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ON THE STRUCTURE OF $(t \mod q)$-ARCS IN FINITE PROJECTIVE GEOMETRIES

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In this paper, we introduce constructions and structure results for $(t \mod q)$-arcs. We prove that all $(2 \mod q)$-arcs in $\mathrm{PG}(r,q)$ with $r \geq 3$ are lifted. We find all $(3 \mod 5)$ plane arcs of small cardinality not exceeding 33 and prove that every $(3 \mod 5)$-arc in $\mathrm{PG}(3,5)$ of size at most 158 is lifted. This result is applied further to rule out the existence of $(104,22)$-arcs in $\mathrm{PG}(3,5)$ which solves an open problem on the optimal size of fourdimensional linear codes over $\mathbb{F}_5$.

Keywords: finite projective geometries, arcs, blocking sets, divisible arcs, quasi-divisible arcs, the Griesmer bound, extendable arcs, minihypers

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1. INTRODUCTION

Consider the geometry $\Sigma = \mathrm{PG}(r,q)$, $r \geq 2$. Denote by $\mathcal{P}$ be the set of points and by $\mathcal{H}$ the set of hyperplanes of Σ. Every mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$ from the pointset of the geometry to the non-negative integers is called a multiset in Σ. This mapping is extended additively to every subset $\mathcal{Q}$ of $\mathcal{P}$ by $\mathcal{K}(\mathcal{Q}) = \sum_{P\in\mathcal{Q}} \mathcal{K}(P)$. The integer $n := \mathcal{K}(\mathcal{P})$ is called the cardinality of $\mathcal{K}$. For every set of points $\mathcal{Q} \subset \mathcal{P}$ we define its characteristic (multi)set $\chi_{\mathcal{Q}}$ by

$$\chi_{\mathcal{Q}}(P) = \begin{cases} 1 & \text{if } P \in \mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Multisets can be viewed as arcs or as blocking sets. A multiset $\mathcal{K}$ in Σ is called an (n,w)-multiarc (or simply (n,w)-arc) if (1) $\mathcal{K}(\mathcal{P}) = n$, (2) $\mathcal{K}(H) \leq w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0) = w$. Similarly, a multiset $\mathcal{K}$ in $\mathrm{PG}(r,q)$ is called an (n,w)-blocking set with respect to the hyperplanes (or (n,w)-minihyper) if (1) $\mathcal{K}(\mathcal{P}) = n$, (2) $\mathcal{K}(H) \geq w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0) = w$.

An (n,w)-arc $\mathcal{K}$ in Σ is called t-extendable, if there exists an $(n+t,w)$-arc $\mathcal{K}'$ in Σ with $\mathcal{K}'(P) \geq \mathcal{K}(P)$ for every point $P \in \mathcal{P}$. An arc is called simply extendable if it is 1-extendable. Similarly, an (n,w)-blocking set $\mathcal{K}$ in Σ is called t-reducible, if there exists an $(n-t,w)$-blocking set $\mathcal{K}'$ in Σ with $\mathcal{K}'(P) \leq \mathcal{K}(P)$ for every point $P \in \mathcal{P}$. A blocking set is called irreducible if it is not reducible.

Given a multiset $\mathcal{K}$ in Σ, we denote by a_i the number of hyperplanes H with $\mathcal{K}(H) = i$. The sequence $(a_i)_{i \geq 0}$ is called the spectrum of $\mathcal{K}$. An (n,w)-arc $\mathcal{K}$ with spectrum (a_i) is said to be divisible with divisor $\Delta > 1$ if $a_i = 0$ for all $i \not\equiv n \pmod{\Delta}$. Given an integer t with $1 \leq t \leq q-1$, we call the (n,w)-arc $\mathcal{K}$ with $w \equiv n+t \pmod{q}$ *t-quasidivisible with divisor* $\Delta > 1$ (or *t-quasidivisible modulo* Δ) if $a_i = 0$ for all $i \not\equiv n, n+1, \ldots, n+t \pmod{\Delta}$.

Let t be a fixed non-negative integer. An arc $\mathcal{K}$ in Σ is called a $(t \mod q)$-arc if

(1) for every point $P \in \mathcal{P}$, $\mathcal{K}(P) \leq t$;

(2) for every subspace S of dimension at least 1, $\mathcal{K}(S) \equiv t \pmod{q}$.

These arcs arise naturally as certain duals of t-quasidivisible arcs. Let $\mathcal{K}$ be a t-quasidivisible (n,w)-arc with divisor q in Σ, $t < q$. Denote by $\widetilde{\mathcal{K}}$ the arc

$$\widetilde{\mathcal{K}} : \begin{cases} \mathcal{H} & \rightarrow \quad \{0,1,\ldots,t\} \\ H & \rightarrow \quad \widetilde{\mathcal{K}}(H) \equiv n+t-\mathcal{K}(H) \pmod{q} \end{cases}, \tag{1.1}$$

where $\mathcal{H}$ is the set of all hyperplanes in Σ. This means that hyperplanes of multiplicity congruent to $n+a \pmod{q}$ become $(t-a)$-points in the dual geometry. In particular, maximal hyperplanes are 0-points with respect to $\widetilde{\mathcal{K}}$. Then $\widetilde{\mathcal{K}}$ is a $(t \mod q)$-arc [7,8]. In the general case the cardinality of $\widetilde{\mathcal{K}}$ cannot be obtained from the parameters of $\mathcal{K}$. Extendability properties of $\mathcal{K}$ can be derived from the structure of $\widetilde{\mathcal{K}}$. In particular, $\mathcal{K}$ is extendable if it contains a hyperplane in its support. For a more detailed introduction to arcs and blocking sets and their relation to linear codes, we refer to[5,8].

The aim of this paper is to present various constructions and structure results for $(t \mod q)$-arcs. Section 2 contains general constructions for $(t \mod q)$-arcs. The most important is the so-called lifting construction, which is partly due to the fact that in dimension higher than 3 the only known $(t \mod q)$-arcs are sums of lifted arcs. In section 3, we prove that every $(2 \mod q)$-arc is lifted. This result implies Maruta's extendability result for linear codes with weights $-2,-1,0 \mod q$ for q odd. In section 4, we characterize the $(3 \mod 5)$-arcs of small cardinality and prove that every $(3 \mod 5)$-arc in $\mathrm{PG}(3,5)$ of size not exceeding 153 is lifted. In section 4, we apply the results from section 3 to rule out the existence of $(104, 22)$-arcs in $\mathrm{PG}(3,5)$, or equivalently, of $[104, 4, 82]_5$-codes.

