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EACH 11-VERTEX GRAPH WITHOUT 4-CLIQUES HAS A TRIANGLE-FREE 2-PARTITION OF VERTICES

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Let G be a graph, $\mathrm{cl}(G)$ denotes the clique number of the graph G. By $G \to (3,3)$ we denote that in any 2-partition $V_1 \cup V_2$ of the set $V(G)$ of his vertices either V_1 or V_2 contains 3-clique (triangle) of the graph G; $\alpha = \min\{|V(G)|,\ G \to (3,3) \text{ and } \mathrm{cl}(G) = 4\}$, $\beta = \min\{|V(G)|,\ G \to (3,3) \text{ and } \mathrm{cl}(G) = 3\}$. In the current article, we consider graphs G with the property $G \to (3,3)$. As a consequence from proven results it follows that $\alpha = 8$ and $\beta \geq 12$.

Keywords: chromatic number, triangle free partition of vertices of graph
1991/95 Math. Subject Classification: 05C55, 05C35

1. INTRODUCTION

We consider only finite, non-oriented graphs without loops and multiple edges. $V(G)$ and $E(G)$ denote respectively the set of the vertices and the set of the edges of the graph G. We say that G is an n-vertex graph when $|V(G)| = n$. If $v, w \in V(G)$ and $[v, w] \in E(G)$, then v and w are called adjacent vertices of the graph G, and the edge $[v, w]$ is called incidental to the vertices v and w. For $v \in V(G)$ we denote by $\mathrm{Ad}(v)$ the set of all vertices adjacent to v, and by $d(v)$ the number of such vertices, i.e. $d(v) = |\mathrm{Ad}(v)|$. For the graph G we put $\delta(G) = \min\{d(v) \mid v \in V(G)\}$ and $\Delta(G) = \max\{d(v) \mid v \in V(G)\}$. The set of vertices of a given graph is called *clique* if arbitrary two of its elements are adjacent vertices. If the number of vertices in a given clique is p, then we call it *p-clique*. The biggest natural number p, such that the graph G contains a p-clique, is called *clique-number* of G and is denoted by $\mathrm{cl}(G)$.

Let $u \in V(G)$ and $[v, w] \in E(G)$. We say that the vertex u is adjacent to the edge $[v, w]$ if $\{u, v, w\}$ is a 3-clique of G.

The set of vertices of a given graph is called *anticlique* if each two of them are not adjacent. The anticlique consisting of p vertices is called *p-anticlique*. The biggest natural number p, for which the graph G has p-anticlique, is called the number of independence of G and is denoted by $\alpha(G)$.

The graph G_1 is called a subgraph of the graph G if $V(G_1) \subset V(G)$ and $E(G_1) \subset E(G)$. Let $M \subset V(G)$. We denote by $\langle M \rangle$ the subgraph generated by M, i.e. $V(\langle M \rangle) = M$, and two vertices of M are adjacent in $\langle M \rangle$ if and only if they are adjacent in G. We denote by $G - M$ the subgraph of G that is produced by taking off the vertices of M and all the edges incidental to the vertices of M.

The partition of $V(G)$ into r pairwise disjoint subsets, $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$, is called r-partition of vertices. If all of V_i, $i = 1, \ldots, r$, are anticliques, then this partition is called *r-chromatic partition*. The smallest natural number r, for which G has an r-chromatic partition, is called *chromatic number* of G and is denoted by $\chi(G)$. The graph G is called *k-chromatic* if $\chi(G) = k$. The graph G is called *vertex-critical k-chromatic* graph if $\chi(G) = k$ and $\chi(G - v) < k$ for arbitrary $v \in V(G)$. We need the following obvious

Proposition 1. *If G is a vertex-critical k-chromatic graph, then $\delta(G) \geq k - 1$.*

The *supplement* $\overline{G}$ of a given graph G is defined by setting $V(G) = V(\overline{G})$; two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. It is clear that $\alpha(G) = \mathrm{cl}(\overline{G})$.

Let p and q be given natural numbers. The number $R(p, q)$ is the minimum of all natural numbers n, such that for arbitrary n-vertex graph G either $\mathrm{cl}(G) \geq p$ or $\alpha(G) \geq q$. The existence of the numbers $R(p, q)$ is proved by F. Ramsey in [14]. Therefore they are refered as *Ramsey numbers*. We need the identity $R(4, 3) = R(3, 4) = 9$, see [3], and more precisely, its obvious consequence:

Proposition 2. *If $|V(G) \geq 9$ and $\mathrm{cl}(G) \leq 3$, then $\alpha(G) \geq 3$.*

If arbitrary two vertices of the given n-vertex graph are adjacent, then it is called *complete* n-vertex graph and is denoted by K_n. The simple cycle of length n is denoted by C_n. Let G_1 and G_2 be two graphs without common vertices, i.e. $V(G_1) \cap V(G_2) = \varnothing$. We denote by $G_1 + G_2$ the graph G, for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[v_1, v_2] \mid v_i \in V(G_i),\ i = 1, 2\}$.

2. MAIN RESULTS

Definition. *The 2-partition $V(G) = V_1 \cup V_2$ of the verteces of the graph G is free of 3-cliques if each of the sets V_1 and V_2 does not contain a 3-clique of the graph G. We write $G \rightarrow (3, 3)$ when there is no 3-cliques free 2-partition of the vertices of G.*

It is obvious that if $\chi(G) \leq 4$, then G has a 3-cliques free 2-partition of vertices. Therefore we have the following

Proposition 3. *If $G \to (3,3)$, then $\chi(G) \geq 5$.*

It is clear that $K_5 \to (3,3)$ and, conversely, if $\mathrm{cl}(G) \geq 5$, then $G \to (3,3)$. The opposite direction is false since it is easy to check that $\overline{C}_9 \to (3,3)$, but $\mathrm{cl}(\overline{C}_9) = 4$.

Definition. *We denote by α the minimum of all natural numbers n such that there exists an n-vertex graph $G \to (3,3)$ with $\mathrm{cl}(G) = 4$. We denote by β the smallest natural n such that there is an n-vertex graph $G \to (3,3)$ with $\mathrm{cl}(G) = 3$.*

We prove in this paper that $\alpha = 8$ and the unique 8-vertex $G \to (3,3)$ with $\mathrm{cl}(G) = 4$ is the graph $K_1 + \overline{C}_7$ (Theorem 1). The existence of the number β is proved by P. Erdös and C. Rogers in [1]. R. Irving shows in [5] that $\beta \leq 17$. N. Nenov constructs in [9] a 14-vertex graph $\Gamma_1 \to (3,3)$ with $\mathrm{cl}(\Gamma_1) = 3$ (see Fig. 1), showing that $\beta \leq 14$. In the paper [10] N. Nenov proves that $\beta \geq 11$. In the present work we prove that $\beta \geq 12$ (Theorem 2).

Theorem 1. *Let the graph G be such that $G \to (3,3)$ and $\mathrm{cl}(G) = 4$. Then $|V(G)| \geq 8$ and $|V(G)| = 8$ only if $G = K_1 + \overline{C}_7$.*

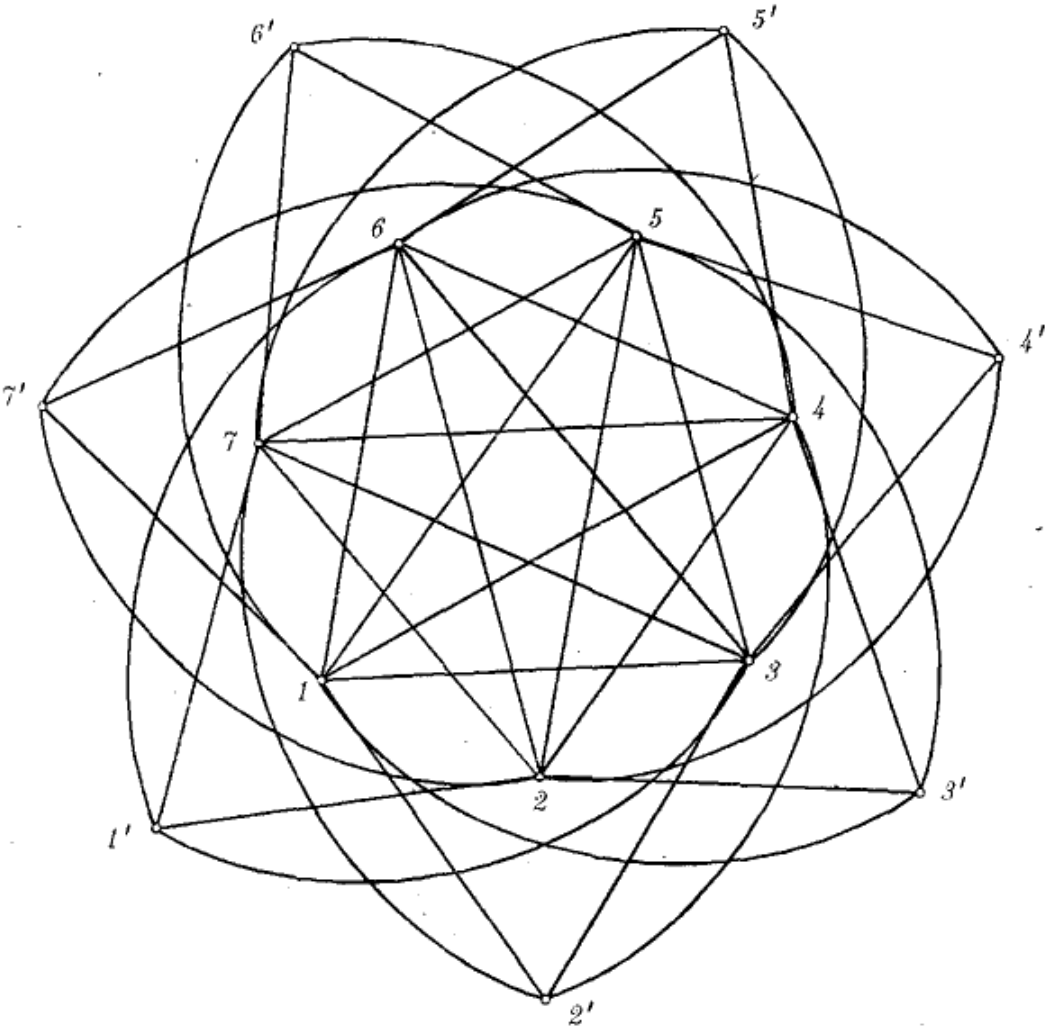

Fig. 1. Graph Γ_1

Theorem 2. *Let the 11-vertex graph G be such that* $\mathrm{cl}(G) = 3$. *Then G has a 3-cliques free 2-partition of vertices.*

Definition. *We say that the graph G is 3-saturated, if for an arbitrary anticlique A of G, the subgraph $G - A$ contains a 3-clique.*

To prove the Theorems 1 and 2 we need also the next assertions.

Theorem 3. *Let G be a 3-saturated graph and* $\mathrm{cl}(G) = 3$. *Then* $|V(G)| \geq 7$ *and* $|V(G)| = 7$ *only if* $G = \overline{C}_7$.

Theorem 4. *Let G be a 3-saturated graph,* $|V(G)| = 8$ *and* $\mathrm{cl}(G) = 3$. *Then either G is isomorphic to one of the graphs L_i, $i = 1, \ldots, 14$, shown at* Fig. 2–15, *or there is $v \in V(G)$ such that $G - v = \overline{C}_7$.*

Theorem 5. *The graphs L_i, $i = 1, \ldots, 14$, are 3-saturated, L_i is not isomorphic to L_j for $i \neq j$ and for arbitrary $v \in V(L_i)$ the graph $L_i - v$ is not isomorphic to $\overline{C}_7$.*

The connection between the 3-saturated graphs and the graphs satisfying $G \to (3,3)$ is given by the following

Proposition 4. *Let $G \to (3,3)$ and B be an anticlique in G. Then the subgraph $G_1 = G - B$ is 3-saturated.*

Proof. Assume that in fact G_1 is not 3-saturated and let A be such anticlique of G_1 that $G_2 = G_1 - A$ contains no 3-cliques. In such case $V(G) = V(G_2) \cup (A \cup B)$ is a 3-clique free 2-partition, which is a contradiction. ∎

If a given graph has a 3-chromatic partition, then obviously it is not 3-saturated. That is why we have

Proposition 5. *If G is 3-saturated, then* $\chi(G) \geq 4$.

