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ADMISSIBILITY IN Σ_n^0-ENUMERATIONS*

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In the paper we introduce the notion of Σ_n^0 *partial enumeration* of an abstract structure $\mathfrak{A}$. Given a $k \leq n$, we obtain a characterization of the subsets of $\mathfrak{A}$ possessing Σ_k^0 pullbacks in all Σ_n^0 partial enumerations of $\mathfrak{A}$.

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1. INTRODUCTION

Let $\mathfrak{A} = (A; R_1, R_2, \ldots, R_l)$ be a countable abstract structure, where each R_i is an a_i-ary predicate on A.

A total mapping f of the set of the natural numbers N onto A is called a *total enumeration* of $\mathfrak{A}$. Every total enumeration f of $\mathfrak{A}$ determines a unique structure $\mathfrak{B}_f = (N; R_1^f, R_2^f, \ldots, R_l^f)$ of the same relational type as $\mathfrak{A}$, where

$$R_i^f(x_1, \ldots, x_{a_i}) \iff R_i(f(x_1), \ldots, f(x_{a_i})).$$

Let $\alpha < \omega_1^{CK}$. A subset M of A^a is said to be Σ_α^0-*admissible* in $\mathfrak{A}$ if for every total enumeration f of $\mathfrak{A}$ the pullback $f^{-1}(M)$ of M is Σ_α^0 in the diagram $D(\mathfrak{B}_f)$ of $\mathfrak{B}_f$.

The notion of Σ_1^0-admissibility with respect to injective total enumerations was introduced in 1964 by Lacombe [3] under the name $\forall$-admissibility. Several modifications and generalizations of this notion have appeared since 1964. Among them

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we would like to mention the Σ_1^0-admissibility in partial enumerations introduced in [5] and the relatively intrinsically Σ_α^0 sets introduced in [1] and [2], which are defined by means of Σ_α^0-admissibility with respect to injective total enumerations.

In [5] the author made the observation that the sets on an abstract structure which are Σ_1^0-admissible with respect to partial enumerations with relatively recursively enumerable (r.e.) domains coincide with the sets which are Σ_1^0-admissible with respect to total enumerations.

In the present paper we are going to study further the interplay between admissibility in total and partial enumerations. For we introduce the notion of Σ_n^0-admissibility in partial enumerations with relatively Σ_n^0 domains, and more generally, for $k \leq n$, Σ_k^0-admissibility with respect to partial enumerations with relatively Σ_n^0 domains. A normal form of the admissible sets is obtained. It turns out that for $k < n$ the admissible sets coincide with those which are Σ_k^0-admissible in all partial enumerations and are described by means of quantifier free recursive Σ_k^0 formulas. If $k = n$, then our notion of admissibility leads to a class of sets, described by means of a simple kind of recursive Σ_n^0 formulas on the abstract structure, in which the quantifiers ranging over the domain of the structure are existential and appear only on the last level.

The arguments use the machinery of the so-called regular enumerations, which seems to have a wide range of other applications.

2. PRELIMINARIES

Consider again the countable structure $\mathfrak{A} = (A; R_1, R_2, \ldots, R_l)$, which from now on we shall suppose fixed.

2.1. Definition. An *enumeration* of $\mathfrak{A}$ is an ordered pair $\langle f, \mathfrak{B}_f \rangle$, where f is a partial surjective mapping of N onto A with an infinite domain, $\mathfrak{B}_f = (N; \sigma_1, \sigma_2, \ldots, \sigma_l)$ is a structure of the same relational type as $\mathfrak{A}$, and the following condition holds for every $i \in [1, l]$ and all $x_1, \ldots, x_{a_i} \in \mathrm{dom}(f)$:

$$\sigma_i(x_1, \ldots, x_{a_i}) \iff R_i(f(x_1), \ldots, f(x_{a_i})).$$

2.2. Definition. Let $n \geq 1$. The enumeration $\langle f, \mathfrak{B}_f \rangle$ is called Σ_n^0 if the domain of f is Σ_n^0 in the diagram $D(\mathfrak{B}_f)$ of $\mathfrak{B}_f$.

2.3. Definition. Let $k \geq 1$. A subset M of A^a is Σ_k^0-*admissible* in $\langle f, \mathfrak{B}_f \rangle$ if there exists a Σ_k^0 in $D(\mathfrak{B}_f)$ subset W of N^a such that for all $x_1, \ldots, x_a \in \mathrm{dom}(f)$

$$(x_1, \ldots, x_a) \in W \iff (f(x_1), \ldots, f(x_a)) \in M.$$

As stated in the introduction, our goal is to obtain an explicit characterization of the sets which are Σ_k^0-admissible in all Σ_n^0 enumerations, $k \leq n$. For we consider two kinds of recursive Σ_k^0 formulas in the language $\mathfrak{L}_{\omega_1\omega}$ of the structure $\mathfrak{A}$, which we call "quantifier-free" and "existential", respectively.

The Σ^0_k, the Π^0_k and the Δ^0_{k+1} quantifier-free formulas are defined simultaneously with their indices by induction on k. We shall suppose that a coding of the formulas in $\mathcal{L}$ is fixed. Given an index v, by Φ^v we shall denote the formula having index v. For every formula Φ, by $\Phi(X_1, \dots, X_a)$ we shall denote that the free variables in Φ are among $X_1, \dots, X_a$.

As usual, by $W_0, \dots, W_e, \dots$ we shall denote the standard enumeration of the r.e. sets of natural numbers.

2.4. Definition.

(i) The logical constant $\mathbb{T}$ and all atomic formulas in $\mathcal{L}$ are Σ^0_0 quantifier-free formulas.

The logical constant $\mathbb{F}$ and all negated atomic formulas in $\mathcal{L}$ are Π^0_0 quantifier-free formulas.

The Δ^0_1 quantifier-free formulas are finite conjunctions of Σ^0_0 and Π^0_0 quantifier-free formulas.

The indices of the Σ^0_0, Π^0_0 and Δ^0_1 quantifier-free formulas are their respective codes as formulas in $\mathcal{L}$.

(ii) If every element of W_e is index of some Δ^0_{k+1} quantifier-free formula with variables among $X_1, \dots, X_a$, then

$$\bigvee_{v \in W_e} \Phi^v(X_1, \dots, X_a)$$

is a Σ^0_{k+1} quantifier-free formula with index $\langle 0, k+1, e\rangle$.

