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BIFURCATIONS OF INVARIANT MANIFOLDS IN A MODEL IN THE RIGID BODY DYNAMICS

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Огнян Христов. БИФУРКАЦИИ ИНВАРИАНТНЫХ МНОГООБРАЗИЙ В ОДНОМ МОДЕЛЕ В ДИНАМИКИ ТВЕРДОВА ТЕЛА

Рассмотрена механическая модель, обобщающая классическая задача о движении твердова тела. Изолирован интегрируемый случай. Описаны топология множества уровня интегралов и все бифуркации Лиувилевых торов и цилиндров.

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A model, generalizing the rigid body problem is considered. An integrable case is isolated. The topology of the real level sets of the motion constants and all bifurcations of the Liouville tori and cylinders are described.

1. INTRODUCTION

Consider the following mechanical problem. A particle, hanged up on a spring, is oscillating in a symmetric heavy body with a fixed point along the axis of symmetry. This is a conservative system with four degrees of freedom. Consider the motion of this system without external forces. It can be described via the Lagrangian [1]

$$(1.1)\qquad L = \frac{1}{2}\left\{\left[(A+mr^2)(\dot\psi^2\sin^2\theta+\dot\theta^2)+C(\dot\varphi+\dot\psi\cos\theta)^2\right]+m\dot r^2-\sigma r^2\right\}.$$

Here ψ, θ, φ are the Euler angles, m is the mass of the particle, r — the deviation of the particle from the fixed point, σ — the stiffness of the spring and the C, A are the inertia moments about the symmetry axis and in the orthogonal plane, respectively. In the next section we shall show that the system (1.1) is completely integrable.

The purpose of this paper is to classify all real level sets. According to the classical Liouville–Arnold theorem [2], we may expect that they are consist of tori, cylinders and planes. In recent papers [3], [4] the generic bifurcations of invariant manifolds are studied for generalized Henon–Heiles system and Gelfand–Dikii system respectively, using their algebraic structure. In quite different way Kharlamov [5] studied the bifurcations of integrable cases of rigid body problem.

However, our case is more simple than others mentioned above. First, the problem is integrated in elliptic functions and, second, the variables are directly separated. So, after reducing the system to two degrees of freedom, in order to study real level sets, it is sufficient to draw the graphs of the effective potentials. Then the real level sets are merely the product of the real level sets of the corresponding integrals.

2. EQUATIONS OF MOTION AND INTEGRALS

Let us simplify the problem. First, let us get rid of unessential constants. After changes of the variables $t \Rightarrow t\sqrt{A}$ and $r \Rightarrow r\sqrt{m/A}$ and denoting $s = \sigma A/m$, $\tilde{C} = C/A$, the Lagrangian (1.1) becomes

$$L = \frac{1}{2}\left\{\left[(1+r^2)(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2) + \tilde{C}(\dot{\varphi} + \dot{\psi}\cos\theta)^2\right] + \dot{r}^2 - sr^2\right\}. \tag{2.1}$$

It is obvious that ψ and φ are cyclic coordinates. Then the corresponding integrals of motion are:

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = \left(1+r^2\right)\dot{\psi}\sin^2\theta + \tilde{C}\left(\dot{\varphi} + \dot{\psi}\cos\theta\right)\cos\theta = a = \text{const}. \tag{2.2}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \tilde{C}\left(\dot{\varphi} + \dot{\psi}\cos\theta\right) = b = \text{const}. \tag{2.3}$$

Second, in order to reduce the system to two degrees of freedom, consider the Routh's function — $R = L - a\dot{\psi} - b\dot{\varphi}$. Simple calculations give

$$R = \frac{1}{2}\left[(1+r^2)\dot{\theta}^2 + \dot{r}^2 - sr^2 - \frac{(a - b\cos\theta)^2}{(1+r^2)\sin^2\theta}\right].$$