We state also the following obvious

Proposition 6. *If* $\mathrm{cl}(G) = 3$, *then* $\mathrm{Ad}(v)$ *does not contain 3-cliques for arbitrary $v \in V(G)$.*

We are going to use the next results.

Theorem A ([6]). *Let G be an 8-vertex graph with* $\mathrm{cl}(G) = 3$ *and* $\alpha(G) = 2$. *Then G is isomorphic to one of the graphs L_1, L_2, L_3 from* Fig. 2–4.

Different proofs of the above theorem could be found in [8], [12] and [13].

Theorem B ([10]). *Let the graph G be such that* $\mathrm{cl}(G) \leq r$ *and* $\chi(G) \geq r + 1$ *for some $r \geq 3$. If* $|V(G)| = r + 4$, *then one of the following two assertions is satisfied:*

(i) *there is a vertex $v \in V(G)$ such that $G - v = K_{r-2} + C_5$;*

(ii) *the graph G is isomorphic to one of the graphs $K_{r-3} + F_i$, $i = 1, \ldots, 7$, where the graphs F_1, ..., F_7 are shown at the* Fig. 16–22.

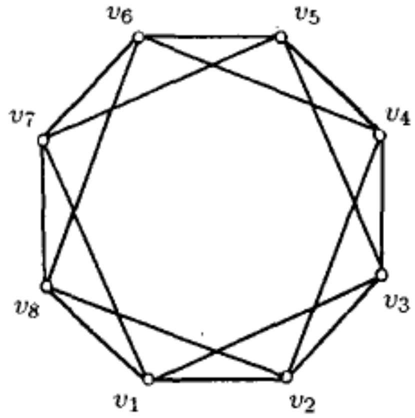

Fig. 2. Graph L_1

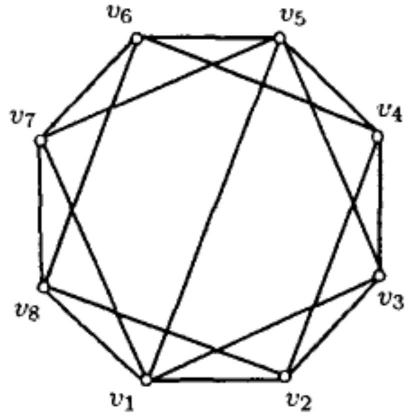

Fig. 3. Graph L_2

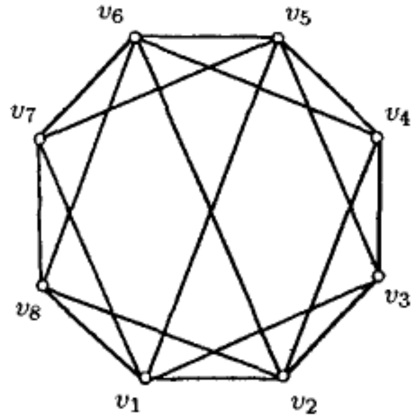

Fig. 4. Graph L_3

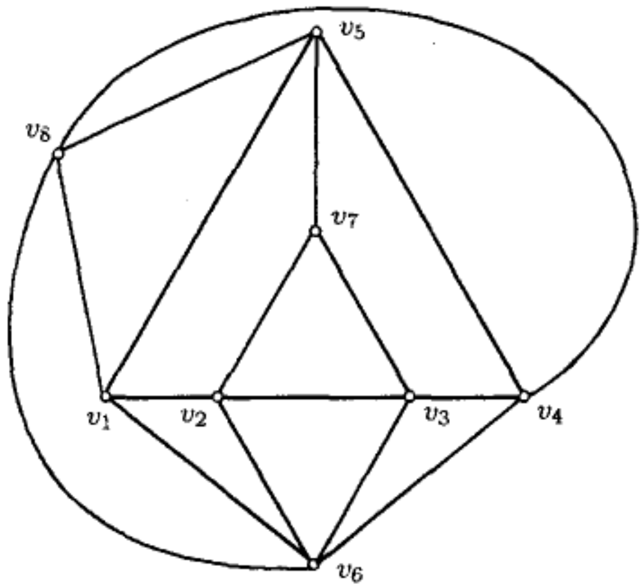

Fig. 5. Graph L_4

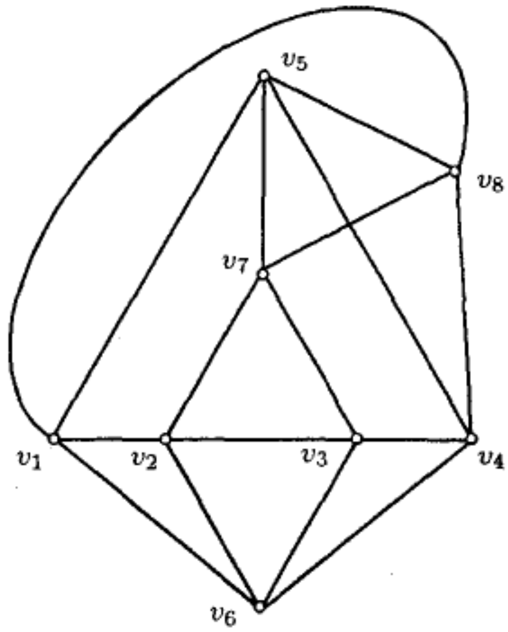

Fig. 6. Graph L_5

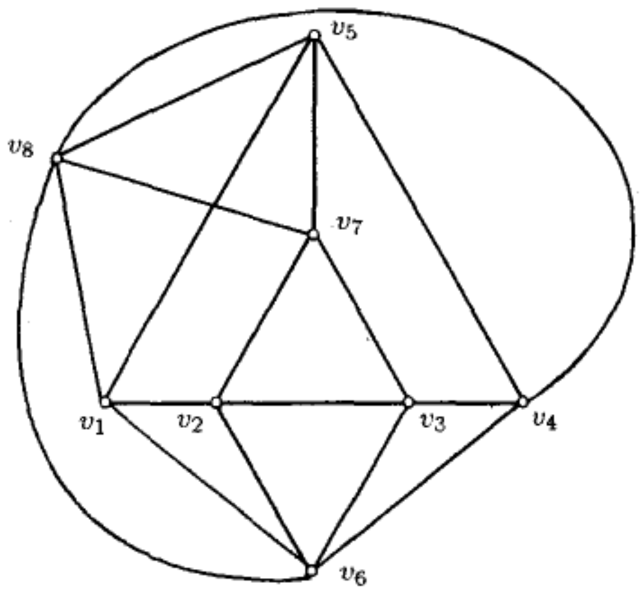

Fig. 7. Graph L_6

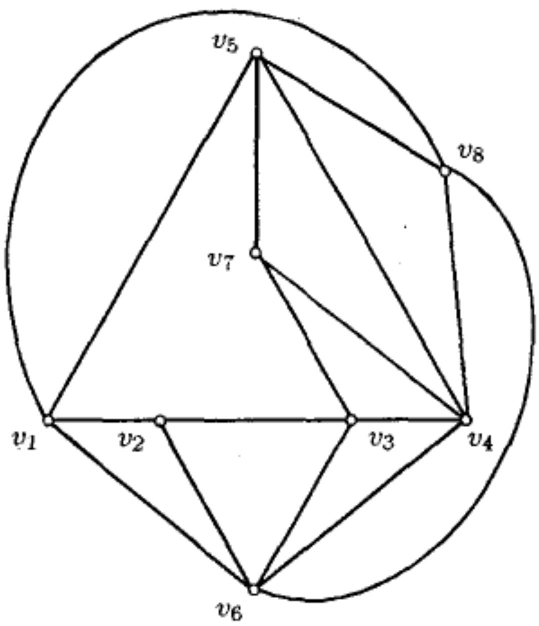

Fig. 8. Graph L_7

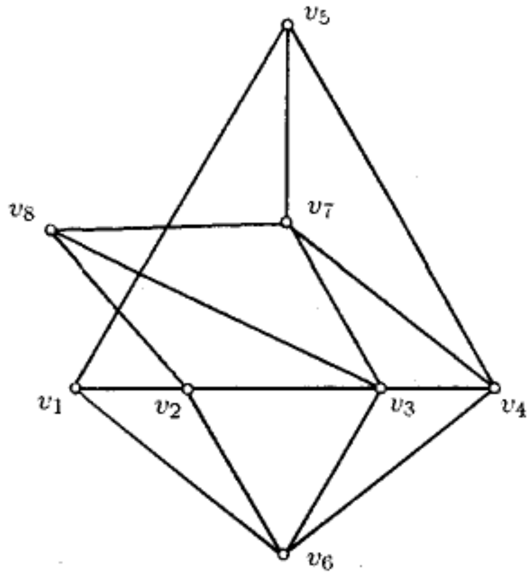

Fig. 9. Graph L_8

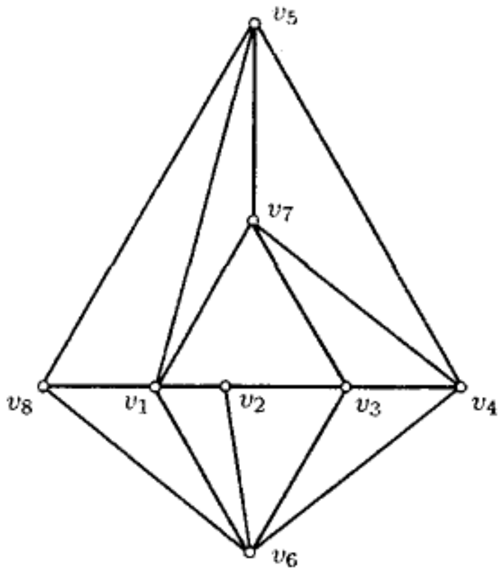

Fig. 10. Graph L_9

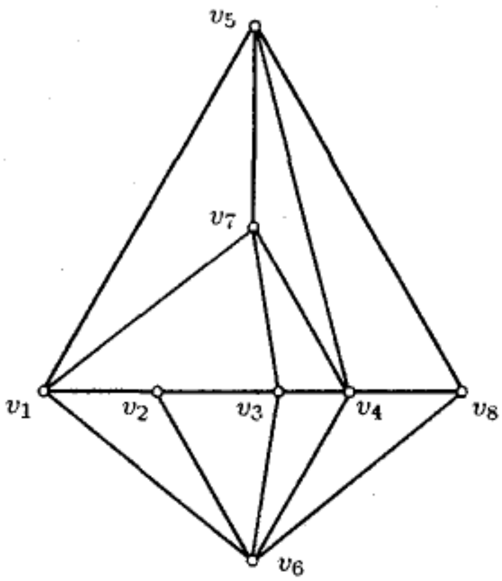

Fig. 11. Graph L_{10}

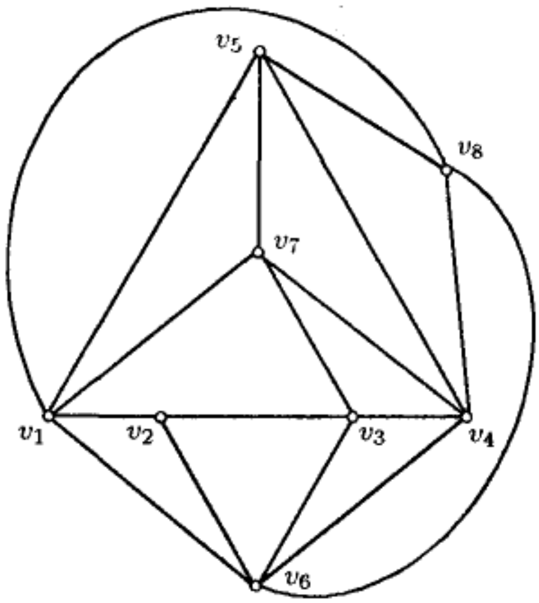

Fig. 12. Graph L_{11}

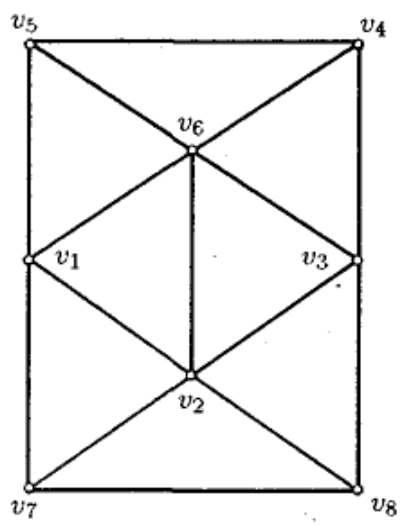

Fig. 13. Graph L_{12}

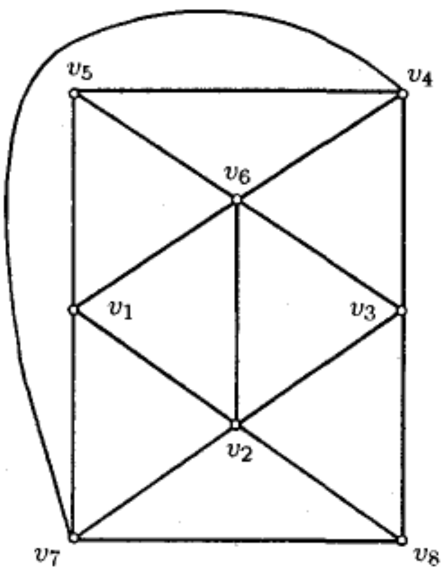

Fig. 14. Graph L_{13}

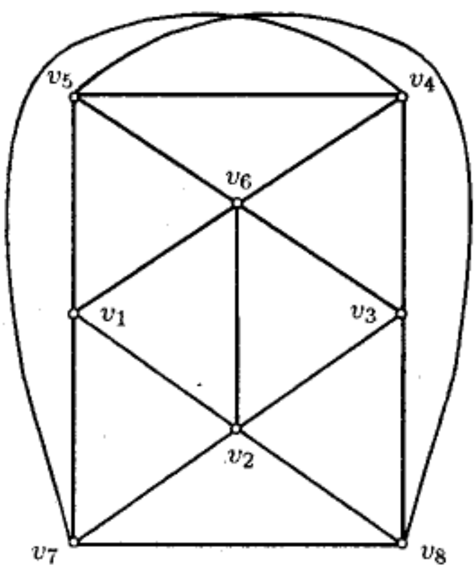

Fig. 15. Graph L_{14}

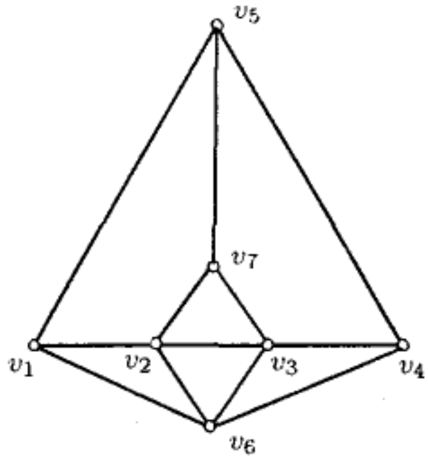

Fig. 16. Graph F_1

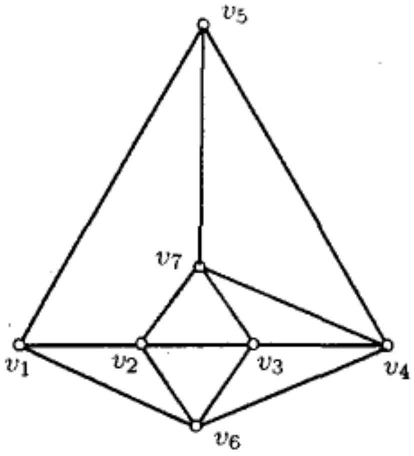

Fig. 17. Graph F_2

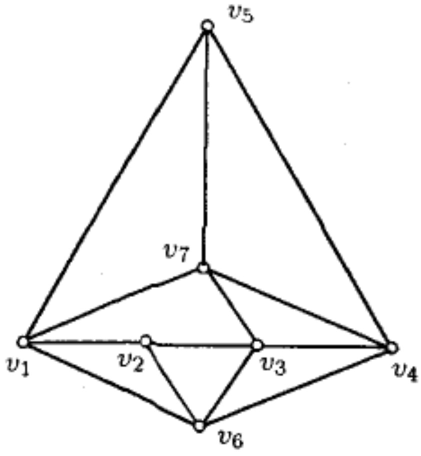

Fig. 18. Graph F_3

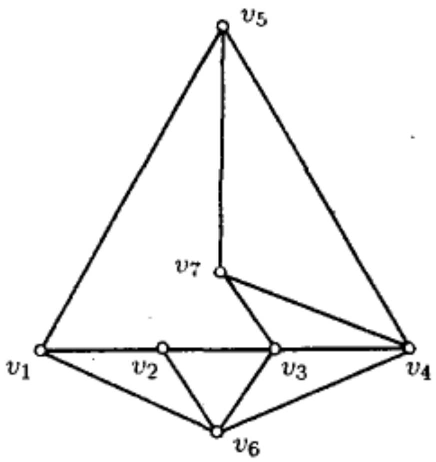

Fig. 19. Graph F_4

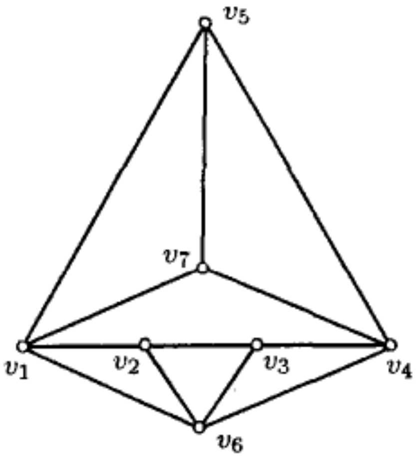

Fig. 20. Graph F_5

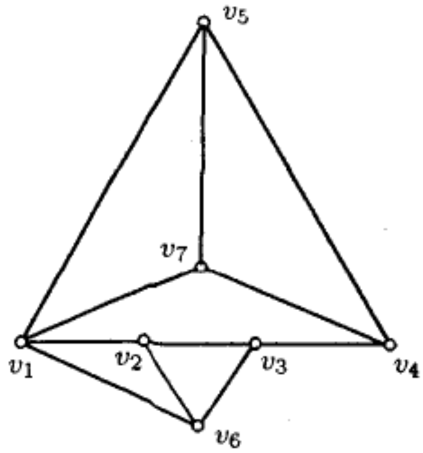

Fig. 21. Graph F_6

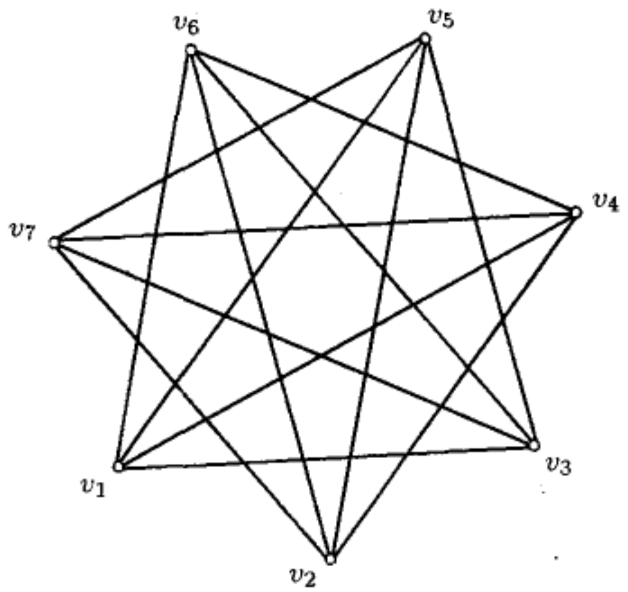

Fig. 22. Graph $F_7 = \overline{C}_7$

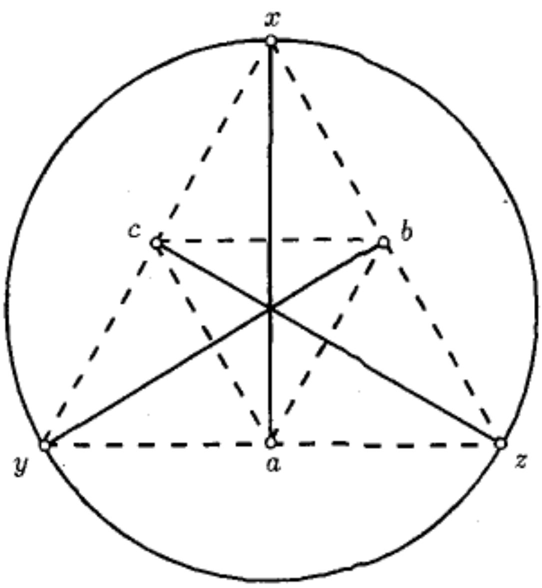

