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AN INEQUALITY OF DUFFIN-SCHAEFFER-SCHUR TYPE*

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It is shown here that the transformed Chebyshev polynomial of the second kind $\overline{U}_n(x) := U_n(x\cos\frac{\pi}{n+1})$ has the greatest uniform norm in $[-1,\ 1]$ of its k-th derivative ($k = 1,\dots,n$) among all algebraic polynomials of degree not exceeding n, which vanish at ±1 and whose absolute value is less than or equal to 1 at the points $\{\cos\frac{j\pi}{n}\ /\ \cos\frac{\pi}{n+1}\}_{j=1}^{n-1}$.

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1. INTRODUCTION AND STATEMENT OF RESULT

Denote by π_n the set of all real algebraic polynomials of degree at most n. As usual, $T_n(x) = \cos n \arccos x$ denotes the n-th Chebyshev polynomial of the first kind. In what follows, $\|\cdot\|$ will mean the uniform norm in $[-1,1]$, $\|f\| := \sup\limits_{x\in[-1,1]} |f(x)|$.

The classical inequality of I. Schur [15] asserts that the transformed Chebyshev polynomial $\overline{T}_n(x) = T_n\left(x\cos\dfrac{\pi}{2n}\right)$ has the greatest uniform norm of its first derivative on $[-1,\ 1]$ among all $f \in \pi_n$, which vanish at the boundary points ±1, and whose uniform norm is less than or equal to 1.

Recently, this result was extended to higher order derivatives by Milev and Nikolov [10] (the special cases $k = 2$ and $k = 3$ have been examined earlier by

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Milev [8, 9]).

Theorem A ([10, Theorem 1.1]). *If $f \in \pi_n$ satisfies*

$$f(-1) = f(1) = 0 \tag{1.1}$$

and

$$\|f\| \le 1, \tag{1.2}$$

then

$$\|f^{(k)}\| \le \|\overline{T}_n^{(k)}\| \tag{1.3}$$

for $k = 1, \ldots, n$. Equality in (1.3) is possible if and only if $f = \pm\overline{T}_n$.

Let $\{y_j^\star\}_{j=1}^{n-1}$ be defined by

$$y_j^\star = \frac{\cos(j\pi/n)}{\cos(\pi/2n)}.$$

For $k \ge 2$ Milev and Nikolov proved the following extension of Theorem A.

Theorem B ([10, Theorem 1.2]). *Let $f \in \pi_n$ satisfy* (1.1) *and*

$$|f(y_j^\star)| \le 1, \quad j = 1, \ldots, n-1. \tag{1.4}$$

Then the inequality (1.3) *holds for $k = 2, \ldots, n$. Moreover, equality is possible if and only if $f = \pm\overline{T}_n$.*

Theorem B asserts that the condition (1.2) in Theorem A is unnecessarily restrictive, and that for $k \ge 2$ the inequality (1.3) remains valid if (1.2) is replaced by the weaker requirement $|f(x)| \le |\overline{T}_n(x)|$ at the extremal points of $\overline{T}_n$, i.e., at $\{y_j^\star\}_{j=1}^{n-1}$. This is very similar to the extension of the Markov inequality, found by Duffin and Schaeffer [4]. For some related results the reader may consult [1, 2, 5, 11–13, 16].

Regarding Theorem B, the following question arises in a natural way: what would happen if the "comparison points" $\{y_j^\star\}_{j=1}^{n-1}$ in (1.4) are replaced by some other points? Answering this question for arbitrary $\{y_j\}_{j=1}^{n-1}$ seems to be a very difficult task.

In this paper we examine completely the case

$$y_j = \frac{\cos(j\pi)}{\cos(\pi/(n+1))}, \quad j = 1, \ldots, n-1.$$

It turns out that in this case the extremizer for all $k \in \{1, \ldots, n\}$ is the transformed Chebyshev polynomial of the second kind $\overline{U}_n$,

$$\overline{U}_n(x) := U_n\left(x \cos\frac{\pi}{n+1}\right).$$

Precisely, we prove the following Duffin-Schaeffer-Schur type inequality:

Theorem 1.1. *Let* $f \in \pi_n$ *satisfy* (1.1) *and*

$$\left| f\left(\frac{\cos(j\pi/n)}{\cos(\pi/(n+1))}\right)\right| \le 1, \quad j = 1, \dots, n-1. \tag{1.5}$$

Then

$$\|f^{(k)}\| \le \|\overline{U}_n^{(k)}\| \tag{1.6}$$

for all $k \in \{1, \dots, n\}$. *Moreover, equality in* (1.6) *is possible if and only if* $f = \pm \overline{U}_n$.

The paper is organized as follows. In Section 2 we prove a pointwise inequality (Theorem 2.1), which is the main ingredient of the proof of Theorem 1.1. The necessary auxiliary results are proven in Section 3, with the exception of Lemma 3.5, the proof of which is the content of Section 5. In Section 4 we prove Theorem 1.1.