If Φ is a Σ^0_{k+1} quantifier-free formula, then $\neg\Phi$ is a Π^0_{k+1} quantifier-free formula. For every index $\langle 0, k+1, e\rangle$ of Φ, the triple $\langle 1, k+1, e\rangle$ is an index of $\neg\Phi$.

If $\Phi_1, \dots, \Phi_b$ are Σ^0_r or Π^0_r, $r \le k+1$, then $\chi = \Phi_1 \& \dots \& \Phi_b$ is a Δ^0_{k+2} quantifier-free formula. If $v_1, \dots, v_b$ are indices of $\Phi_1, \dots, \Phi_b$, respectively, then $\langle 2, v_1, \dots, v_b\rangle$ is an index of χ.

2.5. Definition. A Σ^0_k *existential formula*, $k \ge 1$, is a formula of the form

$$\bigvee_{v \in V} \exists Y_1 \cdots \exists Y_{q_v} \Phi^v(Y_1, \dots, Y_{q_v}, X_1, \dots, X_a),$$

where V is an r.e. set of indices of Δ^0_k formulas.

Let $M \subseteq A^a$ and $\Phi(X_1, \dots, X_a, Z_1, \dots, Z_b)$ be a Σ^0_k quantifier-free or existential formula.

2.6. Definition. The set M is *definable* by Φ on $\mathfrak{A}$ if for some $t_1, \dots, t_b \in A$

$$(\forall s_1, \dots, s_a \in A)((s_1, \dots, s_a) \in M \iff \mathfrak{A} \models \Phi(s_1, \dots, s_a, t_1, \dots, t_b)).$$

In the rest of the paper we are going to prove the next two theorems.

2.7. Theorem. *Let $M \subseteq A^a$ and $1 \leq k < n$. The set M is Σ^0_k-admissible in all Σ^0_n enumerations of $\mathfrak{A}$ if and only if M is definable by some Σ^0_k quantifier-free formula on $\mathfrak{A}$.*

2.8. Theorem. *The set M is Σ^0_n-admissible in all Σ^0_n enumerations of $\mathfrak{A}$ if and only if M is definable by some Σ^0_n existential formula on $\mathfrak{A}$.*

3. GENERIC ENUMERATIONS

The proofs of Theorem 2.7 and Theorem 2.8 use a forcing construction. In this section we shall describe the fundamentals of this construction.

3.1. Satisfaction relation. To simplify the notations we shall consider only the subsets of the domain of the structure $\mathfrak{A}$. All results can be easily proved for subsets of A^a, $a > 1$.

Let $\langle f, \mathfrak{B}_f \rangle$ be a partial enumeration of the structure $\mathfrak{A} = (A; R_1, R_2, \ldots, R_l)$. And suppose that $\mathfrak{B}_f = (N; \sigma_1, \sigma_2, \ldots, \sigma_l)$. We shall identify the diagram $D(\mathfrak{B}_f)$ of $\mathfrak{B}_f$ with the set consisting of the codes of the atomic and the negated atomic formulas which are true on $\mathfrak{B}_f$. In other words, we shall assume that

$$D(\mathfrak{B}_f) = \{\langle i, x_1, \ldots, x_{a_i}, \varepsilon \rangle : \sigma_i(x_1, \ldots, x_{a_i}) = \varepsilon,\ i \in [1, l]\}.$$

If $u \in N$, then define

$$f \models u \iff u \in D(\mathfrak{B}_f).$$

If E is a finite subset of N, then

$$f \models E \iff f \models u \text{ for each } u \in E.$$

Assume also fixed an effective coding of all finite sets of natural numbers. By E_v we shall denote the finite set with the code v.

Let us fix for every $n \geq 1$ and each $e \in N$ a unary predicate letter F^n_e. We adopt the notation $\neg^i F^n_e(x) = F^n_e(x)$ if $i = 0$ and $\neg^i F^n_e(x) = \neg F^n_e(x)$ if $i = 1$. We shall assume that the code of $\neg^i F^n_e(x)$ is $\langle i, n, e, x \rangle$.

For each $x \in N$ and every predicate letter F^n_e the satisfaction relation $f \models \neg^i F^n_e(x)$ is defined by induction on n. Given a finite set E of natural numbers and $n \geq 1$, by $f \models_n E$ we shall denote that every element u of E is of the form $\langle i, n, e, x \rangle$ and $f \models \neg^i F^n_e(x)$.

3.2. Definition.

(i) $f \models F^1_e(x) \iff \exists v(\langle v, x \rangle \in W_e \;\&\; f \models E_v)$;

$f \models \neg F^1_e(x) \iff f \not\models F^1_e(x)$.

(ii) $f \models F_e^{n+1}(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ f \models_n E_v)$;
$f \models \neg F_e^{n+1}(x) \iff f \not\models F_e^{n+1}(x)$.

3.3. Proposition.

(1) *The sets* $\{x : f \models F_e^n(x)\}$ *coincide with the* Σ_n^0 *in* $D(\mathfrak{B}_f)$ *sets.*

(2) *The sets* $\{x : f \models \neg F_e^n(x)\}$ *coincide with the* Π_n^0 *in* $D(\mathfrak{B}_f)$ *sets.*

Proof. The proof is by induction on n.

For $n = 1$ note that from the definition of "$\models$" we have

$$f \models F_e^1(x) \iff x \in \Gamma_e(D(\mathfrak{B}_f)),$$

where Γ_e is the e-th enumeration operator, see [4]

Since $N \setminus D(\mathfrak{B}_f)$ is enumeration reducible to $D(\mathfrak{B}_f)$, the r.e. in $D(\mathfrak{B}_f)$ sets coincide with the sets which are enumeration reducible to $D(\mathfrak{B}_f)$.

The step from n to $n+1$ follows easily by the Strong hierarchy theorem, see [4].

3.4. Corollary. *A set* $M \subseteq A$ *is* Σ_n^0*-admissible in* $\langle f, \mathfrak{B}_f\rangle$ *iff there exists an* $e \in N$ *such that for all* $x \in \text{dom}(f)$

$$f \models F_e^n(x) \iff f(x) \in M.$$

3.5. Finite parts and forcing. The conditions of the forcing are finite mappings of N into A with some additional properties which we call *finite parts*. We use δ, τ, ρ to denote finite parts.

