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ASYMPTOTIC SOLUTION OF DEFINITE CLASS OF SINGULARLY PERTURBED LINEAR BOUNDARY-VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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The singular perturbation for boundary problems for linear systems of ordinary differential equations is considered. Under suitable assumptions using generalized inverse matrix the unique asymptotic expansion with boundary function is constructed.

Keywords: boundary-value problems, singular perturbation, asymptotic solution, boundary functions

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1. INTRODUCTION

The theory of the singularly perturbed systems for ordinary differential equations is primarily due to the works of A. Tikhonov [1, 2] and N. Levinson (see [3, 19]) in the early 1950. The method and results of A. B. Vasil'eva [4, 5] and A. B. Vasil'eva, V. F. Butuzov [6, 7] widely make use of the construction asymptotic solution of a singularly perturbed differential systems. The questions connected with asymptotic calculation of relaxational oscillation are considered in the monographs [8, 9]. The method of regularization of singular perturbation is studied in [10]. A method of separation of differential equations for obtaining asymptotic decomposition similar to regularized decomposition is given in the papers [11,12]. In this paper the behavior of the solution at $\varepsilon \to 0$ is considered for a linear boundary-value problem

$$\varepsilon \dot{x} = Ax + \varepsilon A_1(t)x + \varphi(t), \quad t \in [a,b],\ 0 < \varepsilon < 1, \tag{1}$$

$$l(x) = h, \quad h \in \mathbb{R}^m, \tag{2}$$

where the coefficients of the system (1) and the equality (2) are subordinate to the conditions:

(H1) A is a constant $(n \times n)$-matrix, $\operatorname{Re}\lambda_i < 0$ $(i = \overline{1,n})$, $\lambda_i \in \sigma(A)$;

(H2) $A_1(t)$ is an $(n \times n)$-matrix, $A_1(t) \in C^\infty[a,b]$; φ is an n-vector function, $\varphi(t) \in C^\infty[a,b]$;

(H3) l is a linear m-dimensional bounded functional

$$l = \operatorname{col}(l_1, \dots, l_m), \quad l \in (C([a,b]) \to \mathbb{R}^n, \mathbb{R}^m).$$

The condition (H1) shows that $\det A \neq 0$.

We consider the problem (1), (2) in the class of continuously differentiable functions. Then the domain $D(L_\varepsilon)$ of the operator

$$(L_\varepsilon x)(t) = \varepsilon \dot{x}(t) - Ax(t) - \varepsilon A_1(t)x(t)$$

consists of a continuously differentiable in $[a,b]$ functions, satisfying the boundary condition (2). At $\varepsilon = 0$ we obtain the degenerate equation $Ax_0(t) + \varphi(t) = 0$, which solution $x_0(t) = -A^{-1}\varphi(t)$ for arbitrary $\varphi(t) \in C^\infty[a,b]$ does not belong to the domain $D(L_\varepsilon)$ of the operator L_ε, since, in general, the condition $l(x_0) = h$ is not fulfilled.

Let the equation (1) is solvable for arbitrary $\varphi \in C^\infty[a,b]$. Then the dimension of the kernel of the operator L_ε is equal to the dimension n of the system (1) and the boundary-value problem (1), (2) is the Noetherian problem with index $n - m$: $\operatorname{ind}[L_\varepsilon, l] = n - m \neq 0$. It will be the Fredholm problem ($\operatorname{ind}[L_\varepsilon, l] = 0$) if and only if $m = n$ (see [13]).

We shall consider the case $m \neq n$. We use an asymptotic method of the boundary functions and construct an asymptotic series, satisfying the boundary-value problem (1), (2) at $\det A \neq 0$. The initial research in the case is made in [14].

In the Fredholmian case ($m = n$) an asymptotic integration of boundary-value problems for non-linear and weakly non-linear systems with two-point boundary conditions is studied in [6,7] on the basis of the method of boundary functions, and in [10] — on the basis of the regularization method.

The construction of an asymptotic solution of (1), (2) in the Notherian case ($m \neq n$) is represented on the basis of generalized inverse matrices and projectors [15–17, 13].

2. FORMALLY ASYMPTOTIC EXPANSION

We shall seek a formally asymptotic expansion of the solution of the problem (1), (2) in the form of the series

$$x(t,\varepsilon) = \sum_{i=0}^{\infty} [x_i(t) + \Pi_i(\tau)]\varepsilon^i, \quad \tau = \frac{t-a}{\varepsilon}, \tag{3}$$

where $x_i(t)$ and $\Pi_i(\tau)$ are unknown n vector-functions. By $\Pi_i(\tau)$ (see [6, 7]) we denote the boundary functions in a neighbourhood of the point $t = a$. They will be constructed so that when $0 < \varepsilon \leq \varepsilon_0$, the inequalities

$$\|\Pi_i(\tau)\| \leq \gamma_i \exp(-\alpha_i \tau), \tag{4}$$

where γ_i and α_i are positive constants for $i = 0, 1, 2, \ldots$ and $\tau \geq 0$, hold in $[a, b]$.

Formally, by substituting (3) in (1), for $x_i(t)$ we obtain the recurrent expressions

$$x_i(t) = \begin{cases} -A^{-1}\varphi(t), & i = 0, \\ A^{-1}(Lx_{i-1})(t), & i = 1, 2, \ldots, \end{cases} \tag{5}$$

where L is the differential operator $Lx = \dfrac{d}{dt}x - A_1(t)x$. The boundary functions are solutions of the differential equations

$$\frac{d}{d\tau}\Pi_i(\tau) = A\Pi_i(\tau) + f_i(\tau), \quad \tau \in [0, \tau_b], \ \tau_b = \frac{b-a}{\varepsilon}, \tag{6}$$

where

$$f_i(\tau) = \begin{cases} 0, & i = 0, \\ \displaystyle\sum_{q=i-1}^{0} \frac{1}{q!}\tau^q A_1^{(q)}(a)\Pi_{i-1-q}(\tau), & i = 1, 2, \ldots \end{cases} \tag{7}$$

We substitute (3) in the boundary condition (2). Then the coefficients of the expansion (3) satisfy the boundary conditions

$$l(x_i) + l\left(\Pi_i\left(\frac{(\cdot) - a}{\varepsilon}\right)\right) = \begin{cases} h, & i = 0, \\ 0, & i = 1, 2, \ldots \end{cases} \tag{8}$$

We denote $X(\tau) = \exp(A\tau)$ to be the normal fundamental matrix of the solutions of the linear system $\dfrac{dx}{d\tau} = Ax$, $\tau \in [0, \tau_b]$; $D(\varepsilon) = l(X) = l\left(X(\frac{(\cdot)-a}{\varepsilon})\right)$ is an $(m \times n)$-matrix.

