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AN EXAMPLE OF A FINITE NUMBER OF RECURSIVELY ENUMERABLE m-DEGREES CONTAINING AN INFINITE SEQUENCE OF RECURSIVELY ENUMERABLE SETS

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Елка Божкова. ПРИМЕР КОНЕЧНОГО ЧИСЛА РЕКУРСИВНО ПЕРЕЧИСЛЯЕМЫХ m-СТЕПЕНЕЙ, СОДЕРЖАЩИХ БЕЗКОНЕЧНУЮ ПОСЛЕДОВАТЕЛЬНОСТЬ РЕКУРСИВНО ПЕРЕЧИСЛЯЕМЫХ МНОЖЕСТВ

Цель этой статьи построить для произвольного натурального числа n рекурсивно перечисляемое (р. п.) множество A, такого чтобы множества A, A^2, …, A^n принадлежали разным m-степеням, а множества A^n, A^{n+1}, A^{n+2}, … — одной и той же m-степени. Таким образом мы получаем пример для n разных р. п. m-степеней $d_m(A)$, $d_m(A^2)$, …, $d_m(A^{n-1})$, $d_m(A^n)$, принадлежащих к одной btt-степени (точнее, c-степени — степень $d_c(A)$ множества A совпадающая с $d_c(A^2)$, …, $d_c(A^n)$, …). Для этого достаточно доказать, что множество A^{n+1} m-сводимо к множеству A^n, а множество A^n не m-сводимо к множеству A^{n-1}. Для этой цели создаем схему для построения р. п. множества A, сопоставимого последовательности р. п. множеств A_0, A_1, A_2, …, так чтобы первое условие было выполнено независимо от выбора множеств A_0, A_1, A_2, … После этого мы строим множества A_0, A_1, A_2, … по этапам, пользуясь методом приоритета, чтобы выполнить второе условие.

Elka Bojkova. AN EXAMPLE OF A FINITE NUMBER OF RECURSIVELY ENUMERABLE m-DEGREES CONTAINING AN INFINITE SEQUENCE OF RECURSIVELY ENUMERABLE SETS

The aim of this paper is to construct for an arbitrary natural number n a recursively enumerable (r. e.) set A such that the sets A, A^2, …, A^n belong to different m-degrees and the sets A^n, A^{n+1}, A^{n+2}, … belong to the same m-degree. That will provide an example of n distinct r. e. m-degrees $d_m(A)$, $d_m(A^2)$, …, $d_m(A^n)$ belonging to one r. e. btt-degree (more precisely, c-degree, the degree $d_c(A)$ of the set A coincidental with $d_c(A^2)$, …, $d_c(A^n)$, …). It suffices to prove that the set A^{n+1} is m-reducible to the set A^n, but the set A^n is not m-reducible to the set A^{n-1}. For this purpose we construct a scheme for building a r. e. set A, corresponding to a sequence of r. e. sets A_0, A_1, A_2, ..., such that the first condition holds regardless the choice of A_0, A_1, A_2, ... Further we build the sets A_0, A_1, A_2, ... by steps, using the priority argument to ensure the second condition.

In [1] Fischer proves that the reducibilities $\leqq_m$ and $\leqq_{btt}$ differ on the recursively enumerable (r.e.) non-recursive sets. He constructs an example of a r.e. btt-degree containing infinitely many distinct m-degrees. In [2] Odifreddi asks if every r.e. tt-degree contains one or infinitely many r.e. m-degrees. Downey [3] solves Odifreddi's question by constructing a r.e. tt-degree containing exactly 3 r.e. m-degrees. Rogers [4] and Odifreddi [5] summarize the obtained results about the structure of the different kinds of degrees.

In this paper we construct (for an arbitrary natural number n) a r.e. set A such that

$$A <_m A^2 <_m \cdots <_m A^n \equiv_m A^{n+1} \equiv_m A^{n+2} \equiv_m \cdots$$

That will provide an example of n distinct r.e. m-degrees ($d_m(A)$, $d_m(A^2)$, ..., $d_m(A^{n-1})$ and $d_m(A^n)$) belonging to one r.e. btt-degree (more precisely, c-degree — the degree $d_c(A)$ of the set A coincidental with $d_c(A^2)$, ..., $d_c(A^n)$, ...).

Obviously, all the recursive sets belong to one tt-degree and one m-degree, i.e. for any recursive set B the following equivalencies hold:

$$B \equiv_m B^2 \equiv_m B^3 \equiv_m \cdots$$

Now, we have to find a set A such that the sets A, A^2, ..., A^n belong to different m-degrees and the sets A^n, A^{n+1}, A^{n+2}, ... belong to the same m-degree. It suffices for this purpose to prove

$$A^{n+1} \leqq_m A^n \tag{1}$$

and

$$\neg A^n \leqq_m A^{n-1}. \tag{2}$$

The method elaborated by Ditchev in [6] is used to construct the set responding to both conditions. First, we construct a scheme for building a r.e. set A corresponding to a sequence of r.e. sets A_0, A_1, A_2, ... (answering to some conditions), such that (1) holds regardless the choice of A_0, A_1, A_2, ... After that we build the sets A_0, A_1, A_2, ... by steps, using the priority argument to ensure (2).

* * *

(1.1) **Definition.** Let A and B be sets of natural numbers:

a) We say that the set A is "m-reducible" to the set B ($A \leqq_m B$) if there exists a recursive function $f : N \to N$ such that

$$\forall x\ (x \in A \iff f(x) \in B);$$

b) We say that the sets A and B are "m-equivalent" ($A \equiv_m B$) if

$$A \leqq_m B \quad \text{and} \quad B \leqq_m A.$$

We shall denote by "$A <_m B$" the fact that $A \leqq_m B$ but $A \not\equiv_m B$.