Let $[0, q]$ be an initial segment of N.

3.6. Definition. A *finite part* δ on $[0, q]$ is an ordered triple $\langle \alpha_\delta, H_\delta, D_\delta\rangle$ with the following properties:

(1) α_δ is a partial mapping of $[0, q]$ into A;

(2) $H_\delta \subseteq [0, q]$;

(3) $\text{dom}(\alpha_\delta) \cup H_\delta = [0, q]$ and $\text{dom}(\alpha_\delta) \cap H_\delta = \emptyset$;

(4) D_δ is the diagram of a finite structure of the same relational type as $\mathfrak{A}$ and domain $[0, q]$, and such that if $x_1, \ldots, x_{a_i} \in \text{dom}(\alpha_\delta)$, then

$$\langle i, x_1, \ldots, x_{a_i}, \varepsilon\rangle \in D_\delta \iff R_i(\alpha_\delta(x_1), \ldots, \alpha_\delta(x_{a_i})) = \varepsilon.$$

Let Δ be the set of all finite parts.

3.7. Definition. Given finite parts δ and τ, let

$$\delta \subseteq \tau \iff \alpha_\delta \subseteq \alpha_\tau \ \&\ H_\delta \subseteq H_\tau \ \&\ D_\delta \subseteq D_\tau.$$

If $\langle f, \mathfrak{B}_f\rangle$ is an enumeration, then let

$$\delta \subseteq \langle f, \mathfrak{B}_f\rangle \iff \alpha_\delta \subseteq f \ \&\ H_\delta \cap \text{dom}(f) = \emptyset \ \&\ D_\delta \subseteq D(\mathfrak{B}_f).$$

Let $\delta \in \Delta$.
If $u \in N$, then $\delta \Vdash u$ iff $u \in D_\delta$.
If $E = \{u_1, \dots, u_r\}$ is a finite subset of N, then let

$$\delta \Vdash E \iff \delta \Vdash u_1 \ \& \ \dots \ \& \ \delta \Vdash u_r.$$

Now we are ready to define the forcing relation $\delta \Vdash F_e^n(x)$ for all $e, x \in N$ by induction on $n \geq 1$. As before we shall denote by $\delta \Vdash_n E$ that every element u of the finite set E is in the form $\langle i, n, e, x\rangle$ and $\delta \Vdash \neg^i F_e^n(x)$.

3.8. Definition.

(i) $\delta \Vdash F_e^1(x) \iff \exists v(\langle v, x\rangle \in W_e \ \& \ \delta \Vdash E_v);$

$\delta \Vdash \neg F_e^1(x) \iff \forall \rho(\rho \supseteq \delta \Longrightarrow \rho \not\Vdash F_e^1(x)).$

(ii) $\delta \Vdash F_e^{n+1}(x) \iff \exists v(\langle v, x\rangle \in W_e \ \& \ \delta \Vdash_n E_v);$

$\delta \Vdash \neg F_e^{n+1}(x) \iff \forall \rho(\rho \supseteq \delta \Longrightarrow \rho \not\Vdash F_e^{n+1}(x)).$

From the above definition follows immediately the monotonicity of the forcing, i.e. if $\delta \Vdash F_e^n(x)$ and $\delta \subseteq \tau$, then $\tau \Vdash F_e^n(x)$.

3.9. Definition. Let $Y \subseteq \Delta$. The enumeration $\langle f, \mathfrak{B}_f\rangle$ *meets* Y if for some $\delta \in Y$, $\delta \subseteq f$.

3.10. Definition. A subset $Y \subseteq \Delta$ is *dense in the enumeration* $\langle f, \mathfrak{B}_f\rangle$ if

$$(\forall \delta \subseteq f)(\exists \tau \in Y)(\delta \subseteq \tau).$$

3.11. Definition. Let $\mathfrak{F}$ be a family of subsets of Δ. An enumeration $\langle f, \mathfrak{B}_f\rangle$ is $\mathfrak{F}$-*generic* if whenever $Y \in \mathfrak{F}$ and Y is dense in $\langle f, \mathfrak{B}_f\rangle$, then $\langle f, \mathfrak{B}_f\rangle$ meets Y.

As usual, we have that for every countable family $\mathfrak{F}$ of subsets of Δ and every $\delta \in \Delta$ there exists an $\mathfrak{F}$-generic enumeration $\langle f, \mathfrak{B}_f\rangle$ such that $f \supseteq \delta$.
Let $\mathfrak{F}_0 = \{\emptyset\}$. For $n \geq 1$ set $Y_{e,x}^n = \{\tau : \tau \Vdash F_e^n(x)\}$ and let $\mathfrak{F}_n = (\bigcup_{e,x} Y_{e,x}^n) \cup \mathfrak{F}_{n-1}$.
The following Truth lemma can be proved by induction on n:

3.12. Lemma. *Let $\langle f, \mathfrak{B}_f\rangle$ be an enumeration, $n \geq 0$. Then for all $e, x \in N$:*

(1) *If $\langle f, \mathfrak{B}_f\rangle$ is $\mathfrak{F}_n$-generic, then*

$$f \models F_e^{n+1}(x) \iff (\exists \delta \subseteq f)(\delta \Vdash F_e^{n+1}(x)).$$

(2) *If $\langle f, \mathfrak{B}_f\rangle$ is $\mathfrak{F}_{n+1}$-generic, then*

$$f \models \neg F_e^{n+1}(x) \iff (\exists \delta \subseteq f)(\delta \Vdash \neg F_e^{n+1}(x)).$$

3.13. Definition. Let $\delta \subseteq \tau$. Then τ/δ is the finite part $\langle \alpha_\delta, H_\tau \cup (\mathrm{dom}(\alpha_\tau) \setminus \mathrm{dom}(\alpha_\delta)), D_\tau \rangle$.

By $\delta \preceq \tau$ we shall denote that $\mathrm{dom}(\alpha_\delta) = \mathrm{dom}(\alpha_\tau)$ and $\delta \subseteq \tau$.

3.14. Lemma.

(1) *If* $\delta \subseteq \tau$, *then* $\delta \preceq \tau/\delta$;
(2) *If* $\delta \subseteq \tau_1 \subseteq \tau_2$, *then* $\tau_1/\delta \preceq \tau_2/\delta$;
(3) *If* $\delta \subseteq \tau$ *and* $\tau/\delta \preceq \rho$, *then there exists a finite part* ρ' *such that* $\tau \preceq \rho'$ *and* $\rho'/\delta = \rho$.

