ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 100

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 100

MODAL OPERATORS FOR RATIONAL GRADING[1]

TINKO TINCHEV, MITKO YANCHEV

A generalization of the majority operator based on a rational degree of grading is introduced. In the natural semantics of the language, Kripke frames such that any world can see finitely many worlds, the set of all valid formulae is a non-normal modal logic, RGML. Decidability of RGML and its completeness with respect to the class of all finite tree-like Kripke frames are the main results of the paper.

Keywords: Graded modal logic, majority logic

2000 MSC: main 03B45, secondary 68T27

1. INTRODUCTION

The language of modal logic is widely used to formalize notions like knowledge, possibility and necessity. The modal logics for grading formalize our ability to express assertions about quantity, number or a part of the whole and there are well known results in the integer grading. In this paper we examine grading with rational values. We start with a brief review of two distinctive modal logics for grading.

1.1. GRADED MODAL LOGIC GML

The Graded modal logic formalizes the reasonings about a finite number of objects, i.e. it is connected with integer "grading" of the number of objects.

[1]This paper is partially supported by Sofia University Science Fund, Contract No 27/2006.

Graded modal logic (GML) was introduced for the first time by Kit Fine [2]. Some important results were obtained by Fattorosi-Barnaba and Cerrato [3] and by Caro [4]. We introduce briefly the language of GML, its interpretation and the main results known.

The language of GML contains a countable set of propositional variables $P = \{p_1, p_2, \dots\}$, the Boolean connectives $\neg$ and $\vee$, and a countable set of modal operators $\Diamond_n,\ n \in \mathbb{N}$.

A formula α of GML has the following syntactic form:

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n \alpha,$$

where $p \in P$ and $n \in \mathbb{N}$.

The "integer" modal operators $\Diamond_n$ extend in a natural way the language of normal modal logics, in which $\Diamond\alpha$ says that "α is true in at least one accessible world". The meaning of $\Diamond_n\alpha$ is that "α is true in (strictly) more than n accessible worlds".

The GML formulae are interpreted in the usual Kripke structures. Let $\mathfrak{M} = \langle U, R, V\rangle$ be a Kripke model, where U is a non-empty set of worlds, $R \subseteq U^2$ is an accessibility relation in U and $V : P \to 2^U$ is a valuation function. The propositional variables and the Boolean connectives are evaluated as usual. The evaluation of modal operators is defined in the following way:

$$\mathfrak{M}, x \models \Diamond_n\alpha \Leftrightarrow |\{y | xRy \text{ and } \mathfrak{M}, y \models \alpha\}| > n.$$

A formula α is *valid in a model* $\mathfrak{M}$ iff α is true in any world of the model. α is *valid* iff it is valid in all models (based on a certain class of frames[2]). We will also use the following notation throughout this paper: $R(x) := \{y|xRy\}$ and for an arbitrary formula α (of the corresponding language) $R_\alpha(x) := \{y|\ xRy \text{ and } y \models \alpha\}$. So for the definition above we get the alternative notation:

$$\mathfrak{M}, x \models \Diamond_n\alpha \Leftrightarrow |R_\alpha(x)| > n.$$

GML is shown to be sound and complete with respect to the class of all frames.

GML is shown also to be decidable as it has the finite model property. It is proven that the decidability problem for GML is $PSPACE$-complete.

1.2. MAJORITY LOGIC MJL

The Majority logic was introduced by Eric Pacuit and Samer Salame [5] with the aim to model the concept of majority, i.e. to formalize the reasoning how far a given number of objects is a majority (part) of the whole. The concept of majority plays an important role in different social situations – from taking a decision of a group of friends how to spend the evening, to determining the result of a given

[2] Unless otherwise stated we assume the models, based on the class of all frames.

vote. MJL axiomatizes that concept. Now we give in this section a brief overview of basic ideas, formulations and results from [5].

As an example of the type of reasoning, captured in MJL, a variant of the muddy children puzzle is considered. Suppose that there are $n > 1$ children who have been playing outside and $k > 1$ of them have mud on their forehead. (At that we assume that the children are perfect reasoners, honest, and cannot see the mud on their forehead.) After a while an adult comes and announces: "strictly more than half of you have mud on your forehead". The man then proceeds to ask the children to say if they have mud on their forehead. It is not too hard to see that the $(k - \lfloor \frac{n}{2} \rfloor)^{th}$ time the children are asked, the dirty ones will correctly respond.

The language of MJL extends GML with a new unary modal operator W, where $W\alpha$ has the meaning "α is true in more than or equal to half of the accessible worlds" (Weak majority). Hence the dual $M\alpha$ means α is true in more than half of the accessible worlds (strict Majority). It is shown that the operator W cannot be defined from the standard modal operators ($\Box$ and $\Diamond$), the same is true for the operator $\Diamond_n$. Furthermore, the modal operator M cannot be expressed by the operators of GML. Hence as in GML, in MJL more expressive power of the language is achieved with the new modal operators.

The intuitive semantics of W and M, described above, makes sense only when the half of a finite set "is measured". The key problem is what is the majority (i.e. $\geq \frac{1}{2}$ or at least 50%) of an infinite set. As a decision the so called *majority systems*, which generalize the concept of the ultrafilters, are introduced. Having in mind the majority systems, the valid formulae are axiomatized and soundness and completeness are proven.

1. Syntax and semantics of MJL. A *formula* α of MJL has the syntactic form:

$$\alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \Diamond_n \alpha \mid W\alpha,$$

where $p \in P$, $P = \{p_1, p_2, \dots\}$ is a countable set of propositional variables, and $n \in \mathbb{N}$, $M\alpha := \neg W \neg \alpha$.

MJL formulae are interpreted in the usual Kripke models [1]. If for any accessible world x the set $R(x)$ is a finite set then the natural semantics is the following:

$$\mathfrak{M}, x \models W\alpha \Leftrightarrow |R_\alpha(x)| \geq \frac{1}{2}|R(x)|.$$

But in the common case the set $R(x)$ can be infinite (and that is the case in proving the completeness, for example). The solution of the problem is found by giving to any set $R(x)$ the opportunity to determine which of its subsets are majority ones. That is achieved by defining for each $R(x)$ a family of subsets, called *a majority system*, members of which satisfy properties in accordance with our intuition of majority subset. For the finite sets this definition completely agrees with the well-known properties of the majority subsets. For the infinite sets it was proven that these properties also hold. The connection between the majority systems and ultrafilters, is also proven: namely every non-principal ultrafilter is a majority system; the reverse is not true. Next, the *majority models* are defined by adding to the definition of a standard Kripke model a *majority function*, comparing to any set of accessible worlds $R(x),\ x \in U$, its majority system.

2. The main results, stated and proven in [5], are: *Soundness theorem*, saying that MJL is sound with respect to the class of all majority models, and *canonical model theorem*, proving completeness of MJL by means of canonical majority model.

