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# LEAST FIXED POINTS IN MONOIDAL CATEGORIES WITH CARTESIAN STRUCTURE ON OBJECTS

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*Йордан Зашев.* НАИМЕНЬШИЕ НЕПОДВИЖНЫЕ ТОЧКИ В МОНОИДАЛЬНЫХ КАТЕГОРИЯХ С ДЕКАРТОВОЙ СТРУКТУРЕ НА ОБЪЕКТАХ

В работе предложено обобщение теории рекурсии в итеративных оперативных пространствах Иванова. Обобщение состоит в замены частичного порядка на стрелках в категориях. Для этой цели введено понятие DM-категории. Описан пример DM-категории, в котором третируются некоторые идеализованные недетерминистические программы с доказательствами корректности их работ. Развита теория неподвижных точек определимых функторов в DM-категориях, которая содержит категорных аналогов всех основных результатов абстрактной теории рекурсии в итеративных оперативных пространствах.

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The paper contains a generalization of the recursion theory in iterative operative spaces. The generalization consists in replacing the partial order in an operative space with arrows in a category. For that purpose the notion of a DM-category is introduced. An example of a DM-category is described which deals with some kind of idealized nondeterministic programs together with proofs of the correctness of their work. A theory of fixed points of definable functors in DM-categories is developed which contains categorial analogues of all principal results of the abstract recursion theory in iterative operative spaces.

We present a generalization of the recursion theory in iterative operative spaces in the sense of Ivanov [1]. The generalization consists in replacing the partial order in operative spaces by arrows in a suitable category. The structures obtained in this way are called DM-categories. The method we use is based on an unpublished proof of the first recursion theorem in iterative operative spaces and is essentially a proper generalization of the usual method of coding in the ordinary recursion theory.

## 1. DEFINITIONS

### 1.1. DM-CATEGORIES

A DM-category will be a category $\mathcal{F}$ with two bifunctors $\mathbb{M} : \mathcal{F}^2 \to \mathcal{F}$ and $\mathbb{D} : \mathcal{F}^2 \to \mathcal{F}$, three objects $I$, $L$, $R$, and six natural isomorphisms $\boldsymbol{a}$, $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, $\boldsymbol{l}$, $\boldsymbol{r}$, $\boldsymbol{i}$, satisfying (DM1)–(DM8) below. We shall call $\mathbb{M}$ "multiplication" and we shall write $xy$ for $\mathbb{M}(x, y)$, where $x$ and $y$ are objects or arrows in $\mathcal{F}$. Similarly, we shall call $\mathbb{D}$ "cartesian functor", and we shall write $\langle x, y\rangle$ for $\mathbb{D}(x, y)$. Composition of arrows $f$, $g$ in $\mathcal{F}$ will be denoted by $f \circ g$. Conditions defining a DM-category are the following ones:

(DM1) $\boldsymbol{a}$ is an isomorphism $\boldsymbol{a}(\varphi, \psi, \chi) : (\varphi\psi)\chi \cong \varphi(\psi\chi)$, natural in $\varphi$, $\psi$, $\chi$;
(DM2) $\boldsymbol{\lambda}$ is an isomorphism $\boldsymbol{\lambda}(\varphi) : I\varphi \cong \varphi$, natural in $\varphi$;
(DM3) $\boldsymbol{\rho}$ is an isomorphism $\boldsymbol{\rho}(\varphi) : \varphi I \cong \varphi$, natural in $\varphi$;
(DM4) $\boldsymbol{l}$ is an isomorphism $\boldsymbol{l}(\varphi, \psi) : \langle\varphi, \psi\rangle L \cong \varphi$, natural in $\varphi$, $\psi$;
(DM5) $\boldsymbol{r}$ is an isomorphism $\boldsymbol{r}(\varphi, \psi) : \langle\varphi, \psi\rangle R \cong \psi$, natural in $\varphi$, $\psi$;
(DM6) $\boldsymbol{i}$ is an isomorphism $\boldsymbol{i}(\varphi, \psi, \chi) : \varphi\langle\psi, \chi\rangle \cong \langle\varphi\psi, \varphi\chi\rangle$, natural in $\varphi$, $\psi$, $\chi$;
(DM7) $\boldsymbol{a}(\varphi, \psi, \chi\vartheta) \circ \boldsymbol{a}(\varphi\psi, \chi, \vartheta) = (1_\varphi \boldsymbol{a}(\psi, \chi, \vartheta)) \circ \boldsymbol{a}(\varphi, \psi\chi, \vartheta) \circ (\boldsymbol{a}(\varphi, \psi, \chi)1_\vartheta)$;
(DM8) $\boldsymbol{i}(\varphi\psi, \chi, \vartheta) \circ \overline{\boldsymbol{a}}(\varphi, \psi, \langle\chi, \vartheta\rangle)$

$$= \langle\overline{\boldsymbol{a}}(\varphi, \psi, \chi), \overline{\boldsymbol{a}}(\varphi, \psi, \vartheta)\rangle \circ \boldsymbol{i}(\varphi, \psi\chi, \psi\vartheta) \circ (1_\varphi \boldsymbol{i}(\psi, \chi, \vartheta)),$$

where $\varphi$, $\psi$, $\chi$, $\vartheta$ range over objects of $\mathcal{F}$ and $\overline{\boldsymbol{a}}$ is $\boldsymbol{a}^{-1}$.

Note that condition (DM7) is something less then the usual coherence axioms for monoidal categories (cf. [2] ch. VII). It is rather unexpected that full coherence properties of the functors $\mathbb{M}$ and $\mathbb{D}$ are not necessary for the main Theorem 2.1 below. But since (DM7) seems to be the most essential among the coherence axioms for monoidal categories, we preferred to keep the term "monoidal category" in the title of the paper.

For posets $\mathcal{F}$ the notion of DM-category coincides with that of operative space [1]. A properly categorial example of a DM-category is described below in 1.3.

### 1.2. SOME NOTATIONAL CONVENTIONS

By $\mathcal{F}$ we shall denote usually a DM-category; $\varphi$, $\psi$, $\chi$, $\xi$, $\eta$ etc. will be objects, and $f$, $g$, $h$, $x$, $y$ etc. — arrows in $\mathcal{F}$. In expressions involving arrows we shall usually write $\varphi$ for $1_\varphi$, so if $f : \varphi \to \psi$, then $f\varphi = f = \psi f$, and since $\mathbb{M}$ is a functor we have

$$(\varphi' g) \circ (f\psi) = fg = (f\psi') \circ (\varphi g) \tag{1}$$

for all $f \in \mathcal{F}(\varphi, \varphi')$ and $g \in \mathcal{F}(\psi, \psi')$. We shall often omit brackets in expressions like (1), so in this sense the multiplication is treated as "faster" then the composition "$\circ$". An expression, constructed by means of $\mathbb{D}$, $\mathbb{M}$ and objects of $\mathcal{F}$, defines a functor for both objects and arrows uniformly, the object constants $\varphi$ being interpreted as $1_\varphi$, so in the sequel we shall write such definitions for objects only. We shall write $\overline{\boldsymbol{a}}$, $\overline{\boldsymbol{\lambda}}$, $\overline{\boldsymbol{\rho}}$, $\overline{\boldsymbol{l}}$, $\overline{\boldsymbol{r}}$, $\overline{\boldsymbol{i}}$ for $\boldsymbol{a}^{-1}$, $\boldsymbol{\lambda}^{-1}$, $\boldsymbol{\rho}^{-1}$, $\boldsymbol{l}^{-1}$, $\boldsymbol{r}^{-1}$, $\boldsymbol{i}^{-1}$ respectively and we shall usually omit expressions in brackets after $\boldsymbol{a}$, $\boldsymbol{\lambda}$ etc., so the conditions (DM1)–(DM8) can be written shortly as follows:

(2) $\boldsymbol{a} \circ (fg)h = f(gh) \circ \boldsymbol{a}$;

(3) $\boldsymbol{\lambda} \circ If = f \circ \boldsymbol{\lambda}$;

(4) $\boldsymbol{\rho} \circ fI = f \circ \boldsymbol{\rho}$;

(5) $\boldsymbol{l} \circ \langle f, g\rangle L = f \circ \boldsymbol{l}$;

(6) $\boldsymbol{r} \circ \langle f, g\rangle R = g \circ \boldsymbol{r}$;

(7) $\boldsymbol{i} \circ f\langle g, h\rangle = \langle fg, fh\rangle \circ \boldsymbol{i}$;

(8) $\boldsymbol{a} \circ \boldsymbol{a} = \varphi\boldsymbol{a} \circ \boldsymbol{a} \circ \boldsymbol{a}\vartheta$;

(9) $\boldsymbol{i} \circ \overline{\boldsymbol{a}} = \langle \overline{\boldsymbol{a}}, \overline{\boldsymbol{a}}\rangle \circ \boldsymbol{i} \circ \varphi\boldsymbol{i}$;

where $f \in \mathcal{F}(\varphi, \varphi')$, $g \in \mathcal{F}(\psi, \psi')$, $h \in \mathcal{F}(\chi, \chi')$. Define

$$\langle X_0, \ldots, X_{n-1}\rangle = \langle X_0, \langle X_1, \ldots \langle X_{n-2}, X_{n-1}\rangle \ldots\rangle\rangle,$$

where $X_0, \ldots, X_{n-1}$ are objects in $\mathcal{F}$ or arrows as well, and for $n = 0$ let $\langle X_0, \ldots, X_{n-1}\rangle = I$ (respectively $\langle X_0, \ldots, X_{n-1}\rangle = 1_I$), and for $n = 1$ let $\langle X_0, \ldots, X_{n-1}\rangle = X_0$.

### 1.3. EXAMPLE

Define *types* inductively as follows:

1) 0 is a type;

2) If $a$ and $b$ are types, then $a \to b$ and $a \times b$ are types;

3) If $a_0, a_1, \ldots,$ is an infinite sequence of types, then $a_0 \times a_1 \times a_2 \ldots$ (or shortly $\prod i.a_i$) is a type.

Let $M$ be a set with two disjoint subsets $M_0$ and $M_1$ and three mappings $d_0, d_1, d : M \to M$, s.t. $d_i(x) \in M_i$ and $d(d_i(x)) = x$ for all $x \in M$ and $i < 2$. Denote by $\mathbb{F}_a$ the set of all hereditary partial functionals of type $a$ over $M$, i.e. define $\mathbb{F}_a$ by induction on $a$ as follows: $\mathbb{F}_0$ is $M$; $\mathbb{F}_{a\to b}$ — the set of all partial functions from $\mathbb{F}_a$ to $\mathbb{F}_b$; $\mathbb{F}_{a\times b}$ is $\mathbb{F}_a \times \mathbb{F}_b$; $\mathbb{F}_{\prod i.a_i}$ — the product $\prod_{i=0}^{\infty} \mathbb{F}_{a_i}$.

