ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Приложна математика и информатика
Том 89, 1995

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mathématiques Appliquée et Informatique
Tome 89, 1995

ON THE BRITTLE FRACTURE OF A PIN-JOINTED FRAME

GALJA M. DRAGANOVA, KONSTANTIN Z. MARKOV

The aim of the paper is to report some preliminary results concerning rupture through damage accumulation of a simple pin-jointed frame under tension. Under the elastic and stationary creep conditions (at small strains) this is a well-known problem of strength of material and mathematical theory of creep. Here we assume additionally that damage also evolves in the rods, obeying the classical Kachanov's law, which essentially complicates the problem. In the brittle case, the only one, considered in detail in this paper, the problem is formulated eventually as a coupled nonlinear system of differential equations for the damage variables in the rods. This system, in general, does not admit a close form analytical solution unlike the classical examples of continuum damage mechanics, so that numerical treatment is needed. That is why the special, but realistic case of a common "damage exponent" of the rods is only considered and a simple explicit solution for the damage evolution is found and discussed in more detail.

Keywords: brittle fracture, damage mechanics, pin-jointed frames.
1991 Mathematics Subject Classification: 73M25, 73K05.

1. INTRODUCTION

Consider the pin-jointed frame, shown in Fig. 1. The tensile force **F** is applied in the direction of the rod BD. Finding the stresses in such a frame is a well-known exercise in strength of materials, provided the rods behave elastically, see, e.g., [1] and many other textbooks on the subject. If the rods' behaviour is governed by stationary creep law equations, the stresses in the system and, in particular, the creep rate of the loaded node D, are first found by Kachanov [2], provided the creep deformation is small (so that the so-called elastic analogy applies), see also [3].

Our aim here will be a more detailed investigation of the strain and failure of the frame when damage in the rods appear and evolve following some of the classical schemes of continuum damage mechanics initiated and developed by Kachanov [4, 5], see also [6] for further results and generalizations. Since the rods undergo different stresses, damage within them will reach different levels and will thus lead to a more complicated picture of stress and damage distribution than the ones treated in the classical examples of damage mechanics. In this preliminary stage of our investigation, only the purely brittle case will be dealt with. The problem is rigorously posed in Section 2. But even in this simpler case, unlike the examples of damage mechanics, no simple analytical solution will be possible, since the problem under study will be eventually formulated as a system of two *coupled* nonlinear differential equations governing the damage evolution in the rods which admits, in general, only numerical treatment. That is why the particular, but realistic case of a common "damage exponent" ν of the rods is only considered in Section 3. In this case it appears that the damage parameters of the rods are proportional and a simple explicit solution for the damage accumulation is found in Section 4. This solution is discussed in more detail in the final Section 5.

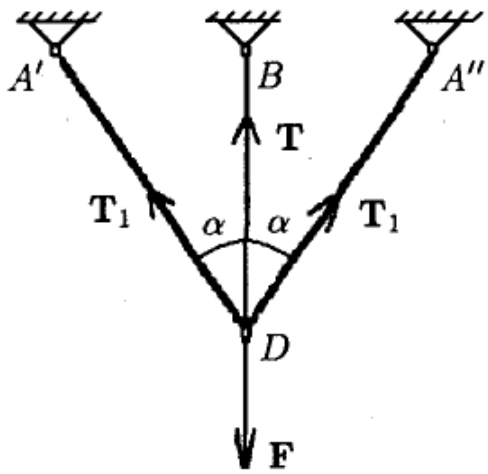

Fig. 1. The pin-jointed frame under study

2. POSING THE PROBLEM

Let all the rods possess in their undamaged state one and the same cross-section S_0 and Young's modulus E^v. Denote as usual by ψ the continuity parameter, so that $\omega = 1 - \psi$ is the damage variable. In the brittle regime under discussion the damage accumulation in a single rod (under uniaxial tension) is governed by the well-known Kachanov's law

$$\dot{\omega} = C\left(\frac{\sigma_0}{\psi}\right)^{\nu}, \tag{2.1}$$

where $\sigma_0 = F/S_0$ is the applied stress, C and ν are material constants [4, 5]. The brittle time-to-rupture, t_b^*, of such a single rod is given then by the known relation

$$t_b^* = \frac{1}{C(1+\nu)\sigma_0^\nu}\,, \tag{2.2}$$

see again [4, 5]. Hereafter the dimensionless time-scale

$$\tau = t/t_b^* \tag{2.3}$$

will be used, since t_b^* is a natural time-unit for the problem under study.