It is pointed out that the main question remains open — the decidability of MJL — with the expectation MJL to possess the finite model property, already proven for GML.

Finally, a possible application of logics for grading is noted and in particular of MJL, in the so called social software, for example in the voting systems.

Now we shall proceed to presenting the modal logic, suggested by us, which introduces modal operators for *rational grading* and which thus develops modal grading, moving the things forward as in comparison with GML, so with MJL. At that we shall follow the semantic approach — we define the language of the new logic and we give the appropriate semantics without axiomatizing the system. Also, we shall consider finite sets of admissible worlds only, i.e. we want the set $R(x)$ to be finite for any $x \in U$. The main results we shall present are: the finite model property with respect to the class of tree-like models, and the decidability of the new logic. The basic idea of the proofs originates in [6] and uses a variant of a theorem from [7].

2. MODAL LOGIC FOR RATIONAL GRADING

2.1. SYNTAX

We define *a modal language* $\mathcal{L}_M$, containing a countable set of propositional variables $P = \{p_1, p_2, \dots\}$, the Boolean connectives $\neg$, $\wedge$ and $\vee$, and the modal operator $M^{p,q}$, where p and q are relatively prime integers and $1 \leq p < q$.

Formulae in $\mathcal{L}_M$ are defined inductively:

1. The elements of P are formulae;

2. If φ and ψ are formulae, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $M^{p,q}\varphi$ are also formulae.

In what follows a *formula* will mean a formula in $\mathcal{L}_M$.

We define, in addition, in the standard manner the rest of the usual Boolean connectives $\rightarrow$ and $\leftrightarrow$, and the dual to $M^{p,q}$ modal operator $W^{p,q} := \neg M^{p,q}\neg$. We also denote: $\top := \varphi \rightarrow \varphi,\ \bot := \neg\top$, where φ is an arbitrary formula.

A frame for $\mathcal{L}_M$ is a tuple $\langle U, R\rangle$, where U is a non-empty set, called *universe*, consisting of *points* or *(possible) worlds*; the elements of U we denote: x, y, dots, eventually with indices. $R \subseteq U^2$ is *an accessibility relation* in U, and we want for any $x \in U$ the set $R(x) := \{y|(x, y) \in R\}$ to be finite. For $(x, y) \in R$ we use the notation $R(x, y)$, also xRy and we say that y is an (R-)*successor* of x or (R-)*accessible* from x.

A model for $\mathcal{L}_M$ is a triplet $\langle U, R, V\rangle$, where $\langle U, R\rangle$ is a frame for $\mathcal{L}_M$ and $V : P \to 2^U$ is *a valuation* of the variables. We call the model $\langle U, R, V\rangle$ *a model based on the frame* $\langle U, R\rangle$. We denote models by $\mathfrak{M}, \mathfrak{N}$.

Truth: The valuation V from a model $\mathfrak{M}$ is inductively extended for an arbitrary formula; so we obtain a valuation of all the formulae in the model. For $x \in V(\varphi)$, $x \in U$, we say that φ *is true in a world* x and we denote also $\mathfrak{M}, x \models \varphi$ or simply $x \models \varphi$, when that makes no confusion.

The set of successors of the world x in which the formula φ is true, i.e. the set $\{y|(x, y) \in R \text{ and } y \models \varphi\}$, we denote by $R_\varphi(x)$.

For the Boolean connectives the inductive definition is standard:

$$V(\neg\varphi) := U\backslash V(\varphi)$$
$$V(\varphi \wedge \psi) := V(\varphi) \cap V(\psi)$$
$$V(\varphi \vee \psi) := V(\varphi) \cup V(\psi)$$

For the modal operator we define:

$$V(M^{p,q}\varphi) := \{x : |R_\varphi(x)| > \frac{p}{q}|R(x)|\}$$

or, with the equivalent notation,

$$\mathfrak{M}, x \models M^{p,q}\varphi \Leftrightarrow |R_\varphi(x)| > \frac{p}{q}|R(x)|,$$

i.e. $M^{p,q}\varphi$ is true in a world x if φ is true in a (strict) greater than $\frac{p}{q}$ part of the successors of x. About the dual modal operator we obtain

$$\mathfrak{M}, x \models W^{p,q}\varphi \Leftrightarrow |R_{\neg\varphi}(x)| \leq \frac{p}{q}|R(x)| \quad \Big(\Leftrightarrow |R_\varphi(x)| \geq \frac{q-p}{q}|R(x)|\Big),$$

i.e. $W^{p,q}\varphi$ is true in a world x if φ is refused in not greater than $\frac{p}{q}$ part of the successors of x.

A formula φ is *valid in a model* $\mathfrak{M} = \langle U, R, V\rangle$, notation $\mathfrak{M} \vDash \varphi$, if φ is true in any world of the model, i.e.

$$\mathfrak{M} \vDash \varphi \Leftrightarrow (\forall x \in U)(\mathfrak{M}, x \vDash \varphi).$$

A formula φ is *valid in a frame* $\mathfrak{F} =\langle U, R\rangle$, denoted by $\mathfrak{F} \vDash \varphi$, if φ is valid in any model on the frame. A formula φ is *valid* (or *tautology*), if φ is valid in any frame (from a given class), notation $\vDash \varphi$.

A formula φ is *satisfiable in a model* if it is true in some world of the model. A formula φ is *satisfiable in a frame* if it is valid in some model, based on the frame. A formula φ *satisfiable* if it is satisfiable in some model.

Definition 1. The logic *RGML is the set of all formulae, valid in the class of all frames.*

2.3. MODAL ELEMENTARY CONJUNCTIONS

Definition 2. *We define by induction* modal depth (md) of a formula*:*

$md(p) = 0$

$md(\neg\varphi) = md(\varphi)$

$md(\varphi \bigtriangleup \psi) = max(md(\varphi), md(\psi)), \bigtriangleup \rightleftharpoons \wedge, \vee$

$md(M^{p,q}\varphi) = md(\varphi) + 1,$ *hence*:

$md(\varphi \Box \psi) = max(md(\varphi), md(\psi)), \Box \rightleftharpoons \rightarrow, \leftrightarrow$

$md(W^{p,q}\varphi) = md(\varphi) + 1$

Using induction on the complexity of formula, it is easy to prove the following properties of the formulae:

Fact 1. $md(\varphi) = 0$ *iff* φ *is a Boolean formula.*

Fact 2. $md(\varphi) \geq md(\chi)$*, for any subformula* χ *of* φ*.*

Definition 3. Modal elementary conjunction *is a formula of the kind:*

$$\theta = p_1^{\varepsilon_1} \wedge \cdots \wedge p_n^{\varepsilon_n} \wedge M^{p,q}\varphi_1 \wedge \cdots \wedge M^{p,q}\varphi_m \wedge W^{p,q}\psi_1 \wedge \cdots \wedge W^{p,q}\psi_l, \qquad (2.1)$$

where $\varepsilon_i, i = 1, \ldots, n$*, are 0 or 1 and for any formula* φ *we use the notation* $\varphi^1 := \varphi,\ \varphi^0 := \neg\varphi$*.*