Let for any natural number n the recursive functions J^n, J_1^n, J_2^n, ..., J_n^n be such that for any n and for any x there exists a single tuple $x_1, \ldots, x_n$ such that

$$(1.2) \qquad x = J^n(x_1, \ldots, x_n)$$

and

$$(1.3) \qquad \forall i_{1 \leqq i \leqq n}[J_i^n(J^n(x_1, \ldots, x_n)) = x_i].$$

(1.4) **Definition.** Let A be an arbitrary set of natural numbers. For any natural number n we define the set A^n:

$$A^n = \{J^n(x_1, \ldots, x_n) \mid x_1 \in A \& \ldots \& x_n \in A\}.$$

Obviously, for any set A

$$A \leqq_m A^2 \leqq_m \cdots \leqq_m A^n \leqq_m A^{n+1} \leqq_m A^{n+2} \leqq_m \cdots,$$

i. e. the essential part of the problem is to find a set A such that:

(1) for every i, $1 \leqq i \leqq n-1$, $\neg A^{i+1} \leqq_m A^i$;

(2) for every i, $i \geqq n$, $A^{i+1} \leqq_m A^i$.

It is easy to prove the following two lemmas:

(1.5) **Lemma.** *Let A be an arbitrary subset of N and n be an arbitrary natural number. Then $A^{n+1} \leqq_m A^n$ iff there exist recursive functions $f_1 : N^{n+1} \to N$, $f_2 : N^{n+1} \to N$, ..., $f_n : N^{n+1} \to N$ such that*

$$(1.6) \qquad x_1 \in A \& \ldots \& x_{n+1} \in A$$

$$\Longleftrightarrow f_1(x_1, \ldots, x_{n+1}) \in A \& \ldots \& f_n(x_1, \ldots, x_{n+1}) \in A.$$

Proof. Using the fact that one unary function g reduces A^{n+1} to A^n iff for the $(n+1)$-ary functions $f_1, \ldots, f_n$, defined as follows:

$$f_i(x_1, x_2, \ldots, x_{n+1}) = J_i^n(g(J^{n+1}(x_1, x_2, \ldots, x_{n+1}))), \quad i = 1, 2, \ldots, n,$$

the condition (1.6) holds, one can easily verify that Lemma (1.5) is true.

(1.7) **Lemma.** a) *If $A^{n-1} <_m A^n$, then for any m, $2 \leqq m \leqq n-1$:*

$$A^{m-1} <_m A^m;$$

b) *If $A^n \equiv_m A^{n+1}$, then for any m, $m > n$:*

$$A^m \equiv_m A^{m+1}.$$

Hence to solve our problem, it is enough to prove

$$A^{n-1} <_m A^n \equiv_m A^{n+1}.$$

* * *

Let n be an arbitrary natural number*, $n > 1$. We shall find a set A such that

$$A <_m A^2 <_m \cdots <_m A^n \equiv_m A^{n+1} \equiv_m A^{n+2} \equiv_m \cdots$$

(2.1) **Definition.** Let L_n be a *language* with the following *alphabet*:

— $c_0, c_1, \ldots$ — infinite sequence of constants;

— $x_1, \ldots, x_n$ — n variables;

— $F_1, \ldots, F_n$ — $(n+1)$-ary functional symbols.

Terms are defined by means of the following inductive clauses:

a) c_i is a term, $i \in N$; x_i is a term, $1 \leqq i \leqq n$;

b) if $\tau^1, \ldots, \tau^{n+1}$ are terms, then $F_i(\tau^1, \ldots, \tau^{n+1})$ is a term, $1 \leqq i \leqq n$.

(2.2) **Definition.** For any term τ we define *subterms* of τ:

a) if $\tau = c_i$, $i \in N$, c_i is a subterm of τ; if $\tau = x_i$, $1 \leqq i \leqq n$, x_i is a subterm of τ;

b) if $\tau = F_i(\tau^1, \ldots, \tau^{n+1})$, $1 \leqq i \leqq n$, then the term τ and the subterms of $\tau^1, \ldots, \tau^{n+1}$ are subterms of τ.

(2.3) **Definition.** For any term τ we define *deepness* $\mathrm{dp}(\tau)$ *of* τ:

a) if $\tau = c_i$, $i \in N$ or $\tau = x_i$, $1 \leqq i \leqq n$, then $\mathrm{dp}(\tau) = 1$;

b) if $\tau = F_i(\tau^1, \ldots, \tau^{n+1})$, $1 \leqq i \leqq n$, and $\mathrm{dp}(\tau^1), \ldots, \mathrm{dp}(\tau^{n+1})$ are defined, then $\mathrm{dp}(\tau) = 1 + \max\limits_{1 \leqq j \leqq n+1} \mathrm{dp}(\tau^j)$.

(2.4) **Definition.** We will call a *partial structure* every ordered $(n+1)$-tuple $U = \langle N; \theta_1, \ldots, \theta_n \rangle$, where $\theta_1, \ldots, \theta_n$ are partial functions of $n+1$ variables.

(2.5) **Definition.** Let $U = \langle N; \theta_1, \ldots, \theta_n \rangle$ be a partial structure. We define the *value* τ_U *of the term* τ *in the partial structure* U:

a) if $\tau = c_i$, $\tau_U(t_1, \ldots, t_n) = i$ for any $t_1, \ldots, t_n$, $i \in N$;

if $\tau = x_i$, $\tau_U(t_1, \ldots, t_n) = t_i$ for any $t_1, \ldots, t_n$, $1 \leqq i \leqq n$;

b) if $\tau = F_i(\tau^1, \ldots, \tau^{n+1})$, $1 \leqq i \leqq n$, $t_1 \in N, \ldots, t_n \in N$ and $\tau_U^1, \ldots, \tau_U^{n+1}$ are defined, then

$$\tau_U(t_1, \ldots, t_n) \cong \theta_i(\tau_U^1(t_1, \ldots, t_n), \ldots, \tau_U^{n+1}(t_1, \ldots, t_n)).$$

(2.6) **Definition.** We define the *cod* $\mathrm{cd}(\tau)$ of the term τ:

a) if $\tau = c_i$, $i \in N$, $\mathrm{cd}(\tau) = J^2(0, i)$; if $\tau = x_i$, $1 \leqq i \leqq n$, $\mathrm{cd}(\tau) = J^2(1, i)$;

b) if $\tau = F_i(\tau^1, \ldots, \tau^{n+1})$, $1 \leqq i \leqq n$, $\mathrm{cd}(\tau) = J^{n+2}(i+1, \mathrm{cd}(\tau^1), \ldots, \mathrm{cd}(\tau^{n+1}))$.