Fig. 23

Theorem C ([11]). *Let the graph G be such that $|V(G)| \leq 10$ and $\mathrm{cl}(G) = 3$. Then $\chi(G) \leq 4$.*

3. PROOFS OF THEOREMS 3, 4 AND 5

Proof of Theorem 3. Assume that G is 3-saturated and $|V(G)| \leq 7$. By adding if necessary few isolated vertices, we may assume that $|V(G)| = 7$. According to Proposition 5, $\chi(G) \geq 4$. As $\mathrm{cl}(G) = 3$, we see that G satisfies the conditions of Theorem B with $r = 3$ and we conclude that there are only two possible cases:

Case 1. $G - v = K_1 + C_5$ for some vertex $v \in V(G)$. Let $V(K_1) = \{u\}$. If u and v are not adjacent, then $G - \{u, v\} = C_5$ and consequently the graph G is not 3-saturated. If u and v are adjacent, then $G - u = \langle \mathrm{Ad}(u) \rangle$. According to Proposition 6, $\mathrm{Ad}(u)$ does not contain 3-cliques and therefore G is not 3-saturated.

Case 2. G coincides with some of the graphs F_i, $i = 1, \dots, 7$ (Fig. 16–22). Each of the graphs F_i, $i = 1, \dots, 6$, satisfies $F_i - \{v_6, v_7\} = C_5$, so these graphs are not 3-saturated. Then the assumption $|V(G)| \leq 7$ leads to $G = \overline{C}_7$. Obviously, $\overline{C}_7$ is 3-saturated, which finishes the proof. ■

To prove Theorem 4, we need some preparation.

Lemma 1. *Let the graph G be such that $|V(G)| = 8$, $\mathrm{cl}(G) = 3$, and $\alpha(G) > 3$. Then G is not 3-saturated.*

Proof. Let $\{v_1, v_2, v_3, v_4\}$ be a 4-anticlique in G and v_5, v_6, v_7, v_8 be the other vertices of G. If $G - \{v_1, v_2, v_3, v_4\}$ contains no 3-cliques, we are done. In the other case, let for example $\{v_5, v_6, v_7\}$ be a 3-clique in G. From $\mathrm{cl}(G) = 3$ it follows that v_8 is non-adjacent to some of the vertices v_5, v_6, v_7. We may assume without a loss of generality that $[v_7, v_8] \notin E(G)$.

Case 1. The vertex v_8 is adjacent to some of v_5 and v_6, for example v_8 is adjacent to v_5. We denote by A the set consisting of the vertex v_5 and these of the vertices v_1, v_2, v_3, v_4, which are not adjacent to v_5. It is clear that A is an anticlique in G. As $G - A = \langle \mathrm{Ad}(v_5) \rangle$, according to Proposition 6 $G - A$ does not contain 3-cliques and the assertion of the lemma is shown to be true in this case.

