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TWO CUT-FREE MODAL SEQUENT CALCULI

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В этой статье рассматривается модальный подход к терминологическим языкам. Вводятся две секвенциальные исчисления для $\mathcal{AL}$ и $\mathcal{ALN}$ и доказывается теорема об устранении сечений для них.

Andrei Arsov. TWO CUT-FREE MODAL SEQUENT CALCULI

In this paper the modal approach to concept languages is considered. Two sequent-style calculi for the modal systems $\mathcal{AL}$ and $\mathcal{ALN}$ are introduced and the cut-elimination property is proved.

1. INTRODUCTION

In recent years there has been a growing interest in presenting formalisms and languages that will be able to express various knowledge in a domain of discourse. One such example are the so-called terminological or concept languages. The brief overview that we will present uses the conventions established in [1]. In the concept languages expressions are built from concepts and roles, which are interpreted as subsets and binary relations on a given universe. Further one can define compound expressions from the primitive concepts using a number of constructs. Two such constructs are intersection and complement of concepts (restricted and unrestricted). Roles are used in the so-called restricted quantification. The restricted quantification of a concept C over a role R gives a concept whose elements x are such that if x is R-connected to an element y, this y is in C.

Another construct used in most concept languages is the number restriction. The number restriction over a role R gives a set of objects or a concept, the elements of which have at least or at most a certain number of R-connections.

As far as we know, the almost obvious connection between modal languages and concept languages was considered for the first time in [5]. In [1] a whole hierarchy of such languages is built — the $\mathcal{AL}$-languages — and their relation with modal languages and systems is used to obtain some complexity results on the satisfaction for the $\mathcal{AL}$-languages. In [4] special modal axiomatic systems are developed for these languages and they are also viewed from the perspective of generalized quantifiers. In the next section we present the precise definition and semantics of the languages we shall be interested in, as well as the axiomatic systems taken from the above mentioned paper.

The axiomatic systems, known up to now, for the terminological languages lack one important feature. What we have in mind is that they are not very suitable for practical derivations. This is quite important since we would like to be able to derive in practice some knowledge that is implicitly embedded on the facts that are known up to a certain moment.

In section 3 we present sequent-style axiomatizations of the validity we have in mind, and we prove one important property of the systems — namely, that the cut-rule can be eliminated from them. It is this feature that makes the derivations in such systems somewhat more feasible.

2. THE MODAL SYSTEMS $\mathcal{AL}$ AND $\mathcal{ALN}$

Definition 2.1. Following Van der Hoek & De Rijke, [4], let us define the basic modal language $\mathcal{AL}$. It has the following elements:

- VAR — a denumerable set of propositional variables;
- $\top$ and $\bot$ — propositional constants;
- $\neg$ and $\wedge$ — classical propositional connectives;
- $\langle R\rangle$ and $[R]$ — modal operators for every binary relation R taken from a collection $\mathcal{R}$.

Now we can define the set Φ of formulas in this language in the following way (we denote the formulas by ϕ, ψ, ...):

- if $p \in VAR$, then $p \in \Phi$ and $\neg p \in \Phi$;
- $\top \in \Phi$ and $\bot \in \Phi$;
- if ϕ_1, $\phi_2 \in \Phi$, then also $(\phi_1 \wedge \phi_2) \in \Phi$;
- if $R \in \mathcal{R}$, then $\langle R\rangle\top \in \Phi$;

- if $R \in \mathcal{R}$ and $\phi \in \Phi$, then $[R]\phi \in \Phi$.