That coding allows us to verify for any natural number whether it is a cod of any term and if so, to find this term.

Theorem 1. *Let* $M = \{2k \mid k \in N\}$. *There exists a partial structure* $U = \langle N, \theta_1, \ldots, \theta_n \rangle$ *such that the functions* $\theta_1, \ldots, \theta_n$ *are recursive and*:

* n is fixed till the end of this paper.

1) $\forall t_1 \in N \ldots \forall t_{n+1} \in N$,

$$(*) \qquad (t_1 \in M \,\&\, \ldots \,\&\, t_{n+1} \in M \iff \theta_1(t_1, \ldots, t_{n+1}) \in M \,\&\, \ldots \,\&\, \theta_n(t_1, \ldots, t_{n+1}) \in M);$$

2) $\forall(\tau^1, \ldots, \tau^{n-1}$ – terms) $\exists t_1 \in N, \ldots, t_n \in N$,

$$(**) \qquad \neg\,(t_1 \in M \,\&\, \ldots \,\&\, t_n \in M \iff \tau_U^1(t_1, \ldots, t_n) \in M \,\&\, \ldots \,\&\, \tau_U^{n-1}(t_1, \ldots, t_n) \in M)).$$

Proof. We shall say that for the partial functions $\omega_1, \ldots, \omega_n$ the condition $(\tilde{*})$ holds if $\operatorname{dom}\omega_1 = \cdots = \operatorname{dom}\omega_n$ and the condition $(*)$ holds for $\operatorname{dom}\omega_1$.

We build the functions $\theta_1, \ldots, \theta_n$ by steps: at any step $s = 0, 1, 2, \ldots$ we build the functions $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ as finite extensions of $\theta_1^{(s-1)}, \ldots, \theta_n^{(s-1)}$ such that for $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ the condition $(\tilde{*})$ holds. For $s = 0$ we accept $\theta_1^{(-1)} = \cdots = \theta_n^{(-1)} = \varnothing$. (The domain of $\varnothing$ is empty.)

Finally, $\theta_i = \bigcup_{s \in N} \theta_i^{(s)}, \quad i = 1, \ldots, n.$

I. By the even steps $s = 2p$ we shall ensure that the functions $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ are defined in $(J_1^{n+1}(p), \ldots, J_{n+1}^{n+1}(p))$ and therefore the functions $\theta_1, \ldots, \theta_n$ will be total.

1) If $(J_1^{n+1}(p), \ldots, J_{n+1}^{n+1}(p)) \notin \operatorname{dom}\theta_1^{(s-1)}$, then for all l, $1 \leqq l \leqq n$,

$$\theta_l^{(s)}(J_1^{n+1}(p), \ldots, J_{n+1}^{n+1}(p)) = \begin{cases} 2, & \text{if } J_1^{n+1}(p) \in M \,\&\ldots\&\, J_{n+1}^{n+1}(p) \in M, \\ 3, & \text{otherwise,} \end{cases}$$

and

$$\theta_l^{(s)}(y_1, \ldots, y_{n+1}) \cong \theta_l^{(s-1)}(y_1, \ldots, y_{n+1}), \quad \text{if } J^{n+1}(y_1, \ldots, y_{n+1}) \neq p.$$

2) If $(J_1^{n+1}(p), \ldots, J_{n+1}^{n+1}(p)) \in \operatorname{dom}\theta_1^{(s-1)}$, then $\forall l_{1 \leqq l \leqq n}\left[\theta_l^{(s)} = \theta_l^{(s-1)}\right]$.

II. By the odd steps $s = 2p+1$, $p \in N$, if there are not terms $\tau^1, \ldots, \tau^{n-1}$ such that $p = J^{n-1}(\operatorname{cd}(\tau^1) \ldots, \operatorname{cd}(\tau^{n-1}))$, we take $\forall l_{1 \leqq l \leqq n}\left[\theta_l^{(s)} = \theta_l^{(s-1)}\right]$. If such terms exist, we shall find $t_1, \ldots, t_n$ such that the condition $(**)$ holds. In addition we obtain that the condition $(**)$ holds for any terms $\tau^1, \ldots, \tau^{n-1}$.

In the second case we first verify whether some of the terms $\tau^1, \ldots, \tau^{n-1}$ is a constant with a value — an odd number. If so, for any n-tuple of even numbers $t_1, \ldots, t_n$ the condition $(**)$ holds and we do nothing; i. e. we take $\theta_l^{(s)} = \theta_l^{(s-1)}$ for all l, $1 \leqq l \leqq n$. If all the terms $\tau^1, \ldots, \tau^{n-1}$, which are constants, have the values — even numbers, we shall ensure $(**)$ by finding $t_1, \ldots, t_n$ and building $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ such that:

— $\tau_{U^{(s)}}^1(t_1, \ldots, t_n) \in M \,\&\ldots\&\, \tau_{U^{(s)}}^{n-1}(t_1, \ldots, t_n) \in M$;

— $t_1 \notin M \vee \ldots \vee t_n \notin M$.

Every term is a constant, a variable, or its deepness is bigger than 1.

(1) For all the terms $\tau^1, \ldots, \tau^{n-1}$, which are constants, we have that their values belong to M.

(2) For these ones, which are variables, we shall choose the corresponding numbers $t_1, \ldots, t_n$ (which are values of such terms) to be even and these terms also will satisfy the condition. Because there is at least one number among $t_1, \ldots, t_n$ which is not a value of any term, let this number be odd and by this way $t_1, \ldots, t_n$ also will satisfy the condition. In both cases we shall take $t_1, \ldots, t_n$ big enough not to belong to the domains of $\theta_1^{(s-1)}, \ldots, \theta_n^{(s-1)}$ and thus to the values of the terms with deepness bigger than 1.