2. A POINTWISE INEQUALITY

For the sake of convenience we examine the usual Chebyshev polynomial of the second kind $U_n(x) := \dfrac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}$ on the interval $[-\eta, \eta]$, where $\eta := \cos\dfrac{\pi}{n+1}$. For this reason the conditions (1.1) and (1.5) are replaced by

$$f(-\eta) = f(\eta) = 0, \tag{2.1}$$

and

$$\left| f\left(\cos\frac{j\pi}{n}\right)\right| \le 1, \quad j = 1, \dots, n-1. \tag{2.2}$$

Throughout, $\|\cdot\|_\star$ will mean the uniform norm in $[-\eta, \eta]$, i.e.,

$$\|f\|_\star := \sup_{x \in [-\eta, \eta]} |f(x)|.$$

Theorem 1.1 is proved with the help of the pointwise inequality, given by the next theorem.

Theorem 2.1. *Let* $f \in \pi_n$ *satisfy the conditions* (2.1)–(2.2). *Then for each* $k \in \{1, \dots, n\}$ *and for every* $x \in [-\eta, \eta]$

$$|f^{(k)}(x)| \le \max\{|U_n^{(k)}(x)|, |Z_{n,k}(x)|\},$$

where

$$Z_{n,k}(x) := \frac{1}{k\eta}\left[\left(x^2 - \frac{n+k}{n}\eta^2\right) T_n^{(k+1)}(x) + kx T_n^{(k)}(x)\right]. \tag{2.3}$$

Proof. Let $x_0 < x_1 < \ldots < x_n$ be the zeros of $\omega(x) := (x^2 - \eta^2)T_n'(x)$, and let $\omega_\nu(x) := \omega(x)/(x - x_\nu)$, $\nu = 0, \ldots, n$. For every polynomial f of degree at most n the Lagrange interpolation formula yields

$$f^{(k)}(x) = \sum_{\nu=0}^{n} \frac{f(x_\nu)}{\omega_\nu(x_\nu)} \omega_\nu^{(k)}(x). \tag{2.4}$$

In particular, for $f \in \pi_n$ satisfying (2.1)–(2.2), (2.4) yields

$$|f^{(k)}(x)| \le \sum_{\nu=1}^{n-1} \left| \frac{\omega_\nu^{(k)}(x)}{\omega_\nu(x_\nu)} \right|. \tag{2.5}$$

According to a well-known result of V. Markov, if the zeros of two polynomials interlace, then the interlacing property is inherited by the zeros of their derivatives (for a proof see, e.g., [14, Lemma 2.7.1]). In particular, for polynomials of the same degree this result could be interpreted as monotone dependence of the zeros of the derivative of a polynomial on its zeros ([1, p. 39]). Since for $i > j$ the zeros of $\omega_i(x)$ are less than or equal to the corresponding zeros of $\omega_j(x)$, we conclude that the same relation remains valid for the zeros of $\omega_i^{(k)}$ and $\omega_j^{(k)}$. Hence, the j-th zeros of the polynomials $\{\omega_i^{(k)}\}_{i=1}^{n-1}$ are located between the j-th zero of $\omega_n^{(k)}$ and the j-th zero of $\omega_0^{(k)}$. Denote by $\{\beta_i\}_{j=1}^{n-k}$ and $\{\alpha_i\}_{i=2}^{n-k+1}$ the zeros of $\omega_n^{(k)}$ and $\omega_0^{(k)}$, respectively, arranged in increasing order. Set $\alpha_1 := -\eta$, $\beta_{n-k+1} := \eta$, then the above reasoning implies that

$$\text{sign}\,\{\omega_\nu^{(k)}(x)\} \quad \text{is the same for all } \nu \in \{1, \ldots, n-1\} \text{ when } x \in [\alpha_j, \beta_j]. \tag{2.6}$$

We observe that the zeros of ω and U_n interlace, and in addition

$$U_n(x_\nu) = \text{sign}\,\{\omega_\nu(x_\nu)\} = (-1)^{n-\nu}, \quad \nu = 1, \ldots, n-1.$$

Therefore, for $x \in [\alpha_j, \beta_j]$ ($j \in \{1, \ldots, n-k+1\}$) the substitution $f = U_n$ in (2.4) yields

$$|U_n^{(k)}(x)| = \left| \sum_{\nu=1}^{n-1} \frac{\omega_\nu^{(k)}(x)}{|\omega_\nu(x_\nu)|} \right| = \sum_{\nu=1}^{n-1} \left| \frac{\omega_\nu^{(k)}(x)}{\omega_\nu(x_\nu)} \right|. \tag{2.7}$$

Comparison of (2.7) and (2.5) implies that if f satisfies the assumptions of Theorem 2.1, then

$$|f^{(k)}(x) \le |U_n^{(k)}(x)| \quad \text{for all } x \in \bigcup_{j=1}^{n-k+1} [\alpha_j, \beta_j]. \tag{2.8}$$

Our next goal is to prove that under the same assumptions

$$|f^{(k)}(x)| \le |Z_{n,k}(x)| \quad \text{for all } x \in \bigcup_{j=1}^{n-k} (\beta_j, \alpha_{j+1}). \tag{2.9}$$

