ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 1 — Математика и механика
Том 89, 1995

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 1 — Mathématiques et Mecanique
Tome 89, 1995

HYPERBOLIC AND EUCLIDEAN DISTANCE FUNCTIONS*

WALTER BENZ

In memory of
Nikola Obreshkoff (1896–1963),
the great Bulgarian mathematician

This is a functional equations approach to the non-negative functions $h(x,y)$ and $e(x,y)$ as defined in formulas (1) and (2). Moreover, all distance functions of $\mathbb{R}^n$ are characterized, which are invariant under linear and orthogonal mappings (see Theorem 1), and, especially, all functions of this type are determined, which satisfy in addition (D_2) (see Theorem 2). Here (D_2) asks for the invariance under euclidean or hyperbolic translations of the x_1-axis. Finally, additivity on the x_1-axis is considered, leading to the distance functions h and e up to non-negative factors (see Theorem 3).

Keywords: hyperbolic distance, invariance of distance functions under special motions, additivity on a line. theorems.
1991 Mathematics Subject Classification: 39B40, 51M10, 51K05.

1. Let $n > 1$ be an integer and let $\mathbb{R}_{\geq 0}$ be the set of all non-negative real numbers. A function

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$$

is then called a *distance function* of $\mathbb{R}_{\geq 0}$. Especially, we are interested in the *hyperbolic distance function* $h(x,y)$ satisfying

$$\cosh h(x,y) = \sqrt{1+x^2}\sqrt{1+y^2} - xy, \tag{1}$$

* Lecture accepted for the Session, dedicated to the centenary of the birth of Nikola Obreshkoff.

and, moreover, in the *euclidean distance function* $e(x, y)$ defined by

$$e(x,y) = \sqrt{(x-y)^2}. \tag{2}$$

In formulas (1) and (2)

$$uv = u_1v_1 + u_2v_2 + \cdots + u_nv_n \quad .$$

denotes the usual scalar product of the elements

$$u = (u_1, \ldots, u_n) \quad \text{and} \quad v = (v_1, \ldots, v_n)$$

of $\mathbb{R}^n$.

We will say that the distance function d of $\mathbb{R}$ is of type (D_1) if, and only if, it satisfies

(D_1) *$d(x,y) = d(\varphi(x), \varphi(y))$ for all $x, y \in \mathbb{R}^n$ and all linear and orthogonal mappings φ of $\mathbb{R}^n$.*

Obviously, distance functions h and e are of type (D_1).

2. It is posible to determine all distance functions d of $\mathbb{R}^n$ which are of type (D_1). We would like to prove the following

Theorem 1. *Define*

$$K := \left\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1, \xi_2 \in \mathbb{R}_{\geq 0} \text{ and } \xi_3^2 \leq \xi_1\xi_2\right\}.$$

Suppose that $f : K \to \mathbb{R}_{\geq 0}$ is chosen arbitrarily. Then

$$d(x,y) = f(x^2, y^2, xy) \tag{3}$$

is a distance function of $\mathbb{R}^n$ of type (D_1). If, vice versa, d is a distance function of $\mathbb{R}^n$ of type (D_1), there exists $f : K \to \mathbb{R}_{\geq 0}$ such that (3) holds true for all $x, y \in \mathbb{R}^n$.

Proof. Since $x^2 = [\varphi(x)]^2$ and $xy = \varphi(x)\varphi(y)$ for all $x, y \in \mathbb{R}^n$ and for every linear and orthogonal mapping φ of $\mathbb{R}^n$ into itself, we get

$$d(x,y) = d(\varphi(x), \varphi(y)).$$

d is hence of type (D_1).

Assume now that d is a distance function of $\mathbb{R}^n$. Suppose that

$$(\xi_1, \xi_2, \xi_3)$$

is an element of K and define

$$e_1 = (1, 0, \ldots, 0) \quad \text{and} \quad e_2 = (0, 1, 0, \ldots, 0)$$

as elements of $\mathbb{R}^n$. Put

$$x_0 = 0 \quad \text{and} \quad y_0 = e_1\sqrt{\xi_2}$$

in the case $\xi_1 = 0$. Observe here $\xi_3 = 0$, in view of $\xi_3^2 \leq \xi_1\xi_2$. Define now

