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MODEL REPRESENTATIONS OF THE LIE ALGEBRA $[A_2, A_1] = iA_1$ OF LINEAR NON-SELFADJOINT OPERATORS

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Construction of functional models of Lie algebra $\{A_1, A_2\}$ ($[A_2, A_1] = iA_1$), one of which is dissipative, was realized earlier. The question of construction of model realizations for the given Lie algebra not containing dissipative operator remained open. This work is dedicated to the construction of model representation of the Lie algebra $\{A_1, A_2\}$ of linear non-selfadjoint operators not containing a dissipative operator which is generated by the commutation relation $[A_2, A_1] = iA_1$. In Paragraph 1 the preliminary information is stated, the definitions of colligation of Lie algebra and corresponding open system on Lie group of affine transformations of the line $M(1)$ are given. Paragraph 2 is dedicated to the construction of triangular model for the Lie algebra $[A_2, A_1] = iA_1$ in the case of finite dimension of the general space of non-hermicity of operator system $\{A_1, A_2\}$. In Paragraph 3 functional model of the Lie algebra $[A_2, A_1] = iA_1$ is presented, it is realized in L. de Branges spaces of whole functions. In the last paragraph of this paper, functional model of the Lie algebra $[A_2, A_1] = iA_1$ on Riemann surface is constructed.

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1. LIE GROUP OF AFFINE TRANSFORMATIONS OF LINE AND COLLIGATION OF LIE ALGEBRA

I. To study a Lie algebra of linear non-selfadjoint operators specified by the commutation relation $[A_2, A_1] = iA_1$, one has [4] to find such Lie group G, vector $\{\partial_1, \partial_2\}$ Lie algebra of which is such that

$$[\partial_2, \partial_1] = \partial_1.$$

Let $\mathbb{R}$ be the real line. Define $G = M(1)$ [7, 8] the group of transformations of $\mathbb{R}$ preserving the orientation. Associate with each $\xi \in \mathbb{R}$ number $\eta = y\xi + x$ ($y > 0$, $x \in \mathbb{R}$). Denote a group element by $g = g(x, y)$. If $\eta = y_1\xi + x_1$ and $\zeta = y_2\eta + x_2$ then

$$\zeta = y_1 y_2 \xi + x_1 y_2 + x_2.$$

Therefore the group operation on G is given by

$$g(x_2, y_2) \circ g(x_1, y_1) = g(x_1 y_2 + x_2, y_2 y_1). \tag{1.1}$$

Hence it follows that the elements $g(x, 1)$ form the subgroup in G, isomorphic to the additive group of real numbers $\mathbb{R}$.

$$g(x_2, 1) \circ g(x_1, 1) = g(x_1 + x_2, 1).$$

And the elements $g(0, y)$ form the subgroup in G equivalent to the multiplicative group of positive numbers in $\mathbb{R}_+$.

$$g(0, y_2) \circ g(0, y_1) = g(0, y_2 y_1).$$

The group G is isomorphic to the group of matrices of the second order given by

$$B_g = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}.$$

This fact immediately follows from the equality

$$B_{g_2} \cdot B_{g_1} = \begin{bmatrix} y_2 & x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 & x_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} y_2 y_1 & y_2 x_1 + x_2 \\ 0 & 1 \end{bmatrix} = B_{g_1 \circ g_2}.$$

Specify two subgroups in G,

$$G_x^1 = \{g(x, 1) \in G\}; \quad G_y^2 = \{g(y, 0) \in G\}; \tag{1.2}$$

as is stated above, they are isomorphic to $\mathbb{R}$ and $\mathbb{R}_+$, respectively. To specify a function $f(g)$ on the group $G = M(1)$, $f\colon G \to \mathbb{C}$, signifies that we define complex-valued function $f(x, y)$ in the upper half-plane $\mathbb{R} \times \mathbb{R}_+$. Calculate vector fields corresponding to the one-parametric semigroups (1.2) [8]. Let $g_t = (t, 1) \in G_x^1$ in (1.2). Then

$$F_t = f(g_t \circ g(x, y)) = f(g(ty + x, y)) = f(ty + x, y).$$

Therefore the derivative by t at unit, $e = e(0, 1) \in G$, of the given function equals

$$\left. \frac{d}{dt} F_t \right|_{t=0} = \partial_1 f$$

where $\partial_1 = y \dfrac{\partial}{\partial x}$. Similarly, consider the functions

$$\tilde{F}_t = f(g_t \circ g(x, y)) = f(x, ty)$$

where $\tilde{g}_t = (0, t) \in G_x^2$ in (1.2). Then

$$\frac{d}{dt}\tilde{F}_t\bigg|_{t_0=0} = \partial_2 f$$

where $\partial_2 = y\dfrac{\partial}{\partial y}$. Thus we construct the Lie algebra of vector fields $m(1)$ of the group $M(1)$ specified by the differential operators of the first order

$$\partial_1 = y\frac{\partial}{\partial x}, \quad \partial_2 = y\frac{\partial}{\partial y}. \tag{1.3}$$

It is easy to see that the Lie algebra $\{\partial_2, \partial_1\}$ is specified by the commutation relation

$$[\partial_2, \partial_1] = \partial_1. \tag{1.4}$$

It is well-known that the simply connected Lie group $M(1)$ is "uniquely" restored by the Lie algebra $m(1)$ of differential operators (1.3) [7, 8].

II. Consider in a Hilbert space H the Lie algebra of linear operators $\{A_1, A_2\}$ satisfying the relation

$$[A_2, A_1] = iA_1. \tag{1.5}$$

Note that A_1 and A_2 cannot be bounded simultaneously, since otherwise (1.5) implies

$$[A_2, A_1^n] = inA_1^n$$

which results in the inequality $2\|A_2\| \geq n$ $(\forall n \in \mathbb{Z}_t)$.

It seems natural to write relation (1.5) in the "integral form" similarly to the Weyl identity in Quantum Mechanics [4]. Let $Z_t(t_k) = \exp(it_k A_k)$ $k = 1, 2$. (1.5) implies

$$Z_1(t_1) A_2 = (A_2 + t_1 A_1) Z_1(t_1). \tag{1.6}$$

Indeed, it is easy to see that $f'(t_1) = iA_1 f(t_1)$ and $f(0) = 0$ where $f(t_1) = Z_1(t_1) A_2 - (A_2 + t_1 A_1) Z_1(t_1)$. Therefore it is obvious that

$$Z_1(t_1) Z_2(t_2) = \exp\{it_2(A_2 + t_1 A_1)\} Z_1(t_1). \tag{1.7}$$

III. Construct the colligation of Lie algebra for the given Lie algebra (1.5) of linear non-selfadjoint operators.

Definition 1.1. *Family*

$$\Delta = \Big(\{A_1, A_2\}; H; \varphi; E; \{\sigma_k\}_1^2; \{\gamma^-, \gamma^+\}\Big), \tag{1.8}$$

where φ*:* $H \to E$*,* σ_k*,* $\gamma^{\pm}$*:* $E \to E$ *(*$\sigma_k^* = \sigma_k$*,* $k = 1, 2$*), is said to be the* colligation *of the Lie algebra* (1.5)*, if*

$$\begin{aligned} &1) \quad [A_2, A_1] = iA_1; \\ &2) \quad 2\mathrm{Im}\,\langle A_k h, h\rangle = \langle \sigma_k \varphi h, \varphi h\rangle\,; \quad \forall h \in \vartheta\,(A_k)\,; \\ &3) \quad \sigma_1 \varphi A_2 - \sigma_2 \varphi A_1 = \gamma^+ \varphi; \\ &4) \quad \gamma^- = \gamma^+ + i\,(\sigma_2 \varphi \varphi^* \sigma_1 - \sigma_1 \varphi \varphi^* \sigma_2)\,. \end{aligned} \tag{1.9}$$

