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FACTORIZATIONS OF THE GROUPS OF LIE TYPE OF LIE RANK THREE OVER FIELDS OF 2 OR 3 ELEMENTS

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Цанко Генчев, Кероnе Чакърян. ФАКТОРИЗАЦИИ ГРУПП ТИПА ЛИ ЛИЕВСКОГО РАНГА 3 НАД ПОЛЯМИ ИЗ ДВУХ ИЛИ ТРЕХ ЭЛЕМЕНТОВ

Доказан следующий результат.

Пусть G — группа типа Ли лиевского ранга 3 над полем из двух или трех элементов. Предположим, что $G = AB$, где A и B — собственные неабелевые простые подгруппы G. Тогда имеет место одно из следующих:

1) $G = L_4(2)$, $A \cong L_3(2)$, $B \cong A_6$ или A_7;

2) $G = L_4(3)$, $A \cong L_3(3)$, $B \cong S_4(3)$;

3) $G = S_6(2)$, $A \cong L_4(2)$, $B \cong L_2(8)$, или $A \cong U_4(2)$, $B \cong L_2(8)$ или $U_3(3)$;

4) $G = U_6(2)$, $A \cong U_5(2)$, $B \cong S_6(2)$, $U_4(3)$ или M_{22};

5) $G = U_6(3)$, $A \cong U_5(3)$, $B \cong S_6(3)$;

6) $G = O_7(3)$, $A \cong L_4(3)$, $B \cong U_3(3)$, $G_2(3)$, $S_6(2)$ или A_9, или $A \cong G_2(3)$, $B \cong S_4(3)$, $S_6(2)$ или A_9.

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The following result is proved.

Let G be a group of Lie type of Lie rank three over a field of 2 or 3 elements. Suppose that $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:

1) $G = L_4(2)$, $A \cong L_3(2)$, $B \cong A_6$ or A_7;

2) $G = L_4(3)$, $A \cong L_3(3)$, $B \cong S_4(3)$;

3) $G = S_6(2)$, $A \cong L_4(2)$, $B \cong L_2(8)$, or $A \cong U_4(2)$, $B \cong L_2(8)$ or $U_3(3)$;

4) $G = U_6(2)$, $A \cong U_5(2)$, $B \cong S_6(2)$, $U_4(3)$ or M_{22};

5) $G = U_6(3)$, $A \cong U_5(3)$, $B \cong S_6(3)$;

6) $G = O_7(3)$, $A \cong L_4(3)$, $B \cong U_3(3)$, $G_2(3)$, $S_6(2)$ or A_9, or $A \cong G_2(3)$, $B \cong S_4(3)$, $S_6(2)$ or A_9.

1. INTRODUCTION

The factorizations (into the product of two simple groups) of all finite groups of Lie type of Lie rank one or two are known (see [3]). In this paper we prove the following

Theorem. Let G be a (finite, simple) group of Lie type of Lie rank three over a field of 2 or 3 elements. Suppose that $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:

1) $G = L_4(2)$, $A \cong L_3(2)$, $B \cong A_6$ or A_7;

2) $G = L_4(3)$, $A \cong L_3(3)$, $B \cong S_4(3)$;

3) $G = S_6(2)$, $A \cong L_4(2)$, $B \cong L_2(8)$, or $A \cong U_4(2)$, $B \cong L_2(8)$ or $U_3(3)$;

4) $G = U_6(2)$, $A \cong U_5(2)$, $B \cong S_6(2)$, $U_4(3)$ or M_{22};

5) $G = U_6(3)$, $A \cong U_5(3)$, $B \cong S_6(3)$;

6) $G = O_7(3)$, $A \cong L_4(3)$, $B \cong U_3(3)$, $G_2(3)$, $S_6(2)$ or A_9, or $A \cong G_2(3)$, $B \cong S_4(3)$, $S_6(2)$ or A_9.

Our notation is standard (see [2]).

2. PROOFS

The groups of Lie type of Lie rank three over a field of 2 or 3 elements are $L_4(2)$, $L_4(3)$, $S_6(2)$, $S_6(3)$, $U_6(2)$, $U_6(3)$, $U_7(2)$, $U_7(3)$, $O_7(3)$, $O_8^-(2)$, $O_8^-(3)$. The factorizations of $L_4(2)$, $L_4(3)$, $S_6(2)$, and $O_8^-(2)$ are known (see [1], [4]); this yields 1)–3) of the theorem.