2. GENERAL CONSTRUCTIONS

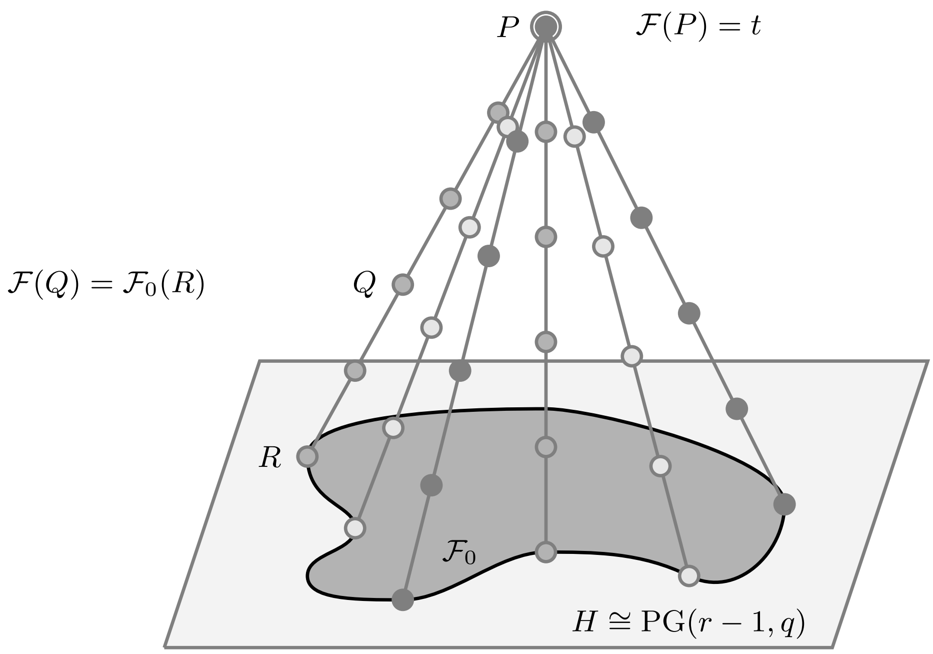
In this section, we describe several constructions for $(t \mod q)$-arcs. We start with a straightforward observation.

Theorem 1. *Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be a $(t_1 \mod q)$-arc (resp. $(t_2 \mod q)$-arc) in $\mathrm{PG}(r,q)$. If $t = t_1 + t_2 < q$, then $\mathcal{F}_1 + \mathcal{F}_2$ is a $(t \mod q)$ arc. In particular, the sum of t (not necessarily different) hyperplanes is a $(t \mod q)$-arc.*

The next construction is less obvious.

Theorem 2. *Let $\mathcal{F}_0$ be a $(t \mod q)$-arc in a hyperplane $H \cong \mathrm{PG}(r-1,q)$ of $\Sigma = PG(r,q)$. For a fixed point $P \in \Sigma \setminus H$, define an arc $\mathcal{F}$ in Σ as follows:*
– $\mathcal{F}(P) = t$;
– for each point $Q \neq P$: $\mathcal{F}(Q) = \mathcal{F}_0(R)$ where $R = \langle P, Q\rangle \cap H$.
Then the arc $\mathcal{F}$ is a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$ of size $q|\mathcal{F}_0| + t$.

Proof. As already noted it is enough to prove that the multiplicity of every line is t modulo q. This is obvious for the lines through the point P. Now consider a line L in Σ which is not incident with P. Let π be the plane defined by P and L: $\pi = \langle L, P\rangle$. Set $L' = \pi \cap H$. Obviously, L contains points of the same multiplicities as L'. The multiplicity of L' is $\mathcal{F}(L') = \mathcal{F}_0(L) \equiv t \pmod q$ which proves the result. The construction is illustrated in the picture below. □



We call the $(t \mod q)$-arcs obtained by Theorem 2 *lifted arcs* and the point P – *lifting point*. We can have a more general notion of lifted arcs replacing the point P by a subspace U. Let $\mathcal{F}_0$ be a $(t \mod q)$-arc in the subspace V of $\Sigma = \mathrm{PG}(r, q)$ and let U be a subspace with $\dim U + \dim V = r - 1$, $U \cap V = \varnothing$. The arc $\mathcal{F}$ in Σ defined by

– $\mathcal{F}(P) = t$ for every point $P \in U$;

– for each point $Q \neq P$: $\mathcal{F}(Q) = \mathcal{F}_0(R)$ where $R = \langle U, Q\rangle \cap V$

is called an arc lifted from the subspace U. Obviously $\mathcal{F}$ is also a $(t \mod q)$-arc. Let us note that if an arc is lifted from a subspace then it can be considered as lifted from any point of that subspace. We have also a partial converse of this observation.

Lemma 1. *Let the arc $\mathcal{F}$ be lifted from the points P and Q, $P \neq Q$. Then $\mathcal{F}$ is also lifted from the line PQ. In particular, the lifting points of a* $(t \mod q)$*-arc $\mathcal{F}$ form a subspace S and $\mathcal{F}$ is lifted from any point of S.*

Proof. All points on the line PQ are t-points. Let R be an arbitrary point in Σ. Then all points on PR (resp. QR) different from P (resp. Q) have the same multiplicity, a say. Then all points in the plane $\langle P, Q, R\rangle$ outside PQ have also multiplicity a, which proves the lemma. □

The sum of t hyperplanes can be viewed as the sum of lifted arcs. Remarkably, we do not know an example of a $(t \mod q)$-arc in $\mathrm{PG}(r, q)$, with $r \geq 3$, that is not the sum of lifted arcs. It turns out that if in the geometry $\mathrm{PG}(r, q)$ there exist only lifted $(t \mod q)$-arcs then every $(t \mod q)$-arc in $\mathrm{PG}(r', q)$, $r' \geq r$, is also lifted.

Theorem 3. *Let $\mathcal{K}$ be a* $(t \mod q)$*-arc in* $\mathrm{PG}(r, q)$ *such that the restriction $\mathcal{K}|_H$ to every hyperplane H of* $\mathrm{PG}(r, q)$ *is also lifted. Then $\mathcal{K}$ is a lifted arc.*

Proof. Consider a $(t \mod q)$-arc $\mathcal{K}$ in $\mathrm{PG}(r, q)$. Let S be an arbitrary subspace of $\mathrm{PG}(r, q)$ of codimension 2. Denote by H_i, $i = 0, \ldots, q$, the hyperplanes through S. The arcs $\mathcal{K}|_{H_i}$ are all lifted $(t \mod q)$-arcs. Let us denote by P_i, $i = 0, \ldots, q$, the corresponding lifting points.

Assume that for some indices, i and j say, $P_i \in S$ and $P_j \in H_j \setminus S$. Clearly, the line P_iP_j consists entirely of t-points. Let L be an arbitrary line in H_j incident with P_j and set $L \cap S = Q_j$. All points on the line P_jQ_j, different from P_j have the same multiplicity a, where $0 \leq a \leq t$. Thus all points in the plane $\langle P_i, P_j, Q_j\rangle$ outside P_iP_j are a points. Now it is clear that $\mathcal{K}|_{H_j}$ can be viewed as lifted from the line P_iP_j and hence from any point of P_iP_j.