Now consider two cases depending on the structure of the matrix $D(\varepsilon)$.

2.1. Let $D(\varepsilon) = D_0 + O\left(\varepsilon^s \exp\left(-\dfrac{\alpha}{\varepsilon}\right)\right)$, where $\alpha > 0$, $s \in \mathbb{N}$, D_0 is an $(m \times n)$-constant matrix.

All the expressions $\varepsilon^s \exp\left(-\dfrac{\alpha}{\varepsilon}\right)$ are exponentially small and it is possible to reject them, because they are of higher order of vanishing than an arbitrary degree of ε.

Let the following condition be fulfilled:

(H4) $\operatorname{rank} D_0 = n_1 < \min(m, n)$.

Denote by P and P^* the matrix orthoprojectors

$$P : \mathbb{R}^n \to \ker(D_0), \quad P^* : \mathbb{R}^m \to \ker(D_0^*), \quad D_0^* = D_0^T.$$

By D_0^+ we denote the unique Moore-Penrose inverse $(n \times m)$ -matrix of the matrix D_0 [15–17, 13]. Let P_d^* be a $(d \times m)$-matrix with $d = m - n_1$ linear independent rows from the matrix P^*, and let P_r be $r = n - n_1$ linear independent columns from the matrix P.

Consider the system (6–8) for $i = 0$. Then the boundary-value problem about $\Pi_0(\tau)$ has the form

$$\frac{d}{d\tau}\Pi_0(\tau) = A\Pi_0(\tau), \quad l(\Pi_0) = h - l(x_0). \tag{9}$$

We substitute the general solution of the system (9) $\Pi_0(\tau) = X(\tau)c_0$ in the boundary condition. Ignoring the exponentially small elements in the matrix $D(\varepsilon)$, we obtain by the algebraic system

$$D_0 c_0 = h_0, \tag{10}$$

where $h_0 = H - l(x_0)$, the n-vector c_0.

When the condition (H4) is fulfilled, the system (10) possesses a family of solutions

$$c_0 = P_r c_0^r + D_0^+ h_0$$

if and only if

$$P^* h_0 = 0 \implies P_d^* h_0 = 0.$$

Substituting c_0 in $\Pi_0(\tau) = X(\tau)c_0$, we obtain

$$\Pi_0(\tau) = X_r(\tau)c_0^r + g_0(\tau), \quad c_0^r \in \mathbb{R}^r, \tag{11}$$

where

$$X_r(\tau) = X(\tau)P_r - (n \times r)\text{-matrix}, \quad g_0(\tau) = X(\tau)D_0^+ h_0. \tag{12}$$

We define the vector $c_0^r \in \mathbb{R}^r$ by obtaining $\Pi_1(\tau)$. Consider the boundary-value problem with respect to $\Pi_1(\tau)$:

$$\frac{d}{d\tau}\Pi_1(\tau) = A\Pi_1(\tau) + f_1(\tau), \quad \tau \in [0, \tau_b], \quad l(\Pi_1) = -l(x_1), \tag{13}$$

where $f_1(\tau) = A_1(a)\Pi_0(\tau)$. Keeping in mind (11), (12), $f_1(\tau)$ will depend on the unknown vector c_0^r:

$$f_1(\tau, c_0^r) = A_1(a)X_r(\tau)c_o^r + A_1(a)g_0(\tau).$$

We substitute the general solution

$$\Pi_1(\tau) = X(\tau)c_1 + \int_0^\tau X(\tau)X^{-1}(s)f_1(s)\, ds \tag{14}$$

of the differential system (13) in the boundary condition and ignoring the exponential small elements in the matrix $D(\varepsilon)$, obtain the system with respect to c_1:

$$D_0 c_1 = h_1(\varepsilon), \quad c_1 \in \mathbb{R}^n, \tag{15}$$

where

$$h_1(\varepsilon) = -l(x_1) - l\left(\int\limits_0^{(\cdot)} X\left(\frac{(\cdot)-a}{\varepsilon}\right) X^{-1}(s) f_1(s, c_0^r)\, ds\right).$$

According to (H4), the system (15) has a solution

$$c_1 = P_r c_1^r + D_0^+ h_1(\varepsilon), \quad c_1^r \in \mathbb{R}^r,$$

if $P_d^* h_1(\varepsilon) = 0$.

