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INFINITESIMAL BENDINGS OF ROTATIONAL SURFACES WITH CHANGING SIGNS CURVATURE*

IVANKA IVANOVA-KARATOPRAKLIEVA

Иванка Иванова-Каратопраклиева. БЕСКОНЕЧНО МАЛЫЕ ИЗГИБАНИЯ ПОВЕРХНОСТЕЙ ВРАЩЕНИЯ ЗНАКОПЕРЕМЕННОЙ КРИВИЗНЫ

Иccследовано множество либманновых параллелей первого порядка нежёсткой поверхности вращения S знакопеременной кривизны K. S — замкнутая (рода 0 либо 1) либо с краем. Доказано, что на S, вне её частей, которые являются круговыми цилиндрами, имеется счетное множество либманновых параллелей, если S имеет бесконечное число нетривиальных фундаментальных полей изгибания. На каждом поясе с $K < 0$ эти параллели расположены везде плотно. На каждом поясе $S_0 = S_{L_0 L_1}$ с $K \geqq 0$, ограниченной асимптотической параллелью L_0, существуют либманновые параллели тогда и только тогда, когда S_0 содержит подпояс $\widehat{S}_0 = S_{L^* L_1}$ (L^* самая правая максимальная параллель на S_0). Все эти параллели образуют счетное множество, принадлежат $\widehat{S}_0$ и сгущаются к L^*. Даны достаточные условия для жесткости S.

Ivanka Ivanova-Karatopraklieva. INFINITESIMAL BENDINGS OF ROTATIONAL SURFACES WITH CHANGING SIGNS CURVATURE

The set of Liebmann's parallels of first order on a non-rigid rotational surface S with changing signs curvature K is investigated. S is closed (of genus 0 or 1) or with a boundary. It is proved that there is a countable set of Liebmann's parallels on S outside of its parts which are circular cylinders if S has got an infinite number non-trivial fundamental fields of bending. On each belt with $K < 0$ these parallels are everywhere densely. On each belt $S_0 = S_{L_0 L_1}$ with $K \geqq 0$, bordered by an asymptotic parallel L_0, there exist Liebmann's parallels if and only if S_0 contains a subbelt $\widehat{S}_0 = S_{L^* L_1}$ (L^* is the most right maximal parallel of S_0). The Liebmann's parallels

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a subbelt $\widehat{S}_0 = S_{L^*L_1}$ (L^* is the most right maximal parallel of S_0). The Liebmann's parallels on S_0 are a countable set, belong to $\widehat{S}_0$ and are condensed to L^*. Some sufficient conditions for rigidity of S are given.

1. PRELIMINARIES

If S is a rotational surface with changing signs curvature, then the domains with positive Gaussian curvature on S are separated from the domains with negative Gaussian curvature by belts with zero curvature or by parabolic parallels, i.e. parallels on which the Gaussian curvature of S is zero. Those parallels are from first, second or third type [1]. A parallel from first type is described by a point of rectification of the meridian c of S (a point of inflection or not) at which the tangent of c is not perpendicular to the rotational axis. The principal curvatures of S at an arbitrary point of a parabolic parallel from first type are $\nu_{\text{mer}} = 0$, $\nu_{\text{par}} \neq 0$. A parabolic parallel from second type is described by such a point of c which is not a point of rectification but the tangent of c at this point is perpendicular to the rotational axis. We have $\nu_{\text{mer}} \neq 0$, $\nu_{\text{par}} = 0$ at an arbitrary point of such a parallel. A parabolic parallel from third type is described by a point of rectification of c (a point of inflection or not) at which the tangent to c is perpendicular to the rotational axis too. Any point of such a parallel is planar one for $S\,(\nu_{\text{mer}} = \nu_{\text{par}} = 0)$. Any parabolic parallel L_0 from second (respectively third) type is an asymptotic line of S because the plane of L_0 is tangent of first (respectively higher) order to the surface at any point of L_0. That is why we shall call the parabolic parallels from second and third type in short asymptotic parallels.

Let S be an arbitrary rotational surface with not more than a finite number of asymptotic parallels. S can be closed (of genus zero or one) or with a boundary (consisting of one or two parallels). Let S be from the class C^q, $q \geqq 2$, out of its poles (if it has such ones).If the surface has not got any planar domains, so its meridian c can be represented as a union of a finite number of arcs such that each of them can be projected one-to-one on the rotational axis. If the surface has got some planar domains, so such a representation is possible for c without those of its parts which are segments, perpendicular to the rotational axis (exactly, they describe the planar domains of S by the rotation of c around the rotational axis).