Models for $\mathcal{AL}$ have the form $\mathcal{M} = (W, \{ R \}_{R \in \mathcal{R}}, V)$, where W is a non-empty set, each R is a binary relation on W, and V is a valuation, that is: a function assigning subsets of W to proposition letters in the language. Next we define the truth value of a formula ϕ in Φ at a given point x in a model $\mathcal{M}$ (this fact is designated by $\mathcal{M}, x \models \phi$) We have that for every x $\mathcal{M}, x \models \top$, and for no x $\mathcal{M}, x \models \bot$. The other elements of Φ are treated as follows:

$$
\begin{aligned}
&\mathcal{M}, x \models p \;\Leftrightarrow\; x \in V(p),\\
&\mathcal{M}, x \models \neg p \;\Leftrightarrow\; \mathcal{M}, x \not\models p,\\
&\mathcal{M}, x \models (\phi_1 \wedge \phi_2) \;\Leftrightarrow\; \mathcal{M}, x \models \phi_1 \text{ and } \mathcal{M}, x \models \phi_2,\\
&\mathcal{M}, x \models [R]\varphi \;\Leftrightarrow\; \forall y (Rxy \Rightarrow \mathcal{M}, y \models \phi),\\
&\mathcal{M}, x \models \langle R \rangle \top \;\Leftrightarrow\; \exists y Rxy.
\end{aligned}
$$

Observe that $\mathcal{AL}$ is a very weak language; it lacks full complementation and full disjunction. It also lacks a full dual to the modal operator $[R]$.

Van der Hoek and De Rijke have proposed an axiomatic system for the language $\mathcal{AL}$ and have proved that it completely captures all validities of the form $\Gamma \vdash \phi$, where $\Gamma \cup \phi$ is a finite set. Such sequents are considered valid in the models if the following is true:

$$\mathcal{M}, x \models \Gamma \vdash \phi \text{ iff } (\mathcal{M}, x \models \Gamma \Rightarrow \mathcal{M}, x \models \phi).$$

They call this the system **AL**. The axioms of this system are:

(A1) $\phi \vdash \phi$,
(A2) $p, \neg p \vdash \bot$,
(A3) $\phi \vdash \top$,
(A4) $\bot \vdash \phi$,
(A5) $\phi, \psi \vdash \phi \wedge \psi$,
(A6) $\phi \wedge \psi \vdash \phi$ and $\phi \wedge \psi \vdash \psi$,
(A7) $[R]\bot, \langle R \rangle \top \vdash \bot$.

Further this system has some rules. Since they also appear in our systems, we refer the reader to the rules (Mon), (Cut), (Distr) and (Compl), presented in subsection 3.1 of section 3.

The next language that we shall consider is given *in the following definition:*

Definition 2.2. The language $\mathcal{ALN}$ has all the elements of $\mathcal{AL}$ plus two other modal operators for every n, which we write as $[R]_n$ and $\langle R \rangle_n$. The set of formulas is further expanded by adding for every n and every relation R in $\mathcal{R}$ the formulas $[R]_n \bot$ and $\langle R \rangle_n \top$ with the following semantics:

$$
\begin{aligned}
\mathcal{M}, x \models [R]_n \bot \quad &\text{iff} \quad |\{ y : Rxy \}| \leq n,\\
\mathcal{M}, x \models \langle R \rangle_n \top \quad &\text{iff} \quad |\{ y : Rxy \}| > n.
\end{aligned}
$$

Observe that in $\mathcal{ALN}$ the standard modal diamond $\langle R \rangle \cdot$ is the special case of $\langle R \rangle_n \cdot$ with $n = 0$.

To capture again all valid sequents, Van der Hoek and De Rijke have devised the following below axiomatic system and called it **ALN**. It has as axioms all the axioms of the system **AL** plus the following ones:

$(A8)$ $\langle R\rangle_{n+1}\top \vdash \langle R\rangle_n\top$ and $[R]_n\bot \vdash [R]_{n+1}\bot$,

$(A9)$ $[R]_n\bot, \langle R\rangle_n\top \vdash \bot$.

The rules of **ALN** are the same as those of **AL** with the exception that the rule (Compl) in section 3.1 is replaced by the respective rule (Compl) in section 3.2.

3. SEQUENT CALCULI AND CUT ELIMINATION

In this section, which is the heart of the paper, we present the promised cut-free systems in the languages $\mathcal{AL}$ and $\mathcal{ALN}$. These new systems are connected to the systems developed by Van der Hoek and De Rijke and further it is shown in what way.