(3) For the terms with deepness bigger than 1 we need an auxiliary lemma — Lemma 1. It will be applied for the partial structure

$$U^{(s-1)} = \langle N, \theta_1^{(s-1)}, \ldots, \theta_n^{(s-1)}\rangle \text{ for } \tau^1, \ldots, \tau^{n-1}, \ \ t_1, \ldots, t_n$$

(obtained from (2) or, if there are no terms with deepness 1, we take $t_1, \ldots, t_n$ big enough and such that $t_1 \notin M \vee \ldots \vee t_n \notin M$) and for one term τ among the terms $\tau^1, \ldots, \tau^{n-1}$, $\mathrm{dp}(\tau) > 1$. Using this lemma we shall obtain the functions $\theta_1^{(s)}, \ldots, \theta_n^{(s)}$ such that $\theta_i^{(s-1)} \leqq \theta_i^{(s)}$, $1 \leqq i \leqq n$, the condition $(\tilde{*})$ holds and the value of the term τ for $(t_1, \ldots, t_n)$ is an even number. If at the same time $(t_1, \ldots, t_n)$ enters in the domain of the value of some other term among $\tau^1, \ldots, \tau^{n-1}$ (except τ), we have to ensure that the value of the other term in $(t_1, \ldots, t_n)$ is also an even number.

Lemma 1. *Let the terms $\tau^1, \ldots, \tau^{n-1}$ be given. For any partial structure $U = \langle N; \theta_1, \ldots, \theta_n\rangle$ with finite functions satisfying $(\tilde{*})$, for any term τ with $\mathrm{dp}(\tau) \geqq 2$, for any natural number m, and for any n-tuple of natural numbers $(t_1, \ldots, t_n)$ such that*

$$(t_1, \ldots, t_n) \notin \mathrm{dom}\ \tau_U \quad \textit{and} \quad t_1 \notin M \vee \ldots \vee t_{n+1} \notin M$$

there exists a partial structure $U' = \langle N, \theta_1', \ldots, \theta_n'\rangle$ with finite recursive functions satisfying $(\tilde{*})$, such that:

(i) $\theta_i' \geqq \theta_i$ *for every* i, $1 \leqq i \leqq n$;

(ii) $(t_1, \ldots, t_n) \in \mathrm{dom}\ \tau_{U'}$ *and* $\tau_{U'}, (t_1, \ldots, t_n) > z$,

where $z = \max(\{m, \max(I_1(\mathrm{dom}\ \theta_1')), \ldots, \max(I_{n+1}(\mathrm{dom}\ \theta_1'))\})$;

(iii) *if* $(t_1, \ldots, t_n) \in (\mathrm{dom}\ \tau_{U'}^j \setminus \mathrm{dom}\ \tau_U^j)$, *then* $\tau_{U'}^j(t_1, \ldots, t_n) \in M$, $j = 1, \ldots, n-1$.

We shall apply this lemma successively for all the terms among $\tau^1, \ldots, \tau^{n-1}$ with deepness bigger than 1 and we shall obtain

$$\tau_{U^{(s)}}^1(t_1, \ldots, t_n) \in M \ \& \ \ldots \ \& \ \tau_{U^{(s)}}^{n-1}(t_1, \ldots, t_n) \in M;$$

$$t_1 \notin M \vee \ldots \vee t_n \notin M.$$

The idea to prove Lemma 1 is to find for the term $\tau = F_i(\sigma^1, \ldots, \sigma^{n+1})$ a partial structure $V = \langle N, \theta_1'', \ldots, \theta_{n+1}''\rangle$ such that $\forall j$, $1 \leqq j \leqq n+1$, $(t_1, \ldots, t_n) \in \mathrm{dom}\ \sigma_V^j$, $z_j = \sigma_V^j(t_1, \ldots, t_n)$ and $(z_1, \ldots, z_{n+1}) \notin \mathrm{dom}\ \theta_{n+1}''$. In this case we shall build the functions $\theta_1', \ldots, \theta_{n+1}'$ such that $\theta_1' \geqq \theta_1'', \ldots, \theta_{n+1}' \geqq \theta_{n+1}''$, $\mathrm{dom}\ \theta_{n+1}' =$

dom $\theta''_{n+1} \cup \{(z_1, \ldots, z_{n+1})\}$, and the values of the functions in $(z_1, \ldots, z_{n+1})$ are defined as follows (z is big enough):

1) If $z_1 \in M \,\&\ldots\&\, z_{n+1} \in M$, the values of all the functions are $2z$;

2) If $z_1 \notin M \vee\ldots\vee z_{n+1} \notin M$, and for some term τ^j among $\tau^1, \ldots, \tau^{n-1}$ there exist a number l, $1 \leqq l \leqq n$, and terms $\varepsilon^1, \ldots, \varepsilon^{n+1}$ such that $\tau^j = F_l(\varepsilon^1, \ldots, \varepsilon^{n+1})$, then the value of the function θ'_l is also $2z$;

3) In the other cases the values of the functions are $2z + 1$.

So we ensure that $(\tilde{*})$ is true for the new partial structure (if $z_1 \notin M \vee\ldots\vee z_{n+1} \notin M$, because the terms $\tau^1, \ldots, \tau^{n-1}$ are $n-1$ and the symbols $F_1, \ldots, F_n$ are n, at least one of the functions is in the case 3) and its value is $2z+1$). Except that we ensure $\tau_{U'}(t_1, \ldots, t_n)$ to be big enough and for any of the terms τ^j, $j = 1, \ldots, n-1$, such that $(t_1, \ldots, t_n) \in (\mathrm{dom}\,\tau^j_{U'} \setminus \mathrm{dom}\,\tau^j_U)$, $\tau^j_{U'}(t_1, \ldots, t_n) \in M$ is true. The last follows from 2), but not evidently. So we need an auxiliary lemma — Lemma (2.7).