Proof. (3) Let $\delta \subseteq \tau$ and $\tau/\delta \preceq \rho$. Then $\tau/\delta = \langle \alpha_\delta, H_\tau \cup (\mathrm{dom}(\alpha_\tau) \setminus \mathrm{dom}(\alpha_\delta)), D_\tau \rangle$. $\tau/\delta \preceq \rho$ implies $\rho = \langle \alpha_\delta, H_\tau \cup (\mathrm{dom}(\alpha_\tau) \setminus \mathrm{dom}(\alpha_\delta)) \cup H', D_\rho \rangle$, where $D_\tau \subseteq D_\rho$ and $H' \cap (\mathrm{dom}(\alpha_\tau) \cup H_\tau) = \emptyset$.

Let $\rho' = \langle \alpha_\tau, H_\tau \cup H', D_\rho \rangle$. Then $\tau \preceq \rho'$ and $\rho'/\delta = \langle \alpha_\delta, H_\tau \cup (\mathrm{dom}(\alpha_\tau) \setminus \mathrm{dom}(\alpha_\delta)) \cup H', D_\rho \rangle = \rho$.

3.15. Stared forcing. We define a stared forcing relation $\delta \Vdash^* F_e^n(x)$ for all $n \geq 1$, $e, x \in N$ by means of the following inductive definition:

3.16. Definition.

(i) $\delta \Vdash^* F_e^1(x) \iff \delta \Vdash F_e^1(x)$;
$\delta \Vdash^* \neg F_e^1(x) \iff \forall \rho (\rho \succeq \delta \Longrightarrow \rho \nVdash^* F_e^1(x))$.

(ii) $\delta \Vdash^* F_e^{n+1}(x) \iff \exists v (\langle v, x \rangle \in W_e \ \&\ \delta \Vdash_n^* E_v)$;
$\delta \Vdash^* \neg F_e^{n+1}(x) \iff \forall \rho (\rho \succeq \delta \Longrightarrow \rho \nVdash^* F_e^{n+1}(x))$.

Here $\delta \Vdash_n^* E_v$ means, as before, that every element of E_v is in the form $\langle i, n, e, x \rangle$ and $\delta \Vdash^* \neg^i F_e^n(x)$.

From the definition above it follows immediately that the stared forcing is monotone with respect to "$\preceq$", i.e. $\delta \Vdash^* F_e^n(x) \ \&\ \delta \preceq \tau \Longrightarrow \tau \Vdash^* F_e^n(x)$.

3.17. Lemma. *Let* $\delta \subseteq \tau$. *Then for all* $e, x \in N$, $n \geq 1$,

(1) $\tau \Vdash F_e^n(x) \iff \tau/\delta \Vdash^* F_e^n(x)$;

(2) $\tau \Vdash \neg F_e^n(x) \iff \tau/\delta \Vdash^* \neg F_e^n(x)$.

Proof. The proof is by induction on n.

Since $D_\tau = D_{\tau/\delta}$, (1) holds for $n = 1$.

Suppose now that (1) is true for some $n \geq 1$.

(2) ($\Rightarrow$). Let $\tau \Vdash \neg F_e^n(x)$. Assume that $\tau/\delta \nVdash^* \neg F_e^n(x)$. Then there is a finite part $\rho \succeq \tau/\delta$ such that $\rho \Vdash^* F_e^n(x)$. By Lemma 3.14 there exists a finite part ρ' such that $\rho' \succeq \tau$ and $\rho'/\delta = \rho$. Then $\rho'/\delta \Vdash^* F_e^n(x)$ and by induction $\rho' \Vdash F_e^n(x)$. Clearly, $\rho' \supseteq \tau$. A contradiction.

(2) ($\Leftarrow$). Let $\tau/\delta \Vdash^* \neg F_e^n(x)$. Assume that $\tau \nVdash \neg F_e^n(x)$. Then there exists $\rho \supseteq \tau$ such that $\rho \Vdash F_e^n(x)$. By induction $\rho/\delta \Vdash^* F_e^n(x)$. By Lemma 3.14 $\rho/\delta \succeq \tau/\delta$. A contradiction.

Now, using the respective definitions, we get immediately that

$$\tau \Vdash F_e^{n+1}(x) \iff \tau \Vdash^* F_e^{n+1}(x).$$

3.18. Lemma. *Let δ be a finite part, $n \geq 1$, $e, x \in N$. Then*

(1) $\delta \Vdash F_e^n(x) \iff \delta \Vdash^* F_e^n(x)$;

(2) $(\exists \tau \supseteq \delta)(\tau \Vdash F_e^n(x)) \iff (\exists \rho \succeq \delta)(\rho \Vdash^* F_e^n(x))$.

Since $\delta/\delta = \delta$, (1) follows from the previous lemma. By the same argument $\delta \Vdash \neg F_e^n(x) \iff \delta \Vdash^* \neg F_e^n(x)$. From here (2) follows by contraposition.

4. REGULAR ENUMERATIONS

Given a finite part δ defined on $[0,q]$, we shall call q the length of δ and denote it by $|\delta|$. If $p \leq q$, then by $\delta\lceil p$ we shall denote the restriction of δ on $[0,p]$, i.e. $\delta\lceil p = \langle \alpha_\delta\lceil[0,p], H_\delta\lceil[0,p], D_\delta\lceil[0,p]\rangle$. Clearly, $\delta\lceil p$ is a finite part and $\delta\lceil p \subseteq \delta$.

Given finite parts τ_1 and τ_2, say that τ_1 is *shorter* than τ_2 if:

(a) $|\tau_1| < |\tau_2|$ or

(b) $|\tau_1| = |\tau_2|$ and the code of the finite set D_{τ_1} is less than the code of D_{τ_2}.

Notice that "being shorter than" is a recursive relation and for every finite part δ it is a well ordering on the set $\{\tau | \delta \preceq \tau\}$.

Let $\mathcal{F}_n^*$ be the sequence $\{X_0^n, X_1^n, \ldots, X_i^n, \ldots\}$ of sets of finite parts, where $X_i^0 = \emptyset$ and $X_i^n = \{\tau : \tau \Vdash^* F_{(i)_0}^n((i)_1)\}$ for $n \geq 1$.