It is obvious that $\gamma^{\pm}$ are non-selfadjoint operators [4] and

$$\gamma^{\pm} - \left(\gamma^{\pm}\right)^* = -i\sigma_1. \tag{1.10}$$

Equations of the open system [2, 3, 4, 5] are given by

$$\begin{cases} i\partial_k h(x,y) + A_k h(x,y) = \varphi^* \sigma_k u(x,y) \quad (k = 1,2) \\ h(e) = h_0 \quad (k = 1,2); \quad (x,y) \in G; \\ v(x,y) = u(x,y) - i\varphi h(x,y). \end{cases} \tag{1.11}$$

Besides, ∂_k in (1.11) are equal to (1.11). It is not hard to show [2, 4, 5] that

$$\left\{\sigma_1 i\partial_2 - \sigma_2 i\partial_1 + \gamma^-\right\} u(x,y) = 0;$$

$$\left\{\sigma_1 i\partial_2 - \sigma_2 i\partial_1 + \gamma_s^+\right\} v(x,y) = 0.$$

2. TRIANGULAR MODEL OF LIE ALGEBRA

I. Consider the colligation Δ (1.8) corresponding to the Lie algebra of linear operators $\{A_1, A_2\}$ assuming that (1.9), (1.10) take place, besides, $\dim E = r < \infty$, operator $\sigma_1 = J$ is involution, and let $\sigma_2 = \sigma$. Define the Hilbert space $L^2_{r,l}(F_x)$ [1, 3] assuming that the measure dF_x is absolutely continuous, $dF_x = a_x dx$, $a_x \leq 0$, $\mathrm{tr} a_x \equiv 1$. Specify in this space the operator system

$$\left(\overset{\circ}{A}_1 f\right)_x = i \int_x^l f_t a_t J dt;$$

$$\left(\overset{\circ}{A}_2 f\right)_x = f'_x b_x + f_x J \gamma_x + i \int_x^l f_t a_t dt \sigma \tag{2.1}$$

$\left(f_x \in L^2_{r,l}(F_x)\right)$ where b_x, γ_x are some operator-functions in E specified on $[0, l]$. Linear span of continuously differentiable functions from $L^2_{2,l}(F_x)$ such that $f'_x b_x \in L^2_{2,l}(F_x)$ and $f_0 = f_l = 0$ is the domain $\mathcal{D}(A_2)$. Note that the structure of A_1 (2.2) coincides with the triangular model [1, 3] when the spectrum $\sigma(A_1) = 0$. Find the necessary and sufficient conditions on a_x, b_x, γ_x, J, σ for this operator system (2.1) to form the Lie algebra,

$$\left[\overset{\circ}{A}_2, \overset{\circ}{A}_1\right] = i\,\overset{\circ}{A}_1 \,. \tag{2.2}$$

It is easy to see that

$$\overset{\circ}{A}_2\overset{\circ}{A}_1\, f_x = -if_x a_x J b_x + i\int_x^l f_t a_t dt\gamma_x - \int_x^l \left(\int_t^l f_s a_s J ds\right) a_t\sigma dt.$$

Similarly,

$$\overset{\circ}{A}_1\overset{\circ}{A}_2\, f_x = i\int_x^l f_t' b_t a_t J dt + \int_x^l f_t J\gamma_t a_t J dt - \int_x^l \left(\int_t^l f_s a_s \sigma ds\right) a_t J dt =$$

$$= -if_x b_x a_x J - i\int_x^l f_t\left(b_t a_t\right)' J dt + i\int_x^l f_t J\gamma_t a_t J dt - \int_x^l \left(\int_t^l f_s a_s \sigma ds\right) a_t J dt$$

by virtue of $f_l = 0$. Suppose that

$$a_x J b_x = b_x a_x J. \tag{2.3}$$

Then

$$\Psi_x \overset{\text{def}}{=} \left(\left[\overset{\circ}{A}_2, \overset{\circ}{A}_1\right] - i\,\overset{\circ}{A}_1\right) f_x = i\int_x^l f_t a_t dt\gamma_x - \int_x^l \left(\int_t^l f_s a_s J ds\right) a_t\sigma dt -$$

$$-i\int_x^l f_t\left\{J\gamma_t a_t J - \left(b_t a_t\right)' J\right\} dt + \int_x^l \left(\int_x^l f_s a_s \sigma ds\right) a_t J dt + \int_x^l f_t a_t J dt.$$

Supposing that γ_x is continuously differentiable operator-function, calculate derivative of the function Ψ_x:

$$\Psi_x' = -if_x a_x\gamma_x + i\int_x^l f_t a_t dt\gamma_x' + \int_x^l f_t a_t dt J a_x\sigma +$$

$$+if_x\left\{J\gamma_x a_x J - \left(b_x a_x\right)' J\right\} - \int_x^l f_t a_t dt\sigma a_x J - f_x a_x J.$$

Hence it follows that $\Psi'_x = 0$, if

$$\begin{cases} i\gamma'_x = \sigma a_x J - J a_x \sigma; \\ a_x \gamma_x J = J \gamma_x a_x - (b_x a_x)' + i a_x. \end{cases} \tag{2.4}$$

Thus $\Psi_x \equiv 0$ since $\Psi_l = 0$.

Lemma 2.1. *Suppose that there exists a family* $\{a_x, \gamma_x, b_x, J, \sigma\}$ *such that* (2.3) *and* (2.4) *take place. Then the operator system* $\left\{ \overset{\circ}{A}_1, \overset{\circ}{A}_2 \right\}$ (2.1) *satisfies the commutation relation* (2.2).

