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A GENERALIZED JACOBI OPERATOR IN THE 4-DIMENSIONAL RIEMANNIAN GEOMETRY*

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Грозьо Станилов, Ирина Петрова. ОБОБЩЕННЫЙ ОПЕРАТОР ЯКОБИ В 4-МЕРНОЙ РИМАНОВОЙ ГЕОМЕТРИИ

Рассмотрим римановое многообразие $(M,\, g)$ размерности n с тензором кривизны R. В точке p берем 2–мерное касательное подпространство E^2 касательного пространство M_p, определенное ортонормальной парой векторов X, Y. Введем в рассмотрении линейный оператор $\lambda_{X,Y} : M_p \to M_p$ при помощи $\lambda_{X,Y}(u) = R(u,X,X) + R(u,Y,Y)$. Это симметрический оператор. Очень важно то, что этот оператор инвариантным относительно ортогональных преобразований в плоскости E^2. Это дает нам возможность определит оператор относительно любой 2–мерной плоскости E^2 в точке p: $\lambda_{E^2} = \lambda_{X,Y}$. В общем случае его собственные значения зависят от точки p и от плоскости E^2. Мы исследуем класс S–римановых многообразий размерности 4, для которых выполняется условие: собственные значения $c_i(p;E^2)$, $i = 1,\, 2,\, 3,\, 4$, всех операторов λ_{E^2} не зависят от плоскости E^2 в точке p.

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We consider a Riemannian manifold $(M,\, g)$ of dimension n with curvature tensor R. At a point $p \in M$ we take a 2–dimensional tangent plane E^2 of the tangent space M_p, spanned by an orthonormal pair of vectors X, Y. We define the linear operator $\lambda_{X,Y} : M_p \to M_p$ by $\lambda_{X,Y}(u) = R(u,X,X) + R(u,Y,Y)$. It is a symmetric operator and a very important fact is namely the assertion it is invariant operator under the orthogonal transformations in E^2. This gives us the possibility to define an operator in respect to any E^2 in M_p: $\lambda_{E^2} = \lambda_{X,Y}$. In

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the general case its eigen values depend of p and of E^2 also. Then we investigate the class of S–Riemannian manifolds of dimension 4 with the property the eigen values $c_i(p; E^2)$, $i = 1, 2, 3, 4$, of the operators λ_{E^2} are independent of E^2.

1. THE CLASS S–RIEMANNIAN MANIFOLDS

Let (M, g) be a Riemannian manifold of dimension n, R — its curvature tensor. For $p \in M$ let X, Y be an orthonormal pair of vectors in M_p. We define a linear operator

$$\lambda_{X,Y} : M_p \to M_p$$

by

$$\lambda_{X,Y}(x) = R(x, X, X) + R(x, Y, Y).$$

It is a symmetrical operator. The following important proposition holds: the operator $\lambda_{X,Y}$ is invariant under the orthogonal transformations in the tangent subspace $E^2 = E^2(p; X \wedge Y)$ spanned by X, Y.

Indeed, if we have another orthonormal pair of vectors $\overline{X}$, $\overline{Y} \in E^2$, then $\lambda_{X,Y} = \lambda_{\overline{X},\overline{Y}}$.

We can define the operator

$$\lambda_{E^2} : M_p \to M_p$$

by

$$\lambda_{E^2} = \lambda_{X,Y}.$$

We call it generalized Jacobi operator in respect to E^2.

Since λ_{E^2} is a symmetric operator, it has a real eigen values $c_i(p, E^2)$. Then we consider the class of 4–dimensional Riemannian manifolds with the following property:

(S) the operator λ_{E^2} has eigen values independent of E^2:
$$c_i(p, E^2) = c_i(p), \quad i = 1, 2, 3, 4.$$

Any 4–dimensional Riemannian manifold with the property (S) we call S–Riemannian manifold.

In this paper we investigate the class S–Riemannian manifolds.

Remarks. 1. We suppose $c_1(p) \leqq c_2(p) \leqq c_3(p) \leqq c_4(p)$.