Observing that the j-th zero of U_n is located between the j-th zero of ω_n and the j-th zero of ω_0 (precisely, the first zeros of U_n and ω_n and the last zeros of U_n and ω_0 coincide), we conclude on the basis of V. Markov's result that each interval (β_j, α_{j+1}), $j = 1, \ldots, n-k$, contains exactly one zero of $U_n^{(k)}$, and consequently

$$\operatorname{sign}\{U_n^{(k)}(\alpha_j)\} = \operatorname{sign}\{U_n^{(k)}(\beta_j)\} = (-1)^{n+1-k-j}, \quad j = 1, \ldots, n-k. \tag{2.10}$$

Using the identity (cf. [17, eqs. (4.7.28)])

$$U_n^{(k)}(x) = xU_{n-1}^{(k)}(x) + (n+k)U_{n-1}^{(k-1)}(x), \tag{2.11}$$

it is not difficult to see that

$$Z_{n,k}(x) - U_n^{(k)}(x) = \frac{1}{k\eta}\left(x - \frac{n+k}{n}\eta\right)\omega_n^{(k)}(x), \tag{2.12}$$

$$Z_{n,k}(x) + U_n^{(k)}(x) = \frac{1}{k\eta}\left(x + \frac{n+k}{n}\eta\right)\omega_0^{(k)}(x), \tag{2.13}$$

whence

$$Z_{n,k}(x) = \begin{cases} -U_n^{(k)}(x) & \text{for } x = \alpha_j,\ j = 2, \ldots, n-k+1, \\ U_n^{(k)}(x) & \text{for } x = \beta_j,\ j = 1, \ldots, n-k. \end{cases} \tag{2.14}$$

If $f \in \pi_n$ satisfies the assumptions of Theorem 2.1, then according to the above reasonings $|f^{(k)}| \le |U_n^{(k)}|$ at the zeros of $\omega_0^{(k)}$ and $\omega_n^{(k)}$. The relations (2.14) and (2.10) then imply that each of the polynomials $Z_{n,k} \pm f^{(k)}$ has at least one zero in each of the intervals $[\alpha_j, \beta_j]$, $j = 2, \ldots, n-k$. Moreover, $\operatorname{sign}\{(Z_{n,k} \pm f^{(k)})(\alpha_{n-k+1})\} = -\operatorname{sign}\{U_n^{(k)}(\alpha_{n-k+1})\} = -1$. Since $Z_{n,k} \pm f^{(k)}$ have positive leading coefficients, it follows that each of them has at least one zero located to the right of α_{n-k+1}. Similar arguments show that $Z_{n,k} \pm f^{(k)}$ must have at least one zero located to the left of β_1. Hence, each of the polynomials $Z_{n,k} \pm f^{(k)}$ has maximal possible number of zeros $(n-k+1)$, and all these zeros lie outside the set $\cup_{j=1}^{n-k}(\beta_j, \alpha_{j+1})$. Now the observation that $|f^{(k)}| \le |Z_{n,k}|$ on the boundary of this set completes the proof of (2.9). Theorem 2.1 is proved. ∎

Remark 1. The claims of both Theorem 1.1 and Theorem 2.1 are trivial when $n \le 2$. Following the proof of Theorem 2.1, one observes that if $k = n$, then $|f^{(k)}(x)| \le |U_n^{(k)}(x)|$ on the whole real axis for every function f satisfying (2.1) and (2.2). Thus, for $k = n$

$$\|f^{(k)}\|_* \le \|U_n^{(k)}\|_*. \tag{2.15}$$

Furthermore, if $k = n-1$, then $f^{(k)}$ is a polynomial of degree 1, and therefore $\|f^{(k)}\|_*$ is attained at $x = -\eta$ or at $x = \eta$. According to (2.8), at these points we have $|f^{(k)}| \le |U_n^{(k)}|$, and therefore again (2.15) holds. The statement of Theorem

1.1 then follows for $k = n - 1, n$ from (2.15) after a linear transformation (see also [16, Corollary 4]).

For this reason, we may restrict our considerations to the case $n \geq k + 2$.

Remark 2. Studying carefully the proof of Theorem 2.1, one can see that for any fixed point $x_0 \in \bigcup_{j=1}^{n-k} (\beta_j, \alpha_{j+1})$ the exact upper bound for $|f^{(k)}(x_0)|$ in (2.5) subject to the constraints (2.1)–(2.2) is attained for a polynomial, which alternates between -1 and 1 at the points $\{x_i\}_{i=1}^{n-1}$ with only one exception (i.e., $|f(x_i)| = 1$, $i = 1, \ldots, n-1$, there is a $\lambda \in \{1, \ldots, n-2\}$ such that $f(x_\lambda)f(x_{\lambda+1}) > 0$ and $f(x_i)f(x_{i+1}) < 0$ for $i \neq \lambda$). Following the notations of Gusev [6], we may call these polynomials as Zolotarev polynomials. The number of the essentially different Zolotarev polynomials is $[(n-1)/2]$, e.g., 1, if $n = 3, 4$; 2, if $n = 5, 6$; 3, if $n = 7, 8$, etc. Hence, for small n one can examine directly all the possible extremal polynomials in Theorem 1.1.