3.1. THE SYSTEM **SAL** AND THE CUT ELIMINATION PROOF FOR IT

As before, we use the capital Greek letters Γ, Δ, ..., to refer to finite sets of formulas, and ϕ, ψ, ..., to refer to single formulas. A *sequent* in the language $\mathcal{AL}$ is an expression $\Gamma \vdash \phi$. The axiomatic system **SAL** is given by the next axioms and rules:

Axioms:

(A1) $\phi \vdash \phi$,

(A2) $p, \neg p \vdash \phi$,

(A3) $[R]\bot, \langle R\rangle\top \vdash \phi$.

Rules:

(Mon) $\dfrac{\Gamma \vdash \chi}{\Gamma \cup \{\phi\} \vdash \chi}$,

(Cut) $\dfrac{\Gamma_1 \vdash \phi \qquad \Gamma_2, \{\phi\} \vdash \chi}{\Gamma_1, \Gamma_2 \vdash \chi}$,

($\wedge$-intr-L) $\dfrac{\Gamma, \phi \vdash \chi}{\Gamma, \phi \wedge \psi \vdash \chi}$, $\qquad \dfrac{\Gamma, \psi \vdash \chi}{\Gamma, \phi \wedge \psi \vdash \chi}$,

($\wedge$-intr-R) $\dfrac{\Gamma_1 \vdash \phi \qquad \Gamma_2 \vdash \psi}{\Gamma_1, \Gamma_2 \vdash \phi \wedge \psi}$,

($\top$-drop) $\dfrac{\Gamma, \top \vdash \phi}{\Gamma \vdash \phi}$, provided Γ is non-empty,

($\bot$-use) $\dfrac{\Gamma \vdash \bot}{\Gamma \vdash \phi}$, (Compl) $\dfrac{\Gamma, p \vdash \phi \qquad \Gamma, \neg p \vdash \phi}{\Gamma \vdash \phi}$, $\qquad \dfrac{\Gamma, [R]\bot \vdash \phi \qquad \Gamma, \langle R\rangle\top \vdash \phi}{\Gamma \vdash \phi}$,

(Distr) $\dfrac{\Gamma \vdash \chi}{[R]\Gamma \vdash [R]\chi}$,

(Comb) $\dfrac{\Gamma \vdash \bot}{[R]\Gamma, \langle R\rangle\top \vdash \bot}$.

Remark 3.1. Using the completeness of **AL** it is easy to see that the both systems **SAL** and **AL** prove the same sequents.

3.1.1. Cut elimination for SAL. Now we turn to presenting the proof for cut elimination of the system **SAL**. Here is a brief outline of the proof. We define a notion of *weight* of a derivation in the system **SAL**, which will associate to every derivation a natural number. By induction on this weight we show that every derivation can be transformed into a cut-free proof of the same conclusion. During the induction we shall need to make quite a few case distinctions, most of which will be left out, however, either because they are trivial or because they are similar to cases that we do consider.

Convention 3.2. If d is an arbitrary derivation, then by $r(d)$ we mean the conclusion, or the last sequent in d.

If Γ is a set of formulas, then $[R]\Gamma = \{[R]\phi : \phi \in \Gamma\}$.

In an application of the cut rule as in the derivation d below we call the sequent $\Gamma_1 \vdash \phi$ the *left premise* of the cut rule, and the sequent $\Gamma_2, \phi \vdash \chi$ its *right premise*:

$$d: \dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2, \phi \vdash \chi}}{\Gamma_1, \Gamma_2 \vdash \chi}.$$

We need the next lemma.

Lemma 3.3. *Assume that the derivation d satisfies the following conditions:*

1. *The last rule applied in d is the cut rule, and this is the only application of cut in d.*
2. *The left premise of the last rule is an axiom.*

Then there is a derivation d' of $r(d)$, which does not use the cut rule.