From the last equality and the form of $h_1(\varepsilon)$ we obtain

$$\overline{D}(\varepsilon) c_0^r = P_d^* b_1(\varepsilon), \tag{16}$$

where

$$\overline{D}(\varepsilon) = P_d^* l\left(\int\limits_0^{(\cdot)} X\left(\frac{(\cdot)-a}{\varepsilon}\right) X^{-1}(s) A_1(a) X_r(s)\, ds\right),$$

$$b_1(\varepsilon) = -l\left(\int\limits_0^{(\cdot)} X\left(\frac{(\cdot)-a}{\varepsilon}\right) X^{-1}(s) A_1(a) g_0(s)\, ds\right) - l(x_1). \tag{17}$$

We assume that $\overline{D}(\varepsilon) = \overline{D}_0 + O\left(\varepsilon^p \exp\left(\frac{-\alpha}{\varepsilon}\right)\right)$, where $\alpha > 0$, $p \in \mathbb{N}$, $\overline{D}_0$ is a $(d \times r)$-constant matrix, and after ignoring the exponentially small elements, the system (16) takes the form

$$\overline{D}_0 c_0^r = P_d^* b_1(\varepsilon). \tag{18}$$

Let the following conditions be satisfied:

(H5) $\operatorname{rank} \overline{D}_0 = r$,

(H6) $\overline{P}_{d_1}^* P_d^* = 0$, $d_1 = d - r$,

where $\overline{P}^* : \mathbb{R}^d \to \ker(\overline{D}_0^*)$. Then the system (18) is always solvable and

$$c_0^r = \overline{D}_0^+ P_d^* b_1(\varepsilon). \tag{19}$$

We substitute (19) in (11) and obtain the resultant expression for $\Pi_0(\tau)$:

$$\Pi_0(\tau) = X_r(\tau) \overline{D}_0^+ P_d^* b_1(\varepsilon) + g_0(\tau). \tag{20}$$

Define the norm of the matrix $B = [b_{ij}]$ by means of the equality $\|B\| = \max\limits_i \sum\limits_{j=1}^n |b_{ij}|$. Keeping in mind the representation $b_1(\varepsilon)$ from (17) and the structure of the matrix

$X(\tau)$, it follows that there exists ε_0 and when $0 < \varepsilon \le \varepsilon_0$, the following inequalities are fulfilled:

$$\|b_1(\varepsilon)\| \le c_4, \quad c_4 > 0; \quad \|X_r(\varepsilon)\| \le c_1 \exp(-\alpha_1 \tau), \quad c_1 > 0, \quad \alpha_1 > 0;$$

$$\|D_0^+\| \le c_2, \quad c_2 > 0; \quad \|P_d^*\| \le c_3, \quad c_3 > 0;$$

$$\|g_0(\tau)\| \le c_5 \exp(-\alpha_2 \tau), \quad c_5 > 0, \quad \alpha_2 > 0.$$

Consequently, we can indicate positive constants γ_0, β_0 such that

$$\|\Pi_0(\tau)\| \le \gamma_0 \exp(-\beta_0 \tau),$$

that is the boundary function $\Pi_0(\tau)$ decreases exponentially.

It is obvious that $\Pi_i(\tau)$ $(i = 1, 2, \ldots)$ will be determined sequentially.

Assume that the boundary functions $\Pi_i(\tau)$ $(\overline{1, i-2})$ are defined. Then the vectors c_i $(\overline{0, i-2})$ are entirely defined. By means of $\Pi_i(\tau)$ we determine the vector c_{i-1}^r, which participates in the boundary function Π_{i-1}:

$$\Pi_{i-1}(\tau) = X_r(\tau) c_{i-1}^r + g_{i-1}(\tau), \quad c_{i-1}^r \in \mathbb{R}^r, \tag{21}$$

where $g_{i-1}(\tau) = g_{i-1}(\tau, c_{i-2}^r, \ldots, c_0^r)$.

We substitute the general solution of the system (6):

$$\Pi_i(\tau) = X(\tau) c_i + \int_0^\tau X(\tau) X^{-1}(s) f_i(s, c_{i-1}^r, \ldots, c_0^r)\, ds, \quad c_{i-1}^r \in \mathbb{R}^n, \tag{22}$$

in (8) and obtain the algebraic system (ignoring the exponentially small elements in $D(\varepsilon)$)

$$D_0 c_i = h_i(\varepsilon, c_{i-1}^r, \ldots, c_0^r), \tag{23}$$

where

$$h_i(\varepsilon, c_{i-1}^r, \ldots, c_0^r)$$

$$= -l \left(\int_0^{(\cdot)} X\left(\frac{(\cdot) - a}{\varepsilon}\right) X^{-1}(s) A_1(a) X_r(s)\, ds \right) c_{i-1}^r + b_i(\varepsilon, c_{i-2}^r, \ldots, c_0^r), \tag{24}$$

$$b_i(\varepsilon) = -l(x_i)$$

$$- l \left(\int_0^{(\cdot)} X(\cdot) X^{-1}(s) \left[\sum_{q=i-1}^{1} \frac{1}{q!} s^q A_1^{(q)}(a) \Pi_{i-1-q}(s) + A_1(a) g_{i-1}(s) \right] ds \right).$$

From the solvability condition of the system (23)

$$P_d^* h_i(\varepsilon, c_{i-1}^r, \ldots, c_0^r) = 0$$

and (24) we get

$$\overline{D}(\varepsilon)c_{i-1}^r = P_d^* b_i(\varepsilon).$$

Let the conditions (H5), (H6) be satisfied. Then

$$c_{i-1}^r = \overline{D}_0^+ P_d^* b_i(\varepsilon). \tag{25}$$

We substitute (25) in (21) and obtain the resultant expression for $\Pi_{i-1}(\tau)$:

$$\Pi_{i-1}(\tau) = X_r(\tau)\overline{D}_0^+ P_d^* b_i(\varepsilon) + g_{i-1}(\tau), \tag{26}$$

where

$$g_{i-1}(\tau) = X(\tau)D_0^+ h_{i-1}(c_{i-2}^r, \ldots, c_0^r) + \int_0^\tau X(\tau)X^{-1}(s) f_{i-1}(s, c_{i-2}^r, \ldots, c_0^r)\, ds.$$

Lemma 2.1. *Let the matrix A satisfy the condition* (H1), *and let the vector function $f(t) \in C[0, +\infty)$ and satisfy the inequality $\|f(t)\| \le c^* \exp(-\alpha^* t)$, where $t \ge 0$, $c^* > 0$, $\alpha^* > 0$. Then there exist positive constants c and γ, so that the system $\dfrac{dx}{dt} = Ax + f(t)$ has a particular solution of the form*

$$\overline{x}(t) = \int_0^{+\infty} K(t,s) f(s)\, ds,$$

satisfying the inequality

$$\|\overline{x}(t)\| \le c e^{(-\gamma t)}, \ t \ge 0, \tag{27}$$

where

$$K(t,s) = \begin{cases} X(t)X^{-1}(s), & \text{if } 0 \le s \le t < \infty, \\ 0, & \text{if } 0 < t < s \le \infty. \end{cases}$$