Case 2. The vertex v_8 is not adjacent neither to v_5 nor to v_6. If A is the anticlique defined in Case 1, then $G - A = \langle \mathrm{Ad}(v_5) \cup \{v_8\} \rangle$. As the vertex v_8 is not adjacent to v_6 and v_7 and $\mathrm{Ad}(v_5)$ does not contain 3-cliques, $G - A$ does not contain 3-cliques, too. ■

Lemma 2. *Let G be a 3-saturated 8-vertex graph and $\mathrm{cl}(G) = 3$. Then $\Delta(G) \leq 5$. Moreover, if v is a vertex of a 3-anticlique of G, then $d(v) \leq 4$.*

Proof. Assume that for some $v \in V(G)$ we have $d(v) = 7$. Then $G - v = \langle \mathrm{Ad}(v) \rangle$. According to Proposition 6, $\mathrm{Ad}(v)$ does not contain 3-cliques of G, which contradicts the fact that G is a 3-saturated graph. If we assume that $d(v) = 6$ and denote by w the vertex of G non-adjacent to v, then $G - \{v, w\} = \langle \mathrm{Ad}(v) \rangle$. Once again the last equality contradicts the 3-saturatedness of G. So, by now we have proved that $\Delta(G) \leq 5$.

Assume now that the second part of the lemma is false and let for example $\{v, u, w\}$ be a 3-anticlique of G and $d(v) > 4$. It follows that $G-\{v,u,w\} = \langle \mathrm{Ad}(v) \rangle$. Again an application of Proposition 6 gets a contradiction to the fact that G is a 3-saturated graph. ■

Lemma 3. *Let G be a vertex-critical 4-chromatic graph, $|V(G)| = 8$, and G contain two 3-anticliques without common vertices. Then G is not a 3-saturated fraph.*

Proof. As $\chi(K_4) = 4$ and G is vertex-critical, $\mathrm{cl}(G) < 4$. Let $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ be the two 3-anticliques given by the condition, and v_7 and v_8 be the other vertices. If $G - \{v_4, v_5, v_6\}$ contains no 3-cliques, then the assertion is proved. Assume that $G - \{v_4, v_5, v_6\}$ contains a 3-clique and let for example $\{v_1, v_7, v_8\}$ be such 3-clique. By a similar argument we may assume that the graph $G - \{v_1, v_2, v_3\}$ contains a 3-clique, say $\{v_4, v_7, v_8\}$. From $\mathrm{cl}(G) < 4$ it follows that $[v_1, v_4] \notin E(G)$.

Assume that v_7 is not adjacent to v_2 and v_3. If v_7 is not adjacent also to v_5 and v_6, then $G - v_8$ does not contain 3-cliques and the lemma is proved. If v_7 is adjacent to both v_5 and v_6, then v_7 is adjacent to each of the vertices of the subgraph $G - \{v_2, v_3, v_7\}$, and from Proposition 6 it follows that $G - \{v_2, v_3, v_7\}$ contains no 3-cliques. If the vertex v_7 is adjacent to only one of v_5 and v_6, for example $[v_5, v_7] \in E(G)$ and $[v_6, v_7] \notin E(G)$, we consider the following two situations:

1. $[v_5, v_8] \notin E(G)$. It is clear that $G - \{v_5, v_8\}$ does not contain 3-cliques and consequently G is not 3-saturated.

2. $[v_5, v_8] \in E(G)$. From $\mathrm{cl}(G) < 4$ it follows that $[v_1, v_5] \notin E(G)$. The subgraph $G - v_8$ does not contain 3-cliques and consequently G is not 3-saturated.

So, in the case when v_7 is not adjacent to the vertices v_2 and v_3 the assertion is proved. Therefore we assume that v_7 is adjacent to some of the vertices v_2 and v_3. Similarly, we may assume also that v_7 is adjacent to some of the vertices v_5 and v_6. We put then without a loss of generality $[v_2, v_7] \in E(G)$ and $[v_5, v_7] \in E(G)$. If the vertex v_7 is adjacent to some of v_3 and v_6, then our assertion is a consequence of Lemma 2, because $d(v_7) \geq 6$. That is why we may and do assume that $[v_3, v_7]$, $[v_6, v_7] \notin E(G)$.

Consider the subgraph $G - \{v_3, v_7\}$. If it does not contain 3-cliques, we are done. Let $G - \{v_3, v_7\}$ contain 3-cliques. Because $G - \{v_3, v_7\} = \langle \mathrm{Ad}(v_7) \cup \{v_6\} \rangle$ and $\mathrm{Ad}(v_7) = \{v_1, v_2, v_4, v_5, v_8\}$ does not contain 3-cliques, certainly $[v_6, v_8] \in E(G)$. By similar argument we conclude that $[v_3, v_8] \in E(G)$. If the vertex v_8 is adjacent also to some of v_2, v_5, then $d(v_8) \geq 6$ and we may apply Lemma 2 to get the conclusion. Therefore we assume that v_8 is not adjacent neither to v_2 nor to v_5.

Let us mention that at least one of the pairs $\{v_2, v_5\}$, $\{v_1, v_5\}$, $\{v_2, v_4\}$ is not adjacent in the graph G, because otherwise we would have $\langle \mathrm{Ad}(v_7) \rangle = C_5$ and $K_1 + C_5 \subset G$, which contradicts to the fact that G is a vertex-critical 4-chromatic graph, since $\chi(K_1 + C_5) = 4$. To conclude, let see that:

If $[v_2, v_5] \notin E(G)$, then $\{v_2, v_5, v_8\}$ is an anticlique and $G - \{v_2, v_5, v_8\}$ does not contain 3-cliques.

If $[v_1, v_5] \notin E(G)$, then $G - \{v_2, v_8\}$ does not contain 3-cliques.

If $[v_2, v_4] \notin E(G)$, then $G - \{v_5, v_8\}$ does not contain 3-cliques. ∎

Lemma 4. *Let G be a vertex-critical 4-chromatic graph and $|V(G)| = 8$. Then $\alpha(G - v) \geq 3$ for arbitrary $v \in V(G)$.*

Proof. It is obvious that if a 7-vertex graph has no 3-anticliques, then its chromatic number is bigger than 3. Therefore $\alpha(G - v) < 3$ implies $\chi(G - v) > 3$, which contradicts the fact that G is a vertex-critical 4-chromatic graph. ∎

Lemma 5. *Let G be a vertex-critical 4-chromatic graph and $|V(G)| = 8$. Then G is not a 3-saturated graph.*

Proof. If $\alpha(G) > 3$, then the assertion follows from Lemma 1. So, we assume that $\alpha(G) < 4$. Taking into account Lemma 3, we may assume that each two 3-anticliques in G have a common vertex.

Case 1. There are two 3-anticliques in G that have exactly one common vertex. We put them to be the 3-anticliques $A = \{a, c, y\}$ and $B = \{a, b, z\}$. Consider the subgraph $G - a$. According to Lemma 4, this subgraph has a 3-anticlique $C = \{u, v, x\}$. Because the sets A and C could not be disjoint, as well as B and C, we may assume that $u = c$ and $v = b$, i.e. $C = \{c, b, x\}$. From the assumption $\alpha(G) < 4$ it follows that $x \neq z$, $x \neq y$, $[a, x] \in E(G)$, $[b, y] \in E(G)$ and $[c, z] \in E(G)$. From the assumption that there are not two disjoint 3-anticliques in G it follows that $[x, y], [x, z], [z, y] \in E(G)$. So we may see that in fact the subgraph generated by the vertices a, b, c, x, y, z coincides with the graph shown at Fig. 23 (the bold lines denote the edges of G and the thin lines — the ones of $\overline{G}$). Let u and v be the last two vertices of G. According to Lemma 2, $\max\{d(x), d(y), d(z)\} \leq 4$. From this inequality we conclude that none of x, y, z can be adjacent to both u and v, hence one of u and v is not adjacent to at least two of x, y, z. We assume without a loss of generality that $[u, x]$, $[u, y] \notin E(G)$.

Subcase 1.a. The vertex u is not adjacent to the vertex z. From $\mathrm{cl}(G) = 3$ it follows that v is not adjacent at least to one of x, y, z. Because of the obvious symmetry we may assume that $[v, x] \notin E(G)$. In the subgraph $G - \{v, x\}$ there are no 3-cliques and consequently the graph G is not 3-saturated.

Subcase 1.b. The vertex u is adjacent to the vertex z. Because $d(z) \leq 4$ (Lemma 2), we have $[z, v] \notin E(G)$. In the subgraph $G - \{z, v\}$ there are no 3-cliques, which shows that G is not 3-saturated.

Case 2. Each two different 3-anticliques in G have two common vertices. Let $A = \{u, v, w\}$ be a 3-anticlique in G. According to Lemma 4, the subgraph $G - w$ contains a 3-anticlique B. Then $B = \{u, v, z\}$, since $|A \cap B| = 2$. Similarly, the subgraph $G - u$ contains 3-anticlique C that has two common vertices with A as well as with B. Then $C = \{z, v, w\}$ and $\{u, v, z, w\}$ is a 4-anticlique and the graph G is not 3-saturated according to Lemma 1. ∎

Lemma 6. *Let G be a 7-vertex graph, $\mathrm{cl}(G) = 3$, $\alpha(G) = 2$ and $\Delta(G) \leq 4$. Then G is isomorphic to one of the graphs F_i, $i = 1, \ldots, 7$ (Fig. 16–22).*

Proof. From $\alpha(G) = 2$ it follows that $\chi(G) \geq 4$. Because $\mathrm{cl}(G) = 3$, we may apply Theorem B with $r = 3$. From $\Delta(G) \leq 4$ it follows that the graph G contains no subgraph isomorphic to $K_1 + C_5$. The only possibility remaining is G to be isomorphic to one of F_i. ∎

Proof of Theorem 4. Theorem C implies that $\chi(G) \leq 4$ and from Proposition 5 we know that $\chi(G) \geq 4$. Consequently, $\chi(G) = 4$. According to Lemma 5, G is not a vertex-critical 4-chromatic graph, i.e. there is a vertex, say $v_8 \in V(G)$, such that $\chi(G - v_8) = 4$. We apply Theorem B with $r = 3$ to the subgraph $G - v_8$ to conclude that either $G - v_8$ is isomorphic to some of F_i, $i = 1, \ldots, 7$ (Fig. 16–22) or there is a $v_7 \in V(G)$ such that $G - \{v_7, v_8\} = K_1 + C_5$. Assume that there is no $v \in V(G)$ such that $G - v \neq \overline{C}_7 = F_7$. The above considerations show that there are the following possibilities:

Case 1. $G - v_8 = F_1$ (Fig. 16). We shall use the following automorphisms of the graph F_1:

$$\varphi(v_2) = v_6, \quad \varphi(v_4) = v_7, \quad \varphi(v_6) = v_2, \quad \varphi(v_7) = v_4, \quad \varphi(v_i) = v_i, \quad i = 1, 3, 5,$$
$$\psi(v_1) = v_7, \quad \psi(v_3) = v_6, \quad \psi(v_6) = v_3, \quad \psi(v_7) = v_1, \quad \psi(v_i) = v_i, \quad i = 2, 4, 5.$$

Subcase 1.a. The vertex v_8 is adjacent to at least one of the vertices v_2, v_3, v_6. Because $\varphi(v_6) = v_2$, $\psi(v_6) = v_3$, we may do assume without a loss of generality that v_8 is adjacent to v_6. From $\mathrm{cl}(G) = 3$ it follows that v_8 is not adjacent to at least one of v_2 and v_3. Because of the symmetry it is enough to consider the case $[v_3, v_8] \notin E(G)$. Certainly, $v_1 \in \mathrm{Ad}(v_8)$, since otherwise $\{v_1, v_3, v_8\}$ would be an anticlique and $G - \{v_1, v_3, v_8\}$ would not contain 3-cliques. From $\mathrm{cl}(G) = 3$ and $v_1, v_6 \in \mathrm{Ad}(v_8)$ it follows that $v_2 \notin \mathrm{Ad}(v_8)$. If we assume that $v_4 \notin \mathrm{Ad}(v_8)$, then $\{v_2, v_4, v_8\}$ is an anticlique and $G - \{v_2, v_4, v_8\}$ does not contain 3-cliques; and if we assume that $v_5 \notin \mathrm{Ad}(v_8)$, then $G - \{v_6, v_7\}$ does not contain 3-cliques. We have got a contradiction in both cases, which means that $v_4, v_5 \in \mathrm{Ad}(v_8)$. Consequently, either $\mathrm{Ad}(v_8) = \{v_1, v_4, v_6, v_5\}$ and G is isomorphic to L_4 (Fig. 5) or $\mathrm{Ad}(v_8) = \{v_1, v_4, v_7, v_5, v_6\}$ and G is isomorphic to L_6 (Fig. 7).