Proof. There are 3 possibilities for the axiom occurring as the left premise of the cut rule. Suppose first that it is (A1). Then the derivation has the form

$$d: \dfrac{\phi \vdash \phi \qquad d': \dfrac{\vdots}{\Gamma, \phi \vdash \chi}}{\Gamma, \phi \vdash \chi}.$$

But then d' is already a cut-free derivation of $r(d)$, as required. The cases when the axiom is (A2) or (A3) are similar, so we consider only one of them: (A2). Then the derivation d has the form displayed below, which can be transformed into a derivation d' with the same conclusion:

$$d: \frac{p, \neg p \vdash \phi \qquad \dfrac{\vdots}{\Gamma, \phi \vdash \chi}}{\Gamma, p, \neg p \vdash \chi} \quad \Rightarrow \quad d': \text{(Mon)}\frac{p, \neg p \vdash \chi}{\Gamma, p, \neg p \vdash \chi}.$$

This completes the proof. ⊣

As announced before, we use a notion of weight to carry through our proof of cut elimination.

Definition 3.4 (weight of a derivation). We define the weight ω of a derivation d in **SAL** by induction:

- If d consists of a single axiom, then $\omega(d) = 1$.
- If the last rule which is applied in d has only one premise, that is if d has the form

  $$d: \frac{d': \dfrac{\vdots}{\Gamma \vdash \chi'}}{\Gamma \vdash \chi},$$

  then $\omega(d) = \omega(d') + 1$.
- If the last rule which is applied in d has two premises, that is if d has the form

  $$d: \frac{d': \dfrac{\vdots}{\Gamma' \vdash \chi'} \qquad d'': \dfrac{\vdots}{\Gamma'' \vdash \chi''}}{\Gamma \vdash \chi},$$

  then $\omega(d) = \omega(d') + \omega(d'') + 1$.

Theorem 3.5. SAL *admits a cut elimination.*

Proof. The proof is by induction on the weight of derivations. As every derivation d has $\omega(d) \geq 1$, the induction starts with $\omega(d) = 1$. In that case the derivation consists of a single cut-free axiom.

Next, suppose that for every derivation d of weight less than n, a cut-free derivation can be found of the same conclusion; we proceed to show that the same is true also for derivations of weight n. Let d be any derivation of weight n. Let (R) be the last rule applied in d. If (R) is not the cut rule, then the derivation without the last rule (R) will be of smaller weight, so by our inductive hypothesis it can be transformed into a cut-free derivation(s) of the same conclusion(s). Subsequently applying (R) yields a cut-free derivation of the conclusion of d.

Now for the main case: the last rule applied in a derivation of weight n is the cut rule. By our inductive hypothesis we can assume that the derivations of the premises of the cut rule are cut-free. We distinguish several cases. If the derivation of the left premise of the cut rule consists of a single axiom, we need only to apply Lemma 3.3 to find a cut-free derivation of the same conclusion.

So assume that the derivation of the left premise does not consist of a single axiom, and consider the different cases for the last rule in this derivation. Let us first consider the case when this rule is such that the formula on the right hand side of the conclusion is the same as the formula on the right hand side of the premises. Such rules are (Mon), ($\wedge$-intr-L), ($\top$-drop), (Compl), and (Comb). Since the required transformations in these cases are similar, we consider only one of them, that of (Mon). In this case we perform the following transformation, denoted by $\Rightarrow$:

$$d: \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \phi}}{\Gamma_1 \cup \{\psi\} \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2, \{\phi\} \vdash \chi}}{\Gamma_1, \Gamma_2, \psi \vdash \chi} \quad \Rightarrow \quad (\mathrm{Mon})\dfrac{d': \dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2, \{\phi\} \vdash \chi}}{\Gamma_1, \Gamma_2 \vdash \chi}}{\Gamma_1, \Gamma_2, \psi \vdash \chi}.$$

We have that $\omega(d') < \omega(d)$, because d' consists of one step less than d, so using the induction hypothesis we can transform d' into a cut-free derivation, but this yields a cut-free derivation of $r(d)$ as well.

The remaining cases are ones in which the last rule applied in the derivation of the left sequent is either ($\bot$-use), ($\wedge$-intr-R), or (Distr). Let us first see what happens when the rule is ($\bot$-use). We can apply the following transformation:

$$d: \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \bot}}{\Gamma_1 \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2, \phi \vdash \chi}}{\Gamma_1, \Gamma_2 \vdash \chi} \quad \Rightarrow \quad (\mathrm{Mon})\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \bot}}{\Gamma_1 \vdash \chi}}{\Gamma_1, \Gamma_2 \vdash \chi}.$$

As the derivation of $\Gamma_1 \vdash \bot$ is cut-free, the derivation d can be transformed into a cut-free one.