Let the meridian c of S be in the co-ordinate plane Ouy and let it has got a finite number of points of inflection. If the point $P_0 \in c$ describes an asymptotic parallel L_0, then either a) P_0 is a point of inflection (see Fig. 1), or b) P_0 is not a point of inflection (see Fig. 2). Let us note that in the case a) there is a two-sided neighbourhood on c which can be projected one-to-one on the rotation axis and in the case b) there is not such a neighbourhood. We denote by c_1 and c_2 the arcs of c bordering on P_0 which can be projected one-to-one on the rotation axis. We shall consider only the case when c_1 and c_2 have not inner points which describe asymptotic parallels because the other case obviously is reduced to that one. We assume that in a neighbourhood of the point $P_0(u_0, r_0)$ the meridian c, i. e. c_1 and

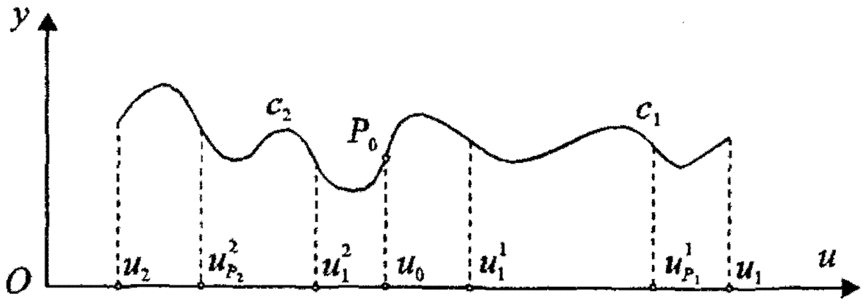

Fig. 1

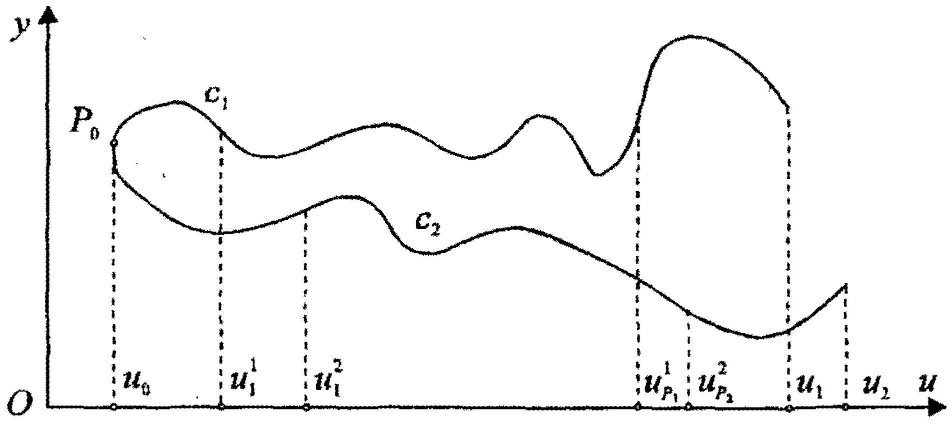

Fig. 2

c_2, has a representation

$$u=(\pm y\mp r_0)^n f_{1,2}(y)+u_0,\ n\geqq 2,\quad f_{1,2}(r_0)\neq 0,$$

(1)

$$f_1\in C^A[r_0,\ r_0+\varepsilon],\quad f_2\in C^A[r_0-\varepsilon,\ r_0].$$

Then we have

(2) $$c_j\ :\ y=r_j(u),\ j=1,\ 2,\ r_1(u_0)=r_2(u_0),\ r_1(u)\in C[u_0,\ u_1]\cap C^q(u_0,\ u_1),$$
$$r_2(u)\in C[u_2,\ u_0]\cap C^q(u_2,\ u_0),\quad \lim_{u\to u_0} r'_{1,2}(u)=+\infty,$$