3. AUXILIARY RESULTS

We begin with listing in a lemma some well-known properties of the ultraspherical polynomials $P_n^{(\lambda)}$ $(\lambda > -1/2)$. Recall that $P_n^{(\lambda)}$ is the n-th orthogonal polynomial in $[-1, 1]$ with respect to the weight $w_\lambda(x) = (1 - x^2)^{\lambda - 1/2}$ and normalized (for $\lambda \neq 0$) by $P_n^{(\lambda)}(1) = \binom{n + 2\lambda - 1}{n}$ (in particular, $P_n^{(1)} = U_n$). In the case $\lambda = 0$ the Chebyshev polynomial T_n is orthogonal and satisfies $T_n(1) = 1$.

Lemma 3.1. (i) *For every* $\lambda > -1/2,\ \lambda \neq 0$,

$$\frac{d}{dx}\{P_n^{(\lambda)}(x)\} = 2\lambda P_{n-1}^{(\lambda+1)}(x)$$

(*the case* $\lambda = 0$ *reads as* $T_n'(x) = nP_{n-1}^{(1)}(x)$).

(ii) *For every* $\lambda \geq \mu > -1/2$, $P_m^{(\lambda)}$ *obeys a representation*

$$P_n^{(\lambda)}(x) = \sum_{m=0}^{n} a_{m,n}(\lambda, \mu) P_m^{(\mu)}(x) \quad \text{with } a_{m,n}(\lambda, \mu) \geq 0,\ m = 0, \ldots, n\,.$$

(iii) *For* $\lambda > 0$ *the absolute values of the local extrema of* $P_n^{(\lambda)}$ *increase as the distance between the points of local extrema and the origin increases.*

(iv) $y = P_n^{(\lambda)}$ *satisfies the differential equation*

$$y'' - (2\lambda + 1)xy' + n(n + 2\lambda)y = 0.$$

For easy reference we formulate in a lemma two simple facts from calculus, which will be used frequently in the sequel.

Lemma 3.2. (i) *For any fixed* $\alpha \in (0,2)$ *the sequence* $a_n = n \sin \frac{\alpha\pi}{n}$, $n = 2, 3, \ldots$, *is monotone increasing.*

(ii) *For any fixed* $0 < \alpha < \beta \le 2$ *the sequence* $b_n := \frac{\sin(\beta\pi/n)}{\sin(\alpha\pi/n)}$, $n = 3, 4, \ldots$, *is monotone increasing.*

Lemma 3.3. *For every natural* $n \ge 2$ *and for* $k = 2, \ldots, n$ *there holds*

$$\max_{x\in[-\cos(\pi/n),\cos(\pi/n)]} |T_n^{(k)}(x)| = T_n^{(k)}(\eta). \tag{3.1}$$

Proof. It suffices to prove only the case $k = 2$. Indeed, if (3.1) is established for $k = 2$, then it follows that $\|T_m''\|_\star = T_m''(\eta)$ for all $m \le n$. For $k \ge 3$, Lemma 3.1(i)–(ii) yields

$$T_n^{(k)}(x) = \sum_{m=2}^{n-k+2} a_m T_m''(x) \quad \text{with } a_m \ge 0,$$

and consequently,

$$\|T_n^{(k)}\|_\star \le \sum_{m=2}^{n-k+2} a_m \|T_m''\|_\star = \sum_{m=2}^{n-k+2} a_m T_m''(\eta) = T_n^{(k)}(\eta).$$

Thus it remains to prove (3.1) for $k = 2$. The cases $n = 2, 3$ are trivial, therefore we suppose that $n \ge 4$. According to Lemma 3.1(i), (iii), we have to compare $T_n''(\eta)$ with $T_n''(z)$, where $z = \cos\tau$ is the last critical point for T_n'', i.e., the last zero of T_n'''. The explicit representation of T_n yields, with $x = \cos\theta$, $0 < \theta < \pi$,

$$T_n''(x) = n\cos\theta \frac{\sin n\theta}{\sin^3\theta} - n^2 \frac{\cos n\theta}{\sin^2\theta}, \tag{3.2}$$

$$T_n'''(x) = \frac{n}{\sin^5\theta}\left\{[3 - (n^2+2)\sin^2\theta]\sin n\theta - 3n\sin\theta\cos\theta\cos n\theta\right\}. \tag{3.3}$$

Putting $\theta = 2\pi/n$ and $\theta = 3\pi/(2n)$ in (3.3), we get

$$T_n'''\left(\cos\frac{2\pi}{n}\right) = -3n^2\frac{\cos(2\pi/n)}{\sin^4(2\pi/n)} \le 0,$$

$$T_n'''\left(\cos\frac{3\pi}{2n}\right) = \frac{n}{\sin^5(3\pi/(2n))}\left[(n^2+2)\sin^2\frac{3\pi}{2n} - 3\right] > 0,$$

hence T_n''' has a zero in the interval $\left[\cos\frac{2\pi}{n}, \cos\frac{3\pi}{2n}\right)$. This zero is readily seen to be unique, and it is the last critical point of T_n''.