In case one of ($\wedge$-intr-R) and (Distr) is the last rule applied to obtain the left premise of the cut rule, we have to dig into the derivation of the right premise of the cut rule. As the arguments for ($\wedge$-intr-R) and (Distr) are similar, we present the details for only one of them, viz. (Distr). First, suppose that the right premise of the cut rule is an axiom. If this axiom is (A1), then the derivation has the form

$$d: \dfrac{d': \dfrac{\dfrac{\vdots}{\Gamma \vdash \chi}}{[R]\Gamma \vdash [R]\chi} \qquad [R]\chi \vdash [R]\chi}{[R]\Gamma \vdash [R]\chi}.$$

Now d' is a cut-free derivation of $r(d)$, so we are done. Clearly, the axiom cannot be (A2), since the left part of the right sequent should contain a formula of the form $[R]\chi$. Hence, the next possibility is (A3), then the derivation has the form of the derivation d below:

$$d: \frac{\dfrac{\overline{\Gamma \vdash \bot}}{[R]\Gamma \vdash [R]\bot} \qquad [R]\bot, \langle R\rangle\top \vdash \chi}{[R]\Gamma, \langle R\rangle\top \vdash \chi} \quad \Rightarrow \quad d': (\bot\text{-use})\frac{(\text{Comb})\dfrac{\overline{\Gamma \vdash \bot}}{[R]\Gamma, \langle R\rangle\top \vdash \bot}}{[R]\bot, \langle R\rangle\top \vdash \chi}.$$

Since the derivation of $\Gamma \vdash \bot$ is cut-free, the above derivation d can be transformed into the cut-free derivation d' of $r(d)$.

Next we proceed to deal with the cases when the right premise of the cut rule is the result of applying one of the rules of **SAL**. We consider all these rules one at a time.

(Mon) The derivation must have the form of the derivation d below.

$$d: \frac{\dfrac{\overline{\Gamma_1 \vdash \phi}}{[R]\Gamma_1 \vdash [R]\phi} \qquad \dfrac{\overline{\Gamma_2 \vdash \chi}}{\Gamma_2 \cup \{\psi\} \vdash \chi}}{[R]\Gamma_1, (\Gamma_2 \cup \{\psi\})\backslash[R]\phi \vdash \chi}.$$

To be able to apply the cut rule, we must have $[R]\phi \in \Gamma_2 \cup \{\psi\}$. We again distinguish two cases: $[R]\phi \in \Gamma_2$ or $[R]\phi = \psi$. Suppose first that $\Gamma_2 = \Gamma_2' \cup \{[R]\phi\}$. Then we can transform d into the derivation d' below. As the sub-derivation d'' of d' has $\omega(d'') < \omega(d)$, it can be transformed into a cut-free derivation, and hence, so can d:

$$d': \frac{d'': \dfrac{\dfrac{\overline{\Gamma_1 \vdash \phi}}{[R]\Gamma_1 \vdash [R]\phi} \qquad \overline{\Gamma_2', [R]\phi \vdash \chi}}{[R]\Gamma_1, \Gamma_2' \vdash \chi}}{[R]\Gamma_1, \Gamma_2', \psi \vdash \chi}.$$

Next, assume that $\psi = [R]\phi$. Then we have $(\Gamma_2 \cup \{\psi\})\backslash[R]\phi = \Gamma_2$, so we can derive $r(d)$ as in the derivation d''':

$$d''': (\text{Mon})\frac{\overline{\Gamma_2 \vdash \chi}}{[R]\Gamma_1, \Gamma_2 \vdash \chi}.$$

