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AN ESTIMATE OF THE PERIOD OF PERIODICAL SOLUTION OF AN AUTONOMOUS SYSTEM OF DIFFERENTIAL EQUATIONS

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In this paper we obtain an estimate for the period of the periodical solution of an autonomous system of differential equations from above.

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We consider the autonomous system

$$\left| \begin{aligned} \dot{x} &= a\left[f(x) - (1+b)\,x - z\right], \\ \dot{y} &= -c\left[f(x) - x - z\right], \\ \dot{z} &= -d\left[y + z\right], \end{aligned} \right. \tag{1}$$

which is found in studying of the oscillations of electrical circuits. In this system $f(x)$ is a twice continuous differentiable function in R such that

$$xf(x) > 0 \ \text{ for } \ x \neq 0, \tag{2}$$

$$|f(x)| < M \text{ for } x \in R, \tag{3}$$

M is a positive constant. The positive constants a, b, c, d are subordinate to the conditions

$$g > \frac{c}{2a} + \frac{d}{2a} + (1+b) - \sqrt{\left(\frac{c}{2a} - \frac{d}{2a}\right)^2 + \frac{bc}{a}}, \quad g = f'(0), \tag{4}$$

$$d > 4.6c \sup_{x \in R} \frac{f(x)}{x} + 9.7c + 5a + 2.4ab. \tag{5}$$

Under these assumptions all solutions of (1) are defined in R and through every point $(t_0, x_0, y_0, z_0) \in R \times R^3$ goes an unique integral curve (see [2] and [3]).

In [1] it has been proved that the system (1) possesses a closed phase curve, different from the degenerate curve, consisting of its unique equilibrium position — the origin of coordinates.

This curve lies in the solid homeomorphic torus V bounded by two cone surfaces

$$\frac{y+z}{\sqrt{2}\sqrt{x^2+y^2+z^2}} = \frac{1}{2}, \tag{6}$$

$$\frac{y+z}{\sqrt{2}\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}, \tag{6'}$$

by the ellipsoid

$$\frac{1}{c^2}y^2 + \frac{1}{b}\left(\frac{x}{a} + \frac{y}{c}\right)^2 + \frac{z^2}{cd} = L, \tag{7}$$

and by the cylindrical surface

$$q_2^2 + q_3^2 = K. \tag{8}$$

Here

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = S \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

L is the smallest and K is the greatest positive constant for which the orbits of (1) cross the contour of V from the outside; S is the matrix, reducing the matrix to the corresponding to (1) linear system

$$\left| \begin{aligned} \dot{x} &= a\left[(g-1-b)x - z\right], \\ \dot{y} &= -c\left[(g-1)x - z\right], \\ \dot{z} &= -d\left[y+z\right], \end{aligned} \right. \tag{1'}$$

of Jordan's normal form.

To each of the existing two possibilities for the roots of the characteristic equation of (1′)

$$\lambda^3 + \left[d + a(1+b) - ag\right]\lambda^2 + d\left[c + a(1+b) - ag\right]\lambda + abcd = 0, \tag{9}$$

namely,

$$\lambda_1 < 0, \quad \lambda_2 = \mu + i\kappa, \quad \lambda_3 = \mu - i\kappa\mu > 0, \tag{10}$$

or

$$\lambda_1 < 0, \quad \lambda_2 = \mu_1 > 0, \quad \lambda_3 = \mu_2 > 0, \quad \mu_1 \neq \mu_2, \tag{11}$$

corresponds a different matrix S.

In both cases (10) and (11) the following estimate is valid:

$$-d < \lambda_1 < -0.876d. \tag{12}$$

We know from [1] that except for the two orbits, lying in the domain $x^2 < y^2 + z^2 + 4yz$, all phase curves of (1) enter V and remain in it with the growth of time t. Furthermore, all orbits, starting from

$$D_1 = V \cap \{x = 0,\ z < 0\},$$

intersect

$$D_2 = V \cap \{x = 0,\ z > 0\}$$

passing from $x > 0$ to $x < 0$, and all orbits, starting from D_2, intersect D_1 passing from $x < 0$ to $x > 0$.

We denote by T the length of the interval of time for which the orbits of (1) with initial points in D_1 intersect D_1 again after their passing through D_2. In the present paper an estimate from above for T is obtained. The estimate does not depend on the initial point of the orbits if this point belongs to D_1.

