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LJUSTERNIK-SCHNIRELMAN CATEGORY OF THE NON-WANDERING SET

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Симеон Стефанов. КАТЕГОРИЯ ЛЮСТЕРНИКА-ШНИРЕЛЬМАНА МНОЖЕСТВА НЕБЛУЖДАЮЩИХ ТОЧЕК

В работе рассмотрены динамические системы на многообразии M, удовлетворяющие некоторое условие, более общее, чем аксиому A + условия отсутствия циклов и значит выполненным для систем Морса–Смейла. Получены оценки снизу для категории Люстерника–Шнирельмана множества неблуждающих точек Ω такой системы. Доказаны неравенства:

а) $\operatorname{cat}(\Omega, M) \geqq \frac{1}{s} \operatorname{cat} M$;

б) $\operatorname{cat}\Omega \geqq \operatorname{cat} M$,

где s обозначает число базисных множеств Ω_i. Получены некоторые применения этого результата.

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The paper deals with dynamical systems in some manifold M satisfying some condition, which is more general than axiom A + no-cycle condition and consequently is fulfilled for Morse-Smale systems. Some low estimates for the Ljusternik-Schnirelman category of the non-wandering set Ω of such a system are obtained. Namely, the following inequalities are proved

a) $\operatorname{cat}(\Omega, M) \geqq \frac{1}{s} \operatorname{cat} M$

b) $\operatorname{cat}\Omega \geqq \operatorname{cat} M$,

where s is the number of the basic sets Ω_i. Some applications of this result are obtained.

We give in this note some low estimates of the Ljusternik-Schirelman category of the non-wandering set Ω for a given flow or diffeomorphism satisfiyng some condition (Theorem 1). This condition is always fulfilled for axiom A + no-cycle condition dynamical systems and for Morse–Smale systems. However, we admit the existence of cycles (only 1-cycles are forbidden). By means of this result we obtain some information and low estimates of the number of critical elements for a given Morse–Smale flow (Theorem 2). Other possible applications are illustrated by the proposition at the end, where we estimate the covering dimension of the critical set of a smooth function with symmetry. The proofs are quite elementary and do not make use of hard algebraic topology arguments.

Recall first some definitions.

Let A be closed subset of M. The Ljusternik–Schnirelman category of A in M is the smallest natural number k such that $A = A_1 \cup \ldots \cup A_k$, where A_i are closed and contractible in M into a point. Then we write

$$\operatorname{cat}(A, M) = k.$$

We shall note for convenience

$$\operatorname{cat} M = \operatorname{cat}(M, M).$$

For the properties of the Ljusternik–Schnirelman category see for example [4]. Given a flow in M, the α and ω-limit sets of a point x are defined as usually:

$$\alpha(x) = \{y \in M \mid t_n x \to y \quad \text{for some } t_n \to -\infty\},$$

$$\omega(x) = \{y \in M \mid t_n x \to y \quad \text{for some } t_n \to +\infty\}$$

(see [3, 5] for details).

The non-wandering set Ω of the flow consists of all points $x \in M$ such that for any open $U \ni x$ and for any $t_0 > 0$ there exists $t \geqq t_0$ such that $tU \cap U \neq \varnothing$.

A subset $V \subset M$ is called *unrevisited*, if $t_1 x \in V$ and $t_2 x \in V$ imply $tx \in V$ for any $t \in [t_1, t_2]$.

We shall suppose that

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_s,$$

where the basic sets Ω_i are disjoint, closed, invariant and for any $x \in M$ we have $\alpha(x) \subset \Omega_i$, $\omega(x) \subset \Omega_j$ for some i, j. Consider the sets

$$N_i = \{x \in M \setminus \Omega \mid \omega(x) \subset \Omega_i\}.$$

We shall formulate now the condition mentioned above.

(*) **Condition.**

(i) $\overline{N}_i \cap N_j = \varnothing$ for $i < j$.

(ii) Each basic set Ω_i has a base of unrevisited open neighbourhoods.

(iii) If $\alpha(x) \subset \Omega_i$ and $\omega(x) \subset \Omega_j$, then $i \neq j$ (no 1-cycle condition).

In the case of a discrete time dynamical system, defined by a diffeomorphism $f : M \to M$, all the definitions are reformulated in an obvious way.

Theorem 1. *Let M be a closed connected manifold with a C^1-flow (diffeomorphism), satisfying Condition (*). Then for the category of the non-wandering set Ω we have the inequalities*

a) $\operatorname{cat}(\Omega, M) \geqq \frac{1}{s} \operatorname{cat} M$;

b) $\operatorname{cat}\Omega \geqq \operatorname{cat} M$;
where s is the number of the basic sets Ω_i.