With this lemma we have to prove that for some l, $1 \leqq l \leqq n$, $\tau^j_{U'}(t_1, \ldots, t_n) = \theta_1(z_1, \ldots, z_{n+1})$ and θ_1 is in case 2), and therefore $\tau^j_{U'}(t_1, \ldots, t_n) = 2z \in M$. In the general case the lemma can not be proved for the term τ^j, it is only true that there exists a subterm of τ^j for which the condition holds, but in this concrete application of Lemma (2.7) in Lemma 1 we can prove that this subterm may be only the term τ^j.

(2.7) **Lemma.** *For any two partial structures* $U = \langle N, \theta_1, \ldots, \theta_n\rangle$ *and* $U' = \langle N, \theta'_1, \ldots, \theta'_n\rangle$, *for any term* σ *and for any natural numbers* $y_1, \ldots, y_{n+1}$; $t_1, \ldots, t_n$, *such that*:

a) $\mathrm{dom}\,\theta_1 \equiv \cdots \equiv \mathrm{dom}\,\theta_n$ *and that is a finite set*;

b) $\mathrm{dom}\,\theta'_i = \mathrm{dom}\,\theta_i \cup \{(y_1, \ldots, y_{n+1})\}$, $i = 1, \ldots, n$, *and* $(y_1, \ldots, y_{n+1}) \notin \mathrm{dom}\,\theta_1$;

c) $\theta_1 \leqq \theta'_1, \ldots, \theta_n \leqq \theta'_n$;

d) $(t_1, \ldots, t_n) \in (\mathrm{dom}\,\sigma_{U'} \setminus \mathrm{dom}\,\sigma_U)$,

there exist a subterm σ' *of* σ, $i \in \{1, \ldots, n\}$, *and terms* $\varepsilon^1, \ldots, \varepsilon^{n+1}$ *such that*:

$$\text{(i)} \quad \sigma' = F_i(\varepsilon^1, \ldots, \varepsilon^{n+1});$$

$$\text{(ii)} \quad \varepsilon^1_{U'}(t_1, \ldots, t_n) = y_1, \ldots, \varepsilon^{n+1}_{U'}(t_1, \ldots, t_n) = y_{n+1}.$$

Proof. By the definitions (2.3) and (2.5) we have

(2.8) $\forall\tau$-term, $\forall U$-partial structure $[\mathrm{dp}(\tau) = 1 \Rightarrow \tau_U$ is a total function$]$.

From condition d) we have $(t_1, \ldots, t_n) \notin \mathrm{dom}\,\sigma_U$. Therefore $\mathrm{dp}(\tau) \geqq 2$ and from (2.3) it follows that there exist k, $1 \leqq k \leqq n$, and terms $\sigma^1, \ldots, \sigma^{n+1}$ such that $F_k(\sigma^1, \ldots, \sigma^{n+1}) = \sigma$.

We have two cases:

C a s e I. $\forall j_{1 \leqq j \leqq n+1}[(t_1, \ldots, t_n) \in \mathrm{dom}\,\sigma^j_U]$. Let $z_j = \sigma^j_U(t_1, \ldots, t_n)$, $j = 1, \ldots, n+1$. From d) it follows that $(z_1, \ldots, z_{n+1}) \notin \mathrm{dom}\,\theta_k$ and $(z_1, \ldots, z_{n+1}) \in \mathrm{dom}\,\theta'_k$ and from b) we obtain $z_1 = y_1, \ldots, z_{n+1} = y_{n+1}$. From c) it follows that

$$\forall j_{1 \leqq j \leqq n+1}[\sigma^j_{U'}(t_1, \ldots, t_n) = \sigma^j_U\,(t_1, \ldots, t_n)].$$

We obtained $\sigma^1_{U'}(t_1, \ldots, t_n) = y_1, \ldots, \sigma^{n+1}_{U'}(t_1, \ldots, t_n) = y_{n+1}$, i.e. the conditions (i) and (ii) hold for $\sigma' = \sigma$ and $\varepsilon^j = \sigma^j$, $j = 1, \ldots, n+1$.

C a s e II. We apply induction on $\mathrm{dp}(\sigma)$:

1) $\mathrm{dp}(\sigma) = 2$. Then $\mathrm{dp}(\sigma^1) = \cdots = \mathrm{dp}(\sigma^{n+1}) = 1$ and the lemma follows from (2.8) and Case I;

2) Let it be true for the terms with deepness less than $\mathrm{dp}(\sigma)$;

3) We shall prove it for $\sigma = F_k(\sigma^1, \ldots, \sigma^{n+1})$. The terms $\sigma^1, \ldots, \sigma^{n+1}$ have smaller deepness. We have $(t_1, \ldots, t_n) \notin \mathrm{dom}\, \sigma_U$. Two cases are possible:

Case 1. $\exists j_{1 \leqq j \leqq n+1}[(t_1, \ldots, t_n) \notin \mathrm{dom}\, \sigma^j_U]$. But $(t_1, \ldots, t_n) \in \mathrm{dom}\, \sigma_{U'} \Rightarrow (t_1, \ldots, t_n) \in \mathrm{dom}\, \sigma^j_{U'}$, and by the induction hypothesis for σ^j we obtain that there exists a σ'-subterm of σ^j, i, $\varepsilon^1, \ldots, \varepsilon^{n+1}$ such that $\sigma' = F_i(\varepsilon^1, \ldots, \varepsilon^{n+1})$ and $\forall m_{1 \leqq m \leqq n+1}[\varepsilon^m_{U'}(t_1, \ldots, t_n) = y_m]$. But σ' is a subterm of σ^j, so σ' is a subterm of σ also and in this case the lemma is proved.

Case 2. $\forall j_{1 \leqq j \leqq n+1}[(t_1, \ldots, t_n) \in \mathrm{dom}\, \sigma^j_U]$. Then Lemma (2.7) follows from Case I.

