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COHOMOLOGIES OF COUNTABLE UNIONS OF CLOSED SETS WITH APPLICATIONS TO CANTOR MANIFOLDS

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Н. Хаджииванов, Е. Щепин. КОГОМОЛОГИИ СЧЕТНОГО ОБЪЕДИНЕНИЯ ЗАМКНУТЫХ МНОЖЕСТВ С ПРИЛОЖЕНИЯМИ ДЛЯ КАНТОРОВЫХ МНОГООБРАЗИЯХ

Основной результат: Пусть X — компакт, A — замкнутое подмножество компакта X, и $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, где F_i — замкнутые в $X \setminus A$ множества, такие что $\dim(F_i \cap F_j) \leqq n-1$ для $i \neq j$. Тогда естественный гомоморфизм $H^r(X, A; G)$ в прямую сумму $\prod_{i=1}^{\infty} H^r(A \cup F_i, A; G)$ является мономорфизмом для $r \geqq n+1$. Получены некоторые применения этого результата для сильных канторовых многообразиях (относительно группы G).

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The main result: Let X be a compact space, A be its closed subset, and $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, where F_i are closed subsets of $X \setminus A$ such that $\dim(F_i \cap F_j) \leqq n-1$ for $i \neq j$. Then the natural homomorphism of $H^r(X, A; G)$ into the direct sum $\prod_{i=1}^{\infty} H^r(A \cup F_i, A; G)$ is a monomorphism for $r \geqq n+1$. Some applications of this result to strong Cantor manifolds (with respect to a group G) are obtained.

Let X be a compact topological space, let A be a closed subset of X and let $X \setminus A = \bigcup_{i=1}^{m} F_i$, where F_i are closed subsets of $X \setminus A$ such that $\dim(F_i \cap F_j) \leqq n-1$ for $i \neq j$. Then by the Meyer–Vietoris sequence we may conclude that for $r \geqq n+1$ there exists a natural isomorphism of $H^r(X, A; G)$ into the direct sum $\prod_{i=1}^{m} H^r(F_i \cup A, A; G)$. (Here we denote by $H^r(X, A; G)$ the r-th relative cohomology group in the sense of Alexandroff–Čech with coefficients in G.).

The same is true under the assumption that the cohomological dimension of $F_i \cap F_j$ with respect to G is less or equal to $n-1$: $\dim_G(F_i \cap F_j) \leqq n-1$.

In case $X \setminus A$ is a countable union $X \setminus A = \bigcup_{i=1}^{\infty} F_i$ such that $\dim_G(F_i \cap F_j) \leqq n-1$ for $i \neq j$, there is a natural homomorphism of $H^r(X, A; G)$ into the direct sum $\prod_{i=1}^{\infty} H^r(F_i \cup A, A; G)$. Generally speaking, this homomorphism is not an isomorphism, but it remains a monomorphism for $r \geqq n+1$. The purpose of this paper is to prove the last result.

In fact we shall prove the following result about extensions of continuous maps:

Theorem 1. *Let X be a compact space, let A be a closed subset of X, and let $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, where F_i are closed in $X \setminus A$. Let furthermore Y be an n-connected CW-complex, i.e. all the homotopy groups of Y up to the n-th are trivial: $\pi_1(Y) = \pi_2(Y) = \cdots = \pi_n(Y) = 0$. Suppose that the inequality $\dim_{G_k}(F_i \cap F_j) \leqq n$ for $i \neq j$, where $G_k = \pi_k(Y)$, holds for any $k \geqq n+1$. Then a continuous map $f : A \to Y$, which is extendable over $A \cup F_i$ for any i, can be extended over X.*

Let us show now that Theorem 1 implies the above result about cohomologies.

It follows from the characteristic property of the Eilenberg–McLane complex $K(G, r)$ that there is an one-to-one correspondence between the cohomology group $H^r(X, A; G)$ and the homotopy classes of maps of X into $K(G, r)$ which are constant on A (cf. [1], p. 550). The natural homomorphism of $H^r(X, A; G)$ into $\prod_{i=1}^{\infty} H^r(F_i \cup A, A; G)$ is a monomorphism if (and only if) each map $f : (X, A) \to (K(G, r), p_0)$ with homotopically trivial restrictions on $(F_i \cup A, A)$ for any i is homotopically trivial globally.

Let $I = [0, 1]$, and let set

$$X_1 = X \times I, \quad A_1 = (A \times I) \cup (X \times \{0\}) \cup (X \times \{1\}), \quad F_i' = F_i \times I,$$

and define $f_1 : A_1 \to K(G, r)$ by $f_1|_{X \times \{0\}} = f$ and $f_1|_{(X \times \{1\}) \cup (A \times I)} = p_0 = \text{const}$. Then applying Theorem 1 to the case X_1, A_1, F_i' and f_1, we get the desired result.