A key role at the estimate of T will be played by the following

Lemma 1. *Suppose that the constants B and γ are defined by*

$$B = \begin{cases} \dfrac{1.2\sqrt{K}\,|\kappa|\,ac}{g\,(d+1)^4} & \text{for (10)}, \\ \dfrac{\sqrt{K}\,|\mu_2 - \mu_1|\,bc}{g^4\,(d+1)^7} & \text{for (11)}, \end{cases} \qquad \gamma = \begin{cases} \dfrac{0.6\sqrt{K}\,|\kappa|\,ac}{g\,(d+1)^3} & \text{for (10)}, \\ \dfrac{0.4\sqrt{K}\,|\mu_2 - \mu_1|\,bc}{g^4\,(d+1)^7} & \text{for (11)}. \end{cases}$$

Then for the third coordinates of all points from $V \cap \{|x| \leq \gamma\}$ the estimate $|z| \geq B$ is valid.

Proof. The intersection of the conical surface (6) with the plane $y + z = 1$ is defined by the system

$$\left| \begin{array}{l} x = \pm\sqrt{1 + 2z - 2z^2}, \\ y = 1 - z, \quad (1 - \sqrt{3})/2 \leq z \leq (1 + \sqrt{3})/2. \end{array} \right. \tag{13}$$

Let $(\tilde{x}, \tilde{y}, \tilde{z})$ be a point from this intersection. The generatrix of (6), passing through $(\tilde{x}, \tilde{y}, \tilde{z})$, intersects (8) in the point $A(x_1, y_1, z_1)$, where

$$x_1 = \pm\sqrt{1 + 2\tilde{z} - 2\tilde{z}^2}\,(S_1)_\pm, \quad y_1 = (1 - \tilde{z})\,(S_1)_\pm, \quad z_1 = \tilde{z}\,(S_1)_\pm. \tag{14}$$

Here

$$(S_1)_\pm = \frac{\sqrt{K}|\det S|}{(W_1)_\pm},$$

$$(W_1)_\pm = \Big\{[\pm A_{12}\sqrt{1+2\tilde{z}-2\tilde{z}^2} + A_{22}(1-\tilde{z}) + A_{32}\tilde{z}]^2 \times [\pm A_{13}\sqrt{1+2\tilde{z}-2\tilde{z}^2} + A_{23}(1-\tilde{z}) + A_{33}\tilde{z}]^2\Big\}^{1/2}, \tag{15}$$

A_{ij} are the algebraic adjuncts to the elements of the matrix S, $1 \leq i, j \leq 3$.

In (13)–(15) there is a correspondence between the signs $\pm$, written before the radicals, and those in the symbol $(S_1)_\pm$.

Investigating $x_1(\tilde{z})$, $y_1(\tilde{z})$ and $z_1(\tilde{z})$ for $\tilde{z} \in [(1-\sqrt{3})/2, (1+\sqrt{3})/2]$ allows us to conclude that the intersection of (6) and (8) represents a simple closed curve. By analogy we come to the same conclusion also for the intersection of (6′) and (8).

Calculating A_{ij} and $\det S$ and estimating them by the following inequalities resulting from (5), (12) and from the relations between the roots of (9) and its coefficients:

$$a < \frac{d}{5}, \quad ab < \frac{d}{2.4}, \quad c < \frac{d}{9.7}, \tag{16}$$

$$abc < \mu^2 + \kappa^2 < 0,05d^2 \quad \text{for the case (10)}, \tag{17}$$

$$abc < \mu_1^2 + \mu_2^2 < a^2 g^2 \quad \text{for the case (11)}, \tag{18}$$

we get that for $(1-\sqrt{3})/2 \leq \tilde{z} \leq (1-\sqrt{3})/4$ the following estimate is valid:

$$|z_1| \geq B. \tag{19}$$

The points

$$A_1\left(\frac{1}{2}\sqrt{4-\sqrt{3}}, \frac{3+\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\right) \quad \text{and} \quad A_2\left(-\frac{1}{2}\sqrt{4-\sqrt{3}}, \frac{3+\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\right)$$

are obtained from (13) for $z = 1-\sqrt{3}$.