and in a neighbourhood of u_0

$$r_1(u)=(u-u_0)^{n_1}\,\tilde\varphi_1(u)+r_0,\quad r_2=(u_0-u)^{n_1}\,\tilde\varphi_2(u)+r_0,\quad n_1=\frac{1}{n},$$

$$\tilde\varphi_j(u_0)\neq 0,\ j=1,\ 2,\quad \tilde\varphi_1(u)\in C^q[u_0,\ u_0+\varepsilon],\quad \tilde\varphi_2(u)\in C^q[u_0-\varepsilon,\ u_0],\ q\geqq 2,$$

when P_0 is a point of inflection, and

(3) $$c_j\ :\ y=r_j(u),\ j=1,\ 2,\ r_1(u_0)=r_2(u_0),\ r_1(u)>r_2(u)$$
$$\text{for } u\in(u_0,\ u_1]\cap(u_0,\ u_2],$$
$$r_{1,2}(u)\in C[u_0,\ u_{1,2}]\cap C^q(u_0,\ u_{1,2}),\ \lim_{u\to u_0} r'_{1,2}(u)=\pm\infty,$$

and in a neighbourhood of u_0

$$r_j(u)=(u-u_0)^{n_1}\,\tilde\varphi_j(u)+r_0,\ \tilde\varphi_j(u_0)\neq 0,\ n_1=\frac{1}{n},\ \tilde\varphi_j(u)\in C^q[u_0,\ u_0+\varepsilon],$$
$$j=1,\ 2,\ q\geqq 2,$$

when P_0 is not a point of inflection.

If the meridian c has other points which describe asymptotic parallels, then in a neighbourhood of any of them we assume that analogical conditions to those in (1) are satisfied. Finally, let us amplify that if the surface S has got one or two poles $P_0^{1,2}(u_0^{1,2},\, 0)$ — smooth or conic, then we assume that in a neighbourhood of any of them analogical conditions to those for c_1 in (1) (see [3, 5]) are satisfied.

We assume that the surface S is non-rigid of first order with a field U of infinitesimal bending (inf.b.) which is continuous on the whole surface and belongs to the class C^1 out of its poles (if S has such ones). It is well-known that such non-rigid of first order rotational surfaces — closed or with a boundary, exist and each of them, which has got asymptotic parallels, is rigid of second order (see for example [1 — 4]).

In this paper we shall investigate the set of Liebmann's parallels of first order on S, i. e. those parallels which remain in their planes by inf.b. of first order. We shall give some sufficient conditions for rigidity of S too.

2. PROPERTIES OF THE FUNDAMENTAL FIELDS $U_k(u,\, v)$, $k \geqq 2$

We represent the parts $S_j \subset S$ obtained by rotation of the arcs $c_j \subset c$, $j = 1, 2$ (see Fig. 1 and 2) with the vectorial parametric equation

$$x(u,\, v) = u.e + r(u).a(v) \tag{4}$$

(here for simplicity we have denoted $r_j(u)$, $j = 1,\, 2$, with $r(u)$), where: u belongs to the indicated in (2) and (3) intervals, $v \in [0,\, 2\pi]$, e is the unit vector of the rotational axis Ou, and $a(v)$ is a unit vector perpendicular to Ou and twisted at an angle v from Oy). Let $U_k(u,\, v)$, $k \geqq 2$ be a non-trivial fundamental field of inf. b. of first order of the surface S. Then [1] we have on S_j, $j = 1,\, 2$,

$$\begin{aligned} U_k(u,\, v) &= e^{ikv}\, [\varphi_k(u).e + \chi_k(u).a + \psi_k(u).a'] \\ &\quad + e^{-ikv}\, [\overline{\varphi}_k(u).e + \overline{\chi}_k(u).a + \overline{\psi}_k(u).a'] , \end{aligned} \tag{5}$$

$$\begin{aligned} &\varphi_k'(u) + r'(u)\, \chi_k'(u) = 0, \\ &\chi_k(u) + i\, k\, \psi_k(u) = 0, \\ &i\, k\, \varphi_k(u) + r'(u)\, [i\, k\, \chi_k(u) - \psi_k(u)] + r(u)\, \psi_k'(u) = 0, \quad k \geqq 2, \end{aligned} \tag{6}$$

from where we obtain for the function $\chi_k(u)$ the differential equation

$$r(u)\, \chi_k''(u) + (k^2 - 1)\, r''(u)\, \chi_k(u) = 0, \quad k \geqq 2. \tag{7}$$