$$f(\xi_1, \xi_2, \xi_3) = d(x_0, y_0).$$

Put $x_0 = e_1\sqrt{\xi_1}$ and

$$y_0 = \frac{e_1\xi_3 + e_2\sqrt{\xi_1\xi_2 - \xi_3^2}}{\sqrt{\xi_1}}$$

in the case $\xi_1 > 0$. Again define

$$f(\xi_1, \xi_2, \xi_3) = d(x_0, y_0).$$

Two things must now be proved. First of all we have to show that the function f is well-established. But since (ξ_1, ξ_2, ξ_3) is in K, there are only these two cases $\xi_1 = 0$ or $\xi_1 > 0$, and in both cases the value under f is uniquely determined. The second thing we have to prove, is that

$$d(x, y) = f(x^2, y^2, xy)$$

holds true for all $x, y \in \mathbb{R}^n$. Let x, y be elements of $\mathbb{R}^n$ and put

$$x^2 =: \xi_1, \quad y^2 =: \xi_2, \quad xy =: \xi_3.$$

Because of the Cauchy–Schwarz inequality, (ξ_1, ξ_2, ξ_3) must be an element of K. If we are able to prove that, there exists a linear and orthogonal mapping

$$\varphi : \mathbb{R}^n \to \mathbb{R}^n$$

satisfying

$$\varphi(x_0) = x \quad \text{and} \quad \varphi(y_0) = y,$$

where x_0, y_0 are the already defined elements with respect to ξ_i, then

$$d(x, y) = d(x_0, y_0) = f(\xi_1, \xi_2, \xi_3) = f(x^2, y^2, xy)$$

holds true and (3) is established. We now make use of the following simple statement: let a_1, a_2, a_3, b_1, b_2, b_3 be points of $\mathbb{R}^n$. Then there exists an orthogonal mapping ψ of $\mathbb{R}^n$ with

$$\psi(a_i) = b_i \quad \text{for all} \quad i \in \{1, 2, 3\}$$

if, and only if,

$$(a_i - a_j)^2 = (b_i - b_j)^2 \tag{4}$$

is satisfied for all $i, j \in \{1, 2, 3\}$ with $i < j$.

In order to apply this statement, we put

$$a_1 = 0 = b_1$$

and $a_2 = x_0$, $a_3 = y_0$, $b_2 = x$, $b_3 = y$. Since the assumptions (4), namely

$$x_0^2 = \xi_1 = x^2, \quad y_0^2 = \xi_2 = y^2$$

and $(x_0 - y_0)^2 = \xi_1 - 2\xi_3 + \xi_2 = (x - y)^2$ are satisfied, ψ exists; which is in addition linear in view of

$$\psi(0) = \psi(a_1) = b_1 = 0. \qquad \blacksquare$$

In the case of the hyperbolic distance function we apply the branch arg ≥ 0 of the inverse function of cosh and we have

$$f(x^2, y^2, xy) = \text{arg}\left(\sqrt{1 + x^2}\sqrt{1 + y^2} - xy\right).$$

In the case of $e(x, y)$ we get

$$f(x^2, y^2, xy) = \sqrt{x^2 + y^2 - 2xy}.$$

3. We would like to prove the following statement. If $z \neq 0$ is an element of $\mathbb{R}^n$, then there exists a bijection γ of $\mathbb{R}^n$ with $\gamma(0) = z$ and

$$h(x, y) = h(\gamma(x), \gamma(y))$$

for all $x, y \in \mathbb{R}^n$.

There definitely exists a linear and orthogonal mapping φ with $\varphi(z) = e_1\sqrt{z^2}$. Take now $t \geq 0$ satisfying

$$\cosh t = \sqrt{1 + z^2}.$$

Then $\tau(x) := \left(x_1 \cosh t + \sqrt{1 + x^2} \sinh t, x_2, \ldots, x_n\right)$ must be a bijection of $\mathbb{R}^n$, transforming 0 into

$$(\sinh t, 0, \ldots, 0) = e_1\sqrt{z^2}.$$

Now put $\gamma = \varphi^{-1}\tau$ and observe that

$$h(x, y) = h(\tau(x), \tau(y))$$

holds true for all $x, y \in \mathbb{R}^n$.

Remark. For more information about the mapping τ see the book [5] of the author.

