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COMPLETE SYSTEMS OF TRICOMI FUNCTIONS IN SPACES OF HOLOMORPHIC FUNCTIONS

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Петр Русев. ПОЛНЫЕ СИСТЕМЫ ФУНКЦИЙ ТРИКОМИ В ПРОСТРАНСТВАХ ГОЛОМОРФНЫХ ФУНКЦИЙ

Пусть $\Psi(a,c;z)$ — главная ветвь вырожденной гипергеометрической функции Трикоми с параметрами a; c и G — произвольная односвязная подобласть комплексной плоскости, разрезанной по вещественной неположительной полуоси. Доказано, что систима вида

$$\{\Psi(n+\lambda+\alpha+1,\alpha+1;z)\}_{n=0}^{\infty}$$

полна в пространстве комплексных функций голоморфных в G, считая что λ и α — вещественные и $\lambda+\alpha>-1$.

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Let $\Psi(a,c;z)$ be the main branch of Tricomi confluent hypergeometric function with parameters a, c and G be an arbitrary simply connected subregion of the complex plane cut along the real non-positive semiaxis. It is proved that a system of the kind

$$\{\Psi(n+\lambda+\alpha+1,\alpha+1;z)\}_{n=0}^{\infty}$$

is complete in the space of the complex functions holomorphic in G provided that λ and α are real and $\lambda+\alpha>-1$.

1. INTRODUCTION

Let G be a region in the complex plane $\boldsymbol{C}$ and $H(G)$ be the $\boldsymbol{C}$-vector space of the complex functions holomorphic in G endowed with the topology of uniform convergence on compact subsets of G. A system $\{\varphi_n(z)\}_{n=0}^{\infty} \subset H(G)$ is called complete in $H(G)$ if for every $f \in H(G)$, every compact set $K \in G$ and every $\varepsilon > 0$ there exists a linear combination $P(z) = \sum_{n=0}^{N} c_n \varphi_n(z)$, $c_n \in \boldsymbol{C}$, $n = 0, 1, 2, \ldots, N$, such that $|f(z) - P(z)| < \varepsilon$ whenever $z \in K$.

Let γ be a Jordan curve in $\boldsymbol{C}$ and C_γ be the closure of its outside with respect to the extended complex plane $\overline{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$. By H_γ we denote the vector space of the complex functions, where each of them is holomorphic in an open set containing C_γ and vanishes at the point ∞. The following statement is a criterion for completeness in spaces of the kind $H(G)$ [2, p. 211, Theorem 17]:

(CC) A system $\{\varphi_n(z)\}_{n=0}^{\infty}$ of complex functions holomorphic in a simply connected region $G \subset \boldsymbol{C}$ is complete in the space $H(G)$ iff for every rectifiable Jordan curve $\gamma \subset G$ the only function $F \in H_\gamma$ which is orthogonal to each of the functions $\{\varphi_n(z)\}_{n=0}^{\infty}$ is identically zero, i.e. the equalities

$$\int_\gamma F(z)\varphi_n(z)\,dz = 0, \quad n = 0, 1, 2, \ldots,$$

imply $F \equiv 0$.

2. TRICOMI FUNCTION

Tricomi function $\Psi(a, c; z)$ is a "second" solution of the confluent hypergeometric equation $zw'' + (c - z)w' - aw = 0$ in the region $\overline{\boldsymbol{C}} \setminus \{0, \infty\}$ [1, 6.5, 6.6]. In general, it is a multivalued (analytic) function with branch points at 0 and ∞ only. The same notation is used for its main branch in the region $\boldsymbol{C} \setminus (-\infty, 0]$. This branch is of course a holomorphic function and if $\operatorname{Re} a > 0$, it has the following representation:

$$\Psi(a, c; z) = \frac{z^{1-c}}{\Gamma(a)} \int_0^{\infty} \frac{u^{a-1}\exp(-u)}{(z+u)^{a-c+1}}\,du. \tag{2.1}$$

Remark. Usually, the main branch of Tricomi function is defined by the equality

$$\Psi(a, c; z) = \frac{1}{\Gamma(a)} \int_{l(\varphi)} \frac{\zeta^{a-1}\exp(-z\zeta)}{(1+\zeta)^{a-c+1}}\,d\zeta, \tag{2.2}$$

where $l(\varphi) = \{\zeta = \exp i\varphi.t, 0 \le t < \infty\}$ provided that $-\pi/2 \le \varphi \le \pi/2$ and $-\pi/2 - \varphi < \arg z < \pi/2 - \varphi$ [1, 6.5, (3)]. But as it is easily seen, if $\varphi = 0$, the right-hand sides of (2.1) and (2.2) are equal when $z = x > 0$. In order to prove this, we put $z = x > 0$ in (1.1) and change the variable u by xu.

