ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 1 — Математика и механика
Том 91, 1997

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 1 — Mathématiques et Mecanique
Tome 91, 1997

K-THEORY OF THE C^*-ALGEBRA OF MULTIVARIABLE WIENER-HOPF OPERATORS ASSOCIATED WITH SOME POLYHEDRAL CONES IN R^n

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We consider the C^*-algebra $WH(R^n, P)$ of the multivariable Wiener-Hopf operators associated with a polyhedral cone in R^n and the extension $0 \to \mathcal{K} \to WH(R^n, P) \to WH(R^n, P)/\mathcal{K} \to 0$.

The main theorem states that if P satisfies suitable geometric conditions (satisfied, e.g., for all simplicial cones and the cones in R^n, $n \leq 3$), then $K_*(WH(R^n, P)) = (0, 0)$; $K_*(WH(R^n, P)/\mathcal{K}) = (0, Z)$, and that the index map is an isomorphism. In the cource of the proof we construct a Fredholm operator in $WH(R^n, P)$ with an index 1. The proof is inductive and uses the Mayer-Vietoris exact sequence and the standart six term exact sequence in K-theory.

Keywords: K-theory, Wiener-Hopf operators

1991/95 Math. Subject Classification: 47A53

0. INTRODUCTION

Let P be a polyhedral cone in R^n. The Wiener-Hopf operators are obtained by compressing the left convolution operators on $L^2(R^n)$ to the $L^2(P)$:

$$W(f)\xi(t) = \int_P f(t-s)\xi(s)\,ds.$$

The C^*-algebra $WH(R^n, P)$, generated by $W(f)$ when f runs through $C_c(R^n)$, is the C^*-algebra of multivariable Wiener-Hopf operators. It is studied with various techniques in [2, 4, 5].

In [4] P. Muhly and J. Renault prove that $WH(R^n, P)$ contains $\mathcal{K} = \mathcal{K}(P)$ — the ideal of the compact operators in $B(L^2(P))$. They obtain a composition series for $WH(R^n, P)$:

$$0 \subset I_0 \cong \mathcal{K} \subset I_1 \subset \ldots \subset I_n \cong WH(R^n, P), \tag{0.1}$$

where $I_k/I_{k-1} \cong C_0(Z) \otimes \mathcal{K}$ and Z is an appropriate locally compact space. They state a problem to calculate the K-theory of $WH(R^n, P)$. Here are calculated $K_*(WH(R^n, P))$ and $K_*(WH(R^n, P)/\mathcal{K})$ when P satisfies suitable geometric conditions (satisfied, e.g., for all simplicial cones and the cones in R^n, $n \leq 3$).

In the present paper we consider the extension

$$0 \to \mathcal{K} \to WH(R^n, P) \to WH(R^n, P)/\mathcal{K} \to 0. \tag{0.2}$$

Our first observation is that if there exists an index 1 Fredholm operator and if $K_*(WH(R^n, P)/\mathcal{K}) = (0, Z)$ (in order to simplify notations, the K-theory is considered to be Z_2-graded theory: $K_*(A) = K_0(A) \oplus K_1(A)$), then we may apply the fundamental six term exact sequence of K-theory:

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \longrightarrow & K_0(WH(R^n, P)) & \longrightarrow & K_0(WH(R^n, P)/\mathcal{K}) \\ \uparrow \text{ind} & & & & \downarrow \\ K_1(WH(R^n, P)/\mathcal{K}) & \longleftarrow & K_1(WH(R^n, P)) & \longleftarrow & K_1(\mathcal{K}) \end{array} \tag{0.3}$$

Then we obtain that $K_*(WH(R^n, P)) = (0, 0)$ and the index map of the extension (0.2):

$$\text{ind} : K_1(WH(R^n, P)/\mathcal{K}) \to K_0(\mathcal{K}), \tag{0.4}$$

is an isomorphism.

Further, the quotient $WH(R^n, P)/\mathcal{K}$ can be represented as a groupoid C^*-algebra. There are groupoid subalgebras, which are more simple (in a K-theory sense). The basic idea is to construct an increasing sequence of such algebras

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \ldots \subset \mathcal{B}_N \cong WH(R^n, P)/\mathcal{K}$$

and to calculate their K-theory applying the Mayer-Vietoris exact sequence in each step.

