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# SOME REMARKS ON THE STRICTLY POSITIVE MEASURES

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*Георги Александров.* НЕСКОЛЬКО ЗАМЕЧАНИЙ О СТРОГО ПОЗИТИВНЫХ МЕР

Показано, что если $K$ компакт Гротендика, тогда $K$ имеет строго позитивную меру тогда и только тогда, когда существует непрерывный линейный иньективный оператор $T: C(K) \to c_0(\Gamma)$.

*George Alexandrov.* SOME REMARKS ON THE STRICTLY POSITIVE MEASURES

It is shown that if $K$ is a Grothendieck compact space, then $K$ admits a strictly positive measure if and only if there exists a linear bounded one-to-one operator $T: C(K) \to c_0(\Gamma)$.

## 1. INTRODUCTION

A finite nonnegative regular Borel measure $\mu$ on the compact Hausdorff space $K$ is called *strictly positive* if $\mu(U) > 0$ for every nonempty open subset $U$ of $K$.

A compact Hausdorff space $K$ is called *Grothendieck-compact* if the space $C(K)$ is the Grothendieck space, i. e. the weak* and weak convergence of sequences coincide in $C(K)^*$.

A compact Hausdorff space $K$ is *extremally disconnected* if the closure of every open subset of $K$ is open-and-closed (for the remaining definitions see below).

Every extremally disconnected compact space is a Grothendieck compact space.

It is known that if a compact space $K$ has a strictly positive measure, then there exists a bounded linear one-to-one operator $T: C(K) \to c_0(\Gamma)$ ([2], p. 179). Therefore the space $C(K)$ admits an equivalent strictly convex norm ([3], p. 101).

The converse is not true. Argyros, Mercourakis and Negrepontis ([1], Theorem 1. 11) proved the existence of a bounded linear one-to-one operator $T: C(K) \to c_0(\Gamma)$ and so $C(K)$ has an equivalent strictly convex norm for the known example of Gaifman (see [2], Theorem 6. 23) of a compact Hausdorff space $K$ without a strictly positive measure.

However, on the class of extremally disconnected compact spaces we have the following

**Theorem** ([1]). *Let $K$ be an extremally disconnected compact space. Then $K$ admits a strictly positive measure if and only if there exists a linear bounded one-to-one operator $T: C(K) \to c_0(\Gamma)$.*

Naturally, the question arises: Is there a class of compact spaces, essentially wider than the class of extremally disconnected compact space, for which this theorem holds?

Here we give the positive answer to this question.

## 2. DEFINITIONS, NOTATIONS AND SOME PROPOSITIONS

If $(X, \|.\|)$ is a Banach space, $X^*$ denotes its *dual*; $X^* = \{x^* : X \to R : x^*$ is linear and continuous $\}$.

The *weak* topology* on $X^*$ is the topology induced on $X^*$ by $X$, i. e. $x_i^* \to x^*$ is weak* convergent in $X^*$ if $x_i^*(x) \to x^*(x)$ for all $x \in X$.

The *unit ball* of a Banach space $X$ is denoted by $B_1(X)$; thus $B_1(X) = \{x \in X : \|x\| \leqq 1\}$.

If $T: X \to Y$ is a linear bounded operator between Banach spaces, then $T^*: Y^* \to X^*$ is the *conjugate* operator of $T$ given by $T^*(y^*) = y^* \circ T$ for all $y^* \in Y^*$. Every conjugate operator $T^*$ is weak*–weak* continuous.

A norm $\|.\|$ of a Banach space $X$ is *strictly convex*, if for all $x, y \in X$ with $\|x\| = \|y\| = 1$ we have $\|(x + y)/2\| < 1$ whenever $x \neq y$.

The subset $A \subset X^*$ is *total* if the linear closure $\text{lin}(A)$ of $A$ is *a total subspace*, i. e. if $x \in X$ and we have $x^*(x) = 0$ for all $x^* \in \text{lin}(A)$, then $x = 0$.

Given a compact Hausdorff space $K$, $C(K)$ denotes the space of all real-valued continuous functions on $K$ with supremum norm.

Given a set $\Gamma$, $l_\infty(\Gamma)$ denotes the Banach space of all bounded functions $f: \Gamma \to \mathbb{R}$, with $\|f\| = \sup_{\gamma\in\Gamma} |f(\gamma)|$, and $c_0(\Gamma) = \{f \in l_\infty(\Gamma) :$ for all $\varepsilon > 0$ $\{\gamma \in \Gamma : |f(\gamma)| > \varepsilon\}$ is finite $\}$. Also, $l_1(\Gamma)$ denotes the Banach space of all functions $f: \Gamma \to \mathbb{R}$ such that $\sum_{\gamma\in\Gamma} |f(\gamma)| < \infty$ with the norm $\|f\| = \sum_{\gamma\in\Gamma} |f(\gamma)|$.

The *support* of a nonnegative regular Borel measure $\mu$ on the compact Hausdorff space $K$, denoted by $\text{supp}(\mu)$, is the set of all $x \in K$ for which $\mu(U) > 0$ for every open set $U$ containing $x$. The support of a measure is a closed subset of $K$. It is clear that if a nonnegative regular Borel measure $\mu$ is strictly positive, then $\text{supp}(\mu) = K$.