* * *

(3.1) **Definition.** The total functions $\varphi_1, \ldots, \varphi_n$ of $n+1$ variables are defined as follows:

$$\varphi_i(t_1, \ldots, t_{n+1}) = J^{n+2}(i, t_1, \ldots, t_{n+1}),\ i = 1, \ldots, n;\ t_1 \in N, \ldots, t_{n+1} \in N.$$

(3.2) **Definition.** Let $N_0 = N \setminus (\bigcup_{i=1}^{n} \text{Range}\,(\varphi_i))$.

(3.3) **Definition.** Let $\{A_i\}_{i \in N}$ be a sequence of disjoint subsets of N_0. The sequence $\{[A_i]\}_{i \in N}$ of disjoint subsets of N and the set A are defined as follows:

a) if $p \in A_i$, then $p \in [A_i]$, $\quad p \in N_0$, $i \in N$;

b) if $p_1 \in [A_{i_1}] \,\&\ldots\&\, p_{n+1} \in [A_{i_{n+1}}] \,\&\, \theta_k(i_1, \ldots, i_{n+1}) = m$, $1 \leqq k \leqq n$, then $\varphi_k(p_1, \ldots, p_{n+1}) \in [A_m]$;

c) $A = \bigcup_{i \in M} [A_i]$. (We remind that $M = \{2k \mid k \in N\}$.)

Note. If the set $\{(p, i) \mid p \in A_i\}$ is r.e., then A is also r.e. In this case we say that we have a r.e. sequence of r.e. sets.

We can prove the following lemma:

(3.4) **Lemma.** *Let $\{A_i\}_{i \in N}$ be a sequence of disjoint subsets of N_0 and the set A be obtained by Definition* (3.3). *Then for any $n+1$ natural numbers $p_1, \ldots, p_{n+1}$ it is true that*

$$p_1 \in A \,\&\ldots\&\, p_{n+1} \in A \iff \varphi_1(p_1, \ldots, p_{n+1}) \in A \,\&\ldots\&\, \varphi_n(p_1, \ldots, p_{n+1}) \in A.$$

(3.5) **Corollary.** *Let $\{A_i\}_{i \in N}$ be a sequence of disjoint subsets of N_0 and the set A be obtained by Definition* (3.3). *Then $A^{n+1} \leqq_m A^n$.*

(3.6) **Definition.** We define a correspondence between two terms τ and σ as follows:

a) if τ has not a subterm that is a constant, then τ corresponds with $\sigma = \tau$;

b) if $c_{i_1}, \ldots, c_{i_k}$ are all the subterms of τ which are constants, $i_1 \in [A_{r_1}], \ldots, i_k \in [A_{r_k}]$, then τ corresponds with the term σ, where σ is obtained from τ by replacing c_{i_j} with c_{r_j} for any $j \in \{1, \ldots, k\}$.

We need this correspondence to have the following

(3.7) **Lemma.** *If $\{A_i\}_{i \in N}$ is a sequence of disjoint subsets of N_0, τ is a term, $p_1, \ldots, p_n$ are arbitrary natural numbers, and:*

a) *for any constant c_j which is a subterm of τ we have a number $i \in N$ such that $j \in [A_i]$;*

b) *τ corresponds with σ by Definition* (3.6);

and

c) $p_1 \in [A_{i_1}] \& \ldots \& p_n \in [A_{i_n}] \& \sigma_U(i_1, \ldots, i_n) = m$,

then $\tau_V(p_1, \ldots, p_n) \in [A_m]$.

(3.8) **Lemma.** *Let $V = \langle N, \varphi_1, \ldots, \varphi_n \rangle$. For any natural number x there is an effective way to find a term τ that have not subterms which are variables (for $x \notin N_0$, $\tau \neq$ constant c_x) such that $x = \tau_V(x_1, \ldots, x_n)$ and the values of all the constants — subterms of τ, belong to N_0.*

We prove this lemma by defining the function $\|z\|$ for any $z \in N$:

$$(3.9) \qquad \begin{aligned} &1) \text{ if } z \in N_0, \quad \text{then } \|z\| = 0; \\ &2) \text{ if } z = \varphi_1(z_1, \ldots, z_{n+1}), \quad \text{then } \|z\| = 1 + \max_{1 \leqq j \leqq n+1} \|z_j\|, \ 1 \leqq i \leqq n, \end{aligned}$$

and applying induction on $\|x\|$.

We shall build a r.e. sequence of disjoint r.e. subsets $A_0, A_1, A_2, \ldots$ of N_0 such that for the set A obtained by Definition (3.3) it holds

$$(3.10) \qquad A^n \not\equiv_m A^{n-1}.$$

Then it follows from Lemma (1.7) and Corollary (3.5) that A is the set we need.

We shall build the sets $\{A_i\}_{i \in N}$ by steps — at any step s we build $\{A_i^{(s)}\}_{i \in N}$, ensuring that $(A^{(s)})^n$ is not m-reducible to $(A^{(s)})^{n-1}$ by the e-th recursive function, $e = J_1^2(s)$. ($A^{(s)}$ is obtained from $\{A_i^{(s)}\}_{i \in N}$ by Definition (3.3).)

At the end we take $A_i = \bigcup_{s=0}^{\infty} A_i^{(s)}$, $i \in N$. For this purpose at any step s we shall find numbers $x_1, \ldots, x_n$ such that if φ_e is the e-th partial recursive function, $e = J_1^2(s)$, $J^{(n)}(x_1, \ldots, x_n) \in \operatorname{dom} \varphi_e$ and $\varphi_e(J^{(n)}(x_1, \ldots, x_n)) = J^{(n-1)}(z_1, \ldots, z_{n-1})$, one of both conditions holds:

(i) $x_1 \in A \& \ldots \& x_n \in A \& \exists i_{1 \leqq i \leqq n-1}(z_i \notin A)$;

(ii) $(x_1 \notin A \vee \ldots \vee x_n \notin A) \& z_1 \in A \& \ldots \& z_{n-1} \in A$.