Using the condition (1), we obtain

$$\begin{aligned} &u'(y)\, \varphi_k'(y) + \chi_k'(y) = 0, \\ &\chi_k(y) + i\, k\, \psi_k(y) = 0, \\ &i\, k\, u'(y)\, \varphi_k(y) + i\, k\, \chi_k(y) - \psi_k(y) + y\, \psi_k'(y) = 0, \quad k \geqq 2, \end{aligned} \tag{6'}$$

and

$$(7') \qquad y\,u'(y)\,\chi_k''(y) - y\,u''(y)\,\chi_k'(y) - (k^2-1)\,u''(y)\,\chi_k(y) = 0, \quad k \geqq 2,$$

in a neighbourhood of the point $P_0(u_0, r_0)$.

From the equalities (6) and from the assumption that the field U of inf. b. of S belongs to class C^1 out of the poles and the meridian $c \in C^q$, $q \geqq 2$, it follows immediately that the fundamental field $U_k(u, v)$, $k \geqq 2$, of S_j, $j = 1, 2$, belongs to the class C^q, $q \geqq 2$, out of the asymptotic parallel L_0. It is seen from (6′) that $\chi_k(y)|_{y=r_0} = \chi_k'(y)|_{y=r_0} = 0$ and therefore the fundamental field $U_k(u, v)$, $k \geqq 2$, satisfies the equality

$$(8) \qquad \chi_k(u_0) = 0, \quad k \geqq 2,$$

i. e.

$$(8') \qquad U_k(u_0, v) = \left[e^{ikv}\,\varphi_k(u_0) + e^{-ikv}\,\overline{\varphi}_k(u_0)\right].e, \quad k \geqq 2,$$

along the asymptotic parallel L_0.

Since the function $\chi_k(u)$, $k \geqq 2$, is a solution of the equation (7), so in the intervals, where $r''(0) \leqq 0$, it is not oscillating, i. e. it has not more than one null, it has neither a positive maximum nor a negative minimum and its graph is convex to the rotational axis Ou. Let us remind that in these intervals the meridian c is convex above and the corresponding belt of the surface S has got Gaussian curvature $K \geqq 0$. The equation (7) has a singularity in the point u_0. Taking y for an independent variable in a neighbourhood of u_0, (7) passes to the equation (7′) which is from Fuchs' type. We have proved in [4] that the problem (7), (8) has got a non-trivial solution $\chi_k(u)$ and in a neighbourhood of u_0 it has the form

$$(9) \qquad \chi_k(u) = (u-u_0)\,\chi_k^0(u), \quad \chi_k^0(u_0) \neq 0,$$

where $\chi_k^0(u) = \tilde{\varphi}_1^n(u)\,\tilde{P}_1\left[r_0 + (u-u_0)^{n_1}\,\tilde{\varphi}_1(u)\right]$, $\tilde{P}_1$ is an analytic function of $y = r_0 + (u-u_0)^{n_1}\,\tilde{\varphi}_1(u)$.

Remark 1. If the surface S has got a planar domain $\tilde{S}_0$, so $\tilde{S}_0$ is a disk or an annulus bounded by asymptotic parallels $L_0^{1,2}$: $u = u_0^{1,2}$ of S (even all the parallels on $\tilde{S}_0$ are asymptotic). In this case $U_k|_{\tilde{S}_0} \perp \tilde{S}_0$ (see [6]), i. e. $\chi_k(y)|_{\tilde{S}_0} = \psi_k(y)|_{\tilde{S}_0} = 0$ and consequently the condition (8) is satisfied on $L_0^{1,2}$.

Remark 2. If the rotational surface S has got poles $P_0^{1,2}$ so the function $\chi_k(u)$, which correspondes to the non-trivial fundamental field $U_k(u, v)$ of S, also satisfies the equality (8) (see for example [3, 5]).