2. If p is a fixed point we write c_i instead of $c_i(p)$.

Proposition. *Every 4–dimensional Riemannian manifold with constant sectional curvature is an S–Riemannian manifold.*

Proposition. *Every S–Riemannian manifold is an Einsteinnian manifold.*

P r o o f. Let X, Y is an orthonormal pair in M_p. It is easy to get

$$\sum_{i=1}^{4} c_i(p) = \rho(X, X) + \rho(Y, Y),$$

where ρ is the Ricci tensor. Hence for each $Z \in M_p$, $|Z| = 1$,

$$\rho(Z, Z) = \frac{1}{2} \sum_{i=1}^{4} c_i(p).$$

From the second proposition and the well-known Herglotzt theorem it follows that

$$\sum_{i=1}^{4} c_i(p) = \text{const}.$$

2. MATRICES OF THE GENERALIZED JACOBI OPERATOR AND A LEMMA FOR THE SPECIAL BASES IN 4-DIMENSIONAL MANIFOLD

Let $(M,\ g)$ be a 4-dimensional Riemannian manifold. We fix $p \in M$ and let $x_1,\ x_2,\ x_3,\ x_4$ be an orthonormal base of M_p, and X, Y be an orthonormal pair in M_p. For the operator $\lambda_{X,Y}$ and for a real c we define the matrix $C^{X,Y}(c)$ in the following way

$$C^{X,Y}(c) = (c_{ik})_{i,k=1,2,3,4}, \quad c_{ik} = g\left(\lambda_{X,Y}(x_i), x_k\right) - \delta_{ik}c.$$

It is a symmetric matrix since $\lambda_{X,Y}$ is a symmetric operator. If c is a k-multiple eigen value of $\lambda_{X,Y}$, we have the relation

$$\operatorname{rang} C^{X,Y}(c) + k = 4.$$

Next we write $C^{i,j}(c)$ and $C^{i,j+k}(c)$ instead of $C^{x_i,x_j}(c)$ and $C^{x_i,\frac{x_j+x_k}{\sqrt{2}}}(c)$, respectively.

Lemma. *Let $(M,\ g)$ be a 4-dimensional manifold. For each $p \in M$ there exists an orthonormal base $x_1,\ x_2,\ x_3,\ x_4$ of M_p, so that*

$$R(x_2, x_1, x_1, x_3) = 0, \quad R(x_2, x_1, x_1, x_4) = 0, \quad R(x_3, x_1, x_1, x_4) = 0.$$

The first vector x_1, $|x_1| = 1$, can be chosen arbitrarily.

Proof. We take $x_1 \in M_p$ with $|x_1| = 1$ and complete x_1 to $x_1,\ x_2,\ x_3,\ x_4$ — an orthonormal base of eigen vectors of the symmetrical Jacobi operator λ_{x_1}, defined by $\lambda_{x_1}(x) = R(x, x_1, x_1)$.

The bases in the Lemma will be called special bases.

3. SOME NOTATIONS FOR S-RIEMANNIAN MANIFOLDS

Let $(M,\ g)$ be an S-Riemannian manifold, $p \in M$, $x_1,\ x_2,\ x_3,\ x_4$ be an orthonormal base of M_p. We set

$$R_{ijst} = R(x_i, x_j, x_s, x_t), \quad r_{ij} = R_{istj} + R_{itsj},$$
$$l^s_{ij} = R(x_i, x_s, x_s, x_j), \quad k_{ij} = R(x_i, x_j, x_j, x_i),$$

for all integers i, j, s, t from 1 to 4, different mutually. We have the following

Lemma. a) $k_{ij} = k_{st}$;

b) $l^s_{ij} + l^t_{ij} = 0$.