Proof. The fact that $\overline{x}(t)$ is a solution is verified directly. From the condition (H1) it follows that $\|X(t)X^{-1}(s)\| \le \overline{c}\exp(-\overline{\alpha}(t-s))$ when $\overline{c} > 0$, $\overline{\alpha} > 0$. We have

$$\|\overline{x}(t)\| \le \int_0^t \|X(t)X^{-1}\|\, \|f(s)\|\, ds \le c^* \overline{c} e^{-\overline{\alpha} t} \int_0^t e^{(\overline{\alpha} - \alpha^*)s}\, ds.$$

If $\alpha^* < \overline{\alpha}$ (or $\alpha^* > \overline{\alpha}$), then choosing $c = \dfrac{2c^*\overline{c}}{\overline{\alpha} - \alpha^*} > 0$ $\left(c = \dfrac{2c^*\overline{c}}{\alpha - \overline{\alpha}}\right)$ and $\gamma \le \alpha^*$ ($\gamma \le \overline{\alpha}$) we get (27).

Let $\alpha^* = \overline{\alpha}$. Then $\|x(t)\| \le c^* \overline{c}\, t\, e^{-\overline{\alpha} t}$. But $\lim\limits_{t\to\infty} t e^{-\overline{\alpha} t} = 0$. Consequently, there exist constants $c > 0$, $\gamma > 0$, so that (27) is fulfilled for $t \ge t_1 > 0$. □

Theorem 2.1. *Let* $D(\varepsilon) = D_0 + O\left(\varepsilon^s \exp\left(-\frac{\alpha}{\varepsilon}\right)\right)$ *and the conditions* (H1)–(H6) *be satisfied. If* $\varphi(t) \in C^\infty[a,b]$ *and* $h \in \mathbb{R}^m$ *satisfies the condition* $P_d^*(h - l(x_0)) = 0$, *the boundary-value problem* (1), (2) *has a unique formal expansion of the form* (3). *The coefficients of the expansion* $x_i(t)$ *and* $\Pi_{i-1}(\tau)$ *have the representations* (5), (20), (26), *respectively, and the boundary functions* $\Pi_i(\tau)$ *decrease exponentially.*

Proof. From the above conclusions and the conditions of the theorem it follows that really the coefficients of the expansion (3) for the boundary-value problem (1), (2) have the representations (5), (20), (26). It will be proved that the functions $\Pi_i(\tau)\,(i = 0, 1, \ldots)$ decrease exponentially. This we have done for $\Pi_0(\tau)$. Let the inequalities (4) be satisfied, that is $\|\Pi_k(\tau)\| \le \gamma_k \exp(-\alpha_k \tau)$ for $\tau \ge 0$ and $k = \overline{1, i-2}$. It is known that for $\beta_{i-1} < \alpha = \max\limits_k \alpha_k$, $c_{i-1}^* = \max\limits_k \gamma_k$ we have

$$(c_{i-1}^* \tau^{i-2} + \cdots + c_{i-1}^* \tau + c_{i-1}^*) \exp(-\alpha\tau) \le c_{i-1}^* \exp(-\beta_{i-1}\tau).$$

Thus $\|f_{i-1}(\tau)\| \le c_{i-1}^* \exp(-\beta_{i-1}\tau)$ for $\tau \ge 0$, where $f_{i-1}(\tau)$ are the functions from (7). Using this inequality, Lemma 2.1 and the estimates $\|b_i(\varepsilon)\| \le c_i$ at $\varepsilon \in (0, \varepsilon_0]$, from (26) we obtain

$$\|\Pi_{i-1}(\tau)\| \le \gamma_{i-1} \exp(-\alpha_{i-1}\tau), \quad \tau \ge 0,$$

that is the boundary functions decrease exponentially. □

Corollary 1. *Let the conditions* (H1)–(H3) *be satisfied and* $\mathrm{rank} D_0 = n_1 = n$. *Then for any function* $\varphi(t) \in C^\infty[a,b]$ *and for any* $h \in \mathbb{R}^m$, *satisfying* $P_d^* h_i(\varepsilon) = 0$, $i = 0, 1, \ldots$, *the boundary-value problem* (1), (2) *has an unique formally asymptotic expansion in the form* (3). *The coefficients* $x_i(t)$ *have the form* (5), *and the boundary functions* $\Pi_i(\tau)$ *have the representations*

$$\Pi_i(\tau) = X(\tau) D_0^+ h_i + \int_0^\tau X(\tau) X^{-1}(s) f_i(s)\, ds.$$

In this case $P = 0$, $c_i = D_0^+ h_i(\varepsilon)$ $(i = 0, 1, \ldots)$, where $h_i(\varepsilon) = b_i(\varepsilon)$, $h_0 = h - l(x_0)$.

Remark 1. If $m = n$ and $\det D_0 \ne 0$, then it is sufficient to replace D_0^+ with D_0^{-1} in Corollary 1. If $m = n$ and $\mathrm{rank} D_0 < m = n$, then all considerations in this case coincide with the mentioned above ones.

Remark 2. If $m \ne n$, $\mathrm{rank} D_0 = n_1 = m$, then $P^* = 0$ and all systems $D_0 c_i = h_i(\varepsilon)$, $i = 0, 1, \ldots$, are always solvable. In this case we get the family of boundary functions.

2.2. Let $D(\varepsilon) = D_0 + D_1\varepsilon + D_2\varepsilon^2 + \cdots + D_s\varepsilon^s + O(\varepsilon^q \exp(-\alpha\varepsilon))$, where D_i are $(m \times n)$-constant matrices, $\alpha > 0$, $q \in \mathbb{N}$.