Proposition 1. *There exists an algorithm* $\mathcal{N}$*, acting on an arbitrary formula* φ*, which terminates in a finite number of steps and the result* $\mathcal{N}(\varphi)$ *is a (finite) disjunction of modal elementary conjunctions (i.e.* $\mathcal{N}$ *transforms an arbitrary formula in a modal equivalent of DNF) and the following holds:*

1. For arbitrary model $\mathfrak{M}$ *and world* x *from it*

$$\mathfrak{M}, x \models \varphi \leftrightarrow \mathcal{N}(\varphi),$$

2. $md(\varphi) \geq md(\mathcal{N}(\varphi))$*.*

The role of $\mathcal{N}$ can play any algorithm, transforming a Boolean formula in DNF, just treating the subformulae with a modal operator on the outer level as variables.

For 2. it is sufficient to note that the transformation in modal DNF cannot increase the modal depth, while a modal operator can be reduced in case of exclusion in disjunction, if a subformula occurs both in positive and negative form.

From Proposition 1 and Fact 2 follows:

Fact 3. *Any modal elementary conjunction from $\mathcal{N}(\varphi)$ has depth, not exceeding the depth of φ, i.e. $md(\theta) \leq md(\varphi)$, for any θ – a modal elementary conjunction from $\mathcal{N}(\varphi)$.*

From Proposition 1 it also follows that the question for the satisfiability of a formula can be reduced to the one for the satisfiability of a finite number of modal elementary conjunctions, each of which with modal depth, not exceeding a constant — the modal depth of the formula itself.

Let us consider one such modal elementary conjunction θ and its satisfiability.

Case 1. The Boolean part of θ, we denote it by $B_\theta := p_1^{\varepsilon_1} \wedge \ldots \wedge p_n^{\varepsilon_n}$, is not satisfiable (when for some $i \neq j \leq n$ it holds $p_i = p_j$ and $\varepsilon_i \neq \varepsilon_j$).

Then θ is not satisfiable.

Case 2. B_θ is satisfiable.

Then we examine the rest (modal) part of θ — the satisfiability of the whole θ depends on it:

$$M^{p,q}\varphi_1 \wedge \ldots \wedge M^{p,q}\varphi_m \wedge W^{p,q}\psi_1 \wedge \ldots \wedge W^{p,q}\psi_l,$$

$m \geq 0,\ l \geq 0$.

Case 2.1. $m = 0$.

Then the modal part is true in any one-world model. Really, let $\mathfrak{M}$ be a model with a single world x. From the valuation of $W^{p,q}$ for any formula ψ holds:

$$\mathfrak{M}, x \models W^{p,q}\psi \Leftrightarrow |R_\psi(x)| \geq \frac{q-p}{q}|R(x)|.$$

As x has no successors, $R_\psi(x) = R(x) = \varnothing$ and the inequality on the right is fulfilled as the equality $0 = 0$. Hence $\mathfrak{M}, x \models W^{p,q}\psi$.

Therefore, as every modal conjunct, so the whole modal part of θ are true in every one-world model, and particularly in every one-world model for B_θ, and such a model exists. Hence θ is satisfiable.

Case 2.2. $m > 0$, i.e. at least one conjunct of the form $M^{p,q}\varphi$ participates in θ.

Let some model $\mathfrak{M}$ and a world x from the model fulfil $\mathfrak{M}, x \models \theta$, then $\mathfrak{M}, x \models M^{p,q}\varphi$. If we assume that $R(x) = \varnothing$, from the evaluation of $M^{p,q}$ we obtain $0 = |R_\varphi(x)| > \frac{p}{q}|R(x)| = 0$ — a contradiction.

Hence, if θ is true in the world x, then x has $a > 0$ successors.

We will formulate and prove a proposition connected with the satisfiability of θ in Case 2.2. First we will give a definition.

Let us consider a model $\mathfrak{M}$ and a world x_0 from the model, having $a > 0$ successors (i.e. $|R(x_0)| = a > 0$), and examine the truth of θ in x_0.

We consider all the conjunctions of the form:

$$\varphi_1^{\varepsilon_1} \wedge \ldots \wedge \varphi_m^{\varepsilon_m} \wedge \psi_1^{\varepsilon_{m+1}} \wedge \ldots \wedge \psi_l^{\varepsilon_{m+l}}, \tag{2.2}$$

where $\varepsilon_j,\ j = 1, \ldots, m + l$, are 0 or 1 and let $\tau_1, \ldots, \tau_t$ are all of them, which are satisfiable, $0 \le t \le 2^{m+l}$, i.e.

$$\begin{aligned} \tau_1 &= \varphi_1^{\varepsilon_{11}} \wedge \ldots \wedge \varphi_m^{\varepsilon_{1m}} \wedge \psi_1^{\varepsilon_{1m+1}} \wedge \ldots \wedge \psi_l^{\varepsilon_{1m+l}} \\ &\vdots \\ \tau_t &= \varphi_1^{\varepsilon_{t1}} \wedge \ldots \wedge \varphi_m^{\varepsilon_{tm}} \wedge \psi_1^{\varepsilon_{tm+1}} \wedge \ldots \wedge \psi_l^{\varepsilon_{tm+l}} \end{aligned}$$

where ε_{ij} is short for $\varepsilon_{i,j}$ and ε_{ij}, $i = 1, \ldots, t$, $j = 1, \ldots, m + l$, are 0 or 1.

We denote $T := \{\tau_1, \ldots, \tau_t\}$.

Let $\mathfrak{M}' = \langle U', R', V' \rangle$ be an arbitrary model. For all formulae in θ, φ_j, $j = 1, \ldots, m$, ψ_{j-m}, $j = m + 1, \ldots, m + l$, consider the corresponding sets $V'(\varphi_j)$, and $V'(\psi_{j-m})$. For $x' \in U'$, we define

$$\varepsilon'_j = \begin{cases} 1, & x' \in V'(\varphi_j) \\ 0, & x' \notin V'(\varphi_j),\ j = 1, \ldots, m, \end{cases}$$

$$\varepsilon'_j = \begin{cases} 1, & x' \in V'(\psi_{j-m}) \\ 0, & x' \notin V'(\psi_{j-m}),\ j = m + 1, \ldots, m + l \end{cases}$$

We consider $\tau' = \varphi_1^{\varepsilon'_1} \wedge \ldots \wedge \varphi_m^{\varepsilon'_m} \wedge \psi_1^{\varepsilon'_{m+1}} \wedge \ldots \wedge \psi_l^{\varepsilon'_{m+l}}$. Then $x' \models \tau'$ and hence $\tau' \in T \neq \varnothing$ and $t > 0$, i.e. $1 \le t \le 2^{m+l}$.