Indeed, the condition $\dim_G(F_i \cap F_j) \leqq n-1$ implies $\dim_G(F_i' \cap F_j') \leqq n$. Then $\dim_{G_k}(F_i' \cap F_j') \leqq n$, where $G_k = \pi_k[K(G, r)]$, since

$$\pi_k[K(G, r)] = \begin{cases} G & \text{for } k = r, \\ 0 & \text{for } k \neq r \end{cases}$$

by definition of the Eilenberg–McLane complexes. Thus we may refer to Theorem 1 and get the following

Theorem 1′. *Let X be a compact space, let A be closed in X, and let* $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, *where F_i are closed subsets of $X \setminus A$ such that* $\dim_G(F_i \cap F_j) \leqq n-1$ *for* $i \neq j$. *Then the natural homomorphism of $H^r(X, A; G)$ into* $\prod_{i=1}^{\infty} H^r(F_i \cup A, A; G)$ *is a monomorphism for $r \geqq n+1$.*

Let us recall that $\dim_{\mathbb{Z}} X = \dim X$ for a finite-dimensional X. Then, by Hu's theorem for obstructions (cf. [2]), it is possible to deduce Theorem 1 from Theorem 1′ as well, in the situation $G = \mathbb{Z}$, $\dim X < \infty$.

Hereafter we shall obtain, by means of Theorem 1′, some results about strong Cantor manifolds.

Let us recall the definition of a strong Cantor n-manifold (see [3]).

The space C is called a *strong Cantor n-manifold* if for an arbitrary representation $C = \bigcup_{i=1}^{\infty} F_i$, where F_i are proper closed subsets of C, we have $\dim(F_i \cap F_j) \geqq n-1$ for some $i \neq j$.

C is called a *strong Cantor n-manifold with respect to a group G* if for any of the above mentioned representations we have $\dim_G(F_i \cap F_j) \geqq n-1$ for some $i \neq j$.

Clearly, if C is a strong Cantor n-manifold with respect to G, then it is a strong Cantor n-manifold as well. The first author has achieved some development of the theory of strong Cantor manifolds (cf. [4]).

Now we shall prove that Theorem 1′ implies the following results:

Theorem 2. *Each compact space X with $\dim_G X = n$ contains a strong Cantor n-manifold (with respect to G).*

Theorem 3. *Let the k-dimensional cycle z^k $(\mathrm{mod}\, G)$ be irreducibly linked with the compact space X in some n-ball $\mathbb{B}^n$. Then X is a strong Cantor $(n-k-1)$-manifold with respect to G.*

Theorem 4. *The ball $\mathbb{B}^n$ is a strong Cantor n-manifold with respect to any group G.*

Theorem 5. *Each absolute boundary in $\mathbb{R}^n$ is a strong Cantor $(n-1)$-manifold with respect to any G. (Recall that C is an absolute boundary in $\mathbb{R}^n$ if it is a common boundary of at least two open domains in $\mathbb{R}^n$.)*

Proof of Theorem 2. The equality $\dim_G X = n$ means that there is a closed subset $A \subset X$ such that $H^n(X, A; G) \neq 0$, where n is the greatest number with this property (cf. [5]). By Zorn's lemma we may find a minimal closed subset $F \subset X$ such that $H^n(F, A \cap F; G) \neq 0$.

We shall show that F is a strong Cantor n-manifold with respect to G. Suppose this is not true, i.e. $F = \bigcup_{i=1}^{\infty} F_i$, where F_i are proper closed subsets of F such that $\dim_G(F_i \cap F_j) \leqq n-2$ for $i \neq j$. Then $H^n(F_i, A \cap F_i; G) = 0$ by the minimal property of F. According to Theorem 1′ the natural homomorphism

$$H^n(F, A \cap F; G) \to \prod_{i=1}^{\infty} H^n(F_i, A \cap F_i; G)$$

is a monomorphism, which is a contradiction. (Here we make use of the fact that $H^n(F, A \cap F; G) = H^n(F \cup A, A; G)$ for $n > 0$.)

Remark. Using the fact that the covering dimension "dim" equals "$\dim_{\mathbb{Z}}$" in the finite-dimensional case (cf. [5]), we obtain a result of the first author about strong Cantor manifolds (cf. [3]).

Proof of Theorem 3. Recall that the k-cycle z^k, lying in $\mathbb{B}^n \setminus X$, is irreducibly linked with X in $\mathbb{B}^n$ if z^k is not homologous to zero in $\mathbb{B}^n \setminus X$, but for any proper closed subset $X' \subset X$ z^k is homologous to zero in $\mathbb{B}^n \setminus X'$.