Let $\mathfrak{F}$ be the set of all relations $\varphi \subseteq \mathbb{F}_a \times M \times M$ for all types $a$; $\varphi \in \mathfrak{F}$ will be called to be of type $a$ iff $\varphi \subseteq \mathbb{F}_a \times M \times M$. An arrow from $\varphi \in \mathfrak{F}$ of type $a$ to $\psi \in \mathcal{F}$ of type $b$ will be called a functional $f \in \mathbb{F}_{a\to(0\to(0\to b))}$, s.t. $fuxy$ is defined iff $\varphi(u, x, y)$ (we are writing $fuxy$ for $f(u)(x)(y)$ etc.), and

$$\forall u, x, y(\varphi(u, x, y) \implies \psi(fuxy, x, y)).$$

Composition $g \circ f$ of arrows $f$ from $\varphi$ to $\psi$ and $g$ from $\psi$ to $\chi$ is defined by $(g \circ f)uxy = g(fuxy)xy$, and for every $\varphi \in \mathfrak{F}$ let $1_\varphi uxy = u$ if $\varphi(u, x, y)$ and let $1_\varphi uxy$ be not defined otherwise. Then $\mathfrak{F}$ is a category w.r.t. the sets of objects and arrows as described, and define functors $\mathbb{M}$ and $\mathbb{D}$ as follows:

$$(\varphi\psi)(w, x, y) = \exists z, u, v(w = \langle z, \langle u, v\rangle\rangle \& \varphi(u, z, y) \& \psi(v, x, z)),$$

so the type of $\varphi\psi$ is $0 \times (a \times b)$ if $a$ and $b$ are the types of $\varphi$ and $\psi$ respectively;

$$(fg)\langle z, \langle u, v\rangle\rangle xy = \langle z, \langle fuzy, gvxz\rangle\rangle,$$

where $f \in \mathcal{F}(\varphi, \varphi')$, $g \in \mathcal{F}(\psi, \psi')$ and $(\varphi\psi)(\langle z, \langle u, v\rangle\rangle, x, y)$ and $(fg)\langle z, \langle u, v\rangle\rangle xy$ is undefined otherwise;

$$\langle\varphi, \psi\rangle(w, x, y) = \exists u, v(w = \langle u, v\rangle \& ((x \in M_0 \& v = o_b \\ \& \varphi(u, d(x), y)) \vee (x \in M_1 \& u = o_a \& \psi(v, d(x), y)))),$$

where $\varphi$ and $\psi$ are objects of types $a$ and $b$ respectively, $o_c \in \mathbb{F}_c$ is any fixed functional for each type $c$; the type of $\langle\varphi, \psi\rangle$ is $a \times b$;

$$\langle f, g\rangle\langle u, v\rangle xy = \begin{cases} \langle fud(x)y, o_b\rangle & \text{if } x \in M_0 \& v = o_b \& \varphi(u, d(x), y) \\ \langle o_a, gvd(x)y\rangle & \text{if } x \in M_1 \& u = o_a \& \psi(v, d(x), y), \end{cases}$$

and $\langle f, g\rangle\langle u, v\rangle xy$ is undefined otherwise. Let $I$ be the object of type 0 defined by $I(u, x, y) \Leftrightarrow u = o_0 \& x = y$, and let $L(u, x, y) \Leftrightarrow I(u, d_0(x), y)$ and $R(u, x, y) \Leftrightarrow I(u, d_1(x), y)$. Then it is straightforward to see that $\mathcal{F}$ is a DM-category w.r.t. $\mathbb{M}$, $\mathbb{D}$, $I$, $L$, $R$ and properly defined $\boldsymbol{a}$, $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, $\boldsymbol{l}$, $\boldsymbol{r}$, $\boldsymbol{i}$.

Informally, this example arises at an attempt to give a more detailed description of some examples of operative spaces, the elements of which represent some idealized programs working on inputs from $M$ and giving for each input a set of outputs from $M$, by taking account for the correctness of the work of the program. In the notations above, while the input–output relation for such a program $\varphi$ is represented by $\exists u\varphi(u, x, y)$, the relation $\varphi(u, x, y)$ is to be conceived as "$u$ is a proof that given the input $x$ to $\varphi$, $y$ will be given as an output".

## 1.4. LEAST FIXED POINTS

Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. By $(F \Rightarrow \mathcal{C})$ we shall denote the category of pairs $(X; x)$ s.t. $X \in \mathcal{C}$ and $x \in \mathcal{C}(F(X), X)$; morphisms $f : (X; x) \to (Y; y)$ in $(F \Rightarrow \mathcal{C})$ are the arrows $f : X \to Y$ in $\mathcal{C}$ s.t. $f \circ x = y \circ F(f)$. Then a *least fixed point* or a "minimal fixed point" (m.f.p.) of $F$ is by definition an initial object $(M; m)$ of $(F \Rightarrow \mathcal{C})$. Some elementary properties of m.f.p. will be used below without special reference. They partially appear in Lambek [3], let us list them:

(i) Suppose an endofunctor $F = F(A)$ in $\mathcal{C}$ depends on a parameter $A \in \mathcal{C}$, i.e. $F$ is a functor from $\mathcal{C}^2$ to $\mathcal{C}$, and $(M(A); m(A))$ is a m.f.p. of $F(A)$ for all $A \in \mathcal{C}$. Then $M$ is a functor from $\mathcal{C}$ to $\mathcal{C}$, where $M(a)$ for $a : A \to B$ is determined uniquely by

$$M(a) \circ m(A) = m(B) \circ F(a, M(a)),$$

since $(M(B); m(B) \circ F(a, M(B))) \in (F(A) \Rightarrow \mathcal{C})$. The same holds for several parameters instead one.

(ii) If $(M; m)$ and $(N; n)$ are two m.f.p. of $F : \mathcal{C} \to \mathcal{C}$, then there is an isomorphism $(M; m) \cong (N; n)$, natural in parameters.

(iii) If $(M(A); m(A))$ is a m.f.p. of an endofunctor $F(A)$ in $\mathcal{C}$ depending on a parameter $A$, then $m(A) : F(A, M(A)) \cong M(A)$ is an isomorphism natural in $A$, and similarly for several parameters.

## 1.5. ITERATIVELY CLOSED DM-CATEGORIES

Let $\mathcal{F}$ be a DM-category, and consider the power category $\mathcal{F}^N$, where $N$ is the set of all natural numbers. A normal functor will be called a functor $H : \mathcal{F} \to \mathcal{F}^N$ of the form $H(\xi) = \lambda i.\varphi(\xi\nu_i)$, where $\nu_i \in \mathcal{F}_0$ for all $i \in N$ and $\mathcal{F}_0$ is the set of all objects of $\mathcal{F}$, produced from $\{L, R\}$ by means of multiplication. Then the category $\mathcal{F}$ will be called an *iteratively closed* DM-category, iff:

(ic1) every normal functor $H : \mathcal{F} \to \mathcal{F}^N$ has right adjoint; and

(ic2) every functor $\Gamma : \mathcal{F} \to \mathcal{F}$ of the form $\Gamma(\xi) = \langle I, \xi\rangle\varphi$ has a m.f.p. $(\mathbb{I}(\varphi); j_\varphi)$ in $\mathcal{F}$.

Condition (ic1) will be used below in the following form:

(ic3) for all $\varphi \in \mathcal{F}$ and all sequences of objects $\psi_i \in \mathcal{F}$ and $\nu_i \in \mathcal{F}_0$ $(i \in N)$ there is $\xi \in \mathcal{F}$ and a sequence of arrows $x_i : \varphi(\xi\nu_i) \to \psi_i$, which is universal in the sense that for every $\eta \in \mathcal{F}$ and all sequences of arrows $y_i : \varphi(\eta\nu_i) \to \psi_i$ there is unique arrow $h : \eta \to \xi$ in $\mathcal{F}$ s.t. $y_i = x_i \circ \varphi(h\nu_i)$ for all $i \in N$.

Note that condition (ic1) follows easily from next two ones:

(ic4) all functors $M_\varphi$, $M^L$ and $M^R$ of the form $M_\varphi(\xi) = \varphi\xi$, $M^L(\xi) = \xi L$ and $M^R(\xi) = \xi R$ have right adjoints;

(ic5) all products $\prod_{i\in N} \varphi_i$ exist in $\mathcal{F}$.

The category $\mathcal{F}$ from 1.3 is an iteratively closed DM-category. Conditions (ic4) and (ic5) can be shown to hold for this category in more or less a straightforward way. A m.f.p. in it for a functor $\Gamma$ of the form of (ic2) can be constructed directly by a proper generalization of the usual method of constructing of iteration, as, for instance, in [4].

## 1.6. TERMS AND VALUES

Let $c_0, \ldots, c_{l-1}$ be a list of symbols called *parameter symbols*, and let we have an infinite list of variables denoted usually by $x$, $y$, $z$ etc. The symbols $\boldsymbol{I}$, $\boldsymbol{L}$, $\boldsymbol{R}$ will be called basic constants, and parameter symbols and basic constants together will be called *constants*. Define *terms* inductively as follows:

a) all constants and variables are terms; they are called *simple* terms;

b) if $t$ and $s$ are terms, then $(ts)$ and $\langle t, s\rangle$ are terms.

If $\mathcal{X}$ is a set of variables, then by **Term**($\mathcal{X}$) we shall denote the set of all terms whose variables belong to $\mathcal{X}$, and **Term** will be the set of all terms.

Let $\mathcal{F}$ be a DM-category and suppose we have an interpretation assigning to each parameter symbol $c_i$ an object (called a parameter) $\gamma_i \in \mathcal{F}$. This interpretation will be fixed throughout the paper. Let $\overline{x} \equiv x_0, \ldots, x_{n-1}$ be a list of distinct variables. Then each term $t \in \mathbf{Term}(\{\overline{x}\})$ defines a functor $[\lambda\overline{x}.t] : \mathcal{F}^n \to \mathcal{F}$ called value of $t$ in an obvious way, namely:

1) if $t$ is a variable $x_i$, $i < n$, then $[\lambda\overline{x}.t](\overline{\xi}) = \xi_i$;

2) if $t$ is a parameter symbol $c_i$, $i < l$, then $[\lambda\overline{x}.t](\overline{\xi}) = \gamma_i$;

3) if $t$ is $\boldsymbol{I}$, $\boldsymbol{L}$ or $\boldsymbol{R}$ then $[\lambda\overline{x}.t](\overline{\xi})$ is $I$, $L$ or $R$, respectively;

4) if $t \equiv (sr)$, then $[\lambda\overline{x}.t](\overline{\xi}) = [\lambda\overline{x}.s](\overline{\xi})[\lambda\overline{x}.r](\overline{\xi})$;

5) if $t \equiv \langle s, r\rangle$, then $[\lambda\overline{x}.t](\overline{\xi}) = \langle[\lambda\overline{x}.s](\overline{\xi}), [\lambda\overline{x}.r](\overline{\xi})\rangle$,

where $\overline{\xi}$ is an arbitrary object of $\mathcal{F}^n$, and the definition of the functor $[\lambda\overline{x}.t]$ for arrows is the same when replacing $\xi_i$, $\gamma_i$, $I$, $L$, $R$ with $1_{\xi_i}$, $1_{\gamma_i}$, $1_I$, $1_L$, $1_R$, respectively.