II. In order to include the operator system $\left\{ \overset{\circ}{A}_1, \overset{\circ}{A}_2 \right\}$ (2.1) in the colligation Δ (1.8), it is necessary to verify that the colligation relations (1.9) are true. It is easy [1, 3] to show that $\overset{\circ}{A}_1 - \overset{\circ}{A}{}_1^* = i \overset{\circ}{\varphi}{}^* J \overset{\circ}{\varphi}$ where the operator $\overset{\circ}{\varphi}$: $L^2_{2,l}(F_x) \to E$ is given by

$$\overset{\circ}{\varphi} f_x = \int_0^l f_t dt. \tag{2.5}$$

Calculate $2\mathrm{Im} \left\langle \overset{\circ}{A}_2 f, f \right\rangle$ where $f \in \mathcal{D}\left(\overset{\circ}{A}_2 \right)$. Then

$$2\mathrm{Im} \left\langle \overset{\circ}{A}_2 f, f \right\rangle = \frac{1}{i} \int_0^l \left(f'_x b_x + f_x J \gamma_x + i \int_x^l f_t a_t \sigma \right) dt a_x f^*_x dx -$$

$$-\frac{1}{i} \int_0^l dx f_x a_x \left(b^*_x (f^*_x)' + \gamma^*_x J f_x - i \int_x^l \sigma a_t f^*_t dt \right) =$$

$$= \frac{1}{i} \int_0^l \left(f'_x b_x a_x f^*_x - f_x a_x b^*_x (f^*_x)' + f_x \left\{ J \gamma_x a_x - a_x \gamma^*_x J \right\} f^*_x \right) dx +$$

$$+ \int_0^l \left(\int_x^l f_t a_t \sigma dt a_x f^*_x + f_x a_x \int_x^l \sigma a_t f^*_t \right) dt.$$

It is easy to see that the second integral after the change of order of integration equals

$$\int_0^l f_t a_t dt \sigma \int_0^l a_t f_t^* dt = \langle \sigma\varphi f, \varphi f\rangle_E.$$

Therefore in order to the colligation relation 2) for $\overset{\circ}{A}_2$ (2.1) take place, it is necessary to ascertain under which conditions the first integral vanishes. The integrand of this integral equals

$$\Phi_x \overset{\text{def}}{=} f_x' b_x a_x f_x^* - f_x a_x b_x^* \left(f_x^*\right)' + f_x \left\{J\gamma_x a_x - a_x \gamma_x^* J\right\} f_x^* =$$

$$= f_x' a_x J b_x J f_x^* - f_x a_x b_x^* \left(f_x^*\right)' + f_x \left\{a_x \gamma_x J + \left(b_x a_x\right)' - i a_x - a_x \gamma_x^* J\right\} f_x^*$$

in virtue of (2.3) and the second equation in (2.4). It is obvious that the solution γ_x of equation (2.4) is given by

$$\gamma_x = \gamma_0 + i \int_0^x \left(J a_t \sigma - \sigma a_t J\right) dt. \tag{2.6}$$

Choose the initial condition $\gamma_0 = \left(\gamma^+\right)^*$. Since the second summand in (2.6) is a selfadjoint operator, then taking into account $\gamma^+ - \left(\gamma^+\right)^* = -iJ$ (1.10) we obtain

$$\gamma_x - \gamma_x^* = \gamma_0 - \gamma_0^* = \left(\gamma^+\right)^* - \gamma^+ = iJ. \tag{2.7}$$

So $\gamma_x^* = \gamma_x - iJ$. Substituting this expression in the formula for Φ_x, we obtain

$$\Phi_x = f_x' a_x J b_x J f_x^* - f_x a_x b_x^* \left(f_x^*\right)' + f_x \left\{a_x \gamma_x J + \left(b_x a_x\right)' - i a_x - ax\left(\gamma_x - iJ\right) J\right\} f_x =$$

$$= f_x' a_x J b_x J f_x^* + f_x a_x \left(-b_x^*\right) f_x^{*\prime} + f_x \left(a_x J b_x J\right)' f_x^*$$

in virtue of (2.3). Let

$$b_x^* = -J b_x J. \tag{2.8}$$

Then $\Phi_x = \left\{f_x a_x J b_x J f_x^*\right\}'$, and hence

$$\int_0^l \Phi_x dx = 0$$

since $f_0 = f_l$ as $f \in \mathcal{D}\left(\overset{\circ}{A}_2\right)$.

Lemma 2.2. *Let the family* $\{a_x, \gamma_x, b_x, J, \sigma\}$ *be such that the relations* (2.3), (2.4) *are true and, moreover,* γ_x*, the solution of the first equation in* (2.4)*, satisfies* *the initial condition* $\gamma_0 = (\gamma^+)^*$, *besides,* $\gamma^+ - (\gamma^+)^* = -iJ$ (1.9). *Then, if* (2.8) *takes place,* $\forall f \in \mathcal{D}\left(\overset{\circ}{A}_2\right)$ *the colligation relation*

$$2\mathrm{Im}\left\langle \overset{\circ}{A}_2 f, f\right\rangle = \left\langle = \sigma \overset{\circ}{\varphi} f, \overset{\circ}{\varphi} f\right\rangle$$

where f *is given by* (2.5).

Verify that the colligation condition 3) (1.9) also is true. Really, find the function Ψ_x,

$$\Psi_x \overset{\text{def}}{=} \left(J \overset{\circ}{\varphi}\overset{\circ}{A}_2 - \sigma \overset{\circ}{\varphi}\overset{\circ}{A}_1 - \gamma^+ \overset{\circ}{\varphi}\right) f_x = \int_0^l \left(f'_x b_x + f_x J\gamma_x + i\int_x^l f_t a_t dt\sigma\right) a_x dx J-$$

$$-\int_0^l i\int_x^l f_t a_t J dt a_x \sigma - \int_0^l f_x a_x dx \gamma^+.$$

Integrating by parts and changing the order of integration, we obtain

$$\Psi_x = \int_0^l dx \left\{-f_x\left(b_x a_x\right)' J + f_x J\gamma_x a_x J + f_x a_x \left[i\int_0^x \left(\sigma a_t J - J a_t \sigma\right) dt\right] - f_x a_x \gamma^+\right\}.$$

Now taking into account (2.6) and the second equality in (2.4), we have

$$\Psi_x = \int_0^l \{f_x a_x \gamma_x - f_x J\gamma_x a_x J - i a_x J + f_x J \gamma_x a_x J + f_x a_x\left(\gamma_0 - \gamma_x\right) - f_x a_x \gamma^+\} dx =$$

$$= \int_0^l f_x a_x dx \left(\gamma_0 - \gamma^+ - iJ\right) = 0$$

in virtue of $\gamma_0 = (\gamma^+)^*$ and condition (1.10). So $\Psi_x \equiv 0$ and relation 3) (1.9) is proved. If one takes into account (2.5), then (2.6) yields

$$\gamma_l = \gamma_0 + i\int_0^l \left(J a_t \sigma - \sigma a_t J\right) dt = \gamma_0 + i\left(J \overset{\circ}{\varphi}\overset{\circ}{\varphi}^* \sigma - \sigma \overset{\circ}{\varphi}\overset{\circ}{\varphi}^* J\right),$$

therefore

$$\gamma_l^* = \gamma^+ + i\left(J \overset{\circ}{\varphi}\overset{\circ}{\varphi}^* \sigma - \sigma \overset{\circ}{\varphi}\overset{\circ}{\varphi}^* J\right).$$

And we obtain the colligation relation 4) (1.9) where $\gamma_l^* = \gamma^-$.