Note that the Routh's function has the Lagrangian form $R = T^* - \Pi^*$. The corresponding Hamiltonian system is defined via the Hamiltonian

$$H = \frac{p_r^2}{2} + \frac{sr^2}{2} + \frac{1}{2(1+r^2)}\left[p_\theta^2 + \frac{(a - b\cos\theta)^2}{\sin^2\theta}\right], \tag{2.4}$$

where $p_r = \dfrac{\partial R}{\partial \dot{r}} = \dot{r}$, $p_\theta = \dfrac{\partial R}{\partial \dot{\theta}} = (1+r^2)\dot{\theta}$ and $H = p_r\dot{r} + p_\theta\dot{\theta} - R$ (the Legandre transformation). There are no problems in deriving the equations of the reduced (two degrees of freedom) system. We shall consider a and b as parameters. The corresponding Hamilton–Jacobi equation to the system (2.4) obviously separates. If we denote

$$f = p_\theta^2 + \frac{(a - b\cos\theta)^2}{\sin^2\theta},$$

we obtain the following first integrals in involution:

$$H = \frac{p_r^2}{2} + \frac{sr^2}{2} + \frac{f}{2(1+r^2)} = h, \tag{2.5}$$

$$F = p_\theta^2 + \frac{(a - b\cos\theta)^2}{\sin^2\theta} = f, \quad \text{where } 0 < \theta < \pi. \tag{2.6}$$

Note that h, f are always $\geqq 0$. In the particular case, when $a = b = 0$, the integrals become

$$H = \frac{p_r^2}{2} + \frac{sr^2}{2} + \frac{f^2}{2(1+r^2)} = h \geqq 0, \tag{2.7}$$

$$F = p_\theta = f, \quad f \text{ arbitrary}. \tag{2.8}$$

3. TOPOLOGICAL ANALYSIS

In this section we shall describe the topological type of the real invariant manifold

$$M = \{H = h,\ F = f\} \subset \mathbb{R}^4.$$

This means (in the context of the present work) that we have to describe

(i) the topological type of M for all values of the constants a, b, f, h,

(ii) how the sets M, fit topologically as a, b, h, f, vary to make up $\mathbb{R}^4(\theta, r, p_\theta, p_r)$.

As it is seen in the previous section, our (reduced) system of two degrees of freedom splits into two one degree of freedom systems. Then the topology of M can be easily described by the product of phase portraits of these systems. The topology of one degree of freedom system can be obtained by investigating the graph of the potential of the system [2]. Of course, topological type may change only in the critical point of the corresponding potential.

Consider first the integral (2.6). Let

$$L(\theta) = \frac{(a - b\cos\theta)^2}{\sin^2\theta}, \quad 0 < \theta < \pi,$$

be a potential of the system. Simple calculations give the shape of the graph of $L(\theta)$ in different cases for a and b (Fig. 1 (a), (b), (c), (d), (e)).

The analysis of the above cases shows that the (a, b)-plane is divided into the following domains: I — $a = b = 0$, II — $\frac{|a|}{|b|} = 1$, III — $\frac{|a|}{|b|} \neq 1$ (Fig. 2). It is seen that only in the domain III there exist ovals (closed phase curves), but in domain I, II there exist only lines.

