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SKORDEV'S CONTRIBUTION TO RECURSION THEORY

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Ladies and gentlemen,

I would like first to thank the organizers of the Logical Biennial for the invitation to briefly share some reflections on the scientific deed of Professor Skordev. I feel greatly honoured by this invitation, indeed. At a jubilee like this it might be a permissible departure from the norm for a student to estimate his teacher's work rather than the opposite. I am not going to discuss so much specific results but rather concentrate on some methodological aspects of Skordev's contribution to Recursion Theory.

A cursory review of Skordev's past scientific activities reveals that a fairly major portion of his research and publications was devoted to Recursion Theory. Following his early papers [18, 19] on computable and mu-recursive operators and recursively complete arithmetical operations, and the subsequent ones [20, 21] on universal functions, Professor Skordev had over 30 publications on Recursion Theory during the period subsequent to 1974. Most of those publications were actually on Algebraic Recursion Theory, including the monographs [23, 24]. Likewise, it was for his research in Algebraic Recursion Theory that Professor Skordev got his Doctor of Sciences Degree and was awarded the *Nikola Obreshkov Prize*, this country's most prestigious award for achievements in the area of mathematics.

Professor Skordev set about his undertaking to generalize and axiomatize Classical Recursion Theory in the early seventies. That happened in the context of particularly interesting developments connected with a number of attempts to expand the scope of Recursion Theory. Probably, the first substantial advancement in that direction were the papers of Kleene [9, 10], affording a presentation of the hyperarithmetic theory via recursion in a second order object embodying quantification over natural numbers. Kleene's generalization was specifically important for not only initiating a new area in Recursion Theory known as Higher Recursion Theory, which was considerably advanced in the sequel, but also for setting a pattern and paving the way for other generalizations, especially those of Platek [16] and Moschovakis [12]. Research on computability over algebraic structures occurred as early as in the sixties, but the appropriate concepts of such computability were devised by Moschovakis [12] and by Friedman [5], the finite algorithmic procedures of the latter accounting for the lightface version. Incidentally, the concepts of prime computability and search computability of Moschovakis had a significant influence on the genesis of Skordev's generalization itself.

It is of interest to clarify the motives behind the various endeavours to generalize Recursion Theory beyond the classical study of effectively computable number-theoretic functions. For instance, recursion on infinite ordinals originated in Takeuti's papers [25, 26] with the necessity of introducing and studying such recursion, arising most naturally out of several areas of Mathematical Logic: Proof Theory, Model Theory and Set Theory. Such recursion was needed in order to deal with concrete problems such as 'effectivity' of proofs and 'arithmetical' undefinability in a generalized sense, as well as to achieve a more precise understanding of set structure, based on which to find solution to some problems already formulated in Set Theory.

Apart of particular problems originating in other areas, the study of effective computability in a more general context was put on the agenda also by certain general principles ensuing from Recursion Theory itself. These comprised the common aims of a mathematical generalization: to design abstract structures that are not only new and support a rich in content theory, but which also clarify Classical Recursion Theory and would possibly prove useful in application. More than that, it was hoped that if successful, such developments would eventually provide an axiomatical foundation of Recursion Theory.

The effort of some of the most brilliant logicians of the sixties and the seventies led to successful generalizations of Classical Recursion Theory in several directions, in the sense that suitable notions of effective computability were identified, providing the means for desired applications in the areas for which the relevant generalizations had been intended. The resulting Generalized Recursion Theory, initially regarded as technically forbidding but for a small community of devoted experts, later got much better and streamlined presentations. The progress in axiomatizing Recursion Theory, however, was less than satisfactory, at least until the invention of Skordev combinatory spaces.

Skordev's ideas of generalizing and axiomatizing Recursion Theory evolved around 1974 by way of extracting certain algorithmic properties of multiple-valued functions which turned out to permit axiomatical treatment. Professor Skordev successfully materialized his ideas by a combination of mathematical intuition and a refined technique based on an excellent command of the apparatus of Classical Recursion Theory and related domains of Logic. In the process, however, he not only achieved the aims he had set, but went far beyond his original goals, taking advantage of the rich opportunities offerred by the very approach invented by him. Actually, within few years Skordev laid the foundations and outlined the scope of a general theory notable for its deepness and elegance combined with an unusually wide scope of application. If the place of Skordev's theory in mathematics is to be described in few words, one might say that from a philosophical viewpoint Skordev's theory captured the nature of effective computability very much in the same way as Group Theory related to the concept of symmetry.