To derive the damage evolution equations in the rods, let us write down first the only non-trivial statics equation for the problem, namely,

$$2T_1 \cos\alpha + T = F\,, \tag{2.4}$$

as well as the equation of the compatibility of the strains in the nod D, namely,

$$\varepsilon_1 = \varepsilon \cos^2 \alpha\,. \tag{2.5}$$

Hereafter all quantities with the subscript '1' refer to the rods $A'D$ or $A''D$, and those without a subscript — to the central rod BD. Hence, in particular,

$$\sigma_1 = T_1/S_0\,, \quad \sigma = T/S_0\,, \quad \sigma_0 = F/S_0 \tag{2.6}$$

are the stresses in the rods, T_1 and T being the respective magnitudes of the tensile forces in them, see Fig. 1; σ_0 would be the stress in any of the rods if they were single and subjected to the same force F_0. Note also that dealing with brittle fracture solely implies that strains are small, so that the angle α in Eqs. (2.4) and (2.5) remains constant — something that does simplify the study (in the ductile and mixed brittle-ductile failure this angle changes considerably during loading and hence an additional non-linear equation involving this angle should be added to the basic equations).

Assume next that the rods $A'D$ and $A''D$ have the same "damage exponent" ν but different material parameter C_1 in the Kachanov's law (2.1) than the central one BD.[1] This means that Eq. (2.1) applies for the central rod BD, but in the two "side" rods $A'D$ and $A''D$ damage accumulates according to the law

$$\dot{\omega}_1 = C_1 \left(\frac{\sigma_1}{\psi_1}\right)^\nu\,, \tag{2.7}$$

where $C \neq C_1$. The reason to take different material parameters C and C_1 is that the well-known elementary elastic solution for the frame under study suggests that the central rod is obviously more stressed than the two "side" ones, i.e. $\sigma > \sigma_1$.

[1] Note that the more general case when the exponents ν of the rods differ as well can also be treated without much effort, though no closed form solution is possible. This case will be considered elsewhere.

This means that the central rod will fail faster. That is why, to make the frame more "damage-resistant", one should accordingly choose for the central rod CD a more "damage-resistant" material which accumulates damage slower, i.e. $C < C_1$ at one and the same fixed damage exponent ν. Hence for a given ν and C_1, the dimensionless time-to-rupture of the frame

$$\tau_f^* = t_f^*/t_b^* = \mathcal{T}(C/C_1) \tag{2.8}$$

will be a function of the dimensionless parameter C/C_1, as we shall see below. As a matter of fact, the function $\mathcal{T}$ will be of central importance in our study, since its behaviour (local extrema if any, monotonic decrease and/or increase, etc.) will allow us to draw non-trivial conclusions, concerning optimal "damage-resistant" design of the frame under study, i.e. to get its time-to-rupture τ_f^* as big as possible through an optimal choice of the damage material constants of the rods.

3. BASIC EQUATIONS

Let us now write down the above formulated basic equations in a dimensionless and more convenient form.

First, the equation of statics (2.4) in such a form reads

$$2s_1 \cos\alpha + s = 1\,, \tag{3.1}$$

where

$$s_1 = \sigma_1/\sigma_0\,, \quad s = \sigma/\sigma_0\,, \tag{3.2}$$

with σ_1 and σ_0 defined in Eq. (2.6).

Next, the damage law (2.1) can be recast as

$$\frac{d\omega}{d\tau} = \frac{1}{1+\nu}\left(\frac{s}{\psi}\right)^{\nu}, \tag{3.3}$$

see Eqs. (2.2), (2.3) and (3.2). In turn, the appropriate damage law for the side-rods becomes

$$\frac{d\omega_1}{d\tau} = \frac{1}{\xi\,(1+\nu)}\left(\frac{s_1}{\psi_1}\right)^{\nu}, \tag{3.4}$$

where the dimensionless quantity

$$\xi = C/C_1 \tag{3.5}$$

determines, so to say, the relative "damage-resistance" of the central rod as compared to that of the side ones (at a fixed "damage exponent" ν for all rods, let us recall).