The groupoid approach gives naturally pullback diagrams of appropriate defined groupoid C^*-algebras:

$$\begin{array}{ccc} \mathcal{B}_k & \longrightarrow & \mathcal{B}_{k-1} \\ \downarrow & & \downarrow \\ \mathcal{D}_k & \longrightarrow & \mathcal{A}_k \end{array}$$

Then the corresponding exact Mayer-Vietoris sequence is

$$\begin{array}{ccccc} K_0(\mathcal{B}_k) & \longrightarrow & K_0(\mathcal{D}_k) \oplus K_0(\mathcal{B}_{k-1}) & \longrightarrow & K_0(\mathcal{A}_k) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{A}_k) & \longleftarrow & K_1(\mathcal{D}_k) \oplus K_1(\mathcal{B}_{k-1}) & \longleftarrow & K_1(\mathcal{B}_k) \end{array}$$

In a general situation it is not sufficient to know only the K-groups of $\mathcal{A}_k$ and $\mathcal{D}_k$. Now we note that when the middle terms in the above exact sequence are trivial, then the maps corresponding to the vertical arrows are isomorphisms. If all these K-groups are trivial, then the same is true for $\mathcal{B}_k$. This fact motivates us to define the class of exhaustible cones — i.e. those cones, for which we can find a sequence of subalgebras as above, but having a trivial K-theory.

The organization of the paper is as follows: In Section 1 we set up the groupoid notations. In Section 2 we prove that there exists a Fredholm operator with index 1 in $WH(R^n, P)^\dagger$ — the algebra with the identity adjoined. As a corollary of the six term exact sequence in the K-theory we show that if $K_*(WH(R^n, P)/\mathcal{K}) = (0, Z)$, then $K_*(WH(R^n, P)) = (0, 0)$ and the index map (0.2) is an isomorphism. Section 3 is concerned with the quotient $WH(R^n, P)/\mathcal{K}$. We define geometrically the property a cone to be exhaustible and we prove the main Theorem 3.5. An example is given.

1. PRELIMINARIES

In this section we collect some facts concerning the groupoid approach to C^*-algebras and the groupoid construction made in [4] of a groupoid whose associated groupoid C^*-algebra is isomorphic to the one generated by the Wiener-Hopf operators.

In the paper P is a polyhedral cone in R^n, i.e. P is generated by its extreme rays. We assume that P contains no line and spans R^n. Let $\mathcal{F}(P)$ denote the set of all faces of P; we count P and $\{0\}$ among the faces of P. For $F \in \mathcal{F}(P)$, $\langle F \rangle$ is the linear subspase $F - F$ generated by F and $St(F)$ is the collection of all faces containing F.

In [4] P. Muhly and J. Renault prove that in a general context (G is a locally compact group and P is its subsemigroup) $\mathcal{B} = WH(G, P)$ is isomorphic with an explicitly constructed groupoid C^*-algebra $C^*(\mathcal{G})$. Here we briefly recall their construction in the case $G = R^n$

First step in this construction is the definition of a locally compact space Y. It may be presented as

$$Y = \{(F, t) : F \in \mathcal{F}(P);\ t \in R^n \ominus \langle F \rangle\}.$$

The space R^n is imbedded in Y ($t \mapsto (\{0\}, t)$) as a dense subset and the space X is defined to be the closure of P in Y. There exists a natural action of R^n on Y and the basic for the constructed groupoid $\mathcal{G}$, whose C^*-algebra yields $WH(R^n, P)$, is a reduction of a transformation group $Y \times R^n$ by the closed subspace X of Y. Explicitly, the elements of $\mathcal{G} = Y \times R^n | X$ are the pairs $(x, s) \in Y \times R^n$ such that $x \in X$ and $x + s \in X$. The family of measures on X:

$$\lambda^x(y, s) = \delta_x(y)\ \chi_X(y)\ \chi_X(y + s)\ ds$$

(here χ_X is the characteristic function of X), is called the left Haar system of measures of $\mathcal{G}$.

The family $C_c(\mathcal{G})$ of the finite functions on X becomes a normed C^*-algebra under the operations and the norm defined as follows:

$$f * g\,(x,t) = \int f(x,s)g(x+s,t-s)\chi_X(x)\chi_X(x+s)\,ds,$$

$$f^*(x,t) = \overline{f(x+s,-s)},$$

$$\|f\|_I = \sup\Big\{\int f\,d\lambda^x,\ \int f^*\,d\lambda^x : x \in X\Big\}.$$

The completion of $C_c(\mathcal{G})$ by the norm $\|\cdot\|_I$ is $L_I(\mathcal{G})$ and $C^*(\mathcal{G})$ is defined as their enveloping C^*-algebra.