We reject the exponentially small elements in $D(\varepsilon)$ and introduce the $(2s+1)m \times (s+1)n$-matrix

$$Q = \begin{bmatrix} D_0 & & & & \\ D_1 & D_0 & & \mathbf{0} & \\ D_2 & D_1 & D_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ D_s & D_{s-1} & D_{s-2} & \dots & D_0 \\ & D_s & D_{s-1} & \dots & D_1 \\ & & D_s & \dots & D_2 \\ & \mathbf{0} & & \ddots & \vdots \\ & & & & D_s \end{bmatrix}.$$

We also introduce the $(s+1)n$-vector $c_i = [c_{i0}\, c_{i1}\, \cdots\, c_{is}]^T$, where c_{ij} are n-vectors and the $(2s+1)m$-vector $b_i = [\underbrace{b_{i0}\, \cdots\, b_{is}}_{(s+1)m} \underbrace{0\, \cdots\, 0}_{sm}]^T$.

Let the following condition be fulfilled:

(H7) $\quad \operatorname{rank} Q = (s+1)n$, $((2s+1)m > (s+1)n)$.

Then $\operatorname{rank} P_1 = 0$, $\operatorname{rank} P_1^* = d_1 = (2s+1)m - (s+1)n$, where

$$P_1 : R^{(s+1)n} \to \ker(Q), \quad P_1^* : R^{(2s+1)m} \to \ker(Q^*), \quad Q^* = Q^T.$$

The algebraic system

$$Qc_i = b_i \tag{28}$$

has the solution

$$c_i = Q^+ b_i \ \text{ or } \ c_{ij} = [Q^+ b_i]_{n_j}, \quad j = \overline{0, s} \tag{29}$$

if and only if $P_1^* b_i = 0$. So we obtain the conditions

(H8) $\quad P_{1d_1}^* b_i = 0$, $i = 0, 1, \ldots$,

where $P_{1d_1}^*$ is a $(d_1 \times (2s+1)m)$-matrix, and $[Q^+ b_i]_{n_0}, [Q^+ b_i]_{n_1}, \ldots, [Q^+ b_i]_{n_s}$ are the first n elements, the second n elements, ..., the last n elements of the $(s+1)n$-vector $Q^+ b_i$, respectively.

In this case we shall seek the solution of the system $D(\varepsilon)c_0 = h_0$ in the form

$$c_0 = c_{00} + \varepsilon c_{01} + \cdots + \varepsilon^s c_{0s},$$

where $c_{0j} \in \mathbb{R}^n$, $j = \overline{1,s}$. We find the vectors c_{0j} from the system (28). From the conditions (H5), (H6) and the equality (29) for $i = 0$ and $\Pi_0(\tau)$ we obtain

$$\Pi_0(\tau) = X(\tau) \sum_{j=0}^{s} \varepsilon^j [Q^+ b_0]_{n_j}, \tag{30}$$

where $b_0 = [h_0 \; 0 \; \cdots \; 0]^T$. Obviously, the boundary function fulfills the requirement $\lim_{\tau\to\infty} \Pi_0(\tau) = 0$.

Analogously, we find $\Pi_1(\tau)$ from (14) and the system $D(\varepsilon)c_1 = h(\varepsilon)$, where

$$h_1(\varepsilon) = -l(x_2) - l\left(\int_0^{(\cdot)} X\left(\frac{(\cdot)-a}{\varepsilon}\right) X^{-1}(s) f_1(s)\, ds\right), \quad f_1(\tau) = A_1(a)\Pi_0(\tau).$$

We seek c_1 in the form $c_1 = c_{10} + \varepsilon c_{11} + \cdots + \varepsilon^s c_{is}$.

Assume that after ignoring the exponentially small elements, $h_1(\varepsilon) = h_{10} + \varepsilon h_{11} + \cdots + \varepsilon^s h_{1s}$. Then we obtain

$$\Pi_1(\tau) = X(\tau) \sum_{j=0}^{s} \varepsilon^j [Q^+ b_1]_{n_j} + \int_0^{\tau} X(\tau) X^{-1}(s) f_1(s)\, ds,$$

where $b_1 = [h_{10} \; \cdots \; h_{1s} \; 0 \; \cdots \; 0]^T$ and $\lim_{\tau\to\infty} \Pi_1(\tau) = 0$.

It is possible to prove (inductively) that the solution of the systems (6)–(8) for an arbitrary i and

$$h_i(\varepsilon) = -l(x_i) - l\left(\int_0^{(\cdot)} X(\cdot) X^{-1}(s) f_i(s)\, ds\right)$$

$$= h_{i0} + \varepsilon h_{i1} + \cdots + \varepsilon^s h_{is} + O(\varepsilon^q \exp(-\alpha\varepsilon))$$

has the form

$$\Pi_i(\tau) = X(\tau) \sum_{j=0}^{s} \varepsilon^j [Q^+ b_i]_{nj} + \int_0^{\tau} X(\tau) X^{-1}(s) f_i(s)\, ds, \tag{31}$$

where $b_i = [h_{i0} \; \cdots \; h_{is} \; 0 \; \cdots \; 0]^T$.

For $\Pi_i(\tau)$ the bound (4) is fulfilled.

So we have proved the following theorem:

Theorem 2.2. *Let $D(\varepsilon) = D_0 + D_1\varepsilon + D_2\varepsilon^2 + \cdots + D_s\varepsilon^s + O(\varepsilon^q \exp(-\alpha\varepsilon))$ and the conditions* (H1)–(H3), (H7), (H8) *be satisfied. Then the solution of the boundary-value problem* (1), (2) *has an unique representation in the form* (3). *The coefficients of the expansion are defined by the equalities* (5), (30), (31).

Remark 3. If rank $Q < (s+1)n$, then we obtain c_i with determination of the boundary function $\Pi_{i+1}(\tau)$.