To find the stresses in the rods and thus the damage accumulation rates by means of Eqs. (3.3) and (3.4), use is to be made now of the strain compatibility condition (2.5). Recall to this end that the rods are assumed to possess, in the virgin state ($\psi = \psi_1 = 1$), one and the same Young's modulus E^v. Two possibilities are open now.

First, as the simplest and rough approximation, one can assume that the Young's modulus is not influenced by damage. Then, from Eq. (2.5) (dividing both its sides by $E^v \sigma_0$), one gets

$$s_1 = s \cos^2 \alpha \,. \tag{3.6}$$

Together with Eq. (3.1), the latter relation yields the well-known elastic stresses in the rods, namely,

$$s = \frac{\sigma}{\sigma_0} = \frac{1}{1 + 2\cos^3 \alpha}\,, \quad s_1 = \frac{\sigma_1}{\sigma_0} = \frac{\cos^2 \alpha}{1 + 2\cos^3 \alpha}\,, \tag{3.7}$$

which therefore are *not affected* by the damage process taking place in the rods. In this way damage accumulation in them is not coupled in the case under study, cf. Eqs. (3.3) and (3.4), and hence they can be solved separately. The failure will have two distinct stages: in the first one all rods will sustain load ($\psi,\ \psi_1 > 0$); in the second stage either the central or the two side rods will already have failed, depending on the ratio ξ, see (3.5), so that the eventual failure will happen when the last of the rods will fail as well. Of course, these two stages will appear in the general case as well, but here, when damage accumulation in the rods is not coupled, the investigation and the appropriate formulae for the time-to-rupture are not difficult to be derived; that is why they will be skipped here.

Instead, let us treat in more detail the more realistic assumption when the current Young's modulus *is influenced* by damage, i.e. $E = E(\psi)$. (This assumption, as well as the idea to measure damage through the observed change in the elastic moduli of a damaging solid, is discussed in detail in [6], where the appropriate references are given as well.) The simplest approximation is to assume that

$$E(\psi) = E^v \psi = E^v (1 - \omega) \tag{3.8a}$$

for the central rod and, accordingly,

$$E(\psi_1) = E^v \psi_1 = E^v (1 - \omega_1) \tag{3.8b}$$

for the two side-rods, E^v denoting the Young's modulus for the virgin rods. It is noted that such an assumption is natural enough if one recalls the original Kachanov's interpretation of the continuity parameter ψ as the fraction of the undamaged rod cross-section area that only sustains load. Also, this assumption, roughly speaking, reflects the well-known Voigt approximation in mechanics of composite media, if the damage parameter ω is treated, somewhat loosely of course, as the void volume fraction in a porous solid. In this case, noting that

$$\sigma_1 = E^v \psi_1 \varepsilon_1\,, \quad \sigma = E^v \psi \varepsilon$$

in virtue of Eqs. (3.8), one finds from Eq. (2.5)

$$\frac{\sigma_1}{\psi_1} = \frac{\sigma}{\psi} \cos^2 \alpha \tag{3.9}$$

which, when coupled with Eq. (3.1), yields

$$s = \frac{\psi}{\psi + 2\psi_1 \cos^3 \alpha}, \quad s_1 = \frac{\psi_1 \cos^2 \alpha}{\psi + 2\psi_1 \cos^3 \alpha}. \tag{3.10}$$

Not surprisingly, for undamaged rods ($\psi = \psi_1 = 1$) the purely elastic solution, Eq. (3.7), is recovered once again from Eq. (3.10).

When inserted into Eqs. (3.3) and (3.4), the stresses from Eq. (3.10) now lead to the basic system of coupled differential equations that describes the damage accumulation of the rods, namely,

$$\frac{d\psi}{d\tau} = -f(\psi, \psi_1), \qquad \frac{d\psi_1}{d\tau} = -\frac{1}{A} f(\psi, \psi_1), \tag{3.11a}$$

with the notations

$$f(\psi, \psi_1) = \frac{1}{1+\nu} \left(\psi + 2\psi_1 \cos^3 \alpha\right)^{-\nu}, \quad A = \frac{\xi}{\cos^{2\nu} \alpha}, \tag{3.11b}$$

since $\omega = 1 - \psi$, $\omega_1 = 1 - \psi_1$. The system (3.11) should be solved under the natural initial conditions

$$\psi = 1, \quad \psi_1 = 1, \quad \text{at} \quad \tau = 0, \tag{3.12}$$

reflecting the fact that the rods are undamaged at the moment $t = 0$ when loading is applied.