Let $p : \mathbb{B}^n \to S^n$ be a map sending $\partial\mathbb{B}^n$ into a point p_0 and the interior of $\mathbb{B}^n$ homeomorphically onto $S^n \setminus \{p_0\}$. Then it is easy to see that

$$H_k(\mathbb{B}^n \setminus X, \partial\mathbb{B}^n \setminus X) = H_k(S^n \setminus p(X))$$

for $k > 0$ and

$$H_0(\mathbb{B}^n \setminus X, \partial\mathbb{B}^n \setminus X) = \widetilde{H}_0(S^n \setminus p(X)),$$

where $\widetilde{H}_0$ is the reduced homology group.

Suppose the assertion of the theorem is not true, i.e. $X = \bigcup_{i=1}^{\infty} X_i$, where $\dim_G(X_i \cap X_j) \leqq n-k-3$ (for $i \neq j$). Then $\dim_G(p(X_i) \cap p(X_j)) \leqq n-k-3$ as well. Consider the commutative diagram

$$\begin{array}{ccccc} H_k(\mathbb{B}^n \setminus X) & \longrightarrow & H_k(\mathbb{B}^n \setminus X, \partial\mathbb{B}^n \setminus X) = H_k(S^n \setminus p(X)) & \longrightarrow & \prod_{i=1}^{\infty} H_k(S^n \setminus p(X_i)) \\ & & \downarrow & & \downarrow \\ & & H^{n-k-1}(p(X)) & \stackrel{q}{\longrightarrow} & \prod_{i=1}^{\infty} H^{n-k-1}(p(X_i)), \end{array}$$

where the vertical maps are the isomorphisms furnished by Alexander duality (cf. [1], p. 381).

Then, analyzing the image of the element $[z^k] \in H_k(\mathbb{B}^n \setminus X)$ and taking into account that q is a monomorphism by Theorem 1′, and having in view the minimal property of X, we arrive to a contradiction as above. (If $k = 0$, we have to consider the reduced groups $\widetilde{H}_0(S^n \setminus p(X))$ at the first row of the diagram.)

Theorems 4 and 5 follow immediately from Theorem 3.

Let us note that Theorem 1 implies directly that $\mathbb{B}^n$ is a strong Cantor n-manifold. To prove this, one has to suppose the contrary and to apply Theorem 1 to the situation $X = \mathbb{B}^n$, $A = \partial\mathbb{B}^n$, $f = \mathrm{id} : \partial\mathbb{B}^n \to \partial\mathbb{B}^n$.

Further the paper is aimed at the proof of Theorem 1.

Lemma 1. *Let X be a compact space, $A \subset X$ be a closed subset, and let $f : A \to Y$ map A into the CW-complex Y. Suppose that f is extendable over both $A \cup F_1$ and $A \cup F_2$ for some closed F_1, F_2. Then there exist a neighbourhood $N(A)$ of A and an extension $f' : N(A) \to Y$ of f, which is still extendable over $N(A) \cup F_1$ and $N(A) \cup F_2$.*

This technical lemma is quite elementary and follows immediately from Borsuk's lemma about extensions of homotopies (cf. [6], p. 231). It remains valid for any Y which is ANE (Absolute Neighbourhood Extensor in the class of normal spaces).

Lemma 2. *Let X be a compact space and let $A \subset X$ be a closed subset such that $X \setminus A = \bigcup_{i=1}^{m} F_i$, where F_i are closed in $X \setminus A$. Let furthermore Y be an n-connected CW-complex and suppose that $\dim_{G_k}(F_i \cap F_j) \leqq n$, where $G_k = \pi_k(Y)$ for any $k \geqq n+1$. Then a map $f : A \to Y$, which is extendable over $A \cup F_i$ for any i, can be extended over X.*

Proof. The first obstruction for extending the map f lies in $H^{n+2}(X, A; \pi_{n+1}(Y))$ (cf. [1], p. 574). The image of this first obstruction in $H^{n+2}(F_i \cup A, A; \pi_{n+1}(Y))$ is the first obstruction for extending f over $F_i \cup A$, which is trivial, since f can be extended over $F_i \cup A$ by hypothesis. But, as we have already noticed, the group $H^{n+2}(X, A; \pi_{n+1}(Y))$, in virtue of the Meyer–Vietoris sequence, is naturally isomorphic to $\prod_{i=1}^{m} H^{n+2}(F_i \cup A, A; \pi_{n+1}(Y))$. Hence, this first obstruction is trivial. We have the same situation for the second, third and higher obstructions. Therefore, there is no obstruction to the extension of f over X.

To go further, we need the following construction.

