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A SEPARATION THEOREM OF Y. TAGAMLITZKI IN ITS NATURAL GENERALITY

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It is shown how the assumptions of a separation theorem of Y. Tagamlitzki can be weakened without any essential change of the proof. In contrast to the original version of the theorem, the obtained thus strengthened version is not an instance of Ellis' separation theorem.

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1. INTRODUCTION

In Y. Tagamlitzki's paper [3] an axiomatization of the notion of segment is used as a basis for an abstract approach to separation of convex sets. The axiomatization looks as follows.

A set K is supposed to be given, and a subset ab of K is supposed to be put into correspondence to any a and b in K in such a way that always $ab = ba$. By definition, $a/b = \{x \in K : a \in bx\}$.[1] The following denotations are adopted for any elements a and b of the set K and any its subsets A and B:

$$aB = \bigcup\{ab : b \in B\},\ Ab = \bigcup\{ab : a \in A\},\ AB = \bigcup\{ab : a \in A, b \in B\},$$

[1] We use this denotation instead of $\frac{a}{b}$ used in [3] (and, similarly, further for a/B, A/b, A/B). Another denotational difference is that we shall designate a set inclusion by $\subseteq$, whereas Tagamlitzki designates it by $\subset$.

$$a/B = \bigcup\{a/b : b \in B\},\ A/b = \bigcup\{a/b : a \in A\},\ A/B = \bigcup\{a/b : a \in A, b \in B\}.$$

The two operations considered so far will be called *multiplication* and *division*, respectively.

Two associativity laws are supposed to hold for any a, b, c in K, namely,

$$(ab)c = a(bc),\ \ a(b/c) \subseteq (ab)/c$$

(the first of these conditions allows freely using expressions of the form abc for arbitrary a, b, c in K).

Remark. After quite a time from the appearance of the paper [3] it became known that a somewhat more restrictive but similar axiomatization of the notion of segment had been given earlier by W. Prenowitz in [2]. It is easy to see that

$$a(b/c) \subseteq (ab)/c$$

for all a, b, c in K iff Prenowitz' transposition law (cf. [2, pp. 4 and 7])

$$(a/b) \cap (c/d) \neq \emptyset \Rightarrow (ad) \cap (bc) \neq \emptyset$$

holds for any a, b, c, d in K. Having this in mind, one sees that Prenowitz's join spaces from [2] coincide with the structures satisfying Tagamlitzki's axioms plus the additional ones (not required in [3]) that $ab \neq \emptyset$, $a/b \neq \emptyset$, $aa = \{a\}$ and $a/a = \{a\}$ for any a, b in K. Therefore any join space is surely a model for Tagamlitzki's axiomatization. In particular, the elements of an arbitrary vector space K form such a model if one sets

$$ab = \{pa + qb\ :\ p > 0,\ q > 0,\ p + q = 1\}.$$

The converse is not true, since the other models indicated in [3] do not satisfy, in general, the whole set of conditions in Prenowitz' definition of join space. We should like especially to mention as an example of such other model the one (indicated on p. 173), where K is again a vector space, but we have

$$ab = \{\lambda a + \mu b\ :\ \lambda > 0,\ \mu > 0\}$$

(the conditions $aa = \{a\}$ and $a/a = \{a\}$ are violated in this model for any non-zero element of K).

To reduce the number of brackets, we accept the convention that multiplication and division have a higher priority than $\cap$ and $\cup$ (thus we could omit the brackets in Prenowitz' transposition law mentioned above).

A subset C of K is called *convex* if the condition $CC \subseteq C$ holds. A *half-space* is a non-empty convex subset S of K such that $K \setminus S$ is also convex and non-empty. The following separation theorem plays a central role in [3]:

Theorem 1.1 (Theorem 1 of [3]). *Let $abb \subseteq ab$ for any a, b in K.*[2] *Then for any two disjoint non-empty convex subsets A and B of K there is a half-space that contains A and does not meet B.*

[2] This condition is surely satisfied in join spaces, since then $abb = a(bb) = ab$. The model mentioned at the end of the remark preceding the theorem also satisfies the condition in question, and we again have the equality $abb = ab$ in this model.

As it became clear later, the above formulated result is an instance of a more general separation theorem of J. W. Ellis published in [1]. Ellis' approach is based on a direct axiomatization of the notion of convex subset of a given set (without axiomatizing the notion of segment), and it turns out that the family of all convex subsets of K in the situation considered in the above theorem satisfies the assumptions of Ellis' one. The present paper aims at showing that Tagamlitzki's proof actually establishes a result stronger than Theorem 1.1 and this result is no more an instance of Ellis' theorem. Namely, a reduction of the assumptions of Theorem 1.1 will be done in the next section without making essential changes in its proof from [3].

2. REDUCTION OF THE ASSUMPTIONS OF TAGAMLITZKI'S SEPARATION THEOREM

We are going to formulate now the stronger result mentioned at the end of the previous section.

