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PARTIAL DIFFERENTIAL EQUATIONS OF TIME-LIKE WEINGARTEN SURFACES IN THE THREE-DIMENSIONAL MINKOWSKI SPACE

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We study time-like surfaces in the three-dimensional Minkowski space with diagonalizable second fundamental form. On any time-like W-surface we introduce locally natural principal parameters and prove that such a surface is determined uniquely (up to motion) by a special invariant function, which satisfies a natural non-linear partial differential equation. This result can be interpreted as a solution of the Lund-Regge reduction problem for time-like W-surfaces with real principal curvatures in Minkowski space. We apply this theory to the class of linear fractional time-like W-surfaces with respect to their principal curvatures and obtain the natural partial differential equations describing them.

Keywords: Time-like W-surfaces in Minkowski space, natural parameters on time-like W-surfaces in Minkowski space, natural PDE's of time-like W-surfaces in Minkowski space.

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1. INTRODUCTION

It has been known to Weingarten [21, 22], Eisenhart [4], Wu [23] that without changing the principal lines on a Weingarten surface in Euclidean space, one can find geometric coordinates in which the coefficients of the metric are expressed by the principal curvatures (or principal radii of curvature).

The geometric parameters on Weingarten surfaces were used in [23] to find the classes of Weingarten surfaces yielding "geometric" $\mathfrak{so}(3)$-scattering systems (real or complex) for the partial differential equations, describing these surfaces.

We have shown that the Weingarten surfaces in Euclidean space [5, 6] and space-like surfaces in Minkowski space [7] admit geometrically determined principal parameters (*natural principal parameters*), which have the following property: all invariant functions on W-surfaces can be expressed in terms of one function ν, which satisfies one *natural* partial differential equation. The Bonnet type fundamental theorem states that any solution to the natural partial differential equation determines a W-surface uniquely up to motion. Thus the description of any class of W-surfaces (determined by a given Weingarten relation) is equivalent to the study of the solution space of their natural PDE. This solves the Lund-Regge reduction problem [13] for W-surfaces in Euclidean space and space-like W-surfaces in Minkowski space.

The relationship between the solutions of certain types of partial differential equations and the determination of various kinds of surfaces of constant curvature has generated many results which have applications to the areas of both pure and applied mathematics. This includes the determination of surfaces of either constant mean curvature or Gaussian curvature. It has long been known that there is a connection between surfaces of negative constant Gaussian curvature in Euclidean $\mathbb{R}^3$ and the sine-Gordon equation. The fundamental equations of surface theory are found to yield a type of geometrically based Lax pair. For instance, given a particular solution of the sinh-Laplace equation, this Lax pair can be integrated to determine the three fundamental vector fields related to the surface. These are also used to determine the coordinate vector field of the surface.

Further results are obtained based on the fundamental equations of surface theory, and it is shown how specific solutions of this sinh-Laplace equation can be used to obtain the coordinates of a surface in either Minkowski $\mathbb{R}^3_1$ or Euclidean $\mathbb{R}^3$ space [9, 10].

In [3] Bracken introduces some fundamental concepts and equations pertaining to the theory of surfaces in three-space, and, in particular, studies a class of sinh-Laplace equation which has the form $\Delta u = \pm \sinh u$.

In this paper we study time-like surfaces with real principal curvatures in the three dimensional Minkowski space $\mathbb{R}^3_1$.

A time-like surface $\mathcal{M}$ with real principal curvatures ν_1 and ν_2 is a Weingarten surface (W-surface) [21, 22] if there exists a function ν on $\mathcal{M}$ and two functions (Weingarten functions) f, g of one variable, such that

$$\nu_1 = f(\nu), \quad \nu_2 = g(\nu).$$

A basic property of W-surfaces in Euclidean space is the following theorem of Lie [12]:

The lines of curvature of any W-surface can be found in quadratures.

This remarkable property is also valid for space-like and time-like W-surfaces in Minkowski space.