Proof. a) It follows from the fact that $(M,\ g)$ is an Einsteinnian manifold [1].

b) Since $(M,\ g)$ is an Einsteinnian manifold, the Ricci tensor is proportional to the metric tensor. Hence

$$0 = \rho(x_i, x_j) = R(x_i, x_s, x_s, x_j) + R(x_i, x_t, x_t, x_j) = l_{ij}^s + l_{ij}^t.$$

If $x_1,\ x_2,\ x_3,\ x_4$ is a special base of M_p, by lemma in section 2 we obtain

$$l_{23}^1 = l_{24}^1 = l_{34}^1 = 0.$$

For such a base the substantial components of R are $R_{1234},\ R_{1342},\ l_{12}^3,\ l_{13}^2,\ l_{14}^2,\ k_{12},\ k_{13},\ k_{14}$. In these notations and for such base we write the matrix $C^{1,2}(c)$. From section 2 we know

$$C^{1,2} = (c_{st})_{s,t=1,2,3,4},$$

$$c_{st} = g\left(\lambda_{x_1,x_2}(x_s), x_t\right) - \delta_{st}c,$$

$$c_{st} = R(x_s, x_1, x_1, x_t) + R(x_s, x_2, x_2, x_t) - \delta_{st}c.$$

Hence,

$$c_{11} = R(x_1, x_1, x_1, x_1) + R(x_1, x_2, x_2, x_1) - \delta_{11}c = k_{12} - c,$$

$$c_{12} = R(x_1, x_1, x_1, x_2) + R(x_1, x_2, x_2, x_2) - \delta_{12}c = 0,$$

$$c_{13} = R(x_1, x_1, x_1, x_3) + R(x_1, x_2, x_2, x_3) - \delta_{13}c = l_{13}^2,$$

$$c_{14} = R(x_1, x_1, x_1, x_4) + R(x_1, x_2, x_2, x_4) - \delta_{14}c = l_{14}^2,$$

$$c_{22} = R(x_2, x_1, x_1, x_2) + R(x_2, x_2, x_2, x_2) - \delta_{22}c = k_{12} - c,$$

$$c_{23} = R(x_2, x_1, x_1, x_3) + R(x_2, x_2, x_2, x_3) - \delta_{23}c = l_{23}^1 = 0,$$

$$c_{24} = R(x_2, x_1, x_1, x_4) + R(x_2, x_2, x_2, x_4) - \delta_{24}c = l_{24}^1 = 0,$$

$$c_{33} = R(x_3, x_1, x_1, x_3) + R(x_3, x_2, x_2, x_3) - \delta_{33}c = k_{13} + k_{23} - c = k_{13} + k_{14} - c,$$

$$c_{34} = R(x_3, x_1, x_1, x_4) + R(x_3, x_2, x_2, x_4) - \delta_{34}c = l_{34}^1 + l_{34}^2 = 0,$$

$$c_{44} = R(x_4, x_1, x_1, x_4) + R(x_4, x_2, x_2, x_4) - \delta_{44}c = k_{14} + k_{24} - c = k_{14} + k_{13} - c.$$

Then $C^{1,2}(c)$ is a symmetrical matrix of the form

$$C^{1,2}(c) = \begin{pmatrix} k_{12} - c & 0 & l_{13}^2 & l_{14}^2 \\ 0 & k_{12} - c & 0 & 0 \\ l_{13}^2 & 0 & k_{13} + k_{14} - c & 0 \\ l_{14}^2 & 0 & 0 & k_{13} + k_{14} - c \end{pmatrix}.$$

In the same way we find the matrices

$$C^{1,3}(c) = \begin{pmatrix} k_{13} - c & l_{12}^3 & 0 & -l_{14}^2 \\ l_{12}^3 & k_{12} + k_{14} - c & 0 & 0 \\ 0 & 0 & k_{13} - c & 0 \\ -l_{14}^2 & 0 & 0 & k_{12} + k_{14} - c \end{pmatrix},$$

$$C^{1,4}(c) = \begin{pmatrix} k_{14} - c & -l_{12}^3 & -l_{13}^2 & 0 \\ -l_{12}^3 & k_{12} + k_{13} - c & 0 & 0 \\ -l_{13}^2 & 0 & k_{12} + k_{13} - c & 0 \\ 0 & 0 & 0 & k_{14} - c \end{pmatrix},$$