First of all, we reduce the assumptions from the beginning of Section 1 by omitting the first associativity law. For the reader's convenience, we formulate now what is remaining from those assumptions. Namely, we suppose in/the present section a set K to be given and a subset ab of K to be put into correspondence to any a and b in K in such a way that always the equality $ab = ba$ and the inclusion $a(b/c) \subseteq (ab)/c$ hold, adopting the denotations introduced in Section 1 before the formulation of the associativity laws.

Clearly, the definition of convex set remains the same as in Section 1, but the absence of the first associativity law obliges us now to write all brackets in the expressions that are built up by more than one application of multiplication. In particular, Theorem 1.1 does not make sense now without specifying the meaning of its assumption that $abb \subseteq ab$ for any a, b in K. (Does abb mean $(ab)b$ or $a(bb)$?) The following modification of the theorem can be established with almost no change in the proof of Theorem 1 from [3].

Theorem 2.1. *Let $(ab)b \subseteq a(bb) \subseteq ab$ for any a, b in K. Then for any two disjoint non-empty convex subsets A and B of K there is a half-space that contains A and does not meet B.*

To see the workability of the mentioned proof in the new situation considered now, it is sufficient to note that there are only two steps in the proof needing a revision: the first of them is in the transition from $\xi \in S/(xx)$ to $\xi x \cap S \neq \emptyset$ (cf. p. 174), where one has to apply now the inclusion $\xi(xx) \subseteq \xi x$, and the second one is in proving that $((SS)/x)/x$ is a subset of $(SS)/(xx)$ — to prove this inclusion (used on p. 175), one should consider an arbitrary element ξ of $((SS)/x)/x$ and apply the inclusion $(\xi x)x \subseteq \xi(xx)$. Of course, as in the original proof one uses many times the equivalence

$$BX \cap A \neq \emptyset \Leftrightarrow X \cap A/B \neq \emptyset,$$

where A, B, X can be arbitrary subsets of K.[3]

A further reduction of the assumptions is possible at the cost of a quite small change that in fact even simplifies the proof. The change consists in using the set $(S/x)/x$ instead of the set $S/(xx)$. We shall formulate now a result obtainable thanks to the admissibility of this change, and we shall present its proof following Tagamlitzki's one as close as possible, making only the necessary changes (except for small differences in the denotations).

Theorem 2.2. *Let $(ab)b \subseteq ab$ for any a, b in K. Then for any two disjoint non-empty convex subsets A and B of K there is a half-space that contains A and does not meet B.*

Proof. Let A and B be disjoint non-empty convex subsets of K. By Zorn's Lemma, there is some maximal convex subset S of K containing A and not intersecting B. We set $T = K \setminus S$ for short.

Let x be an arbitrary element of K. We shall firstly prove that

$$(S/x)/x \subseteq S/x\,. \tag{2.1}$$

In fact, let $\xi \in (S/x)/x$. Then $\xi x \cap S/x \neq \emptyset$, and hence $(\xi x)x \cap S \neq \emptyset$. Making use of the inclusion $(\xi x)x \subseteq \xi x$, we conclude that $\xi x \cap S \neq \emptyset$, hence $\xi \in S/x$.

It is easy to see now that the set $S \cup S/x$ is convex. Indeed, we have

$$\begin{aligned}(S \cup S/x)(S \cup S/x) &= SS \cup S(S/x) \cup (S/x)S \cup (S/x)(S/x)\\ &= SS \cup S(S/x) \cup (S/x)(S/x) \subseteq SS \cup (SS)/x \cup ((S/x)S)/x\\ &\subseteq SS \cup (SS)/x \cup ((SS)/x)/x \subseteq S \cup S/x \cup (S/x)/x \subseteq S \cup S/x\end{aligned}$$

(the first two inclusions follow from the second associativity law, the convexity of S implies the inclusion next to the last, and (2.1) is applied to obtain the last one).

Let us consider now the particular case when $x \in B$. We shall show that S/x does not meet B in this case. In fact, if $S/x \cap B \neq \emptyset$, then $S \cap xB \neq \emptyset$, hence $S \cap BB \neq \emptyset$, and from here, by the convexity of B, the false conclusion $S \cap B \neq \emptyset$ follows. So S/x does not meet B and therefore the set $S \cup S/x$ also does not. By the convexity of $S \cup S/x$ and the maximality of S, we get the inclusion $S/x \subseteq S$. Thus we see that $S/x \cap T = \emptyset$, hence $x \notin S/T$. Since x can be any element of B, it follows that

$$S/T \cap B = \emptyset\,. \tag{2.2}$$

Consider now the convex set $S \cup S/x$, where x is an arbitrary element of T. By (2.2), this convex set does not meet B and hence, by the maximality of S, the inclusion $S/x \subseteq S$ holds. Since x can be an arbitrary element of T, we get the

[3] This equivalence has been observed by Ivan Prodanov about 1962, but in fact it is indicated earlier in [2] (cf. Theorem 5 of that paper).