It it well-known that $\mathbb{R}^n$ is a metric space with respect to the distance function $e(x, y)$. We would like to show the following

Proposition. *$\mathbb{R}^n$ is a metric space with respect to the distance function $h(x, y)$.*

Proof. Suppose that x, y are elements of $\mathbb{R}^n$. The inequality of Cauchy–Schwarz

$$(xy)^2 \leq x^2y^2$$

then implies $(xy)^2 \leq x^2y^2 + (x - y)^2$, i.e.

$$(xy)^2 + 2xy + 1 \leq (1 + x^2)(1 + y^2)$$

and hence $xy + 1 \leq |xy + 1| \leq \sqrt{1 + x^2}\sqrt{1 + y^2}$. We thus have

$$\sqrt{1 + x^2}\sqrt{1 + y^2} - xy \geq 1,$$

so that (1) determines $h(xy) \geq 0$ uniquely. In view of (1), obviously,

$$h(x, y) = h(y, x)$$

holds true for all $x, y \in \mathbb{R}^n$. Observe, moreover, $h(x, x) = 0$ for all $x \in \mathbb{R}^n$. Suppose now that $h(x, y) = 0$. Then (1) implies

$$(xy)^2 = (x - y)^2 + x^2y^2.$$

If x were $\neq y$, we would have the contradiction

$$(xy)^2 \leq x^2y^2 < (x - y)^2 + x^2y^2.$$

In order to prove the triangle inequality

$$h(x, z) \leq h(x, y) + h(y, z), \tag{5}$$

take a bijection γ of $\mathbb{R}^n$ satisfying $\gamma(0) = y$ and

$$h(p, q) = h(\gamma(p), \gamma(q)) \tag{6}$$

for all $p, q \in \mathbb{R}^n$. Put $a = \gamma^{-1}(x)$ and $b = \gamma^{-1}(z)$. Then we shall prove

$$h(a,b) \le h(a,0) + h(0,b), \tag{7}$$

which leads to (5) by applying (6). Now observe

$$-ab \le |ab| \le \sqrt{a^2}\sqrt{b^2},$$

i.e. $\sqrt{1+a^2}\sqrt{1+b^2} - ab \le \sqrt{1+a^2}\sqrt{1+b^2} + \sqrt{a^2}\sqrt{b^2}$. Hence

$$\cosh h(a,b) \le \cosh h(a,0).\cosh h(0,b) + \sinh h(a,0).\sinh h(0,b)$$

by observing

$$0 \le \sinh h(a,0) = \sqrt{\cosh^2 h(a,0) - 1} = a^2$$

and $0 \le \sinh h(0,b) = b^2$. Thus

$$\cosh h(a,b) \le \cosh(h(a,0) + h(0,b)).$$

This implies (7) since $\cosh t_1 \le \cosh t_2$ leads to $t_1 \le t_2$ for non-negative real numbers t_1, t_2.

Remark. Observe that $\mathbb{R}^n$ is also a metric space under the rather strange distance function

$$d(x,y) := h(x,y) + e(x,y)$$

(for all $x, y \in \mathbb{R}^n$) which is of type (D_1) as well.

4. We shall call a distance function $d(x,y)$ an *euclidean* (or a *hyperbolic*) distance function if it admits all euclidean (or all hyperbolic) motions.

Define for a distance function d the property (D_2), as follows:

(D_2) *$d(x,y) = d(\tau(x), \tau(y))$ for all $x, y \in \mathbb{R}^n$ and all euclidean (or hyperbolic) translations of the x_1-axis.*

The euclidean translations of the x_1-axis are the mappings

$$(x_1, \ldots, x_n) \to (x_1 + t, x_2, \ldots, x_n)$$

for $t \in \mathbb{R}$; the hyperbolic translations of the same axis are the already defined mappings

$$x \to \left(x_1 \cosh t + \sqrt{1+x^2}\sinh t, x_2, \ldots, x_n\right). \tag{8}$$

Theorem 2. *Let g be a function from $\mathbb{R}_{\ge 0}$ into $\mathbb{R}_{\ge 0}$. Then*

$$d(x,y) = g(e(x,y))$$

is an euclidean distance function, and

$$d(x,y) = g(h(x,y))$$

is a hyperbolic distance function. There are no other distance functions satisfying (D_1) and (D_2).