Let $G = S_6(3)$. As $3^9 \mid |G|$, we can assume that $3^5 \mid |A|$. However, $S_6(3)$ has no proper simple subgroup of such an order ([2]).

If $G = U_7(2)$, we can assume that $43 \mid |A|$ (and $|A| \mid |G|$). This leads to $A \cong L_2(43)$. Then $|G : A| = 2^{19}.3^7.5$ divides $|B|$ (and $|B| \mid |G|$). However, there is no such simple group B.

If $G = U_7(3)$, there is no simple group A with $547 \mid |A|$ and $|A| \mid |G|$.

Let $G = O_8^-(3)$. Assuming that $41 \mid |A|$, we have $A \cong L_2(41)$ or $L_2(81)$. Then $|B|$ is divisible by $2^7.3^{11}.13$ or $2^6.3^8.7.13$, respectively. This yields $A \cong L_2(81)$, $B \cong S_6(3)$ or $O_7(3)$, and $|A \cap B| = 120$. But $L_2(81)$ has no subgroup of order 120.

Now we treat the remaining groups $U_6(2)$, $U_6(3)$, and $O_7(3)$.

$G = U_6(2)$ (order $2^{15}.3^6.5.7.11$). We use the character table and the maximal subgroup list of G [2]. The proper simple subgroups of G are A_5, A_6, A_7, A_8, $L_2(7)$, $L_2(8)$, $L_2(11)$, $L_3(4)$, $U_3(3)$, $U_4(2)$, $U_4(3)$, $U_5(2)$, $S_6(2)$, M_{22}. This leads to the following possibilities: $A \cong S_6(2)$, $B \cong M_{22}$ and $A \cong U_5(2)$, $B \cong A_8$, $U_3(3)$, $L_3(4)$, $U_4(3)$, $S_6(2)$, or M_{22}. In the first case $|A \cap B| = 70$, hence $A \cap B$ has an element of order 35 which is impossible in M_{22} (as well as in $S_6(2)$).

Thus $A \cong U_5(2)$. If $B \cong A_8$ or $L_3(4)$, then $|A \cap B| = 30$, hence $A \cap B$ has an element of order 15. This contradicts the structure of $L_3(4)$, so $B \cong A_8$. Now $A \cap B$ contains an element t from the class $(3A)$ of G, as the remaining elements of order 3 of G do not commute with elements of order 5. An inspection of the centralizers of those elements of order 3 in A and B which commute with elements of order 5 implies $C_A(t) = C_G(t)$, $|C_B(t)| = 180$. Hence $|A \cap B| \geqq 180$, a contradiction.

Let $B \cong U_3(3)$. Then $|A \cap B| = 9$. Any involution in $U_3(3)$ is a square (of an element of order 4) and the only involutions of G with this property are those in the classes $(2A)$ and $(2B)$. Thus B contains an element from $(2A)$ or $(2B)$. On the other hand, G has a single class of $U_5(2)$ subgroups and the permutation character of G on the cosets of such a subgroup implies that it contains involutions from both classes $(2A)$ and $(2B)$. It follows that $2 \mid |A \cap B|$, a contradiction.

Thus $A \cong U_5(2)$, $B \cong S_6(2)$, $U_4(3)$, or M_{22} and we reach 4) of the theorem.

The existence of the first factorization in 4) is known [4]. Next, take subgroups $A \cong U_5(2)$ and $B \cong M_{22}$ of G such that $11 \mid |A \cap B|$. In M_{22}, every proper subgroup of order divisible by 11 is contained in a (maximal) subgroup isomorphic to $L_2(11)$. Hence $|A \cap B| \leqq |L_2(11)|$ which produces $|AB| \geqq |G|$, i. e. $G = AB$ thus proving the third factorization in 4).

Lastly, let $A \cong U_5(2)$, $B \cong U_4(3)$ be subgroups of G and $C = A \cap B$. We can assume that $3^5 \mid |C|$, as B contains a Sylow 3-subgroup of G. Further, $|C| \geqq |A||B|/|G| = 2^2.3^5.5$ and $|C|$ divides $(|A|, |B|) = 2^7.3^5.5$. Now the subgroup structure of $U_4(3)$ implies that C is contained in a (maximal) subgroup D of B which is isomorphic to a split extension of E_{81} by A_6. Then $|D : C| = 3$ or 6. As D obviously has no subgroup of index 3, it follows that $|C| = 2^2.3^5.5$, whence $G = AB$ and the second factorization in 4) is proved.