The equation (3.2) can be rewritten as

$$T_n''(\cos\theta) = \frac{n}{2\sin^3\theta}\varphi(\theta), \tag{3.4}$$

where $\varphi(\theta) = (n+1)\sin(n-1)\theta - (n-1)\sin(n+1)\theta$. The points in $(0,\pi)$, at which the function φ has local extrema, are $\theta_k = \dfrac{k\pi}{n}$, $k = 1, \ldots, n-1$, and

$$|\varphi(\theta_k)| = 2n \sin\frac{k\pi}{n}. \tag{3.5}$$

Taking into account the inequalities $\cos\dfrac{2\pi}{n} < \cos\tau < \cos\dfrac{3\pi}{2n}$, we obtain from (3.4) and (3.5)

$$|T_n''(\cos\tau)| < n^2 \frac{\sin(2\pi/n)}{\sin^3(3\pi/(2n))}. \tag{3.6}$$

The substitution $\theta = \dfrac{\pi}{n}$ in (3.2) yields

$$\left|T_n''\left(\cos\frac{\pi}{n}\right)\right| = \frac{n^2}{\sin^2(\pi/n)} \tag{3.7}$$

and the lemma will be proved if we succeed to show that the right-hand side of (3.7) is greater than the right-hand side of (3.6), which is equivalent to

$$\left(\frac{\sin(3\pi/(2n))}{\sin(\pi/n)}\right)^3 \geq 2\cos\frac{\pi}{n}. \tag{3.8}$$

According to Lemma 3.2(i), the left-hand side of (3.8) is increasing with respect to n, and for $n \geq 4$ it is greater than 2. This completes the proof of Lemma 3.3. ∎

Remark 3. A more precise examination of equation (3.3) shows that for $n \geq 5$ the last critical point of T_n'' is located in $\left(\cos\dfrac{2\pi}{n}, \cos\dfrac{7\pi}{4n}\right)$ (see the proof of Lemma 5.1 below).

As an immediate consequence from Lemma 3.3 we get

Corollary 3.1. *For all natural $k \leq n$ there hold:*

(a) $||U_n^{(k)}||_\star = U_n^{(k)}(\eta)$;

(b) $||T_n^{(k)}||_\star = T_n^{(k)}(\eta)$ for $k \geq 2$.

The function $Z_{n,k}(x)$ appearing in Theorem 2.1 can be represented as

$$Z_{n,k}(x) = c_k[u_{n,k}(x) - v_{n,k}(x)], \tag{3.9}$$

where $c_k := 1/(k\eta)$ and

$$u_{n,k}(x) := (x^2-1)T_n^{(k+1)}(x) + kxT_n^{(k)}(x), \tag{3.10}$$

$$v_{n,k}(x) := \left(\frac{n+k}{n}\eta^2 - 1\right) T_n^{(k+1)}(x). \tag{3.11}$$

We quote without proof the following simple lemma:

Lemma 3.4. *The inequality*

$$\frac{n+k}{n}\eta^2 \geq 1$$

holds for every $n \geq 9$ *if* $k=1$, *for every* $n \geq 5$ *if* $k=2$, *and for every* $n, k \geq 3$.

Corollary 3.2. *For* n *and* k *as in Lemma 3.4 there holds*

$$\|v_{n,k}\|_\star = v_{n,k}(\eta).$$

The next lemma shows that a similar conclusion holds for the function $u_{n,k}(x)$.

Lemma 3.5. *For all natural* $k \geq 3$ *there holds*

$$\|u_{n,k}\|_\star = u_{n,k}(\eta). \tag{3.12}$$

The proof of this lemma requires more work, and we put it off to the last section.

4. PROOF OF THEOREM 1.1

According to Remark 1, we may assume that $n \geq k+2$. We exclude also the cases $k=1,\ 3 \leq n \leq 8$, and $k=2,\ n=4$, which are verified directly as indicated in Remark 2. For the remaining n and k we shall prove the inequality

$$\|Z_{n,k}\|_\star < \|U_n^{(k)}\|_\star. \tag{4.1}$$

Having established (4.1), we can readily deduce Theorem 1.1 as a corollary of Theorem 2.1. Indeed, if $f \in \pi_n$ satisfies (1.1) and (1.5), then $p(x) := f(x/\eta)$ will satisfy (2.1)–(2.2). It follows then from Theorem 2.1 that for every $x \in [-\eta, \eta]$

$$|p(x)| \leq \max\{|U_n^{(k)}(x)|, |Z_{n,k}(x)|\} \leq \|U_n^{(k)}\|_\star = U_n^{(k)}(\eta). \tag{4.2}$$

Then $f(x) = p(x\eta)$ will satisfy

$$\|f^{(k)}\| = \eta^k \|p^{(k)}\|_\star \leq \eta^k U_n^{(k)}(\eta) = \|\overline{U}_n^{(k)}\|,$$

whence the inequality of Theorem 1.1 follows. To clarify the cases in which equality holds, we observe that equality in (4.2) is possible only if $x = \pm\eta$ and $p = \pm U_n$. This completes the proof of Theorem 1.1.