Sometimes we shall write $t_0t_1\dots t_n$ for $(\dots(t_0t_1)\dots t_{n-1})t_n$, where $t_0,\dots,t_n$ are terms.

## 1.7. REDUCTIONS AND B-NORMAL TERMS

A formal expression of one of the forms

(a) $t(sr) \to (ts)r$

or

(i) $t\langle s,r\rangle \to \langle ts, tr\rangle$,

where $t$, $s$, $r$ are terms, will be called a basic contraction. As usual the notion of basic contraction gives rise to a reduction notion: we shall write $t \mapsto s$ for "$s$ is obtained by replacing an occurrence in $t$ on the left hand side of a basic contraction with the corresponding occurrence on the right hand side of the same basic contraction" and the symbol "$\mapsto\!\!\!\!\mapsto$" for the reflexive transitive closure of the relation "$\mapsto$". A term $t$ will be called b-normal (or simply normal), if $t \mapsto s$ is impossible for any $s$; $s$ will be called b-normal form of $t$ iff $t \mapsto\!\!\!\!\mapsto s$ and $s$ is normal.

**Lemma 1.** *For every term $t$ there is unique b-normal form $t^b$ of $t$.*

Indeed, let $\mathrm{lh}(t)$ be the length of the term $t$, and let $\mathrm{lh}'(t)$ be the number $\mathrm{lh}(s_0)+\dots+\mathrm{lh}(s_{n-1})$, where $t=(\dots(ps_0)\dots s_{n-2})s_{n-1}$ and $p$ is a term which is not of the form $p_0p_1$ for some terms $p_0$ and $p_1$. Define $\mu(t)=\mathrm{lh}(t)\omega+\mathrm{lh}'(t)$. Then using induction on the ordinal $\mu(t)$ we may see that the following equalities define uniquely a total operation on terms denoted by $t^b$ for a term $t$:

(1) $t^b = t$, if $t$ is a simple term;

(2) $t^b = p^b s$, if $t = ps$ and $s$ is simple;

(3) $t^b = ((ps)r)^b$, if $t = p(sr)$;

(4) $t^b = \langle (ps)^b, (pr)^b\rangle$, if $t = p\langle s,r\rangle$;

(5) $t^b = \langle t_0^b, t_1^b\rangle$, if $t = \langle t_0, t_1\rangle$.

Again, we have for all terms $t$ and $s$:

(6) $t^b$ is b-normal;

(7) $t \mapsto\!\!\!\!\mapsto t^b$;

(8) if $t \mapsto s$, then $t^b = s^b$;

which can be seen straightforwardly by induction on $\mu(t)$. ∎

Now we shall define for every term $t \in \mathrm{Term}(\{x_0,\dots,x_{n-1}\})$ an isomorphism

(9) $b_t(\overline{\xi}) : [\lambda\overline{x}.t](\overline{\xi}) \cong [\lambda\overline{x}.t^b](\overline{\xi})$,

natural in $\overline{\xi}$, where $\overline{\xi} = (\xi_0,\dots,\xi_{n-1})$ and $\overline{x} = (x_0,\dots,x_{n-1})$.

Writing for short $b(t)$ for $b_t(\overline{\xi})$ and $t^*$ for $[\lambda\overline{x}.t](\overline{\xi})$ for any $t \in \mathrm{Term}(\{x_0,\dots,x_{n-1}\})$, define $b_t(\overline{\xi})$ as follows:

(b1) $b(t) = t^*$ if $t$ is normal;

(b2) $b(ts) = b(t)s^*$ if $s$ is simple and $t$ is not normal;

(b3) $b(ts) = b(tp)q^* \circ \overline{a}$ if $s = pq$ is normal;

(b4) $b(ts) = \langle b(ts_0), b(ts_1)\rangle \circ i$, if $s = \langle s_0, s_1\rangle$ is normal;

(b5) $b(ts) = b(ts^b) \circ t^* b(s)$ if $s$ is not normal;

(b6) $b(\langle t_0, t_1\rangle) = \langle b(t_0), b(t_1)\rangle$ if $\langle t_0, t_1\rangle$ is not normal.

Note that (b2), (b5) and (b6) hold for any terms $t$, $t_0$, $t_1$. In (b3) and (b4) $\overline{a}$ is $\overline{a}(t^*, p^*, q^*)$ and $i$ is $i(t^*, s_0^*, s_1^*)$.

**Lemma 2.** *For all terms $t$, $r$ and every normal $s$ we have:*

$$b(ts) = b(t^b s) \circ b(t)s^*, \tag{10}$$

*and*

$$b(t(rs)) = b((tr)s) \circ \overline{a}. \tag{11}$$

P r o o f. Induction on $s$ for both (10) and (11). If $s$ is simple, then

$$b(t^b s) \circ b(t)s^* = (t^b)^* s^* \circ b(t)s^* = b(ts).$$

If $s = pq$, then $q$ is simple since $s$ is normal, and by (b3), 1.2.(2) and the induction hypothesis for $p$ we get:

$$b(t^b s) \circ b(t)s^* = b(t^b p)q^* \circ \overline{a} \circ b(t)(p^* q^*) = b(t^b p)q^* \circ b(t)p^* q^* \circ \overline{a}$$
$$= b(tp)q^* \circ \overline{a} = b(ts).$$

If $s = \langle s_0, s_1\rangle$ then similarly

$$b(t^b s) \circ b(t)s^* = \langle b(t^b s_0), b(t^b s_1)\rangle \circ i \circ b(t)\langle s_0^*, s_1^*\rangle$$
$$= \langle b(t^b s_0) \circ b(t)s_0^*, b(t^b s_1) \circ b(t)s_1^*\rangle \circ i$$
$$= \langle b(ts_0), b(ts_1)\rangle \circ i = b(ts).$$

This proves (10). If $rs$ is normal, then (11) is received immediately from (b2) and (b3). Suppose $rs$ is not normal. Then by (b5) we have

$$b(t(rs)) = b(t(rs)^b) \circ t^* b(rs). \tag{12}$$

Consider cases for $s$. If $s$ is simple, then

$$b(t(rs)) = b(t(rs)^b) \circ t^*(b(r)s^*) = b(t(r^b s)) \circ t^*(b(r)s^*)$$
$$= b(tr^b)s^* \circ \overline{a} \circ t^*(b(r)s^*) = b(tr^b)s^* \circ (t^* b(r))s^* \circ \overline{a} = b(tr)s^* \circ \overline{a}$$
$$= b((tr)s) \circ \overline{a}.$$

If $s = pq$, then $q$ is simple and using (12), the induction hypothesis for $p$, and 1.2.(8), we have:

$$b(t(rs)) = b(t((rp)^b q)) \circ t^*(b(rp)q^*) \circ t^* \overline{a}$$
$$= b(t(rp)^b)q^* \circ \overline{a} \circ t^*(b(rp)q^*) \circ t^* \overline{a} = b(t(rp)^b)q^* \circ (t^* b(rp))q^* \circ \overline{a} \circ t^* \overline{a}$$
$$= b(t(rp))q^* \circ \overline{a} \circ t^* \overline{a} = b((tr)p)q^* \circ \overline{a} q^* \circ \overline{a} \circ t^* \overline{a} = b((tr)p)q^* \circ \overline{a} \circ \overline{a}$$
$$= b((tr)s) \circ \overline{a}.$$

Finally, if $s = \langle s_0, s_1\rangle$, then $s_0$ and $s_1$ are normal, and using (12), the induction hypothesis for $s_0$ and $s_1$, and 1.2.(9), we have:

$$b(t(rs)) = b(t\langle (rs_0)^b, (rs_1)^b\rangle) \circ t^*\langle b(rs_0), \overline{b}(rs_1)\rangle \circ t^* i$$
$$= \langle b(t(rs_0)^b), b(t(rs_1)^b)\rangle \circ i \circ t^*\langle b(rs_0), \overline{b}(rs_1)\rangle \circ t^* i$$
$$= \langle b(t(rs_0)), b(t(rs_1))\rangle \circ i \circ t^* i = \langle b((tr)s_0) \circ \overline{a}, b((tr)s_1) \circ \overline{a}\rangle \circ i \circ t^* i$$
$$= \langle b((tr)s_0), b((tr)s_1)\rangle \circ \langle \overline{a}, \overline{a}\rangle \circ i \circ t^* i = \langle b((tr)s_0), b((tr)s_1)\rangle \circ i \circ \overline{a} = b((tr)s) \circ \overline{a}. \blacksquare$$

A *term system* in $\mathfrak{F}$ is a pair $(\overline{s};\overline{x})$, where $\overline{x} \equiv x_0,\dots,x_{n-1}$ is a list of variables and $\overline{s} \equiv s_0,\dots,s_{n-1}$ is a list of terms $s_i$ from $\mathbf{Term}(\{\overline{x}\})$. Each term system $S \equiv (\overline{s};\overline{x})$ defines a functor $\overline{S} : \mathfrak{F}^n \to \mathfrak{F}^n$ by $\overline{S}(\xi) = (S_0(\xi),\dots,S_{n-1}(\xi))$, where $S_i = [\lambda\overline{x}.s_i]$ for all $i < n$, and $\xi$ is an arbitrary object or arrow in $\mathfrak{F}^n$ as well. A term system $S = (\overline{s};\overline{x})$ will be called *normal* iff all terms $s_0,\dots,s_{n-1}$ are normal.

Let $\mathbf{Term2}(\mathfrak{X})$ be the set of all sequences $t_0,\dots,t_{n-1}$ of terms $t_i \in \mathbf{Term}(\mathfrak{X})$ with length $n \leq 2$. For every term system $S \equiv (\overline{s};\overline{x})$ we shall define a mapping $S^{\#} : \mathbf{Term}(\{\overline{x}\}) \to \mathbf{Term2}(\{\overline{x}\})$ (called $S$-reduction function) as follows:

(#1) if $t$ is a constant, then $S^{\#}(t)$ is the empty sequence $\Lambda$;

(#2) if $t = pc$, $t$ is normal and $c$ is a constant, then $S^{\#}(t)$ is (the one membered sequence) $p$;

(#3) if $t = \langle t_0, t_1\rangle$ and $t$ is normal, then $S^{\#}(t)$ is $t_0, t_1$;

(#4) if $t = x_i$, then $S^{\#}(t)$ is $s_i$, $i < n$;

(#5) if $t = px_i$ and $t$ is normal, then $S^{\#}(t)$ is $(ps_i)^b$.

A set $T \subseteq \mathbf{Term}(\{\overline{x}\})$ will be called closed w.r.t. $S^{\#}$ iff for any $t \in T$ all members of $S^{\#}(t)$ belong to $T$. If $f$ is a mapping $f : \mathbf{Term}(\{\overline{x}\}) \to \mathbf{Term}(\{\overline{x}\})$, then we shall write $f(S^{\#}(t))$ for the term $\langle f(t_0),\dots,f(t_{n-1})\rangle$, where $S^{\#}(t)$ is $t_0,\dots,t_{n-1}$.