$G = U_6(3)$ (order $2^{13}.3^{15}.5.7^2.13.61$). Checking the known simple groups A with $61 \mid |A|$ and $|A| \mid |G|$, we conclude that $A \cong U_5(3)$. Then $|G : A| \mid |B|$ and $|B| \mid |G|$ imply $B \cong L_3(9)$, $S_6(3)$, $G_2(3)$, or $O_7(3)$. However, G has no $G_2(3)$ subgroups and hence (as $O_7(3)$ contains $G_2(3)$) no $O_7(3)$ subgroups. Indeed, if $U_6(3)$ contains $G_2(3)$ then $G_2(3)$ must embed into $SU_6(9)$, as $G_2(3)$ has a Schur multiplier of odd order. But $G_2(3)$ has 2-rank three and a single class of involutions, i. e. it has an E_8 subgroup all of whose involutions are conjugate. This is impossible in $SL_6(9)$ by [4], Lemma 4.3. Thus $B \cong L_3(9)$ or $S_6(3)$. Suppose $B \cong L_3(9)$. Each of the groups $U_5(3)$ and $L_3(9)$ has $GL_2(9)$ subgroups, so has an $SL_2(9)$ subgroup centralized by some element of order 8. On the other hand, as shown below, G has a single conjugacy class of $SL_2(9)$ subgroups with this property. It follows that $A \cap B$ contains an $SL_2(9)$ subgroup which contradicts (by order considerations) the assumption $G = AB$. Thus $A \cong U_5(3)$, $B \cong S_6(3)$ and we reach 5) of the theorem; the factorization is known [4].

Now we proceed to prove the claim that G has exactly one conjugacy class of $SL_2(9)$ subgroups centralized by elements of order 8. We use the bar convention to denote homomorphic images of elements and subgroups of

$$SU_6(9) = \left\{ x \in GL_6(9) \mid x^{*'} x = E,\ \det x = 1 \right\}$$

in $G = SU_6(9)/\langle -E \rangle$; here, E is the identity matrix and $x^{*'}$ denotes the transpose of the matrix x^* whose entries are the cubes of the entries of a matrix x. Let ω be a generator of the multiplicative group of $GF(9)$, $\omega^2 = \omega + 1$, and

$$I = \begin{pmatrix} & & & \omega^2 \\ & & \omega^2 & \\ & -\omega^2 & & \\ -\omega^2 & & & \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -\omega^{-1} & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 1 & 0 & 0 & \omega \\ 0 & 1 & \omega^{-1} & 0 \end{pmatrix}.$$

If $X, Y \in GL_4(9)$ and $X = T^{-1}YT$ then $X^{*^t}X = E$ if and only if $Y^{*^t}IY = Y$. Let

$$z = \begin{pmatrix} T^{-1}\begin{pmatrix} U & \\ & U^* \end{pmatrix} T & \\ & E \end{pmatrix}, \quad \text{where } U = \begin{pmatrix} 0 & \omega \\ -\omega^{-1} & \omega^{-1} \end{pmatrix}.$$

Then $\overline{z}$ is a representative of the single conjugacy class of elements of order 5 in G. Furthermore, $C_G(\overline{z}) = \overline{C}$, where

$$C = \left\{ \begin{pmatrix} T^{-1}\begin{pmatrix} UV & \\ & -\omega U^*V^* \end{pmatrix}^k T & \\ & S \end{pmatrix} \;\middle|\; 1 \leqq k \leqq 80,\, S \in GL_2(9), \right.$$

$$\left. S^{*^t}S = E,\ \det S = \omega^{2k} \right\},$$

$V = \begin{pmatrix} 1 & \omega^2 \\ -1 & -1 \end{pmatrix}$. All the elements $\overline{y}$ of order 8 in $C_G(\overline{z})$ are given by

$$y = \begin{pmatrix} W^l & \\ & S \end{pmatrix}, \quad W = T^{-1}\begin{pmatrix} -\omega & & & \\ & -\omega & & \\ & & \omega & \\ & & & \omega \end{pmatrix} T, \qquad l = 1,\ 3,\ 5,\ \text{or } 7,$$

$$S = \begin{pmatrix} \pm 1 & \\ & \mp 1 \end{pmatrix}, \quad \begin{pmatrix} \pm\omega^2 & \\ & \pm\omega^2 \end{pmatrix}, \quad \begin{pmatrix} & \pm 1 \\ \pm 1 & \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} & \pm\omega^2 \\ \mp\omega^2 & \end{pmatrix}.$$