If for this purpose we put the numbers $x_1, \ldots, x_n$ in some sets $A_{i_1}, \ldots, A_{i_n}$, we create a positive e-requirement $\{x_1, \ldots, x_n\}$, and if some numbers $y_1, \ldots, y_k$

must not belong to some set, we create a negative e-requirement $\{y_1, \dots, y_k\}$. We shall use the priority argument: if at the step s we need one number x to belong to some set and at a step t — not to belong, the smaller between $J_1^2(s)$ and $J_1^2(t)$ has a priority. So, when we choose $x_1, \dots, x_n$ at the step s, they must not belong to any negative requirements created at some steps $t < s$ such that $J_1^2(t) < J_1^2(s)$, but they may belong to negative requirements created at steps $r < s$ such that $J_1^2(r) > J_1^2(s)$. In the second case the $J_1^2(r)$-requirement is injured and we need at some later step r', $J_1^2(r') = J_1^2(r)$, to create a new $J_1^2(r)$-requirement.

If one $J_1^2(s)$-requirement is not injured at a step $r > s$, it is called active at this step. If it is active at every step $r > s$, it is called constant. At the end we shall prove that for any e the condition ((i) $\vee$ (ii)) is injured only finite times.

Now we shall describe the construction of the sets $\{A_i\}_{i \in N}$.

(3.11) S t e p $s = 0$. Let $N_0 = N_1 \cup N_2$, where N_1 and N_2 are infinite disjoint recursive sets and $N_2 = \{a_0 < a_1 < \cdots\}$. Let r' be a monotonically increasing function such that $\mathrm{Ran}\,(r') = N_1$ and $r(x) = r'(n.2^x + x)$. Let

$$\varphi_{e,s}(x) \cong \begin{cases} \varphi_e(x), & \text{if } x \in \operatorname{dom} \varphi_e \text{ and } \varphi_e(x) \text{ is countable for less than } s \text{ steps,} \\ \text{not defined, otherwise.} \end{cases}$$

Let $A_i^{(0)} = \{a_i\}$, $i \in N$. So, all the sets are not empty.

S t e p $s > 0$. $e = J_1^2(s)$. First we verify whether there exists an active e-requirement. If such a requirement exists, then we do nothing, i.e. we take $A_i^{(s)} = A_i^{(s-1)}$, $i \in N$. Otherwise we verify whether there exist $x_1 \in N_1, \dots, x_n \in N_1$, $x_1 > r(e) \,\&\dots\&\, x_n > r(e)$, $J^n(x_1, \dots, x_n) \in \operatorname{dom} \varphi_{e,s}$, belonging neither to $\bigcup_{i \in N} A_i^{(s-1)}$ nor to any active negative requirement, created at a step $t < s$ such that $J_1^2(t) < J_1^2(s)$. If such numbers do not exist, we do nothing. If there are such numbers $x_1, \dots, x_n$, we take the smallest — $x_1^{(e)}, \dots, x_n^{(e)}$. From the choice of J^n there exist $z_1, \dots, z_{n-1}$ such that

$$\varphi_e(J^n(x_1^{(e)}, \dots, x_n^{(e)})) = J^{n-1}(z_1, \dots, z_{n-1}).$$

It follows from Lemma (3.8), applied for $z_1, \dots, z_{n-1}$, that there exist terms $\psi^1, \dots, \psi^{n-1}$ such that

(3.12) $$z_i = \psi_V^i(x_1, \dots, x_n), \quad i = 1, \dots, n-1.$$

We consider these numbers among $z_1, \dots, z_{n-1}$ for which the constants-subterms of the corresponding terms $\psi^1, \dots, \psi^{n-1}$ already belong to some sets $A_i^{(s-1)}$, $i \in N$. Let that be $z_1, \dots, z_q$.

C a s e I. $\exists i_{1 \leqq i \leqq q}\, \exists k \in N\,(z_i \in [A_{2k+1}^{(s-1)}])$, i.e. $z_i \notin A$ and in this case we satisfy (i). Let $A_{2j}^{(s)} = A_{2j}^{(s-1)} \cup \{x_j^{(e)}\}$, $j = 1, \dots, n$, and $A_l^{(s)} = A_l^{(s-1)}$ for $l \notin \{2, 4, 6, \dots, 2n\}$. We create a positive e-requirement $\{x_1^{(e)}, \dots, x_n^{(e)}\}$, which is also constant.

C a s e II. $z_1 \in A^{(s)} \,\&\dots\&\, z_q \in A^{(s)}$.

II.1. $q = n - 1$. Then we satisfy (ii).

Let $A_j^{(s)} = A_j^{(s-1)} \cup \{x_j^{(e)}\}$, $j = 1, \ldots, n$, and $A_l^{(s)} = A_l^{(s-1)}$ for $l \notin \{1, \ldots, n\}$. We create a positive e-requirement $\{x_1^{(e)}, \ldots, x_n^{(e)}\}$, which is also constant.

II.2. $q < n - 1$.

II.2.1. $\exists p_{1 \leqq p \leqq n-1}$ (ψ_p has a subterm-constant with a value y_j^p,

$$y_j^p \notin (\bigcup_{i \in N} A_i^{(s-1)} \cup \{x_1^{(e)}, \ldots, x_n^{(e)}\})).$$

For simplicity let $p = n - 1$. We satisfy (i) — we put $x_1^{(e)}, \ldots, x_n^{(e)}$ into A and create a negative requirement such that z_{n-1} stays always out of A.