**Definition.** A *coding* for a term system $S$ in $\mathfrak{F}$ (w.r.t. a given interpretation of the parameters) is a quadruple $\langle K, k, \boldsymbol{k}, \rho\rangle$, s.t.: $K \subseteq \mathbf{Term}(\{\overline{x}\})$, $K$ is closed w.r.t. $S^{\#}$ and $x_i \in K$ for all $i < n$; $k : K \to \mathfrak{F}_0$ is a mapping ($\mathfrak{F}_0$ is defined in 1.5); $\rho \in \mathfrak{F}$; $\boldsymbol{k}$ is a mapping assigning to each $t \in K$ an isomorphism in $\mathfrak{F}$, namely:

$$\text{(c)} \qquad \boldsymbol{k}(t) : \rho k(t) \cong F_t(k(S^{\#}(t))),$$

where $F_t$ is an endofunctor in $\mathfrak{F}$ defined for each term $t$ as follows:

(F1) $F_t(\xi) = L\gamma$, if $t$ is a constant with value $\gamma$;

(F2) $F_t(\xi) = R(\xi\gamma)$, if $t \equiv pc$, where $c$ is a constant with value $\gamma$ and $t$ is normal;

(F3) $F_t(\xi) = R\xi$ in all other cases.

## 2. RECURSION THEORY IN DM-CATEGORIES

**2.1. Theorem.** *Let $\mathfrak{F}$ be an iteratively closed DM-category and $S = (\overline{s};\overline{x})$ be a normal term system in $\mathfrak{F}$, where $\overline{x} \equiv (x_0,\dots,x_{n-1})$. Suppose $\langle K, k, \boldsymbol{k}, \rho\rangle$ is a coding for $S$ in $\mathfrak{F}$ and $(\omega; m)$ is a m.f.p. of the endofunctor $U$ in $\mathfrak{F}$ defined by $U(\xi) = \langle I, \xi\rangle\rho$. Then there is an arrow $\overline{w}$ in $\mathfrak{F}^n$ s.t. $(\omega k(x_0),\dots,\omega k(x_{n-1}); \overline{w})$ is a m.f.p. of $\overline{S}$ in $\mathfrak{F}^n$.*

This is the main result of the paper and a detailed sketch of a proof of Theorem 2.1 will be given in section 3 below. In the present section we shall see how it can be used to extend principal results of recursion theory in iterative operative spaces in the sense of Ivanov [1] for DM-categories. Up to the end of the section we shall suppose that $\mathfrak{F}$ is an iteratively closed DM-category.

Define for every natural number $n$ an object $n^+ \in \mathcal{F}$ inductively as follows: $0^+ = L$; $(n+1)^+ = Rn^+$ . Then by (DM4) and (DM5) we have

$$\langle \alpha_0, \ldots, \alpha_{n-1} \rangle i^+ \cong \alpha_i \quad \text{for all } i < n,$$

where $\alpha_0, \ldots, \alpha_{n-1}$ are objects or arrows in $\mathcal{F}$.

**Definition.** A functor $T : \mathcal{F} \to \mathcal{F}$ is called a translation (this term is adopted by Ivanov [1]) iff for every object $\varphi \in \mathcal{F}$ and for each natural $n$

$$T(\varphi)n^+ \cong n^+\varphi$$

and the last isomorphism is natural in $\varphi$.

**Example.** If for all functors $\Psi(\xi)$ of the form $\Psi(\xi) = \langle L\varphi, R\xi \rangle$ there is a m.f.p. $(T(\varphi); \boldsymbol{t}(\varphi))$ in $\mathcal{F}$, then $T$ is a translation in $\mathcal{F}$. Indeed, by 1.4.(iii) we have a natural isomorphism $\langle L\varphi, RT(\varphi) \rangle \cong T(\varphi)$, whence by induction on $n$ we see that $T(\varphi)n^+ \cong n^+\varphi$.

Every translation $T$ gives rise to a bifunctor $T'(\varphi, \psi) = [T(\psi)]T(\varphi)$ (cf. Ivanov [1]), called a *primitive recursion* (or better a $T$-primitive recursion). Objects $\varphi \in \mathcal{F}$, produced from constants $L$, $R$, $I$ (respectively $L$, $R$, $I$, $\gamma_0$, ..., $\gamma_{l-1}$) by means of the functors $T'$, $\mathbb{M}$ and $\mathbb{D}$ are called $T$-primitive recursive (respectively $T$-primitive recursive in $\{\gamma_0, \ldots, \gamma_{l-1}\}$).

**Proposition 1.** *If $T$ is a translation in $\mathcal{F}$, then for every primitive recursive function $f$ there is a $T$-primitive recursive $\varphi \in \mathcal{F}$, s.t. $\varphi n^+ \cong (f(n))^+$ for all natural $n$.*

This proposition is a corollary to corresponding results in [1] or [5]. ∎

## 2.3. FINITE CODINGS

Theorem 2.1 reduces the problem of expressing m.f.p. to that of constructing a suitable coding. The last construction is easy when the domain $K$ of the coding can by chosen finite. Let us consider this case first, it will give us a description of m.f.p. functors, produced by means of the functors $\mathbb{M}$, $\mathbb{D}$ and the *iteration functor* $\mathbb{I}$ defined in 1.5.(ic2).

**Definition.** A coding $\langle K, k, \boldsymbol{k}, \rho \rangle$ for a system of terms $S = (\bar{s}; \bar{x})$ is *finite*, iff $K$ is a finite set. A system $(\bar{s}; \bar{x})$ is called *finitary*, iff there is a finite coding for it. An object $\alpha \in \mathcal{F}$ is called *finitely recursive* (in $\{\gamma_0, \ldots, \gamma_{l-1}\}$), iff there is a finitary system $(\bar{s}; \bar{x}) \equiv (s_0, \ldots, s_{n-1}; \bar{x})$ and a m.f.p. $(\bar{\xi}; \bar{x}) \equiv (\xi_0, \ldots; \bar{x})$ of the functor $\overline{S}$, s.t. $\xi_0 \cong \alpha$.

**Remark.** It is clear by the definition of coding in 1.8 that a system

$$(\bar{s}; \bar{x}) \equiv (s_0, \ldots, s_{n-1}; x_0, \ldots, x_{n-1})$$

is finitary iff the set $K$ of terms, produced in an obvious sense from $\{x_0, \ldots, x_{n-1}\}$ by means of the mapping $S^\#$, is finite. Indeed, if that is the case and $K = \{t_0, \ldots, t_m\}$, where $n < m$ and $t_i = x_i$ for all $i < n$, then define $k(t_i) = i^+$ for all $i \leqq m$ and

$$\rho = \langle \rho_0, \ldots, \rho_m \rangle, \tag{1}$$

where:

$$\rho_i = \begin{cases} L\gamma & \text{if } t_i = c \\ R(j^+\gamma) & \text{if } t_i = t_j c \\ R\langle j^+, k^+\rangle & \text{if } t_i = \langle t_j, t_k\rangle \\ Rj^+ & \text{if } i < n \text{ and } s_i = t_j \\ Rj^+ & \text{if } t_i = px_k \text{ and } t_j = (ps_k)^b \text{ for some } k < n, \end{cases}$$

for all $i \leqq m$. Then $\langle K, k, \boldsymbol{k}, \rho\rangle$ is a coding for a suitable $\boldsymbol{k}$. So by Theorem 2.1 we have

**Corollary 1.** *If $\varphi \in \mathcal{F}$ is finitely recursive in $\{\gamma_0, \ldots, \gamma_{l-1}\}$ then there is $\rho \in \mathcal{F}$ s.t. $\varphi \cong \mathbb{I}(\rho)L$, and $\rho$ is of the form*

(2) $$\rho = \langle I, n_0^+\beta_0, \ldots, n_{l'-1}^+\beta_{l'-1}\rangle\alpha,$$

*where $n_0, \ldots, n_{l'-1}$ are natural numbers, $\beta_i \in \{\gamma_0, \ldots, \gamma_{l-1}\}$ for all $i < l'$, and $\alpha$ belongs to the set, produced from $\mathcal{F}_0$ by means of $\mathbb{D}$.*

Indeed, (2) can be obtained from (1) by some easy transformations using (DM4), (DM5) and (DM6). ■

Note that the isomorphism $\varphi \cong \mathbb{I}(\rho)L$ is natural in parameters.

**Lemma 1.** *If $\varphi$ and $\psi$ are finitely recursive then $\varphi\psi$, $\langle\varphi, \psi\rangle$ and $\mathbb{I}(\varphi)$ are finitely recursive.*

P r o o f. Suppose $\varphi$ and $\psi$ are defined through systems $S = (\overline{s}; \overline{x})$ and $S' = (\overline{r}; \overline{y})$ respectively, i.e. there are m.f.p. $(\overline{\mu}; \overline{m})$ and $(\overline{\nu}; \overline{n})$ of the functors $\overline{S}$ and $\overline{S'}$ respectively, s.t. $\varphi \cong \mu_0$ and $\psi \cong \nu_0$, where $\overline{\mu} = \mu_0, \ldots$ and $\overline{\nu} = \nu_0, \ldots$ We may suppose that all variables in $\overline{x}$, $\overline{y}$ are distinct and let $\overline{x} = x_0, \ldots$ and $\overline{y} = y_0, \ldots$ Then $(\mu_0\nu_0, \overline{\mu}, \overline{\nu}; \mu_0\nu_0, \overline{m}, \overline{n})$ is a m.f.p. of $\overline{S}_1$, where $S_1 = (x_0y_0, \overline{s}, \overline{r}; z, \overline{x}, \overline{y})$, and $z$ is a variable not occurring in $\overline{x}$, $\overline{y}$. But the last system is finitary, provided $S$ and $S'$ are. Indeed, if there are finite codings $\langle K, \ldots\rangle$ and $\langle K', \ldots\rangle$ for $S$ and $S'$ respectively, then the set

$$K'' = K \cup K' \cup \{(x_0t)^b \mid t \in K'\} \cup \{z\}$$

is finite and closed under $S_1^{\#}$ and $\{\overline{x}, \overline{y}, z\} \subseteq K''$. Therefore by the Remark above the last system is finitary and $\varphi\psi \cong \mu_0\nu_0$ is finitely recursive. We can see in a similar way that the systems $(\langle x_0, y_0\rangle, \overline{s}, \overline{r}; z, \overline{x}, \overline{y})$ and $(\langle I, z\rangle x_0, \overline{s}; z, \overline{x},)$ define $\langle\varphi, \psi\rangle$ and $\mathbb{I}(\varphi)$ respectively and are finitary, provided $(\overline{s}; \overline{x})$ and $(\overline{r}; \overline{y})$ are. ■

**Corollary 2.** (i) *An object $\varphi \in \mathcal{F}$ is finitely recursive iff $\varphi$ is isomorphic to a member of the set, produced from the constants $\gamma_0$, …, $\gamma_{l-1}$, $I$, $L$, $R$ by means of the functors $\mathbb{M}$, $\mathbb{D}$ and $\mathbb{I}$.*

(ii) *Any functor defined explicitly by means of the constants and functors in* (i) *is naturally isomorphic to a functor $\Gamma$ of the form*

$$\Gamma(\xi) = \mathbb{I}(\langle I, n_0^+\xi, \ldots, n_{k-1}^+\xi\rangle\alpha)L,$$

*where $n_0$, …, $n_{k-1}$ are natural numbers and $\alpha$ belongs to the set produced from the constants by means of $\mathbb{M}$ and $\mathbb{D}$.* ■

For operative spaces the item (ii) in the last corollary is essentially a result of Georgieva [6].