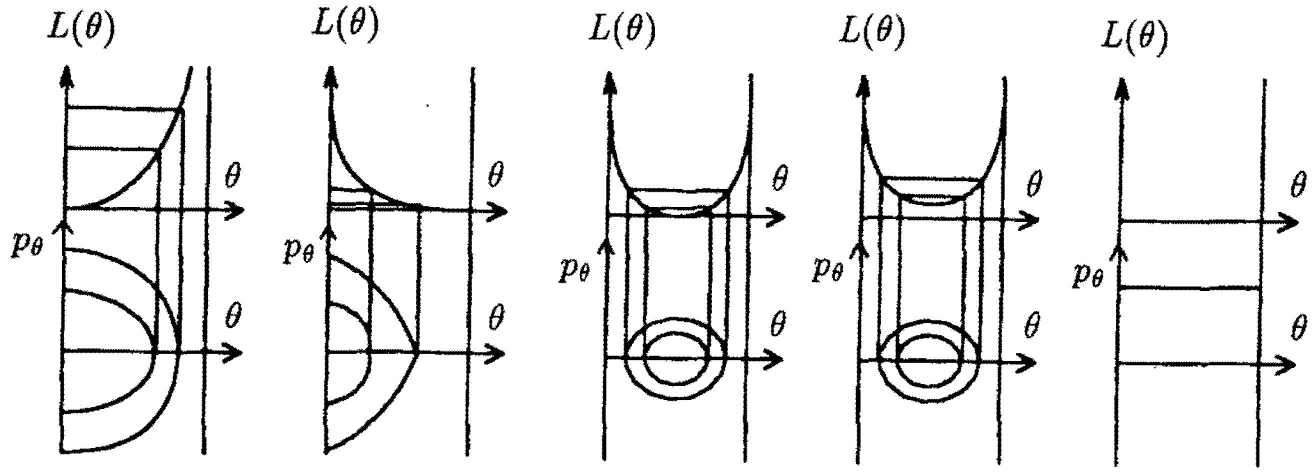

Fig. 1 (a) Fig. 1 (b) Fig. 1 (c) Fig. 1 (d) Fig. 1 (e)

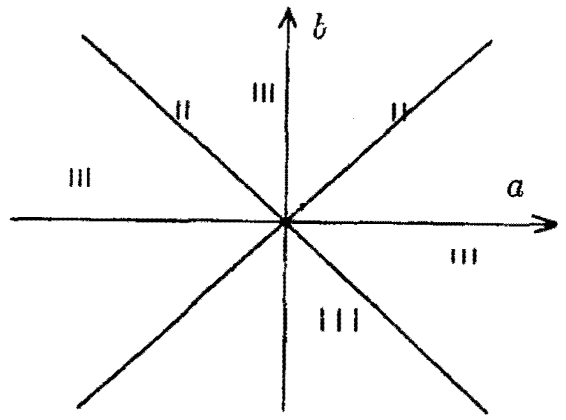

Fig. 2

Now consider the integral (2.5). Denote by $V(r)$ the effective potential

$$V(r) = \frac{sr^2}{2} + \frac{f}{2(1+r^2)}.$$

Simple calculations give the shape of the graph of $V(r)$ for different cases for f (Fig. 3 (a), (b), (c)).

Consider the set of critical values of the integrals (2.5) and (2.6) in the (h, f) plane. One should note that in the these values integral becomes dependent, i. e. $dH \wedge dF = 0$. Then combining the information of Fig. 1 and Fig. 3, we may see what is the type of the invariant manifold in concrete domain (Fig. 4). Denote by $\mathbb{T}$, $\mathbb{C}$ two dimensional torus and cylinder, respectively, by S^1, L the circle and line, respectively, and $\mathbb{T}.\mathbb{T}$ ($\mathbb{C}.\mathbb{C}$) denotes two sticked tori (cylinders).

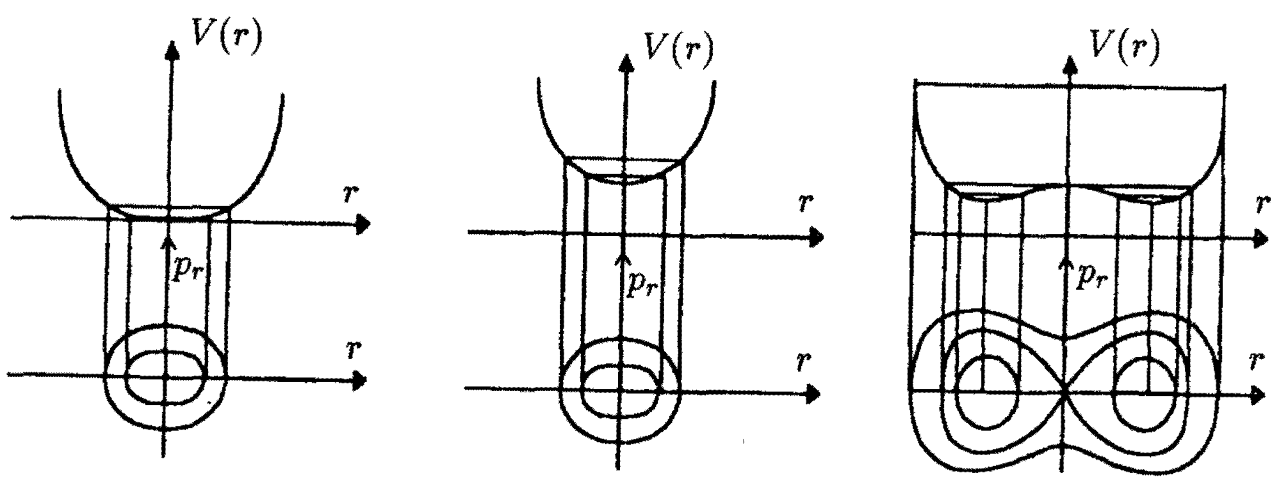

Fig. 3 (a) Fig. 3 (b) Fig. 3 (c)