Subcase 1.b. The vertex v_8 is adjacent to none of v_2, v_3, v_6. If we assume that $v_1 \notin \mathrm{Ad}(v_8)$, then $\{v_1, v_3, v_8\}$ is an anticlique and $G - \{v_1, v_3, v_8\}$ does not contain 3-cliques; if $v_4 \notin \mathrm{Ad}(v_8)$, then $\{v_2, v_4, v_8\}$ is an anticlique and $G - \{v_2, v_4, v_8\}$ does not contain 3-cliques; if we assume that $v_5 \notin \mathrm{Ad}(v_8)$, $G - \{v_6, v_7\}$ does not contain 3-cliques, and if $v_7 \notin \mathrm{Ad}(v_8)$, then the subgraph $G - \{v_6, v_7, v_8\}$ does not contain 3-cliques. Thus we have proved that $\{v_1, v_4, v_5, v_7\} \subset \mathrm{Ad}(v_8)$. Because $v_2, v_3, v_6 \notin \mathrm{Ad}(v_8)$, we compute $\mathrm{Ad}(v_8) = \{v_1, v_4, v_5, v_7\}$, and G is isomorphic to the graph L_5 (Fig. 6).

Case 2. $G - v_8 = F_2$ (Fig. 17). We shall use the following automorphism of the graph F_2:

$$\varphi(v_1) = v_5, \quad \varphi(v_2) = v_4, \quad \varphi(v_3) = v_3, \quad \varphi(v_4) = v_2,$$
$$\varphi(v_5) = v_1, \quad \varphi(v_6) = v_7, \quad \varphi(v_7) = v_6.$$

The vertex v_8 is adjacent to at least one of the vertices v_6, v_7, since otherwise $\{v_6, v_7, v_8\}$ would be an anticlique and $G - \{v_6, v_7, v_8\}$ would contain no 3-clique.

Because of the certain symmetry ($\varphi(v_6) = v_7$) we may assume that $v_6 \in \mathrm{Ad}(v_8)$. From $\mathrm{cl}(G) = 3$ it follows that v_8 is not adjacent to the edges $[v_1, v_2]$, $[v_2, v_3]$, $[v_3, v_4]$. Because $G - \{v_6, v_7\}$ contains 3-cliques, we have two possibilities:

Subcase 2.a. The vertex v_8 is adjacent to the edge $[v_1, v_5]$. From $\mathrm{cl}(G) = 3$ it follows that $v_2 \notin \mathrm{Ad}(v_8)$. Certainly, $v_4 \in \mathrm{Ad}(v_8)$, since otherwise $\{v_2, v_4, v_8\}$ would be an anticlique and $G - \{v_2, v_4, v_8\}$ would contain no 3-clique. So, $\{v_1, v_4, v_5, v_6\} \subset \mathrm{Ad}(v_8)$. Because $\mathrm{cl}(G) = 3$, we have $\mathrm{Ad}(v_8) = \{v_1, v_4, v_5, v_6\}$. We see that $\alpha(G) = 2$ and then by Theorem A the graph G is isomorphic to the graph L_2 (Fig. 3).

Subcase 2.b. The vertex v_8 is adjacent to the edge $[v_4, v_5]$. From $\mathrm{cl}(G) = 3$ it follows that $v_3, v_7 \notin \mathrm{Ad}(v_8)$. If $v_1 \notin \mathrm{Ad}(v_8)$, then $G - \{v_2, v_4\}$ contains no 3-clique. If $v_1 \in \mathrm{Ad}(v_8)$, then as in subcase 2.a we conclude that the graph G is isomorphic to the graph L_2 (Fig. 3).

Case 3. $G - v_8 = F_3$ (Fig. 18). If $v_6, v_7 \notin \mathrm{Ad}(v_8)$, then $\{v_6, v_7, v_8\}$ is an anticlique and $G - \{v_6, v_7, v_8\}$ contains no 3-clique, which is a contradiction. Thus the vertex v_8 is adjacent to at least one of v_6, v_7. Because of the symmetry we may assume that $v_6 \in \mathrm{Ad}(v_8)$. From $\mathrm{cl}(G) = 3$ it follows that v_8 is not adjacent to the edges $[v_1, v_2]$, $[v_2, v_3]$, $[v_3, v_4]$. Because $G - \{v_6, v_7\}$ contains 3-cliques, v_8 is adjacent to at least one of the edges $[v_1, v_5]$, $[v_4, v_5]$.

Subcase 3.a. The vertex v_8 is adjacent to the edge $[v_1, v_5]$ and is not adjacent to the edge $[v_4, v_5]$, i.e. $v_1, v_5 \in \mathrm{Ad}(v_8)$ and $v_4 \notin \mathrm{Ad}(v_8)$. From $\mathrm{cl}(G) = 3$ it follows that $v_2, v_7 \notin \mathrm{Ad}(v_8)$. So, $\{v_1, v_5, v_6\} \subset \mathrm{Ad}(v_8)$ and $v_2, v_4, v_7 \notin \mathrm{Ad}(v_8)$. That is why either $\mathrm{Ad}(v_8) = \{v_1, v_5, v_6\}$ or $\mathrm{Ad}(v_8) = \{v_1, v_5, v_6, v_3\}$. If $\mathrm{Ad}(v_8) = \{v_1, v_5, v_6\}$, then the graph G is isomorphic to the graph L_9 (Fig. 10). If $\mathrm{Ad}(v_8) = \{v_1, v_5, v_6, v_3\}$, then $\alpha(G - v_2) = 2$ and $\Delta(G - v_2) = \delta(G - v_2) = 4$. From Lemma 6 it follows that $G - v_2$ is isomorphic to some of the graphs F_i, $i = 1, \dots, 7$. Because $\delta(F_i) = 3$ for $i = 1, \dots, 6$, we have that $G - v_2 = \overline{C}_7 = F_7$, which contradicts the assumption at the top of the proof.

Subcase 3.b. The vertex v_8 is adjacent to the edge $[v_4, v_5]$ and is not adjacent to the edge $[v_1, v_5]$, i.e. $v_4, v_5 \in \mathrm{Ad}(v_8)$ and $v_1 \notin \mathrm{Ad}(v_8)$. From $\mathrm{cl}(G) = 3$ it follows that $v_3, v_7 \notin \mathrm{Ad}(v_8)$. If $v_2 \in \mathrm{Ad}(v_8)$, then $G - v_6$ is isomorphic to the graph F_1 and we are back to the case 1. If $v_2 \notin \mathrm{Ad}(v_8)$, then G is isomorphic to the graph L_{10} (Fig. 11).

Subcase 3.c. The vertex v_8 is adjacent to the both edges $[v_1, v_5]$ and $[v_4, v_5]$. From $\mathrm{cl}(G) = 3$ it follows that v_8 is not adjacent to any of v_2, v_3, v_7. We take the conclusion that G is isomorphic to the graph L_{11} (Fig. 12).

Case 4. $G - v_8 = F_4$ (Fig. 19). We use the following automorphism of the graph F_4:

$$\varphi(v_1) = v_5, \quad \varphi(v_2) = v_7, \quad \varphi(v_3) = v_3, \quad \varphi(v_4) = v_6,$$
$$\varphi(v_5) = v_1, \quad \varphi(v_6) = v_4, \quad \varphi(v_7) = v_2.$$

We consider three subcases:

Subcase 4.a. The vertices $v_4, v_6 \in \mathrm{Ad}(v_8)$. From $\mathrm{cl}(G) = 3$ and $v_6 \in \mathrm{Ad}(v_8)$ it follows that the vertex v_8 is not adjacent to the edges $[v_1, v_2]$, $[v_2, v_3]$, $[v_3, v_4]$. From this fact we conclude that $v_5 \in \mathrm{Ad}(v_8)$ (otherwise v_8 is not adjacent to any of the edges of the 5-cycle v_1, v_2, v_3, v_4, v_5, v_1 and $G - \{v_6, v_7\}$ contains no 3-cliques). From $\text{cl}(G) = 3$ and $v_4 \in \text{Ad}(v_8)$ it follows that the vertex v_8 is not adjacent to the edges $[v_3, v_6]$, $[v_3, v_7]$, $[v_7, v_5]$. Hence $v_1 \in \text{Ad}(v_8)$ (otherwise v_8 is not adjacent to any of the edges of the 5-cycle v_1, v_6, v_3, v_7, v_5, v_1 and $G - \{v_2, v_4\}$ contains no 3-cliques). So, $\{v_1, v_4, v_5, v_6\} \subset \text{Ad}(v_8)$. Because $\text{cl}(G) = 3$, we compute $\text{Ad}(v_8) = \{v_1, v_4, v_5, v_6\}$, and G is isomorphic to the graph L_7 (Fig. 8).

Subcase 4.b. The vertex v_8 is adjacent to only one of the vertices v_4, v_6. Because of the certain symmetry ($\varphi(v_6) = v_4$) we may assume that $v_6 \in \text{Ad}(v_8)$ and $v_4 \notin \text{Ad}(v_8)$. If $v_2 \notin \text{Ad}(v_8)$, then $\{v_2, v_4, v_8\}$ is an anticlique and $G - \{v_2, v_4, v_8\} = C_5$ contains no 3-cliques — a contradiction. If $v_2 \in \text{Ad}(v_8)$, then from $\text{cl}(G) = 3$ it follows that v_8 is not adjacent to v_1 and v_3. Since v_8 is not adjacent also to v_4, we have that v_8 is not adjacent to any of the edges of the 5-cycle v_1, v_2, v_3, v_4, v_5, v_1. This is a contradiction, because $G - \{v_6, v_7\}$ does not contain 3-cliques.

Subcase 4.c. The vertices $v_4, v_6 \notin \text{Ad}(v_8)$. Certainly, $v_2, v_7 \in \text{Ad}(v_8)$: if $v_2 \notin \text{Ad}(v_8)$, then $G - \{v_2, v_4, v_8\}$ contains no 3-cliques; if $v_7 \notin \text{Ad}(v_8)$, then $G - \{v_6, v_7, v_8\}$ contains no 3-cliques. If v_8 is adjacent to the edge $[v_1, v_5]$, then $\alpha(G) = 2$ and from Theorem A it follows that the graph G is isomorphic to the graph L_1 (Fig. 2). Assume now that the vertex v_8 is not adjacent to the edge $[v_1, v_5]$. Then either $v_1 \notin \text{Ad}(v_8)$ or $v_5 \notin \text{Ad}(v_8)$. From the reasons of symmetry ($\varphi(v_1) = v_5$) we may assume that $v_1 \notin \text{Ad}(v_8)$. The subgraph $G - \{v_6, v_7\}$ contains a 3-clique and thus $v_3 \in \text{Ad}(v_8)$. If $v_5 \notin \text{Ad}(v_8)$, then $\text{Ad}(v_8) = \{v_2, v_3, v_7\}$, and G is isomorphic to the graph L_8 (Fig. 9). If $v_5 \in \text{Ad}(v_8)$, then the subgraph $G - v_7$ is isomorphic to F_1, which is the case 1.