Suppose τ_i is satisfiable in a_i worlds from $R(x_0)$, i.e. $|R_{\tau_i}(x_0)| = a_i,\ 0 \le a_i \le a,\ i = 1, \ldots, t$. We form the following system of $m + l$ linear inequalities, in which we consider $a_1, \ldots, a_t$ as unknowns:

$$(\sigma_\theta)\ \left|\ \begin{array}{ll} \sum\limits_{i=1}^{t} \varepsilon_{ij} a_i > \frac{p}{q}(a_1 + \cdots + a_t), & j = 1, \ldots, m \\ \\ \sum\limits_{i=1}^{t} \varepsilon_{ij} a_i \ge \frac{q-p}{q}(a_1 + \cdots + a_t), & j = m + 1, \ldots, m + l \end{array}\right. \tag{2.3}$$

Definition 4. *We call the above system of linear inequalities* corresponding to the modal elementary conjunction θ*.*

We also define a system of linear inequalities, corresponding to a modal elementary conjunction θ, consisting just of a (satisfiable!) Boolean part, i.e. we define a system of linear inequalities, corresponding to a satisfiable Boolean formula B, in the following way:

$$(\sigma_B) \mid a \geq a$$

Note 1. In the condition of Case 2.2 ($m > 0$), if (σ_θ) has a solution, then this solution is not zero, as (σ_θ) contains at least one strict inequality.

Note 2. In Case 2.1 ($m = 0$) we can also consider a system (σ_θ), corresponding to θ, but that system has always the zero solution as a system of non-strict homogeneous linear inequalities. This exactly corresponds to the fact that the modal part of θ is always satisfiable, but we already know that. That is why the interesting case is when (σ_θ) has (only) non-zero solution.

Now we formulate a proposition related to the satisfiability of θ in Case 2.2.

Proposition 2. *For any modal elementary conjunction θ with a satisfiable Boolean part and a modal part which is either empty or has at least one conjunct with modal operator $M^{p,q}$, the following three statements are equivalent:*

(i) θ is satisfiable;

(ii) θ is satisfiable in the root of a finite tree-like model;

(iii) the system of linear inequalities (σ_θ), corresponding to θ, has non-negative (non-zero) integer solution.

Proof. We use induction on the modal depth of θ.

1. For $md(\theta) = 0$, θ is a Boolean formula. Then θ is satisfiable iff it is satisfiable in a single-world model. As it is satisfiable under the terms of the proposition, it is also satisfiable in a single-world (tree-like) model. (σ_θ) has (a trivial) solution 1. So (i), (ii) and (iii) are fulfilled, and hence are equivalent.

2. (ih) Let, for any modal elementary conjunction θ with $md(\theta) \leq n$, (i), (ii) and (iii) be equivalent.

3. Consider θ: $md(\theta) = n + 1$.

3.1. (ii) $\Rightarrow$ (i) is always (trivially) fulfilled;

3.2. (i) $\Rightarrow$ (iii)

Let θ be satisfiable. Then there exist a model $\mathfrak{M}$ and a world x_0 from the model in which θ is true and let the number of successors of x_0, $|R(x_0)|$, is $a^0 > 0$.

Let $\tau_1, \ldots, \tau_t$ be all the conjunctions from T (as defined above), and let τ_i be true in a_i^0 in number worlds from $R(x_0)$, i.e. $|R_{\tau_i}(x_0)| = a_i^0$, $0 \leq a_i^0 \leq a^0$, $i = 1, \ldots, t$.

For any $x \in R(x_0)$ there exists $\tau \in T$: $x \models \tau$, i.e. $x \in R_\tau(x_0)$, (τ can be constructed as in the proof of $T \neq \varnothing$), i.e. $x \in R_{\tau_i}(x_0)$ for some i, $1 \leq i \leq t$. Hence $R(x_0) \subseteq \bigcup_{i=1}^t R_{\tau_i}(x_0)$. The reverse inclusion obviously holds, so $R(x_0) = \bigcup_{i=1}^t R_{\tau_i}(x_0)$. Then

$$a^0 = |R(x_0)| = |\bigcup_{i=1}^{t} R_{\tau_i}(x_0)| \leq \sum_{i=1}^{t} |R_{\tau_i}(x_0)| = a_1^0 + \cdots + a_t^0$$

Let $x' \in R_{\tau_i}(x_0)$. As for $i \neq j$, τ_j differs from τ_i at least in one conjunct, $x' \notin R_{\tau_j}(x_0)$. Hence $R_{\tau_i}(x_0) \cap R_{\tau_j}(x_0) = \varnothing$, for all $i \neq j$, $i, j \in \{1, \ldots, t\}$, and $|\bigcup_{i=1}^{t} R_{\tau_i}(x_0)| = \sum_{i=1}^{t} |R_{\tau_i}(x_0)|$. Hence

$$a^0 = a_1^0 + \cdots + a_t^0 > 0$$

Now, as θ is true in x_0, any of $M^{p,q}\varphi_j$, $j = 1, \ldots, m$, and any of $W^{p,q}\psi_{j-m}$, $j = m+1, \ldots, m+l$, is also true in x_0. Now consider any of the sets $R_{\varphi_j}(x_0)$, $j = 1, \ldots, m$. From the truth definition for $M^{p,q}$ follows:

$$|R_{\varphi_j}(x_0)| > \frac{p}{q} a^0 = \frac{p}{q}(a_1^0 + \cdots + a_t^0) \tag{2.4}$$

But $R_{\varphi_j}(x_0)$ is a union of sets $R_{\tau_i}(x_0)$ for those τ_i in which φ_j is in positive form. As these sets are pairwise disjoint and taking into account how the coefficients ε_{ij} are defined, we get that the following holds:

$$|R_{\varphi_j}(x_0)| = \sum_{i=1}^{t} \varepsilon_{ij} |R_{\tau_i}(x_0)| = \sum_{i=1}^{t} \varepsilon_{ij} a_i^0 \tag{2.5}$$

From (2.4) and (2.5) we obtain that $|R_{\tau_i}(x_0)| = a_i^0$, $i = 1, \ldots, t$, satisfy the inequalities from the system (σ_θ), corresponding to θ, for $j = 1, \ldots, m$.

In the same way, considering any of the sets $R_{\psi_{j-m}}(x_0)$, $j = m+1, \ldots, m+l$, from the truth definition for $W^{p,q}$, we get

$$\sum_{i=1}^{t} \varepsilon_{ij} a_i^0 = \sum_{i=1}^{t} \varepsilon_{ij} |R_{\tau_i}(x_0)| = |R_{\psi_{j-m}}(x_0)| \geq \frac{q-p}{q} a^0 = \frac{q-p}{q}(a_1^0 + \cdots + a_t^0) \tag{2.6}$$

From (2.6) we obtain that $|R_{\tau_i}(x_0)| = a_i^0$, $i = 1, \ldots, t$, satisfy also the inequalities from (σ_θ) for $j = m+1, \ldots, m+l$.