Under the assumpton that $a + c + 1 \neq 0, -1, -2, \ldots$ we define the function $\widetilde{\Psi}(a, c; z)$ by

$$\widetilde{\Psi}(a, c; z) = \Gamma(a + c + 1)\Psi(a + c + 1, c + 1; z). \tag{2.3}$$

If a, as well as $a-c+1$, is different from $0,-1,-2,\dots$, the following relation is valid [1, 6.5, (6)]:

$$\Psi(a,c;z)=z^{1-c}\Psi(a-c+1,2-c;z). \tag{2.4}$$

By using it, we come to the representation

$$\widetilde{\Psi}(a,c;z)=\frac{\Gamma(a+c+1)}{\Gamma(a+1)}\int_0^\infty \frac{u^a\exp(-u)}{(z+u)^{a+c+1}}\,du, \tag{2.5}$$

which is valid in the region $\boldsymbol{C}\setminus(-\infty,0]$ provided that $\operatorname{Re}a>-1$ and $a+c+1\neq 0,-1,-2,\dots$

3. AUXILIARY STATEMENTS

If $a+1\neq 0,-1,-2,\dots$, the function

$$T(a;z,w)=\frac{\Gamma(a+1)}{(z+w)^{a+1}} \tag{3.1}$$

is holomorphic with respect to w in the region $\boldsymbol{C}\setminus l_z$, where l_z denotes the ray $\{w=-z-t, 0\leq t<\infty\}$.

Let $z\in\boldsymbol{C}\setminus(-\infty,0]$, then the equation $\operatorname{Re}(-w)^{1/2}=\operatorname{Re}z^{1/2}$ defines a parabola p_z passing through the point $-z$ and having its vertex at the point $-(\operatorname{Re}z^{1/2})^2$. We denote by Δ_z the inside of p_z. Evidently, $\Delta_z\in\boldsymbol{C}\setminus l_z$ for every $z\in\boldsymbol{C}\setminus(-\infty,0]$.

Lemma I. *Let $\lambda>-1, \lambda+\alpha+1\neq 0,-1,-2,\dots$ and $z\in\boldsymbol{C}\setminus(-\infty,0]$. Then for every $w\in\Delta_z$ the equality*

$$T(\lambda+\alpha;z,w)=\sum_{n=0}^{\infty}\widetilde{\Psi}(n+\lambda,\alpha;z)L_n^{(\lambda)}(w) \tag{3.2}$$

holds, where $\left\{L_n^{(\lambda)}(w)\right\}_{n=0}^{\infty}$ are the Laguerre's polynomials with parameter λ.

Proof. By the generalized Pollard's theorem [3, Theorem A] the function $T(\lambda+\alpha;z,w)$ has an expansion in the Laguerre's polynomials with parameter λ in the region Δ_z, i.e.

$$T(\lambda+\alpha;z,w)=\sum_{n=0}^{\infty}A_n(\lambda,\alpha;z)L_n^{(\lambda)}(w).$$

For the coefficients of the series in the right-hand side we have the following representations:

$$A_n(\lambda,\alpha;z)=\frac{\Gamma(\lambda+\alpha+1)\Gamma(n+1)}{\Gamma(n+\lambda+1)}\int_0^\infty\frac{u^\lambda\exp(-u)L_n^{(\lambda)}(u)}{(z+u)^{\lambda+\alpha+1}}\,du,\quad n=0,1,2,\dots$$

Futher, the Rodrigues' formula for the Laguerre's polynomials [1, 10.12, (5)] gives

$$A_n(\lambda,\alpha;z) = \frac{(-1)^n\Gamma(\lambda+\alpha+1)}{\Gamma(n+\lambda+1)} \int_0^\infty \frac{\{u^{n+\lambda}\exp(-u)\}^{(n)}}{(z+u)^{\lambda+\alpha+1}}\,du, \quad n=0,1,2,\ldots,$$

and after integration by parts we obtain

$$A_n(\lambda\alpha;z) = \frac{\Gamma(n+\lambda+\alpha+1)}{\Gamma(n+\lambda+1)} \int_0^\infty \frac{u^{n+\lambda}\exp(-u)}{(z+u)^{n+\lambda+\alpha+1}}\,du, \quad n=0,1,2,\ldots,$$

i.e. $A_n(\lambda,\alpha;z) = \widetilde{\Psi}(n+\lambda,\alpha;z), n=0,1,2,\ldots$

If $\mu>0$, by $\tilde{\Delta}_\mu$ we denote the closed set defined by the inequality $\operatorname{Re} z^{1/2} \geq \mu$ provided that $z \in \boldsymbol{C}\setminus(-\infty,0]$. In other words, $\tilde{\Delta}_\mu$ is the closure of the outside of the parabola with focus at the origin and vertex at the point μ^2.