For the proof of this theorem we need two technical lemmas.

Lemma 1. *Let A be a compact and invariant (with respect to some given flow) subset of the closed manifold M and there is no such a point $x \in M$ that $\alpha(x) \cap A \neq \varnothing$ and $\omega(x) \cap A \neq \varnothing$. Let U and V be open subsets of M such that U is unrevisited, $U \supset A$, $U \cap V = \varnothing$ and for any $x \in M$ either $\omega(x) \subset A$ or $\omega(x) \subset V$. Let, finally $F \subset M \setminus V$ be closed and $\omega(x) \subset A$ for any $x \in F$. Then there exists t_0 such that $tF \subset U$ for $t \geqq t_0$.*

Proof. Since F is compact, it is enough to show, that for every $x \in F$ there are a neighbourhood Ox and t_0 such that $t(Ox \cap F) \subset U$ for $t \geqq t_0$. Consider the sets

$$F' = \{x \in F \mid tx \in M \setminus V \text{ for } t \geqq 0\}, \quad F^0 = \{tx \mid x \in F', t \geqq 0\}.$$

Clearly, F^0 is positively invariant (i. e. $tF^0 \subset F^0$ for $t \geqq 0$). Consider the closure $\Phi = \overline{F^0}$. It is compact, positively invariant and $\Phi \subset M \setminus V$. Evidently $\omega(x) \subset A$ for $x \in \Phi$.

We shall prove first the local assertion about Φ — that for any $x \in \Phi$ there are Ox and t_0 such that $t(Ox \cap \Phi) \subset U$ for $t \geqq t_0$. Suppose, this is not true. Then we can find a sequence $x_n \to x_0$, $x_n \in \Phi$, and positive numbers $t_n \to \infty$ such that $t_n x_n \notin U$. Passing to a subsequence we may suppose that $t_n x_n \to z_0$. Then $z_0 \in \Phi$ since Φ is positively invariant and closed.

Now we shall show that $\alpha(z_0) \cap A \neq \varnothing$. Consider the arcs $[x_n,\, t_n x_n]$, where $x_n \to x_0$, $t_n x_n \to z_0$. Suppose first, that the limit set of these arcs does not intersect $\alpha(z_0)$, then x_0 and z_0 lie in one and the same trajectory and $[x_n,\, t_n x_n] \to [x_0,\, z_0]$. Since $\omega(x_0) \subset A$, there exists a sequence $y_n = \theta_n x_n$ such that $0 < \theta_n < t_n$ and $y_n \to y_0 \in A$. But then $y_0 \in [x_0,\, z_0]$, i.e. $y_0 = \theta_0 x_0$ which is a contradiction, since $x_0 \notin A$ and A is invariant.

So, the limit set of the arcs $[x_n,\, t_n x_n]$ intersects $\alpha(z_0)$. Then we can find a sequence $y_n = \theta_n x_n$ with $0 < \theta_n < t_n$ such that $y_n \to y_0 \in \alpha(z_0)$. But $x_n \in \Phi$, hence $y_n \in \Phi$ (Φ is positively invariant). Then $y_0 \in \Phi$ and therefore $\omega(y_0) \subset A$. But $\alpha(z_0)$ is invariant and $y_0 \in \alpha(z_0)$, consequently $\omega(y_0) \subset \alpha(z_0)$ and $\alpha(z_0) \cap A \neq \varnothing$. On the other hand, $z_0 \in \Phi$ hence $\omega(z_0) \subset A$ which contradicts the conditions of the lemma.

Since Φ is compact, $t_1\Phi \subset U$ for some $t_1 > 0$. Then $t_1(O\Phi) \subset U$ for some open neighbourhood $O\Phi$. Note that $t_2 F \subset O\Phi$ for some $t_2 > 0$. Really, if $x \in F$ then $tx \in \Phi$ for some $t \geqq 0$ and there is an open neighbourhood Ox such that $tOx \subset O\Phi$. Therefore $t_2 F \subset O\Phi$ for some $t_2 \geqq 0$. Then $t_1 t_2 F \subset t_1 O\Phi \subset U$ and $t_1 t_2 \geqq 0$. If now $t \geqq t_1 t_2$, then $tF \subset U$ since U is unrevisited and for any $x \in F$ we have $\omega(x) \subset A \subset U$.

The lemma is proved.