The finite part τ *decides* X_i^n if $\tau \in X_i^n$ or $(\forall \rho \succeq \tau)(\rho \notin X_i^n)$. Clearly, for every δ and i there exists a $\tau \succeq \delta$ such that τ decides X_i^n. By Lemma 3.18, if τ decides X_i^n and $\tau \subseteq \rho$, then ρ also decides X_i^n.

Let

$$\mu_n(i,\delta) = \begin{cases} \delta & \text{if } (\forall \tau \succeq \delta)(\tau \notin X_i^n), \\ (\textit{the shortest } \tau)(\delta \preceq \tau \ \&\ \tau \in X_i^n) & \text{otherwise.} \end{cases}$$

Clearly, $\mu_n(i,\delta)$ decides X_i^n. Notice also that the length of $\mu_n(i,\delta)$ depends only on the length $|\delta|$ of δ and on its diagram D_δ. Moreover, there exists a recursive in $\emptyset^{(n)}$ function λ_n such that

$$\forall i \forall \delta (\lambda_n(i, |\delta|, D_\delta) = |\mu_n(i,\delta)|).$$

4.1. Definition. Let δ be a finite part on $[0,q]$. Then δ is *n-regular* if $0 \in \mathrm{dom}(\alpha_\delta)$, and if $q_0 < q_1 < \ldots < q_r$ are the elements of $\mathrm{dom}(\alpha_\delta)$, then:

(a) $(\forall i < r)(\delta\lceil(q_{i+1}-1) = \mu_n(i, \delta\lceil q_i))$;

(b) $\delta = \mu_n(r, \delta\lceil q_r)$.

We shall denote the number r from the above definition by $\|\delta\|$.

4.2. Lemma. *Let δ be an n-regular finite part, where* $\mathrm{dom}(\alpha_\delta) = \{q_0 < q_1 < \ldots < q_r\}$. *Then for each $i < r$, $\delta\lceil(q_{i+1} - 1)$ is n-regular.*

4.3. Definition. An enumeration $\langle f, \mathfrak{B}_f \rangle$ of $\mathfrak{A}$ is called *n-regular* if for each finite part $\delta \subseteq f$ there exists an n-regular finite part τ such that $\delta \subseteq \tau \subseteq f$.

4.4. Lemma. *Let $\langle f, \mathfrak{B}_f \rangle$ be an n-regular enumeration of $\mathfrak{A}$. Then for each natural number r there exists an n-regular finite part $\delta \subseteq f$ such that $\|\delta\| = r$.*

Proof. Given an r, consider the first $r+1$ elements $q_0 < q_1 < \ldots < q_r$ of $\mathrm{dom}(f)$. Let δ be the shortest n-regular finite part such that $\{q_0, \ldots, q_r\} \subseteq \mathrm{dom}(\alpha_\delta)$ and $\delta \subseteq f$. Assume that $\|\delta\| > r$. Then there exists an element q_{r+1} of $\mathrm{dom}(\alpha_\delta)$ such that $q_r < q_{r+1}$. By Lemma 4.2 $\delta\lceil(q_{r+1} - 1)$ is n-regular. Clearly, $\delta\lceil(q_{r+1} - 1)$ is shorter than δ and $\{q_0, \ldots, q_r\} \subseteq \mathrm{dom}(\alpha_{\delta\lceil(q_{r+1}-1)})$. The last contradicts the choice of δ.

Recall the family $\mathfrak{F}_n$. Notice that by Lemma 3.18 $\mathfrak{F}_n = \mathfrak{F}_n^*$.

4.5. Proposition. *Let $\langle f, \mathfrak{B}_f \rangle$ be an n-regular enumeration of $\mathfrak{A}$. Then $\langle f, \mathfrak{B}_f \rangle$ is $\mathfrak{F}_n$-generic.*

Proof. Skipping the trivial case $n = 0$, suppose that $n \geq 1$. We shall show that $\langle f, \mathfrak{B}_f \rangle$ is generic with respect to the family $\mathfrak{F}_n^*$. Suppose that X_i^n is dense in $\langle f, \mathfrak{B}_f \rangle$. We have to prove that $\langle f, \mathfrak{B}_f \rangle$ meets X_i^n, i.e. there is a $\delta \subseteq f$ such that $\delta \in X_i^n$. By the previous lemma there exists an n-regular $\delta \subseteq f$ such that $\|\delta\| = i$. Clearly, δ decides X_i^n. Assume that $\delta \notin X_i^n$. Then $\delta \Vdash^* \neg F^n_{(i)_0}((i)_1)$ and hence, by Lemma 3.17, $\delta \Vdash \neg F^n_{(i)_0}((i)_1)$. The last contradicts the density of X_i^n.

4.6. Proposition. *Let $\langle f, \mathfrak{B}_f \rangle$ be an n-regular enumeration of $\mathfrak{A}$. Then* $\mathrm{dom}(f)$ *is Δ^0_{n+1} relative to $D(\mathfrak{B}_f)$.*

Proof. We have the following recursive in $D(\mathfrak{B}_f) \oplus \emptyset^{(n)}$ procedure, which lists the elements of $\mathrm{dom}(f)$ in an increasing order.

We start by printing out 0. Suppose that the first $r + 1$ elements $q_0, \ldots, q_r$ of $\mathrm{dom}(f)$ are listed. Consider the finite part $\delta_r \subseteq f$ on $[0, q_r]$. Using the oracle $D(\mathfrak{B}_f)$, we can obtain the diagram D_{δ_r}. Let q_{r+1} be the first element of $\mathrm{dom}(f)$ greater than q_r. Clearly, there exists an n-regular finite part τ such that $\delta_r \subseteq \tau$ and $q_{r+1} \in \mathrm{dom}(\alpha_\tau)$. By Definition 4.1 $q_{r+1} = \lambda_n(r, q_r, D_{\delta_r}) + 1$.

5. THE NORMAL FORM THEOREMS

In this section we shall obtain a normal form of the Σ^0_k-admissible in all Σ^0_n enumerations of $\mathfrak{A}$ sets for $k \leq n$. We start with the case $k = n$.

Let δ be a finite part, $x = |\delta| + 1$ and $s \in A$. By $\delta * s$ we shall denote the finite part $\langle \alpha', H_\delta, D \rangle$, where $\mathrm{dom}(\alpha') = \mathrm{dom}(\alpha_\delta) \cup \{x\}$, $\alpha_\delta \subseteq \alpha'$, $\alpha'(x) \simeq s$, and D is the appropriate extension of the diagram D_δ.