Let $A \subset \mathcal{F}(P)$ and $X(A)$ consist of those $x = (F,t) \in X$ such that the face F belongs to A. Then $\mathcal{G}(A)$ is defined to be the groupoid obtained by a reduction of $\mathcal{G}$ by $X(A)$ and $C^*(\mathcal{G}(A))$ to be the corresponding C^*-algebra. We denote some often used groupoids as follows: $\mathcal{G}(F) = \mathcal{G}(\{F\})$, $\mathcal{G}_0 = \mathcal{G}(\{0\})$ and $\mathcal{G}_\infty = \mathcal{G}(\mathcal{F}(P)\setminus\{0\})$.

1.1. Proposition ([4, § 4.7]). *There exists an isomorphism between the C^*-algebra $WH(R^n,P)$ and the groupoid C^*-algebra $C^*(\mathcal{G})$. $WH(R^n,P)$ contains $\mathcal{K} = \mathcal{K}(L^2(P))$, which is isomorphic to $C^*(\mathcal{G}_0)$, and the quotient $WH(R^n,P)/\mathcal{K}$ is isomorphic to $C^*(\mathcal{G}_\infty)$.*

Let F be a face of P. The set $P-F$ is a cone containing the linear space $\langle F\rangle$ and $P_F = (P-F)/\langle F\rangle$ denotes the cone in $R^n \ominus \langle F\rangle$ determined by F. More generally, if $F_1 \in St(F)$, then $F-F_1$ contains $\langle F\rangle$ and the map

$$F_1 \mapsto (F_1 - F)/\langle F\rangle$$

is an order preserving bijection between $ST(F)$ and $\mathcal{F}(P_F)$. The next proposition describes the groupoid C^*-algebra $C^*(\mathcal{G}(St(F))) \cong WH(R^n, P-F)$.

1.2. Proposition ([4, § 3.7.1]). *$WH(R^n,P-F)$ is isomorphic to $WH(R^n \ominus \langle F\rangle, P_F) \otimes C^*_{\text{red}}(\langle F\rangle)$, where the tensor product is endowed with the least C^*-cross norm.*

We note that $C^*_{\text{red}}(\langle F\rangle) \cong C_0(\langle F\rangle)$ and that all the algebras considered here are postliminal ([2]) and there exists an unique C^*-cross norm. The above fact and the Bott periodicity say that $K_i(WH(R^n,P-F)) = K_{i+l\,(\text{mod }2)}(WH(R^n\ominus\langle F\rangle, P_F))$, where $l = \dim(\langle F\rangle)$.

1.3. Observation. Here we describe a construction which allows us to use often the Mayer-Vietoris exact sequence.

Let choose subsets A, B, C and D of $\mathcal{F}(P)$ such that $B = C\cup D$ and $A = C\cap D$. Let denote the corresponding groupoids by $\mathcal{G}(A)$, $\mathcal{G}(B)$, $\mathcal{G}(C)$, $\mathcal{G}(D)$ and their groupoid C^*-algebras by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.

When one glues the groupoids $\mathcal{G}(C)$, $\mathcal{G}(D)$ along $\mathcal{G}(A)$, the result is $\mathcal{G}(B)$. Then the following diagram of C^*-algebras is commutative:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\psi_2} & \mathcal{C} \\ \psi_1 \downarrow & & \downarrow \varphi_2 \\ \mathcal{D} & \xrightarrow{\varphi_1} & \mathcal{A} \end{array}$$

The C^*-algebra $\mathcal{B}$ is a pullback of $(\mathcal{C}, \mathcal{D})$ along φ_1, φ_2 (i.e. $\mathcal{B} \cong \{(c,d) : \varphi_1(d) = \varphi_2(c)\} \subseteq \mathcal{C} \oplus \mathcal{D}$ (cf. [1, § 15.3])).

By [1, § 18.12.4], when a pullback diagram of C^*-algebras is given as above, then we may write the corresponding exact Mayer-Vietoris sequence

$$\begin{array}{ccccc} K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{D}) \oplus K_0(\mathcal{C}) & \longrightarrow & K_0(\mathcal{A}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{A}) & \longleftarrow & K_1(\mathcal{D}) \oplus K_1(\mathcal{C}) & \longleftarrow & K_1(\mathcal{B}) \end{array}$$

2. CONSTRUCTION OF A FREDHOLM OPERATOR WITH INDEX 1

In this section an one-dimensional projector $E(x,s)$ in $WH(R^n, P)$ and an essentially unitary operator S in $WH(R^n, P)^\dagger$ — the algebra with the identity adjoined, are given explicitly.