We take $A_{2j}^{(s)} = A_{2j}^{(s-1)} \cup \{x_j^{(e)}\}$, $j = 1, \ldots, n$, and $A_l^{(s)} = A_l^{(s-1)}$ for $l \notin \{2, 4, 6, \ldots, 2n\}$. We create a positive e-requirement $\{x_1^{(e)}, \ldots, x_n^{(e)}\}$ and a negative e-requirement $\{y_j^{n-1}\}$;

II.2.2. $\forall p_{1 \leqq p \leqq n-1}$ (ψ_p has a subterm-constant with value y_j^p, $y_j^p \notin \bigcup_{i \in N} A_i^{(s-1)}$ $\Rightarrow y_j^p \in \{x_1^{(e)}, \ldots, x_n^{(e)}\}$). We find the terms $\psi^1, \ldots, \psi^{n-1}$ corresponding to the terms $\tau^1, \ldots, \tau^{n-1}$ according to Definition (3.6). From Theorem 1 for the terms $\tau^1, \ldots, \tau^{n-1}$ there exist natural numbers $i_1, \ldots, i_n$ such that

$$i_1 \in M \,\&\, \ldots \,\&\, i_n \in M \iff \tau_U^1(i_1, \ldots, i_n) \in M \,\&\, \ldots \,\&\, \tau_U^{n-1}(i_1, \ldots, i_n) \in M$$

does not hold, i.e. one of the following holds:

(3.13) $i_1 \in M \,\&\, \ldots \,\&\, i_n \in M \,\&\, (\tau_U^1(i_1, \ldots, i_n) \notin M \vee \ldots \vee \tau_U^{n-1}(i_1, \ldots, i_n) \notin M)$,

(3.14) $(i_1 \notin M \vee \ldots \vee i_n \notin M) \,\&\, \tau_U^1(i_1, \ldots, i_n) \in M \,\&\, \ldots \,\&\, \tau_U^{n-1}(i_1, \ldots, i_n) \in M$.

Let $m_j = \tau_U^j(i_1, \ldots, i_n)$, $1 \leqq j \leqq n - 1$. We take: $A_j^{(s)} = A_j^{(s-1)} \cup \{x_j^{(e)}\}$, $j \in \{i_1, \ldots, i_n\}$, and $A_l^{(s)} = A_l^{(s-1)}$ for $l \notin \{i_1, \ldots, i_n\}$.

If (3.13) holds, then by Definition (3.3c) we have

$$x_1^{(e)} \in A^{(s)} \,\&\, \ldots \,\&\, x_n^{(e)} \in A^{(s)}$$

and from Lemma (3.7):

$$z_i = \psi_V^i(x_1^{(e)}, \ldots, x_n^{(e)}) \in [A_{m_j}^{(s)}],$$

and therefore $z_1 \notin A^{(s)} \vee \ldots \vee z_{n-1} \notin A$, so in this case (i) is true.

If (3.14) holds, then $m_1 \in M \,\&\, \ldots \,\&\, m_{n-1} \in M$ and therefore $z_1 \in A^{(s)} \,\&\, \ldots \,\&\, z_{n-1} \in A^{(s)}$. We have also $i_1 \notin M \vee \ldots \vee i_n \notin M$ and then $x_1^{(e)} \notin A^{(s)} \vee \ldots \vee x_n^{(e)} \notin A^{(s)}$, i.e. (ii) is true.

We create a positive e-requirement $\{x_1^{(e)}, \ldots, x_n^{(e)}\}$, which is also constant.

At the end we take $A_i = \bigcup_{s=0}^{\infty} A_i^{(s)}$, $i \in N$. Now we have to prove that the set A satisfies the condition of our problem for n.

(3.15) **Lemma.** *For any e-number of a p.r. function the condition ((i) $\vee$ (ii)) is injured only $2^e - 1$ times, i.e. we create not more than 2^e e-requirements.*

Proof. We shall use induction on e.

1) For $e = 0$ we have that the e-requirement can not be injured, because there is no requirement with higher priority and $2^0 - 1 = 0$.

For $e = 1$ we can injure the e-requirement only once when we create the single 0-requirement (if it exists) and $2^1 - 1 = 1$.

2) Let the statement hold for all the numbers smaller than e.

3) Let e be a number of a p.r. function. The condition $((\mathrm{i}) \vee (\mathrm{ii}))$ for e is injured when a requirement with number between 0 and $e-1$ is created, i.e. not more than $2^0 + 2^1 + \cdots + 2^{e-1} = 2^e - 1$ times.

The lemma is proved.

(3.16) **Lemma.** *The set $N_1 \setminus A$ is infinite.*

Proof. Let $(N_1)_x = \{y \mid y \in N_1 \,\&\, y < x\}$. We shall prove that the set $(N_1)_{r(x)} \cap (N_1 \setminus A)$ contains at least x elements.

$$(N_1)_{r(x)} = \{y \mid y \in N_1 \,\&\, y < r(x)\} = \{y \mid y \in N_1 \,\&\, y < r'(n.2^x + x)\}$$

and because $r'(0) < r'(1) < \cdots < r'(n.2^x + x - 1) < r'(n.2^x + x) < \cdots$, the elements of the set $(N_1)_{r(x)}$ are $n.2^x + x$. Between them only 0-, 1-, ..., $x-1$-requirements may be elements of A (the others are bigger than $r(x)$), i.e. not more than $n.2^0 + n.2^1 + \cdots + n.2^{x-1} = 2^x$. Therefore, the elements of $(N_1)_{r(x)} \cap (N_1 \setminus A)$ are at least $n.2^x + x - n.2^x = x$ and the lemma is proved.

(3.17) **Lemma.** *For any natural number e such that $N_1 \subseteq \operatorname{dom}\varphi_e$ (and especially for any e which is a number of recursive function) there exists a constant e-requirement.*

Proof. Let $e \in N$ and $N_1^n \subseteq \operatorname{dom}\varphi_e$. Let us assume that there is not a constant e-requirement. We find s_0 such that at the step s_0 all e_1-requirements for $e_1 < e$ are already built.

From Lemma (3.16) we have that there exist $x_1 \in N_1 \setminus A, \ldots, x_n \in N_1 \setminus A$ such that $x_1 > r(e) \,\&\, \ldots \,\&\, x_n > r(e)$. Let $s > s_0$ and $J^n(x_1, \ldots, x_n) \in \operatorname{dom}\varphi_{e,s}$. There at the step s a constant e-requirement is created.

The lemma is proved.

Let A be obtained from $\{A_i\}_{i \in N}$ according to Definition (3.3). According the construction, A is a r.e. set, $A^n \not\equiv_m A^{n-1}$, and $A^{n+1} \leq_m A^n$, i.e. the needed set is built.

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