An appropriate conjugation in $C_G(\overline{z})$ implies that every cyclic subgroup of order 8 of $C_G(\overline{z})$ is conjugate to $\langle\overline{y}_1\rangle$ or $\langle\overline{y}_2\rangle$, where

$$y_1 = \begin{pmatrix} W & & \\ & \pm 1 & \\ & & \mp 1 \end{pmatrix}, \quad y_2 = \begin{pmatrix} W & & \\ & \pm\omega^2 & \\ & & \pm\omega^2 \end{pmatrix}.$$

Next, $C_G(\overline{y}_1) = \overline{C}_1$ and $C_G(\overline{y}_2) = \overline{C}_2$, where

$$C_1 = \left\{ \begin{pmatrix} T^{-1}\begin{pmatrix} A & \\ & \Delta^{-3}A^* \end{pmatrix} T & & \\ & a & \\ & & a^{-1}\Delta^2 \end{pmatrix}, \begin{pmatrix} T^{-1}\begin{pmatrix} & A \\ \Delta^{-3}A^* & \end{pmatrix} T & & \\ & & a \\ & -a^{-1}\Delta^2 & \end{pmatrix} \;\middle|\; \right.$$

$$\left. A \in GL_2(9), \Delta = \det A,\ a \in GF(9),\ a^4 = 1 \right\},$$

$$C_2 = \left\{ \begin{pmatrix} T^{-1}\begin{pmatrix} A & \\ & \Delta^{-3}A^* \end{pmatrix} T & \\ & B \end{pmatrix} \;\middle|\; A, B \in GL_2(9), \right.$$

$$\left. \Delta = \det A,\ B^{*^t} B = E,\ \det B = \Delta^2 \right\}.$$

Hence $|C_G(\overline{y}_1)| = 2^9.3^2.5$, $|C_G(\overline{y}_2)| = 2^9.3^3.5$. Let

$$L = \left\{ \begin{pmatrix} T^{-1}\begin{pmatrix} A & \\ & A^* \end{pmatrix} T & \\ & E \end{pmatrix} \;\middle|\; A \in SL_2(9) \right\}.$$

Then $\overline{L} \cong L \cong SL_2(9)$ and $\overline{L}$ is a normal subgroup of both $C_G(\overline{y}_1)$, $C_G(\overline{y}_2)$. If $\overline{L}_i$ is another $SL_2(9)$ subgroup in $C_G(\overline{y}_i)$ then $\overline{L} \cap \overline{L}_i$ is a proper normal subgroup of $\overline{L}_i$ of order at least $2^3.3^{2-i}$ $(i = 1, 2)$ which is, of course, impossible. Thus $\overline{L}$ is the unique $SL_2(9)$ subgroup in each of $C_G(\overline{y}_1)$ and $C_G(\overline{y}_2)$.

Now let $H \cong SL_2(9)$ be an arbitrary subgroup of G centralized by some element $\overline{y}$ of order 8. We can assume, up to conjugacy, that $\overline{z} \in H$ whence $\overline{y} \in C_G(\overline{z})$. Then $\langle \overline{y} \rangle$ is conjugate to $\langle \overline{y}_1 \rangle$ or $\langle \overline{y}_2 \rangle$, so H is conjugate to an $SL_2(9)$ subgroup of $C_G(\overline{y}_1)$ or $C_G(\overline{y}_2)$, i. e. H is conjugate to the subgroup $\overline{L}$. This proves the claim.

$G = O_7(3)$ (order $2^9.3^9.5.7.13$). We use the information in [2]. The proper simple subgroups of G are A_5, A_6, A_7, A_8, A_9, $L_2(7)$, $L_2(8)$, $L_2(13)$, $L_3(3)$, $L_4(3)$, $U_3(3)$, $S_4(3)$, $G_2(3)$, and $S_6(2)$. This leads exactly to the possibilities listed in 6) of the theorem.

Now we proceed to prove the existence of these seven factorizations.

Let $A \cong L_4(3)$, $B \cong G_2(3)$ be subgroups of G. We can assume that $13 \mid |A \cap B|$. Now $A \cap B$ is a proper subgroup of both A, B of order at least $2^4.3^3.13$. The subgroup structure of $L_4(3)$ implies that $A \cap B$ is contained in a subgroup of A isomorphic to a split extension of E_{27} by $L_3(3)$ while the structure of $G_2(3)$ implies that $A \cap B$ is contained in a subgroup of B isomorphic to a split extension of $L_3(3)$ by C_2. It follows that $A \cap B \cong L_3(3)$. This produces $G = AB$.