**Definition.** (i) An object $\alpha \in \mathcal{F}$ is called *recursive* (in the parameters $\gamma_0, \ldots, \gamma_{l-1}$), iff there is a system of terms $S = (\overline{s}; \overline{x})(s_0, \ldots, s_{n-1}; \overline{x})$ and a m.f.p. $(\overline{\xi}; \overline{x}) \equiv (\xi_0, \ldots; \overline{x})$ of the functor $\overline{S}$, s.t. $\xi_0 \cong \alpha$.

(ii) Writing **Syst** for the set of all normal systems of terms, an *universal coding* is a triple $\langle k, \mathbf{k}, \rho \rangle$, s.t. $k : \mathbf{Syst} \times \mathbf{Term} \to \mathcal{F}_0$ is a function, $\mathbf{k}$ is a function assigning to each pair $(S, t) \in \mathbf{Syst} \times \mathbf{Term}$ an isomorphic arrow in $\mathcal{F}$, and for every $S \in \mathbf{Syst}$ the quadruple $\langle \mathbf{Nterm}, \lambda t.k(S, t), \lambda t.\mathbf{k}(S, t), \rho \rangle$ is a coding for $S$, where **Nterm** is the set of all normal terms.

**Lemma 2.** *Suppose $T$ is a translation functor in $\mathcal{F}$ and $S \in$ **Syst**. Then there is an universal coding $\langle k, \mathbf{k}, \rho \rangle$, s.t. $k(S, x_0) = L$, where $S \equiv (\overline{s}; x_0, \ldots)$, and*

$$\rho = \langle I, T(\gamma_0), \ldots, T(\gamma_{l-1}) \rangle \alpha,$$

*for a $T$-primitive recursive object $\alpha \in \mathcal{F}$.*

The proof of Lemma 2 is more or less a standard one, using a Gödel numbering of **Syst** × **Term** and Proposition 1. Formally, it is a corollary to the special case for operative spaces instead of DM-categories $\mathcal{F}$, and the last one is a special case of Proposition 3 in [5]. ∎

By Lemma 2 and Theorem 2.1 we have immediately:

**Corollary 3.** *Suppose that each functor $\Psi(\xi) = \langle L\varphi, R\xi \rangle$ has a m.f.p. in $\mathcal{F}$ and $T$ is a least fixed point functor $T(\varphi) \cong \langle L\varphi, RT(\varphi) \rangle$ in the sense of the Example in 2.2. Then:*

(i) *Every object $\varphi \in \mathcal{F}$, recursive in $\{\gamma_0, \ldots, \gamma_{l-1}\}$, is naturally (in $\gamma_0, \ldots, \gamma_{l-1}$) isomorphic to an object, which can be expressed explicitly by means of $\gamma_0, \ldots, \gamma_{l-1}$, $I$, $L$, $R$, $\mathbb{M}$, $\mathbb{D}$, $\mathbb{I}$, $T$.*

(ii) *Any functor, defined explicitly by means of the constants and the functors in* (i) *is naturally isomorphic to a functor $\Gamma$ of the form*

$$\Gamma(\xi) = \mathbb{I}(\langle I, T(\xi) \rangle \alpha) L,$$

*where $\alpha$ is a $T$-primitive recursive in $\{\gamma_0, \ldots, \gamma_{l-1}\}$ object of $\mathcal{F}$.*

(iii) *There is an object $\omega \in \mathcal{F}$, recursive in $\{\gamma_0, \ldots, \gamma_{l-1}\}$, which is universal among all objects recursive in $\{\gamma_0, \ldots, \gamma_{l-1}\}$, i.e.*

(a) *for every recursive in $\{\gamma_0, \ldots, \gamma_{l-1}\}$ object $\varphi \in \mathcal{F}$ there is a natural number $n$ such that $\varphi \cong \omega n^+$ and*

(b) *there is a primitive recursive function $s(n, m)$, s.t. $\omega(s(n, m))^+ \cong \omega n^+ m^+$ for all natural $n$, $m$.*

## 3. PROOF OF THE MAIN THEOREM

Assume the suppositions of Theorem 2.1. Up to the end of the proof $c$ will denote an arbitrary constant and $\gamma$ will be the value of $c$ in $\mathcal{F}$; the letters $t, s, p, t_0$ etc. will be used to denote terms. We shall write $t^\#$ for $S^\#(t)$ and we shall suppose that all terms in $K$ are normal. This is not a loss of generality, since otherwise we may take the set $\{t \in K : t \text{ is normal}\}$ instead of $K$. We shall adopt some rules for omitting brackets in long expressions, e.g. $\varphi\psi\chi\vartheta$ will be a short notation for

$((\varphi\psi)\varphi)\vartheta$. This rule of "association to the left" will apply to objects, arrows and terms as well, as mentioned before in 1.6.

### 3.1. DEFINITION OF THE ARROWS $\boldsymbol{m}$

For every term $t \in K$ define an endofunctor $G_t$ in $\mathcal{F}$ as follows:

$$\text{(G)} \qquad G_t(\xi) = \begin{cases} \gamma & \text{if } t \equiv c \\ \xi k(p)\gamma & \text{if } t \equiv pc \text{ and } t \text{ is normal} \\ \xi k(t^{\#}) & \text{otherwise.} \end{cases}$$

**Lemma 1.** *For each $t \in K$ there is an isomorphism*

$$(n) \qquad \boldsymbol{n}_t(\xi) : U(\xi)k(t) \cong G_t(\xi),$$

*natural in $\xi$.*

Indeed, by 1.1.(DM1) and 1.8.(c):

$$(1) \qquad U(\xi)k(t) = ((\langle I,\xi\rangle\rho)k(t) \cong \langle I,\xi\rangle(\rho k(t)) \cong \langle I,\xi\rangle F_t(k(t^{\#})).$$

Consider cases for $t \in K$:

1) $t = c$; then by (1), 1.8.(F1), 1.1.(DM1), 1.1.(DM4) and 1.1.(DM2)

$$U(\xi)k(t) \cong \langle I,\xi\rangle(L\gamma) \cong (\langle I,\xi\rangle L)\gamma \cong I\gamma \cong \gamma = G_t(\xi);$$

2) $t = pc$ and $t$ is normal; then by (1), 1.1.(DM1) and 1.1.(DM5)

$$U(\xi)k(t) \cong \langle I,\xi\rangle((Rk(p))\gamma) \cong (\langle I,\xi\rangle(Rk(p)))\gamma \cong ((\langle I,\xi\rangle R)k(p))\gamma$$
$$\cong (\xi k(p))\gamma = G_t(\xi);$$

3) all other cases; by (1), 1.1.(DM1), 1.1.(DM5) we have

$$U(\xi)k(t) \cong \langle I,\xi\rangle(Rk(t^{\#})) \cong (\langle I,\xi\rangle R)k(t^{\#}) \cong \xi k(t^{\#}) = G_t(\xi).\blacksquare$$

We shall write $\overline{\boldsymbol{n}}_t(\xi)$ for $\boldsymbol{n}_t^{-1}(\xi)$. Since $(\omega; m)$ is a m.f.p. of $U$, the arrow $m : U(\omega) \to \omega$ is an isomorphism. Therefore by $(n)$ we have an isomorphism

$$(m) \qquad \boldsymbol{m}(t) = mk(t) \circ \overline{\boldsymbol{n}}_t(\omega) : G_t(\omega) \cong \omega k(t),$$

and we shall write $\overline{\boldsymbol{m}}(t)$ for $\boldsymbol{m}^{-1}(t)$.

### 3.2. CONSTRUCTION OF THE ARROWS $M$

We shall define for all $t, s \in K$, s.t. $(ts)^b \in K$, an arrow

$$(M) \qquad M(t,s) : \omega k(t)(\omega k(s)) \to \omega k((ts)^b).$$

Fix $t \in K$. Since $\mathcal{F}$ is iteratively closed, (ic3) in 1.5 holds. Therefore there is an object $\vartheta \in \mathcal{F}$ and a family of arrows

$$(X) \qquad X(t,s) : \omega k(t)(\vartheta k(s)) \to \omega k((ts)^b) \quad (s, (ts)^b \in K),$$

which is universal in the sense of (ic3). Then for all $s \in K$, s.t. $(ts)^b \in K$, define:

$$X'(t,s) = \begin{cases} \boldsymbol{m}(ts) \circ \omega k(t)\boldsymbol{n}_s(\vartheta), & s = c \\ \boldsymbol{m}((ts)^b) \circ X(t,p)\gamma \circ \overline{\boldsymbol{a}} \circ \omega k(t)\boldsymbol{n}_s(\vartheta), & s = pc \\ \boldsymbol{m}((ts)^b) \circ \overline{\boldsymbol{i}} \circ \langle X(t,p_0), X(t,p_1)\rangle \circ \boldsymbol{i} \circ \omega k(t)\boldsymbol{i} \circ \omega k(t)\boldsymbol{n}_s(\vartheta), & s = \langle p_0, p_1\rangle \\ \boldsymbol{m}(ts) \circ X(t,s_i) \circ \omega k(t)\boldsymbol{n}_s(\vartheta), & s = x_i,\ i < n \\ \boldsymbol{m}((ts)^b) \circ X(t,(ps_i)^b) \circ \omega k(t)\boldsymbol{n}_s(\vartheta), & s = px_i,\ i < n. \end{cases}$$

Using $(n)$, $(G)$, $(m)$, $(X)$, 1.1.(DM1), 1.1.(DM6) and Lemma 1 in 1.7, we may see that $X'(t,s)$ is an arrow

$$X'(t,s) : \omega k(t)(U(\vartheta)k(s)) \to \omega k((ts)^b),$$

whence by the universal property (ic3) of the family $(X)$ it follows that there is unique arrow $g : U(\vartheta) \to \vartheta$, s.t.

$$(g) \qquad X'(t,s) = X(t,s) \circ \omega k(t)(gk(s))$$

for all $s \in K$, s.t. $(ts)^b \in K$. Since $(\omega; m)$ is a m.f.p. of the functor $U$, we have:

$(ug)$ there is unique $u_g : \omega \to \vartheta$, s.t. $u_g \circ m = g \circ U(u_g)$.

Finally, defining

$$(dM) \qquad M(t,s) = X(t,s) \circ \omega k(t)(u_g k(s)),$$

we obtain $(M)$.