Lemma 1. *Let $u = \alpha$ and $u = \beta$ be two sequential nulls of $\chi_k(u)$, $k \geqq 2$.*

a) *If the belt $S_{\alpha\beta}$ of S does not contain a subbelt with extremal parallels of S, so the function $\varphi_k(u)$, $k \geqq 2$, has got exactly one null in (α, β).*

b) *If S has got a subbelt $S_{u_1^* u_2^*}$ with extremal parallels, then either $\varphi_k(u)$, $k \geqq 2$, has got exactly one null in (α, β) and it is in $(\alpha, \beta) \setminus [u_1^*, u_2^*]$ or $\varphi_k(u) \equiv 0$, $k \geqq 2$, in $[u_1^*, u_2^*]$ but $\varphi_k(u) \neq 0$ in $(\alpha, \beta) \setminus [u_1^*, u_2^*]$.*

Proof. From (6) we find

$$(10) \qquad \varphi_k(u) = -\frac{r(u)\,\chi_k(u)}{k^2}\,f_k(u), \quad f_k(u) = \frac{\chi_k'(u)}{\chi_k(u)} + \frac{(k^2-1)\,r'(u)}{r(u)}$$

in the interval (α, β), wherefrom we obtain directly

$$f_k'(u) = -\left[\left(\frac{\chi_k'(u)}{\chi_k(u)}\right)^2 + (k^2-1)\left(\frac{r'(u)}{r(u)}\right)^2\right]. \tag{11}$$

From here and from $f_k(\alpha+0) = +\infty$, $f_k(\beta-0) = -\infty$ it follows that in the case a) $\varphi_k(u)$ has got exactly one null in the interval (α, β). The statement in b) follows directly from (6′), (10) and (11).

Lemma 2. *Let the belt $S_{\overline{u}_1\overline{u}_2} \subset S$ has got negative Gaussian curvature and u', u'' are two arbitrary points from the interval $[\overline{u}_1, \overline{u}_2]$. There exists $k_0 \geqq 2$ such that the function $\varphi_k(u)$, $k \geqq k_0$, has got a null in (u', u'') if U_k, $k \geqq k_0$, is a non-trivial field of bending of S.*

Proof. We write the equation (7) for the interval $[\overline{u}_1, \overline{u}_2]$ in the form

$$\chi_k''(u) + G_k(u)\chi_k(u) = 0, \quad k \geqq 2, \tag{12}$$

where

$$G_k(u) = \frac{(k^2-1)\,r''(u)}{r(u)}.$$

We have

$$\min_{\overline{u}_1 \leqq u \leqq \overline{u}_2} G_k(u) \geqq \frac{(k^2-1)\,m}{M}, \tag{13}$$

where $m = \min r''(u)$, $M = \max r(u)$ when $u \in [\overline{u}_1, \overline{u}_2]$. We choose k_0 so that

$$\frac{(k_0^2-1)\,m}{M} > \left(\frac{N\Pi}{u''-u'}\right)^2, \tag{14}$$

where $N \geqq 2$. We consider the equation

$$Y''(u) + \mu^2 Y(u) = 0 \tag{15}$$

with $\mu = \dfrac{N\Pi}{u''-u'}$. Since the solution $Y = \sin\mu(u-u')$ of (15) has got $N+1 \geqq 3$ nulls in $[u', u'']$ and $G_k(u) > \mu^2$ holds for $k \geqq k_0$ in $[u', u'']$ because of (13) and (14), then from the Sturm's theorem it follows that every solution $\chi_k(u)$, $k \geqq k_0$, of (12) has $M_k \geqq N \geqq 2$ nulls in $[u', u'']$. Then from Lemma 1 it follows that every function $\varphi_k(u)$, $k \geqq k_0$, has got at least one null in (u', u'').

Lemma 3. *Let the belt $S_0 \subset S$ has a non-negative Gaussian curvature and $\partial S_0 = L_0 \cup L_1$, where L_0 is the asymptotic parallel described by the point $P_0(u_0, r_0)$ and L_1 is the parallel*[1] *described by the neighbour point of inflection $P_1(u_1^1, r_1^1)$ of P_0. At each fixed $u \in (\tilde{u}_0, u_1^1]$, $\tilde{u}_o > u_0$, for which $r(u) > r(\tilde{u}_0)$ the following property is true:*

$$\text{the number sequence } \frac{\chi_k'(u)}{\sqrt{(k^2-1)}\,\chi_k(u)} \text{ is bounded} \tag{16}$$

[1] L_1 is a non-asymptotic parallel since the surface S is non-rigid (see [2]).

if each U_k, $k \geqq 2$, is a non-trivial field of bending of S.