Lemma II. *Whatever $\mu>0$, a_0 and $c\in\boldsymbol{R}$ be, the inequality*

$$(3.3)\qquad |\widetilde{\Psi}(a,c;z)| = O\left(|z|^{-c/2-1/4}a^{c/2-1/4}\exp(-2\mu\sqrt{a})\right)$$

holds uniformly with respect to $z\in\tilde{\Delta}_\mu$ and $a\geq a_0$.

Remark. The exact meaning of (3.3) is that there exists a constant $M = M(\mu,a_0,c)$ (i.e. M depends on μ, a_0 and c only) such that

$$(3.4)\qquad |\widetilde{\Psi}(a,c;z)| \leq M|z|^{-c/2-1/4}a^{c/2-1/4}\exp(-2\mu\sqrt{a})$$

when $z\in\tilde{\Delta}_\mu$ and $a\geq a_0$.

Proof. Suppose that $c\geq 1/2$. The integral representation [1, 6.11, (10)] and the equality (1.3) give that if $z\in\boldsymbol{C}\setminus(-\infty,0]$ and $a>0$, then

$$(3.5)\qquad \widetilde{\Psi}(a,c;z) = \frac{2z^{-c/2}}{\Gamma(a+1)}\int_0^\infty u^{a+c/2}\exp(-u)K_c(2\sqrt{zu})\,du,$$

where K_c is the modified Bessel function of the third kind with index c.

From the defining equalities [1, 7.2, (13), (37)] and the asymptotic formula [1, 7.4, (4)] it follows that there exists a positive constant $A=A(c,\mu)$ such that if $\zeta\in\boldsymbol{C}\setminus(-\infty,0]$, then $|K_c(2\zeta)|\leq A|\zeta|^{-c}$ when $0<|\zeta|\leq\mu^2$, and $|K_c(2\zeta)|\leq A|\zeta|^{-1/2}\exp(-2\operatorname{Re}\zeta)$ when $|\zeta|\geq\mu^2$. Then (3.5) leads to the conclusion that for every $z\in\tilde{\Delta}_\mu$ and every $a>0$ the following inequality holds:

$$|\widetilde{\Psi}(a,c;z)| \leq 2A|z|^{-c/2-1/4}\left\{I_1(\mu,a,c;z)+I_2(\mu,a,c;z)\right\},$$

where

$$I_1(\mu,ac;z) = \frac{|z|^{-c/2+1/4}}{\Gamma(a+1)}\int_0^{\mu^2/|z|} u^a\exp(-u)\,du$$

and

$$I_2(\mu, a, c; z) = \frac{1}{\Gamma(a+1)} \int_{\mu^2/|z|}^{\infty} u^{a+c/2-1/4} \exp(-u - 2\mu\sqrt{u})\, du.$$

Since $|z| \geq \mu^2$ for every $z \in \tilde{\Delta}_\mu$ and, moreover, $c/2 \geq 1/4$, we have that if $z \in \tilde{\Delta}_\mu$ and $a > 0$, then

$$I_1(\mu, ac; z) \leq \mu^{-c+1/2}/\Gamma(a+2).$$

Further from the equality

$$\lim_{n\to\infty} a^{-c/2+1/4} \exp(2\mu\sqrt{a})(\Gamma(a+2))^{-1} = 0$$

it follows that for every $a_0 > 0$ there exists $B_1 = B_1(\mu, a_0, c)$ such that the inequality

$$I_1(\mu, a, c; z) \leq B_1 a^{c/2-1/4} \exp(-2\mu\sqrt{a})$$

holds for every $a \geq a_0$ and every $z \in \tilde{\Delta}_\mu$.

If we change u by $u^2/2$ and take into consideration the integral representation [1, 8.3, (3)] for the Weber-Hermite function $D_\nu(z)$, then we obtain that the inequality

$$\begin{aligned} &\exp(-\mu^2/2) I_2(\mu, a, c; z) \\ &\leq 2^{a+c/2+3/4}\Gamma(2a+c+3/2)(\Gamma(a+1))^{-1} D_{-(2a+c+3/2)}(\mu\sqrt{2}) \end{aligned}$$

is valid for every $z \in \tilde{\Delta}_\mu$ and every $a > 0$.