Let us choose points y_i, $i = 1, 2, \ldots, N$, on the extreme rays of P such that $|y_i| = 1$. We may assume that y_i, $i = 2, 3, \ldots, n$, determine extreme rays of $P_1 = (P - F_1)/\langle F_1 \rangle$ and let P' be the cone spaned on y_i, $i = 1, 2, 3, \ldots, n$. We define

$$E(x,s) = C \prod_{k=1}^{n} e^{-(x,y_k)} e^{-\frac{1}{2}(s,y_k)} \chi_{P'}(x) \chi_{P'}(x+s),$$

$$F(x,s) = C\, e^{\frac{1}{2}(s,y_1)} \chi_{(-\infty,0]}(s,y_1) \prod_{k=2}^{n} e^{-(x,y_k)} e^{-\frac{1}{2}(s,y_k)} \chi_{P'}(x) \chi_{P'}(x+s).$$

2.1. Lemma. (i) *E is an one-dimensional projection in $WH(R^n, P)$.*
(ii) *F is in $WH(R^n, P)$ and satisfies the equalities*

$$F^* * F = F + F^* \quad \textit{and} \quad F * F^* = F + F^* - E.$$

Proof. Let first assume that $P = R^n_+$. Then we rewrite $E^*(x,s)$ and $F(x,s)$:

$$E(x,s) = \prod_{k=1}^{n} e^{-x_k} e^{-\frac{1}{2}s_k} \chi_{R^n_+}(x) \chi_{R^n_+}(x+s),$$

$$F(x,s) = e^{\frac{1}{2}s_1} \chi_{(-\infty,0]}(s) \prod_{k=2}^{n} e^{-x_k} e^{-\frac{1}{2}s_k} \chi_{R^n_+}(x) \chi_{R^n_+}(x+s) \chi_X(x) \chi_X(x+s).$$

The elements of $L_I(\mathcal{G})$ are the measurable functions on $\mathcal{G}$ with a finite norm $\|\cdot\|_I$. We observe that $E(x,s) = \overline{E(x+s,-s)} = E^*(x,s)$. Using the Fubini theorem and the fact that

$$\int e^{-(x+s)}\chi_{[0,\infty)}(x+s)\,ds = 1, \tag{2.1}$$

where $x, s \in R$, we obtain

$$|E|_I = \sup\{\int E(x,s)\,ds,\ \int E(x,s)\,ds : x \in X\} \le 1$$

and E belongs to $WH(R^n, P)$. Similar estimate proves that F is in $WH(R^n, P)$ and we omit it.

To prove that E is an one-dimensional projector, we have to check the equalities $E = E^*$, $E = E*E$ and $\mathrm{tr}(E) = 1$. The first one is obvious. Using again the Fubini theorem and (2.1), we get

$$E*E(x,t) = \int E(x,s)E(x+s,t-s)\chi_X(x+s)\,ds$$

$$= E(x,t)\int\prod_{k=1}^{n} e^{-(x_k+s_k)}\chi_{[0,\infty)}(x+s)\,ds = E(x,t).$$

By [4] $E(x, x-s)$, where $x \in P$, may be considered as a kernel of a selfadjoint integral operator in $L^2(R_+)$. Using the well-known formula for the trace of a selfadjoint integral operator with a continuous kernel, we obtain

$$\mathrm{tr}(E) = \int E(x,0)dx = \int\prod_{k=1}^{n} e^{-x_k}\chi_{[0,\infty)}(x)\,dx = 1$$

and hence E is an one-dimensional projector.

We rewrite F as follows:

$$F(x,s) = e^{\frac{1}{2}s_1}\chi_{(-\infty,0])}(s)E_{n-1},$$

$$F*(x,s) = e^{-\frac{1}{2}s_1}\chi_{[0,\infty))}(s)E_{n-1},$$

and then easy but tedious calculations prove the equalities of (ii).

Further, let Φ be the linear map determined by the matrice $(y_{i,j})$. Then the map $(x,t) \mapsto (\Phi(x), \Phi(t))$ may be extended to a topological isomorphism between $\mathcal{G}(R^n, R^n_+)$ and $\mathcal{G}(R^n, P')$. The measures in the left-hand Haar systems differ with a constant $C = |\det(y_{i,j})|$ and the statement is true if $P = P'$.