Theorem 2.1. *Suppose that an operator family* $\{a_x, \gamma_x, b_x, J, \sigma\}$ *is such that*

$$\begin{aligned} &1) \quad a_x J b_x = b_x a_x J; \\ &2) \quad b_x^* = -J b_x J; \\ &3) \quad i\gamma_x^- = \sigma a_x J - J a_x \sigma; \quad \gamma_0 = (\gamma^+)^*; \\ &4) \quad (b_x a_x)' = J\gamma_x a_x - a_x \gamma_x J + i a_x; \end{aligned} \tag{2.9}$$

besides, $\gamma^+ - (\gamma^+)^* = -iJ$. *Then the set*

$$\overset{\circ}{\Delta} = \left(\left\{ \overset{\circ}{A}_1, \overset{\circ}{A}_2 \right\}; L^2_{2,l}(F_x); \overset{\circ}{\varphi}; E; \{J, \sigma_k\}; \{\gamma^-, \gamma^+\} \right) \tag{2.10}$$

is the colligation of Lie algebra (1.8)–(1.9) *where* $\overset{\circ}{A}_1$, $\overset{\circ}{A}_2$ *are given by* (2.1), *the operator* $\overset{\circ}{\varphi}$ *equals* (2.5) *and* $\gamma^- = \gamma_l^*$.

Now use the Theorem on unitary equivalence [1, 3, 4].

Theorem 2.2. *Let* Δ *be a simple colligation* (1.8), (1.9). *If the spectrum of operator* A_1 *is concentrated at zero and the characteristic function* $S_1(\lambda) = I - i\varphi(A_1 - \lambda I)^{-1} \varphi^* J$ *is given by*

$$S_1(\lambda) = \overset{\leftarrow}{\int_0^l} \exp \frac{iJdF_t}{\lambda},$$

besides, $dF_x = a_x dx$ *and* a_x *is such that for the family* $\{a_x, \gamma_x, b_x, J, \sigma\}$ ($\sigma_1 = J$ *is involution and* $\sigma = \sigma^*$) *the equation system* 1) – 4) (2.9) *is solvable. Then the colligation* Δ *is unitary equivalent to the simple part of colligation* $\overset{\circ}{\Delta}$ (2.10).

Observation 2.1. 1), 2) (2.9) *imply*

$$a_x b_x^* + b_x a_x = 0, \tag{2.11}$$

$\forall x \in [0, l]$.

3. FUNCTIONAL MODEL IN L. DE BRANGES SPACE

This section is concerned with the construction of functional model of the studied in this paper Lie algebra in L. de Branges space [3]. Consider the triangular model of the colligation of Lie algebra (2.10) assuming that $r = 2$ and J is given by

$J = j_N$ (2.1). Under the action of L. de Branges transformation [3], the operator $\overset{\circ}{A}_1$ (2.1) changes into the shift operator since

$$\mathcal{B}_L \left(\overset{\circ}{A}_1 \ f_t \right) = \frac{1}{\pi} \int_0^l \left\{ i \int_t^l f_s dF_s J \right\} dF_t L_t^* \left(\bar{z} \right) =$$

$$= \frac{1}{\pi} \int_0^l f_t dF_t \left\{ - \frac{L_t^* \left(\bar{z} \right) - L_t^*(0)}{z} \right\}$$

and thus

$$\mathcal{B}_L \left(\overset{\circ}{A}_1 \ f_1 \right) = \frac{F(z) - F(0)}{z} \tag{3.1}$$

where $F(z) \in \mathcal{B}_L \left(f_t \right)$. In order to find $\mathcal{B}_L \left(\overset{\circ}{A}_1 \ f_t \right)$, first of all note that

$$L_t(z) = \left(I - z \overset{\circ}{A_1^*} \right)^{-1} \overset{\circ}{\varphi^*} (1, 0). \tag{3.2}$$

Since

$$\mathcal{B}_L \left(\overset{\circ}{A}_2 \ f_t \right) = \left\langle \overset{\circ}{A}_2 \ f_t, L_t \left(\bar{z} \right) \right\rangle = \left\langle f_t, \overset{\circ}{A_2^*} \ L_t \left(\bar{z} \right) \right\rangle,$$

then, taking into account (3.2), we have to calculate the expression

$$\overset{\circ}{A_2^*} \left(I - z \overset{\circ}{A_1^*} \right)^{-1} \overset{\circ}{\varphi^*} (1, 0). \tag{3.3}$$

(2.2) implies

$$\overset{\circ}{A}_2 \left(I - z \overset{\circ}{A}_1 \right) - \left(I - z \overset{\circ}{A}_1 \right)^{-1} = -iz \overset{\circ}{A}_1,$$

therefore

$$\left(I - z \overset{\circ}{A}_1 \right)^{-1} \overset{\circ}{A}_2 - \overset{\circ}{A}_2 \left(I - z \overset{\circ}{A}_1 \right)^{-1} = -iz \overset{\circ}{A}_1 \left(I - z \overset{\circ}{A}_1 \right)^{-2}.$$

Thus

$$\overset{\circ}{A_2^*} \left(I - \bar{z} \overset{\circ}{A_1^*} \right)^{-1} - \left(I - z \overset{\circ}{A_1^*} \right) \overset{\circ}{A_2^*} = -i\bar{z} \overset{\circ}{A_1^*} \left(I - \bar{z} \overset{\circ}{A_1^*} \right)^{-2}. \tag{3.4}$$

Using (3.4), we obtain

$$\overset{\circ}{A_2^*} \left(I - \bar{z} \overset{\circ}{A_1^*} \right)^{-1} \overset{\circ}{\varphi^*} = \left(I - \bar{z} \overset{\circ}{A_1^*} \right)^{-1} \overset{\circ}{A_2^*} \overset{\circ}{\varphi^*} + i\bar{z} \overset{\circ}{A_1^*} \left(I - \bar{z} \overset{\circ}{A_1^*} \right)^{-2} \overset{\circ}{\varphi^*}.$$

The colligation relation $J \overset{\circ}{\varphi}\overset{\circ}{A}_2 = \sigma \overset{\circ}{\varphi}\overset{\circ}{A}_1 + \gamma^+ \overset{\circ}{\varphi}$ yields

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* = \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* \overset{\circ}{\varphi}{}^* \sigma J + \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* \left(\gamma^+\right)^* J +$$

$$+ i\bar{z} \overset{\circ}{A}{}_1^* \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{\varphi}{}^* .$$

Now taking into account $\left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{A}{}_1^* = \dfrac{1}{z}\left\{\left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} - I\right\}$, we finally obtain

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* = \frac{1}{z}\left\{\left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* \sigma J - \overset{\circ}{\varphi}{}^* \sigma J\right\} +$$

$$+ \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* \left(\gamma^+\right)^* J + i\bar{z} \overset{\circ}{A}{}_1^* \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{\varphi}{}^*$$

Thus expression (3.3) has the form

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi} (1,0) = \frac{1}{z}\left\{\left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* - \overset{\circ}{\varphi}{}^*\right\} \sigma J(1,0) +$$

$$+ \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* \left(\gamma^+\right)^* J(1,0) + i\bar{z} \overset{\circ}{A}{}_1^* \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{\varphi}{}^* (1,0). \qquad (3.5)$$

Expand the vectors $\sigma J(1,0)$ and $\left(\gamma^+\right)^* J(1,0)$ by the basis $(1,0)$ and $(0,1)$ in E^2.