The hard core of Skordev's axiomatic approach was based on the algebraic structure of combinatory space. The principal characteristics of those spaces comprised: first, dealing with more general mathematical objects, members of a partially ordered semigroup rather than just functions or functionals; and second, choosing few basic or initial operations and setting forth their fundamental properties by means of a small number of elegant algebraically-styled axioms including a mu-induction principle. The basic operations of a combinatory space corresponded both intuitively and in a direct way to certain constructions to be found in structural programming or to certain patterns of combining computational devices, namely composition, branching or if-then-else statement, loop or while-do statement. Their axioms were first order axioms and also a first order mu-induction axiom sufficed for the bulk of the theory.

It is instructive to notice that in essence the basic operations of combinatory spaces occurred independently in other works, mainly in Computer Science, e.g. in the functional programming structures of Backus [1] and the schemes of Böhm and Jacopini [2], where, however, their mathematical potential had not been profitably exploited due to a number of reasons. The method of mu-induction, too, could be found in Computer Science; indeed the mu-induction axiom of combinatory space was a particular instance of Scott's mu-induction rule. A comparison shows that, due to the right choice of basic operations and initial elements, mu-induction in combinatory spaces was a powerful technical device, while the general Scott's rule was not, precisely because the system of Scott [17] lacked such suitable basic elements and operations.

Owing to the combination of aptly chosen basic operations and the mu-induction technique, a fairly non-trivial results were obtained in the general theory of combinatory spaces. Typical of that theory are assertions such as the Normal Form Theorem, the Enumeration Theorem and the First and Second Recursion Theorems, abstract Rice and Rogers Theorems. Needless to say, representation of the ordinary partial recursive functions was available too, hence the Classical Recursion Theory was not just a particular instance (i.e. model of the general theory) but at the same time was always imbedded as a minimum component. This was very much the case of Kleene-recursiveness in finite type objects, which was both a particular instance of relative recursiveness within a suitable combinatory space, and was also represented (and thus imbedded) in hierarchies of spaces and in a certain kind of more sophisticated spaces studied in Ivanov [6, 8].

On the other hand, the appropriate choice — or one might say design — of the basic operations and their axiomatically captured properties resulted in a surprising variety of models or particular spaces with essentially different semantics of various order. Apart of the standard case of single-valued and multiple-valued functions forming first order spaces, and monotonous functionals and second order relations forming second order spaces, these included also spaces of first and higher order related to certain concepts of everywhere-definedness and complexity of data processing, or comprising functions with finite types arguments, ordinal functions, probabilistic functions, fuzzy relations and the like. This abundance of spaces made it possible, first, to generalize via Skordev's approach already existing notions of effective computability, thereby paving the way for ample applications of the general theory. And second, it allowed to introduce notions of effective computability in areas which had not supported such notions before.

The approach initiated by Skordev provided a good illustration to another aspect of generalization by contributing to better understanding of Classical Recursion Theory and Generalized Recursion Theory. Certain phenomena which in Classical Recursion Theory were muted by 'too much arithmetic', i.e. by the availability of uninherently strong tools, had been known to emerge even in Generalized Recursion Theory. Such was, for instance, the distinction between lightface and boldface versions of the theory; also the understanding that Classical Recursion Theory traditionally employed operations which fitted better in arithmetic, but belong less naturally in Recursion Theory. Indeed, unsuccessful attempts to make use of minimization (or least number) operator in Generalized Recursion Theory had shown that operation to be inadequate for the purposes of prime or search computability or, as a matter of fact, recursion in higher types or recursion on ordinals. In contrast, the iteration operation of Skordev that superseded the least number operator was always suitable, because it was defined by its properties needed for the theory.

Of course, this universality of the axioms of combinatory space had most interesting semantical implications for its operations, resulting in semantic multiplicity even within a single higher order space. That applies particularly to multiplication and, as a consequence, to iteration operation. The semigroup multiplication would usually be a sort of composition, executed however in an opposite order, respectively in first and higher order spaces. The first order semantics of iteration was more or less of a loop nature, while in higher order spaces iteration at the higher level was nothing else but the least fixed point operator over the preceding level. Thus in the context of Algebraic Recursion Theory one could ascertain a sort of identity between seemingly completely different operations: the least fixed point operator was a particular instance of iteration which in turn was a particular instance of the least fixed point operator. Another similar phenomenon in the axiomatic theory was explicated by Skordev's pairing operation, which drew the lightface-boldface division line in the theory. Its first order semantics dealt with coding of pairs of data other than natural numbers, while its higher order semantics, as shown in Ivanov [8], took care of lambda abstraction.