Finally, in the general case for P, the supports of $E(x,s)$ and $F(x,s)$ are in the reduction of $\mathcal{G}(R^n, P')$ by the $X(\{0\}) \cup X(F_1)$, which is a subgroupoid of $\mathcal{G}$. Thus $E(x,s)$ and $F(x,s)$ are in $C^*(\mathcal{G})$ and the above equalities are satisfied.

2.2. Theorem. (i) *There exists a Fredholm operator* $S \in WH(R^n, P)$ *such that* $\mathrm{ind}S = 1$.

Proof. Let $S = 1 - F$. Then by Lemma 2.1 we have $S^*S = 1$ and $SS^* = 1 - E$.

2.3. Corollary. *If* $K_*(WH(R^n, P)/\mathcal{K}) = (0, Z)$, *then*:

(i) $K_*(WH(R^n,P)) = (0,0)$ *and*

(ii) *the index map of the extension* (2)

$$\text{ind} : K_1(WH(R^n,P)/\mathcal{K}) \to K_0(\mathcal{K}) \tag{2.2}$$

is an isomorphism.

Proof. Let us consider the fundamental six-term exact sequence of K-theory corresponding to the extension (2):

$$\begin{array}{ccccc} K_0(\mathcal{K}) & \longrightarrow & K_0(WH(R^n,P)) & \longrightarrow & K_0(WH(R^n,P)/\mathcal{K}) \\ \uparrow \text{ind} & & & & \downarrow \\ K_1(WH(R^n,P)/\mathcal{K}) & \longleftarrow & K_1(WH(R^n,P)) & \longleftarrow & K_1(\mathcal{K}) \end{array}$$

Let $[S]$ be the generator of $K_1(WH(R^n,P)/\mathcal{K}) = Z$ and let $[E]$ be the generator of $K_0(\mathcal{K}) = Z$. If $\text{ind}([S]) = m[E]$, then the image of the morphism "ind" is $mZ \subset Z = K_0(\mathcal{K})$. But by Theorem 2.2 $[E]$ belongs to the image of ind. Thus $|m| = 1$ and ind is an isomorphism. We know that the right-hand groups of (6) are equal to 0, thus $K_*(WH(R^n,P)) = (0,0)$.

3. K-THEORY OF THE QUOTIENT ALGEBRA

3.1. Proposition. *Let* $n \leq 3$. *Then* $K_*(WH(R^n,P)/\mathcal{K}) = (0,Z)$.

Proof. Let $n = 1$ and $P = R_+$. We write the extension (2) as follows:

$$0 \to \mathcal{K} \to WH(R,R_+) \to C_0(R) \to 0.$$

The fact that $K_*(WH(R^n,P)/\mathcal{K}) = K_*(C_0(R)) = (0,Z)$ is well known.

Let $n = 2$ and P be a polyhedral cone in R^2 (the quarter plane-case). The faces of P are $\{0\}$, P and two one-dimentional faces F_1 and F_2.

Let us denote:

$$\mathcal{G}_1 = \mathcal{G}|X(F_1) \cup X(P) \quad \text{and} \quad \mathcal{B}_1 = C^*(\mathcal{G}_1);$$

$$\mathcal{G}_2 = \mathcal{G}|X(F_2) \cup X(P) \quad \text{and} \quad \mathcal{B}_2 = C^*(\mathcal{G}_2);$$

$$\mathcal{G}_{1,2} = \mathcal{G}|X(P) \quad \text{and} \quad \mathcal{B}_{1,2} = C^*(\mathcal{G}_{1,2}).$$

We recall that $\mathcal{G}_\infty = \mathcal{G}|X(F_1) \cup X(F_2) \cup X(P)$ and by Proposition 1.1 $WH((R^2,P)/\mathcal{K} \cong C^*(\mathcal{G}_\infty)$.

There exists an isomorphism of groupoids $\mathcal{G}_1 \cong R \times \mathcal{G}(R,R_+)$, where $\mathcal{G}(R,R_+)$ is the groupoid corresponding to the Wiener-Hopf algebra with $P = R_+ \subset R$. Then $\mathcal{B}_1 = C^*(\mathcal{G}_1) \cong C_0(R) \otimes WH((R,R_+)$ and therefore

$$\begin{aligned} K_*(\mathcal{B}_1) &= K_*(C^*(\mathcal{G}_1)) = K_*(C_0(R)) \times K_*(WH((R,R_+)) \\ &= (0,Z) \times (0,0) = (0,0). \end{aligned}$$

Analogously, $K_*(\mathcal{B}_2) = K_*(C^*(\mathcal{G}_2)) = (0,0)$.