$$C^{1,2+3}(c) = \frac{1}{2}\begin{pmatrix} k_{12}+k_{13}-2c & l_{12}^3-l_{13}^2 & -\left(l_{12}^3-l_{13}^2\right) & r_{14} \\ l_{12}^3-l_{13}^2 & 2k_{12}+k_{14}-2c & -k_{14} & 0 \\ -\left(l_{12}^3-l_{13}^2\right) & -k_{14} & 2k_{13}+k_{14}-2c & 0 \\ r_{14} & 0 & 0 & k_{12}+k_{13}+2k_{14}-2c \end{pmatrix},$$

$$C^{1,2+4}(c) = \frac{1}{2}\begin{pmatrix} k_{12}+k_{14}-2c & -\left(l_{12}^3+l_{14}^2\right) & r_{13} & l_{12}^3+l_{14}^2 \\ -\left(l_{12}^3+l_{14}^2\right) & 2k_{12}+k_{13}-2c & 0 & -k_{13} \\ r_{13} & 0 & k_{12}+2k_{13}+k_{14}-2c & 0 \\ l_{12}^3+l_{14}^2 & -k_{13} & 0 & k_{13}+2k_{14}-2c \end{pmatrix},$$

$$C^{1,3+4}(c) = \frac{1}{2}\begin{pmatrix} k_{13}+k_{14}-2c & r_{12} & -\left(l_{13}^2-l_{14}^2\right) & l_{13}^2-l_{14}^2 \\ r_{12} & 2k_{12}+k_{13}+k_{14}-2c & 0 & 0 \\ -\left(l_{13}^2-l_{14}^2\right) & 0 & k_{12}+2k_{13}-2c & -k_{12} \\ l_{13}^2-l_{14}^2 & 0 & -k_{12} & k_{12}+2k_{14}-2c \end{pmatrix}.$$

4. THEOREM FOR THE LOCAL POSSIBILITIES. A CONSEQUENCE

Theorem. *Let $(M,\ g)$ be an S-Riemannian manifold. Then for every $p \in M$ one of the following possibilities holds:*

a) $0 = c_1 = c_2 = c_3 \leqq c_4$ *or* $c_1 \leqq c_2 = c_3 = c_4 = 0$;

b) $c_1 = c_2 < c_3 = c_4$.

If x_1, x_2, x_3, x_4 is an orthonormal base of M_p, so that

$$l_{23}^1 = l_{24}^1 = l_{34}^1 = 0,$$

then a) *implies* $k_{12} = k_{13} = k_{14} = 0$, *and* b) *implies* $k_{12} + k_{13} + k_{14} = c_1 + c_3$, $(k_{12} - c_1)(k_{12} - c_3) = 0$, $(k_{13} - c_1)(k_{13} - c_3) = 0$, $(k_{14} - c_1)(k_{14} - c_3) = 0$.

Proof. For c_i $(i = 1,\ 2,3,4)$ we have the following logical possibilities:

P_1. There are at least three equal among them;

P_2. They are two by two equal;

P_3. There are at least three different among them.

We prove that P_1 implies a), P_2 implies b) and P_3 is impossible.

4.1. The case P_1. We have

$$c_1 = c_2 = c_3 \leqq c_4 \quad \text{or} \quad c_1 \leqq c_2 = c_3 = c_4.$$

Hence c_2 is at least 3–multiple eigen value. Let x_1, x_2, x_3, x_4 be an orthonormal base of M_p, so that $l_{23}^1 = l_{24}^1 = l_{34}^1 = 0$. Then rang $C^{1,i}(c_2) \leqq 1$ $(i = 2,\ 3,4)$. All minors of order 2 of these matrices are 0 and then $k_{12} = k_{13} = k_{14} = c_2 = 0$.