Proof. Let $\chi_{k_1}(u)$ and $\chi_{k_2}(u)$ are solutions of the problem (7), (8) at $k = k_1$ and $k = k_2$, correspondingly, $k_1 < k_2$. We multiply (7) at $k = k_1$ by $(k_2^2 - 1)\,\chi_{k_2}(u)$ and (7) at $k = k_2$ by $(k_1^2 - 1)\,\chi_{k_1}(u)$, subtract the obtained equalities and integrate the result from u_0 to $u \in (u_0,\, u_1^1]$. We obtain

$$(k_2^2 - 1)(k_1^2 - 1)\,\chi_{k_2}(u)\,\chi_{k_1}(u)\left(\frac{\chi'_{k_1}(u)}{(k_1^2 - 1)\,\chi_{k_1}(u)} - \frac{\chi'_{k_2}(u)}{(k_2^2 - 1)\,\chi_{k_2}(u)}\right)$$
$$= (k_2^2 - k_1^2)\int\limits_{u_0}^{u} \chi'_{k_1}(u)\,\chi'_{k_2}(u)\,du.$$

Since $\chi_k(u)\,\chi'_k(u) > 0$ in $(u_0,\, u_1^1]$, we conclude from here that at each fixed $u \in (u_0,\, u_1^1]$ the inequality

$$\frac{\chi'_{k_1}(u)}{(k_1^2 - 1)\,\chi_{k_1}(u)} > \frac{\chi'_{k_2}(u)}{(k_2^2 - 1)\,\chi_{k_2}(u)} \quad \text{for } k_1 < k_2 \tag{17}$$

holds.

Multiplying (7) by $2\chi'_k(u)$ we obtain

$$(r(u)\,\chi_k'^{\,2}(u))' = \left[r'(u)\,\frac{\chi'_k(u)}{(k^2 - 1)\,\chi_k(u)} - 2\,r''(u)\right](k^2 - 1)\,\chi_k(u)\,\chi'_k(u). \tag{18}$$

Let $\tilde{u}_0 > u_0$, $\overline{u} \in (\tilde{u}_0,\, u_1^1]$, and $N \geqq 2$ be an integer. Since $\chi_k(u)$, $k \geqq 2$, has not a null in $(u_0,\, u_1^1]$, so there exists a constant $M > 0$ such that

$$\frac{\chi'_N(u)}{(N^2 - 1)\,\chi_N(u)} < M \quad \text{in } [\tilde{u}_0,\, \overline{u}].$$

From here because of (17) we have

$$\frac{\chi'_k(u)}{(k^2 - 1)\,\chi_k(u)} < M \quad \text{in } [\tilde{u}_0,\, \overline{u}] \text{ for each } k \geqq N. \tag{19}$$

From (18) and (19) we obtain

$$\begin{aligned}(r(u)\,\chi_k'^{\,2}(u))' &< (|r'(u)|\,M - 2\,r''(u))\,(k^2 - 1)\,\chi_k(u)\,\chi'_k(u)\\ &< 2\,M_1(k^2 - 1)\,\chi_k(u)\,\chi'_k(u),\end{aligned} \tag{20}$$

where M_1 is a suitable constant. Integrating (20) from $\tilde{u}_0$ to $u \in (\tilde{u}_0,\, u_1^1]$ we find

$$\frac{\chi_k'^{\,2}(u)}{(k^2 - 1)\,\chi_k^2(u)}\left[1 - \frac{r(\tilde{u}_0)}{r(u)}\,\frac{\chi_k'^{\,2}(\tilde{u}_0)}{\chi_k'^{\,2}(u)}\right] < \frac{M_1}{r(u)}\left[1 - \frac{\chi_k^2(\tilde{u}_0)}{\chi_k^2(u)}\right], \quad k \geqq N. \tag{21}$$

From here and from

$$\frac{\chi_k'^{\,2}(\tilde{u}_0)}{\chi_k'^{\,2}(u)} < 1, \quad \frac{\chi_k^2(\tilde{u}_0)}{\chi_k^2(u)} < 1$$

we obtain

$$\frac{\chi_k'^{\,2}(u)}{(k^2 - 1)\,\chi_k^2(u)}\,[r(u) - r(\tilde{u}_0)] < M_1, \quad u \in (\tilde{u}_0,\, u_1^1],\ k \geqq N, \tag{22}$$

from where the statement in Lemma 3 follows immediately.