$$\sigma J(1,0) = \bar{\alpha}(1,0) + \bar{\beta}(0,1);$$

$$\left(\gamma^+\right)^* J(1,0) = \bar{\mu}(1,0) + \bar{\nu}(0,1) \qquad (3.6)$$

where

$$\bar{\alpha} = (1,0)\sigma J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\beta} = (1,0)\sigma J \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{\mu} = (1,0) \left(\gamma^+\right)^* J \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \bar{\nu} = (1,0) \left(\gamma^+\right)^* J \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (3.7)$$

As a result, we obtain that expression (3.5) is written in the following form:

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* (1,0) = \bar{\alpha}\frac{1}{z}\left\{\left(I - \bar{z} \overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* - \overset{\circ}{\varphi}{}^*\right\} (1,0) +$$

$$+\bar{\beta}\frac{1}{z}\left\{\left(I-\bar{z}\overset{\circ}{A_1^*}\right)^{-1}\overset{\circ}{\varphi^*}-\overset{\circ}{\varphi^*}\right\}(0,1)+\bar{\mu}\left(I-\bar{z}\overset{\circ}{A_1^*}\right)^{-1}\overset{\circ}{\varphi^*}(1,0)+$$

$$+\bar{\nu}\left(I-\bar{z}\overset{\circ}{A_1^*}\right)^{-1}\overset{\circ}{\varphi^*}(0,1)+i\bar{z}\overset{\circ}{A_1^*}\left(I-\bar{z}\overset{\circ}{A_1^*}\right)^{-2}\overset{\circ}{\varphi^*}(1,0). \tag{3.8}$$

Along with the integral equation for $L_x(z)$

$$L_x(z)+iz\int_0^x L_t(z)dF_tJ=(1,0), \tag{3.9}$$

consider [3] the integral equation for $N_x(z)$

$$N_x(z)+iz\int_0^x N_t(z)dF_tJ=(0,1). \tag{3.10}$$

So we can rewrite expression (3.8) as

$$\overset{\circ}{A_2^*}L_t(\bar{z})=\bar{\alpha}\frac{L_t(\bar{z})-L_t(0)}{\bar{z}}+\bar{\beta}\frac{N_t(\bar{z})-N_t(0)}{\bar{z}}+\bar{\mu}L_t(\bar{z})+\bar{\nu}N_t(\bar{z})-i\bar{z}\frac{d}{dz}L_t(\bar{z}). \tag{3.11}$$

By the vector-row $N_x(z)=[C_x(z);D_x(z)]$, similar to [3], construct the L. de Branges space $\mathcal{B}(C,D)$ and specify the L. de Branges transform from $L^2_{2,l}(F_x)$ on $\mathcal{B}(C,D)$ by the formula

$$G(z)\overset{\text{def}}{=}\mathcal{B}_N(f_t)=\frac{1}{\pi}\int_0^l f_t dF_t N_t^*(\bar{z}). \tag{3.12}$$

A function $G(z)\in\mathcal{B}(C,D)$ is said to be the dual to $F(z)\in\mathcal{B}(A,B)$ if

$$F(z)=\mathcal{B}_L(f_t),\quad G(z)=\mathcal{B}_N(f_t). \tag{3.13}$$

Using the notation (3.11) and (3.13), we obtain

$$\mathcal{B}_L\left(\overset{\circ}{A_2}f_t\right)=\alpha\frac{F(z)-F(0)}{z}+\beta\frac{G(z)-G(0)}{z}+\mu F(z)+\nu G(z)-iz\frac{d}{dz}F(z). \tag{3.14}$$

Thus the Lie algebra (2.2) of linear operators $\left\{\overset{\circ}{A_1},\overset{\circ}{A_2}\right\}$ (2.1) after the L. de Branges transform $\mathcal{B}_L$ changes into the following operator system

$$\tilde{A}_1F(z)=\frac{F(z)-F(0)}{z};$$

$$\tilde{A}_2 F(z) = \frac{\alpha F(z) + \beta G(z) - \alpha F(0) - \beta G(0)}{z} +$$

$$+\mu F(z) + \nu G(z) - iz\frac{d}{dz}F(z) \tag{3.15}$$

where the numbers α, β, μ, ν are given by (3.7) and the functions $F(z)$ and $G(z)$ are equal to(3.13).

Observation 3.1. *The dual function* $G(z)$ (3.13) *does not necessarily belong to the space* $\mathcal{B}(A, B)$*, nevertheless, under such selection of* α, β, μ, ν (3.7)*, the expressions*

$$\mu F(z) + \nu G(z); \qquad \frac{\alpha F(z) + \beta G(z) - \alpha F(0) - \beta G(0)}{z}$$

already belong to $\mathcal{B}(A, B)$*. Note that the numbers* α, β, μ, ν *do not depend on* $F(z)$*.*

Specify now the operator $\tilde{\varphi}$ from $\mathcal{B}(A, B)$ into E^2 using the formula

$$\tilde{\varphi} F(z) = \langle F(z), e_1(z)\rangle (1, 0) + \langle F(z), e_2(z)\rangle (0, 1) \tag{3.16}$$

where

$$\hat{e}_1(z) = \frac{B_l^*(\bar{z})}{z}; \quad \hat{e}_2(z) = \frac{1 - A_l^*(\bar{z})}{z}. \tag{3.17}$$

Theorem 3.1. *Let* Δ *be the simple colligation of Lie algebra* (1.8), (1.9)*, besides, the spectrum of operator* A *is concentrated at zero and the characteristic function* $S_1(\lambda) = I - i\varphi (A_1 - \lambda I)^{-1} \varphi^* J$ *is given by*

$$S_1(\lambda) = \int_0^l \exp\left\{\frac{iJdF_t}{\lambda}\right\},$$

besides, the measure dF_x *is absolutely continuous,* $dF_x = a_x dx$, $a_x \geq 0$, a_x *is a matrix-function in* E^2 *and* J *is given by (2.1) [3]. And, moreover, a selfadjoint operator* σ *and operators* $\gamma^{\pm}$ *are given in* E^2 *such that* $\gamma^{\pm} - (\gamma^{\pm})^* = iJ$*. Then the colligation* Δ (1.8) *is unitary equivalent to the functional model*

$$\tilde{\Delta} = \left(\left\{\tilde{A}_1, \tilde{A}_2\right\}; \mathcal{B}(A, B); \tilde{\varphi}, E^2, \{J, \sigma\}; \{\gamma^+, \gamma^-\}\right) \tag{3.18}$$

where $\tilde{A}_1$, $\tilde{A}_2$ *are given by* (3.15)*; the operator* $\tilde{\varphi}$ *equals* (3.16)*; the numbers* α, β, μ, $\nu \in \mathbb{C}$ *are given by the formulas* (3.7)*;* $G(z)$ *is the dual function of* $F(z)$*, and, finally,* $\{e_k(z)\}_1^2$ *are given by* (3.17)*.*

4. FUNCTIONAL MODELS ON RIEMANN SURFACE

I. Let $r = 2n$ be even, and let a_x be given by

$$a_x = I_n \times \hat{a}_x \tag{4.1}$$

where I_x is the unit operator in E^n and $\hat{a}_x$ is such non-negative matrix (2×2) that $\mathrm{tr}\hat{a}_x = n^{-1}$. It is obvious that the Hilbert space $L^2_{2n,l}(F_x)$ [1, 3] is formed by the vector-functions $f(x) = (f_1(x), \ldots, f_n(x))$ such that

$$\int_0^l f_k(x)\hat{a}_x f_k^*(x)dx < \infty$$

for all k $(1 \geq k \geq n)$ where $f_k(x) \in E^2$ for every $x \in [0, l]$.