Remark 4. If $D(\varepsilon) = l(X) = l(e^{A\tau})$

$$= l(E) + \varepsilon^{-1} l\left(A\frac{((\cdot)-a)}{1!}\right) + \varepsilon^{-2} l\left(A^2\frac{((\cdot)-a)^2}{2!}\right) + \cdots$$

$$= D_0 + \varepsilon^{-1}D_{-1} + \varepsilon^{-2}D_{-2} + \cdots,$$

then we seek c_i in the form $c_i = c_{i0} + \varepsilon^{-1}c_{i1} + \cdots$ From the structure of the matrix $X(\tau)$ it follows that $\lim\limits_{\tau\to\infty} \Pi_i(\tau) = 0$.

3. A BOUND OF THE REMAINDER TERM OF THE ASYMPTOTIC SERIES

The solution of the boundary-value problem (1), (2) we seek in the form

$$x(t,\varepsilon) = X_n(t,\varepsilon) + \varepsilon^{n+1}\xi(t,\varepsilon), \tag{32}$$

where

$$X_n(t,\varepsilon) = \sum_{i=0}^{n}[x_i(t) + \Pi_i(\tau)]\varepsilon_i.$$

We shall proof that in [a,b], when $\varepsilon \to 0$, the function $\xi(t,\varepsilon)$ fulfills the inequality $\|\xi(t,\varepsilon)\| \le K$, where K is a positive constant.

We substitute (32) in (1), (2), where $x_i(t)$ and $\Pi_i(\tau)$ are defined in Section 2. After some transformations we obtain that the function $\xi(t,\varepsilon)$ satisfies the boundary-value problem

$$\varepsilon\,\dot{\xi}(t,\varepsilon) = A\xi(t,\varepsilon) + \varepsilon A_1(t)\xi(t,\varepsilon) + H(t,\varepsilon), \;\; l(\xi(\cdot,\varepsilon)) = 0, \tag{33}$$

where

$$H(t,\varepsilon) = \frac{1}{\varepsilon^{n+1}}(H_1(t,\varepsilon) + H_2(t,\varepsilon)),$$

$$H_1(t,\varepsilon) = -\varepsilon^{n+1}Ax_{n+1}(t), \;\; H_2(t,\varepsilon) = \varepsilon^{n+1}F_1(t,\varepsilon), \tag{34}$$

$$F_1(t,\varepsilon) = \sum_{k=0}^{n}\frac{1}{(n-k)!}A_1^{(n-k)}(a)\tau^{n-k}\Pi_k(\tau)$$

$$+\sum_{i=1}^{n}\varepsilon^i\sum_{k=0}^{n-i}\frac{1}{(n-k)!}A_1^{(n-k)}(a)\tau^{n-k}\Pi_{k+i}(\tau)$$

$$+\frac{1}{(n+1)}A_1^{(n+1)}(a+\theta\tau\varepsilon)\tau^{n+1}\sum_{i=1}^{n+1}\varepsilon^i\Pi_{i-1}(\tau), \quad 0<\theta<1.$$

Since $x_i(t)$, $i = 0, 1, \ldots$, are continuous functions in $[a,b]$, then $\|x_i(t)\| \le \eta_i$, where η_i are positive constants.

So we have

$$\left\| \frac{1}{\varepsilon^{n+1}} H_1(t,\varepsilon) \right\| \le \|A\|\,\|x_{n+1}(t)\| \le \|A\|\eta_{n+1}. \tag{35}$$

Let

$$K_1 = \max_{k=\overline{0,n}} \left(\frac{1}{(n-k)!}\|A_1^{(n-k)}(a)\|,\ \frac{1}{(n+1)!}\|A_1^{(n+1)}(a+\theta\tau\varepsilon)\| \right),$$

when $0<\theta<1$ and $t\in[a,b]$, $\|\Pi_i(\tau)\| \le p_i e^{-\alpha_i\tau}$, $p_i>0$, $\alpha_i>0$ $(i=\overline{0,n})$ and $\alpha = \min_i(\alpha_i)$, $p = \max_i(p_i)$.

When $\varepsilon\in(0,\varepsilon_0]$, let denote $c = \max_{i=\overline{0,n+1}}(c_i)$, where $c_0 = 1$, $c_j = 1 + \sum_{k=1}^{j}\varepsilon^k$, $j=\overline{1,n}$, $c_{n+1} = \sum_{k=1}^{n+1}\varepsilon^k$.

By (34) we obtain

$$\|F_1(t,\varepsilon)\| \le K_1[c_{n+1}\tau^{n+1} + c_n\tau^n + \cdots + c_1\tau + c_0]\,p\,e^{-\alpha\tau}$$
$$\le K_1\,c\,p[\tau^{n+1}+\cdots+\tau+1]e^{-\alpha\tau}.$$

Let $K_2 = K_1 c\, p$. There exists $\overline{\alpha}$, $0<\overline{\alpha}<\alpha$, such that $(\tau^{n+1}+\cdots+\tau+1)e^{-\alpha\tau} \le e^{-\overline{\alpha}\tau}$.

Consequently,

$$\left\| \frac{1}{\varepsilon^{n+1}} H_2(t,\varepsilon) \right\| = \|F_1(t,\varepsilon)\| \le K_2 e^{-\overline{\alpha}\tau} \le K_3 = \text{const}.$$

Keeping in mind (35) and the last inequality, we have

$$\|H(t,\varepsilon)\| \le \left\| \frac{1}{\varepsilon^{n+1}} H_1(t,\varepsilon) \right\| + \left\| \frac{1}{\varepsilon^{n+1}} H_2(t,\varepsilon) \right\| \le \|A\|\eta_{n+1} + K_3 = \eta,$$

that is $\|H(t,\varepsilon)\| \le \eta$, $\eta>0$.