Assume that $P_i \in H_i \setminus S$ for all $i = 0, \ldots, q$. If the points $P_0, \ldots, P_q$ are collinear then $\mathcal{K}$ is lifted from the line $\langle P_i \mid i = 0, \ldots, q\rangle$.

Now assume that the points P_i are not collinear. Then there exists a hyperplane H in $\mathrm{PG}(r, q)$ that does not contain any of the points P_i. Set $T = H \cap S$. If we denote $G_i = H \cap H_i$ then all the arcs $\mathcal{K}|_{G_i}$ are projectively equivalent to $\mathcal{K}_S$.

Let us first assume that the lifting point Q of $\mathcal{K}|_H$ is contained in $G_i \setminus T$. Set $Q_i = S \cap QP_i$. Obviously, P_iQ_i is a line of t-points. Consider an arbitrary line L

in H_i through P_i. If the points on L different from P_i are a-points then all points on the line through Q and $L \cap G_i$ different from Q are also a-points. Hence all points in the plane $\langle L, Q_i \rangle$ outside P_iQ_i are a-points and $\mathcal{K}|_{H_i}$ is lifted from P_iQ_i. Therefore it can be viewed as lifted from any point on P_iQ_i, in particular Q_i.

We have proved so far that without loss of generality we can assume that all points P_i are contained in S.

Consider the subspace T of S generated by the points P_i, $T = \langle P_i \mid i = 0, \ldots, q\}$. All points in T are of maximal multiplicity. Let $Q \in S \setminus T$ be a point of multiplicity a. All points from $\langle T, Q \rangle \setminus T$ also have multiplicity a. Hence the restriction $\mathcal{K}|_S$ is lifted from the subspace T. Since S was fixed arbitrarily, the restriction of $\mathcal{K}$ to any subspace of codimension 2 is a lifted arc.

We repeat this argument for the subspaces of smaller dimension. For subspaces of dimension 2 this means that all planes contain a line of t-points with all the remaining points of multiplicity a. It is easily checked that in such case we have a hyperplane of t-points and all the remaining points outside this hyperplane are a-points. But such an arc is obviously a lifted arc. □

In the plane case, non-trivial ($t \mod q$)-arcs can be constructed as σ-duals of certain blocking sets. Let $\mathcal{K}$ be a multiset in Σ. Consider a function σ such that $\sigma(\mathcal{K}(H))$ is a non-negative integer for all hyperplanes H. The multiset

$$\widetilde{\mathcal{K}}^\sigma : \begin{cases} \mathcal{H} & \to & \mathbb{N}_0 \\ H & \mapsto & \sigma(\mathcal{K}(H)) \end{cases} \tag{2.1}$$

in the dual space $\widetilde{\Sigma}$ is called the σ-*dual* of $\mathcal{K}$. If σ is a linear function, the parameters of $\widetilde{\mathcal{K}}^\sigma$, as well as its spectrum, are easily computed from the parameters and the spectrum of $\mathcal{K}$ (cf. [1,10]).

Theorem 4. *[7,8] Let $\mathcal{F}$ be a ($t \mod q$)-arc in* PG$(2, q)$ *of size $mq + t$. Then the arc $\mathcal{F}^\sigma$ with $\sigma(x) = (x - t)/q$ is a $((m - t)q + m, m - t)$-blocking set in the dual plane. Moreover the multiplicities of the lines with respect to this blocking set belong to $\{m - t, m - t + 1, \ldots, m\}$.*

3. (2 mod Q)-ARCS

Let us start by noting that an ($1 \mod q$) arc is projective and hence either a hyperplane or the complete space [3,4]. For $t = 2$ and odd $q \geq 5$, the ($t \mod q$)-arcs were characterized by Maruta [13]. These are the following:

(I) a lifted arc from a 2-line; such an arc has $2q + 2$ points and there exist two possibilities

(I-1) a double line, or

(I-2) a sum of two different lines;

(II) a lifted arc from a $(q+2)$-line; such a line has i double points, $q-2i+2$ single points and $i-1$ 0-points, where $i=1,\dots,\frac{q+1}{2}$; we say that such an arc is of type (II-i) if it is lifted from a line with i double points;

(III) a lifted arc from a $(2q+2)$-line, i.e. the sum of two copies of the same plane;

(IV) an exceptional $(2 \mod q)$-arc for q odd; it consists of the points of an oval, a fixed tangent to this oval, and two copies of each internal point of the oval.

Now we are going to prove that in higher dimensions every $(2 \mod q)$-arc is a lifted arc. Consider a projection φ from a 2-point P onto some plane not incident with that point. Let L be a line incident with P. We have the following possibilities for the image of L:

type of L	multiplicity of L	type of $\varphi(L)$
$(2,0,\dots,0)$	2	ω
$(2,1,\dots,1)$	$q+2$	α
$(2,2,\dots,2)$	$2q+2$	β
$(2,\underbrace{2,\dots,2}_{i},\underbrace{1,\dots,1}_{q-2i},\underbrace{0,\dots,0}_{i})$	$q+2$	γ_i

Note that in type γ_i we have $i=1,\dots,\frac{q-1}{2}$. Now the images of the plane $(2 \mod q)$-arcs under φ are the following:

Type	the image of the plane arc	Remark
(I-1)	$(\beta,0,\dots,0)$	
(I-2)	$(\alpha,\alpha,0,\dots,0)$	
(II-i)	$(\underbrace{\beta,\dots,\beta}_{i},\underbrace{\alpha,\dots,\alpha}_{q-2i+1},\underbrace{0,\dots,0}_{i-1})$	projection from the exceptional 2-point
	$(\beta,\gamma_i,\gamma_i,\dots,\gamma_i)$ $i=1,\dots,\frac{q-1}{2}$	projection from all other 2-points
(III)	$(\beta,\beta,\dots,\beta)$	
(IV)	$(\alpha,\gamma_{\frac{q-1}{2}},\dots,\gamma_{\frac{q-1}{2}})$	from the 2-point on the oval
	$(\underbrace{\gamma_{\frac{q-1}{2}},\dots,\gamma_{\frac{q-1}{2}}}_{\frac{q+3}{2}},\underbrace{\gamma_{\frac{q-3}{2}},\dots,\gamma_{\frac{q-3}{2}}}_{\frac{q-1}{2}})$	from an internal point to the oval

Assume a $(2 \mod q)$-arc $\mathcal{K}$ in $\mathrm{PG}(3,q)$, q odd, is given and consider a projection from a 2-point P. The table above implies that

(i) no line in the projection plane is incident with points of type ω and points of type γ_i;

(ii) if on a line in the projection plane there exist points of type γ_i and points of type γ_j, $i\neq j$, then $i=\frac{q-3}{2}$, $j=\frac{q-1}{2}$.