We denote by α_j the plane containing the axis of the cone (6) and passing through the point A_j, $j = 1, 2$:

$$\alpha_1 : x = \frac{\sqrt{4-\sqrt{3}}}{1+\sqrt{3}}(y-z),$$

$$\alpha_2 : x = -\frac{\sqrt{4-\sqrt{3}}}{1+\sqrt{3}}(y-z).$$

From the position of V in the space it follows that for the third coordinates of all points from

$$V_1 = V \cap \left\{-\frac{\sqrt{4-\sqrt{3}}}{1+\sqrt{3}}(y-z) \leq x \leq \frac{\sqrt{4-\sqrt{3}}}{1+\sqrt{3}}(y-z)\right\}$$

the estimate (19) is valid as well.

By analogy, considering (6'), one proves similarly that the estimate (19) is valid also for the third coordinates of all points from

$$V_2 = V \cap \left\{ \frac{\sqrt{4-\sqrt{3}}}{1+\sqrt{3}} (y-z) \le x \le -\frac{\sqrt{4-\sqrt{3}}}{1+\sqrt{3}} (y-z) \right\}.$$

We consider the generatrix of the cones (6) and (6'), lying in α_1 and α_2, and their intersection points with the surface (8).

Estimating A_{ij}, $\det S$ and using the inequalities (12), (16)–(18), we obtain for the first coordinates of these points the inequality

$$|x| \ge \gamma.$$

Then $V \cap \{|x| \le \gamma\} \subset V_1 \cup V_2$. The lemma is proved. □

Let $x(t, x_0, y_0, z_0)$ denote the x-component of the orbit of (1), corresponding to the initial condition $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$.

We introduce now the constants

$$M_1 = \max_{|x| \le a\sqrt{L}(\sqrt{b}+1)} |f'(x)|, \quad M_2 = \max_{|x| \le a\sqrt{L}(\sqrt{b}+1)} |f''(x)|.$$

Lemma 2. *Let δ_1, δ_2, t_0 be defined as follows:*

$$\delta_1 = \min\left(\gamma, \frac{\sqrt{g^2 + BM_2} - g}{M_2}\right), \quad \delta_2 = \frac{a}{4} t_0 B \left(1 - \frac{1}{4.10^n}\right),$$
$$t_0 = \frac{1}{10^n} \frac{Ba}{2d^2 \left\{ M(M_1+1) + d\sqrt{L}\left[d(1+M_1)\left(1+\sqrt{b}\right)+1\right]\right\}}. \tag{20}$$

Then:

a) *$x(t, x_0, y_0, z_0)$ is an increasing function in the interval $|t| \le t_0$, $\forall (x_0, y_0, z_0) \in V_1 \cap \{|x| \le \delta_1\}$;*

b) *$x(t_0/2) - x_0 \ge \delta_2$ and $x_0 - x(-t_0/2) \ge \delta_2$, $\forall (x_0, y_0, z_0) \in V_1 \cap \{|x| \le \delta_1\}$.*

Proof. We develop $f(x)$ by Taylor's formula about $x = 0$:

$$f(x) = gx + \frac{x^2}{2} f''(\theta x), \quad \theta \in (0,1), \ g = f'(0).$$

Let (x_0, y_0, z_0) be an arbitrary point from $V_1 \cap \{|x| \le \gamma\}$. Then $-z_0 > B$. We develop $\dot{x}(t, x_0, y_0, z_0)$ about $t = 0$:

$$\dot{x}(t, x_0, y_0, z_0) = a\left[(g-1-b)x_0 - z_0 + \frac{x_0^2}{2} f''(\theta x_0)\right] + t\ddot{x}(\theta_1 t), \quad \theta_1 \in (0,1).$$

Suppose that $|x_0| \leq \delta_1$. Then

$$\left|(g-1-b)\,x_0 + \frac{x_0^2}{2} f''(\theta x_0)\right| \leq \frac{B}{2}$$

and

$$(g-1-b)\,x_0 - z_0 + \frac{x_0^2}{2} f''(\theta x_0) \geq \frac{B}{2}. \tag{21}$$

The orbit with an origin (x_0, y_0, z_0) lies in V and the following estimates are valid:

$$|x| \leq a\sqrt{L}\left(\sqrt{b}+1\right), \quad |y| \leq c\sqrt{L}, \quad |z| \leq \sqrt{cd}\sqrt{L}, \quad \forall (x,y,z) \in V. \tag{22}$$

Then

$$|\ddot{x}(t, x_0, y_0, z_0)| \leq d^2 \left\{ M(M_1+1) + d\sqrt{L}\left[d(1+M_1)\left(1+\sqrt{b}\right)+1\right]\right\}$$

for any $t \in R$ and any $(x_0, y_0, z_0) \in V$. Therefore, for every $t \in R$ for which $|t| \leq t_0$ and for every point $(x_0, y_0, z_0) \in V_1 \cap \{|x| \leq \delta_1\}$ the following two inequalities are valid:

$$|t\ddot{x}(\theta_1 t, x_0, y_0, z_0)| \leq \frac{1}{10^n}\frac{Ba}{2}, \quad \dot{x}(t, x_0, y_0, z_0) > 0,$$

where n is a suitable positive integer.