We use four invariant functions (two principal normal curvatures ν_1, ν_2 and two principal geodesic curvatures γ_1, γ_2) and divide time-like W-surfaces into two classes with respect to these invariants:

(1) the class of *strongly regular* time-like surfaces defined by

$$(\nu_1 - \nu_2)\,\gamma_1\,\gamma_2 \neq 0;$$

(2) the class of time-like surfaces defined by

$$\gamma_1 = 0, \quad (\nu_1 - \nu_2)\,\gamma_2 \neq 0.$$

The basic tool to investigate the relation between time-like surfaces and the partial differential equations describing them, is Theorem 2.1. This theorem is a reformulation of the fundamental Bonnet theorem for the class of strongly regular time-like surfaces in terms of the four invariant functions. Further, we apply this theorem to time-like W-surfaces.

In Section 3 we prove (Proposition 3.3) that any time-like W-surface admits locally special principal parameters (*natural principal parameters*).

Theorem 3.6 is the basic theorem for time-like W-surfaces of type (1):

Any strongly regular time-like W-surface is determined uniquely up to motion by the functions f, g and the function ν, satisfying the natural PDE (3.3).

Theorem 3.7 is the baic theorem for time-like Weingarten surfaces of type (2):

Any time-like W-surface with $\gamma_1 = 0$ *is determined uniquely up to motion by the functions f, g and the function ν, satisfying the natural ODE* (3.8).

In natural principal parameters the four basic invariant functions, which determine time-like W-surfaces uniquely up to motions in $\mathbb{R}^3_1$, are expressed by a single function, and the system of Gauss-Codazzi equations reduces to a single partial differential equation (the Gauss equation). Thus, the number of the four invariant functions, which determine time-like W-surfaces, reduces to one invariant function, and the number of Gauss-Codazzi equations reduces to one *natural* PDE. This result gives a solution to the Lund-Regge reduction problem [13] for the time-like W-surfaces in $\mathbb{R}^3_1$. The Lund-Regge reduction problem has been analyzed and discussed from several view points in the paper of Sym [18].

In Proposition 4.1 we prove that

The natural principal parameters of a given time-like W-surface $\mathcal{M}$ are natural principal parameters for all parallel time-like surfaces $\overline{\mathcal{M}}(a)$, $a = \text{const} \neq 0$ *of* $\mathcal{M}$.

Theorem 4.2 states that (cf. [6, 7]):

The natural PDE of a given time-like W-surface $\mathcal{M}$ is the natural PDE of any parallel time-like surface $\overline{\mathcal{M}}(a)$, $a = \text{const} \neq 0$, *of* $\mathcal{M}$.

In [14, 16] Milnor studies surface theory in Euclidean and Minkowski space, considering harmonic maps and various relations between the Gauss curvature K, the mean curvature H and the curvature $H' = \dfrac{\nu_1 - \nu_2}{2}$. In [15, 6] it is proved that any surface in $\mathbb{R}^3_1$, whose Gauss curvature K and mean curvature H satisfy the linear relation

$$\delta K = \alpha H + \gamma, \quad \alpha, \gamma, \delta - \text{constants}; \quad \alpha^2 + 4\gamma\delta \neq 0, \tag{1.1}$$

is parallel to a surface, satisfying one of the following conditions: $H = 0$, $K = 1$ or $K = -1$.

There arises the following question: what are the natural PDE's describing the surfaces, whose curvatures satisfy the relation (1.1)?

Since any time-like surface $\mathcal{M}$, whose invariants K and H satisfy the linear relation (1.1), is (locally) parallel to one of the following three types of basic surfaces: a surface with $H = 0$; a surface with $K = 1$; a surface with $K = -1$, from Theorem 4.2 it follows that

Up to similarity, the time-like surfaces, whose curvatures satisfy the linear relation (1.1)*, are described by the natural PDE's of the basic surfaces.*

A. Ribaucour [17] has proved that *a necessary condition for the curvature lines of the first and second focal surfaces of* $\mathcal{M}$ *to correspond to each other resp. to a conjugate parametric lines on* $\mathcal{M}$ *is* $\rho_1 - \rho_2 = \text{const}$ *resp.* $\rho_1 \rho_2 = \text{const}$.