Further, $\mathcal{B}_{1,2} = C^*(\mathcal{G}_{1,2}) \cong C_0(R^2)$ and $K_*(\mathcal{B}_{1,2}) = K_*(C_0(R^2)) = (Z,0)$.

There is a pullback diagram of C^*-algebras by (1.3):

$$\begin{array}{ccc} C^*(\mathcal{G}_\infty) & \longrightarrow & \mathcal{B}_1 \\ \downarrow & & \downarrow \\ \mathcal{B}_2 & \longrightarrow & \mathcal{B}_0 \end{array}$$

Further, the corresponding Mayer-Vietoris exact sequence is

$$\begin{array}{ccccc} K_0(WH((R^2,P)/\mathcal{K}) & \longrightarrow & K_0(\mathcal{B}_1)\oplus K_0(\mathcal{B}_2) & \longrightarrow & K_0(C_0(R^2)) \\ \uparrow & & & & \downarrow \\ K_1(C_0(R^2)) & \longleftarrow & K_1(\mathcal{B}_1)\oplus K_1(\mathcal{B}_2) & \longleftarrow & K_1(WH((R^2,P)/\mathcal{K}) \end{array}$$

The middle terms equal $\{0\}$ and hence the vertical maps are isomorphisms:

$$K_0(WH(R^2,P)/\mathcal{K}) = K_1(C_0(R^2)) = 0,$$
$$K_1(WH(R^2,P)/\mathcal{K}) = K_0(C_0(R^2)) = Z.$$

Let us recall that the set $St(F_l)$ of faces of P containing F_l is bijective to the set of faces of $P - F_l$. Therefore each subset A_l of $St(F_l)$ determines a subset $\widetilde{A_l}$ of $\mathcal{F}(P_l)$, where P_l is the lower-dimensional cone

$$(P - F_l)/(\langle F_l\rangle) \subset R^n \ominus \langle F_l\rangle\rangle.$$

The next definition is recursive and outlines the cones with which we deal.

3.2. Definition. Let P be a polyhedral cone in R^n, $n \geq 2$. We say that $L \subset \mathcal{F}(P)$ satisfies the condition (C) iff:

(i) there exists an one-dimensional face which does not belong to L;

(ii) L is an union of stars of some one-dimensional faces of P;

(iii) there is an ordering $F_1, \ldots, F_k$ of these one-dimensional faces such that for each $l = 2, \ldots, k$

$$A_l = St(F_l) \cap [St(F_1) \cup \ldots \cup St(F_{l-1})]$$

determines a subset $\widetilde{A_l} \subset \mathcal{F}(P_l)$ which satisfies the condition (C).

If $n = 2$, we count $St(F_1)$ and $St(F_2)$ among the sets satisfying the condition (C).

3.3 Definition. We say that P is exhaustible iff there exists an one-dimensional face F of P such that $L = \mathcal{F}(P)\backslash\{\{0\}, F\}$ satisfies the condition (C).

3.4. Lemma. *The cones in R^2 and R^3 and the simplicial cones in R^n are exhaustible.*

Proof. When $n = 2$, by the definitions P is exhaustible.

Let $n = 3$ and P be a polyhedral cone in R^3. Let us choose the custumary ordering $F_1, F_2, \ldots, F_N$ of the one-dimensional faces of P (i.e. the extreme rays of P) . Two neighbouring one-dimensional faces F_k and F_{k+1} of P (the calculations with the indices are mod N) span the two-dimensional face $F_{k,k+1}$. The rest faces of P are $\{0\}$ and P.

It is sufficient to prove that $F_1, F_2, \ldots, F_{N-1}$ satisfy the condition (C). It is evident that

$$A_l = St(F_l) \cap [St(F_1) \cup \ldots \cup St(F_{l-1})] = St(F_{l-1,l})$$

for $l = 2, \ldots, N-1$. The associated with $St(F_{l-1,l}) \subset \mathcal{F}_P$ family of faces of the cone $P_l = (P - F_l)/\langle F_l \rangle \subset R^2$ satisfies the condition (C) by the definition and this proves the case $n = 3$.