5.1. Theorem. *Let $M \subseteq A$, $n \geq 1$, and M be a Σ_n^0-admissible in all Σ_n^0 enumerations of $\mathfrak{A}$ set. Then there exists a finite part δ and a natural number e such that for each $s \in A$ if $x = |\delta| + 1$, then*

$$s \in M \iff (\exists \tau \supseteq \delta * s)(\tau \text{ is } (n-1)\text{-regular } \ \& \ \tau \Vdash^* F_e^n(x)). \tag{5.1}$$

Proof. Assume the opposite. We shall construct an $(n-1)$-regular enumeration $\langle f, \mathfrak{B}_f \rangle$ of $\mathfrak{A}$ such that M is not admissible in it.

The construction of $\langle f, \mathfrak{B}_f \rangle$ will be carried out by steps. On each step j we shall define an $(n-1)$-regular finite part δ_j, so that $\delta_j \subseteq \delta_{j+1}$, and take $f = \bigcup \alpha_{\delta_j}$ and $\mathfrak{B}_f$ to be the structure with diagram $\bigcup D_{\delta_j}$.

On the even steps we shall ensure that f is onto A. On the odd steps we shall ensure that M is not admissible in $\langle f, \mathfrak{B}_f \rangle$.

Let $t_0, t_1, \ldots, t_i, \ldots$ be a fixed enumeration of the elements of A.

Let δ_0 be the shortest $(n-1)$-regular finite part such that $\alpha_{\delta_0}(0) = t_0$.

Step $j = 2e+1$. Let $x = |\delta_{2e}| + 1$. By the assumption there exists an $s \in A$ such that

$$\neg\left[s \in M \iff (\exists \tau \supseteq \delta_{2e} * s)(\tau \text{ is } (n-1)\text{-regular } \ \& \ \tau \Vdash^* F_e^n(x))\right].$$

We have two possibilities:

Case (i). $s \in M$ and $(\forall \tau \supseteq \delta_{2e} * s)(\tau \text{ is } (n-1)\text{-regular} \Rightarrow \tau \nVdash^* F_e^n(x))$. In this case let δ_{2e+1} be the shortest $(n-1)$-regular finite part τ such that $\tau \supseteq \delta_{2e} * s$;

Case (ii). $s \notin M$ and $(\exists \tau \supseteq \delta_{2e} * s)(\tau$ is $(n-1)$-regular and $\tau \Vdash^* F_e^n(x))$. In this case let δ_{2e+1} be the shortest such τ.

Step $j = 2e+2$. Let t be the first $t_i \in A$ such that $t \notin \text{range}(\alpha_{\delta_{2e+1}})$. Let δ_{2e+2} be the shortest $(n-1)$-regular finite part τ such that $\tau \supseteq \delta_{2e+1} * t$.

Clearly, the enumeration $\langle f, \mathfrak{B}_f \rangle$ is $(n-1)$-regular and hence dom(f) is Σ_n^0 relative to $D(\mathfrak{B}_f)$ and $\langle f, \mathfrak{B}_f \rangle$ is $\mathfrak{F}_{n-1}$-generic.

Towards a contradiction assume that M is Σ_n^0-admissible in $\langle f, \mathfrak{B}_f \rangle$. Then there exists an $e \in N$ such that for all $x \in \text{dom}(f)$

$$f(x) \in M \iff f \models F_e^n(x).$$

Consider the stage $j = 2e+1$ of the construction. Let $x = |\delta_{2e}| + 1$. Using the Truth lemma (Lemma 3.12), we get that

$$f(x) \in M \iff (\exists \tau)(\delta_{2e+1} \subseteq \tau \subseteq f \ \& \ \tau \Vdash F_e^n(x)).$$

On the other hand, according to our construction this is not the case. So, M is not Σ_n^0-admissible in $\langle f, \mathfrak{B}_f \rangle$.

5.2. Theorem. *Let $k < n$, $M \subseteq A$ and let M be Σ_k^0-admissible in all Σ_n^0 enumerations of $\mathfrak{A}$. Then there exists a finite part δ and a natural number e such that for each $s \in A$ if $x = |\delta| + 1$, then*

$$s \in M \iff (\exists \tau \succeq \delta * s)(\tau \Vdash^* F_e^k(x)). \tag{5.2}$$

Proof. Assume the contrary. We shall construct an enumeration $\langle f, \mathfrak{B}_f \rangle$ of $\mathfrak{A}$ with the following properties:

(1) $\langle f, \mathfrak{B}_f \rangle$ is $\mathcal{F}_{n-1}$-generic;
(2) $\mathrm{dom}(f)$ is Σ^0_n relative to $D(\mathfrak{B}_f)$;
(3) the set M is not Σ^0_k-admissible in $\langle f, \mathfrak{B}_f \rangle$.

The construction of the enumeration $\langle f, \mathfrak{B}_f \rangle$ is very similar to that used in the proof of the previous theorem. Again it will be carried out by steps. On steps $j = 3e+1$ we shall satisfy that $\langle f, \mathfrak{B}_f \rangle$ is an $\mathcal{F}_{n-1}$-generic enumeration. On steps $j = 3e+2$ — that M is not Σ^0_k-admissible in $\langle f, \mathfrak{B}_f \rangle$. And on steps $j = 3e+3$ we shall ensure that f is a mapping onto A.

Let $t_0, t_1, \dots, t_i, \dots$ be a fixed enumeration of the elements of A and let δ_0 be the shortest $(n-1)$-regular finite part such that $\alpha_{\delta_0}(0) = t_0$.

Step $j = 3e+1$. Let $\delta_{3e+1} = \mu_{n-1}(e, \delta_{3e})$.

Step $j = 3e+2$. Let $x = |\delta_{3e+1}| + 1$. According to the assumption there exists an $s \in A$ such that

$$\neg\left[s \in M \iff (\exists \tau \succeq \delta_{3e+1} * s)(\tau \Vdash^* F^k_e(x))\right].$$

We have two possibilities:

Case (i). $s \in M$ and $(\forall \tau \succeq \delta_{3e+1} * s)(\tau \nVdash^* F^k_e(x))$.