4.2. The case P_2. We have $c_1 = c_2 < c_3 = c_4$. Let x_1, x_2, x_3, x_4 be an orthonormal base of M_p, so that $l^1_{23} = l^1_{24} = l^1_{34} = 0$. We have $\operatorname{rang} C^{1,2}(c) = 2$ for $c \in \{c_1, c_3\}$. It follows that $(k_{12} - c)(k_{13} + k_{14} - c) = 0, \quad c \in \{c_1, c_3\}$. Then

$$(k_{12} - c_1)(k_{13} + k_{14} - c_1) = (k_{12} - c_3)(k_{13} + k_{14} - c_3).$$

The developing of this identity gives $k_{12} + k_{13} + k_{14} = c_1 + c_3$. Then follows

$$(k_{12} - c_1)(k_{12} - c_3) = 0.$$

Considering the matrices $C^{1,3}(c)$ and $C^{1,4}(c)$ we get the last two relations in the theorem.

4.3. The case P_3. Let $\Omega = \{c_1, c_2, c_3, c_4\}$. Because of P_3 Ω has at least 3 different elements. Let again x_1, x_2, x_3, x_4 be an orthonormal base of M_p, so that $l^1_{23} = l^1_{24} = l^1_{34} = 0$. We set $\lambda_1 = l^3_{12}$, $\lambda_2 = l^2_{13}$, $\lambda_3 = l^2_{14}$. Since $\det C^{1,3+4}(c) = 0$ $(c \in \Omega)$, we have

$$f(c) - (\lambda_2 - \lambda_3)^2 g(c) = 0 \quad (c \in \Omega),$$

where

$$g(c) = (k_{13} + k_{14} - 2c)(2k_{12} + k_{13} + k_{14} - 2c),$$
$$f(c) = (2(k_{13} - c)(k_{14} - c) + k_{12}(k_{13} + k_{14} - 2c))\left(g(c) - r^2_{12}\right).$$

Taking the base x_1, x_2, x_3, x_4 of M_p we get

$$f(c) - (\lambda_2 + \lambda_3)^2 g(c) = 0 \quad (c \in \Omega).$$

Then

$$4\lambda_2\lambda_3 g(c) = 0 \quad (c \in \Omega).$$

The polynomial $g(c)$ is of degree 2. It has no more than two different roots. But Ω has at least three different elements. Hence $\lambda_2\lambda_3 = 0$. $C^{1,2+3}(c)$ and $C^{1,2+4}(c)$ provide analogously $\lambda_1\lambda_2 = 0$, $\lambda_1\lambda_3 = 0$. Hence at least two among the $\lambda_1, \lambda_2, \lambda_3$ are 0. Supposing that $\lambda_2 = \lambda_3 = 0$, from the equality $\det C^{1,2}(c) = 0$ for $c \in \Omega$ we get

$$(k_{12} - c)(k_{13} + k_{14} - c) = 0, c \in \Omega.$$

It is impossible because Ω has at least three different elements.

4.4. A consequence. Let M_p be S–Riemannian manifold, $p \in M$, so that $c_1 = c_2 < c_3 = c_4$. Let x_1, x_2, x_3, x_4 be an orthonormal base of M_p with $l^1_{23} = l^1_{24} = l^1_{34} = 0$. Then

1) If $c_1 = 0$ then $\{k_{12}, k_{13}, k_{14}\} = \{0, 0, c_3\}$;

2) If $c_3 = 0$ then $\{k_{12}, k_{13}, k_{14}\} = \{0, 0, c_1\}$;

3) If $c_1 \neq 0$, $c_3 \neq 0$ then $|c_1| \neq |c_3|$ and $k_{12} = k_{13} = k_{14} = \varkappa$, where

$$\varkappa = \begin{cases} c_1, & \text{if } |c_1| < |c_3|, \\ c_3, & \text{if } |c_3| < |c_1|. \end{cases}$$

Remark. $\{k_{12}, k_{13}, k_{14}\} = \{0, 0, c_3\}$ means that among the numbers k_{12}, k_{13}, k_{14} two of them are 0 and the third is c_3.

The proof uses only the curvature relations in the theorem in this section.

