'_{i+1}-p_{i+1}\right|\le2\omega_2\left(y',x_{i+1/2};h/2\right).$$

Since $p_{i+1/2}-\frac{1}{2}p_i-\frac{1}{2}p_{i+1}=0$, p being of first degree, we obtain

$$\left|y'_{i+1/2}-\frac{1}{2}(y'_i+y'_{i+1})\right|\le\frac{1}{2}\omega_2(y',x_{i+1/2};h/2)\le\frac{1}{2}\omega_2(y',x_{i+1/2};h).$$

Similarly we get

$$\left|z'_{i+1/2}-\frac{1}{2}(z'_i+z'_{i+1})\right|\le\frac{1}{2}\omega_2(z',x_{i+1/2};h/2)\le\frac{1}{2}\omega_2(z',x_{i+1/2};h/2).$$

Now

$$\begin{aligned}
&\left|f(x_{i+1}, y_{i+1}, z_{i+1}) - f(x_{i+1}, y_i + hf(x_i, y_i, z_i), z_i + hg(x_i, y_i, z_i))\right| \\
&\qquad \leqq K\left\{|y_{i+1} - y_i - hy_i'| + |z_{i+1} - z_i - hz_i'|\right\} \\
&\qquad \leqq \ldots \leqq Kh\left(\left|\frac{y_{i+1} - y_i}{h} - y_i'\right| + \left|\frac{z_{i+1} - z_i}{h} - z_i'\right|\right) \\
&\qquad \leqq Kh\left\{\left|\frac{1}{h}\int_{x_i}^{x_{i+1}} [y'(t) - y_i']\, dt\right| + \left|\frac{1}{h}\int_{x_i}^{x_{i+1}} [z'(t) - z_i']\, dt\right|\right\}
\end{aligned}$$

$$\leqq Kh\{\omega(y', x_{i+1/2}; h) + \omega(z', x_{i+1/2}; h)\}. \tag{8}$$

From (4), (6), (5) and (8) we obtain

$$\begin{aligned}
|y_{i+1} - \widetilde{y}_{i+1}| + |z_{i+1} - \widetilde{z}_{i+1}| &\leqq \left[1 + Kh\left(1 + \frac{Kh}{2}\right)\right]|y_i - \widetilde{y}_i| + \frac{h}{2}\omega_2\left(y', x_{i+1/2}; \frac{h}{2}\right) \\
&+ \frac{Kh^2}{2}\omega(y', x_{i+1/2}; h) + 2h\omega_2(y', x_{i+1/2}; h) + \left[1 + Kh\left(1 + \frac{Kh}{2}\right)\right]|z_i - \widetilde{z}_i| \\
&+ \frac{h}{2}\omega_2\left(z', x_{i+1/2}; \frac{h}{2}\right) + \frac{Kh^2}{2}\omega\left(z', x_{i+1/2}; h\right) + 2h\omega_2\left(z', x_{i+1/2}; h\right).
\end{aligned}$$

Applying the above inequality recursively on i, we obtain

$$\begin{aligned}
&|y_{i+1} - \widetilde{y}_{i+1}| + |z_{i+1} - \widetilde{z}_{i+1}| \\
&\leqq \sum_{k=0}^{i}\left[1 + Kh\left(1 + \frac{Kh}{2}\right)\right]^{i-k}\left[\frac{h}{2}\omega_2\left(y', x_{k+1/2}; \frac{h}{2}\right) + \frac{Kh^2}{2}\omega(y', x_{k+1/2}; h)\right. \\
&\left.+ 2h\omega_2(y', x_{k+1/2}; h)\right] + \sum_{k=0}^{i}\left[1 + Kh\left(1 + \frac{Kh}{2}\right)\right]^{i+k}\left[\frac{h}{2}\omega_2\left(z', x_{k+1/2}; \frac{h}{2}\right)\right. \\
&\left.+ \frac{Kh^2}{2}\omega(z', x_{k+1/2}; h) + 2h\omega_2(z', x_{k+1/2}; h)\right].
\end{aligned} \tag{9}$$