Von Lilienthal [19] (cf. [20, 1, 2, 4]) has proved in $\mathbb{R}^3$ that a surface with a relation $\rho_1 - \rho_2 = \dfrac{1}{R}$, $R = \text{const} \neq 0$, between its principal radii of curvature $\rho_1 = \dfrac{1}{\nu_1}$ and $\rho_2 = \dfrac{1}{\nu_2}$ has first and second focal surfaces $\widetilde{\mathcal{M}}$ of constant Gauss curvature $-R^2$ and vice versa. The involute surfaces $\overline{\mathcal{M}}(a)$, $a \in \mathbb{R}$ of $\widetilde{\mathcal{M}}$ are parallel surfaces of $\mathcal{M}$ with the property $\rho_1 - \rho_2 = \text{const}$. This implies that the family $\overline{\mathcal{M}}(a)$ are integrable surfaces as a consequence of the integrability of $\widetilde{\mathcal{M}}$. The curvatures of the above surfaces $\mathcal{M}$ satisfy the relation $K = \beta\, H'$, $\beta = \text{const} \neq 0$.

In $\mathbb{R}^3_1$ one can prove in a similar way the corresponding property: The first focal surface of a time-like surface with $K = \beta\, H'$, $\beta \neq 0$, is space-like of constant Gauss curvature $\beta^2/4$, and its second focal surface is time-like of constant Gauss curvature $-\beta^2/4$.

Obviously the time-like surfaces with $K = \beta\, H'$, $\beta = \text{const} \neq 0$, are not included in the class characterized by (1.1).

These surfaces belong to the classes of time-like W-surfaces, defined by the following more general linear relation

$$\delta K = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants}; \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0 \tag{1.2}$$

between the Gauss curvature K, the mean curvature H and the curvature H'. We denote this class by $\mathfrak{K}$.

We show that the class $\mathfrak{K}$ is the class of linear fractional time-like W-surfaces with respect to the principal curvatures (cf. [6, 7]). Furthermore, if $\mathcal{M}$ is a time-like surface in $\mathfrak{K}$, then its parallel surfaces $\overline{\mathcal{M}}(a)$, $a = \text{const}$, belong to $\mathfrak{K}$ too.

In the main Theorem 5.3 in this paper we determine ten basic relations with respect to the constants in (1.2) and each of them generates a *basic subclass of surfaces* of $\mathfrak{K}$. Any time-like surface $\mathcal{M}$, whose invariants K, H and H' satisfy the linear relation (1.2) is (locally) parallel to one of these basic surfaces.

In [10] Hu has cleared up the relationship between the PDE's

$$\begin{aligned}
&\alpha_{uu} - \alpha_{vv} = \pm \sin\alpha && (\sin -\text{Gordon PDE}),\\
&\alpha_{uu} - \alpha_{vv} = \pm \sinh\alpha && (\sinh -\text{Gordon PDE}),\\
&\alpha_{uu} + \alpha_{vv} = \pm \sin\alpha && (\sin -\text{Laplace PDE}),\\
&\alpha_{uu} + \alpha_{vv} = \pm \sinh\alpha && (\sinh -\text{Laplace PDE})
\end{aligned}$$

and the construction of various kinds of surfaces of constant curvature in $\mathbb{R}^3$ or $\mathbb{R}^3_1$.

In [11] by using Darboux transformations, from a known solution to the sinh-Laplace (resp. sin-Laplace) equation have been obtained explicitly new solutions to the sin-Laplace (resp. sinh-Laplace) equation.

Time-like surfaces with positive Gauss curvature and imaginary principal curvatures have been constructed in [8].