Corollary 1. *For each fixed* $u \in (\tilde{u}_0,\, u_1^1]$, $\tilde{u}_0 > u_0$, *such that* $r(u) > r(\tilde{u}_0)$ *it is valid*

$$\lim_{k\to\infty} \frac{\chi_k'(u)}{(k^2-1)\,\chi_k(u)} = 0. \tag{23}$$

Remark 3. If we multiply (7) at $k = k_1$ and at $k = k_2$ by $\chi_{k_2}(u)$ and $\chi_{k_1}(u)$, correspondingly, subtract the obtained equalities and integrate the result from u_0 to $u \in (u_0,\, u_1^1]$, we obtain that the inequality

$$\frac{\chi_{k_1}'(u)}{\chi_{k_1}(u)} < \frac{\chi_{k_2}'(u)}{\chi_{k_2}(u)}, \quad k_1 < k_2, \tag{24}$$

holds at each fixed $u \in (u_0,\, u_1^1]$.

Remark 4. The properties (24), (17), (16) and (23) of the fundamental fields $U_k(u,\,v)$, $k \geqq 2$, are proved by E. Rembs [7] for the case when the belt S_0 is simply connected, i. e. when $\partial S_0 = L_1$, and the point P_0 is a smooth non-parabolic pole of S_0. They are also valid in the case when the pole P_0 is parabolic or conic (see [5] and for a generalization see [2]). Proving these properties here for the doubly-connected belt S_0, we have used the equalities (8). That is why these properties will also be valid in the case when the tangent at P_0 to c is not perpendicular to the rotational axis, i.e. when the parallel L_0 is not asymptotic but the fields of the bending satisfy the conditions (8). From Remark 1 it is clear that we can ensure the conditions (8) sticking the boundary of a disk $\tilde{S}_0$ along L_0 and assuming that the field of inf.b. of the surface $S \cup \tilde{S}_0$ is continuous on it and from class C^1 on S and $\tilde{S}_0$.

Remark 5. If the belt S_0 is obtained for $u \in [u_1^1,\, u_0]$, i. e. if instead of c_1 and c_2 at Fig. 1 and 2 we have theirs orthogonally symmetric curves with respect to the line by P_0, which is parallel to the axis Oy, then obviously Lemma 3 and Corollary 1 — the properties (16) and (23), are valid too, but $\chi_k(u)\,\chi_k'(u) < 0$ and the inequalities (24) and (17) are inverted.

3. MAIN RESULTS

If $U_k(u,\,v)$, $k \geqq 2$, is a fundamental field of inf.b. of the surface S for which the parallel $\widehat{L} : u = \widehat{u}$ is Liebmann's, i.e. $\widehat{L}$ remains in its plane, then we say that $U_k(u,\,v)$ is a field of inf.b. with sliding along $\widehat{L}$. It is clear from (5) and (6) that $U_k(u,\,v)$ is a field of inf.b. with sliding along $\widehat{L}$ exactly when

$$\varphi_k(u)|_{\widehat{L}} = 0, \quad k \geqq 2. \tag{25}$$

The following statements are valid:

Theorem 1. *On the rotational surface S, outside of her belts of extremal parallels (if S has got such belts) there exists a countable set of Liebmann's parallels. Moreover:*

a) *On each belt with negative Gaussian curvature the Liebmann's parallels are everywhere densely;*

b) *There are Liebmann's parallels on every belt S_0 with non-negative Gaussian curvature, which belt is simply connected with a pole and a boundary $\partial S_0 = L_1$, or doubly connected with a boundary $\partial S_0 = L_0 \cup L_1$, where L_0 is an asymptotic parallel, if and only if S_0 contains a subbelt $\widehat{S}_0 = S_{L^* L_1}$ (respectively, $\widehat{S}_0 = S_{L_1 L^*}$) bounded by the most right (respectively, the most left) maximal parallel L^* of S_0 and the parallel L_1. All these Liebmann's parallels are a countable set, belong to $\widehat{S}_0$ and are condensed to L^* if S has got an infinite number non-trivial fundamental fields of bending*[2].