Let us first assume that there exists a plane π such that $\mathcal{K}|_\pi$ is the exceptional arc (IV). Denote by φ a projection from the 2-point on the oval. Then the image of the plane π is of type $(\alpha, \gamma_{\frac{q-1}{2}}, \ldots, \gamma_{\frac{q-1}{2}})$. Denote by L the line of type α and fix a 1-point Q on this line. Assume there is a point of type β in the projection plane. Then the projection plane contains a line of type $(\beta, \alpha, \ldots, \alpha)$ and all the remaining lines through the type β point are of type $(\beta, \gamma_{\frac{q-1}{2}}, \ldots, \gamma_{\frac{q-1}{2}})$. Hence the lines in the projection plane through a point of type α are of the following types:

- $(\alpha, \alpha, \ldots, \alpha, \beta)$ – there is one such line;
- $(\alpha, \gamma_{\frac{q-2}{2}}, \ldots, \gamma_{\frac{q-2}{2}}, \gamma_{\frac{q-2}{2}})$ – there are q such lines.

Denote the points on the $2(q+1)$-line (the preimage of the point of type β) by $P_0, P_1, \ldots, P_q$. Assume that there is a point of type P_i such that all planes through QP_i (different from π_0) are not of type (IV). Then $\mathcal{K}$ is obviously lifted. If for all P_i there is a plane through QP_i such that the restriction of $\mathcal{K}$ to this plane is of type (IV) then projecting from each P_i we must get the same types of the lines in the projection plane (described above). Therefore no three of the q^2 1-points contained in the ovals, are collinear. Hence we can construct a $q^2 + 2$-cap taking these q^2 1-points and P_0, P_1, say. This is a contradiction since the maximal size of a cap in $\mathrm{PG}(3, q)$ is $q^2 + 1$.

We have proved that if there is a point of type β in the projection plane, then $\mathcal{K}$ is lifted. But there always must be a point of type β since the types $\gamma_{\frac{q-1}{2}}$ and ω are not compatible. Thus we have proved that if there exists a plane π such that $\mathcal{K}|_\pi$ is of type (IV) then $\mathcal{K}$ is a lifted arc.

Now assume that there is no plane such that $\mathcal{K}|_\pi$ is of type (IV). Now the restriction of $\mathcal{K}$ to any plane is a lifted arc and by Theorem 3 $\mathcal{K}$ is again lifted. We have proved the following lemma.

Lemma 2. *Let $\mathcal{K}$ be a $(2 \bmod q)$-arc in $\mathrm{PG}(3, q)$, q odd. Then $\mathcal{K}$ is a lifted arc.*

Now we proceed by induction on the dimension. Again by Theorem 3, we get that every $(2 \mod q)$-arc in a geometry of dimension at least 3 is lifted.

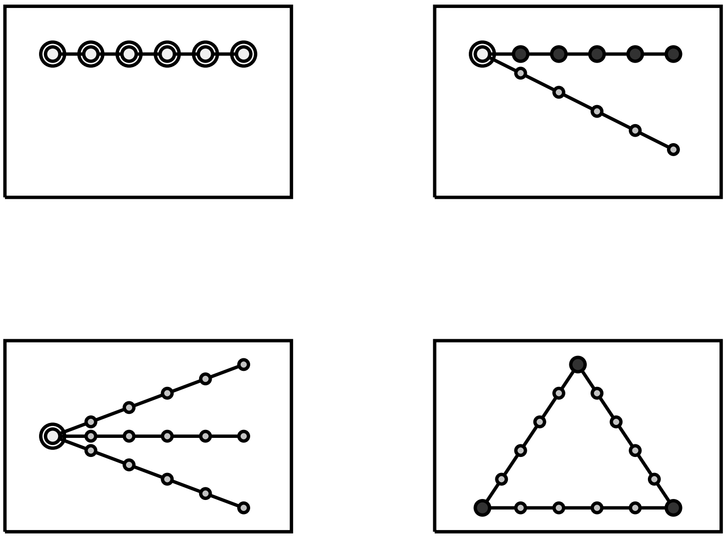
Theorem 5. *Let $\mathcal{K}$ be a $(2 \mod q)$-arc in $\mathrm{PG}(r, q)$, q odd, $r \geq 3$. Then $\mathcal{K}$ is a lifted arc. In particular, every $(2 \mod q)$-arc in $\mathrm{PG}(r, q)$, $r \geq 2$, has a hyperplane in its support.*

Remark. Theorem 5 provides alternative proof of Maruta's theorem on the extendability of codes with weights $-2, -1, 0 \pmod q$ [13]. The existence of such a code is equivalent to that of an arc $\mathcal{K}$ which is 2-quasidivisible modulo q. It was pointed out in [7,8] that for every t-quasidivisible arc $\mathcal{K}$ in Σ it is possible to define uniquely a $(t \mod q)$- arc $\widetilde{\mathcal{K}}$ in the dual geometry. If $\widetilde{\mathcal{K}}$ contains a hyperplane in its support then $\mathcal{K}$ is extendable. This is the fact established in Theorem 5.

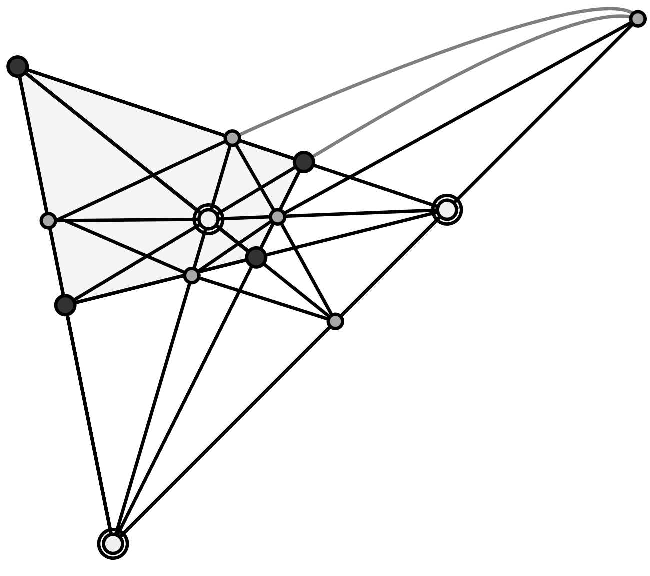
4. $(3 \mod Q)$-ARCS

For values of t larger than 2 complete classification seems out of reach. However, it is still possible to obtain partial results on the structure of such arcs. In this section we classify some small (3 mod 5)-arcs in PG(2, 5). Due to Theorem 4, the classification of such arcs is equivalent to the classification of certain blocking sets with an additional restriction on the line multiplicities.

Arcs of cardinality 18. These arcs are (18, 3)-blocking sets and hence the sum of three not necessarily different lines [9,11]. It is an easy check that there exist four (3 mod 5)-arcs of cardinality 18. They are given in the pictures below.