**Lemma 2.** *For all $t, s \in K$, s.t. $(ts)^b \in K$, we have:*
(1) $M(t,s) \circ \omega k(t)(mk(s)) = \boldsymbol{m}(ts) \circ \omega k(t)\boldsymbol{n}_s(\omega)$, *if* $s = c$;
(2) $M(t,s) \circ \omega k(t)(mk(s)) = \boldsymbol{m}((ts)^b) \circ M(t,p)\gamma \circ \overline{\boldsymbol{a}} \circ \omega k(t)\boldsymbol{n}_s(\omega)$, *if* $s = pc$;
(3) $M(t,s) \circ \omega k(t)(mk(s))$
$= \boldsymbol{m}((ts)^b) \circ \overline{\boldsymbol{i}} \circ \langle M(t,p_0), M(t,p_1)\rangle \circ \boldsymbol{i} \circ \omega k(t)\boldsymbol{i} \circ \omega k(t)\boldsymbol{n}_s(\omega)$, *if* $s = \langle p_0, p_1\rangle$;
(4) $M(t,s) \circ \omega k(t)(mk(s)) = \boldsymbol{m}(ts) \circ M(t,s_i) \circ \omega k(t)\boldsymbol{n}_s(\omega)$, *if* $s = x_i$, $i < n$;
(5) $M(t,s) \circ \omega k(t)(mk(s)) = \boldsymbol{m}((ts)^b) \circ M(t,(ps_i)^b) \circ \omega k(t)\boldsymbol{n}_s(\omega)$, *if* $s = px_i$, $i < n$.

Proof. A direct calculation, using $(dM)$, $(ug)$, $(g)$, the naturality of $\boldsymbol{n}_s$ (Lemma 1), the definition of $X'$ and $(G)$. ■

### 3.3. DEFINITION OF THE ARROWS $w$

In the sequel we shall write $\omega k(\overline{x})$ for the object $(\omega k(x_0), \ldots, \omega k(x_{n-1}))$ of $\mathfrak{F}^n$, and $t^*(\overline{\xi})$ for $[\lambda\overline{x}.t](\overline{\xi})$ for any $\overline{\xi} \in \mathfrak{F}$ and $t \in \mathbf{Term}$. Define by induction on $t \in K$ an arrow

$$w(t) : t^*(\omega k(\overline{x})) \to \omega k(t),$$

as follows:

$$w(t) = \begin{cases} \boldsymbol{m}(t) & \text{if } t = c \\ \boldsymbol{m}(t) \circ w(p)\gamma & \text{if } t = pc \\ \boldsymbol{m}(t) \circ \overline{\boldsymbol{i}} \circ \langle w(t_0), w(t_1)\rangle & \text{if } t = \langle t_0, t_1\rangle \\ \omega k(t) = 1_{\omega k(t)} & \text{if } t = x_i,\ i < n \\ M(p, x_i) \circ w(p)(\omega k(x_i)) & \text{if } t = px_i,\ i < n; \end{cases}$$

and define for all $i < n$:

$$w_i = \boldsymbol{m}(x_i) \circ w(s_i).$$

Then by $(m)$ we have that $w_i$ is an arrow

$$w_i : s_i^*(\omega k(\overline{x})) \to \omega k(x_i),$$

i.e. $\overline{w} : \overline{S}(\omega k(\overline{x})) \to \omega k(\overline{x})$ in $\mathfrak{F}^n$, where $\overline{w} = (w_0, \ldots, w_{n-1})$. We shall show that $(\omega k(\overline{x}); \overline{w})$ is a m.f.p. of the functor $\overline{S}$.

Let $(\overline{\xi};\overline{x})$ be an arbitrary object of the category $(\overline{S} \Rightarrow \mathcal{F}^n)$ (defined in 1.4), i.e. $\overline{\xi} = (\xi_0, \ldots, \xi_{n-1}) \in \mathcal{F}^n$, and $\overline{x}$ is a tuple $(x_0, \ldots, x_{n-1})$ of arrows $x_i : s_i^*(\overline{\xi}) \to \xi_i$ in $\mathcal{F}$. We shall define for every $t \in K$ an arrow

$$(v) \qquad v_t(\overline{\xi};\overline{x}) : \omega k(t) \to t^*(\overline{\xi}).$$

By 1.5.(ic3) there is an object $\eta \in \mathcal{F}$ and a family of arrows

$$(Y) \qquad Y((\overline{\xi};\overline{x}),t) : \eta k(t) \to t^*(\overline{\xi}), \quad t \in K,$$

which is universal in the sense of 1.5.(ic3), i.e. for every other family of arrows $y_t : \eta' k(t) \to t^*(\overline{\xi})$ there is unique $f : \eta' \to \eta$, s.t. $y_t = Y((\overline{\xi};\overline{x}),t) \circ fk(t)$ for all $t \in K$. We shall write $Y(t)$ for $Y((\overline{\xi};\overline{x}),t)$; the object $(\overline{\xi};\overline{x})$ will be usually fixed below.

Define for all $t \in K$:

$$Y'(t) = \begin{cases} \boldsymbol{n}_t(\eta) & \text{if } t = c \\ Y(p)\gamma \circ \boldsymbol{n}_t(\eta) & \text{if } t = pc \\ \langle Y(t_0), Y(t_1)\rangle \circ \boldsymbol{i} \circ \boldsymbol{n}_t(\eta) & \text{if } t = \langle t_0, t_1\rangle \\ x_i \circ Y(s_i) \circ \boldsymbol{n}_t(\eta) & \text{if } t = x_i,\ i < n \\ p^*(\overline{\xi})x_i \circ \overline{b}(ps_i) \circ Y((ps_i)^b) \circ \boldsymbol{n}_t(\eta) & \text{if } t = px_i,\ i < n, \end{cases}$$

where $\overline{b}$ is $b^{-1}$ and $b$ is defined in 1.7. We leave to the reader to show that $Y'(t) : U(\eta)k(t) \to t^*(\overline{\xi})$ for all $t \in K$. Hence by the universal property of the family $(Y)$ there is unique $h : U(\eta) \to \eta$, s.t.

$$(h) \qquad Y'(t) = Y(t) \circ hk(t) \quad \text{for all } \ t \in K.$$

Since $(\omega; m)$ is m.f.p. of $U$, we have:

$$(uh) \qquad \text{there is unique } \ u_h : \omega \to \eta, \ \text{s.t. } u_h \circ m = h \circ U(u_h).$$

Then we may define

$$(dv) \qquad v_t(\overline{\xi};\overline{x}) = Y(t) \circ u_h k(t).$$

We shall write $v(t)$ for $v_t(\overline{\xi};\overline{x})$.

**Lemma 3.** *For all $t \in K$ we have*

(1) $v(t) \circ mk(t) = \boldsymbol{n}_t(\omega)$ *if* $t = c$;
(2) $v(t) \circ mk(t) = v(p)\gamma \circ \boldsymbol{n}_t(\omega)$ *if* $t = pc$;
(3) $v(t) \circ mk(t) = \langle v(t_0), v(t_1)\rangle \circ \boldsymbol{i} \circ \boldsymbol{n}_t(\omega)$ *if* $t = \langle t_0, t_1\rangle$;
(4) $v(t) \circ mk(t) = x_i \circ v(s_i) \circ \boldsymbol{n}_t(\omega)$ *if* $t = x_i$, $i < n$;
(5) $v(t) \circ mk(t) = p^*(\overline{\xi})x_i \circ \overline{b}(ps_i) \circ v((ps_i)^b) \circ \boldsymbol{n}_t(\omega)$ *if* $t = px_i$, $i < n$.

Proof. A direct calculation similar to the proof of Lemma 2, using $(dv)$, $(uh)$, $(h)$, the naturality of $\boldsymbol{n}_s$ (Lemma 1), the definition of $Y'$ and $(G)$. ■

3.5. **Lemma 4.** *For all $t, s \in K$ we have*

$$v((ts)^b) \circ M(t,s) = b(ts) \circ v(t)v(s).$$

Proof. Fix $t \in K$. By 1.5.(ic3) there is $\zeta \in \mathcal{F}$ and a family of arrows

$$Z(t,s) : \omega k(t)(\zeta k(s)) \to (ts)^{b*}(\overline{\xi}), \quad s \in K,$$

which is universal in the sense of 1.5.(ic3). Define :

$$Z'(s) = \begin{cases} v(t)\gamma \circ \omega k(t)n_s(\zeta), & s = c \\ Z(t,p)\gamma \circ \overline{a} \circ \omega k(t)n_s(\zeta), & s = pc \\ \langle Z(t,p_0).Z(t,p_1)\rangle \circ i \circ \omega k(t)i \circ \omega k(t)n_s(\zeta), & s = \langle p_0, p_1\rangle \\ t^*(\overline{\xi})x_i \circ \overline{b}(ts_i) \circ Z(t,s_i) \circ \omega k(t)n_s(\zeta), & s = x_i,\ i < n \\ (tp)^{b*}(\overline{\xi})x_i \circ \overline{b}((tp)^b s_i) \circ Z(t,(ps_i)^b) \circ \omega k(t)n_s(\zeta), & s = px_i,\ i < n. \end{cases}$$

It is left to the reader to show that

$$Z'(s) : \omega k(t)(U(\zeta)k(s)) \to (ts)^{b*}(\overline{\xi}).$$

Then by the universal property of the family $Z(t,s)$ there is unique $z : U(\zeta) \to \zeta$, s.t.

(z) $$Z'(s) = Z(t,s) \circ \omega k(t)(zk(s)) \quad \text{for all } s \in K,$$

and since $(\omega; m)$ is a m.f.p. of $U$, we have:

(uz) there is unique $u_z : \omega \to \zeta$, s.t. $u_z \circ m = z \circ U(u_z)$.

We shall prove Lemma 4 by showing that

(1) $$Z(t,s) \circ \omega k(t)(u_z k(s)) = b(ts) \circ v(t)v(s)$$

and

(2) $$Z(t,s) \circ \omega k(t)(u_z k(s)) = v((ts)^b) \circ M(t,s)$$

for all $s \in K$.

### 3.6. PROOF OF 3.5.(1)

By the universal property of the family $Z(t,s)$ there is unique $y : \omega \to \zeta$, s.t.