Lemma 2. *Let A be a closed invariant subset of M which has a base of unrevisited neighbourhoods and F be a closed set such that for any neighbourhood $U \supset A$ we have $tF \subset U$ for t sufficiently large. Then there exists an open $V \supset A$, such that*

$$\operatorname{cat}(F \cup \overline{V},\, M) = \operatorname{cat}(A,\, M)$$

Proof. Take such open unrevisited U, V that $A \subset V \subset \overline{V} \subset U$ and $\operatorname{cat}(\overline{U}, M) = \operatorname{cat}(A, M)$. Let $\lambda : [0, 1] \to M$ be a continuous map, such that $\lambda(\overline{V}) = 0$, $\lambda(M \setminus U) = 1$. We have $tF \subset V$ for some $t \geqq 0$. Let $\varphi(x) = t\lambda(x).x$. It is easy to see, that $\varphi(F \cup \overline{V}) \subset U$. Really, for $x \in \overline{V}$ we have $\varphi(x) = x$ and for $x \in F \setminus U$ — $\varphi(x) = tx \in V$. If $x \in U \cap F$, then $\varphi(x) \in U$ since $\omega(x) \subset U$ and U is unrevisited.

But evidently φ is homotopic to the identity, $\varphi \sim id_M$ and by the elementary properties of the Ljusternik-Schnirelman category we obtain

$$\operatorname{cat}(F \cup \overline{V}, M) \leqq \operatorname{cat}(\overline{U}, M) = \operatorname{cat}(A, M).$$

The inverse inequality is obvious.

Proof of Theorem 1. Take open $V_i \supset \Omega_i$ satisfying the conditions of Lemma 2. Then the set $F_1 = N_1 \setminus \bigcup_{i=2}^{s} V_i$ is closed in M as following from Condition (i). According to Lemma 1, we have that for any open $U \supset \Omega_1$ there exists t_0 such that $tF_1 \subset U$ for $t \geqq t_0$. Then Lemma 2 implies $\operatorname{cat}(F_1 \cup \overline{V_1}, M) = \operatorname{cat}(\Omega_1, M)$. We may find an open neighbourhood $W_1 \supset F_1 \cup \overline{V_1}$ with $\operatorname{cat}(\overline{W_1}, M) = \operatorname{cat}(\Omega_1, M)$. Set $F_2 = N_2 \setminus (\bigcup_{i \neq 2} V_i \cup W_1)$. We shall prove, that it is closed in M. Really, if $x \in \overline{F_2}$, then $x \in \overline{N_2}$ and $x \notin \bigcup_{i \neq 2} V_i \cup W_1$. If we suppose $x \notin N_2$ then $x \in N_1$ (as following from (i)), hence $x \in F_1$ therefore $x \in W_1$, which is a contradiction.

By the same reasoning we obtain from Lemma 1 and Lemma 2, that $\operatorname{cat}(F_2 \cup \overline{V_2}, M) = \operatorname{cat}(\Omega_2, M)$. Take $W_2 \supset F_2 \cup \overline{V_2}$ such that $\operatorname{cat}(\overline{W_2}, M) = \operatorname{cat}(\Omega_2, M)$. Proceeding by induction we define closed sets F_k and their open neighbourhoods $F_k = N_k \setminus (\bigcup_{i \neq k} V_i \cup \bigcup_{j=1}^{k-1} W_j)$, $W_k \supset F_k \cup \overline{V_k}$ such that

$$\operatorname{cat}(\overline{W_k}, M) = \operatorname{cat}(\Omega_k, M).$$

It is easy to show, that $\bigcup W_i = M$. Really, if we suppose, that $x \notin \bigcup W_i$, then $x \notin \bigcup V_i$, and if now $x \in N_k$, then $x \in F_k$ whereby $x \in W_k$ — contradiction. So

$$\sum_i \operatorname{cat}(\Omega_i, M) = \sum_i \operatorname{cat}(\overline{W_i}, M) \geqq \operatorname{cat} M.$$

The second inequality is an obvious property of the category. But since Ω_i are disjoint and M is connected,

$$\operatorname{cat}(\Omega, M) = \max \operatorname{cat}(\Omega_i, M)$$

hence

$$\operatorname{cat}(\Omega, M) \geqq \frac{1}{s} \sum_i \operatorname{cat}(\Omega_i, M) \geqq \frac{1}{s} \operatorname{cat} M$$

so a) is proved and b) follows from the inequalities

$$\operatorname{cat} \Omega = \sum \operatorname{cat} \Omega_i \geqq \sum \operatorname{cat}(\Omega_i, M) \geqq \operatorname{cat} M.$$

The theorem is proved.

For diffeomorphisms the proof works with little modifications — we have only to make use of the trivial fact, that $\operatorname{cat} A = \operatorname{cat} f(A)$ for any diffeomorphism $f : M \to M$.