Set $1 + \frac{kA}{2} = c_3$, then from (9) we get

$$\begin{aligned}
&\max\{|y_i - \widetilde{y}_i| + |z_i - \widetilde{z}_i| \ :\ 0 \leqq i \leqq n\} \\
&\leqq \left(1 + \frac{c_3AK}{n}\right)^n \sum_{k=1}^{n-1}\left[\frac{h}{2}\omega_2\left(y', x_{k+1/2}; \frac{h}{2}\right) + \frac{Kh^2}{2}\omega\left(y', x_{k+1/2}; h\right)\right. \\
&\left.+ 2h\omega_2(y', x_{k+1/2}; h)\right] + \left(1 + \frac{c_3AK}{n}\right)^n \sum_{k=1}^{n-1}\left[\frac{h}{2}\omega_2\left(z', x_{k+1/2}; \frac{h}{2}\right)\right. \\
&\left.+ \frac{Kh^2}{2}\omega(z', x_{k+1/2}; h) + 2h\omega_2(z', x_{k+1/2}; h)\right]
\end{aligned}$$

$$\le e^{c_3AK}\sum_{k=0}^{n-1}\left[\frac{1}{2}\int_{x_k}^{x_{k+1}}\omega_2(y',x;h)\,dx+\frac{Kh}{2}\int_{x_k}^{x_{k+1}}\omega(y',x;h)\,dx+2\int_{x_k}^{x_{k+1}}\omega_2(y',x;h)\,dx\right]$$

$$+\,e^{c_3AK}\sum_{k=0}^{n-1}\left[\frac{1}{2}\int_{x_k}^{x_{k+1}}\omega_2(z',x;h)\,dx+\frac{Kh}{2}\int_{x_k}^{x_{k+1}}\omega(z',x;h)\,dx+2\int_{x_k}^{x_{k+1}}\omega_2(z',x;h)\,dx\right]$$

$$=e^{c_3AK}\int_0^A\left[\frac{1}{2}\omega_2(y',x;h)+\frac{Kh}{2}\omega(y',x;h)+2\omega_2(y',x;h)\right]dx$$

$$+\,e^{c_3AK}\int_0^A\left[\frac{1}{2}\omega_2(z',x;h)+\frac{Kh}{2}\omega(z',x;h)+2\omega_2(z',x;h)\right]dx$$

$$=e^{c_3AK}A\left[\frac{1}{2}\tau_2(y';h)_{L_1}+\frac{Kh}{2}\tau(y';h)_{L_1}+2\tau_2(y';h)_{L_1}\right]$$

$$+\,e^{c_3AK}A\left[\frac{1}{2}\tau_2(z';h)_{L_1}+\frac{Kh}{2}\tau(z';h)_{L_1}+2\tau_2(z';h)_{L_1}\right]$$

$$=e^{c_3AK}A\left[\frac{1}{2}\left(\tau_2(y';h)_{L_1}+\tau_2(z';h)_{L_1}\right)+\frac{Kh}{2}\left(\tau(y';h)_{L_1}+\tau(z';h)_{L_1}\right)\right.$$

$$\left.+\,2\left(\tau_2(y';h)_{L_1}+\tau_2(z';h)_{L_1}\right)\right].$$

Therefore we have proved the following theorem.

Theorem 2. *For the solution of the problem (1) the estimation*

$$\max\{\,|y_i-\tilde{y}_i|+|z_i-\tilde{z}_i|\;:\;0\le i\le n\}$$

$$\le c\left[\tau_2(y';h)_{L_1}+\tau_2(z';h)_{L_1}+h\left[\tau(y';h)+\tau(z';h)\right]\right]$$

holds, where c is a constant depending on A and K only.

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