4. SOLUTION OF THE BASIC SYSTEM OF EQUATIONS (3.11)

The solution of the basic initially-value problem (3.11) – (3.12) is elementary. First, dividing equations (3.11a) gives

$$\frac{d\psi}{d\psi_1} = A, \quad \text{i.e.} \quad \psi = A(\psi_1 - 1) + 1,$$

or

$$\omega = 1 - \psi = A\omega_1, \quad \omega_1 = 1 - \psi_1. \tag{4.1}$$

Hence an important consequence of the assumption of common damage exponent ν of the rods is the fact that their damage parameters are proportional, with the proportionality factor A, given in Eq. (3.11b). In this way it turns out that the value of the factor A, i.e. of the dimensionless ratio $\xi = C/C_1$, determines which of the rods will fail first. More precisely:

a) if $A < 1$, i.e. $A = \dfrac{C/C_1}{\cos^{2\nu} \alpha} < 1$ or $C < C_1 \cos^{2\nu} \alpha$, (4.2a) then the two side-rods fail simultaneously first;

$$\text{b) if } A = 1, \text{ i.e. } A = \frac{C/C_1}{\cos^{2\nu}\alpha} = 1 \text{ or } C = C_1 \cos^{2\nu}\alpha, \tag{4.2b}$$

then all rods fail simultaneously;

$$\text{c) if } A > 1, \text{ i.e. } A = \frac{C/C_1}{\cos^{2\nu}\alpha} > 1 \text{ or } C > C_1 \cos^{2\nu}\alpha, \tag{4.2c}$$

then the central rod fails first.

It is noted that these results are natural enough since, e.g., the inequality (4.2a) means that the "damage-resistance" of the central rod is considerably higher than that of the side ones because the constant C of the former is considerably less than that of the latter. The central rod accumulates thus damage slower than the other two and, not surprisingly, it ruptures last.

Now, introducing Eq. (4.1) into the second of Eqs. (3.11a) gives

$$\frac{d\psi_1}{d\tau} = -\frac{1}{A(1+\nu)} \left(A' + A''\psi_1\right)^{-\nu} \tag{4.3}$$

with the constants

$$A' = 1 - A, \quad A'' = A + 2\cos^3\alpha. \tag{4.4}$$

The integration of Eq. (4.3) gives

$$\tau = \frac{A}{A + 2\cos^3\alpha} \left[\left(1 + 2\cos^3\alpha\right)^{\nu+1} - \left(A' + A''\psi_1\right)^{\nu+1}\right]. \tag{4.5}$$

Solving Eq. (4.5) with respect to ψ_1 and using Eq. (4.1) lead to the needed explicit time-dependence of the rods' damage parameters during the loading in the frame under study.

5. DISCUSSION AND CONCLUDING REMARKS

Consider now in more detail the above mentioned three particular cases a) — c), see Eqs. (4.2), in order to determine the eventual time-to-rupture τ_f^* of the frame.

Let first $A < 1$, i.e. the case a) takes place. Then, at the end of the first stage of failure of the frame, when $\psi_1 = 0$ and the side-rods fail, the damage parameter of the central rod has the value $\omega_I = A < 1$, see Eq. (4.1). As it follows from Eq. (4.5), this happens at the moment

$$\tau_I^* = \frac{A}{A + 2\cos^3\alpha} \left[\left(1 + 2\cos^3\alpha\right)^{\nu+1} - \left(1 - A\right)^{\nu+1}\right] \quad (A < 1). \tag{5.1a}$$

In the second failure stage, when $\tau > \tau_I^*$, only the central rod "works", so that one should solve Eq. (2.3) with the initial condition $\omega = \omega_I$ at $\tau = \tau_I^*$ in order to find the final time-to-rupture, τ_f^*, of the whole frame, corresponding to the moment when $\omega = 1$. Elementary calculations give

$$\tau_f^* = \mathcal{T}^{(a)}(\xi) = \tau_I^* + (1-A)^{\nu+1} \tag{5.1b}$$

with τ_I^* given in Eq. (5.1a). (Note that in this second failure stage $T = F$, so that $s = 1$.)