It remains to prove (4.1). Equation (3.9) and Corollary 3.2 yield

$$\begin{aligned}
\|Z_{n,k}\|_\star &\leq c_k[\|u_{n,k}\|_\star + \|v_{n,k}\|_\star] = c_k[v_{n,k}(\eta) + \|u_{n,k}\|_\star] \\
&= -Z_{n,k}(\eta) + c_k[u_{n,k}(\eta) + \|u_{n,k}\|_\star] \\
&= U_n^{(k)}(\eta) - c_k\frac{k(2n+k)}{n}\eta T_n^{(k)}(\eta) + c_k[u_{n,k}(\eta) + \|u_{n,k}\|_\star]
\end{aligned}$$

(for the last equality we used equation (2.13)). Clearly, the inequality (4.1) will hold if we succeed in showing that

$$u_{n,k}(\eta) + \|u_{n,k}\|_\star < \frac{k(2n+k)}{n}\eta T_n^{(k)}(\eta). \tag{4.3}$$

In the proof of (4.3) we shall distinguish between the cases $k = 1$, $k = 2$ and $k \geq 3$. Note that

$$T_n(\eta) = -\cos\frac{\pi}{n+1}, \quad T_n'(\eta) = n, \quad T_n''(\eta) = n(n+1)\frac{\cos(\pi/(n+1))}{\sin^2(\pi/(n+1))}. \tag{4.4}$$

Case $k = 1$. Lemma 3.1(iv) implies that $u_{n,1}(x) = n^2 T_n(x)$ and the inequality (4.3) in this case reduces to

$$\cos\frac{\pi}{n+1} > \left(\frac{n}{n+1}\right)^2,$$

the verification of which causes no difficulties.

Case $k = 2$. Using Lemma 3.1(iv) we obtain $u_{n,2}(x) = (n^2-1)T_n'(x) - xT_n''(x)$. The explicit form of T_n' and Lemma 3.3 yield the estimate

$$\|u_{n,2}\|_\star \leq (n^2-1)\|T_n'\|_\star + \|xT_n''(x)\|_\star \leq \frac{n(n^2-1)}{\sin(\pi/(n+1))} + \eta T_n''(\eta),$$

and consequently,

$$u_{n,2}(\eta) + \|u_{n,2}\|_\star \leq n(n^2-1)\left(1 + \sin^{-1}\frac{\pi}{n+1}\right).$$

Therefore, (4.3) will follow with $k = 2$ if

$$n(n-1)\left[\sin\frac{\pi}{n+1} + \sin^2\frac{\pi}{n+1}\right] < 4(n+1)\cos^2\frac{\pi}{n+1}.$$

Putting $A_n := (n+1)\sin\dfrac{\pi}{n+1}$, we rewrite the latter inequality in the following form

$$(n-2)A_n + A_n^2 + (A_n+2)\sin\frac{\pi}{n+1} + 2\sin^2\frac{\pi}{n+1} < 4(n+1). \tag{4.5}$$

According to Lemma 3.2(i), $A_n < A_\infty = \pi$, and we increase the left-hand side of (4.5) to obtain the inequality

$$(n-2)\pi + \pi^2 + (\pi+2)\sin\frac{\pi}{n+1} + 2\sin^2\frac{\pi}{n+1} < 4(n+1),$$

which is easily seen to be true for all $n \geq 4$.

Case $k \geq 3$. According to Lemma 3.5, in this case $\|u_{n,k}\|_\star = u_{n,k}(\eta)$ and the inequality (4.3) becomes

$$(1-\eta^2)T_n^{(k+1)}(\eta) + \frac{k^2}{2n}\eta T_n^{(k)}(\eta) \geq 0,$$

which is obviously true, since η is located to the right from the right-most zero of $T_n^{(k)}$, $k = 1, \ldots, n$. With this (4.3) is proved and therefore Theorem 1.1. ∎

5. PROOF OF LEMMA 3.5

We first observe that the general case is a consequence of the case $k = 3$. Indeed, let

$$\|u_{n,3}\|_\star = u_{n,3}(\eta). \tag{5.1}$$

It is readily seen that $u_{n,3}$ is strictly monotone increasing to the right of $x = \eta$. This implies that (5.1) follows also with n replaced by m, $m \le n$. Then for $k \ge 4$ Lemma 3.1(ii) and Corollary 3.1(b) yield

$$T_n^{(k)}(x) = \sum_{m=3}^{n-k+3} b_m T_m'''(x) \ \text{ with non-negative } b_m,\ m = 3, \ldots, n-k+3,$$

$$\begin{aligned}
\|u_{n,k}\|_\star &= \|(x^2-1)T_n^{(k+1)}(x) + 3xT_n^{(k)}(x) + (k-3)xT_n^{(k)}(x)\|_\star \\
&= \|\sum_{m=3}^{n-k+3} b_m u_{m,3}(x) + (k-3)xT_n^{(k)}(x)\|_\star \\
&\le \sum_{m=3}^{n-k+3} b_m \|u_{m,3}\|_\star + (k-3)\eta T_n^{(k)}(\eta) \\
&= \sum_{m=3}^{n-k+3} b_m u_{m,3}(\eta) + (k-3)\eta T_n^{(k)}(\eta) = u_{n,k}(\eta).
\end{aligned}$$