C a s e 5. $G - v_8 = F_5$ (Fig. 20). We consider the following two possibilities:

Subcase 5.a. The vertex $v_6 \notin \text{Ad}(v_8)$. Here we surely have $v_5, v_7 \in \text{Ad}(v_8)$: if $v_5 \notin \text{Ad}(v_8)$, then $\{v_5, v_6, v_8\}$ is an anticlique and $G - \{v_5, v_6, v_8\}$ contains no 3-cliques; if $v_7 \notin \text{Ad}(v_8)$, then $\{v_6, v_7, v_8\}$ is an anticlique and $G - \{v_6, v_7, v_8\}$ contains no 3-cliques. From $\text{cl}(G) = 3$ it follows that $v_1, v_4 \notin \text{Ad}(v_8)$. Because $G - \{v_6, v_7\}$ contains a 3-clique, the vertex v_8 is adjacent to the edge $[v_2, v_3]$. Thus the subgraph $G - v_7$ is isomorphic to F_1, which is the case 1.

Subcase 5.b. The vertex $v_6 \in \text{Ad}(v_8)$. From $\text{cl}(G) = 3$ it follows that the vertex v_8 is not adjacent to the edges $[v_1, v_2]$, $[v_2, v_3]$ and $[v_3, v_4]$. Because $G - \{v_6, v_7\}$ contains a 3-clique, the vertex v_8 is adjacent to at least one of the edges $[v_1, v_5]$ and $[v_4, v_5]$. For the symmetry we may assume that v_8 is adjacent to the edge $[v_1, v_5]$. From $\text{cl}(G) = 3$ we have that $v_7 \notin \text{Ad}(v_8)$ and thus v_8 is not adjacent to the edges $[v_1, v_7]$ and $[v_4, v_7]$. But v_8 is not adjacent also to the edges $[v_1, v_2]$, $[v_2, v_3]$ and $[v_3, v_4]$ and the subgraph $G - \{v_5, v_6\}$ contains no 3-cliques, a contradiction.

C a s e 6. $G - v_8 = F_6$ (Fig. 21). We shall use the following automorphisms of the graph F_6:

$$\varphi(v_2) = v_6, \quad \varphi(v_6) = v_2, \quad \varphi(v_5) = v_7, \quad \varphi(v_7) = v_5, \quad \varphi(v_i) = v_i \quad i = 1, 3, 4,$$
$$\psi(v_1) = v_1, \quad \psi(v_2) = v_5, \quad \psi(v_3) = v_4, \quad \psi(v_4) = v_3,$$
$$\psi(v_5) = v_2, \quad \psi(v_6) = v_7, \quad \psi(v_7) = v_6,$$
$$\nu(v_1) = v_1, \quad \nu(v_2) = v_7, \quad \nu(v_3) = v_4, \quad \nu(v_4) = v_3,$$
$$\nu(v_5) = v_6, \quad \nu(v_6) = v_5, \quad \nu(v_7) = v_2.$$

Subcase 6.a. The vertex v_8 is not adjacent to some of the vertices v_2, v_5, v_6, v_7. Because of the symmetry ($\varphi(v_2) = v_6$, $\psi(v_2) = v_5$, $\nu(v_2) = v_7$) it is enough to consider only the situation when $v_2 \notin \mathrm{Ad}(v_8)$. In this situation certainly $v_5, v_7 \in \mathrm{Ad}(v_8)$ (if $v_5 \notin \mathrm{Ad}(v_8)$, then $\{v_2, v_5, v_8\}$ is an anticlique and $G-\{v_2, v_5, v_8\}$ contains no 3-clique; if $v_7 \notin \mathrm{Ad}(v_8)$, then $\{v_2, v_7, v_8\}$ is an anticlique and $G - \{v_2, v_7, v_8\}$ contains no 3-clique). From $\mathrm{cl}(G) = 3$ and $v_5, v_7 \in \mathrm{Ad}(v_8)$ it follows that $v_1, v_4 \notin \mathrm{Ad}(v_8)$. The subgraph $G - \{v_6, v_7\}$ contains no 3-cliques, a contradiction.

Subcase 6.b. The vertex v_8 is adjacent to all vertices v_2, v_5, v_6, v_7. From $\mathrm{cl}(G) = 3$ it follows that v_8 is not adjacent to the vertices v_1, v_3, v_4. We get the conclusion that the subgraph $G - \{v_6, v_7\}$ contains no 3-cliques, a contradiction.

Case 7. There are $v_7, v_8 \in V(G)$ such that $G - \{v_7, v_8\} = K_1 + C_5$. Let $V(K_1) = \{v_6\}$ and $C_5 = v_1, v_2, v_3, v_4, v_5, v_1$. From Lemma 2 and the fact that $d(v_6) \geq 5$ we conclude that $\Delta(G) = 5$ and $v_7, v_8 \notin \mathrm{Ad}(v_6)$. Certainly, $[v_7, v_8] \in E(G)$ (or, otherwise, $\{v_6, v_7, v_8\}$ is an anticlique and $G - \{v_6, v_7, v_8\}$ contains no 3-clique). If we assume that $\alpha(G - v_7) = \alpha(G - v_8) = 2$, then from $[v_7, v_8] \in E(G)$ it follows that $\alpha(G) = 2$ and according to Theorem A the graph G is isomorphic to some of L_1, L_2, L_3.

Let us now assume that at least one of the numbers $\alpha(G - v_7)$, $\alpha(G - v_8)$ is bigger than 2. Without a loss of generality, $\alpha(G - v_7) > 2$. This means that the vertex v_7 together with two non-adjacent vertices of the cycle v_1, v_2, v_3, v_4, v_5, v_1 form a 3-anticlique. Let, for example, $\{v_3, v_5, v_7\}$ be a 3-anticlique. Then $v_1, v_2 \in \mathrm{Ad}(v_7)$, since $G - \{v_6, v_8\}$ contains a 3-clique. From $\mathrm{cl}(G) = 3$ it follows that v_8 is not adjacent to at least one of the vertices v_1, v_2. Let v_8 be non-adjacent to v_1.

Assume first that v_8 is not adjacent also to v_2. Put $V_5(G) = \{v \in V(G) \mid d(v) = 5\}$. Then $V_5(G) \subset \{v_4, v_6\}$. Because $G - \{v_6, v_7\}$ contains a 3-clique, it follows that the vertex v_8 is adjacent to at least one of the edges $[v_3, v_4]$, $[v_4, v_5]$. For the symmetry we may assume that v_8 is adjacent to the edge $[v_4, v_5]$. Then $\alpha(G - v_3) = 2$ and from $V_5(G) \subset \{v_4, v_6\}$ it folows that $\Delta(G - v_3) = 4$. According to Lemma 6, $G - v_3$ is isomorphic to some of the graphs F_i for $i = 1, \ldots, 7$. By our assumption $G - v_3 \neq F_7$ and thus we turn to one of the cases 1–6.

Assume now that v_8 is adjacent to v_2.

Subcase 7.a. The vertex $v_4 \in \mathrm{Ad}(v_8)$. It is clear that $\alpha(G - v_3) = 2$. Note that $\Delta(G - v_3) = 4$ since $V_5(G) \subset \{v_2, v_4, v_6, v_8\}$. According to Lemma 6, $G - v_3$ is isomorphic to some of the graphs F_i for $i = 1, \ldots, 7$. By our assumption $G - v_3 \neq F_7$ and thus we turn to one of the cases 1–6.

Subcase 7.b. The vertex $v_4 \notin \mathrm{Ad}(v_8)$. Because we have also $v_1 \notin \mathrm{Ad}(v_8)$, the vertex v_8 is adjacent to none of the edges $[v_1, v_5]$, $[v_1, v_2]$, $[v_3, v_4]$, $[v_4, v_5]$. But the subgraph $G - \{v_6, v_7\}$ contains a 3-clique and therefore the vertex v_8 is adjacent to the edge $[v_2, v_3]$. So, we proved that the vertex v_7 is adjacent to the edge $[v_1, v_2]$ and, eventually, to the vertex v_4, and the vertex v_8 is adjacent to the edge $[v_2, v_3]$ and, eventually, to the vertex v_5. Now, if $v_4 \notin \mathrm{Ad}(v_7)$ and $v_5 \notin \mathrm{Ad}(v_8)$, then the graph G is isomorphic to the graph L_{12} (Fig. 13). If $v_4 \in \mathrm{Ad}(v_7)$ and $v_5 \notin \mathrm{Ad}(v_8)$ or $v_4 \notin \mathrm{Ad}(v_7)$ and $v_5 \in \mathrm{Ad}(v_8)$, then the graph G is isomorphic to the graph L_{13}

(Fig. 14). If $v_4 \in \mathrm{Ad}(v_7)$ and $v_5 \in \mathrm{Ad}(v_8)$, then the graph G is isomorphic to the graph L_{14} (Fig. 15). ■

Proof of Theorem 5. We fix some notations:

$e(G) = |E(G)|$,

$t(G)$ is the number of the 3-cliques of the graph G,

$\bar{t}(G)$ is the number of the 3-anticliques of the graph G,

$n(G)$ is the number of the pairs of 3-anticliques that have only one common vertex,

$m(G)$ is the number of the pairs of 3-anticliques that have no common vertex;

$i =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$e(L_i)$	16	17	18	16	16	17	16	15	16	16	17	15	16	17
$t(L_i)$	8	10	12	8	7	9	9	7	8	8	10	8	8	8
$\bar{t}(L_i)$	0	0	0	1	1	1	1	2	2	2	1	2	2	2
$n(L_i)$	0	0	0	0	0	0	0	1	0	1	0	0	0	0
$m(L_i)$	0	0	0	0	0	0	0	0	0	0	0	1	1	1

From these relations we see that each two of the graphs L_i, $i = 1, \ldots, 14$, are not isomorphic. As $\alpha(L_i) \leq 3$, for proving that the graphs L_i, $i = 1, \ldots, 14$, are 3-saturated, we need to show that:
(1) $t(L_i - v) \geq 1$ for an arbitrary $v \in L_i$, $i = 1, \ldots, 14$;
(2) $t(L_i - \{u, v\}) \geq 1$, $i = 1, \ldots, 14$, for each two non-adjacent vertices u and v from L_i;
(3) $t(L_i - \{u, v, w\}) \geq 1$, $i = 1, \ldots, 14$, for an arbitrary 3-anticlique $\{u, v, w\}$ of L_i.

We need the following assertions:

Proposition 7 ([2], see also [7]). *Let $|V(G)| = 6$. Then $t(G) + \bar{t}(G) \geq 2$.*

Proposition 8 ([4], see also [7]). *Let $|V(G)| = 6$, $\bar{t}(G) = 2$ and the both 3-anticliques of G have only one common vertex. Then $t(G) \geq 1$.*

For arbitrary $i = 1, \ldots, 14$ and for arbitrary vertex of L_i there is non-adjacent vertex of L_i, therefore (2) implies (1). Because $\bar{t}(L_i) \leq 2$, the check of (3) is easy. We only show the 3-anticliques of the graphs L_i. The graphs L_1, L_2 and L_3 have not 3-anticliques. The graphs L_4, L_5 and L_6 have the unique 3-anticlique $\{v_1, v_4, v_7\}$. The graph L_7 has the unique 3-anticlique $\{v_2, v_7, v_8\}$. The graph L_8 has two 3-anticliques — $\{v_1, v_4, v_8\}$ and $\{v_5, v_6, v_8\}$. The graph L_9 has two 3-anticliques — $\{v_2, v_4, v_8\}$ and $\{v_2, v_7, v_8\}$. The graph L_{10} has two 3-anticliques — $\{v_1, v_3, v_8\}$ and $\{v_2, v_7, v_8\}$. The graph L_{11} has the unique 3-anticlique $\{v_2, v_7, v_8\}$. Each of the graphs L_{12}, L_{13} and L_{14} has only these two 3-anticliques — $\{v_3, v_5, v_7\}$ and $\{v_1, v_4, v_8\}$.