Let us point out that $\tau_I^* = 0$ at $A = 0$, i.e. $\tau_f^* = 1$ at $A = 0$ as it should be. The reason is that $A = 0$ means that $C_1 = \infty$, so that the two side-rods fail instantaneously and from the very beginning only the central rod sustains load. Moreover, the time-to-rupture τ_f^* as a function of A should be increasing in the interval $A \in [0,1]$, since increasing A at fixed C and ν (and thus at fixed t_b^*) implies that the parameter C_1 decreases; hence the side-rods become more "damage-resistant" which increases, naturally enough, the life-time of the frame.

Next, from Eqs. (5.1) one immediately finds the time-to rupture τ_f^* in the case b) when all the rods fail simultaneously, just putting $A = 1$ in them:

$$\tau_f^* = \mathcal{T}^{(b)}(\xi) = \left(1 + 2\cos^3\alpha\right)^{\nu} \quad (A = 1)\,. \tag{5.2}$$

Let now $A > 1$, i.e. the case c) takes place. Then, at the end of the first stage of failure of the frame, when $\psi = 0$ and the central rod fails, the damage parameter of the side-rods has the value $\omega_I = 1/A < 1$, see Eq. (4.1). This happens at the moment

$$\tau_I^* = \frac{A}{A + 2\cos^3\alpha}\left[\left(1 + 2\cos^3\alpha\right)^{\nu+1} - \left(2\cos^3\alpha\,\frac{A-1}{A}\right)^{\nu+1}\right] \quad (A > 1)\,, \tag{5.3a}$$

as it again follows from Eq. (4.5). In the second failure stage, when $\tau > \tau_I^*$, only the side-rods "work", so that one should solve Eq. (3.4) with the initial condition $\omega = \omega_I$ at $\tau = \tau_I^*$ in order to find the final time-to-rupture, τ_f^*, of the whole frame, corresponding to the moment when $\omega_1 = 1$. Elementary calculations give

$$\tau_f^* = \mathcal{T}^{(c)}(\xi) = \tau_I^* + 2^{\nu}(1-A)^{\nu+1}\cos^{3\nu}\alpha \tag{5.3b}$$

with τ_I^* given this time by Eq. (5.3a). (Note that in this second failure stage $2T\cos\alpha = F$, so that $s_1 = 1/2\cos\alpha$.)

It is noted that $\tau_f^* \to \infty$ at $A \to \infty$, which again is natural. Indeed, at fixed $C > 0$ (in order that the basic time-unit t_b^* makes sense, cf. Eq. (2.2)) $A \to \infty$ only if $C_1 \to 0$, so that in the limit $A = \infty$ the side-rods do not accumulate damage. The only damage phenomenon will be in this case the failure of the central rod which will happen at the moment

$$\lim_{A\to\infty} \tau_f^* = (1 + 2\cos^3\alpha)^{\nu+1} - 2^{\nu+1}\cos^{3(\nu+1)}\alpha\,,$$

as it follows from Eq. (5.3a).

Combining now the formulae (5.1) to (5.3) gives

$$\tau_f^* = \mathcal{T}(\xi) = \begin{cases} \mathcal{T}^{(a)}(\xi), & \text{if } \xi < \cos^{2\nu}\alpha, \\ \mathcal{T}^{(b)}(\xi), & \text{if } \xi = \cos^{2\nu}\alpha, \\ \mathcal{T}^{(c)}(\xi), & \text{if } \xi > \cos^{2\nu}\alpha, \end{cases} \tag{5.4}$$

which accomplishes our aim — analytic evaluation of the function $\mathcal{T}(\xi)$, see Eq. (2.8), that gives the time-to-rupture τ_f^* of the whole frame for a given dimensionless ratio $\xi = C/C_1$ of Kachanov's material parameters of the rods (with a fixed and common "damage exponent" ν). The superscripts in Eq. (5.4) correspond obviously to the three different situations a) — c) of frame failure, discussed in Section 4, see Eq. (4.2).