inclusion $S/T \subseteq S$. Consequently, $S/T \cap T = \emptyset$, therefore $S \cap TT = \emptyset$, i.e. $TT \subseteq T$. So the convexity of T is established, and it remains to notice that S and T are non-empty because $A \subseteq S$ and $B \subseteq T$. □

The improved versions Theorem 2.1 and Theorem 2.2 of Tagamlitzki's Theorem 1 are not instances of the separation theorem from [1]. For any a, b in K let $[a, b]$ (*the convex closure* of the set $\{a, b\}$) be the intersection of all convex subsets of K containing both a and b as elements. In order the separation theorem from [1] to be applicable to the family $\mathcal{G}$ of the convex subsets of K, this family must satisfy the following condition: for any set X belonging to $\mathcal{G}$ and any a in K the union of all convex closures $[a, x]$, where $x \in X$, must belong to $\mathcal{G}$ too. We shall give now an example showing the existence of cases when this condition is not satisfied, but nevertheless the assumptions of Theorems 2.1 and 2.2 are fulfilled (of course, it is sufficient to check only the stronger assumptions — those of Theorem 2.1).

Example. Let K consist of five distinct elements p_1, p_2, p_3, p_4, p_5, and let the multiplication in K be defined by the condition that $x \in yz$ iff some of the three cases below is present:

(α) $x \in \{y, z\}$;
(β) $x = p_3$, $\{y, z\} = \{p_1, p_2\}$;
(γ) $x = p_5$, $\{y, z\} = \{p_3, p_4\}$.

The commutativity of the multiplication is obvious. To check the validity of the second associativity law, suppose a, b, c are elements of K, and x is an element of $a(b/c)$. We shall prove that x belongs to $(ab)/c$. We have $x \in ay$ for some y such that $b \in cy$, and we must show that $cx \cap ab \neq \emptyset$. If $x \in ay$ holds according to case (α), i.e. $x \in \{a, y\}$, then $a \in cx \cap ab$ in the case of $x = a$, and $b \in cx \cap ab$ in the case of $x = y$. The situation is similar if $b \in cy$ holds according to case (α). Now suppose that each of the statements $x \in ay$ and $b \in cy$ holds according to some of the cases (β), (γ). Since $\{a, y\} \cap \{c, y\} \neq \emptyset$, it is not possible that one of the both statements holds according to (β) and the other one holds according to (γ). Therefore $x = b$, hence the condition $cx \cap ab \neq \emptyset$ is satisfied again. We obviously have $bb = \{b\}$ for any b in K, therefore $a(bb) = ab$ for any a, b in K. We shall prove the inclusion $(ab)b \subseteq a(bb)$ by proving that $(ab)b \subseteq ab$. Suppose $x \in (ab)b$ for some a, b in K; we shall prove that $x \in ab$. We have $x \in yb$ for some $y \in ab$. But the cases of $x \in \{y, b\}$ or $y \in \{a, b\}$ are easy, and, on the other hand, it turns out to be not possible that each of the statements $x \in yb$ and $y \in ab$ holds according to some of the cases (β), (γ). So we have shown that all assumptions of Theorems 2.1 and 2.2 are satisfied in this example. Let us now consider the convex set $X = \{p_1, p_4\}$ and the union of all convex closures $[p_2, x]$, where $x \in X$. The union in question is $\{p_1, p_2, p_3\} \cup \{p_2, p_4\} = \{p_1, p_2, p_3, p_4\}$, and it is not convex due to $p_5 \in p_3p_4$.

Remark. Theorem 2.2 remains true if the inclusion $(ab)b \subseteq ab$ is replaced by the weaker one $(ab)b \subseteq ab \cup \{a, b\}$. To see this, it is sufficient to make the following changes in the proof:

- The sentence "Let x be an arbitrary element of K" must be replaced by "Let x be an arbitrary element of T".
- The inclusion (2.1) must become $(S/x)/x \subseteq S \cup S/x$.
- The third sentence after (2.1) must become "Making use of the inclusion $(\xi x)x \subseteq \xi x \cup \{\xi, x\}$, we conclude that $\xi x \cap S \neq \emptyset$ or $\xi \in S$, hence $\xi \in S \cup S/x$".

3. CONCLUDING REMARKS

We think it is quite possible that in the time of writing [3] Professor Tagamlitzki had already been aware of the possibility to prove a version of Theorem 1 in the absence of the first associativity law. In our opinion, he could have the following reasons not to mention this possibility in his paper:

- a lack of known interesting applications of such a generalization of the theorem;
- the fact that the rest of the paper anyway needs the first associativity law (Theorem 1 being mainly a tool for the considerations there);
- the lack of information about Ellis' separation theorem at that time.

There is, however, a chance that a generalization of this kind could be possibly applied in the future to some problems of interest, and also the other considerations from [3] perhaps could be generalized in some way for the case of absent first associativity law. If this happens, then the fact that Ellis' theorem does not completely cover the content of Tagamlitzki's result will turn out to be more essential than it could seem at the present moment.

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