Let us note, that the conditions of Theorem 1 are fulfilled for any axiom A + no-cycle condition flow or diffeomorphism, or any flow (diffeomorphism) whit a Morse-Smale decomposition and consequently for "gradient-like" flows (diffeomorphisms). In all these examples there are no cycles, but the theorem covers quite more general situations, where cycles are admitted. A simple example is given by the flow in S^1 with 2 nonhyperbolic stationary points. On the other hand, Theorem 1 easily implies the classical Ljusternik-Schnirelman theorem, which states that any smooth function on M has at least $\operatorname{cat} M$ critical points. (We only have to consider its gradient flow on M). Theorem 1 also enables us to give low estimates for the covering dimension of the non-wandering set Ω. If M is a closed connected manifold and A is a closed subset, the following inequality holds

$$\operatorname{cat}(A, M) \leqq \dim A + 1.$$

Really, let $\dim A = k$, then for any $\varepsilon > 0$ there is a ε-map $\varphi_\varepsilon : A \to P_k$ into a k-dimensional polyhedron. We have only to note, that P_k may be represented as the union $P_k = \bigcup_{i=1}^{k+1} F_i$ of $k+1$ closed subset, each F_i being a finite union of sufficiently small nonintersecting closed sets. Then, evidently $\operatorname{cat}(\varphi_\varepsilon^{-1}(F_i), M) = 1$, therefore $\operatorname{cat}(A, M) \leqq k + 1$. This inequality and Theorem 1 imply the following

Corollary. *Let M be as in Theorem 1. Then*

$$\dim \Omega \geqq \frac{1}{s} \operatorname{cat} M - 1.$$

Another corollary is obtained for a flow with a finite number of critical elements (stationary points and periodic trajectories). The following theorem gives some low estimates for their number. Its conditions are obviously fulfilled for Morse-Smale flows.

Theorem 2. *Let M be a closed connected manifold with a flow, satisfying Condition (*), whose non-wandering set consists of s_1 stationary points and s_2 periodic trajectories. Let $s = s_1 + s_2$. Then*

1) $s \geqq \frac{1}{2} \operatorname{cat} M$.

2) $s_1 + 2s_2 \geqq \operatorname{cat} M$.

3) *If $s < \operatorname{cat} M$, then there is a periodic trajectory nonhomotopic to zero in M.*

P r o o f. 1) Evidently $\operatorname{cat}(\Omega, M) \leqq 2$ and Theorem 1 gives

$$2 \geqq \operatorname{cat}(\Omega, M) \geqq \frac{1}{s_0} \operatorname{cat} M \geqq \frac{1}{s} \operatorname{cat} M,$$

where s_0 is the number of the basic sets, i. e. $s \geqq \frac{1}{2} \operatorname{cat} M$.

2) Clearly, $\operatorname{cat} \Omega = s_1 + 2s_2$ and again by Theorem 1

$$s_1 + 2s_2 = \operatorname{cat} \Omega \geqq \operatorname{cat} M.$$

3) We have $\operatorname{cat}(\Omega, M) \geqq \frac{1}{s} \operatorname{cat} M > 1$, i.e. $\operatorname{cat}(\Omega, M) \geqq 2$ which means that Ω is not contractible into a point in M, consequently there is a periodic trajectory in Ω, which is nonhomotopic to zero in M.

Another application of Theorem 1 is the estimate of the critical set of a smooth function. It may be ilustrated by the following proposition (this kind of estimates are typical in the Ljusternik-Schnirelman-Morse theory – see for example [2]).

Proposition. *Let G be a finite abelian group, or $G = S^1$, or S^3, acting freely in the n-sphere S^n. Let $f : S^n \to \mathbf{R}^1$ be a smooth function such that $f(gx) = f(x)$ for any $x \in S^n$, $g \in G$, with exactly s critical values. Consider the critical set, where the Jacobian Df vaniches*

$$\Omega = \{x \in S^n \mid Df(x) = 0\}.$$

Then

$$\dim \Omega \geqq \frac{n+1}{s(\dim G + 1)} + \dim G - 1.$$

Proof. It is well-known, that for the category of the orbit space S^n/G (which is a closed manifold) we have

$$\operatorname{cat} S^n/G = \frac{n+1}{\dim G + 1}$$

(see for example [1]).

Clearly, f induces a function on the orbit space $f_G : S^n/G \to \mathbf{R}^1$. Consider now the gradient flow defined by $\operatorname{grad} f_G$. Its non-wandering set coincides with the critical set Ω/G of f_G. Since f_G has s critical values, Ω/G is decomposed into s basic sets. Now, the corollary from Theorem 1 gives

$$\dim \Omega \setminus G \geqq \frac{n+1}{s(\dim G + 1)} - 1$$

which implies, naturally,

$$\dim \Omega \geqq \frac{n+1}{s(\dim G + 1)} + \dim G - 1.$$

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