Furthermore, it is known that $L_4(3) = L_3(3)S_4(3)$ (see [4]). As $L_4(3)$ has a single conjugacy class of $L_3(3)$ subgroups, it follows that there is a subgroup $C \cong S_4(3)$ of A such that $A = (A \cap B)C$. Then $|(A \cap B) \cap C| = 24$, i. e. $|B \cap C| = 24$ which produces $G = BC$.

Similarly, it is known that $G_2(3) = L_3(3)U_3(3)$ (see [4]). As $G_2(3)$ has two conjugacy classes of $L_3(3)$ subgroups which are interchanged by an outer automorphism of $G_2(3)$, it follows that there is a subgroup $D \cong U_3(3)$ of B such that $B = (A \cap B)D$. This yields $|A \cap D| = 8$ whence $G = AD$.

Next, let $F \cong S_6(2)$ be a subgroup of G. The permutation character of G on the cosets of F shows that the three conjugacy classes of elements of order 3 of F are contained in three distinct conjugacy classes of G. Hence any two elements of order 3 of F which are conjugate in G are conjugate also in F.

Now $|A \cap F|$ divides $(|A|, |F|) = 2^7.3^4.5$. Suppose that $9 \mid |A \cap F|$. The relevant permutation characters imply that the only common nonidentity 3-elements of A

and F are from the class $(3B)$ of G. Then $A \cap F$ contains an E_9 subgroup all of whose nonidentity elements are from $(3B)$ and hence (by the above paragraph) are conjugate in F. However, $S_6(2)$ has no such E_9 subgroup (by the irreducible character of degree 7 of $S_6(2)$). Thus $9 \nmid |A \cap F|$ whence $|A \cap F| \mid 2^7.3.5$. This yields $G = AF$.

Similarly, we can choose subgroups $B \cong G_2(3)$ and $F \cong S_6(2)$ of G such that the only common nonidentity 3-elements of B and F are from the class $(3F)$ of G. Then (just as above) $9 \nmid |B \cap F|$. As $|B \cap F| \mid 2^6.3^4.7$, it follows that $|B \cap F| \mid 2^6.3.7$. This yields $G = BF$.

The group G has two conjugacy classes of subgroups $H \cong A_9$. Using various arguments, it is not difficult to determine the class structure of H in G for elements of order 2 and 3 (notation for the classes of H is as in [2], p. 37):

Class in H	$(2A)'$	$(2B)'$	$(3A)'$	$(3B)'$	$(3C)'$
Class in G	$(2B)$	$(2C)$	$(3B)$	$(3D)$ or $(3E)$	$(3F)$.

Now let $A \cong L_4(3)$, $H \cong A_9$ be subgroups of G. Then $|A \cap H| \geqq 2^4.3.5$ and $|A \cap H| \mid 2^6.3^4.5$. The above paragraph and the permutation character of G on A imply that the only common elements of order 3 of A and H are from the class $(3B)$. Suppose that $9 \mid |A \cap H|$. As the elements of $(3B)$ are not cubes in G, it follows that $A \cap H$ contains an E_9 subgroup all of whose nonidentity elements are from $(3B)$. This is, however, impossible, as H has no E_9 subgroup with nonidentity elements from the class $(3A)'$ only. Thus $|A \cap H| \mid 2^6.3.5$. The subgroup structure of A_9 implies that $|A \cap H| = 2^4.3.5$ or $2^5.3.5$. In the latter case, $A \cap H$ is contained in a subgroup of H isomorphic to a split extension of $A_5 \times A_4$ by C_2 and then it is easy to see that $A \cap H$ must intersect this $A_5 \times A_4$ in an $A_5 \times E_4$ subgroup. However, this contradicts the structure of A as the elements of order 5 in $L_4(3)$ are not centralized by E_4 subgroups. Thus $|A \cap H| = 2^4.3.5$ whence $G = AH$.

Lastly, we can choose subgroups $B \cong G_2(3)$ and $H \cong A_9$ of G such that the only common elements of order 3 of B and H are from the class $(3F)$ of G. Then (as in the above paragraph) $9 \nmid |B \cap H|$, as H has no E_9 subgroup with nonidentity elements from the class $(3C)'$ only. Further, all the involutions of B are from the class $(2C)$ while H has no subgroup of order 16 containing involutions from the class $(2B)'$ only (by the irreducible character of A_9 of degree 8). Hence $16 \nmid |B \cap H|$. It follows that $|B \cap H| \mid 2^3.3.7$ which leads to $G = BH$.

This completes the proof of the theorem.

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