(∧-intr-R), (⊤-drop), (⊥-use), (Compl) Since the transformations we perform on the derivations in these cases are similar, we consider only one of them. Suppose the rule that is last applied in the derivation of the right premise of the cut rule is (∧-intr-R), i.e. the derivation d has the form

$$d: \frac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \chi}}{[R]\Gamma \vdash [R]\chi} \qquad \dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2 \vdash \psi}}{\Gamma_1, \Gamma_2 \vdash \phi \wedge \psi}}{[R]\Gamma, (\Gamma_1 \cup \Gamma_2)\backslash[R]\chi \vdash \phi \wedge \psi}.$$

As the last step in d is an application of the cut rule, we must have that $[R]\chi \in \Gamma_1$ or $[R]\chi \in \Gamma_2$. Assume first that $[R]\chi \in \Gamma_1$ and $[R]\chi \notin \Gamma_2$; then Γ_1 has the form $\Gamma_1 = \Gamma_1' \cup \{[R]\chi\}$ and $(\Gamma_1 \cup \Gamma_2)\backslash[R]\chi = \Gamma_1' \cup \Gamma_2$. Now we can derive $r(d)$ as follows:

$$d': \frac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash \chi}}{[R]\Gamma \vdash [R]\chi} \qquad \dfrac{\vdots}{\Gamma_1', [R]\chi \vdash \phi}}{[R]\Gamma, \Gamma_1' \vdash \phi} \qquad \dfrac{\vdots}{\Gamma_2 \vdash \psi}}{[R]\Gamma, \Gamma_1', \Gamma_2 \vdash \phi \wedge \psi}.$$

As before we can transform d' into a cut-free derivation, and thus get a cut-free derivation of $r(d)$. For the cases when $[R]\chi \notin \Gamma_1$, $[R]\chi \in \Gamma_2$, and $[R]\chi \in \Gamma_1$ and $[R]\chi \in \Gamma_2$, we can perform similar transformations to arrive at the required result.

(∧-intr-L) We only consider one of the rules of this kind, the first one. The form of our initial derivation d in this case is

$$d: \frac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \lambda}}{[R]\Gamma_1 \vdash [R]\lambda} \qquad \dfrac{\dfrac{\vdots}{\Gamma_2, \phi \vdash \chi}}{\Gamma_2, \phi \wedge \psi \vdash \chi}}{[R]\Gamma_1, \Gamma_2\backslash[R]\lambda, \phi \wedge \psi \vdash \chi}.$$

Since $[R]\lambda \neq \phi \wedge \psi$, we must have $\Gamma_2 = \Gamma_2' \cup \{[R]\lambda\}$. Using this, we can also derive $r(d)$ in the following way:

$$d': \frac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \lambda}}{[R]\Gamma_1 \vdash [R]\lambda} \qquad \dfrac{\vdots}{\Gamma_2' \cup \{[R]\lambda\}, \phi \vdash \chi}}{[R]\Gamma_1, \Gamma_2', \phi \vdash \chi}}{[R]\Gamma_1, \Gamma_2', \phi \wedge \psi \vdash \chi}.$$

As before, this derivation can be turned into a cut-free one of $r(d)$.

(Distr), (Comb) In these cases the transformations that we perform on the derivations are again similar, so we treat only one of them: (Distr). Then d has the form

$$d: \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \chi_1}}{[R]\Gamma_1 \vdash [R]\chi_1} \qquad \dfrac{\dfrac{\vdots}{\Gamma_2 \vdash \chi_2}}{[R]\Gamma_2 \vdash [R]\chi_2}}{[R]\Gamma_1,\, [R]\Gamma_2 \backslash [R]\chi_1 \vdash [R]\chi_2}.$$

Since $[R]\chi_1 \in [R]\Gamma_2$, we must have $\chi_1 \in \Gamma_2$, so we can transform the derivation to the following one, which can be turned into a cut-free one using the induction hypothesis:

$$d': \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \chi_1} \qquad \dfrac{\vdots}{\Gamma_2 \cup \{\chi_1\} \vdash \chi_2}}{\Gamma_1,\, \Gamma_2' \vdash \chi_2}}{[R]\Gamma_1,\, [R]\Gamma_2' \vdash [R]\chi_2}\,(\text{Distr}).$$

Summing up, any derivation in the system **SAL** can be transformed into a derivation of the same conclusion in which the cut rule is not used. ⊣

3.2. THE SYSTEM **SALN** AND ITS CUT ELIMINATION

In this section we present a cut-free sequential system, the logic **ALN** in [4]. Since the system **ALN** is in fact the system **AL** with added a few more axioms, our system **SALN** will be very much like the system **SAL**. We shall only present in detail the axioms and rules that are new or differ from the corresponding ones in **SAL**.