Put $\delta_{3e+2} = \delta_{3e+1} * s$;

Case (ii). $s \notin M$ and $(\exists \tau \succeq \delta_{3e+1} * s)(\tau \Vdash^* F^k_e(x))$.

In this case let δ_{3e+2} be the shortest such τ.

Step $j = 3e+3$. Find the first $t \in A$ such that $t \notin \mathrm{range}(\alpha_{\delta_{3e+2}})$. Let $\delta_{3e+3} = \delta_{3e+2} * t$.

The enumeration $\langle f, \mathfrak{B}_f \rangle$ is constructed as in Theorem 5.1, i.e. $f = \bigcup \alpha_{\delta_j}$ and $D(\mathfrak{B}_f) = \bigcup D_{\delta_j}$.

Arguments very similar to those used in the previous section show that $\langle f, \mathfrak{B}_f \rangle$ is $\mathcal{F}_{n-1}$-generic and $\mathrm{dom}(f)$ is Δ^0_n in $D(\mathfrak{B}_f)$.

Assume that M is Σ^0_k-admissible in $\langle f, \mathfrak{B}_f \rangle$. Then there is an $e \in N$ such that for all $x \in \mathrm{dom}(f)$

$$f \models F^k_e(x) \iff f(x) \in M.$$

Consider the stage $j = 3e+2$ of our construction and let $x = |\delta_{3e+1}| + 1$. There exists an $s \in A$ such that:

Case (i). $s \in M$ and $(\forall \tau \succeq \delta_{3e+1} * s)(\tau \nVdash^* F^k_e(x))$.

Since $\delta_{3e+2} \subseteq f$, $f(x) \in M$. Then $f \models F^k_e(x)$. Clearly, $\langle f, \mathfrak{B}_f \rangle$ is $\mathcal{F}_{k-1}$-generic. By Lemma 3.12 and Lemma 3.18 there exists a finite part τ such that $\delta_{3e+1} * s \preceq \tau$ & $\tau \Vdash^* F^k_e(x)$. A contradiction;

Case (ii). $s \notin M$ and $(\exists \tau \succeq \delta_{3e+1} * s)(\tau \Vdash^* F^k_e(x))$. Since $\delta_{3e+2} \subseteq f$, $f(x) = s$. Using again Lemma 3.12 and Lemma 3.18, we get $f \models F^k_e(x)$. A contradiction.

If a subset M of A is definable by a Σ^0_k quantifier-free formula on $\mathfrak{A}$, then it is clear that M is Σ^0_k-admissible in all enumerations of $\mathfrak{A}$. It is easy to verify also that if a set M is definable by a Σ^0_n existential formula on $\mathfrak{A}$, then M is Σ^0_n-admissible in all Σ^0_n enumerations of $\mathfrak{A}$.

The proofs of both theorems in the non-trivial directions make use of the respective normal form theorems.

Suppose that the first order language $\mathfrak{L}$ consists of the predicate letters $\{P_1, \ldots, P_l\}$ and let var be a recursive one to one mapping of the natural numbers onto the set of all variables.

6.1. Lemma. *Let K, H, D be finite sets and $K = \{z_1, \ldots, z_r\}$. Let $Z_1 =$ $\mathrm{var}(z_1), \ldots, Z_r = \mathrm{var}(z_r)$. There exists a uniform effective way to define a Δ^0_1 quantifier-free formula $\Pi_{K,H,D}(Z_1, \ldots, Z_r)$ such that for all $t_1, \ldots, t_r \in A$*

$$\mathfrak{A} \models \Pi_{K,H,D}(Z_1/t_1, \ldots, Z_r/t_r) \iff \exists\delta(\mathrm{dom}(\alpha_\delta) = K \ \&\ H_\delta = H \ \&\ D_\delta = D \ \&\ \alpha_\delta(z_i) \simeq t_i).$$

Proof. If $K \cap H \neq \emptyset$ or $K \cup H$ is not an initial segment $[0, q]$ or D is not a diagram of a finite structure of the language $\mathfrak{L}$ with domain $K \cup H$, then set $\Pi_{K,H,D} = \mathbb{F}$. Otherwise, let $\{u_1, \ldots, u_v\}$ be all elements of D such that if $u_j =$ $\langle i, x_1, \ldots, x_{a_i}, \varepsilon\rangle, i \in [1, l]$, then $\{x_1, \ldots, x_{a_i}\} \subseteq K$. For every such u_j let $L_j =$ $\neg^\varepsilon P_i(\mathrm{var}(x_1), \ldots, \mathrm{var}(x_{a_i}))$ and define $\Pi_{K,H,D} = L_1 \& \ldots \& L_v$.

6.2. Corollary. *There exists a uniform effective way, given finite sets K, H, D and E, to define a Δ^0_1 quantifier-free formula $\Pi_{K,H,D,E}$ with free variables among $\{\mathrm{var}(z) : z \in K\}$ such that if $K = \{z_1, \ldots, z_r\}$ and $\mathrm{var}(z_i) = Z_i$, then for all $t_1, \ldots, t_r \in A$*

$$\mathfrak{A} \models \Pi_{K,H,D,E}(Z_1/t_1, \ldots, Z_r/t_r) \iff \exists\delta(\mathrm{dom}(\alpha_\delta) = K \ \&\ H_\delta = H \ \&\ D_\delta = D \ \&(\forall i \in [1, r])(\alpha_\delta(z_i) \simeq t_i) \ \&\ \delta \Vdash^* E)).$$

Proof. Set $\Pi_{K,H,D,E} = \mathbb{F}$ if $E \not\subseteq D$ and let $\Pi_{K,H,D,E} = \Pi_{K,H,D}$ otherwise.