Proof. a) Let us assume that d satisfies (D_1) and (D_2) with respect to euclidean translations. Then d admits all congruent mappings of $\mathbb{R}^n$, in view of (D_1) and (D_2). Hence

$$d(x,y) = d(x + (-y), y + (-y)) = d(x - y, 0)$$

and thus $d(x,y) = f((x-y)^2, 0, 0)$ because of Theorem 1. Define

$$g(\xi) := f(\xi^2, 0, 0)$$

for all real $\xi \geq 0$. Hence

$$d(x,y) = g\left(\sqrt{(x-y)^2}\right) = g(e(x,y)).$$

b) Suppose that d is a distance function satisfying (D_1) and (D_2) with respect to hyperbolic translations. From

$$(x_1, \ldots, x_n) \in \mathbb{R}^n$$

we go over to Weierstrass co-ordinates

$$\left(x_1, \ldots, x_n, \sqrt{1 + x^2}\right).$$

The mapping (8) then reads

$$\tau\left(x_1, \ldots, x_n, \sqrt{1+x^2}\right) = \left(x_1, \ldots, x_n, \sqrt{1+x^2}\right) H(t)$$

with the $(n+1, n+1)$-matrix

$$H(t) = \begin{pmatrix} \cosh t & & & & \sinh t \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \sinh t & & & & \cosh t \end{pmatrix}$$

with zeros elsewhere. Let

$$B(p_1, \ldots, p_n; k)$$

be an arbitrary Lorentz boost (see [3, Sections 6.10, 6.11]). We hence have $k \geq 1$,

$$\begin{aligned} &p_1^2 + \cdots + p_n^2 < 1, \\ &k^2\left(1 - p_1^2 - \cdots - p_n^2\right) = 1. \end{aligned} \tag{9}$$

Set $\cosh t := k$, $t \geq 0$, and

$$(a_{11}, a_{21}, \ldots, a_{n1}) := \frac{\cosh t}{\sinh t}(p_1, \ldots, p_n)$$

for $t > 0$. (For $t = 0$, i.e. $k = 1$, the matrix B must be the identity matrix E, and we are not interested in this case.) Observe

$$a_{11}^2 + \cdots + a_{n1}^2 = \frac{k^2}{k^2 - 1} \sum_{i=1}^{n} p_i^2 = 1$$

from (9). Extend

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}$$

to an orthogonal matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

of $\mathbb{R}^n$. Define the so-called *induced Lorentz matrix*

$$\hat{A} := \left(\begin{array}{ccc|c} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array}\right)$$

and observe

$$B(p_1, \dots, p_n; k) = \hat{A} H(t) \hat{A}^{-1}.$$

(In the case $B = E$ we have $E = EH(0)E^{-1}$.) Because of A.10.1 (see [3. p. 249]), an arbitrary orthochronous Lorentz matrix of $\mathbb{R}^{n+1}$ can be written as the product of a Lorentz boost and an induced Lorentz matrix. This implies that the group $\overset{(n)}{H}$ of all motions of n-dimensional hyperbolic geometry (that is the group of all orthochronous Lorentz matrices of $\mathbb{R}^{n+1}$, see [4, Sections 2.6 and 5.7]), can be generated by $H(t)$, $t \in \mathbb{R}$, and the induced Lorentz matrices, i.e. by linear orthogonal mappings of $\mathbb{R}^n$ and hyperbolic translations concerning the x_1-axis. We now would like to define a function

$$g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

as follows: for $\xi \geq 0$ set

$$g(\xi) := d(0, e_1 \sinh \xi).$$

We then have to prove

$$d(x, y) = g(h(x, y))$$

for all $x, y \in \mathbb{R}^n$. Put $h(x, y) =: \xi$. Hence

$$h(x, y) = h(0, e_1 \sinh \xi).$$

Take a linear and orthogonal mapping φ_1 of $\mathbb{R}^n$ that transforms x in $e_1\sqrt{x^2}$, then a τ which maps this latter point into 0. With another φ_2 we get

$$\varphi_2 \tau \varphi_1(x) = 0 \quad \text{and} \quad \varphi_2 \tau \varphi_1(y) =: e_1 \eta$$

with $\eta \geq 0$. Because of

$$\xi = h(x, y) = h(0, e_1 \eta),$$

it follows $\cosh \xi = \cosh h(0, e_1 \eta) = \sqrt{1 + \eta^2}$, i.e.