Suppose that the operators $\sigma_1 = J$, $\sigma_1 = \sigma$, $\gamma^{\pm}$ are given by

$$\sigma_1 = J = I_n \otimes J_N; \quad \sigma = \tilde{\sigma} \otimes J_N; \quad \gamma^{\pm} = \tilde{\gamma} \otimes J_N \tag{4.2}$$

where $\tilde{\sigma}$ is a selfadjoint operator in E^n and $\tilde{\gamma}$ is such operator in E^n that

$$\tilde{\gamma} - (\tilde{\gamma})^* = -iI_n. \tag{4.3}$$

Realize the L. de Branges transform $\mathcal{B}_L$ [3] of each component $f_k(x) \in L^2_{2,l}(\hat{a}_x dx)$ of the vector-function $f(x)$ from $L^2_{2n,l}(F_x)$ assuming that a_x is given by (4.1),

$$F_k(x) \stackrel{\text{def}}{=} \mathcal{B}_L(f_k) = \frac{1}{\pi}\int_0^l f_k(x)\hat{a}_x L_x^*(\bar{z})\,dx \tag{4.4}$$

where $L_x(z)$ is the solution of the integral equation (3.9) by the measure $\hat{a}_x dx$.

As a result, we obtain the Hilbert space $\mathcal{B}^n(A, B) = E^n \otimes \mathcal{B}(A, B)$ which is formed by the vector-functions $F(z) = (F_1(z), \ldots, F_n(z))$,

$$\mathcal{B}^n(A, B) = \{F(z) = (F_1(z), \ldots, F_n(z)) : F_k(z) \in \mathcal{B}(A, B)\,(1 \leq k \leq n)\}, \tag{4.5}$$

besides, the scalar product in $\mathcal{B}^n(A, B)$ is given by

$$\langle F(z), G(z)\rangle_{\mathcal{B}^n(A,B)} = \sum_{k=1}^n \langle F_k(z), G_k(z)\rangle_{\mathcal{B}(A,B)}. \tag{4.6}$$

Taking into account the form of a_x (4.1) and J (4.2), we obtain that the L. de Branges transform $\mathcal{B}_L$ [3] translates the triangular model $\overset{\circ}{A}_1$ (2.1) into the shift operator

$$\left(\tilde{A}_1 F\right) = \frac{1}{z}(F(z) - F(0)); \quad \forall F(z) \in \mathcal{B}^N(A, B). \tag{4.7}$$

To obtain the model representation $\overset{\circ}{A}_2$ (2.1), use the formula

$$\overset{\circ}{A}{}_2^* \left(I - \bar{z}\,\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^* = \frac{1}{z}\left\{\left(I - \bar{z}\,\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^*\,\sigma J\right\} +$$

$$+\left(I - \bar{z}\,\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^*\left(\gamma^+\right)^* J + i\bar{z}\,\overset{\circ}{A}{}_1^*\left(I - \bar{z}\,\overset{\circ}{A}{}_1^*\right)^{-2} \overset{\circ}{\varphi}{}^* \tag{4.8}$$

and the fact that $L_x^*(\bar{z}) = \left(I - \bar{z}\,\overset{\circ}{A}{}_1^*\right)^{-1} \overset{\circ}{\varphi}{}^*(1,0)$. Taking into account the concrete form of the operators J, σ, γ^+ (4.2), we obtain

$$\sigma J = \tilde{\sigma} \otimes I_2, \quad \left(\gamma^+\right)^* J = \tilde{\gamma}^* \otimes I_2. \tag{4.9}$$

Therefore, after the L. de Branges transform (2.24), the operator $\overset{\circ}{A}_2$ (2.1) is given by

$$\left(\tilde{A}_2 F\right)(z) = \frac{1}{z}(F(z) - F(0))\tilde{\sigma} + F(z)\tilde{\gamma} - iz\frac{d}{dz}F(z). \tag{4.10}$$

Thus

$$\tilde{A}_2 F(z) = \frac{1}{z}\left\{F(z)\left(\tilde{\sigma} + z\tilde{\gamma}\right) - F(z)\left.\left(\tilde{\sigma} + z\tilde{\gamma}\right)\right|_0\right\} + iz\frac{d}{dz}F(z) \tag{4.11}$$

where $F(z)\left.\left(\tilde{\sigma} + z\tilde{\gamma}\right)\right|_0 = F(0)\sigma$.

Now define the colligation of Lie algebra

$$\tilde{\Delta} = \left(\left\{\tilde{A}_1, \tilde{A}_2\right\}; \mathcal{B}^n(A,B); \tilde{\varphi}, E^{2n}; J, \sigma, \gamma^+, \gamma^-\right), \tag{4.12}$$

besides, J, σ, γ^+ are given by (4.2), $\gamma^- = \gamma^+$, and the operator $\tilde{\varphi}$ on each component of $F_k(z)$ acts in a standard way [1, 3].

Theorem 4.1. *Let* Δ (1.8) *be the simple colligation of Lie algebra such that* $\dim E = 2n$, $\sigma_1 = J$, $\sigma_1 = \sigma$, γ^+ *are given by* (4.2), *spectrum of the operator* A_1 *lies at zero, and the characteristic function of operator* A_1 *is such that the measure* dF_x *in multiplicative representation of* $S_1(\lambda)$ *(see Theorem* 3.1*) is absolutely continuous,* $dF_x = a_x dx$ *and* a_x *equals* (4.1). *Then the colligation* Δ *is unitarily equivalent to the simple part of the functional model* $\tilde{\Delta}$ (4.12) *where the operators* $\tilde{A}_1$ *and* $\tilde{A}_2$ *are given by the formulas* (4.7) *and* (4.11), *respectively.*

II. Consider the linear bundle

$$\tilde{\sigma} + z\tilde{\gamma} = \sigma + z\tilde{\gamma}_R - \frac{i}{2}zI_n \tag{4.13}$$

in view of (4.3) where $\tilde{\gamma}_R = \tilde{\gamma}_R^* = \frac{1}{2}\left(\tilde{\gamma} + \tilde{\gamma}^*\right)$.