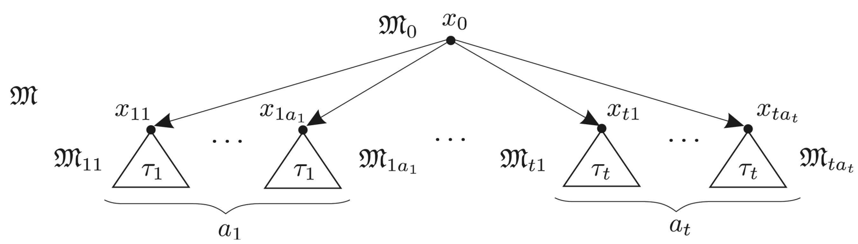
It follows that $(|R_{\tau_1}(x_0)|, \ldots, |R_{\tau_t}(x_0)|)$ is a solution of (σ_θ), moreover non-negative (non-zero) integer one, i.e. (iii) holds.

3.3. (iii) $\Rightarrow$ (ii)

Let (σ_θ) be the system, corresponding to θ (i.e. it is formed in the way, described above), and let it have at least one non-negative (non-zero) integer solution, i.e. there exist numbers $a_1^0, \ldots, a_t^0$, $a_i^0 \in \mathbb{N}$, $i = 1, \ldots, t$, $a^0 := \sum_{i=1}^{t} a_i^0 > 0$, and $(a_1^0, \ldots, a_t^0)$ is a solution of (σ_θ).

So, there exist just t different conjunctions of form (2) — we denote them by $\tau_1, \ldots, \tau_t$ — which are got from θ and which are satisfiable. As $md(\tau_i) \leq md(\theta) - 1 = n$, by (ih) the proposition holds for τ_i, $i = 1, \ldots, t$. Hence any τ_i is satisfiable in the root of a finite tree-like model. For any τ_i we consider a_i^0 finite tree-like models $\mathfrak{M}_{ik} = \langle U_{ik}, R_{ik}, V_{ik} \rangle$, $k = 1, \ldots, a_i^0$, with roots, denoted by $x_{i1}, \ldots, x_{ia_i^0}$ respectively, in each of which τ_i is true. In addition, we want the universes of the models $\mathfrak{M}_{ik}$, $i = 1, \ldots, t$, $k = 1, \ldots, a_i^0$, to be pairwise disjoint (we can obtain these models by some procedure of "copying" or "colouring").

Let the Boolean part of θ, B_θ, be true in a single-world model $\mathfrak{M}_0 = \langle U_0, R_0, V_0\rangle$, where $U_0 = \{x_0\}$, $R_0 = \varnothing$ and V_0 is an evaluation in $\mathfrak{M}_0$ for which $x_0 \models B_\theta$; such a model exists as B_θ is satisfiable. We define a model $\mathfrak{M} = \langle U, R, V\rangle$ as a natural union of the above models in the following way:



$$U = \{x_0\} \cup \bigcup_{i=1}^{t} \bigcup_{k=1}^{a_i^0} U_{ik},$$

$$\begin{aligned} R &= \{(x_0, x_{ik}),\ 1 \le i \le t,\ 1 \le k \le a_i^0\} \cup \bigcup_{i=1}^{t} \bigcup_{k=1}^{a_i^0} R_{ik}, \\ V &= V_0 \cup \bigcup_{i=1}^{t} \bigcup_{k=1}^{a_i^0} V_{ik}, \end{aligned}$$

and the definition of V is extended to the set of all formulae.

From the definition of $\mathfrak{M}$ follow:

Fact 4. *$\mathfrak{M}$ is a finite tree-like model with a root — the world x_0, which has $a^0 = a_1^0 + \cdots + a_t^0 > 0$ successors.*

Fact 5. *For any Boolean formula A: $x_0 \in V(A) \;\Leftrightarrow\; x_0 \in V_0(A)$, i.e.*

$$\mathfrak{M}, x_0 \models A \Leftrightarrow \mathfrak{M}_0, x_0 \models A$$

So for the modal part B_θ of θ holds:

$$\mathfrak{M}, x_0 \models B_\theta \tag{2.7}$$

Fact 6. *For any world $x \in U$, $x \neq x_0$, there exist $1 \le i \le t$, $1 \le k \le a_i^0$ such that $x \in U_{ik}$, and for any formula φ: $x \in V(\varphi) \;\Leftrightarrow\; x \in V_{ik}(\varphi)$, i.e.*

$$\mathfrak{M}, x \models \varphi \Leftrightarrow \mathfrak{M}_{ik}, x \models \varphi$$

so that

$$\mathfrak{M}, x_{ik} \models \tau_i,\ i = 1, \ldots, t,\ k = 1, \ldots, a_i^0. \tag{2.8}$$

Now we examine the truth of φ_j from θ, $j = 1, \ldots, m$. Any φ_j is true just in those worlds, in which are true these τ_i, $i = 1, \ldots, t$, in which φ_j takes part in positive form, i.e. $\varepsilon_{ij} = 1$. For the number of R-successors of x_0, in which φ_j is true, we have:

$$|R_{\varphi_j}(x_0)| = \sum_{i=1}^{t} \varepsilon_{ij} |R_{\tau_i}(x_0)| = \sum_{i=1}^{t} \varepsilon_{ij} a_i^0 \overset{\text{from } (\sigma_\theta)}{>} \frac{p}{q} \sum_{i=1}^{t} a_i^0 = \frac{p}{q} |R(x_0)|. \tag{2.9}$$

From (2.9), using the truth definition for $M^{p,q}$, we get:

$$\mathfrak{M}, x_0 \models M^{p,q} \varphi_j, \quad j = 1, \ldots, m. \tag{2.10}$$

In the same way, for the number of R-successors of x_0, in which ψ_{j-m}, $j = m + 1, \ldots, m + l$, are true, holds:

$$\begin{aligned} |R_{\psi_{j-m}}(x_0)| &= \sum_{i=1}^{t} \varepsilon_{ij} |R_{\tau_i}(x_0)| \\ &= \sum_{i=1}^{t} \varepsilon_{ij} a_i^0 \overset{\text{from } (\sigma_\theta)}{\geq} \frac{q-p}{q} \sum_{i=1}^{t} a_i^0 = \frac{q-p}{q} |R(x_0)|. \end{aligned} \tag{2.11}$$

From (2.11), using the truth definition for $W^{p,q}$, we get:

$$\mathfrak{M}, x_0 \models W^{p,q} \psi_{j-m}, \quad j = m + 1, \ldots, m + l. \tag{2.12}$$

From (2.7), (2.10) and (2.12) we get:

$$\mathfrak{M}, x_0 \models \theta,$$

which, taking into account Fact 4, means that θ is satisfiable in the root of a finite tree-like model, i.e. (ii) holds. □

The equivalence of (i) and (ii) from the Proposition 2 gives, as an immediate corollary, the following proposition:

Proposition 3. *The logic RGML coincides with the set of formulae, valid in the finite trees.*

Later on we will use Proposition 2 to prove that the logic RGML is decidable. But first we state briefly some elements from the theory of systems of linear inequalities and we prove a proposition connected with them.