Finally, let P be a simplicial cone in R^n. Note that each collection of extreme rays uniquely determines a face of P and for each one-dimensional faces F_k and F_l of P follows that $St(F_k) \cap St(F_l) = St(F_{k,l})$. A trivial induction on the dimension n proves that for each ordering $F_1, F_2, \ldots, F_n$ of the one-dimensional faces of P and for each $l < n$ the subset $F_1, F_2, \ldots, F_l$ satisfies the condition (C).

3.5. Theorem. *Let P be an exhaustible polyhedral cone in R^n, $n \geq 2$. Then:*
(i) $K_*(WH(R^n, P)) = (0,0)$;
(ii) $K_*(WH(R^n, P)/\mathcal{K}) = (0, Z)$;
(iii) *the index map of the extension* (0.2)

$$\text{ind} : K_1(WH(R^n, P)/\mathcal{K}) \to K_0(\mathcal{K}) \tag{3.1}$$

is an isomorphism;
(iv) *if $A \subset \mathcal{F}(P)$ satisfies the condition* (C), *then* $K_*(C^*(\mathcal{G}(A))) = (0,0)$.

Proof. We shall prove the theorem by induction on the dimension n. If $n = 2$, Lemma 3.1 and Theorem 2.3 prove the statements (i)–(iv). Now suppose that they are true for $2, \ldots, n-1$.

Let P be an exhaustible polyhedral cone in R^n. By Definition 3.3 there exists an ordering $F_1, \ldots, F_N$ of the one-dimensional faces of P such that $B_{N-1} = St(F_1) \cup \ldots \cup St(F_{N-1}) \subset \mathcal{F}(P)$ satisfies the condition (C) given in Definition 3.2.

Now let us consider some subsets of $\mathcal{F}(P)$ and the corresponding C^*-algebras:
$D_k = St(F_k)$ and $\mathcal{D}_k = C^*(\mathcal{G}(D_k))$ for $k = 1, 2, \ldots, N$;
$B_k = St(F_k) \cup \ldots \cup St(F_k)$ and $\mathcal{B}_k = C^*(\mathcal{G}(B_k))$ for $k = 1, 2, \ldots, N$;
$A_k = D_k \cap B_k$ and $\mathcal{A}_k = C^*(\mathcal{G}(A_k))$ for $k = 2, 3, \ldots, N$.
We note that $\mathcal{B}_1 = \mathcal{D}_1$ and $\mathcal{B}_N = WH(R^n, P)/\mathcal{K}$ by Proposition 1.1.

Our first aim is to compute the K-theory of these algebras, in particular to prove that $K_*(\mathcal{B}_k) = (0,0)$ for $k = 1, 2, \ldots, N-1$.

By Proposition 1.2 there is an isomorphism $\mathcal{D}_k \cong C_0(\langle F_k \rangle) \otimes WH(R^n \ominus \langle F_k \rangle, P_k)$. Since by the condition (i) of the inductive supposition $K_*(WH(R^n \ominus \langle F_k \rangle, P_k)) = (0,0)$, then for $k = 1, 2, \ldots, N$

$$K_*(\mathcal{D}_k) = (0,0).$$

Further, $A_k \subset St(F_k)$ and by Proposition 1.2 it determines a family $\widetilde{A_k} \subset \mathcal{F}(P_k)$ of faces of $\mathcal{F}(P_k)$ and associated with it groupoid C^*-algebra $\widetilde{\mathcal{A}_k}$ such that $\mathcal{A}_k \cong C_0(\langle F_k \rangle) \otimes \widetilde{\mathcal{A}_k}$.

If $1 < k < N$, then by Definition 3.2 (iii) A_k satisfies the condition (C), and therefore by the condition (iv) of the inductive supposition it follows $K_*(\widetilde{\mathcal{A}_k}) = (0,0)$ and hence

$$K_*(\mathcal{A}_k) = (0,0), \quad k = 2,3,\ldots,N-1. \tag{3.2}$$

Now we shall show that $\mathcal{A}_N$ has a non-trivial K-theory. Indeed, $A_N = St(F_N)\setminus \{F_N\}$ and by Proposition 1.2

$$\mathcal{A}_N \cong C_0(\langle F_N \rangle) \otimes [WH(R^n \ominus \langle F_N \rangle, P_N)/\mathcal{K}].$$