**Proposition** (Rosenthal [6]). *Let $K$ be a compact Hausdorff space. Then $K$ admits a strictly positive measure if and only if $C(K)^*$ contains a weakly compact total subset.*

A compact Hausdorff space $K$ is called Rosenthal-compact if $K$ is homeomorphic to a subspace of the space of functions of the first Baire class with the pointwise convergence for some complete separable metric space.

The Rosenthal-compacts are introduced by H. Rosenthal in connection with the characterization of the Banach spaces isomorphically containing $l_1(\mathbf{N})$ ($\mathbf{N}$ denote the set of positive integers) [7]. The class of these compacts extends in a natural way the class of metrizable compacts.

**Proposition** (Godefroy [4]). *Let $K$ be a Rosenthal-compact space and $\mu$ is a nonnegative regular Borel measure, then the* $\operatorname{supp}(\mu)$ *is a separable space.*

## 3. RESULTS

**Theorem 1.** *Let $K$ be a Grothendieck-compact. Then $K$ admits a strictly positive measure if and only if there exists a linear bounded one-to-one operator $T: C(K) \to c_0(\Gamma)$.*

Proof. We need to prove only the "if" part. Let $T: C(K) \to c_0(\Gamma)$ be a linear bounded one-to-one operator and let $T^*: l_1(\Gamma) \to C(K)^*$ be the conjugate operator of $T$.

The unite ball $B_1 = B_1(l_1(\Gamma))$ of the space $l_1(\Gamma)$ is weak*-sequentially compact. Then $T^*(B_1)$ is also weak*-sequentially compact, because the conjugate operator $T^*$ is weak*–weak* continuous. Since $K$ is a Grothendieck-compact space, then on the set $T^*(B_1)$ the weak* and weak convergence of sequences coincide and, consequently, $T^*(B_1)$ is a weak compact.

On the other hand, $T^*(B_1)$ is a total set, because $T^*(l_1(\Gamma))$ ($T^*(l_1(\Gamma))$ is a linear closure of $T^*(B_1)$) is a total subspace in $C(K)^*$.

Really, let $f \in C(K)$ and $(T^*g)(f) = 0$ for all $g \in l_1(\Gamma)$. Since

$$(T^*g)(f) = g(Tf) = 0, \quad \forall g \in l_1(\Gamma),$$

and $l_1(\Gamma)$ is a dual space to the space $c_0(\Gamma)$, then $Tf = 0$. However, the operator $T$ is one-to-one and therefore $f = 0$.

Thus we have a weak compact total subset $T^*(B_1)$ in $C(K)^*$ and then the assertion follows from the result of Rosenthal.

**Corollary 1.** *There exists a Grothendieck-compact $K$ which is not an extremally disconnected compact with a strictly positive measure.*

Haydon [5] constructed a Grothendieck-compact $K$ which is not extremally disconnected, such that the space $C(K)$ is isomorphic to a subspace of $l_\infty(\mathbf{N})$. Consequently, there is a linear bounded one-to-one operator $T: C(K) \to c_0(\mathbf{N})$.

Really, let $T_1$ be an isomorphism from $C(K)$ into $l_\infty(\mathbf{N})$. The map $T_2: l_\infty(\mathbf{N}) \to c_0(\mathbf{N})$, defined by the equality

$$T_2(x) = \{x_n/n\}_{n=1}^{\infty}, \quad x = \{x_n\}_{n=1}^{\infty} \in l_\infty(\mathbf{N}),$$

is obviously bounded linear one-to-one operator. Then our operator is $T = T_1 \circ T_2$.

**Corollary 2.** *If $K$ is a Grothendieck-compact with a strictly positive measure, then the space $C(K)$ admits an equivalent strictly convex norm.*

**Theorem 2.** *Let $K$ be a Rosenthal-compact. Then $K$ admits a strictly positive measure if and only if $K$ is separable.*

Proof. Let $K$ be a separable compact, in particular, a separable Rosenthal-compact, and let $\{x_n\}_{n=1}^{\infty}$ be a dense subset in $K$, then we define a srictly positive measure $\mu$ on $K$ by

$$\mu(U) = \sum_{x_n \in U} \frac{1}{2^n}$$

for all open sets $U$.

If now a separable Rosenthal-compact space $K$ has a strictly positive measure $\mu$, then the support of $\mu$ is equal to $K$. Therefore, the assertion follows from a result of Godefroy.

**Corollary 3.** *If $K$ is a separable Rosenthal-compact, then the space $C(K)$ admits an equivalent strictly convex norm.*

## 4. QUESTION

A norm $\|.\|$ of a Banach space $X$ is *locally uniformly convex* if for every sequence $\{x_n\}_{n=1}^{\infty} \subset X$ and every $x \in X$, such that $\|x_n\| = \|x\| = 1$, if $\lim \|(x + x_n)/2\| = 1$ then $\lim \|x_n - x\| = 0$.

If the norm is locally uniformly convex then this norm is strictly convex.

**Question.** Let $K$ be a separable Rosenthal-compact. Does the space $C(K)$ admit an equivalent locally uniformly convex norm?

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