It is essential to note that the natural PDE's of the time-like W-surfaces from the class $\mathfrak{K}$ are expressed in the form $\delta\lambda = f(\lambda)$, where δ is one of the operators (cf. [6, 7]):

$$\begin{aligned}
\Delta\lambda &:= \lambda_{xx} + \lambda_{yy}, & \bar{\Delta}\lambda &:= \lambda_{xx} - \lambda_{yy};\\
\Delta^*\lambda &:= \lambda_{xx} + (\lambda^{-1})_{yy}, & \bar{\Delta}^*\lambda &:= \lambda_{xx} - (\lambda^{-1})_{yy}.
\end{aligned}$$

2. PRELIMINARIES

Let $\mathbb{R}^3_1$ be the three dimensional Minkowski space with the standard flat metric $\langle\, ,\, \rangle$ of signature $(2,1)$. We assume that the following orthonormal coordinate system $Oe_1e_2e_3:\ e_1^2 = e_2^2 = -e_3^2 = 1,\ \langle e_i, e_j\rangle = 0,\ i \neq j$ is fixed and gives the orientation of the space.

Let $\mathcal{M}:\ z = z(u,v),\ (u,v) \in \mathcal{D}$ be a time-like surface in the three dimensional Minkowski space $\mathbb{R}^3_1$ and ∇ be the flat Levi-Civita connection of the metric $\langle\, ,\, \rangle$. The unit normal vector field to $\mathcal{M}$ is denoted by l and E, F, G; L, M, N stand for the coefficients of the first and the second fundamental forms, respectively. Then we have

$$E = z_u^2 < 0, \quad F = z_u\, z_v, \quad G = z_v^2 > 0, \quad EG - F^2 < 0, \quad l^2 = 1.$$

The coefficients of the second fundamental form are given as follows:

$$L = l\, z_{uu} = -l_u\, z_u, \quad M = l\, z_{uv} = -l_u\, z_v = -l_v\, z_u, \quad N = l\, z_{vv} = -l_v\, z_v.$$

The linear Weingarten map γ is determined by the conditions

$$\gamma(z_u) = l_u, \quad \gamma(z_v) = l_v.$$

Then the mean curvature H and the Gauss curvature K of $\mathcal{M}$ are given in the standard way

$$H = -\frac{1}{2}\,\mathrm{tr}\,\gamma, \quad K = \det \gamma.$$

While the Weingarten map of a space-like surface satisfies the inequality $H^2 - K \geq 0$ and is always diagonalizable, the Weingarten map on a time-like surface can satisfy the inequalities $H^2 - K \geq 0$ or $H^2 - K < 0$.

Throughout this paper we deal with time-like surfaces satisfying the inequality $H^2 - K \geq 0$, i.e. time-like surfaces with real principal curvatures.

We suppose that the surfaces under consideration are free of points with $H^2 - K = 0$, i.e. satisfy the strong inequality

$$H^2 - K > 0 \tag{2.1}$$

and denote by H' the invariant curvature

$$H' = \sqrt{H^2 - K}.$$

Under the above condition the theory of time-like surfaces can be developed in a way similar to the theory of surfaces in Euclidean space or space-like surfaces in Minkowski space.

Time-like surfaces satisfying the condition (2.1) can be locally parameterized by principal parameters. Further we assume that the parametric net is principal, i.e.

$$F(u,v) = M(u,v) = 0, \quad (u,v) \in \mathcal{D}.$$

Then the principal curvatures ν_1, ν_2 and the principal geodesic curvatures (geodesic curvatures of the principal lines) γ_1, γ_2 are given by

$$\nu_1 = \frac{L}{E}, \quad \nu_2 = \frac{N}{G}; \qquad \gamma_1 = \frac{E_v}{2E\sqrt{G}}, \quad \gamma_2 = \frac{-G_u}{2G\sqrt{-E}}, \tag{2.2}$$

and ν_1, ν_2 satisfy the Rodrigues' formulas:

$$l_u = -\nu_1\, z_u, \quad l_v = -\nu_2\, z_v.$$

We consider the tangential frame field $\{X, Y\}$ determined by

$$X := \frac{z_u}{\sqrt{-E}}\,, \qquad Y := \frac{z_v}{\sqrt{G}}$$

and suppose that the moving frame field XYl is positive oriented.