(1) $$Z(t,s) \circ \omega k(t)(yk(s)) = b(ts) \circ v(t)v(s) \quad \text{for all } s \in K.$$

Therefore it is enough to show that $y = u_z$. By $(uz)$ the last will follow from $y \circ m = z \circ U(y)$ or $y = y'$, where $y' = z \circ U(y) \circ m^{-1}$. Since $y$ is the unique arrow satisfying (1), it is enough to show that

$$Z(t,s) \circ \omega k(t)(y'k(s)) = bts) \circ v(t)v(s) \quad \text{for all } s \in K.$$

By the definition of $y'$ the last equality is equivalent to

$$Z(t,s) \circ \omega k(t)(zk(s)) \circ \omega k(t)(U(y)k(s)) = b(ts) \circ v(t)v(s) \circ \omega k(t)(mk(s)),$$

but by $(z)$ and Lemma 1

$$Z(t,s) \circ \omega k(t)(zk(s)) \circ \omega k(t)(U(y)k(s)) = \Psi,$$

where

$$\Psi = Z'(s) \circ \omega k(t)\overline{n}_s(\zeta) \circ \omega k(t)G_s(y) \circ \omega k(t)n_s(\omega).$$

We shall prove that

$$\Psi = b(ts) \circ v(t)v(s) \circ \omega k(t)(mk(s)).$$

Consider cases for $s$ as follows:

*Case 1.* $s = c$. Then

$$\Psi = v(t)\gamma \circ \omega k(t)\gamma \circ \omega k(t)\boldsymbol{n}_s(\omega) \qquad \text{(by definition of } Z'(s) \text{ and } (G))$$
$$= v(t)\gamma \circ \omega k(t)(v(s) \circ mk(t)) \qquad \text{(by Lemma 3, (1))}$$
$$= v(t)v(s)\circ\omega k(t)(mk(t)) = b(ts)\circ v(t)v(s)\circ\omega k(t)(mk(t)) \text{ (by 1.2.(1) and 1.7.(b1))}.$$

*Case 2.* $s = pc$. Then similarly

$$\Psi = Z(t,p)\gamma \circ \overline{\boldsymbol{a}} \circ \omega k(t)(yk(p)\gamma) \circ \omega k(t)\boldsymbol{n}_s(\omega) \quad \text{(by definition of } Z'(s) \text{ and } (G))$$
$$= (Z(t,p) \circ \omega k(t)(yk(p)))\gamma \circ \overline{\boldsymbol{a}} \circ \omega k(t)\boldsymbol{n}_s(\omega) \quad \text{(by 1.2.(2))}$$
$$= b(tp)\gamma \circ v(t)v(p)\gamma \circ \overline{\boldsymbol{a}} \circ \omega k(t)\boldsymbol{n}_s(\omega) \qquad \text{(by (1))}$$
$$= b(tp)\gamma \circ \overline{\boldsymbol{a}} \circ v(t)(v(p)\gamma) \circ \omega k(t)\boldsymbol{n}_s(\omega) \qquad \text{(by 1.2.(2))}$$
$$= b(ts) \circ v(t)(v(p)\gamma \circ \boldsymbol{n}_s(\omega)) = b(ts) \circ v(t)(v(s) \circ mk(s)) \quad \text{(by Lemma 3, (2))}$$
$$= b(ts) \circ v(t)v(s) \circ \omega k(t)(mk(s)).$$

*Case 3.* $s = \langle p_0, p_1\rangle$. Similarly

$$\Psi = \langle Z(t,p_0).Z(t,p_1)\rangle \circ \boldsymbol{i} \circ \omega k(t)\boldsymbol{i} \circ \omega k(t)(y\langle k(p_0), k(p_1)\rangle) \circ \omega k(t)\boldsymbol{n}_s(\omega)$$
$$= \langle b(tp_0) \circ v(t)v(p_0), b(tp_1) \circ v(t)v(p_1)\rangle \circ \boldsymbol{i} \circ \omega k(t)\boldsymbol{i} \circ \omega k(t)\boldsymbol{n}_s(\omega)$$
$$= \langle b(tp_0), b(tp_1)\rangle \circ \boldsymbol{i} \circ v(t)\langle v(p_0), v(p_1)\rangle \circ \omega k(t)\boldsymbol{i} \circ \omega k(t)\boldsymbol{n}_s(\omega)$$
$$= b(ts) \circ v(t)(\langle v(p_0), v(p_1)\rangle \circ \boldsymbol{i} \circ \boldsymbol{n}_s(\omega)) = b(ts) \circ v(t)(v(s) \circ mk(s))$$
$$= b(ts) \circ v(t)v(s) \circ \omega k(t)(mk(s)).$$

*Case 4.* $s = x_i$, $i < n$.

$$\Psi = t^*(\overline{\xi})x_i \circ \overline{b}(ts_i) \circ Z(t,s_i) \circ \omega k(t)(yk(s_i)) \circ \omega k(t)\boldsymbol{n}_s(\omega)$$
$$= t^*(\overline{\xi})x_i \circ \overline{b}(ts_i) \circ b(ts_i) \circ v(t)v(s_i) \circ \omega k(t)\boldsymbol{n}_s(\omega)$$
$$= v(t)(x_i \circ v(s_i) \circ \boldsymbol{n}_s(\omega)) = v(t)v(s) \circ \omega k(t)(mk(s))$$
$$= b(ts) \circ v(t)v(s) \circ \omega k(t)(mk(s)).$$

*Case 5.* $s = px_i$, $i < n$. As before

$$\Psi = (tp)^{b*}(\overline{\xi})x_i \circ \overline{b}((tp)^b s_i) \circ Z(t,(ps_i)^b) \circ \omega k(t)(yk((ps_i)^b)) \circ \omega k(t)\boldsymbol{n}_s(\omega)$$
$$= (tp)^{b*}(\overline{\xi})x_i \circ \overline{b}((tp)^b s_i) \circ b(t(ps_i)^b) \circ v(t)v((ps_i)^b) \circ \omega k(t)\boldsymbol{n}_s(\omega).$$

But

$$(tp)^{b*}(\overline{\xi})x_i \circ \overline{b}((tp)^b s_i) \circ b(t(ps_i)^b)$$
$$= (tp)^{b*}(\overline{\xi})x_i \circ b(tp)s_i^*(\overline{\xi}) \circ \overline{b}(tps_i) \circ b(t(ps_i)^b) \qquad \text{(by 1.7.(10))}$$
$$= (tp)^{b*}(\overline{\xi})x_i \circ b(tp)s_i^*(\overline{\xi}) \circ \overline{\boldsymbol{a}} \circ \overline{b}(t(ps_i)) \circ b)t(ps_i)^b) \quad \text{(by 1.7.(11))}$$
$$= b(tp)\xi_i \circ (tp)^*(\overline{\xi})x_i \circ \overline{\boldsymbol{a}} \circ t^*(\overline{\xi})\overline{b}(ps_i) \qquad \text{(by 1.2.(1) and 1.7.(b5))}$$
$$= b(tp)\xi_i \circ \overline{\boldsymbol{a}} \circ t^*(\overline{\xi})(p^*(\overline{\xi})x_i) \circ t^*(\overline{\xi})\overline{b}(ps_i) \qquad \text{(by 1.2.(2))}$$
$$= b(ts) \circ t^*(\overline{\xi})(p^*(\overline{\xi})x_i) \circ t^*(\overline{\xi})\overline{b}(ps_i) \qquad \text{(by 1.7.(b3))}.$$

Therefore

$$\Psi = b(ts) \circ t^*(\overline{\xi})(p^*(\overline{\xi})x_i) \circ t^*(\overline{\xi})\overline{b}(ps_i) \circ v(t)v((ps_i)^b) \circ \omega k(t)\boldsymbol{n}_s(\omega)$$
$$= b(ts) \circ v(t)(p^*(\overline{\xi})x_i \circ \overline{b}(ps_i) \circ v((ps_i)^b) \circ \boldsymbol{n}_s(\omega))$$

$$= b(ts) \circ v(t)(v(s) \circ mk(s)) \qquad \text{(by Lemma 3, (5))}$$
$$= b(ts) \circ v(t)v(s) \circ \omega k(t)(mk(s)).$$

This finishes the proof of 3.5.(1).

### 3.7. PROOF OF 3.5.(2)

This proof is similar but simpler then that in 3.6. We replace $b(ts) \circ v(t)v(s)$ by $v((ts)^b) \circ M(t,s)$ and use Lemma 2. The properties of the isomorphisms $b$ are not used in this proof. We leave it to the reader.

This finishes the proof of Lemma 4. As a corollary we have:

If $tx_i \in K$, where $i < n$, then

(1) $$v(tx_i) \circ M(t, x_i) = v(t)v(x_i).$$

We shall write $v(\overline{x})$ for the arrow $(v(x_0), \ldots, v(x_{n-1})) : \omega k(\overline{x}) \to \overline{\xi}$ in $\mathcal{F}^n$.

3.8. **Corollary 1.** *For all $t \in K$ we have $v(t) \circ w(t) = t^*(v(\overline{x}))$.*

P r o o f. Induction on $t$. Consider cases for $t$ as in the definition of $w(t)$. All of them are easy to be proved, but in the last one $t = px_i$ and Lemma 4 is used:

$$v(t) \circ w(t) = v(px_i) \circ M(p, x_i) \circ w(p)(\omega k(x_i)) \qquad \text{(by the definition of } w(t)\text{)}$$
$$= v(p)v(x_i) \circ w(p)(\omega k(x_i)) \qquad \text{(by 3.7.(1))}$$
$$= (v(p) \circ w(p))v(x_i) = p^*(v(\overline{x}))v(x_i) \qquad \text{(by the induction hypothesis)}$$
$$= t^*(v(\overline{x})). \blacksquare$$

**Corollary 2.** *$v(\overline{x})$ is an arrow $v(\overline{x}) : (\omega k(\overline{x}); \overline{w}) \to (\overline{\xi}; \overline{x})$ in the category $(\overline{S} \Rightarrow \mathcal{F}^n)$, i.e.*

(1) $$v(x_i) \circ w_i = x_i \circ s_i^*(v(\overline{x})) \quad \textit{for all } i < n.$$

Indeed, by the definition of $w_i$, Lemma 3, 3.1.$(m)$ and Corollary 1:

$$v(x_i) \circ w_i = x_i \circ v(s_i) \circ n_t(\omega) \circ m^{-1}k(t) \circ m(x_i) \circ w(s_i) = x_i \circ v(s_i) \circ w(s_i) = x_i \circ s_i^*(v(\overline{x})).$$

It remains to show that $v(\overline{x})$ is the unique arrow $(\omega k(\overline{x}); \overline{w}) \to (\overline{\xi}; \overline{x})$ in $(\overline{S} \Rightarrow \mathcal{F}^n)$. For that suppose that $\overline{v} : (\omega k(\overline{x}); \overline{w}) \to (\overline{\xi}; \overline{x})$ is an arbitrary arrow in the last category, i.e. $\overline{v} = (v_0, \ldots, v_{n-1})$ and

(2) $$v_i \circ w_i = x_i \circ s_i^*(\overline{v}) \quad \text{for all } i < n.$$

We shall write $v_\omega(t)$ for $v_t(\omega k(\overline{x}); \overline{w})$ (for definition see 3.4).

3.9. **Lemma 5.** *For all $t \in K$ we have $v(t) = t^*(\overline{v}) \circ v_\omega(t)$.*

P r o o f. By 1.5.(ic3) there is unique $y : \omega \to \eta$ s.t.