Denote by $h(z,w)$ the eigenvectors of selfadjoint (when $z \in \mathbb{R}$) bundle $\tilde{\sigma} + z\tilde{\gamma}_R$,

$$h(P)\left(\tilde{\sigma} + z\tilde{\gamma}_R\right) = wh(P) \tag{4.14}$$

where $P = (z,w)$ belongs to the algebraic curve

$$\mathbb{Q} = \left\{P = (z,w) \in \mathbb{C}^2 : \mathbb{Q}(z,w) = 0\right\} \tag{4.15}$$

specified by the polynomial

$$\mathbb{Q}(z,w) = \det\left(\tilde{\sigma} + z\tilde{\gamma}_R - wT_n\right). \tag{4.16}$$

Suppose that the curve $\mathbb{Q}$ (4.15) is nonsingular [4, 9], then $z = z(P)$ and $w = w(P)$ are "l-valued" and, respectively, "n-valued" functions on $\mathbb{Q}$ ($l = \operatorname{rank}\tilde{\gamma}_R$). We normalize the rational function $h(P)$ (4.14) using the condition $h_n(P) = 1$ where $h_n(P)$ is the nth component of the vector $h(P)$.

It is easy to see [4] that the quantity of poles, taking into account the multiplicity, of vector-function $h(P)$, equals $N = g + n - 1$ where g is the type of Riemann surface $\mathbb{Q}$ (4.15). Specify on $\mathbb{Q}$ analogues of halfplanes $\mathbb{C}_\pm$ and $\mathbb{R}$,

$$\mathbb{Q}_\pm = \{P = (z,w) \in \mathbb{Q} : \pm\operatorname{Im} z(P) > 0\}; \quad \mathbb{Q}^0 = \partial\mathbb{Q}_\pm. \tag{4.17}$$

Expand every function $F(z) \in \mathcal{B}^n(A,B)$ by the basis $h\left(P_k\right)$ ($z \in \mathbb{R}$),

$$F(z) = \sum_{k=1}^{n} g\left(P_k\right)\left\|h\left(P_k\right)\right\|_{E^n}^{-2} h\left(P_k\right)$$

where $P_k = \left(z, w^k(z)\right) \in \mathbb{Q}$ and $w^k(z)$ are different roots of the polynomial $\mathbb{Q}(z,w) = 0$ (4.16); $g\left(P_k\right) = \left\langle F(z), h\left(P_k\right)\right\rangle_{E^n}$ ($1 \le k \le n$). It is obvious that $w^k(P)$, along with $h\left(P_k\right)$, $g\left(P_k\right)$, represents branches of "n-valued" algebraic functions $w(P)$, $h(P)$, $g(P)$. Therefore the last equality signifies that

$$F(P) = F(z(P)) = g(P)\|h(P)\|_{E^n}^{-2} h(P). \tag{4.18}$$

And since the basis $h(P)$ in E^n is constant, the vector-function $F(P)$ is defined by the scalar component $g(P)$. The function $g(P)$ is a meromorphic function on $\mathbb{Q}$ (4.15), the poles of which may lay only in the poles of $h(P)$ (4.14), and their joint multiplicity could not exceed $N = g + n - 1$.

Define [3] the L. de Branges space $\mathcal{B}_{\mathbb{Q}}(A,B,h)$ on the Riemann surface $\mathbb{Q}$ (4.15). It is easy to see that the operator $\tilde{A}_1$ (4.7) in L. de Branges space $\mathcal{B}_{\mathbb{Q}}(A,B,h)$ acts in the following way:

$$\left(\tilde{A}_1 g\right)(P) = \frac{g(P) - \psi\left(P,P_0\right) g\left(P_0\right)}{z(P) - z\left(P_0\right)} \tag{4.19}$$

where $\psi\left(P,P_0\right)$ is given by

$$\psi\left(P,P_0\right) = \left\langle h\left(P_0\right), h(P)\right\rangle_{E^n} \|h(P)\|_{E^n}^{-2}, \tag{4.20}$$

besides, $P_0 = (0, w) \in \mathbb{Q}$.

Now study how the operator $\tilde{A}_2$ (4.11) acts in the space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$. (4.11), (4.13) imply

$$\left(\tilde{A}_2 F\right)(P) = \frac{1}{z(P) - z(P_0)} \left\{ g(P) \left[w(P) + \frac{i}{2} z(P) \right] \cdot \|h(P)\|_E^{-2} h(P) - \right.$$

$$\left. -g(P_0) \left[w(P_0) + \frac{i}{2} z(P_0) \right] \cdot \|h(P_0)\|_{E^n}^{-2} h(P_0) \right\} -$$

$$-iz(P) \left\{ \frac{d}{dz} g(P) \cdot \|h(P)\|_{E^n}^{-2} h(P) - 2g(P) \cdot \|h(P)\|_{E^n}^{-3} \frac{d}{dz} \|h(P)\|_{E^n} h(P) + \right.$$

$$\left. + g(P) \cdot \|h(P)\|_{E^n}^{-2} \frac{d}{dz} h(P) \right\}.$$

Therefore we arrive at the following structure of the operator $\tilde{A}_2$ in L. de Branges space $\mathcal{B}_{\mathbb{Q}}(A, B, h)$:

$$\left(\tilde{A}_2 g\right)(P) = \frac{1}{z(P) - z(P_0)} \left\{ g(P) \left[w(P) + \frac{i}{2} z(P) \right] - \right.$$

$$\left. -\psi(P, P_0)\, g(P_0) \left[w(P_0) + \frac{i}{2} z(P_0) \right] \right\} - iz(P) \frac{d}{dz} g(P) - iz(P) b(P) g(P) \quad (4.21)$$

where the function $b(P)$ equals

$$b(P) = \left\langle \frac{d}{dz} h(P), h(P) \right\rangle_{E^n} \|h(P)\|_{E^n}^{-4} - 2\|h(P)\|_{E^n}^{-3} \frac{d}{dz} \|h(P)\|_{E^n}. \quad (4.22)$$

Now construct the colligation of Lie algebra

$$\tilde{\Delta} = \left(\left\{ \tilde{A}_1, \tilde{A}_2 \right\}; \mathcal{B}_{\mathbb{Q}}(A, B, h); \tilde{\varphi}, E^{2n}; J, \sigma, \gamma^+, \gamma^- \right) \quad (4.23)$$

where the operators $\tilde{A}_1$ and $\tilde{A}_2$ are given by (4.19), (4.21); the functions $\psi(P, P_0)$ and $b(P)$ are given by the formulas (4.20) and (4.22); the operators J, σ, γ^+ are represented by (4.2), $\gamma^- = \gamma^+$; and the operator $\tilde{\varphi}$ acts on the function $g(P)$ in the following way:

$$\tilde{\varphi} g(P) = \sum_{k=1}^{2} \langle g(P), e_k(z(P)) \rangle_{\mathcal{B}_{\mathbb{Q}}(A,B,h)} \cdot e_k,$$

besides, $e_k(z)$ are given by

$$e_1(z) = \frac{1 - \alpha z}{z} B^*(\bar{z}); \quad e_2(z) = \frac{1 - \alpha z}{z} \left(1 - A^*(\bar{z})\right); \quad e_1 = (1, 0); \quad e_2 = (0, 1). \quad (4.24)$$

Theorem 4.2. *Let there be given such simple colligation* Δ (1.8) *of Lie algebra that* $\dim E = 2n$, $\sigma_1 = J$, $\sigma_1 = \sigma$, γ^+ *is given by* (4.2), *spectrum of* A_1 *is concentrated at zero, and the characteristic function of operator* A_1 *is such that the measure* dF_x *in the multiplicative representation of* $S_1(\lambda)$ (*see Theorem* 3.1) *is absolutely continuous,* $dF_x = a_x dx$, *besides,* a_x *equals* (4.1). *Then the colligation* Δ (1.8) *is unitarily equivalent to the simple part of functional model* $\tilde{\Delta}$ (4.23).