Let $W(t,s,\varepsilon)$ be a fundamental matrix for the homogeneous system

$$\varepsilon\frac{d\xi}{dt} = A\xi, \quad W(t,s,\varepsilon) = E_n, \quad E_n \text{ — } (n\times n)\text{-unit matrix.}$$

Lemma 3.1 [18, 19]. *For the matrix $W(t,s,\varepsilon)$, when $a\le s\le t\le b$, $0<\varepsilon\le\varepsilon_0$ the exponential bound*

$$\|W(t,s,\varepsilon)\| \le \beta\exp\left(-\frac{\alpha(t-s)}{\varepsilon}\right) \tag{36}$$

is fulfilled, where $\alpha>0$, $\beta>0$ are any constants.

Lemma 3.2 [18, 19]. *Any continuous solution of the system* (33) *is a solution of the system of integral equations*

$$\xi(t,\varepsilon) = W(t,a,\varepsilon)\xi(a,\varepsilon) + \int_a^t W(t,s,\varepsilon)\frac{1}{\varepsilon}[\varepsilon A_1(s)\xi(s,\varepsilon) + H(s,\varepsilon)]\, ds, \tag{37}$$

and conversely.

Lemma 3.3 [18, 19]. *When* $\varepsilon \to 0$, *the integral* $\int\limits_a^t \left\| \frac{1}{\varepsilon} W(t,s,\varepsilon) \right\| ds$ *is uniformly bounded in the segment* $[a,b]$.

Lemma 3.3 reveals that there exists a constant $M > 0$ such that for $\varepsilon \to 0$ and $t \in [a,b]$ the inequality

$$\int_a^t \left\| \frac{1}{\varepsilon} W(t,s,\varepsilon) \right\| ds \leq M$$

holds.

The system (37) will be solved by the method of successive approximations. Let

$$\xi_0(t,\varepsilon) = 0,$$

$$\xi_j(t,\varepsilon) = F(t,\varepsilon) + \int_a^t W(t,s,\varepsilon)\frac{1}{\varepsilon}[\varepsilon A_1(s)\xi_{j-1}(s,\varepsilon) + H(s,\varepsilon)]\, ds \tag{38}$$

be the Picard successive approximations, where $F(t,\varepsilon) = W(t,a,\varepsilon)\xi(a,\varepsilon)$.

Theorem 3.1. *Let the conditions of Theorem* 2.1 *(or Theorem* 2.2*) be fulfilled. Let* $\overline{\beta}$, h, h_1, h_2, h_3, h_4, ε_0 *be positive constants such that*

$$\|W(t,a,\varepsilon)\| \leq \overline{\beta}; \quad \|F(t,\varepsilon)\| \leq h_1, \quad \textit{where } h_1 = 2\overline{\beta}h,\ 0 < 2\overline{\beta} < 1;$$

$$\|A_1(t)\| \leq h_2, \quad \textit{where } t \in [a,b]; \quad \left\| \overline{\overline{D}}_0^+ + \right\| \leq h_3;$$

$$\|l(\psi)\| \leq h_4\|\psi\|, \quad h_3h_4 < 2, \quad \varepsilon_0 \leq \frac{1}{2Mh_2}.$$

If $\dfrac{M\eta}{1-2\overline{\beta}} \leq h \leq \dfrac{2-(1-2\overline{\beta})h_3h_4}{h_3h_4 2\overline{\beta}} h_2$, *then the asymptotic solution of the boundary-value problem* (1), (2) *has the representation* (32), *where* $\xi(t,\varepsilon)$ *satisfies the inequality* $\|\xi(t,\varepsilon)\| \leq 2h$. *The vector* $\xi(a,\varepsilon)$ *is defined by the algebraic system* $\overline{\overline{D}}(\varepsilon)\xi(a,\varepsilon) = g(\varepsilon)$, *where* $\overline{\overline{D}}(\varepsilon) = l(W(\cdot,a,\varepsilon))$ *is an* $(m \times n)$*-matrix,*

$$g(\varepsilon) = -l\left(\int_a^{(\cdot)} W(\cdot,s,\varepsilon)\frac{1}{\varepsilon}[\varepsilon A_1(s)\xi(s,\varepsilon) + H(s,\varepsilon)]\, ds \right). \tag{39}$$

Besides, $x(t,\varepsilon)$ approaches the degenerating system at $\varepsilon \to 0$ and $t \in (a,b]$.

Proof. Using (38), we shall prove that the system (37) has an unique continuous solution, which does not leave the domain

$$\Omega = \{(t,\xi) \mid a \le t \le b,\ \|\xi\| \le 2h\},$$

depending on an arbitrary vector $\xi(a,\varepsilon)$.

By the equalities (38), for the first approximation we have

$$\|\xi_1 - \xi_0\| \le \|F(t,\varepsilon)\| + \int_a^t \left\| W(t,s,\varepsilon)\frac{1}{\varepsilon}\right\| \|H(t,s)\|\, ds \le h_1 + M\eta \le h.$$

If $0 < \varepsilon \le \varepsilon_0$ and $\varepsilon_0 \le \dfrac{1}{2Mh_2}$, we obtain

$$\|\xi_j - \xi_{j-1}\| \le \varepsilon \int_a^t \left\| W(t,s,\varepsilon)\frac{1}{\varepsilon}\right\| ds\, \|A_1(t)\|\, \|\xi_{j-1}(t,\varepsilon) - \xi_{j-2}(t,\varepsilon)\|$$

$$\le \varepsilon M h_2\, \|\xi_{j-1}(t,\varepsilon) - \xi_{j-2}(t,\varepsilon)\| \le \frac{1}{2}\, \|\xi_{j-1} - \xi_{j-2}\|,\ j = 2,3,\ldots$$

This reveals that in the segment $[a,b]$, when ε is sufficiently small, the successive approximations (38) are absolutely and uniformly convergent. We shall show that the successive approximations do not leave the domain Ω. We have

$$\|\xi_k(t,\varepsilon)\| \le \sum_{j=1}^{k} \|\xi_j(t,\varepsilon) - \xi_{j-1}(t,\varepsilon)\| \le h + \frac{h}{2} + \frac{h}{2^2} + \cdots + \frac{h}{2^{n-1}} \le 2h.$$

Let $\lim\limits_{k\to\infty} \xi_k(t,\varepsilon) = \xi(t,\varepsilon)$ satisfy (37) identically. Then in the interval $[a,b]$ for $\varepsilon \to 0$ the inequality $\|\xi(t,\varepsilon)\| \le 2h$ is fulfilled.