Arcs of cardinality 23. These arcs correspond to (9, 1)-blocking sets with lines of multiplicity 1, 2, 3, 4. Hence blocking sets containing a full line do not give (3 mod 5)-arcs. Thus the only possibility is the projective triangle. Dualizing we get a (3 mod 5)-arc in which the 2-points form a complete quadrangle, the intersections of the diagonals are 3-points and the intersections of the diagonals with the sides of the quadrangle are 1-points. This arc is presented in the picture below. The doubly circled points are 3-points; the big black points are 2-points and the small gray points are 1-points.

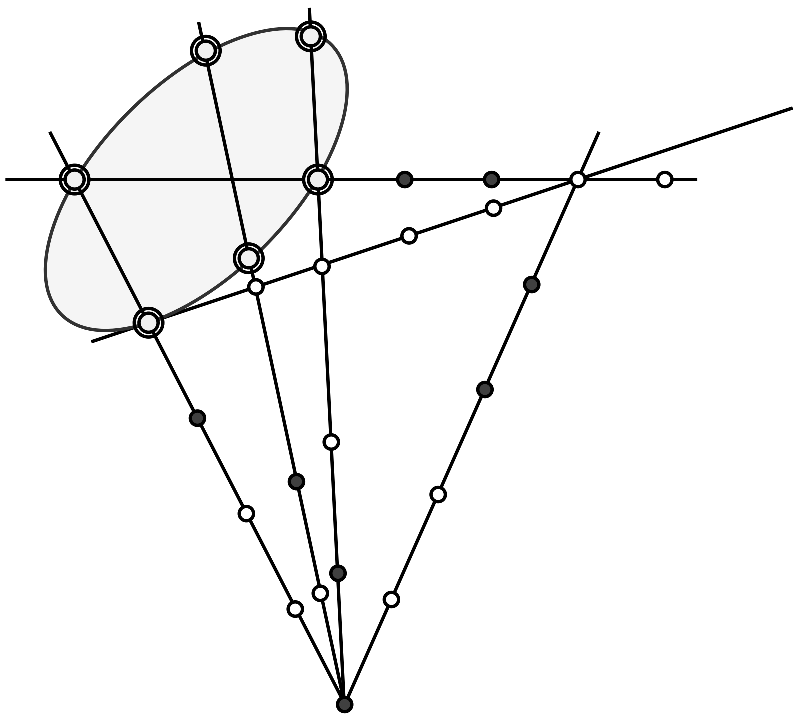


Arcs of cardinality 28. These arcs are obtained from (15,2)-blocking sets with lines of multiplicity 2, 3, 4, or 5. If such a blocking set does not have multiple points it is obtained as the complement of a (16,4)-arc. Such an arc should not have external lines since the maximal multiplicity of a line with respect to the blocking set is 5. The classification of the (16,4)-arcs is well-known. There exists exactly one such arc without external lines obtained by deleting the common points of six lines in general position from the plane. Now we are going to prove that a (15,2)-blocking set having points of multiplicity greater than 1 always has a line of multiplicity 6 and hence does not give a $(3 \mod 5)$-arc.

Let us note that such a blocking set cannot have a point of multiplicity 3. In this case the remaining 12 points would form a (12,2)-blocking set which is the sum of two lines and therefore has a line of multiplicity greater than 6.

Denote by Λ_i, $i = 0, 1, 2$, the number of i-points of a $(15, 2)$-blocking set. Clearly $\Lambda_2 \leq 6$ since the colinearity of three 2-points implies the existence of a 6-line. In the case of $\Lambda_2 = 4, 5, 6$, it is easily checked that the remaining 1-points cannot block twice each of the external lines. The remaining possibilities $1 \leq \Lambda_2 \leq 3$ are ruled out using additional arguments.

Thus the only $(3 \mod 5)$-arc of cardinality 28 has six 3-points forming an oval and ten 1-points that are the internal points to this oval.



Arcs of cardinality 33. If $\mathcal{F}$ is such an arc then $\mathcal{F}^\sigma$ is a $(21, 3)$-blocking set with line multiplicities $3, 4, 5, 6$. Again such a blocking set cannot have points of multiplicity 3 or larger since this would impose lines of multiplicity larger than 6 in $\mathcal{F}$.

Denote by Λ_i the number of points of multiplicity i. Since there cannot be five collinear 0-points we have $\Lambda_0 \leq 16$ and therefore $\Lambda_2 \leq 6$. We are going to rule out the case $\Lambda_2 = 6$. Assume there exist three collinear 2-points. There exist two lines containing three 2-points. They must necessarily meet in a 2-point. Now since a 0-point on a 6-line is incident with 3-lines only, a simple counting gives that the sixth point of multiplicity 2 is incident with three 2-lines. Counting the multiplicities through the exceptional 2-point, we get $21 \geq 3 \cdot 3 + 3 \cdot 6 - 5 \cdot 2 = 17$. Hence the 2-points form an oval. Now the ten external lines to the oval have to be blocked at least three times each by the 1-points. Since each 1-point blocks at most three external lines we need at least $3 \cdot 10/3$ such points, a contradiction since $\Lambda_1 = 9$.

The cases $3 \leq \Lambda_2 \leq 5$ are ruled out in a similar fashion.

For $\Lambda_2 = 0, 1, 2$ constructions are possible. In such case, $\mathcal{F}^\sigma$ is one of the following:

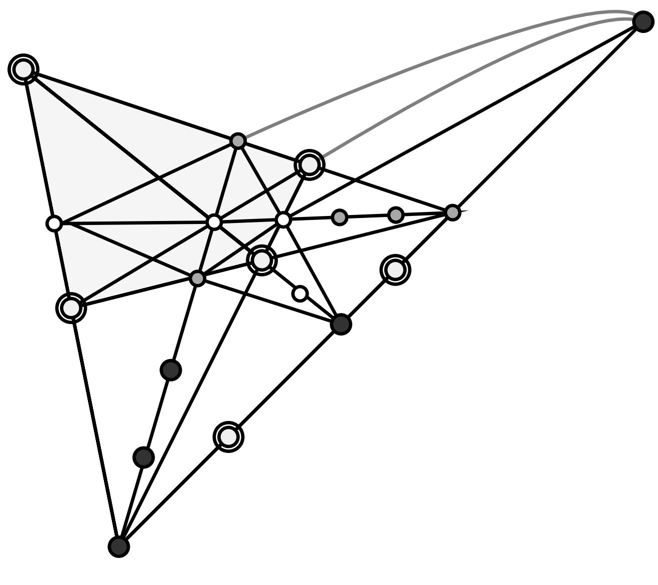
(1) the complements of the seven non-isomorphic $(10, 3)$-arcs; $\Lambda_2 = 0$;

(2) the complement of the $(11, 3)$-arc with four external lines and a double point – a point not on an external line, $\Lambda_2 = 1$;