For illustration the plot of the function $\tau_f^* = \mathcal{T}(\xi)$ for a typical angle $\alpha = \pi/4$ and $\nu = 3$ is shown in Fig. 2.

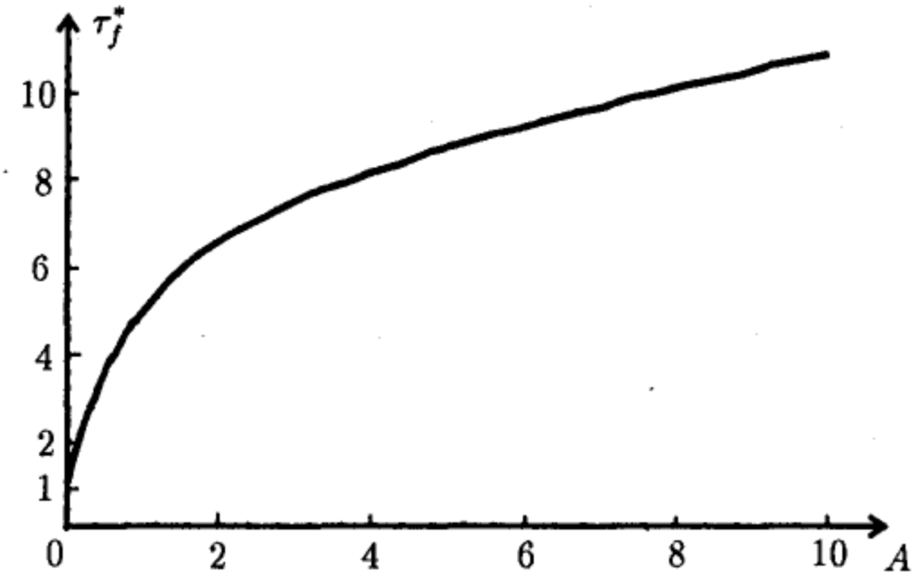

Fig. 2. Dimensionless time-to-rupture τ_f^* of the frame as a function of the parameter $A = (C/C_1)/\cos^{2\nu}\alpha$ at $\alpha = \pi/4$ and $\nu = 3$

A more detailed numerical investigation shows that τ_f^* is always a monotonically increasing function of the ratio C/C_1. This means in the damage mechanics context that for a given central rod one should add side-rods for which C_1 is as small as possible, i.e. their "damage-resistance" is as high as possible. Of course, this result should have been expected qualitatively. The above analysis allows us, however, to draw quantitative conclusions as well, i.e. to evaluate simply the relative time-to-rupture increase of the frame as compared to that of the central rod if it were a single one and subjected to the same tensile force **F**.

ACKNOWLEDGEMENTS. The support of this work by the Bulgarian Ministry of Education, Science and Technology under Grant No MM416-94 is gratefully acknowledged.

REFERENCES

1. Gastev, V. A. A brief course on strength of materials. 2nd ed., Nauka, Moscow, 1977 (in Russian).
2. Kachanov, L. M. Theory of creep. Gos. Izd. Fiz.-mat. Lit., Moscow, 1977 (in Russian).
3. Odqvist, F. K. G. Mathematical theory of creep and creep rupture. Oxford University Press, 1974.
4. Kachanov, L. M. T On the time to rupture in creep. *Izv. AN SSSR, Otd. Tehn. Nauk*, **8**, 1958, 26–31 (in Russian).
5. Kachanov, L. M. Introduction to continuum damage Mechanics. Klüwer Acad. Publ., 1986.
6. Lemaitre, J., J.-L. Chaboche. Mécanique des Materiaux Solids. Dunod, 2-ème ed., Paris, 1988.

Received on July 17, 1996
Revised on October 21, 1997

Galja M. Draganova
Institute of Chemical Technology
and Biotechnology
3 blvd. Aprilsko vastanie
POB 110, BG-7200 Razgrad
Bulgaria

Konstantin Z. Markov
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5 blvd J. Bourchier
BG-1164 Sofia, Bulgaria
e-mail: kmarkov@fmi.uni-sofia.bg