To prove the second part of the lemma, we develop $x(t, x_0, y_0, z_0)$ about $t = 0$:

$$x(t, x_0, y_0, z_0) = x_0 + a\left[f(x_0) - (1+b)\,x_0 - z_0\right] t + \frac{t^2}{2}\ddot{x}(\theta_2 t, x_0, y_0, z_0), \quad \theta_2 \in (0,1).$$

On the one hand, it follows from (20) that

$$\left|\frac{t_0^2}{8}\ddot{x}(\theta_2 t, x_0, y_0, z_0)\right| \leq \frac{t_0}{16}\frac{Ba}{10^n}$$

and, from the other hand, (21) implies

$$\left[f(x_0) - (1+b)\,x_0 - z_0\right]\frac{t_0}{2} \geq \frac{B}{4} t_0.$$

These two estimates together with (21) yield

$$x\left(\frac{t_0}{2}\right) - x_0 \geq \delta_2, \quad \forall (x_0, y_0, z_0) \in V_1 \cap \{|x_0| \leq \delta_1\}.$$

In a similar way the estimate for $x_0 - x(-t_0/2)$ is obtained. The lemma is proved. □

Theorem 1. *For $\delta = \min(\delta_1, \delta_2)$ we have:*

a) *All orbits of (1) going from points $(x_0, y_0, z_0) \in V_1 \cap \{-\delta \le x \le 0\}$ intersect the plane $x = 0$ for an interval of time not greater than $t_0/2$;*

b) *All orbits of (1) going from points $(x_0, y_0, z_0) \in V_1 \cap \{0 \le x \le \delta\}$ intersect the plane $x = \delta$ for an interval of time not greater than $t_0/2$;*

c) *All orbits of (1) going from points $V_2 \cap \{0 \le x \le \delta\}$ intersect the plane $x = 0$ for an interval of time with length not greater than $t_0/2$;*

d) *All orbits of (1) going from points $V_2 \cap \{-\delta \le x \le 0\}$ intersect the plane $x = -\delta$ for an interval of time not greater than $t_0/2$.*

The proof follows from Lemma 2 and the repetition of reasoning for the points from $V_2 \cap \{|x| \le \gamma\}$.

We are already prepared to prove the following

Theorem 2. *All orbits of equation* (1), *starting from D_1, intersect again D_1 for the interval of time not greater than $T = 2t_0 + 4\sqrt{L}/(b\delta)$.*

Proof. Let $(0, y_0, z_0)$ be an arbitrary point from D_1. Denote with t_1 the first moment when the orbit beginning at this point intersects $x = \delta$, and with T_1 the second moment. Then

$$x\,(t, 0, y_0, z_0) \ge \delta, \quad \forall t \in [t_1, T_1]\,. \tag{23}$$

Multiply the first equation of (1) by c, the second by a and sum the results:

$$c\frac{dx}{dt} + a\frac{dy}{dt} = -abcx\,(t)\,.$$

Integrate this equation on the given orbit from t_1 to T_1. We obtain

$$bc\int_{t_1}^{T_1} x\,(t, 0, y_0, z_0)\,dt = y\,(t_1) - y(T_1).$$

Estimate the left-hand side of this equation from below with the help of (23), and the right-hand one above with the help of (22). This yields the inequality

$$T_1 - t_1 \le \frac{2\sqrt{L}}{b\delta}. \tag{24}$$

In a similar way we obtain that if t_2 is the first moment in which the considered orbit intersects $x = -\delta$, and T_2 is the second such moment, then

$$T_2 - t_2 \le \frac{2\sqrt{L}}{b\delta}. \tag{25}$$

The obtained estimates do not depend of the point $(0, y_0, z_0) \in D_1$.

Finally, we take into account Lemma 2, the results of Theorem 1 and eqs. (24) and (25). The theorem is proved. □

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