$$\eta = \sinh \xi.$$

Hence with $\gamma := \varphi_2 \tau \varphi_1$

$$d(x,y) = d(\gamma(x), \gamma(y)) = d(0, e_1 \sinh \xi) = g(\xi) = g(h(x,y)).$$

With respect to the first part of Theorem 2 we know that e and h admit the corresponding mappings mentioned in (D_1) and (D_2). But those mappings already generate the automorphism groups of the geometries in question. ∎

A distance function d of $\mathbb{R}^n$ will be called *additive* on the x_1-axis if, and only if, the following property holds true:

(D_3) *Let α, β, γ be real numbers with $\alpha \leq \beta \leq \gamma$. Then*

$$d(\alpha e_1, \gamma e_1) = d(\alpha e_1, \beta e_1) + d(\beta e_1, \gamma e_1). \tag{10}$$

Theorem 3. *Let d be a distance function of $\mathbb{R}^n$ satisfying* (D_1), (D_2), (D_3). *Then*

$$d(x,y) = ke(x,y)$$

or

$$d(x,y) = kh(x,y)$$

holds true with a fixed real number $k \geq 0$.

Proof. a) *Euclidean case.* Taking into account Theorem 5 (see [4, Section 5.1]) we only need to prove that (D_3) carries over to every euclidean line of $\mathbb{R}^n$. Let x, z be distinct elements of $\mathbb{R}^n$ and let y be the element

$$y = \lambda x + (1 - \lambda) z$$

with $0 \leq \lambda \leq 1$. We then transform x, y, z in

$$\alpha e_1, \; \beta e_1, \; \gamma e_1$$

with $\alpha = 0$, $\beta = (1 - \lambda) e(x,z)$, $\gamma = e(x,z)$. Now with Theorem 2

$$d(x,y) = g(e(x,y)) = g(e(0, \beta e_1)) = d(0, \beta e_1)$$

and so on. Hence (10) yields

$$d(x,z) = d(x,y) + d(y,z).$$

Then everything else depends on the solution of the functional equation

$$g(\alpha + \beta) = g(\alpha) + g(\beta)$$

for all $\alpha, \beta \in \mathbb{R}_{\geq 0}$ (see [1]).

b) *Hyperbolic case.* We have to apply Theorem 9 (Section 2.6 in [4]) and a similar procedure as in part a). ∎

Remarks. 1) It is possible now to determine all distance functions d satisfying (D_1), (D_2), constituting a metric. By applying Theorem 2 the reader might verify the next statement which we shall formulate for the hyperbolic case. The situation in question is characterized by all functions

$$g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

satisfying

(i) $g(\xi) = 0 \iff \xi = 0$;

(ii) Let α, β, γ be real numbers such that there exists a triangle xyz with $\alpha = h(x,y)$, $\beta = h(y,z)$, $\gamma = h(z,x)$, then

$$g(\gamma) \leq g(\alpha) + g(\beta).$$

2) For general information about hyperbolic geometry compare [5–8].

REFERENCES

1. Aczél, J. Lectures on functional equations and their applications. Academic Press, New York – London, 1966.
2. Aczél, J., J. Dhombres. Functional Equations in several variables. New Tork, 1989.
3. Benz, W. Geometrische Transformationen. BI Wissenschaftsverlag, Mannheim, Leipzig, Wien, Zürich, 1992.
4. Benz, W. Real Geometries. BI Wissenschaftsverlag, Mannheim, Leipzig, Wien, Zürich, 1994.
5. Benz, W. Ebene Geometrie. Eine Einfhrung in Theorie und Anwendungen (to appear).
6. Karzel, H., K. Sörensen, D. Windelberg. Einführung in die Geometrie. UTB Vandenhoeck, Göttingen, 1973.
7. Nevanlinna, R., P. Kustaanheimo. Grundlagen der Geometrie. Basel, 1976.
8. Schwerdtfeger, H. Geometry of complex numbers. Circle Geometry, Moebius Transformation, Noneuclidean Geometry. Dover Publications, New York, 1979.

Received on 10.06.1996

E-mail address: benz@math.uni-hamburg.de