5. THE CLASS F–RIEMANNIAN MANIFOLDS

Let $(M,\ g)$ be a 4–dimensional Riemannian manifold, $p \in M$, $X \in M_p$, $|X| = 1$. The well-known Jacobi operator is defined in the following way:

$$\lambda_X(x) = R(x, X, X).$$

It is a symmetrical operator and its eigen values are real numbers depending of p and X. We consider the class of the 4–dimensional Riemannian manifolds with the property

(F) For each $p \in M$ and for every (unit) $X \in M_p$ the Jacobi operator λ_X has eigen values $d_i(p, X)$ independent of X, i. e. $d_i(p, X) = d_i(p)$, $i = 1, 2, 3, 4$.

Any 4–dimensional Riemannian manifold with this property will be called F–Riemannian manifold. This class is investigated in [2], [3], [4]. It is well-known that every 4–dimensional Riemannian manifold of constant sectional curvature is F–Riemannian manifold and every F–Riemannian manifold is Einsteinnian manifold. For such a manifold we have the relations $k_{ij} = k_{st}$, $l^s_{ij} + l^t_{ij} = 0$.

To every operator λ_X we can associate the matrices $D^X(d)$ in a way like that in section 2:

$$D^X(d) = (d_{st})_{s,t=1,2,3,4}, \quad d_{st} = g\left(\lambda_X(x_s), x_t\right) - \delta_{st} d.$$

If d is k–multiple eigen value of λ_X, then rang $D^X(d) + k = 4$. Instead of $D^{x_i}(d)$ and $D^{\frac{x_i + x_j}{\sqrt{2}}}(d)$ we shall write $D^i(d)$ and $D^{i+j}(d)$ respectively.

6. COMPARISON OF BOTH CLASSES

Theorem. *If $(M,\ g)$ is an S–Riemannian manifold, then it is also an F–Riemannian manifold. The converse is not true.*

Proof. 1) Let $(M,\ g)$ be an S–Riemannian manifold, $p \in M$. We take $X \in M_p$ with $|X| = 1$. Let $d_i(p)$, $i = 1, 2, 3, 4$, are the eigen values of λ_X. We set

$$\Psi(p, X) = \left\{d_i(p, X) \,|\, i = 1,\ 2,\ 3,\ 4\right\}.$$

Then we consider the orthonormal base $X = x_1$, x_2, x_3, x_4 of eigen vectors of λ_X in M_p. Hence $l^1_{23} = l^1_{24} = l^1_{34} = 0$. Let $d_i(p, X)$ corresponds to x_i, $i = 1, 2, 3, 4$. Then

$$d_1(p, X) = 0, \quad d_2(p, X) = k_{12}, \quad d_3(p, X) = k_{13}, \quad d_4(p, X) = k_{14}.$$

Hence

$$\Psi(p, X) = \{0, k_{12}, k_{13}, k_{14}\}.$$

From the theorem for the local possibilities in p one of the following cases holds:

a) $0 = c_1 = c_2 = c_3 \leqq c_4$ or $c_1 \leqq c_2 = c_3 = c_4 = 0$;

b) $0 = c_1 = c_2 < c_3 = c_4$;

c) $c_1 = c_2 < c_3 = c_4 = 0$;

d) $0 \neq c_1 = c_2 < c_3 = c_4 \neq 0$.

Using this theorem and the consequence of it we obtain for the cases a), b), c) and d) correspondingly:

a′) $\Psi(p, X) = \{0, 0, 0, 0\}$,

b′) $\Psi(p, X) = \{0, 0, 0, c_3\}$,

c′) $\Psi(p, X) = \{0, 0, 0, c_1\}$,

d′) $\Psi(p, X) = \{0, \varkappa, \varkappa, \varkappa\}$,

where $\varkappa$ is the smaller by module number between c_1 and c_3. It means that the elements of $\Psi(p, X)$ are independent of X and hence $(M,\ g)$ is an F–Riemannian manifold.

2) The 4–dimensional Kaehlerian manifolds with constant non-zero holomorphic sectional curvature are F–Riemannian manifolds, but they are not S–Riemannian manifolds.