6.3. Lemma. *Let $k \geq 0$, $\delta = \langle \alpha_\delta, H_\delta, D_\delta\rangle$ be a finite part, $\mathrm{dom}(\alpha_\delta) =$ $\{z_1, \ldots, z_r\}$ and $\alpha_\delta(z_1) \simeq t_1, \ldots, \alpha_\delta(z_r) \simeq t_r$. Suppose that $\mathrm{var}(z_i) = Z_i$. Then there exists a uniform in $\mathrm{dom}(\alpha_\delta), H_\delta, D_\delta$ effective way, given natural numbers e, x and finite set E of natural numbers, to define:*

(1) *A Δ^0_{k+1} quantifier-free formula $\Gamma^k_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,E}(Z_1, \ldots, Z_r)$ such that*

$$\mathfrak{A} \models \Gamma^k_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,E}(Z_1/t_1, \ldots, Z_r/t_r) \iff \delta \Vdash^*_k E;$$

(2) *A* Σ^0_{k+1} *quantifier-free formula* $\Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1,\ldots,Z_r)$ *such that*

$$\mathfrak{A} \models \Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1/t_1,\ldots,Z_r/t_r) \iff \delta \Vdash^* F^{k+1}_e(x);$$

(3) *A* Σ^0_{k+1} *quantifier-free formula* $\Psi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1,\ldots,Z_r)$ *such that*

$$\mathfrak{A} \models \Psi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1/t_1,\ldots,Z_r/t_r) \iff (\exists\tau \succeq \delta)(\tau \Vdash^* F^{k+1}_e(x));$$

(4) *A* Π^0_{k+1} *quantifier-free formula* $\Phi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1,\ldots,Z_r)$ *such that*

$$\mathfrak{A} \models \Phi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1/t_1,\ldots,Z_r/t_r) \iff \delta \Vdash^* \neg F^{k+1}_e(x).$$

Proof. Induction on k. Using Corollary 6.2, we shall suppose that (1) is true for k and proceed to prove (2), (3) and (4). After that we shall show the validity of (1) for $k+1$. Let $R_{e,x} = \{v : \langle v, x\rangle \in W_e\}$. Following the definition of the stared forcing, we get

$$\Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x} = \bigvee_{v\in R_{e,x}} \Gamma^k_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,E_v},$$

$$\Psi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x} = \bigvee_{H\supseteq H_\delta, D\supseteq D_\delta} \Pi_{\mathrm{dom}(\alpha_\delta),H,D} \mathbin{\&} \Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H,D,e,x},$$

$$\Phi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x} = \neg\Psi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}.$$

So it remains to construct $\Gamma = \Gamma^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,E}$. Set $\Gamma = \mathbb{F}$ if not all elements u of E are of the form $\langle i, k+1, e, x\rangle, i \in \{0,1\}$. Otherwise, for every element $u = \langle i, k+1, e, x\rangle$ of E let $L^u = \Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}$ if $i = 0$, and let $L^u = \Phi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}$ if $i = 1$. Put $\Gamma = \bigwedge_{u\in E} L^u$.

As a corollary we obtain the proof of Theorem 2.7. Indeed, suppose that $M \subseteq A$, $1 \le k < n$, and M be Σ^0_k-admissible in all Σ^0_n enumerations. Using Theorem 5.2, we obtain that there exist δ and e such that if $x = |\delta| + 1$, then for all $s \in A$

$$s \in M \iff (\exists\tau \succeq \delta * s)(\tau \Vdash^* F^k_e(x)).$$

Let $\mathrm{dom}(\alpha_\delta) = \{z_1,\ldots,z_r\}$, $\mathrm{var}(z_i) = Z_i$, $\mathrm{var}(x) = X$. Denote by K the finite set $\mathrm{dom}(\alpha_\delta) \cup \{x\}$. Put $\Psi = \Psi^k_{K,H_\delta,D_\delta,e,x}$. Clearly, the variables of Ψ are among $\{Z_1,\ldots,Z_r,X\}$. Let $\alpha_\delta(z_i) \simeq t_i$. Notice that $\alpha_{\delta * s}(x) \simeq s$ for all $s \in A$. Then

$$s \in M \iff \mathfrak{A} \models \Psi(Z_1/t_1,\ldots,Z_r/t_r,X/s).$$

Using Lemma 6.3 and the definition of the regular finite parts, one can easily prove the following

6.4. Lemma. *For every* $n \ge 0$ *there exists a uniform effective way to construct, given finite sets* $K = \{z_1,\ldots,z_r\}$, H *and* D, *a finite disjunction* $\Omega^n_{K,H,D}$ *of*

Δ^0_{n+1} *quantifier-free formulas with variables among* $\mathrm{var}(z_1), \ldots, \mathrm{var}(z_r)$ *such that if* $\mathrm{var}(z_i) = Z_i$ *and* $t_1, \ldots, t_r$ *are elements of* A, *then*

$$\mathfrak{A} \models \Omega^n_{K,H,D}(Z_1/t_1, \ldots, Z_r/t_r) \iff \exists\delta(\delta \text{ is n-regular } \& \operatorname{dom}(\alpha_\delta) = K \;\&\; H_\delta = H \;\&\; D_\delta = D \;\&\; (\forall i \in [1,r])(\alpha_\delta(z_i) \simeq t_i)).$$

Now we are ready to prove Theorem 2.8. Let $n \geq 1$, $M \subseteq A$. Suppose that M is Σ^0_n admissible in all Σ^0_n enumerations. By Theorem 5.1 there exist δ and e such that if $x = |\delta| + 1$, then for all $s \in A$

$$s \in M \iff (\exists \tau \supseteq \delta * s)(\tau \text{ is } (n-1)\text{-regular } \& \; \tau \Vdash^* F^n_e(x)).$$

Let $\operatorname{dom}(\alpha_\delta) = \{z_1, \ldots, z_r\}$ and $\alpha_\delta(z_i) = t_i$. Let $\mathrm{var}(z_i) = Z_i$ and $\mathrm{var}(x) = X$. Given any formula Φ and finite set $K = \{y_1 < \ldots < y_q\}$, by $\exists(y \in K)\Phi$ we shall denote the formula $\exists \mathrm{var}(y_1) \ldots \exists \mathrm{var}(y_q)\Phi$. Let $K_\delta = \operatorname{dom}(\alpha_\delta) \cup \{x\}$. Define

$$\Phi(Z_1, \ldots, Z_r, X) = \bigvee_{K \supseteq K_\delta, H \supseteq H_\delta, D \supseteq D_\delta} \exists(y \in K \setminus K_\delta)(\Omega^{n-1}_{K,H,D} \;\&\; \Theta^n_{K,H,D,e,x}).$$

Clearly, Φ is a Σ^0_n existential formula and

$$\mathfrak{A} \models \Phi(Z_1/t_1, \ldots, Z_r/t_r, X/s) \iff (\exists \tau \supseteq \delta * s)(\tau \text{ is } (n-1)\text{-regular } \& \; \tau \Vdash^* F^n_e(x)).$$

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