(1) $$t^*(\overline{v}) \circ v_\omega(t) = Y(t) \circ yk(t) \quad \text{for all } t \in K.$$

Let $y' = h \circ U(y) \circ m^{-1}$. We shall show that $y = y'$, whence by 3.4.$(uh)$ it will follow that $y = u_h$ and by 3.4.$(dv)$ the lemma will be proved. Since the arrow $y$ satisfying (1) is unique, it is enough to show that

$$t^*(\overline{v}) \circ v_\omega(t) = Y(t) \circ y'k(t) \quad \text{for all } t \in K.$$

The last equality is equivalent to

$$t^*(\overline{v}) \circ v_\omega(t) \circ mk(t) = Y(t) \circ hk(t) \circ U(y)k(t)$$

By 3.4.($h$) and Lemma 1

$$Y(t) \circ hk(t) \circ U(y)k(t) = Y'(t) \circ \overline{n}_t(\eta) \circ G_t(y) \circ n_t(\omega).$$

Denote the last expression by $\chi$ and consider cases for $t$ to show that $\chi = t^*(\overline{v}) \circ v_\omega(t) \circ mk(t)$. We shall treat only the case, in which the supposition 3.8.(2) is used. This is the case when $t = px_i$, $i < n$. Then by the definition of $Y'$ and $(G)$

$$\begin{aligned}
\chi &= p^*(\overline{\xi})x_i \circ \overline{b}(ps_i) \circ Y((ps_i)^b) \circ yk((ps_i)^b) \circ n_t(\omega) \\
&= p^*(\overline{\xi})x_i \circ \overline{b}(ps_i) \circ (ps_i)^{b*}(\overline{v}) \circ v_\omega((ps_i)^b) \circ n_t(\omega) \qquad \text{(by (1))} \\
&= p^*(\overline{\xi})x_i \circ (ps_i)^*(\overline{v}) \circ \overline{b}(ps_i) \circ v_\omega((ps_i)^b) \circ n_t(\omega) \quad \text{(by the naturality of } b) \\
&= p^*(\overline{v})(x_i \circ s_i^*(\overline{v})) \circ \overline{b}(ps_i) \circ v_\omega((ps_i)^b) \circ n_t(\omega) \\
&= p^*(\overline{v})(v_i \circ w_i) \circ \overline{b}(ps_i) \circ v_\omega((ps_i)^b) \circ n_t(\omega) \qquad \text{(by 3.8.(2))} \\
&= p^*(\overline{v})v_i \circ p^*(\omega k(\overline{x}))w_i \circ \overline{b}(ps_i) \circ v_\omega((ps_i)^b) \circ n_t(\omega) \\
&= t^*(\overline{v}) \circ v_\omega(t) \circ mk(t) \qquad \text{(by Lemma 3, (5))}. \blacksquare
\end{aligned}$$

**3.10. Lemma 6.** *For all $t, s, r \in K$, s.t. $(ts)^b \in K$ and $(sr)^b \in K$ we have $M(t,(sr)^b) \circ \omega k(t)M(s,r) \circ \boldsymbol{a} = M((ts)^b, r) \circ M(t,s)(\omega k(r))$.*

This is the most complicated lemma in the proof but quite similar to Lemma 4: using 1.5.(ic3) we construct a family of arrows

$$M'(t,s,r) : \omega k(t)(\omega k(s))(\omega k(r)) \to \omega k((\mathrm{tsr})^b)$$

and prove separately that $M(t,(sr)^b) \circ \omega k(t)M(s,r) \circ \boldsymbol{a} = M'(t,s,r)$, $M((ts)^b, r) \circ M(t,s)(\omega k(r)) = M'(t,s,r)$. We leave this to the reader.

We shall write $\boldsymbol{b}_\omega(t)$ for $\boldsymbol{b}_t(\omega k(\overline{x}))$ (see the definition of $\boldsymbol{b}_t(\overline{\xi})$ in 1.7) and $\overline{\boldsymbol{b}}_\omega(t)$ for $\boldsymbol{b}_\omega^{-1}(t)$.

**3.11. Lemma 7.** *For all $t, s \in K$, s.t. $(ts)^b \in K$ we have*

$$M(t,s) \circ w(t)w(s) = w((ts)^b) \circ \boldsymbol{b}_\omega(ts).$$

P r o o f. Induction on $s$. Consider cases for $s$ as in the definition of $w(s)$. We shall treat only two of the cases: $s = pc$ and $s = px_i$.

Let $s = pc$. Then by Lemma 2 and the definition of $w(s)$

$$\begin{aligned}
M(t,s) \circ w(t)w(s) &= \boldsymbol{m}((ts)^b) \circ M(t,p)\gamma \circ \overline{\boldsymbol{a}} \circ \omega k(t)\overline{\boldsymbol{m}}(s) \circ w(t)(\boldsymbol{m}(s) \circ w(p)\gamma) \\
&= \boldsymbol{m}((ts)^b) \circ M(t,p)\gamma \circ \overline{\boldsymbol{a}} \circ w(t)(w(p)\gamma) = \boldsymbol{m}((ts)^b) \circ M(t,p)\gamma \circ w(t)w(p)\gamma \circ \overline{\boldsymbol{a}} \\
&= \boldsymbol{m}((ts)^b) \circ w((tp)^b)\gamma \circ \boldsymbol{b}_\omega(tp)\gamma \circ \overline{\boldsymbol{a}} \quad \text{(by the induction hypothesis)} \\
&= w((ts)^b) \circ \boldsymbol{b}_\omega(ts) \qquad \text{(by the definition of } w((ts)^b) \text{ and 1.7.(b3))}.
\end{aligned}$$

Let $s = x_i$, $i < n$. Then by the definition of $w(s)$

$$\begin{aligned}
M(t,s) \circ w(t)w(s) &= M(t,s) \circ w(t)(M(p,x_i) \circ w(p)(\omega k(x_i))) \\
&= M(t,s) \circ \omega k(t)M(p,x_i) \circ \omega k(t)(w(p)(\omega k(x_i))) \circ w(t)s^*(\omega k(\overline{x})) \quad \text{(by 1.2.(1))}
\end{aligned}$$

$$= M((tp)^b, x_i) \circ M(t,p)(\omega k(x_i)) \circ \overline{a} \circ \omega k(t)(w(p)(\omega k(x_i))) \circ w(t)s^*(\omega k(\overline{x}))$$

(by Lemma 6)

$$= M((tp)^b, x_i) \circ M(t,p)(\omega k(x_i)) \circ w(t)w(p)(\omega k(x_i)) \circ \overline{a}$$

$$= M((tp)^b, x_i) \circ w((tp)^b)(\omega k(x_i)) \circ b_\omega(ts)(\omega k(x_i)) \circ \overline{a}$$

(by the hypothesis of the induction)

$$= w((ts)^b) \circ b_\omega(ts). \blacksquare$$

**3.12. Lemma 8.** *For all $t \in K$ we have $w(t) \circ v_\omega(t) = \omega k(t) = 1_{\omega k(t)}$.*

Proof. Similar to that of Lemma 4. By 1.5.(ic3) and 1.1.(DM2) there is $\varepsilon \in \mathfrak{F}$ and a family of arrows $E(t) : \varepsilon k(t) \to \omega k(t)$, $t \in K$, which is universal in an obvious sense. Define for $t \in K$:

$$E'(t) = \begin{cases} m(t) \circ n_t(\varepsilon) & \text{if } t = c \\ m(t) \circ E(p)\gamma \circ n_t(\varepsilon) & \text{if } t = pc \\ m(t) \circ \overline{i} \circ \langle E(t_0), E(t_1)\rangle \circ i \circ n_t(\varepsilon) & \text{if } t = \langle t_0, t_1\rangle \\ m(t) \circ E(s_i) \circ n_t(\varepsilon) & \text{if } t = x_i,\ i < n \\ m(t) \circ E((ps_i)^b) \circ n_t(\varepsilon) & \text{if } t = px_i,\ i < n. \end{cases}$$

By the universal property of the family $E(t)$ there is unique $e : U(\varepsilon) \to \varepsilon$, s.t. $E'(t) = E(t) \circ ek(t)$ for all $t \in K$. Since $(\omega; m)$ is a m.f.p. of the functor $U$, there is unique $u_e : \omega \to \varepsilon$, s.t.

$$u_e \circ m = e \circ U(u_e).$$

We shall prove the lemma by showing that

(1) $$w(t) \circ v_\omega(t) = E(t) \circ u_e k(t)$$

and

$$\omega k(t) = E(t) \circ u_e k(t)$$

for all $t \in K$. By the universal property of the family $E(t)$ there is unique $y : \omega \to \varepsilon$, s.t.

$$w(t) \circ v_\omega(t) = E(t) \circ yk(t) \quad \text{for all } t \in K.$$

To prove (1) it is enough to show that

$$w(t) \circ v_\omega(t) = E(t) \circ y'k(t) \quad \text{for all } t \in K,$$

where $y' = e \circ U(y) \circ m^{-1}$. For that consider cases for $t$ as in the definition of $E'(t)$. We shall treat only the last one of the cases in order to show how Lemma 7 is used. Let $t = px_i$, $i < n$. Then

$$E(t) \circ y'k(t) = E(t) \circ ek(t) \circ U(y)k(t) \circ m^{-1}k(t)$$

$$= E'(t) \circ \overline{n}_t(\varepsilon) \circ G_t(y) \circ n_t(\omega) \circ m^{-1}k(t) \qquad \text{(using 3.1.(n))}$$

$$= m(t) \circ E((ps_i)^b) \circ yk((ps_i)^b) \circ \overline{m}(t) \quad \text{(definition of } E', 3.1.(m), (G))$$

$$= m(t) \circ w((ps_i)^b) \circ v_\omega((ps_i)^b) \circ \overline{m}(t) \quad \text{(definition of } y)$$

$$= m(t) \circ M(p, s_i) \circ w(p)w(s_i) \circ \overline{b}_\omega(ps_i) \circ v_\omega((ps_i)^b) \circ \overline{m}(t) \qquad \text{(Lemma 7)}$$

$$= M(p, x_i) \circ \omega k(p)m(x_i) \circ w(p)w(s_i) \circ \overline{b}_\omega(ps_i) \circ v_\omega((ps_i)^b) \circ \overline{m}(t) \quad \text{(Lemma 2, (4))}$$

$$= M(p, x_i) \circ w(p) w_i \circ \overline{b}_\omega(ps_i) \circ v_\omega((ps_i)^b) \circ \overline{m}(t) \qquad \text{(definition of } w_i)$$

$$= M(p, x_i) \circ w(p) \omega k(x_i) \circ p^*(\omega k(\overline{x})) w_i \circ \overline{b}_\omega(ps_i) \circ v_\omega((ps_i)^b) \circ \overline{m}(t) \quad (1.2.(1))$$

$$= w(t) \circ v_\omega(t) \qquad \text{(definition of } w(t) \text{ and Lemma 3, (5))}.$$

The rest of the proof of Lemma 8 is left to the reader.

### 3.13. FINAL OF THE PROOF OF THE THEOREM

By Lemma 8 $\omega k(x_i) = w(x_i) \circ v_\omega(x_i) = \omega k(x_i) \circ v_\omega(x_i) = v_\omega(x_i)$, whence by Lemma 5 $v(x_i) = v_i \circ v_\omega(x_i) = v_i$. ■

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