3. SYSTEMS OF LINEAR INEQUALITIES

3.1. A METHOD OF SOLVING OF SYSTEMS OF LINEAR INEQUALITIES BY CONSECUTIVE REDUCTION OF THE NUMBER OF UNKNOWNS

Here we present a modified version of the method stated in [7]. Consider a system of linear inequalities (σ):

$$(\sigma) \left| \begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n + a_1 \geqslant 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n + a_m \geqslant 0 \end{array} \right. \tag{3.1}$$

where the sign $\geqslant$ stands for $\geq$ or $>$.

We associate with (σ) a system (σ'), called *attendant on* (σ), which has just one unknown less than (σ) and for definiteness let it be the one with the greatest index — x_n. Let

$$b_1x_1 + \cdots + b_nx_n + b \geqslant 0 \tag{3.2}$$

be an arbitrary inequality from (σ). The following possibilities for b_n exist:

1) $b_n = 0$ — in that case we do not change the inequality (3.2);

2) $b_n > 0$ — in that case we divide (3.2) by b_n and take all the members except x_n from the right side; we get the inequality

$$x_n \geqslant c_1x_1 + \cdots + c_{n-1}x_{n-1} + c \tag{3.3}$$

3) $b_n < 0$ — in that case we take the member with x_n on the right side and divide (3.2) by $-b_n$, so we get:

$$d_1x_1 + \cdots + d_{n-1}x_{n-1} + d \geqslant x_n. \tag{3.4}$$

Applying this procedure to every inequality from (σ), we get (with possible change in the order of the inequalities) the system (σ^*), equivalent to (σ) and having the form

$$(\sigma^*) \left| \begin{array}{l} P_1 \geqslant x_n \\ \vdots \\ P_p \geqslant x_n \\ \\ x_n \geqslant Q_1 \\ \vdots \\ x_n \geqslant Q_q \end{array} \right. \tag{3.5}$$

$$\left|\begin{array}{l} R_1 \geqslant 0 \\ \vdots \\ R_r \geqslant 0 \end{array}\right.$$

The first block of (σ^*) includes the inequalities from (σ), falling in case 3), the second — in case 2 and the third — in case 1, and obviously P_α, $1 \leq \alpha \leq p$, Q_β, $1 \leq \beta \leq q$ and R_γ, $1 \leq \gamma \leq r$, are linear functions of $x_1, \ldots, x_{n-1}$, not containing x_n.

(σ^*) can be written shortly:

$$\left|\begin{array}{ll} P_\alpha \geqslant x_n \geqslant Q_\beta, & \alpha = \overline{1,p},\ \beta = \overline{1,q} \\ R_\gamma \geqslant 0, & \gamma = \overline{1,r} \end{array}\right.$$

We consider the system (σ'):

$$(\sigma')\ \left|\begin{array}{ll} P_\alpha \geqslant Q_\beta, & \alpha = \overline{1,p},\ \beta = \overline{1,q} \\ R_\gamma \geqslant 0, & \gamma = \overline{1,r} \end{array}\right.$$

Definition 5. *The system (σ') considered as derived from system (σ^*) is called* attendant on the system (σ)*.*

Obviously (σ') has $n-1$ unknowns $x_1, \ldots, x_{n-1}$.

Note. If there is no inequality in (σ), falling in case 1, then the second group of inequalities in (σ') is missing. If there is no inequality in (σ), falling in case 2 or if there is no inequality in (σ), falling in case 3, then the first group of inequalities in (σ') is missing.

The following theorem about the systems above holds:

Theorem 1. *For any solution $(x_1, \ldots, x_n)$ of (σ), $(x_1, \ldots, x_{n-1})$ is a solution of (σ'). Conversely, for any solution $(x_1, \ldots, x_{n-1})$ of (σ') there is a number x_n such that $(x_1, \ldots, x_{n-1}, x_n)$ is a solution of (σ), i.e. any solution of the attendant system can be extended to a solution of the initial one.*

Proof. The proof follows the steps of the one stated in [7], with just one additional check to ensure that everything goes well in both cases – in a non-strict as well as in a strict inequality. □

The next proposition is an immediate corollary of Theorem 1.

Proposition 4. *The system (σ) has a solution iff the system (σ') has a solution.*

Definition 6. Admissible vector *for the first k unknowns of (σ) is the vector of numbers $(x_1^0, \ldots, x_k^0)$, if it can be extended to a solution of (σ), i.e. if there exist* *numbers* $x^0_{k+1}, \dots, x^0_n$ *such that* $(x^0_1, \dots, x^0_k, x^0_{k+1}, \dots, x^0_n)$ *is a solution of* (σ).

Now we denote with $\mathcal{C}'$ the algorithm which, for a given system of linear inequalities, constructs (in a finite number of steps) its attendant system, excluding the unknown with the greatest index. So $\mathcal{C}'((\sigma)) = (\sigma')$, and if (σ) has unknowns $x_1, \dots, x_n$, then (σ') has unknowns $x_1, \dots, x_{n-1}$. We can use again $\mathcal{C}'$ to act on (σ'). After $n-1$ usages of $\mathcal{C}'$ we get $\mathcal{C}'^{n-1}((\sigma)) = (\sigma^{n-1})$, where (σ^{n-1}) is a system of linear inequalities with just one unknown x_1.

Using Proposition 4 $n-1$ times we get that (σ) is compatible iff (σ^{n-1}) is compatible. In case that (σ^{n-1}) is compatible, we can easily find a solution x^0_1 of (σ^{n-1}), which is also an admissible vector for (σ^{n-2}). Substituting x^0_1 for x_1 in (σ^{n-2}) and solving (σ^{n-2}) regarding x_2, we get x^0_2. (x^0_1, x^0_2) is a solution of (σ^{n-2}) and an admissible vector for (σ^{n-3}). Continuing in that reverse way for $n-1$ steps we get a solution $(x^0_1, \dots, x^0_n)$ of (σ).