Corollary 2. *The surface S is rigid with respect to inf.b. with sliding along an asymptotic parallel of S.*

Corollary 3. *The surface S is rigid with respect to inf.b. with sliding along a parallel $\widehat{L} \in S_0$ if the belt S_0 has not got a subbelt $\widehat{S}_0$, and along a parallel $\widehat{L} \in S_0 \backslash \widehat{S}_0$ if the belt S_0 has a subbelt $\widehat{S}_0$.*

Proof. These statements follow directly from the lemmas. In fact, the existance of a countable set of Liebmann's parallels on S follows from the condition (25), Lemma 1 and from the facts that the non-trivial fundamental fields $U_k(u, v)$, $k \geqq 2$, of S are a countable set and any function $\chi_k(u)$, $k \geqq 2$, can have only a finite number nulls in a closed interval. The statement a) follows from (25) and Lemma 2. We shall pause in detail on the proof of the statement b).

For concreteness let S_0 be obtained by $u \in (u_0,\, u_1^1]$. If the belt S_0 is simply connected, so the statement b) is well-known (see [3, 5, 7]). Let S_0 be a doubly connected belt. It is seen from (10) that the function $\varphi_k(u)$, $k \geqq 2$, is annuled in $(u_0,\, u_1^1)$ if and only if $f_k(u)$, $k \geqq 2$, is annuled. Because of (11) the function

$$f_k(u) = (k^2 - 1)\left[\frac{\chi_k'(u)}{(k^2-1)\,\chi_k(u)} + \frac{r'(u)}{r(u)}\right] \tag{26}$$

is monotonically decreasing, as $f_k(u_0 + 0) = +\infty$ and $f_k(u^*) > 0$ for each $k \geqq 2$, where $L^* \,:\, u = u^*$ is the most right maximal parallel of S_0. Since (26) and Corollary 1 hold and $\dfrac{r'(u)}{r(u)} < 0$ in $(u^*,\, \overline{u}]$, $\overline{u} \leqq u_1^1$, it follows that for each fixed $\overline{u} \in (u^*,\, u_1^1]$ such that $r(\overline{u}) > r(\tilde{u}_0)$ there exists an integer $N_1 \geqq 2$ such that $f_k(\overline{u}) < 0$ for any $k \geqq N_1$. Consequently, for each $k \geqq N_1$ there exists $u_k \in (u^*,\, \overline{u})$ such that $f_k(u_k) = 0$, i. e. $\varphi_k(u_k) = 0$. Moreover, if $k_1 < k_2$, then from (17) and from the fact that $\dfrac{r'(u)}{r(u)}$ is a monotonically decreasing function in $[u^*,\, u_1^1]$ it follows $u_{k_2} < u_{k_1}$. Thus for each $k \geqq N_1$ there exists a Liebmann's parallel L_k in $(u^*,\, \overline{u})$. In addition, for $k_2 > k_1$ the Liebmann's parallel L_{k_2}, corresponding to

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[2] Such surfaces exist — for example, if S is simply connected and it has not an asymptotic parallel, or S is doubly connected with not more than one asymptotic parallel, then it has got a countable number non-trivial fundamental fields of bending (see [2], [3]).

the fundamental field $U_{k_2}(u, v)$, is situated more to the left than the Liebmann's parallel L_{k_1}, corresponding to the fundamental field $U_{k_1}(u, v)$. All these Liebmann's parallels condense to the most right maximal parallel $L^* : u = u^*$ of S_0. In order to verify this it is sufficient to take $\overline{u} = u^* + \varepsilon < u_1^1$, where ε is a small enough positive number, and to repeat the considerations which we have just done.

Remark 6. The statement a) for a rotational surface S with a negative curvature is proved in [8]. There are such investigations in [9] and [11] too, but the formulated results contradict to [8] and to our statement a) here.

Remark 7. The statement in Corollary 2 follows from the well-known lemma of Minagawa and Rado (see [2]). It is proved in [10] (see also [3, 9, 11]) and here we formulate it for completeness. That statement is proved in [9] (see Theorem 5 there) by the method "a, b, c" under a lot of restrictions on the surface.

Remark 8. The statement in Corollary 3 is proved as well in [12] and [9]. In [12] the asymptotic parallel is not of second type (as it is said there) — it is of third type, and in [9] (see Corollary 3 there) the statement is proved by the method "a, b, c" under a lot of restrictions on the surface.

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