Consequently, the system (37) has an unique continuos solution, which does not leave the domain Ω and depends on an arbitrary vector $\xi(a,\varepsilon)$.

We define $\xi(a,\varepsilon)$ by the algebraic system

$$\overline{\overline{D}}(\varepsilon)\xi(a,\varepsilon) = g(\varepsilon), \tag{40}$$

where $\overline{\overline{D}}(\varepsilon)$ and $g(\varepsilon)$ are the expressions from (39). The system (40) is obtained substituting $\xi(t,\varepsilon)$ in the boundary condition $l(\xi) = 0$ of (33).

Let $\overline{\overline{D}}(\varepsilon) = \overline{\overline{D}}_0 + O\left(\varepsilon^s \exp\left(-\frac{\gamma}{\varepsilon}\right)\right)$, $\gamma > 0$, $s \in \mathbb{N}$, where $\overline{\overline{D}}_0$ is $(m \times n)$-constant matrix. Then if $\operatorname{rank}\overline{\overline{D}}_0 = n$, for $\varepsilon \in (0,\varepsilon_0]$ the system (40) has an unique solution

$$\xi(a,\varepsilon) = \overline{\overline{D}}_0^{\,+} g(\varepsilon)$$

if and only if

$$P_3^* g(\varepsilon) = 0 \quad \text{and} \quad P_3^* : R^n \to \ker(\overline{\overline{D}}_0^{\,*}).$$

The inequality $\|\xi(a,\varepsilon)\| \le 2h$ is fulfilled for $\xi(a,\varepsilon)$. Really,

$$\|\xi(a,\varepsilon)\| = \|\overline{\overline{D}}_0^{\,+}\| \, \|g(\varepsilon)\|$$

$$\le h_3 h_4 \int_a^t \left\| W(t,s,\varepsilon)\frac{1}{\varepsilon} \right\| \left[\varepsilon \|A_1(s)\| \, \|\xi(s,\varepsilon)\| + \|H(s,\varepsilon)\|\right] ds$$

$$\le h_3 h_4 M(2\varepsilon h_1 h + \eta) \le h_3 h_4 M \left(2\frac{1}{2Mh_2} 2\overline{\beta} h^2 + \frac{h(1-2\overline{\beta})}{M} \right)$$

$$\le h h_3 h_4 \left(2\overline{\beta}\frac{h}{h_2} + 1 - 2\overline{\beta} \right)$$

$$\le h h_3 h_4 \left(\frac{2\overline{\beta}}{h_2} \frac{2-(1-2\overline{\beta})h_3 h_4}{h_3 h_4 2\overline{\beta}} h_2 + 1 - 2\overline{\beta} \right) = 2h. \quad \square$$

4. EXAMPLE

We consider the two-point boundary-value problem

$$\varepsilon \dot{x} = Ax + \varphi(t), \quad t \in [0,1], \quad l(x) = Mx(0) + Nx(1) = h,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}, \quad \varphi(t) = \begin{bmatrix} t-1 \\ t \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 10 \\ 6 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 6 \\ 31 \\ 25 \end{bmatrix}.$$

If $\varepsilon = 0$, then $x_0(t) = -A^{-1}\varphi(t) = \begin{bmatrix} 3t+1 \\ 2t+1 \end{bmatrix}$. It is obvious that $l(x_0) = [5 \ 31 \ 25]^T \ne h$. Since $\lambda_{1,2} = -1$ and the normal fundamental matrix has the form $X(t) = \begin{bmatrix} 1-2t & 4t \\ -t & 1+2t \end{bmatrix} e^{-t}$, then $D(\varepsilon) = MX(0) + NX\left(\frac{1}{\varepsilon}\right)$ has the representation

$$D(\varepsilon) = M + N \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^{-\frac{1}{\varepsilon}} + N \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}}$$

$$= D_0 + O\left(e^{-\frac{1}{\varepsilon}} + \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}} \right),$$

where $D_0 = M$ and $\operatorname{rank} D_0 = 2$.

We obtain sequentially

$$D_0^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad P_1 = \frac{1}{2}[0\ 1\ -1], \quad h_0 = [1\ 0\ 0]^T, \quad h_1 = [10\ 33\ 33]^T.$$

In this case the conditions $P_1^* h_i = 0$, $i = 0, 1, \ldots$, are fulfilled.

According to Corollary 1, we obtain

$$c_0 = D_0^+ h_0 = [1\ 0]^T, \quad c_1 = D_0^+ h_1 = [10\ 33]^T,$$

$$\Pi_0(\tau) = X(\tau)c_0 = \begin{bmatrix} 1-2\tau \\ \tau \end{bmatrix} e^{-\tau}, \quad \Pi_1(\tau)c_1 = X(\tau)c_1 = \begin{bmatrix} 10+112\tau \\ 33-23\tau \end{bmatrix} e^{-\tau}.$$

The asymptotic solution of the two-point boundary-value problem has the form

$$x(t,\varepsilon) = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} t-1 \\ t \end{bmatrix} + \begin{bmatrix} 1 - \dfrac{2t}{\varepsilon} \\ \dfrac{t}{\varepsilon} \end{bmatrix} e^{-\frac{t}{\varepsilon}}$$

$$+\varepsilon \left(\begin{bmatrix} -5 \\ -3 \end{bmatrix} + \begin{bmatrix} 10 + 112\dfrac{t}{\varepsilon} \\ 33 - 23\dfrac{t}{\varepsilon} \end{bmatrix} e^{-\frac{t}{\varepsilon}} \right) + O(\varepsilon^2).$$

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