The axioms (A1) and (A2) are the same as in **SAL**, only (A3) is changed to the following form:

(A3) $\quad [R]_n\bot,\, \langle R\rangle_n\top \vdash \phi, \quad n \geq 1.$

The rules (Mon), ($\wedge$-intr-L), ($\wedge$-intr-R), ($\bot$-drop), ($\top$-use) and (Distr) are again the same in **SALN** as in **SAL**. The second part of the rule (Compl) and the rule (Comb) are changed to the following forms:

(Compl) $\quad \dfrac{\Gamma, [R]_n\bot \vdash \chi \qquad \Gamma, \langle R\rangle_n\top \vdash \chi}{\Gamma \vdash \chi}.$

(Comb) $\quad \dfrac{\Gamma \vdash \bot}{[R]_n\Gamma,\, \langle R\rangle_n\top \vdash \bot}, \quad n \geq 1.$

Now the next three rules that we present are new ones, specific of the system **SALN**.

($\langle R\rangle$+) $\quad \dfrac{\Gamma, \langle R\rangle_n\top \vdash \phi}{\Gamma, \langle R\rangle_{n+1}\top \vdash \phi}, \quad n \geq 1,$

([R]-) $\quad \dfrac{\Gamma, [R]_{n+1}\top \vdash \phi}{\Gamma, [R]_n\top \vdash \phi}, \quad n \geq 1,$

([R]+) $\quad \dfrac{\Gamma \vdash [R]_n\bot}{\Gamma \vdash [R]_{n+1}\bot}, \quad n \geq 1.$

Before beginning the proof that the system **SALN** admits a cut elimination, let us briefly outline how we are going to proceed. First, we proof a lemma which in effect claims that a somewhat restricted form of the cut rule can be eliminated and then we proof that also the general form of the (Cut) is dispensable. To prove both of these claims, we use the technique that we used to prove the cut elimination for the system **SAL**, namely induction on the weight of the derivation containing (Cut). Since the most cases will emerge as similar to the respective ones in the cut elimination proof of **SAL**, we shall be more concise in our exposition and we shall treat in detail only some of the different cases.

Lemma 3.6. *The following rule can be eliminated from every* **SALN**-*derivation.*

$$(\mathrm{Cut}_{mn}) \quad \frac{\Gamma_1 \vdash [R]_m\bot \qquad \Gamma_2, [R]_n\bot \vdash \phi}{\Gamma_1, \Gamma_2 \vdash \phi}, \quad \textit{where } m \leq n.$$

Proof. We prove this claim following the same pattern we used before. Suppose d is some derivation and (Cut_{mn}) is applied only at the last step of this derivation. Further suppose that the left premise of (Cut_{mn}) is an axiom. The only case of any interest is the case of (A1). Then d has the form

$$d\colon \frac{[R]_m\bot \vdash [R]_m\bot \qquad \dfrac{\vdots}{[R]_n\bot,\, \Gamma \vdash \phi}}{[R]_m\bot,\, \Gamma \vdash \phi}.$$

Now, since we have applied the rule (Cut_{mn}), we have $m \leq n$. If $m = n$, then we can use the derivation of the right premise as the Cut_{mn}-free derivation of $r(d)$. If $m < n$, then to the right premise we apply $(n - m)$ times the rule ([R]-) and we shall have the desired derivation.

Further we have to consider the cases when the left premise appeared by an application of some rule. The transformations we do are similar to those presented so far, so to diminish the risk of becoming boring we shall present only those we think most unusual.