The proof of (5.1) goes through several lemmas. For the sake of simplicity we suspend the indices in $u_{n,3}(x)$ and simply write $u(x)$, where

$$u(x) = (x^2-1)T_n^{\mathrm{IV}}(x) + 3xT_n'''(x) = (n^2-4)T_n''(x) - 2xT_n'''(x).$$

It is not difficult to verify that (5.1) is true for $n \le 10$ and we suppose in what follows $n \ge 11$.

We shall need information about the location of the last critical points of $u(x)$ (i.e., the last zero of $u'(x)$), which we denote by ξ. As a first, we observe that the zeros of $u'(x)$ and $T_n^{(4)}(x)$ interlace and a brief examination shows that $\xi \in \left(\cos\frac{2\pi}{n}, \cos\frac{3\pi}{2n}\right)$. Sharper bounds are given in the next lemma.

Lemma 5.1. *For every natural $n > 10$ there holds*

$$\cos\frac{7\pi}{4n} < \xi < \cos\frac{5\pi}{3n}. \tag{5.2}$$

Proof. Putting $x = \cos\theta$, after some straightforward calculations we obtain

$$u'(x) := \frac{-n\sin n\theta}{\sin^7\theta}\left[(5+\sigma_n)t_n^3 - 30t_n\right]\left[g(t_n,\sigma_n) + \cos\theta\cot n\theta\right], \tag{5.3}$$

where $\sigma_n := 4/n^2$, $t_n = t_n(\theta) := n\sin\theta$, and

$$g(t,\sigma) := \frac{(1+2\sigma)t^4 - (15+6\sigma)t^2 + 30}{(5+\sigma)t^3 - 30t}. \tag{5.4}$$

Since $\xi \in \left(\cos\frac{2\pi}{n}, \cos\frac{3\pi}{2n}\right)$ and $n \geq 11$, we may assume that $t_n(\theta) > 11\sin\frac{3\pi}{22} > 4.5$ and $0 < \sigma_n < 1/30$, i.e., $(t_n,\sigma_n) \in \mathcal{A}$, where

$$\mathcal{A} := \{(t,\sigma) \mid t > 4.5,\ 0 \leq \sigma < 1/30\}.$$

The function $g(t,\sigma)$ has continuous derivatives in $\mathcal{A}$ and $\dfrac{\partial g}{\partial\sigma} > 0$ therein. This implies for $(t,\sigma) \in \mathcal{A}$

$$g_1(t) := \frac{t^4 - 15t^2 + 30}{5t^3 - 30t} \leq g(t,\sigma) \leq \frac{32t^4 - 456t^2 + 900}{151t^3 - 900t} := g_2(t), \tag{5.5}$$

where $g_1(t) = g(t,0)$ and $g_2(t) = g(t,1/30)$. Moreover, $g_1(t)$ and $g_2(t)$ are monotone increasing for $t > 4.5$. Looking at (5.3) and taking into account $\theta \in (3\pi/(2n), 2\pi/n)$, we observe that

$$\operatorname{sign}\{u'(\cos\theta)\} = \operatorname{sign}\{g(t_n,\sigma_n) + \cos\theta\cot n\theta\} := \operatorname{sign}\{h(\theta)\}. \tag{5.6}$$

For $\theta_1 = \dfrac{7\pi}{4n}$ Lemma 3.2(i) yields $t_n(\theta_1) \leq t_\infty(\theta_1) < 5.498$, hence for $n \geq 11$

$$h(\theta_1) \leq g_2(5.498) - \cos\frac{7\pi}{44} = -0.065 < 0.$$

For $\theta_2 = \dfrac{5\pi}{3n}$ and $n \geq 11$ Lemma 3.2(i) asserts $t_n(\theta_2) \geq t_{11}(\theta_2) > 5.04$ and therefore

$$h(\theta_2) \geq g_1(5.04) + \cot\frac{5\pi}{3} = 0.024 > 0.$$

Consequently, we obtain $u'\left(\cos\dfrac{7\pi}{4n}\right) < 0$ and $u'\left(\cos\dfrac{5\pi}{3n}\right) > 0$. This completes the proof of Lemma 5.1. ∎

Lemma 5.2. *The local maxima of $|u(x)|$ increase as $|x|$ increases.*

For the proof of Lemma 5.2 we apply the following result (see, e.g., [17, (7.31)]):

Lemma 5.3 (Theorem of Sonin-Pòlya). *Let $y(x)$ be a non-trivial solution of the differential equation*

$$(py')' + Py = 0, \tag{5.7}$$

where $p(x)$ and $P(x)$ are continuously differentiable and positive in the interval (a, b), and let the function $p(x)P(x)$ be non-decreasing (non-increasing) on (a, b). Then the relative maxima of $|y|$ in (a, b) form a non-increasing (non-decreasing) sequence.