By the condition (ii) of the inductive supposition $K_*(WH(R^n) \ominus \langle F_N \rangle, P_N)/\mathcal{K}) = (0,Z)$ and hence

$$K_*(\mathcal{A}_N) \cong K_*(C_0(R)) \otimes K_*(WH(R^n \ominus \langle F_N \rangle, P_N)/\mathcal{K}) = (0,Z) \times (0,Z) = (Z,0).$$

The equalities $B_k = B_{k-1} \cup D_k$, $A_k = B_{k-1} \cap D_k$ and Proposition 1.4 imply that there are pullbacks of the corresponding C^*-algebras for $k = 1,2,\ldots,N$:

$$\begin{array}{ccc} \mathcal{B}_k & \longrightarrow & \mathcal{B}_{k-1} \\ \downarrow & & \downarrow \\ \mathcal{D}_k & \longrightarrow & \mathcal{A}_k \end{array}$$

Now we shall prove that

$$K_*(\mathcal{B}_k) = (0,0); \; k = 1,2,\ldots,N-1. \tag{3.3}$$

Indeed, $\mathcal{B}_1 = \mathcal{D}_1$ and $K_*(\mathcal{B}_1) = K_*(\mathcal{D}_1) = (0,0)$. Suppose that the above holds for $1,\ldots,k-1$ and we write the Mayer-Vietoris exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{B}_k) & \longrightarrow & K_0(\mathcal{B}_{k-1}) \oplus K_0(\mathcal{D}_k) & \longrightarrow & K_0(\mathcal{A}_k) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{A}_k) & \longleftarrow & K_1(\mathcal{B}_{k-1}) \oplus K_1(\mathcal{D}_k) & \longleftarrow & K_1(\mathcal{B}_k) \end{array}$$

The middle terms in this exact sequence are the groups $\{0\}$, hence the vertical arrows maps are isomorphisms. For $k = 1,2,\ldots,N$ it follows that $K_0(\mathcal{B}_k) = K_1(\mathcal{A}_k)$ and $K_1(\mathcal{B}_k) = K_0(\mathcal{A}_k)$. Using $(N-2)$ times the Mayer-Vietoris exact sequence, we obtain that $K_*(\mathcal{B}_k) = (0,0)$ for $k = 1,2,\ldots,N-1$. Here we note that the proof of the condition (iv) is the same as the above fragment and we omit it. Further, the final Mayer-Vietoris exact sequence gives

$$K_*(\mathcal{B}_N) = (0,Z). \tag{3.4}$$

So, the condition (ii) is verified for n. The left standing for n conditions (i) and (iii) follow from Theorem 2.3.

It is attractive to conjecture that all the polyhedral cones in R^n are exhaustible. However, we are unable to prove it. The next example shows that the ordering of the one-dimensional faces in Definition 3.2 (iii) is essential. We construct $L \subset \mathcal{F}(P)$ which is an union of stars of some one-dimensional faces, but which does not satisfy the condition (C), because some of the corresponding C^*-algebras have non-trivial K-groups.

3.6. Example. Let P be a cone in R^4 such that the cut Q through P determined of a hyperplane α is a cube. We denote the extreme points of Q (ordered in the custumary way) by $A_1, \ldots, A_8$ and the corresponding one-dimensional faces of P by $F_1, \ldots, F_8$:

$L_1 = St(F_1)$, $K_*(C^*(\mathcal{G}(L_1))) = (0,0)$,
$L_2 = St(F_2) \cup L_1$, $K_*(C^*(\mathcal{G}(L_2))) = (0,0)$,
$L_3 = St(F_6) \cup L_2$, $K_*(C^*(\mathcal{G}(L_3))) = (0,0)$,
$L_4 = St(F_7) \cup L_3$, $K_*(C^*(\mathcal{G}(L_4))) = (0,0)$,
$L_5 = St(F_8) \cup L_4$, $K_*(C^*(\mathcal{G}(L_5))) = (Z,0)$,
$L_6 = St(F_4) \cup L_5$, $K_*(C^*(\mathcal{G}(L_6))) = (0,Z)$,
$L_7 = St(F_3) \cup L_6$, $K_*(C^*(\mathcal{G}(L_7))) = (0,0)$.

Clearly, L_7 with the above order of the one-dimensional faces is not exhaustible. It can be verified that the customary order of the extreme points of the cube determines an order of the one-dimensional faces of P such that L_7 is exhaustible.

ACKNOWLEDGEMENTS. I wish to express my indebtedness to R. Levi for the numerous and fruitful comments at the various stages of this paper.

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Received March 17, 1998

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