3.2. ALGORITHM FOR SYSTEMS OF LINEAR HOMOGENEOUS INEQUALITIES WITH RATIONAL COEFFICIENTS

Proposition 5. *There exists an algorithm, acting on systems of linear homogeneous inequalities with rational coefficients, which terminates for any such a system* (σ) *in a finite number of steps, giving a result* yes *if* (σ) *has a non-negative non-zero integer solution, and* no *in the opposite case.*

Let, for definiteness, (σ) has n unknowns $x_1, \dots, x_n$. We expand the system with the inequalities

$$\begin{aligned} &x_1 \geq 0 \\ &\vdots \\ &x_n \geq 0 \\ &x_1 + \cdots + x_n > 0 \end{aligned}$$

Note. If at least one of the inequalities in (σ) is strict, the last one of the upper inequalities is redundant and we do not add it.

We denote the expanded system by (σ^+). Obviously the system (σ) has a non-negative non-zero solution iff (σ^+) has a solution.

Let $\mathcal{D}$ be an algorithm, acting on systems of linear homogeneous inequalities with just one unknown, which terminates in a finite number of steps with a result *yes* if the system is compatible, and *no* in the opposite case. It is easy to see that such an algorithm exists and it can be easily constructed.

Then we put $\mathcal{C}((\sigma)) = \mathcal{D}(\mathcal{C}'^{n-1}(\sigma^+))$ and state that $\mathcal{C}$ is the algorithm we ask for.

Proof of Proposition 5.

1. $\mathcal{C}((\sigma))$ is well defined and terminates in a finite number of steps.

Really, (σ^+) is a system of linear (homogeneous) inequalities with n unknowns, so $\mathcal{C}'((\sigma^+))$ is defined and, after being applied $n-1$ times, $\mathcal{C}'$ transforms (in a finite number of steps) (σ^+) into a system (σ^{+n-1}) with just one unknown (from subsection). Then the algorithm $\mathcal{D}(\mathcal{C}'^{n-1}(\sigma^+))$ is also defined and in a finite number of steps gives a result *yes* or *no*.

2. $\mathcal{C}((\sigma)) = no \Rightarrow (\sigma)$ has no non-negative non-zero integer solution.

Let $\mathcal{D}(\mathcal{C}'^{n-1}(\sigma^+)) = \mathcal{D}((\sigma^{+n-1})) = no$. Then (σ^{+n-1}) is incompatible and by Proposition 4 (σ^+) is also incompatible, i.e. has no solution. Hence (σ) has no non-negative non-zero (integer) solution.

3. $\mathcal{C}((\sigma)) = yes \Rightarrow (\sigma)$ has non-negative non-zero integer solution.

Let $\mathcal{D}(\mathcal{C}'^{n-1}(\sigma^+)) = \mathcal{D}((\sigma^{+n-1})) = yes$. Then (σ^{+n-1}) has a solution and by Proposition 4 (σ^+) is also has a solution.

Now consider the system (σ^{+n-1}). It contains the inequality

$$x_1 \geq 0 \tag{3.6}$$

and (eventually) other inequalities of that kind and of the following kinds:

$$x_1 > 0 \tag{3.7}$$

$$-x_1 \geq 0 \tag{3.8}$$

$$-x_1 > 0 \tag{3.9}$$

As (σ^{+n-1}) has a solution, it has no inequalities of the kind (3.9) and also it has no together inequalities of kinds (3.7) and (3.8), i.e. it contains, except the inequality (3.6), (eventually) inequalities of kind (3.7) or inequalities of kind (3.8). Hence the set of solutions of (σ^{+n-1}) is either the point $x_1^0 = 0$ or a positive half-line with the beginning at the point 0 (eventually not including the point 0 itself). In the second case we can chose $x_1^0 = 1$ (or arbitrary rational number). Thus obtained x_1^0 is an admissible vector for (σ^{+n-2}). Substituting it for x_1 in (σ^{+n-2}) we can get x_2^0, by Theorem 1. Besides, as all the coefficients in (σ^+), and finally consecutively obtained from it attendant systems, are rational and x_1^0 is also rational, we can get x_2^0 also to be rational. (x_1^0, x_2^0) is an admissible vector for (σ^{+n-3}) and, continuing in that way, we get (in $n-1$ steps) $X^0 = (x_1^0, \dots, x_n^0)$ — a (non-negative, non-zero) rational solution of (σ^+). Hence X^0 is a non-negative non-zero rational solution of (σ).

Let k be the lowest common denominator of the integers $x_1^0, \dots, x_n^0$. Then $kX^0 = (kx_1^0, \dots, kx_n^0)$ is also a solution of (σ), moreover kX^0 is a non-negative non-zero integer solution of (σ). □

4. DECIDABILITY OF THE LOGIC RGML

Theorem 2. *There exists an algorithm $\mathcal{A}$, acting on formulae, which, applied on an arbitrary formula φ, terminates in a finite number of steps with result* yes *or* no *such that*

$$\mathcal{A}(\varphi) = yes \quad \textit{iff } \varphi \textit{ is satisfiable.} \tag{4.1}$$

Proof. We construct $\mathcal{A}$ by induction on the modal depth of the formulae, namely we build a sequence of algorithms $\mathcal{A}_0, \mathcal{A}_1, \ldots$ such that the algorithm $\mathcal{A}_n$ acts only on formulae with modal depth not greater than n and for these formulae it carries out the equivalence (4.1).

1. $\mathcal{A}_0$ is an algorithm, acting on the Boolean formulae.

2. (ih) Let $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ be defined and let the assertion of the theorem hold for them.

3. We define $\mathcal{A}_{n+1}$ in the following way:

3.1. Let φ be such that $md(\varphi) \leq n$. We put $\mathcal{A}_{n+1}(\varphi) := \mathcal{A}_n(\varphi)$.

3.2. Let φ be such that $md(\varphi) = n + 1$. We present φ in the form $\bigvee_{i=1}^{k} \theta_i$, where θ_i, $i = 1, \ldots, k$, are modal elementary conjunctions. By Proposition 1 there exists an algorithm $\mathcal{N}$, transforming φ in this form in a finite number of steps, and $md(\theta_i) \leq n + 1,\ i = 1, \ldots, k$.

We define the algorithm $\mathcal{B}_{n+1}$, acting on modal elementary conjunctions with $md \leq n + 1$, which, for θ — a modal elementary conjunction with $md(\theta) \leq n + 1$, gives the result *yes*, if θ is satisfiable, or *no* in the opposite case.

3.2.a. $md(\theta) \leq n$, $\mathcal{B}_{n+1}(\theta) := \mathcal{A}_n(\theta)$.

3.2.b. $md(\theta) = n + 1$. Let B_θ is the Boolean part of θ. Then:

b.1. If $\mathcal{A}_0(B_\theta) = no$, i.e. if B_θ is not satisfiable, then θ is not satisfiable also, and we put $\mathcal{B}_{n+1}(\theta) = no$.

b.2. Let $\mathcal{A}_0(B_\theta) = yes$, i.e. B_θ is satisfiable, and the modal part of θ contains only conjuncts with $W^{p,q}$ on the outer level. Then the whole θ is satisfiable (in a single-world model), so we put $\mathcal{B}_{n+1}(\theta) = yes$.

b.3. Let $\mathcal{A}_0(B_\theta) = yes$, i.e. B_θ is satisfiable, and the modal part of θ contains at least one conjunct with $M^{p,q}$ on the outer level. Then we consider all the conjunctions of the form (2.2) for θ. They are with modal depth not greater than n and the algorithm $\mathcal{A}_n$ acts on them. We implement $\mathcal{A}_n$ on any of them consecutively and pick out which of them are satisfiable: let these be $\tau_1, \ldots, \tau_t,\ 1 \leq t \leq 2^{m+l}$.