We now show the inequalities (2). If $i = 1, 2, 3, 4, 5, 6, 7, 11$, then $\bar{t}(L_i) \leq 1$ and the inequality (2) follows from Proposition 7. Let $i = 8, 10$. If at least one of the vertices u, v is a vertex of a 3-anticlique of the graph L_i, then (2) follows from Proposition 7. If none of the vertices u, v is a vertex of a 3-anticlique of the graph L_i, then the subgraph $L_i - \{u, v\}$ satisfies the conditions of Proposition 8 and hence (2) is satisfied. Let $i = 9$. The graph L_9 has only the 3-anticliques $\{v_2, v_4, v_8\}$ and $\{v_2, v_7, v_8\}$. If $u, v \in V(L_9)$, $[u, v] \notin E(L_9)$, and at least one of the vertices u, v is a vertex of a 3-anticlique, then $\bar{t}(L_9 - \{u, v\}) \leq 1$ and the inequality $t(L_9 - \{u, v\}) \geq 1$ follows from Proposition 7. If none of the vertices u, v is a vertex of a 3-anticlique, then the pair $\{u, v\}$ coincedes with one of the following pairs of non-adjacent vertices of L_9: $\{v_1, v_3\}$, $\{v_5, v_6\}$, $\{v_3, v_5\}$, for which (2) is obvious.

Consider the graphs L_{12}, L_{13} and L_{14}. It is enough to prove (2) for L_{12}, since L_{12} is a subgraph of L_{13} and L_{14}. The only vertices of L_{12} that do not take part in 3-anticliques are v_2 and v_6. Since v_2 and v_6 are adjacent vertices of the graph L_{12}, if the vertices u and v are not adjacent, it follows that one of them is a vertex of a 3-anticlique of L_{12}. Therefore $\bar{t}(L_{12} - \{u, v\}) \leq 1$. From Proposition 7 we get $t(L_{12} - \{u, v\}) \geq 1$.

We can see that $L_i - v \neq \overline{C}_7$, $\forall v \in V(L_i)$, comparing the inequalities $\delta(L_i - v) \leq 3$ and $\delta(\overline{C}_7) = 4$. ■

4. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. Assume that $|V(G)| \leq 8$. By adding if necessary isolated vertices, we may consider only the case $|V(G)| = 8$. According to Proposition 3, we have $\chi(G) \geq 5$. We apply Theorem B ($r = 4$) to conclude that either $G = K_1 + F_i$, $i = 1, \dots, 7$, or there exists $v \in V(G)$ such that $G - v = K_2 + C_5$. We are going to prove that in the second case we can also find a vertex that is adjacent to all other vertices of the graph G. Let $G - v = K_2 + C_5$ and $V(K_2) = \{x, y\}$. If the vertex v is not adjacent to the edge $[x, y]$, then $\{x, y, v\} \cup V(C_5)$ is a 3-cliques free 2-partition of the vertices of G, which is impossible. Hence the vertex v is adjacent to the edge $[x, y]$ and then x is adjacent to all other vertices of the graph G. So, if the graph G satisfies the conditions of Theorem 1, then there is a vertex $v_0 \in V(G)$ which is adjacent to all other vertices of the graph G. Proposition 4 implies that $G - v_0$ is a 3-saturated 7-vertex graph. It is clear that $\mathrm{cl}(G - v_0) = 3$. According to Theorem 3, $G - v_0 = \overline{C}_7$ and since v_0 is adjacent to all vertices of $\overline{C}_7$, it follows that $G = K_1 + \overline{C}_7$. ■

We need the next lemmas.

Lemma 7. *Let A be an anticlique of the graph G, $G_1 = G - A$; and $V(G_1) = B \cup C$ be a 3-cliques free 2-partition of vertices of G_1 such that: each vertex of A, that is adjacent to some edge of the subgraph $\langle B \rangle$, is not adjacent to any edge of the subgraph $\langle C \rangle$. Then G has a 3-cliques free 2-partition of vertices.*

Proof. Let $A_1 = \{v \in A \mid v \text{ is not adjacent to any edge of } \langle B \rangle\}$. Put $V_1 = A_1 \cup B$ and $V_2 = (A \setminus A_1) \cup C$. Consider the 2-partition $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$. It is clear that V_1 does not contain 3-cliques of the graph G. If $v \in V_2 \cap A$, then v is adjacent to some edge of the subgraph $\langle B \rangle$ and therefore is not adjacent to any edge of the subgraph $\langle C \rangle$. That is why V_2 does not contain 3-cliques, too. ■

Lemma 8. *Let G be a graph, $|V(G)| = n$, $\mathrm{cl}(G) = 3$, and A be an anticlique of G, $|A| = n-8$. Put $G_1 = G - A$. If $G \rightarrow \underline{(3,3)}$, then either $G_1 = L_{14}$* (Fig. 15) *or there exists $v \in V(G_1)$ such that $G_1 - v = \overline{C}_7$.*

Proof. According to Proposition 4, the subgraph G_1 is a 3-saturated graph. Since $|V(G_1)| = 8$ and $\mathrm{cl}(G) = 3$, we can apply Theorem 4 to the subgraph G_1. If we assume that the assertion of Lemma 8 is false, then G_1 is isomorphic to one of the graphs L_i, $i = 1, \ldots, 13$. We shall consider all these cases:

Case 1. G_1 is some of the graphs L_1, L_2, L_3. We put $B = \{v_3, v_4, v_7, v_8\}$ and $C = \{v_1, v_2, v_5, v_6\}$. For any of L_1, L_2, L_3 we have $E(\langle B \rangle) = \{[v_3, v_4], [v_7, v_8]\}$.

For any of L_1, L_2, L_3 it is true that each edge of $\langle C \rangle$ belongs either to $E(\langle \mathrm{Ad}(v_3) \rangle)$ or to $E(\langle \mathrm{Ad}(v_4) \rangle)$. Therefore, if we assume that some $v \in A$ is adjacent to the edge $[v_3, v_4]$, then $\mathrm{cl}(G) = 3$ implies that v is not adjacent to any of the edges of $\langle C \rangle$. Similarly, if some $v \in A$ is adjacent to $[v_7, v_8]$, then v is not adjacent to any of the edges of $\langle C \rangle$. We see from Lemma 7 that G has a 3-cliques free 2-partition of vertices, which is a contradiction.

Case 2. G_1 is some of the graphs L_4, L_5, L_6. We put $B = \{v_2, v_3, v_5, v_8\}$ and $C = \{v_1, v_4, v_6, v_7\}$. For any of L_4, L_5, L_6 we have $E(\langle B \rangle) = \{[v_2, v_3], [v_5, v_8]\}$ and $E(\langle C \rangle) = \{[v_1, v_6], [v_4, v_6]\}$. If some of the vertices of the anticlique A is adjacent to the edge $[v_2, v_3]$, then $\mathrm{cl}(G) = 3$ implies that this vertex is not adjacent to the edges $[v_1, v_6]$, $[v_4, v_6]$, i.e. it is not adjacent to any of the edges of $\langle C \rangle$. If any of the vertices of the anticlique A is adjacent to the edge $[v_5, v_8]$, then from $\mathrm{cl}(G) = 3$ it follows that this vertex is not adjacent to the vertices v_1 and v_4. Consequently, it is not adjacent to the edges $[v_1, v_6]$ and $[v_4, v_6]$ of the subgraph $\langle C \rangle$. We see then from Lemma 7 that G has a 3-cliques free 2-partition of vertices, which is a contradiction.

Case 3. G_1 is some of the graphs L_7, L_8, L_{10} L_{11}. We put $B = \{v_1, v_3, v_4\}$ and $C = \{v_2, v_5, v_6, v_7, v_8\}$. For any of L_7, L_8, L_{10}, L_{11} we have $E(\langle B \rangle) = \{[v_3, v_4]\}$. Also, for L_7, L_{10}, L_{11} we denote $E_1 = E(\langle C \rangle) = \{[v_2, v_6], [v_6, v_8], [v_8, v_5], [v_5, v_7]\}$, and for L_8 — $E_2 = E(\langle C \rangle) = \{[v_2, v_6], [v_2, v_8], [v_8, v_7], [v_5, v_7]\}$.

Let the vertex $u \in A$ be adjacent to the edge $[v_3, v_4]$. For the graphs L_7, L_{10}, L_{11} we have that $\{v_2, v_6\} \subset \mathrm{Ad}(v_3)$ and $\{v_5, v_6, v_7, v_8\} \subset \mathrm{Ad}(v_4)$. Therefore $\mathrm{cl}(G) = 3$ implies that the vertex u is not adjacent to any of the edges from E_1. For the graph L_8 we have that $\{v_2, v_6, v_7, v_8\} \subset \mathrm{Ad}(v_3)$ and $\{v_5, v_7\} \subset \mathrm{Ad}(v_4)$. Therefore $\mathrm{cl}(G) = 3$ implies that the vertex u is not adjacent to any of the edges from E_2.

So, the conditions of Lemma 7 are satisfied and we conclude that in the considered case the graph G has a 3-cliques free 2-partition of vertices, which is a contradiction.

Case 4. G_1 coincides with the graph L_9. We put $B = \{v_1, v_3, v_4, v_8\}$ and $C = \{v_2, v_5, v_6, v_7\}$. We have that $E(\langle B \rangle) = \{[v_1, v_8], [v_3, v_4]\}$. If some of the vertices of the anticlique A is adjacent to the edge $[v_1, v_8]$, then $\mathrm{cl}(G) = 3$ and $C \subset \mathrm{Ad}(v_1)$ imply that this vertex is not adjacent to the edges $[v_2, v_6]$ and $[v_5, v_7]$, i.e. it is not adjacent to any of the edges of $\langle C \rangle$. If the anticlique A contains a vertex that is adjacent to the edge $[v_3, v_4]$, then from $\mathrm{cl}(G) = 3$ it follows that this vertex is not adjacent to the edges $[v_2, v_6]$, $[v_5, v_7]$, i.e. it is not adjacent to the edges of $\langle C\rangle$. We see from Lemma 7 that G has a 3-cliques free 2-partition of vertices, which is a contradiction.

Case 5. G_1 is some of the graphs L_{12}, L_{13}. We put $B = \{v_1, v_2, v_4\}$ and $C = \{v_3, v_5, v_6, v_7, v_8\}$. We have that $E(\langle B\rangle) = \{[v_1, v_2]\}$ and $E(\langle C\rangle) = \{[v_7, v_8], [v_3, v_8], [v_3, v_6], [v_5, v_6]\}$. Let some of the vertices of the anticlique A be adjacent to the edge $[v_1, v_2]$. From $\mathrm{cl}(G) = 3$ and $\{v_3, v_6, v_7, v_8\} \subset \mathrm{Ad}(v_2)$ it follows that this vertex is not adjacent to the edges $[v_7, v_8]$, $[v_3, v_8]$ and $[v_3, v_6]$; from $\mathrm{cl}(G) = 3$ and $\{v_5, v_6\} \subset \mathrm{Ad}(v_1)$ it follows that this vertex is not adjacent to the edge $[v_5, v_6]$.

The above reasoning shows that the conditions of Lemma 7 are satisfied and we conclude that the graph G has a 3-cliques free 2-partition of vertices, which is a contradiction. ■

Lemma 9. *Let G be an 11-vertex graph, $\mathrm{cl}(G) = 3$, and G have three 3-anticliques, each two of which have an empty intersection. Then the graph G has a 3-cliques free 2-partition of vertices.*

Proof. Let A, B and C be the anticliques given by the condition. Assume the contrary, i.e. $G \to (3,3)$. We put $G_1 = G - A$. Because G_1 has two anticliques B and C with empty intersection and $\alpha(\overline{C}_7) = 2$, we have that $G_1 - v \neq \overline{C}_7$, $\forall v \in V(G_1)$. From Lemma 7 it follows that $G_1 = L_{14}$ (Fig. 15). Let $A = \{v_9, v_{10}, v_{11}\}$. At least one of the vertices v_9, v_{10}, v_{11} is adjacent to the edge $[v_2, v_6]$ (if not, $\{v_2, v_6, v_9, v_{10}, v_{11}\} \cup \{v_1, v_7, v_8, v_3, v_4, v_5\}$ is a 3-cliques free 2-partition of the vertices of G). Thus we assume that v_9 is adjacent to $[v_2, v_6]$. At least one of the vertices v_9, v_{10}, v_{11} is adjacent to the edge $[v_1, v_2]$ (if not, $\{v_1, v_2, v_4, v_9, v_{10}, v_{11}\} \cup \{v_3, v_5, v_6, v_7, v_8\}$ is a 3-cliques free 2-partition of the vertices of G). The vertex v_9 is not adjacent to the edge $[v_1, v_2]$, since otherwise $\{v_1, v_2, v_6, v_9\}$ would be a 4-clique. Hence we may assume that v_{10} is adjacent to the edge $[v_1, v_2]$. Surely, one of the vertices v_9, v_{10}, v_{11} is adjacent to the edge $[v_1, v_6]$ (if not, $\{v_1, v_6, v_8, v_9, v_{10}, v_{11}\} \cup \{v_2, v_3, v_4, v_5, v_7\}$ is a 3-cliques free 2-partition of the vertices of G). $\mathrm{cl}(G) = 3$ implies that both vertices v_9 and v_{10} are not adjacent to the edge $[v_1, v_6]$, thus v_{11} is adjacent to the edge $[v_1, v_6]$.