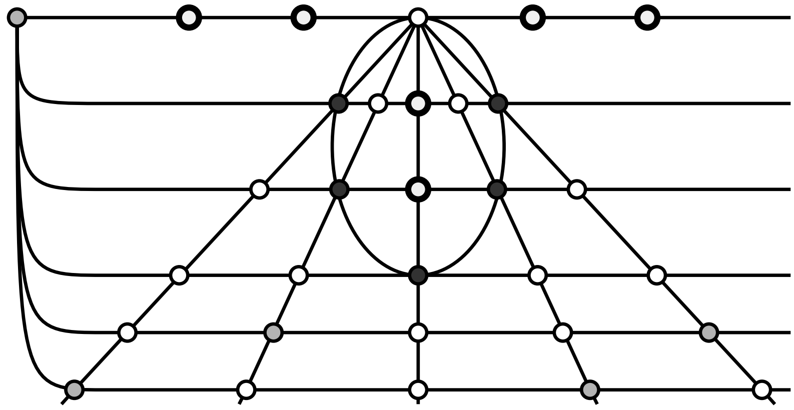
(3) one double point which forms an oval with five of the 0-points; the tangent in the 2-point is a 3-line, $\Lambda_2 = 1$;

(4) PG$(2, 5)$ minus a triangle with vertices of multiplicity $2, 2, 1$; $\Lambda_2 = 2$.

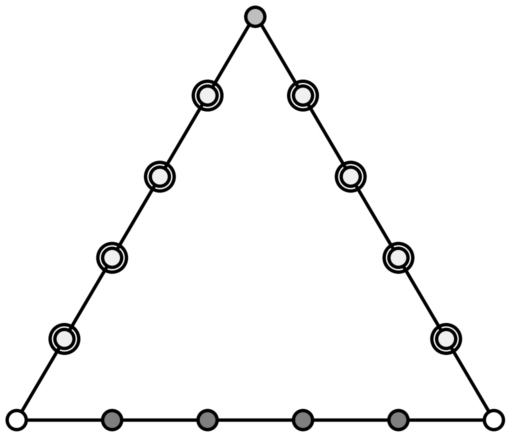
(2) The first $(3 \mod 5)$-arc of cardinality 33 with one 13-line



(3) the second $(3 \mod 5)$-arc of cardinality 33 with one 13-line



(4) $(33, \{3, 8, 13\})$-arc with two 13-lines



Arcs of cardinality 38. The $(3 \mod 5)$-arcs of cardinality 38 can be derived from the (27,4)-blocking sets with line multiplicities $4, 5, 6, 7$ in $\mathrm{PG}(2, 5)$. Such a blocking set does not have 3-points. Otherwise, removing a 3-point would give a $(24, 4)$-blocking set which is a sum of line. This forces a line of multiplicity greater than 7. If there exist three collinear 2-points then $\Lambda_2 = 3$ and the corresponding line is a 7-line.

There exist a lot of such blocking sets and, consequently, $(3 \mod 5)$-arcs of cardinality 38. In all cases, such arcs have a 13-line with a 0-point or an 8-line of type $(2, 2, 2, 2, 0, 0)$, $(2, 2, 2, 1, 1, 0)$ or $(3, 3, 2, 0, 0, 0)$.

For instance, in the case of $\Lambda_2 = 0$ the blocking set consists of all points in the plane minus four points in general position. The corresponding $(3 \mod 5)$-arc has a line of type $(2, 2, 2, 1, 1, 0)$. In the case $\Lambda_2 = 6$ the 2-points form an oval. Th external points to this oval have to be blocked at least four times by the fifteen 1-points. An easy counting gives that we should take necessarily the ten internal points plus five external points. But now the six tangents cannot be blocked twice by six points not on the oval. The remaining cases are treated using similar arguments.

Now we can prove our main result for this section. The following observation turns out to be very useful. Let $\mathcal{F}$ be a $(3 \mod 5)$-arc in $\mathrm{PG}(3,5)$ and consider a projection φ from a 0-point P onto some plane π not incident with P. Set

$$\mathcal{G} = \frac{1}{5}(\mathcal{F}^{\varphi} - 3). \tag{4.1}$$

Now lines through P of multiplicity $3+5i$, $i = 0, 1, 2$, become i-points. The following lemma restricts the possible structure of $\mathcal{G}$.

Lemma 3. *Let XY and XZ be 2-lines in π with respect to $\mathcal{G}$ and let there exist an 1-point $U \neq X, Y, Z$ which is incident with a 2-line. Then U does not lie on a tangent of $\mathcal{G}$.*

Proof. Let t be the tangent through U and let $t \cap XY = V$, $t \cap XZ = W$. Obviously $V \neq X, Y$, $W \neq X, Z$. Since U, V, W are on 2-lines that are the image of 28-planes, they are the image of 2-lines without 2-points. Then the preimage of t is a 23-plane with at most three 2-points, a contradiction since a 23-plane contains four 2-points. □

Theorem 6. *Every $(3 \mod 5)$-arc $\mathcal{F}$ in $\mathrm{PG}(3,5)$ with $|\mathcal{F}| \leq 158$ is a lifted arc. In particular, $|\mathcal{F}| = 93, 118,$ or 143.*

Proof. Assume there exists a 13-line L with 0-point. By the classification of the plane $(3 \mod 5)$-arcs we have that all planes through such a line have multiplicity at least 33. If there exists a 33-plane, π say, through L then it must be of type (2), (3), or (4). In the first two cases π is incident with an 8-line of type (3,3,2,0,0,0), while in the third case it is incident with a line of type (2,2,2,2,0,0). Planes of multiplicity less than 33 do not contain such lines. Hence $|\mathcal{F}| \geq 8 + 6 \cdot 25 = 158$. If all planes through L are of cardinality ≥ 38, then again $|\mathcal{F}| \geq 13 + 6 \cdot 25 = 163$, a contradiction.

If $|\mathcal{F}| = 158$ then there exists a 33-plane of the type (2), (3), or (4). Assume there exists a 33-plane of type (2). It contains a line of type (2,2,2,1,1,0). Consider a projection from the 0-point on this line. The induced arc has thirteen 8-points and eighteen 3-points. We cannot have a line incident with one, two or six 8-points. Now by an easy counting we get that there are no lines with four or five 8-points, a contradiction since the number of 8-points is 13 and the largest $(n, 3)$-arc has 11 points.

The case of planes with 33 points of type (3) and (4) are ruled out in a similar way. Thus 0-points are incident with 3- or 8-lines only.

Further, a line containing a 0-point has multiplicity at most 48. It is easily checked that 48-planes are impossible. In such a plane each 8-line is incident with exactly two 0-points and the 0-points must form an oval. But an oval in $PG(2,5)$ has 6 points while a 48-plane has seven 0-points.