([R]+) In this case we do the following transformation:

$$d\colon \frac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash [R]_m\bot}}{\Gamma_1 \vdash [R]_{m+1}\bot} \qquad \dfrac{\vdots}{\Gamma_2,\, [R]_n\bot \vdash \phi}}{\Gamma_1, \Gamma_2 \vdash \phi}$$

$$\Rightarrow \quad d'\colon \frac{\dfrac{\vdots}{\Gamma_1 \vdash [R]_m\bot} \qquad \dfrac{\vdots}{\Gamma_2,\, [R]_n\bot \vdash \phi}}{\Gamma_1, \Gamma_2 \vdash \phi}.$$

Now, since we have applied (Cut_{mn}) in the first place, it is true that $m+1 \leq n$, so $m \leq n$, *and the application of* (Cut_{mn}) in d' is legal. Using also that $\omega(d') < \omega(d)$, we can conclude the result.

(Distr) Our derivation which we want to turn into a Cut_{mn}-free one in this case has the form

$$d: \quad \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \bot}}{[R]\Gamma_1 \vdash [R]\bot} \qquad \dfrac{\vdots}{\Gamma_2, [R]_n\bot \vdash \phi}}{[R]\Gamma_1, \Gamma_2 \vdash \phi}.$$

In this case as before we have to dig into the derivation of the right premise. To deal with the axioms (A1) and (A3), we use the rules ([R]+) and (Comb), respectively. Next we should turn to considering the cases when we have applied one of the rules in the derivation of the right premise. As an example we shall consider only one case which is the most instructive. Suppose the last rule applied in the derivation of the right premise is the rule ([R]–). We do the following transformation on the derivation d:

$$d: \quad \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \bot}}{[R]\Gamma_1 \vdash [R]\bot} \qquad \dfrac{\dfrac{\vdots}{\Gamma_2, [R]_{n+1}\bot \vdash \phi}}{\Gamma_2, [R]_n\bot \vdash \phi}}{[R]\Gamma_1, \Gamma_2 \vdash \phi}$$

$$\Rightarrow \quad d': \quad \dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash \bot}}{[R]\Gamma_1 \vdash [R]\bot} \qquad \dfrac{\vdots}{\Gamma_2, [R]_{n+1}\bot \vdash \phi}}{[R]\Gamma_1, \Gamma_2 \vdash \phi}.$$

We have that the derivation d' is of lesser weight than d, so we can apply the induction hypothesis to get the result.

Now all the cases for the type of the last rule in the derivation of the left premise of the (Cut_{mn}) are considered and the proof is finished. ⊣

Further we turn to the proof that the whole version of the cut rule can be eliminated in the system **SALN**, that is to the proof of the next theorem.

Theorem 3.7. SALN *admits a cut elimination.*

Proof. We shall use again in the proof our well-worked induction on the weight of the derivation with the different cases for the structure of the last steps of the derivations of the premises of the cut rule.

First, we consider the form of teh derivation of the left premise. The case when this derivation consists of a single axiom presents no difficulty. Let us turn to the case when the last step in the derivation of the left premise consists of applying some rule. The rules that are the same as in **SAL** or the formula on the right hand side of the conclusion is the same as that on the right hand side of the premise(s), create no difficulty. Such rules are (Mon), ($\wedge$-intr-L), ($\wedge$-intr-R), ($\top$-drop), ($\bot$-use), (Compl), ($\langle R\rangle$+), ([R]−), and (Comb).

If the last rule is ([R]+), then since the cut formula is of the form $[R]_m\bot$, we must only apply Lemma 3.6.

Now we turn to the most difficult to treat rule, namely (Distr). As usual, in this case we consider the cases for the form of the derivation of the right premise of the cut rule. The axioms and the rules that are present in **SAL** are treated as in this system. We need only to show that nothing goes wrong if the last rule in the derivation of the right premise is one of the new or changed rules. For ($\langle R\rangle$+), ([R]+) and (Comb) we do quite obvious transformations, using the fact that we can locate where the cut formula belongs. The only case that remains is that of ([R]+). But we can use the fact that the cut formula in this case is of the form $[R]_m\bot$, and applying Lemma 3.6 we conclude the proof. $\dashv$

So we can now state that the presented sequent system **SALN** indeed admits elimination of the cut rule.

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