Consider the 2-partition $V(G) = V_1 \cup V_2$, where $V_1 = \{v_6, v_7, v_8, v_{10}\}$ and $V_2 = \{v_1, v_2, v_3, v_4, v_5, v_9, v_{11}\}$. Since v_{10} is adjacent to the vertex v_2 and $\mathrm{cl}(G) = 3$, the vertex v_{10} is not adjacent to the edge $[v_7, v_8]$. That is why V_1 contains no 3-cliques. From $\mathrm{cl}(G) = 3$ and the fact that v_9 is adjacent to the edge $[v_2, v_6]$ it follows that v_9 is not adjacent neither to the vertices v_1, v_3 nor to the edge $[v_4, v_5]$. Thus v_9 is not adjacent to any of the edges of the 5-cycle v_1, v_2, v_3, v_4, v_5, v_1. From $\mathrm{cl}(G) = 3$ and the fact that v_{11} is adjacent to $[v_1, v_6]$ it follows that v_{11} is not adjacent neither to the vertices v_2 and v_5 nor to the edge $[v_3, v_4]$. This shows that v_{11} is adjacent to none of the edges of the 5-cycle v_1, v_2, v_3, v_4, v_5, v_1. Since v_9 and v_{11} are not adjacent, V_2 does not contain 3-cliques. We have proved that $V(G) = V_1 \cup V_2$ is a 3-cliques free 2-partition of the vertices of G. This contradiction completes the proof. ■

Proof of Theorem 2. Assume the contrary, i.e. $G \rightarrow (3,3)$. According to Proposition 2, $\alpha(G) \geq 3$. Let $A = \{v_9, v_{10}, v_{11}\}$ be a 3-anticlique of G. Put $G_1 = G - A$; $V(G_1) = \{v_1, \ldots, v_8\}$. Because L_{14} (Fig. 15) has two disjoint 3-anticliques, Lemma 9 implies that $G_1 \neq L_{14}$. From Lemma 8 it follows that there exists $v \in V(G_1)$ such that $G_1 - v = \overline{C}_7$. Let, for example, $G_1 - v_8 = \overline{C}_7 = F_7$ (Fig. 22).

We shall prove first that the vertex v_8 together with some two vertices of $\overline{C}_7$ form a 3-anticlique of the graph G. From $\mathrm{cl}(G) = 3$ it follows that the vertex v_8 is not adjacent to some of the vertices of $\overline{C}_7$. Let, for example, v_8 be not adjacent to v_1 (Fig. 22). If the vertex v_8 is not adjacent to v_2 or v_7, then $\{v_1, v_2, v_8\}$ or, respectively, $\{v_1, v_7, v_8\}$ is a 3-anticlique of G. If v_8 is adjacent to both v_2 and v_7, then $\mathrm{cl}(G) = 3$ implies that $\{v_4, v_5, v_8\}$ is a 3-anticlique of G.

So, we may assume that $\{v_1, v_2, v_8\}$ is a 3-anticlique of the graph G. From $\mathrm{cl}(G) = 3$ it follows that v_8 is not adjacent to one of the vertices of the 3-clique $\{v_3, v_5, v_7\}$. We shall consider the following two cases.

Case 1. The vertex v_8 is not adjacent to v_3 or v_7, for example v_8 is not adjacent to v_3. One of the vertices v_9, v_{10}, v_{11} is adjacent to the edge $[v_1, v_3]$ (if not, $\{v_1, v_2, v_3, v_8, v_9, v_{10}, v_{11}\} \cup \{v_4, v_5, v_6, v_7\}$ is a 3-cliques free 2-partition). Let, for example, v_9 be adjacent to the edge $[v_1, v_3]$. From $\mathrm{cl}(G) = 3$ it follows that $\{v_5, v_6, v_9\}$ is a 3-anticlique. One of the vertices v_9, v_{10}, v_{11} is adjacent to the edge $[v_1, v_6]$ (if not, $\{v_1, v_6, v_7, v_9, v_{10}, v_{11}\} \cup \{v_2, v_3, v_4, v_5, v_8\}$ is a 3-cliques free 2-partition). From $\mathrm{cl}(G) = 3$ it follows that v_9 is not adjacent to the edge $[v_1, v_6]$. Therefore we may assume that v_{10} is adjacent to the edge $[v_1, v_6]$. From $\mathrm{cl}(G) = 3$ it follows that $\{v_3, v_4, v_{10}\}$ is a 3-anticlique. We obtain that G contains the pairwise disjoint 3-anticliques $\{v_1, v_2, v_8\}$, $\{v_5, v_6, v_9\}$ and $\{v_3, v_4, v_{10}\}$, which contradicts Lemma 9.

Case 2. The vertex v_8 is not adjacent to v_5. Surely, one of the vertices v_9, v_{10}, v_{11} is adjacent to the edge $[v_1, v_6]$ (if not, $\{v_1, v_6, v_7, v_9, v_{10}, v_{11}\} \cup \{v_2, v_3, v_4, v_5, v_8\}$ is a 3-cliques free 2-partition). Let, for example, v_9 be adjacent to $[v_1, v_6]$. From $\mathrm{cl}(G) = 3$ it follows that $\{v_3, v_4, v_9\}$ is a 3-anticlique. One of the vertices v_9, v_{10}, v_{11} is adjacent to the edge $[v_2, v_4]$ (if not, $\{v_2, v_3, v_4, v_9, v_{10}, v_{11}\} \cup \{v_1, v_5, v_6, v_7, v_8\}$ is a 3-cliques free 2-partition). Because v_9 is adjacent to v_6 and $\mathrm{cl}(G) = 3$, we know that v_9 is not adjacent to the edge $[v_2, v_4]$. Consequently, we may assume that the vertex v_{10} is adjacent to the edge $[v_2, v_4]$. From $\mathrm{cl}(G) = 3$ it follows that $\{v_6, v_7, v_{10}\}$ is a 3-anticlique. We have obtained that G contains the pairwise disjoint 3-anticliques $\{v_1, v_2, v_8\}$, $\{v_3, v_4, v_9\}$ and $\{v_6, v_7, v_{10}\}$, which contradicts Lemma 9.

The proof of Theorem 2 is completed. ■

5. AN EXAMPLE

We consider the graph L_{14} (Fig. 15) and the following subsets of $V(L_{14})$: $M_1 = \{v_2, v_4, v_6, v_7\}$, $M_2 = \{v_2, v_5, v_6, v_8\}$, $M_3 = \{v_1, v_2, v_5, v_8\}$, $M_4 = \{v_3, v_5, v_6, v_8\}$, $M_5 = \{v_2, v_3, v_4, v_7\}$, $M_6 = \{v_1, v_4, v_6, v_7\}$, $M_7 = \{v_4, v_5, v_7, v_8\}$. We denote by Γ_2

the extension of the graph L_{14} that is obtained by adding to $V(L_{14})$ new 7 vertices $u_1, \ldots, u_7$, none of which are adjacent and such that $\mathrm{Ad}(u_i) = M_i$, $i = 1, \ldots, 7$.

Proposition 9. $\Gamma_2 \to (3,3)$ *and* $\mathrm{cl}(\Gamma_2) = 3$.

Proof. The equality $\mathrm{cl}(\Gamma_2) = 3$ is true, because $\mathrm{cl}(L_{14}) = 3$, $\{u_1, \ldots, u_7\}$ is an anticlique, and $\mathrm{Ad}(u_i)$ does not contain 3-cliques for $i = 1, \ldots, 7$.

Let $V(\Gamma_2) = V_1 \cup V_2$ be an arbitrary 2-partition of the vertices of Γ_2.

Case 1. v_2 and v_6 belong to only one of the sets V_1 and V_2, for example $v_2, v_6 \in V_1$. From $v_2, v_6 \in V_1$ it follows that at least one of the vertices v_7, v_8 belongs to V_2. Let, for example, $v_7 \in V_2$. From $v_2, v_6 \in V_1$ it follows also that at least one of the vertices v_4, v_5 belongs to V_2. Therefore we have only two possibilities:

Subcase 1.a. $v_4 \in V_2$. If $u_1 \in V_1$, then $\{u_1, v_2, v_6\}$ is a 3-clique of Γ_2, contained in V_1. If $u_1 \in V_2$, then $\{u_1, v_4, v_7\}$ is a 3-clique of Γ_2, contained in V_2.

Subcase 1.b. $v_5 \in V_2$. From $v_2, v_6 \in V_1$ it follows also that $v_1 \in V_2$. Let $v_8 \in V_1$. If $u_3 \in V_1$, then $\{u_3, v_2, v_8\}$ is a 3-clique of Γ_2, contained in V_1. If $u_3 \in V_2$, then $\{u_3, v_1, v_5\}$ is a 3-clique of Γ_2, contained in V_2. Assume that $v_8 \in V_2$. If $u_2 \in V_1$, then $\{u_2, v_2, v_6\}$ is a 3-clique of Γ_2, contained in V_1. If $u_2 \in V_2$, then $\{u_2, v_5, v_8\}$ is a 3-clique of Γ_2, contained in V_2.

Case 2. One of the vertices v_2, v_6 belongs to V_1 and the other one belongs to V_2. Let, for example, $v_2 \in V_1$, $v_6 \in V_2$.

Subcase 2.a. One of the vertices v_7, v_8 belongs to V_1, for example $v_7 \in V_1$. If $v_8 \in V_1$ or $v_1 \in V_1$, then V_1 will contain respectively the 3-clique $\{v_2, v_7, v_8\}$ or the 3-clique $\{v_1, v_2, v_7\}$. Therefore we assume that $v_1, v_8 \in V_2$.

Let $v_4 \in V_1$. If $u_6 \in V_1$, then $\{u_6, v_4, v_7\}$ is a 3-clique of Γ_2, contained in V_1. If $u_6 \in V_2$, then $\{u_6, v_1, v_6\}$ is a 3-clique of Γ_2, contained in V_2.

Let $v_4 \in V_2$. If $u_1 \in V_1$, then $\{u_1, v_2, v_7\}$ is a 3-clique of Γ_2, contained in V_1. If $u_1 \in V_2$, then $\{u_1, v_4, v_6\}$ is a 3-clique of Γ_2, contained in V_2.

Subcase 2.b. $v_7, v_8 \in V_2$. Assume first that at least one of the vertices v_4, v_5 belongs to V_2 and let, for example, $v_4 \in V_2$. If $v_3 \in V_2$, then $\{v_3, v_4, v_6\}$ is a 3-clique of Γ_2, contained in V_2. Thus we assume that $v_3 \in V_1$. Now, if $u_5 \in V_1$, then $\{u_5, v_2, v_3\}$ is a 3-clique of Γ_2, contained in V_1. If $u_5 \in V_2$, then $\{u_5, v_4, v_7\}$ is a 3-clique of Γ_2, contained in V_2.

Finally, we consider the case when $v_4, v_5 \in V_1$. If $u_7 \in V_1$, then $\{u_7, v_4, v_5\}$ is a 3-clique of Γ_2, contained in V_1. If $u_7 \in V_2$, then $\{u_7, v_7, v_8\}$ is a 3-clique of Γ_2, contained in V_2. ∎

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