Let X be a locally compact space and let $\sigma = \{F_i\}_{i=1}^{\infty}$ be a covering of X by closed sets. For any $A \subset X$ let us set

$$A(\sigma) = A \setminus \bigcup_{i=1}^{\infty} \mathrm{Int}_A(F_i \cap A).$$

It follows from Baire's theorem that $A \neq A(\sigma)$ for any non-empty closed set A. We may define by transfinite induction a decreasing transfinite sequence of closed sets B_α as follows:

$$B_1 = X, \quad B_\alpha = B_{\alpha-1}(\sigma) \text{ for a non-limit ordinal } \alpha,$$
$$B_\alpha = \bigcap_{\beta<\alpha} B_\beta \text{ for a limit ordinal } \alpha.$$

We call the family $\{B_\alpha\}$ *filtration of X generated by σ*. Furthermore we shall have to manage with the following situation: X is a compact space, A is its closed subset, and $\sigma = \{F_i\}_{i=1}^{\infty}$ is a covering of $X \setminus A$ by closed in $X \setminus A$ sets F_i. The main property of the filtration of $X \setminus A$ generated by σ is the following:

(P) For any neighbourhood N of $A \cup B_{\alpha+1}$ in X there exists such an m that $A \cup B_\alpha$ is contained in the union $N \cup F_1 \cup \ldots \cup F_m$.

Indeed, $B_\alpha \setminus N$ is a compact space covered by the interiors of F_i with respect to B_α, so we may choose a finite subcover and take m greater than the maximal index of elements of this subcover.

Lemma 3. *Let* $X \setminus A = \bigcup_{i=1}^{\infty} F_i$, *where* A *is closed in* X *and* F_i *are closed in* $X \setminus A$, *and let* $\dim_{G_k}(F_i \cap F_j) \leqq n$ *for* $i \neq j$, $k \geqq n+1$, *where* $G_k = \pi_k(Y)$ *for a given* n*-connected* CW*-complex* Y. *Suppose that the map* $f : A \to Y$ *is extendable over both* $A \cup F_1$ *and* $\left[A \cup \bigcup_{i=2}^{\infty} F_i\right]$. *Then* f *is extendable over* X.

Proof. Let $\{B_\alpha\}$ be the filtration of $X \setminus A$ generated by $\{F_i\}_{i=1}^{\infty}$. Let α be the smallest ordinal such that it is still possible to construct a continuous map $f_\alpha : A \cup B_\alpha \to Y$ which is extendable over both F_1 and $\left[\bigcup_{i=2}^{\infty} F_i\right]$. It follows from the compactness of X and from Lemma 1 that α cannot be a limit ordinal. Hence there exists $\alpha - 1$, or $\alpha = 1$. The second case concludes the proof. It is sufficient now to lead the first case to a contradiction. Lemma 1 provides us with an extension $f'_\alpha : N(A \cup B_\alpha) \to Y$ over some neighbourhood of $A \cup B_\alpha$, which is still extendable over both F_1 and $\left[\bigcup_{i=2}^{\infty} F_i\right]$. By property (P) of the filtration we have $B_{\alpha-1} \subset N(A \cup B_\alpha) \cup \bigcup_{i=2}^{m} F_i$ for some m.

To obtain the needed contradiction, it suffices to prove that f'_α is extendable over $\bigcup_{i=1}^{m} F_i$. According to Lemma 2 it is sufficient to prove that f'_α is extendable over F_i for any i. But this is true by the hypothesis.

Proof of Theorem 1. Let $X \setminus A = \bigcup_{i=1}^{\infty} F_i$ and $\{B_\alpha\}$ be the filtration generated by $\{F_i\}_{i=1}^{\infty}$. Suppose that α is the smallest ordinal such that the extension of f on $A \cup B_\alpha$ is possible. It is possible to extend f on some neighbourhood $N(A \cup B_\alpha)$. If we assume that α is a limit ordinal, then $B_\alpha = \bigcap_{\beta<\alpha} B_\beta$ and in virtue of the compactness of X one may conclude that for some $\beta < \alpha$ we have $A \cup B_\beta \subset N(A \cup B_\alpha)$ in contradiction with the minimal property of α. If $\alpha = 1$, the theorem is proved. Suppose that $\alpha \neq 1$. Then $\alpha - 1$ exists and we have some extension $f' : N(A \cup B_\alpha) \to Y$. For any i we may extend f over F_i by hypothesis. According to Lemma 3 we may extend f over $F_i \cup B_\alpha$, and by Lemma 2 we may extend f over $\bigcup_{i=1}^{m} F_i \cup B_\alpha$ for any m. Therefore by the property (P) of the filtration we may extend f over $A \cup B_{\alpha-1}$ *in contradiction with the minimal property of* α.

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