So, we have

Proposition 1. All possible types of invariant manifold M are described in Table I and Fig. 4.

Table I

Domain	1	2	3	4	5	6	7	8	9	10	11
Type of M	0	$2S^1$	$2\mathbb{T}$	$\mathbb{T}.\mathbb{T}$	$\mathbb{T}$	S^1	$\mathbb{C}$	$\mathbb{C}.\mathbb{C}$	$2\mathbb{C}$	$2L$	L

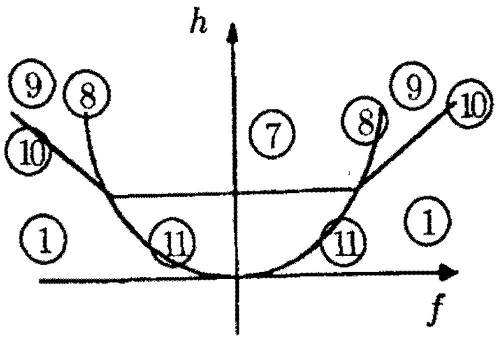

Fig. 4 (a) $a = b =.0$

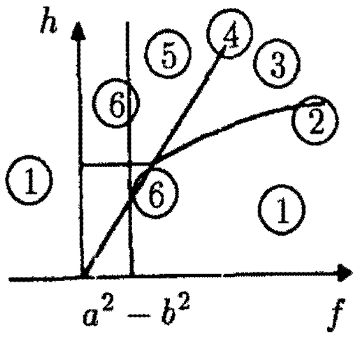

Fig. 4 (b) $\left|\frac{a}{b}\right| > 1$

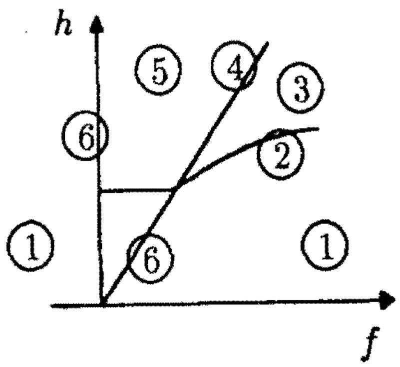

Fig. 4 (c) $\left|\frac{a}{b}\right| < 1$

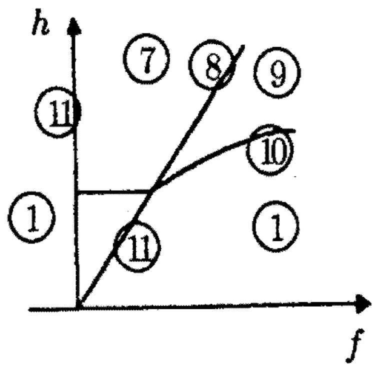

Fig. 4 (d) $|a| = |b| = 1$

Having in hand all information from Table I and Fig. 4, it is easy to describe all generic bifurcations when a, b, f, h vary.

Consider the following bifurcations:

(i) $\mathbb{T} \Rightarrow 0$: the torus collapses along its "axes" — the S^1, and then vanishes, i. e. $\mathbb{T} \Rightarrow S^1 \Rightarrow 0$,

(ii) $2\mathbb{T} \Rightarrow 0$: i. e. $2\mathbb{T} \Rightarrow 2S^1 \Rightarrow 0$,

(iii) $\mathbb{C} \Rightarrow 0$: i. e. $\mathbb{C} \Rightarrow L \Rightarrow 0$,

(iv) $2\mathbb{C} \Rightarrow 0$: i. e. $2\mathbb{C} \Rightarrow 2L \Rightarrow 0$,