The restriction of $\mathcal{F}$ to a 43-plane in which every line through a 0-point is necessarily lifted from an 8-line. The planes through a line of type (3,1,1,1,1,1) in a 43-plane are either lifted 43-planes or 18-planes that are again lifted. Hence in such case $\mathcal{F}$ is necessarily a lifted arc of size 118 or 143 since the 3-point on the line of type (3,1,1,1,1,1) is the lifting point of the 43- as well as of the 18-planes.

Finally, the fact that 38-planes have either a 13-line with a 0-point or an 8-line of type $(2,2,2,2,0,0)$, $(2,2,2,1,1,0)$ or $(3,3,2,0,0,0)$ implies that such planes are impossible if $|\mathcal{F}| \leq 158$. Thus we can assume with no loss of generality that every plane incident with a 0-point has multiplicity 18, 23, 28, or 33. Moreover, a 33-plane should necessarily be of type (1).

Now consider the arc $\mathcal{G}$ defined in (4.1). Since it does not have 4-lines $|\mathcal{G}| \leq 11$ and $\mathcal{F} \leq 148$. These cases are ruled out easily by Lemma 3. □

5. AN EXAMPLE FROM CODING THEORY

One of the forms of the main problem of coding theory is to determine the minimal length of an $[n,k,d]_q$-code for fixed q,k and d [10]. For codes over $\mathbb{F}_5$ of dimension 4 there exist four values of d for which $n_5(4,d)$ is not decided [12]. The results from the previous section enable us to solve one of the four open cases. We can rule out the existence of codes with parameters $[104,4,82]_5$ which implies that $n_5(4,82) = 105$.

The approach to this problem is geometric. The existence of a $[104,4,82]_5$-code is equivalent to the existence of a $(104,22)$-arc in $PG(3,5)$ (cf. [2,6,10]). Such a hypothetical arc will turn out to be non-extendable.

Assume that $\mathcal{K}$ is a $(104,22)$-arc in $PG(3,5)$. Let us denote by δ_i, $i = 0,1,2$, the maximal multiplicity of an i-dimensional subspace in $PG(3,5)$. In the following lemma, we summarize the straightforward properties of $(104,22)$-arcs.

Lemma 4. *Let $\mathcal{K}$ be a $(104,22)$-arc with spectrum (a_i). Then*

(a) $\delta_0 = 1$, $\delta_1 = 5$, $\delta_2 = 22$;

(b) The maximal multiplicity of a line in an m-plane is $\lfloor (6+m)/5 \rfloor$;

(c) There do not exist planes with 2, 3, 7, 8, 12, 13, 17, 18 *points.*

(d) $a_0 = 0$.

(e) $a_1 = 0$.

(f) $a_4 = a_5 = 0$

(g) The spectrum of $\mathcal{K}$ satisfies the following identity

$$\sum_{i=0}^{20} \binom{22-i}{2} a_i = 468. \tag{5.1}$$

By Lemma 4, a $(104,22)$-arc $\mathcal{K}$ is 3-quasidivisible. Moreover, 0-points with respect to the dual arc $\widetilde{\mathcal{K}}$ must come necessarily from maximal planes. This forces certain restrictions on the structure of $\widetilde{\mathcal{K}}$ described in the lemma below.

Lemma 5. *Let $\mathcal{K}$ be a $(104,22)$-arc in $\mathrm{PG}(3,5)$. Then there exists no plane $\widetilde{P}$ in the dual space such that $\widetilde{\mathcal{K}}|\widetilde{P}$ is $3\chi_{\widetilde{L}}$ for some line $\widetilde{L}$ in the dual space.*

Proof. Let X be a point in $\mathrm{PG}(3,5)$. Summing up the multiplicities of all planes through X, we have:

$$\sum_{H:H\ni X} \mathcal{K}(H) = 6|\mathcal{K}| + 25\mathcal{K}(X).$$

On the other hand, a point $\widetilde{H}$ in the dual space with $\widetilde{K}(\widetilde{H}) = 0$ comes necessarily from a maximal plane. For the points on the line L with $\widetilde{K}(\widetilde{L}) = 18$ we have

$$\sum_{\widetilde{P}:\widetilde{P}\in\widetilde{L}} \mathcal{K}(\widetilde{P}) = |\mathcal{K}| + 5\mathcal{K}(L).$$

This implies that

$$6|\mathcal{K}| + 25\mathcal{K}(X) = 25\cdot 22 + |\mathcal{K}| + 5\mathcal{K}(L),$$

which gives

$$649 \geq 6|\mathcal{K}| + \mathcal{K}(X) = 654 + 5\mathcal{K}(L),$$

a contradiction. □

Lemma 6. *Let $\mathcal{K}$ be a $(104,22)$-arc in $\mathrm{PG}(3,5)$. Then $|\widetilde{\mathcal{K}}| \geq 163$.*

Proof. This follows by Lemma 5, Theorem 6 and the fact that a $(104,22)$-arc is not extendable. □

We can use Lemma 6 together with the necessary condition (5.1) to restrict further the possible multiplicities of planes. Our key observation is that if a 5-tuple of planes through a a line L in H_0 gives a high contribution to the left-hand side of (5.1) then $\widetilde{K}(\widetilde{L})$ is small.

Lemma 7. *Let $\mathcal{K}$ be a $(104,22)$-arc in $\mathrm{PG}(3,5)$. Then $a_6 = 0$.*

Proof. Let H_0 be a 6-plane. Then $\mathcal{K}|_{H_0}$ is a $(6,2)$-arc and has spectrum $a_2 = 15, a_1 = 6, a_0 = 10$. Consider an arbitrary line L in H_0. By Theorem 4, if L is a 2-line with respect to $\mathcal{K}$, then it is a 3-line with respect to $\widetilde{\mathcal{K}}$; similarly, if L is a 1-line it is a 3-line with respect to $\widetilde{\mathcal{K}}$ (since 22-planes do not have 1-lines) and, finally, if it is a 0-line with respect to $\mathcal{K}$, it is a 3-, 8- or 13-line with respect to $\widetilde{\mathcal{K}}$.