By means of the asymptotic formula [1, 8.4, (5)] as well as Stirling's formula we come to the conclusion that for every $a_0 > 0$ there exists a constant $B_2 = B_2(\mu, a_0, c)$ such that the inequality

$$I_2(\mu, a, c; z) \leq B_2 a^{c/2-1/4} \exp(-2\mu\sqrt{a})$$

is valid whenever $z \in \tilde{\Delta}_\mu$ and $a \geq a_0$.

So far the validity of the inequality (2.4) with $M = 2A\max(B_1, B_2)$ is proved under the assumption that $c \geq 1/2$. By means of the relation (2.4) we prove that it holds also in the case when $c < 1/2$.

4. MAIN RESULT

Theorem. *Let λ and α be real and let $\lambda + \alpha > -1$. Then for every simply connected region $G \subset \boldsymbol{C} \setminus (-\infty, 0]$ the system*

$$\{\Psi(n+\lambda+\alpha+1, \alpha+1; z)\}_{n=0}^{\infty} \tag{4.1}$$

is complete in the space $H(G)$.

Proof. It is sufficient to prove that the system

$$\left\{\tilde{\Psi}(n+\lambda, \alpha; z)\right\}_{n=0}^{\infty} \tag{4.2}$$

has the desired property.

Let $\gamma \subset G$ be a rectifiable Jordan curve and let the function F be in the space H_γ. We define the function f in the region $C \setminus \bigcup_{z\in\gamma} l_z$ by

$$f(w) = \int_\gamma T(\lambda + \alpha; z, w)\, dz.$$

Let ζ be a point of γ such that $\operatorname{Re}\zeta^{1/2} \le \operatorname{Re} z^{1/2}$ for every $z \in \gamma$. In the region Δ_ζ we have the representation

$$f(w) = \sum_{n=0}^{\infty} T_n(F) L_n^{(\lambda)}(w), \tag{4.3}$$

where

$$T_n(F) = \int_\gamma F(z)\widetilde{\Psi}(n+\lambda, \alpha; z)\, dz, \quad n = 0, 1, 2, \ldots$$

In order to prove this, it is sufficient to show that the series in (3.2) is uniformly convergent on the curve γ whenever $w \in \Delta_\zeta$.

Let $\mu = \operatorname{Re}\zeta^{1/2}$ and $w \in \Delta_\zeta$, then the inequality $\operatorname{Re}(-w)^{1/2} < \mu$ holds. By using the asymptotic formulas for the Laguerre's polynomials [4, (8.22.1), (8.22.3)] as well as the inequality (3.3) we come to the conclusion that there exists a positive integer n_0 such that the inequality

$$|\widetilde{\Psi}(n+\lambda, \alpha; z) L_n^{(\lambda)}(w)| = O\left(n^{(\alpha+\lambda-1)/2} \exp[-2(\mu - \operatorname{Re}(-w)^{1/2})\sqrt{n}]\right)$$

holds uniformly with respect to $z \in \gamma$ and $n \ge n_0$. In other words, the series in (3.2) is majorized on γ by a convergent series and therefore it is uniformly convergent on γ. After integrating it term by term on γ, we get the representation (4.3).

Suppose that $T_n(F) = 0$ for every $n = 0, 1, 2, \ldots$ Then (4.3) gives that $f \equiv 0$ in the region Δ_ζ, i.e.

$$f(w) = \int_\gamma \frac{F(w)}{(z+w)^{\lambda+\alpha+1}}\, dz = 0$$

for every $w \in \Delta_\zeta$. Further the equalities $f^{(n)}(0) = 0$, $n = 0, 1, 2, \ldots$, give that

$$\int_\gamma F(z) z^{-\lambda-\alpha-1} z^{-n}\, dz = 0, \quad n = 0, 1, 2, \ldots$$

Since by Runge's theorem [5, p. 176, (2.1)] the system $\{z^{-n}\}_{n=0}^{\infty}$ is complete in $H(G)$, from the completeness criterion (CC) it follows that $F \equiv 0$.

Corollary. *Under the conditions of the Theorem the system*

$$\{\Psi(n+\lambda+1, 1-\alpha; z)\}_{n=0}^{\infty}$$

is complete in $H(G)$.

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