(v) $\mathbb{C} \Rightarrow 2\mathbb{C}$ (Fig. 5a),

(vi) $\mathbb{T} \Rightarrow 2\mathbb{T}$ (Fig. 5b),

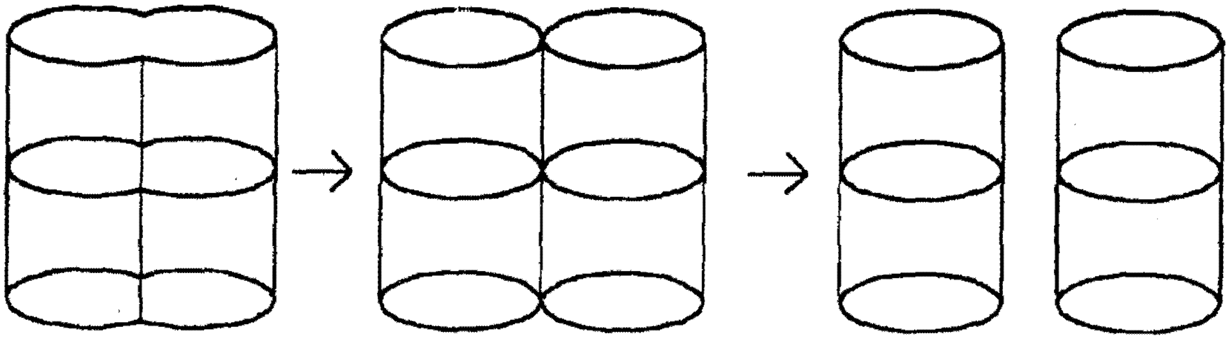

Fig. 5 (a)

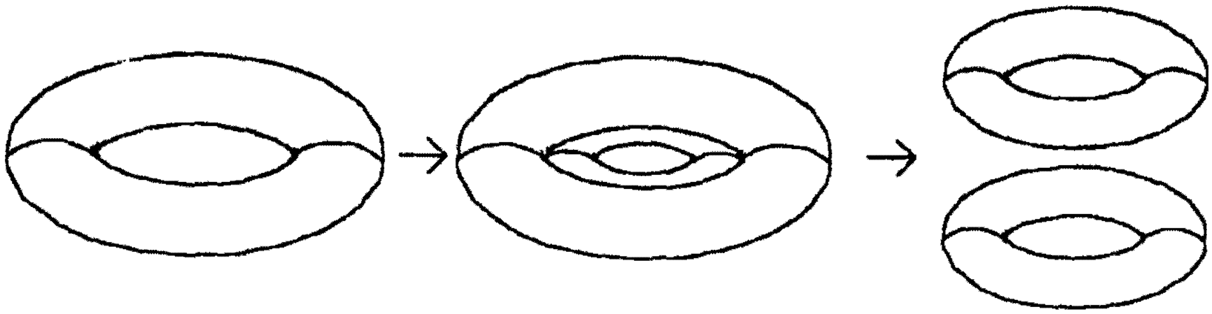

Fig. 5 (b)

(vii) $\mathbb{T} \Rightarrow \mathbb{C}$ ($2\mathbb{T} \Rightarrow 2\mathbb{C}$) when in Fig. 2 a, b cross the line $|a| = |b|$.

If $M_1 \Rightarrow M_2$ is an already defined bifurcation, then we denote by $M_2 \Rightarrow M_1$ the "inverse" bifurcation.

So, we have the obvious

Proposition 2. Any generic bifurcation of connected components of the invariant manifold M can be found among the bifurcations (i)—(vii). The precise description of all generic bifurcations of M is given in Table II.

Table II

$1 \Rightarrow 5$	$1 \Rightarrow 7$	$1 \Rightarrow 3$	$1 \Rightarrow 9$	$3 \Rightarrow 5$	$9 \Rightarrow 7$
$0 \Rightarrow \mathbb{T}$	$0 \Rightarrow \mathbb{C}$	$0 \Rightarrow 2\mathbb{T}$	$0 \Rightarrow 2\mathbb{C}$	$2\mathbb{T} \Rightarrow \mathbb{T}$	$2\mathbb{C} \Rightarrow \mathbb{C}$

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