7. LEMMA FOR THE SPECIAL BASES IN AN S–RIEMANNIAN MANIFOLD

From the theorem in section 6 we get the following

Consequence. *If $(M,\ g)$ is an S–Riemannian manifold and in $p \in M$ holds:*

a) $0 = c_1 = c_2 = c_3 \leqq c_4$ *or* $c_1 \leqq c_2 = c_3 = c_4 = 0$,

then every λ_X has 4–multiple eigen value 0;

b) $0 = c_1 = c_2 < c_3 = c_4$ *or* $c_1 = c_2 \leqq c_3 = c_4 = 0$,

then every λ_X has 3–multiple eigen value 0.

Using it, we can prove the following

Lemma. *Let $(M,\ g)$ be an S–Riemannian manifold, $p \in M$, and in p holds* a) *or* b) *of the above consequence.*

Then there is an orthonormal base of M_p, so that $l^s_{ij} = 0$ for all integers i, j, s from 1 to 4, unequal mutually. The first vector x_1 can be chosen arbitrarily.

P r o o f. Let x_1, x_2, x_3, x_4 be an orthonormal base so that $l^1_{23} = l^1_{24} = l^1_{34} = 0$. If in p a) holds, we have

$$\operatorname{rang} D^2(0) = \operatorname{rang} D^3(0) = 0,$$

and if in p b) holds, we have

$$\operatorname{rang} D^2(0) = \operatorname{rang} D^3(0) = 1.$$

In both cases we obtain by considering suitable minors that $l^3_{12} = l^2_{13} = l^2_{14} = 0$.

8. A MORE PRECISE VARIANT OF THE THEOREM FOR THE LOCAL POSSIBILITIES

Theorem. *Let $(M,\ g)$ be an S–Riemannian manifold, $p \in M$. For p holds one of the following possibilities:*

a) $c_1 = c_2 = c_3 = c_4 = 0$.

In such case the sectional curvature of M in p is 0;

b) $0 = c_1 = c_2 < c_3 = c_4$.

In this case there is an orthonormal base x_1, x_2, x_3, x_4 of M_p, so that the only substantial components of R are

$$R_{1234} = R_{1342} = \frac{c_3}{3}, \quad k_{14} = c_3;$$

c) $c_1 = c_2 < c_3 = c_4 = 0$.
In this case there is an orthonormal base x_1, x_2, x_3, x_4 of M_p, so that the substantial components of R are

$$R_{1234} = R_{1342} = \frac{c_1}{3}, \quad k_{14} = c_1;$$

d) $0 < c_1 = c_2 < c_3 = c_4, \quad c_3 = 2c_1$.
In this case the sectional curvature of M in p is c_1;
e) $c_1 = c_2 < c_3 = c_4 < 0, \quad c_1 = 2c_3$.
In this case the sectional curvature of M in p is c_3.

Proof. In p holds one of the following possibilities:
a) $0 = c_1 = c_2 = c_3 \leqq c_4$ or $c_1 \leqq c_2 = c_3 = c_4 = 0$,
b) $0 = c_1 = c_2 < c_3 = c_4$,
c) $c_1 = c_2 < c_3 = c_4 = 0$,
d) $0 \neq c_1 = c_2 < c_3 = c_4 \neq 0$.

1) Let in p holds a). We shall prove that if three of the numbers c_1, c_2, c_3, c_4 are 0, then the forth is also 0. Let suppose $0 = c_1 = c_2 = c_3 \leqq c_4$. By the lemma in section 7 we take an orthonormal base x_1, x_2, x_3, x_4 of M_p such that $l^s_{ij} = 0$ for i, j, s from 1 to 4, unequal mutually. From the theorem for the local possibilities we have $k_{12} = k_{13} = k_{14} = 0$. The eigen value c_4 is at least 1–multiple. Hence $\det C^{1,2}(c_4) \neq 0$ and then $c_4 = 0$. Then $c_1 = c_2 = c_3 = c_4 = 0$.