In the case of $\mathcal{K}(L) = 0$ and $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$ the maximal contribution of the planes through L is 66 obtained for

$$(\mathcal{K}(H_0), \dots, \mathcal{K}(H_5)) = (6, 22, 22, 22, 22, 10);$$

if $\widetilde{\mathcal{K}}(\widetilde{L}) = 8$ the maximal contribution of the planes through L is 31 obtained for

$$(\mathcal{K}(H_0), \dots, \mathcal{K}(H_5)) = (6, 22, 22, 21, 19, 14);$$

and if $\widetilde{\mathcal{K}}(\widetilde{L}) = 13$ the maximal contribution of the planes through L is 13 obtained for

$$(\mathcal{K}(H_0), \dots, \mathcal{K}(H_5)) = (6, 22, 19, 19, 19, 19).$$

Let us denote by x the number of 0-lines L of H_0 with $\widetilde{\mathcal{K}}(H_0) = 3$ and by y the number of such lines with $\widetilde{\mathcal{K}}(H_0) = 8$. Counting the contribution of the different planes through the lines of L we get

$$\binom{16}{2} + 15 \cdot 1 + 6 \cdot 3 + 66x + 31y + 13(10 - x - y) \geq 468,$$

whence $53x + 18y \geq 185$. On the other hand, we have

$$|\widetilde{\mathcal{K}}| = 121 \cdot 2 + 2x + 7y + 12(10 - x - y) = 163 - 5x - 10y.$$

Since $\mathcal{K}$ is not extendable we have $|\widetilde{\mathcal{K}}| \geq 163$, and hence $x + 2y \leq 0$, i.e. $x = y = 0$, a contradiction to $53x + 18y \geq 185$. □

Lemma 8. *Let $\mathcal{K}$ be a $(104, 22)$-arc in $\mathrm{PG}(3, 5)$. Then $a_9 = a_{10} = a_{11} = 0$.*

Proof. We use the classification of the (9,3), (10,3)- and (11,3)-arcs made in [5]. We will demonstrate only the non-existence of 9-planes of type C4 (we use the notation from [5]). The non-existence of 9-planes of the other three types, as well as the non-existence of 10- and 11-planes, is done analogously.

Let H_0 be a 9-plane and let $\mathcal{K}|_{H_0}$ be a $(9,3)$-arc of type C4. For a arbitrarily fixed line L in H_0 we denote by $H_1, \dots, H_5$ the other 5 planes through L. We have the following possibilities:

$\mathcal{K}(L)$	$\widetilde{\mathcal{K}}(\widetilde{L})$	η_i	$(\mathcal{K}(H_0), \dots, \mathcal{K}(H_5))$
3	3	0	(22,22,22,22,22,9)
2	8	4	(22,22,22,2,19,9)
1	8	15	(21,21,21,21,16,9)
1	13	7	(21,21,20,19,19,9)
0	8	79	(22,22,22,19,10,9)
0	13	34	(22,21,19,19,14,9)
0	18	15	(19,19,19,19,19,9)

Counting the contribution of the planes through the different lines in H_0 to the left-hand side of (5.1), we get

$$\binom{12}{3} + 7 \cdot 0 + 15 \cdot 4 + 15x + 7 \cdot (3 - x) + 79u + 34v + 15(6 - u - v) \geq 468,$$

whence $8x + 64u + 19v \geq 219$.

On the other hand, computing the cardinality of $\widetilde{\mathcal{K}}$ and taking into account that $\widetilde{\mathcal{K}}(\widetilde{H}) = 3$, we get

$$3 + 7 \cdot 0 + 15 \cdot 5 + 5x + 10(3 - x) + 5u + 10v + 15(6 - u - v) \geq 163,$$

whence $x + 2u + v \leq 7$. Now we have the chain of inequalities

$$224 \geq 32x + 64u + 32v \geq 8x + 64u + 19v \geq 219.$$

This implies that $x = v = 0$, which in turn gives $224 \geq 64u \geq 219$, a contradiction since u is an integer. □

Now using once more the same idea we can prove the nonexistence of $(104, 22)$-arcs.

Theorem 7. *There is no* $(104, 22)$*-arc in* $\mathrm{PG}(3, 5)$.

Proof. We apply the above technique to the three non-isomorphic $(22, 5)$-arcs. Their spectra are given below.

Type	a_0	a_1	a_2	a_3	a_4	a_5
D1	1	0	1	0	15	14
D2	1	0	0	3	12	15
D3	0	0	3	4	6	18

Let H_0 be a fixed 22-plane. For a line L in H_0 we have the following possibilities:

$\mathcal{K}(L)$	$\widetilde{\mathcal{K}}(\widetilde{L})$	η_i	$(\mathcal{K}(H_0), \ldots, \mathcal{K}(H_5))$
5	3	3	(22,22,22,22,22,19)
4	3	28	(22,22,22,22,22,14)
4	8	7	(22,22,22,22,22,14)
3	3	36	(22,22,22,22,16,15)
3	8	32	(22,22,22,20,19,14)
3	13	13	(22,20,20,19,19,19)
2	3	45	(22,22,22,16,16,16)
2	8	57	(22,22,22,20,14,14)
2	13	37	(22,21,19,19,19,14)
0	8	86	(22,22,16,16,14,14)
0	13	87	(22,21,19,14,14,14)

(D1) Denote by x the number of lines L in H_0 of multiplicity 4 for which $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$. Counting the contribution of the planes through the different lines in H_0 to the left-hand side of (5.1), we get

$$14 \cdot 3 + 28x + 7(15 - x) + 1 \cdot 57 + 1 \cdot 87 \geq 468,$$

whence $21x \geq 177$, i.e. $x \geq 9$. On the other hand,

$$|\widetilde{\mathcal{K}}| \leq 14 \cdot 3 + x \cdot 3 + (15 - x) \cdot 8 + 13 = 13,$$

whence $|\widetilde{\mathcal{K}}| \leq 188 - 5x$. This implies $188 - 5x \geq 163$, i.e. $x \leq 5$, a contradiction.

(D2) Denote by x the number of 4-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$; by u – the number of 3-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$, and by v – the number of 3-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 8$. Again counting the contribution to the left-hand side of (5.1), we have

$$15 \cdot 3 + x \cdot 28 + (12 - x) \cdot 7 + u \cdot 36 + v \cdot 32 + (3 - u - v)12 + 1 \cdot 87 \geq 468,$$

whence $21x + 24u + 20v \geq 216$. On the other hand,

$$|\widetilde{\mathcal{K}}| = 15 \cdot 3 + 3x + 8(12 - x) + 3u + 8v + (3 - u - v) \cdot 13 + 13 \geq 163,$$

$x + 2u + v \leq 6$. Now we get

$$126 \geq 21x + 42u + 21v \geq 21x + 24u + 20v \geq 216,$$

a contradiction.

(D3) Let x, u and v be as above. Denote also by s the number of 2-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$, and by t – the number of 2-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 8$. Once again:

$$18 \cdot 3 + x28 + (6 - x) \cdot 7 + u36 + v32 + (4 - u - v)12 + s \cdot 45 + t \cdot 57 + (3 - s - t)37 \geq 468,$$

whence $21x + 24u + 20v + 8s + 20t \geq 213$. On the other hand

$$|\widetilde{\mathcal{K}}| = 18 \cdot 3 + 3x + 8(6 - x) + 3u + 8v + 13(4 - u - v) + 3s + 8t + (13(3 - s - t) \geq 163,$$

hence $x + 2u + v + 2s + t \leq 4$. This implies

$$84 \geq 21x + 42u + 21v + 42s + 21t \geq 21x + 24u + 21v + 42s + 21t \geq 213,$$

a contradiction. □

Corollary 1. *There exists no* $[104, 4, 82]_5$*-code and, consequently,* $n_5(4, 82) = 105$.

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