An important aspect of any mathematical theory are not just its statements but their proofs as well. Here we see one of the unmistakable symptoms of a non-trivial generalization in the fact that quite a few of the proofs in Skordev's general theory were new rather than just modified proofs extracted from particular instances. More often than not those proofs tended to be streamlined and elegant on account of avoiding the temptation to solve problems 'by force' due to availability of excessive tools. At the same time, Skordev's axiomatic theory established common proofs and direct links between theorems belonging to different theories which otherwise seemed to be analogous, but actually proved to be particular instances of one and the same abstract proposition of Algebraic Recursion Theory; the situation earlier discussed for operations applies here to statements. For example, the First Recursion Theorem of Skordev generalized both the Kleene First Recursion Theorem and the Moschovakis Induction Completeness Theorem.

One of the popular and quite natural approaches to generalizing Recursion Theory was by way of employing inductive definability as a foundation, an idea stemming from Moschovakis [14] and supported by Feferman [3], too. The interesting try of Moschovakis [15] was further aimed at elevating the theory of inductive definability to a more abstract axiomatic level comparable with that of Skordev's setting for Recursion Theory. From the point of view of Recursion Theory however, Skordev's approach had the advantage of being not transplanted but intrinsic to that theory. Moreover, his approach made it possible for the inductive definability itself to be dealt with as a particular instance of relative recursiveness in a suitable combinatory space, i.e. within Recursion Theory, thus showing that Recursion Theory was just as fundamental as Inductive Definability Theory.

Returning to the strive for building axiomatic foundations of Recursion Theory, the attempts prior to Skordev's one might be regarded as partially successful, as far as their results and acceptance by the logician community were concerned. It was true that considerable effort had been allotted to the detailed elaboration of certain axiomatic approaches to Recursion Theory; typical example of that were the so-called computation theories of Moschovakis [13] studied extensively by Fenstad [4]. It turned out eventually that it was possible to embrace a number of notions of Generalized Recursion Theory in the framework of the computation theories and to reaffirm once again the relevant results from particular theories, leaving however the feeling of a persisting necessity to readapt the general setting, i.e. lack of true uniform general approach. Combined with the domination of modified proofs, that hinted at a certain creative potential deficiency.

Needless to say, there are still many open problems in Algebraic Recursion Theory. One of the major challenges at this stage appears to be the necessity to identify a reasonable concept of 'finite' in Algebraic Recursion Theory, needed, e.g., to deepen the study of abstract degrees initiated by Ivanov [7]. The importance of such a step in any generalization of Recursion Theory was stressed by Kreisel [11].

With its undoubted quality of good mathematics the approach of Skordev inspired natural interest among a number of other logicians as well as computer scientists. That resulted in dozens of publications, M.Sc. and Ph.D. theses by N. Georgieva, J. Zashev, O. Ignatov, L. Ivanov, R. Lukanova, S. Nikolova, E. Pazova, V. Petrov, A. Radenski, I. Soskov, M. Tabakov and others. Most interesting are the works of Zashev [27–29] in a related new area, Recursion Theory on partially ordered combinatory algebras and further generalizations at categorial level. During the last two decades Professor Skordev worked out a new portion of Recursion Theory which, with the contribution of his followers, evolved to form an original school in the Theory of Effective Computability. Apart of that, ideas and methods originating in Skordev's approach were applied to other areas of Recursion Theory and to Non-Classical Logic by A. Dichev, I. Soskov, A. Soskova, D. Vakarelov, G. Gargov, S. Passy, T. Tinchev and V. Goranko. As a matter of fact, a good deal of Bulgarian logicians have had a more than passing interest in this subject matter.

In conclusion, as a witness of these developments during the last twenty years or so, in which I was honoured to participate, I would like to take this opportunity to most cordially congratulate Professor Skordev as my teacher, on the occasion of his anniversary, and wish him best health and further twenty years of tireless and fruitful work.

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