Let E^2 be any 2–dimensional tangent subspace in M_p and X, Y be an orthonormal base of M_p. We set $x_1 = X$ and complete x_1 to x_1, x_2, x_3, x_4 — an orthonormal base of M_p with the property $l^1_{23} = l^1_{24} = l^1_{34} = 0$. From the theorem for the local possibilities we have $k_{12} = k_{13} = k_{14} = 0$. Using $Y = \alpha^i x_i$, $\sum_{i=2}^{4}(\alpha^i)^2 = 1$, one can check easily that $K(E^2) = 0$.

2) Let in p holds b), i. e.

$$0 = c_1 = c_2 < c_3 = c_4.$$

We use again an orthonormal base x_1, x_2,x_3, x_4 of M_p, so that $l^s_{ij} = 0$ for all i, j, s from 1 to 4 different mutually. From the consequence in section 4 $\{k_{12}, k_{13}, k_{14}\} = \{0, 0, c_3\}$. If it is necessary we can renumber the vectors x_1, x_2, x_3, x_4 to get $k_{12} = k_{13} = 0$, $k_{14} = c_3$. From the consequence in section 7 we know that every operator λ_X has 3–multiple eigen value 0. Then $\operatorname{rang} D^{1+2}(0) = \operatorname{rang} D^{1+4}(0) = 1$. This implies

$$r_{12} = -c_3 \quad \text{or} \quad r_{12} = c_3, \quad r_{14} = 0.$$

If it is necessary we can change x_1 with $-x_1$ to get $r_{12} = c_3$. Using the first Bianki identity we obtain $R_{1234} = R_{1342} = \dfrac{c_3}{3}$.

3). Let in p holds d), i.e.

$$0 \neq c_1 = c_2 < c_3 = c_4 \neq 0.$$

We take the orthonormal base x_1, x_2, x_3, x_4 of M_p such that $l^1_{23} = l^1_{24} = l^1_{34} = 0$. From the consequence in section 4 $k_{12} = k_{13} = k_{14} = \varkappa$, where

$$\varkappa = \begin{cases} c_1, & \text{if } |c_1| < |c_3| \\ c_3, & \text{if } |c_3| < |c_1|. \end{cases}$$

In p we have $\varkappa = c_1$ or $\varkappa = c_3$.

Let in p holds $\varkappa = c_1$. Then $k_{12} = k_{13} = k_{14} = c_1$. From the theorem of the local possibilities we have $k_{12} + k_{13} + k_{14} = c_1 + c_3$. Hence $c_3 = 2c_1$. But $c_1 < c_3$. Hence $c_1 > 0$. Then $0 < c_1 = c_2 < c_3 = c_4$, $c_3 = 2c_1$. It is easy to check that for every E^2 in M_p

$$K(E^2) = c_1.$$

Consequence 1. *Let (M, g) be an S–Riemannian manifold. If one of $c_i(p)$, $i = 1, 2, 3, 4$, is a global constant the others are also global constants.*

Proof. We have $\sum_{i=1}^{4} c_i(p) = \text{const}$ (from the consequence in section 1) and $c_1(p) = c_2(p)$, $c_3(p) = c_4(p)$ (from the above theorem).

Consequence 2. *Let (M, g) be an S–Riemannian manifold and for every point $p \in M$ holds $c_2 = c_3$. Then M is flat.*

Proof. At every point $p \in M$ holds a) from the above theorem.

Consequence 3. *Let (M, g) be an S–Riemannian manifold and let for every $p \in M$ holds $c_i \neq 0$, $i = 1, 2, 3, 4$. Then M is non-flat with constant sectional curvature.*

Proof. In every point $p \in M$ holds e) or d) from the above theorem. Then we apply the well-known Schur's theorem.

Consequence 4. *Let (M, g) be an S–Riemannian manifold with sectional curvature $K(p, E^2)$ and let $K(p, E^2) \neq 0$ for every point p and for every E^2 in M_p. Then M is non-flat with constant sectional curvature.*

Proof. In every point $p \in M$ holds e) or d) from the above theorem because in the cases a), b), c) there are planes E^2 with sectional curvature 0.

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