The application of Lemma 5.3 with $y = u$ is possible because of the next lemma.

Lemma 5.4. *The function $u(x)$ satisfies a differential equation of the type* (5.7) *with*

$$p(x) = \frac{(1-x^2)^{7/2}}{n^2(1-x^2)-6} \tag{5.8}$$

and

$$P(x) = \frac{(1-x^2)^{5/2}\left[(n^2-4)n^2(1-x^2)-10n^2+48\right]}{\left[n^2(1-x^2)-6\right]^2}. \tag{5.9}$$

The proof of Lemma 5.4 is by direct verification, applying Lemma 3.1(iv). For the proof of Lemma 5.2 one only have to check that the functions p and P defined by (5.8) and (5.9) are positive in $\left(-\cos\frac{5\pi}{3n}, \cos\frac{5\pi}{3n}\right)$ and that $(pP)'$ is negative in $\left(0, \cos\frac{5\pi}{3n}\right)$. This is an easy exercise if the inequality $n^2(1-x^2) \geq t_{11}^2\left(\frac{5\pi}{3n}\right) > 25$ is taken into account. ∎

Now we are in a position to prove (5.1). According to Lemma 5.2,

$$\|u\|_\star = \max\{|u(\xi)|, u(\eta)\}$$

and it suffices to show that

$$|u(\xi)| \leq u\left(\cos\frac{\pi}{n+1}\right). \tag{5.10}$$

We have

$$u\left(\cos\frac{\pi}{n+1}\right) = \frac{n(n+1)\cos(\pi/(n+1))}{\sin^4(\pi/(n+1))}\left[n(n+2)\sin^2\frac{\pi}{n+1} - 6\right]$$

and for $n \geq 11$ the application of Lemma 3.2(i) yields the estimate

$$u\left(\cos\frac{\pi}{n+1}\right) > 3.457\frac{n(n+1)}{\sin^4(\pi/(n+1))}. \tag{5.11}$$

At the point ξ we have $(n^2-6)T_n^{(3)}(\xi) = 2\xi T_n^{(4)}(\xi)$, and using Lemma 3.1(iv) repeatedly, we obtain

$$u(\xi) = \left[1 + \frac{4\xi^2}{(n^2+4)(1-\xi^2)-10}\right](n^2-4)T_n''(\xi). \qquad (5.12)$$

According to Remark 3, $T_n''(x)$ is monotone increasing in $\left(\cos\frac{7\pi}{4n}, \cos\frac{5\pi}{3n}\right)$ and equation (3.2) shows that T_n'' is negative therein. Therefore $|T_n''(\xi)|$ is bounded from above by $\left|T_n''\left(\cos\frac{7\pi}{4n}\right)\right|$. For $n \geq 11$, an upper bound for the first factor in equation (5.12) is given by $1 + 4/(121\sin^2 5\pi/33 - 10)$. Substituting these bounds in (5.12), we obtain

$$|u(\xi)| < 5.788\frac{n(n^2-4)}{\sin^3(7\pi/(4n))}. \qquad (5.13)$$

In view of (5.11) and (5.13), (5.10) will hold if

$$\left(\frac{\sin(7\pi/(4n))}{\sin(\pi/(n+1))}\right)^3 \geq 1.6743\frac{n^2-4}{(n+1)^2}\left[(n+1)\sin\frac{\pi}{n+1}\right],$$

or if the following stronger inequality holds:

$$\left(\frac{\sin(7\pi/(4n))}{\sin(\pi/n)}\right)^3 \geq 5.26\frac{n^2-4}{(n+1)^2}.$$

According to Lemma 3.2(ii), the left-hand side of the latter inequality increases monotonically as n increases and for $n \geq 24$ it is greater than 5.26. By verification its validity is seen also for $11 \leq n \leq 23$. This proves (5.10), (5.1) and Lemma 3.5. ∎

Remark 4. It is not difficult to see that Theorem 1.1 remains true even if the polynomials under consideration are allowed to have complex coefficients (the same applies to Theorem B). Indeed, let p be the extremal polynomial from this larger class, and let

$$\sup_f\{\|f^{(k)}\|\} = |p^{(k)}(\tau)| = e^{i\theta}p^{(k)}(\tau), \quad \tau \in [-1, 1]$$

with some real θ. Then the polynomial $g(x) = \mathrm{Re}\{e^{i\theta}p(x)\}$ also belongs to the class under consideration and satisfies $g^{(k)}(\tau) = |p^{(k)}(\tau)|$. Thus we found another extremal polynomial, which, in addition, has real coefficients. Following the proof of Theorem 1.1, we conclude that this is only possible if $\tau = \pm 1$ and $g = \pm\overline{U}_n$.

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