Then we consider the system (σ_θ), attendant on θ and having the form (2.3). By Proposition 2 the formula θ is satisfiable iff (σ_θ) has non-negative (non-zero) integer solution. Let $\mathcal{C}$ be the algorithm from Proposition 5 (which for any system of the above kind tells if the system has non-negative non-zero integer solution or not). In this case we put $\mathcal{B}_{n+1}(\theta) := \mathcal{C}((\sigma_\theta))$.

Now we proceed with $\mathcal{A}_{n+1}$ in case 3.2. Having the algorithm $\mathcal{B}_{n+1}$ just defined, we define $\mathcal{A}_{n+1}$ as an implementation of the algorithm $\mathcal{N}$, followed the implementations of $\mathcal{B}_{n+1}$ on each of the modal elementary conjunctions $\theta_i,\ i = 1,\ldots,k$, of φ. If for some $i,\ 1 \le i \le k,\ \mathcal{B}_{n+1}(\theta_i) = yes$, we put $\mathcal{A}_{n+1}(\varphi) = yes$; in the opposite case, i.e. if all the results are *no*, we put $\mathcal{A}_{n+1}(\varphi) = no$.

Thus the inductive definition of $\mathcal{A}_n$ for any natural number n is finished. Let $\mathcal{M}$ be an algorithm, calculating the modal depth of the formulae, i.e. $\mathcal{M}$ acts on an arbitrary formula and gives (in a finite number of steps) as a result a natural number, so that $\mathcal{M}(\varphi) = n$ iff $md(\varphi) = n$. It is easy to see that, as the formulae are of finite length and the modal depth is inductively defined, such an algorithm exists.

Now we define the algorithm $\mathcal{A}$ for an arbitrary formula φ in the following way: first we implement the algorithm $\mathcal{M}$ on φ, next we implement the algorithm $\mathcal{A}_{\mathcal{M}(\varphi)}$ on φ, and then we put

$$\mathcal{A}(\varphi) := \mathcal{A}_{\mathcal{M}(\varphi)}(\varphi)$$

It is clear from the above definitions that $\mathcal{A}$ always terminates in a finite number of steps and satisfies the equivalence (4.1). □

As a corollary of Theorem 2 we obtain the main theorem of this paper:

Theorem 3. *The logic RGML is decidable.*

Proof. Let us use the notation $\overline{yes} := no$ and $\overline{no} := yes$. We define the algorithm $\mathcal{R}$, acting on formulae, in the following way: $\mathcal{R}(\varphi) := \overline{\mathcal{A}(\neg\varphi)}$. We state that $\mathcal{R}$ is a decision method for RGML and the formula φ belongs to RGML iff $\mathcal{R}(\varphi) = yes$.

Indeed, for an arbitrary formula φ the algorithm $\mathcal{R}$ terminates in a finite number of steps and $\mathcal{R}(\varphi) = yes$ iff $\mathcal{A}(\neg\varphi) = no$, i.e. just when $\neg\varphi$ is not satisfiable. But that holds iff φ is valid or — what is the same — φ belongs to RGML. □

5. SOME EXTENSIONS OF THE LANGUAGE

We extend the language $\mathcal{L}_M$ with additional "rational" modal operators.

1. We consider n fractions $\frac{p_1}{q_1},\ldots,\frac{p_n}{q_n}$ and the corresponding to them modal operators M^{p_k,q_k}, $1 \le p_k < q_k$, p_k and q_k are relatively prime integers, $k = 1,\ldots,n$, with the interpretation in a model

$$\mathfrak{M}, x \models M^{p_k,q_k}\varphi \Leftrightarrow |R_\varphi(x)| > \frac{p_k}{q_k}|R(x)|, \quad k = 1,\ldots,n.$$

We consider also the dual modal operators W^{p_k,q_k}, respectively with the interpretation

$$\mathfrak{M}, x \models W^{p_k,q_k}\varphi \Leftrightarrow |R_{\neg\varphi}(x)| \le \frac{p_k}{q_k}|R(x)|, \quad k = 1,\ldots,n.$$

The logic, consisting of all valid formulae, we denote by RGML^n.

The common form of a modal elementary conjunction is:

$$\theta = B \wedge \bigwedge_{j=1}^{m_1} M^{p_1,q_1}\varphi_j \wedge \ldots \wedge \bigwedge_{j=m_{n-1}+1}^{m_n} M^{p_n,q_n}\varphi_j \wedge$$

$$\bigwedge_{j=1}^{l_1} W^{p_1,q_1}\psi_j \wedge \ldots \wedge \bigwedge_{j=l_{n-1}+1}^{l_n} W^{p_n,q_n}\psi_j.$$

For the system of linear inequality, attendant on θ, we get:

$$(\sigma_\theta)\quad \left| \begin{array}{l} \sum_{i=1}^{t} \varepsilon_{ij} a_i > \frac{p_k}{q_k} \sum_{i=1}^{t} a_i, \\ \quad k = 1, \ldots, n,\ j = 1, \ldots, m_1 + \cdots + m_n \\ \\ \sum_{i=1}^{t} \varepsilon_{ij} a_i \geq \frac{q_k - p_k}{q_k} \sum_{i=1}^{t} a_i, \\ \quad k = 1, \ldots, n,\ j = m_1 + \cdots + m_n + 1, \ldots, m_1 + \cdots + m_n + l_1 + \cdots + l_n \end{array} \right.$$

Besides, the tree-like model which we build in Proposition 2, is still finite and all the propositions from the case with just one modality hold.

2. If we consider all fractions $\frac{p}{q}$ with relatively prime integers p, q with $1 \leq p < q$, we obtain the logic RGML^ω.

As in the language under consideration there are only finite formulae, any modal elementary conjunction θ is finite and therefore contains only finitely many modal operators. Hence we can consider the attendant on it (finite) system (σ_θ). The tree-like frame and model are finite again, and all the propositions from the case with a single modality, including the decidability, hold.

The following proposition holds for the above defined logics:

Theorem 4. *The logic $RGML^n$ ($RGML^\omega$) is decidable. It coincides with the set of formulae, valid in the finite trees.*

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Received on March 28, 2010

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier Blvd, 1164 Sofia
BULGARIA
E-mail: tinko@fmi.uni-sofia.bg, yanchev@fmi.uni-sofia.bg