III. Consider the following example. Let $\dim E = 6$, the operators $\tilde{\sigma}$ and $\tilde{\gamma}$ in E^3 be equal

$$\tilde{\sigma} = \begin{bmatrix} -\frac{1}{k} & 0 & 0 \\ 0 & 1 & b \\ 0 & b & \frac{1}{k} - 2 \end{bmatrix}; \quad \tilde{\gamma} = \begin{bmatrix} -\frac{i}{2} & 0 & a \\ 0 & -\frac{i}{2} & 0 \\ a & & -\frac{i}{2} \end{bmatrix}; \tag{4.25}$$

where $a > 0$; $k \in (0,1)$; $b = \sqrt{2\left(\frac{1}{k} - 1\right)}$. In this case the curve $\mathbb{Q}$ is given by the polynomial

$$k^2 a^3 z^2 (1 - w) = (1 + w)\left(1 - k^2 w^2\right). \tag{4.26}$$

Assuming that $\xi = ka\lambda(1 - w)$, we obtain the Legendre algebraic curve

$$\xi^2 = \left(1 - w^2\right)\left(1 - k^2 w^2\right). \tag{4.27}$$

The two-sheeted Riemann surface (4.27) has the genus $g = 1$ and is formed by the "crosswise" gluing of two w planes along the cut $\left(-\infty, -\frac{1}{k}\right] \cup [-1,] \cup \left(\frac{1}{k}, \infty\right)$. The imaginary part

$$ka\mathrm{Im}z = \mathrm{Im}\sqrt{\frac{1 + w}{1 - w}\left(1 - k^2 w^2\right)}$$

changes its sign on the cuts, therefore $\mathbb{Q}^+$ and $\mathbb{Q}^-$ (4.17) are sheets of the Riemann surface (4.15) and $\mathbb{Q}^0 = \partial\mathbb{Q}^\pm$ coincides with the mentioned cuts. On surface (4.27) there exists the Abelian differential of genus one [9],

$$\bar{\omega} = \frac{dw}{\sqrt{\left(1 - w^2\right)\left(1 - k^2 w^2\right)}}. \tag{4.28}$$

Using the elliptic integral

$$u(P) = \int_{P_1}^{P} \omega \quad (P = (\lambda, w) \in \mathbb{Q}), \tag{4.29}$$

specify the conform map [9] between (4.27) and the rectangle

$$\Gamma = \{u \in \mathbb{C} : \mathrm{Re}u \in [-2k, 2k]; \mathrm{Im}u \in [-k', k]\} \tag{4.30}$$

where $P_1 = (0, 1)$ and the numbers $4k$ and $2ik'$ are the periods of the closed differential ω (4.28). Inversion of the elliptic integral (4.29) results in the uniformization of curve (4.27) in terms of the elliptic Jacobi functions [9]. Therefore for (4.26) we obtain

$$z(u) = \frac{\mathrm{sn}'u}{ka(1 - \mathrm{sn}u)}; \quad w(u) = \mathrm{sn}u. \tag{4.31}$$

The eigenvectors $h(P) = h(u)$ of linear bundle $h(P)\left(\tilde{\sigma} + z\tilde{\gamma}_R\right) = wh(P)$ are given by

$$h(P) = \left[\frac{kaz}{1+kw}, \frac{b}{w-1}, 1\right]; \quad h(u) = \left[\frac{\mathrm{sn}'u}{(1-\mathrm{sn}u)(1+k\mathrm{sn}u)}, \frac{b}{\mathrm{sn}u - 1}, 1\right]. \tag{4.32}$$

It is easy to show that the function $\psi\left(P, P_0\right)$ (4.20) equals 1, $\psi\left(P, P_0\right) = 1$. The function $b(P)$ (4.22) is given by

$$b(P) = \|h(P)\|^{-4}_{E^n} \left\{\frac{b^2}{(w-1)^2} - \frac{k^2a^2z}{(1+kw)^2}\right\}. \tag{4.33}$$

Thus in this case the functional model of Lie algebra is

$$\left(\tilde{A}_1 g\right)(P) = \frac{g(P) - g\left(P_0\right)}{z(P) - z\left(P_0\right)};$$

$$\left(\tilde{A}_2 g\right)(P) = \frac{g(P)\left[w(P) + \frac{i}{2}z(P)\right] - g\left(P_0\right)\left[w\left(P_0\right) + \frac{i}{2}z\left(P_0\right)\right]}{z(P) - z\left(P_0\right)} - \tag{4.34}$$

$$-iz(P)\frac{d}{dz}g(P) - iz(P)b(P)g(P).$$

Thus,

1) for the Lie algebra $\{A_1, A_2\}$ ($[A_2, A_1] = iA_1$), the triangular model (2.1) in the space of functions $L^2_{r,l}\left(F_x\right)$ is constructed (see Theorem 2.2);

2) functional model (3.15) for the studied in this paper Lie algebra $\{A_1, A_2\}$ in spaces of entire L. de Branges functions is determined (Theorem 3.1);

3) model realization of the given Lie algebra on Riemann surface is presented (Theorem 4.1 and Theorem 4.2).

REFERENCES

1. Livšic, M. S., A. A. Yantsevich. Theory of operator colligations in Hilbert spaces. Kharkov, Kharkov University Publising House, 1971 (in Russian).
2. Livšic, M. S., N. Kravitsky, A. Markus, V. Vinnikov. Theory of commuting non-selfadjoint operators. Kluwer acad. publ., Dordrecht – London, 1995.
3. Zolotarev, V. A. Analytic methods of spectral representations of non-selfadjoint and nonunitary operators. MagPress, Kharkov, 2003 (in Russian).
4. Zolotarev, V. A. Lax–Fillips scattering scheme on groups and functional models of Lie algebra. *Mat. sb.* **183**, 1992, No 5, 115–144 (in Russian).
5. Zolotarev, V. A. Functional model for Lie algebra ISO(1,1) of linear non-selfadjoint operators. *Mat. sb.* **186**, 1995, No 1, 76–106 (in Russian).
6. Vaksman, L. L. On characteristic operator-functions of Lie aldebras. *Vestnik Khark. un-ta. Ser. mat i mech.* No. **33**, Issue 37, 1972, 41–45 (in Russian).
7. Vilenkin, N. Ya. Special functions and theory of group representations. Nauka, M., 1991, (in Russian).
8. Vilenkin, N. Ya., A. U. Klimyk. Representations of Lie groups and special functions. *Itogi nauki i techniki. Ser. Sovr. probl. mat. Fundam. napravleniya.* VINITI, Vol. **59**, 1990, 145–264.
9. Springer, G. Introduction into theory of Riemann sufaces. Foreign Literature Publishing House, M., 1960 (in Russian).
10. Serr, Sh.-P. Lie algebras and Lie groups. Mir, M, 1969 (in Russian).

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