The following Frenet type formulas for the frame field XYl are valid

$$\left|\begin{array}{lcl} \nabla_X X & = & \gamma_1 Y - \nu_1 l, \\ \nabla_X Y & = & \gamma_1 X, \\ \nabla_X l & = & -\nu_1 X, \end{array}\right. \qquad \left|\begin{array}{lcl} \nabla_Y X & = & -\gamma_2 Y, \\ \nabla_Y Y & = & -\gamma_2 X + \nu_2 l, \\ \nabla_Y l & = & -\nu_2 Y. \end{array}\right. \tag{2.3}$$

The Codazzi equations have the form

$$\gamma_1 = \frac{-Y(\nu_1)}{\nu_1 - \nu_2} = \frac{-(\nu_1)_v}{\sqrt{G}(\nu_1 - \nu_2)}, \qquad \gamma_2 = \frac{-X(\nu_2)}{\nu_1 - \nu_2} = \frac{-(\nu_2)_u}{\sqrt{-E}\,(\nu_1 - \nu_2)}, \tag{2.4}$$

and the Gauss equation can be written as follows:

$$X(\gamma_2) + Y(\gamma_1) + \gamma_1^2 - \gamma_2^2 = -\nu_1\nu_2 = -K,$$

or

$$\frac{(\gamma_2)_u}{\sqrt{-E}} + \frac{(\gamma_1)_v}{\sqrt{G}} + \gamma_1^2 - \gamma_2^2 = -\nu_1\nu_2 = -K. \tag{2.5}$$

A time-like surface $\mathcal{M}: \; z = z(u,v),\; (u,v) \in \mathcal{D}$ parameterized by principal parameters is said to be *strongly regular* if (cf. [5, 6, 7])

$$(\nu_1(u,v) - \nu_2(u,v))\gamma_1(u,v)\gamma_2(u,v) \neq 0, \quad (u,v) \in \mathcal{D}.$$

The Codazzi equations (2.4) imply that

$$\gamma_1\gamma_2 \neq 0 \iff (\nu_1)_v(\nu_2)_u \neq 0.$$

Because of (2.4) the formulas

$$\sqrt{-E} = \frac{-(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} > 0, \quad \sqrt{G} = \frac{-(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} > 0 \tag{2.6}$$

are valid on strongly regular time-like surfaces.

Taking into account (2.6), for strongly regular time-like surfaces formulas (2.3) become

$$\left| \begin{array}{l} X_u = -\dfrac{\gamma_1\,(\nu_2)_u}{\gamma_2(\nu_1-\nu_2)}\,Y + \dfrac{\nu_1\,(\nu_2)_u}{\gamma_2(\nu_1-\nu_2)}\,l, \; Y_u = -\dfrac{\gamma_1\,(\nu_2)_u}{\gamma_2(\nu_1-\nu_2)}\,X, \; l_u = \dfrac{\nu_1\,(\nu_2)_u}{\gamma_2(\nu_1-\nu_2)}\,X; \\[2ex] X_v = \dfrac{\gamma_2\,(\nu_1)_v}{\gamma_1(\nu_1-\nu_2)}\,Y, \quad Y_v = \dfrac{\gamma_2\,(\nu_1)_v}{\gamma_1(\nu_1-\nu_2)}\,X - \dfrac{\nu_2\,(\nu_1)_v}{\gamma_1(\nu_1-\nu_2)}\,l, \quad l_v = \dfrac{\nu_2\,(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)}\,Y. \end{array} \right. \tag{2.7}$$

Finding the compatibility conditions for the system (2.7), we reformulate the fundamental Bonnet theorem for strongly regular time-like surfaces in terms of the invariants of the surface.

Theorem 2.1. *Let the four functions $\nu_1(u,v)$, $\nu_2(u,v)$, $\gamma_1(u,v)$, $\gamma_2(u,v)$ be defined in a neighborhood $\mathcal{D}$ of (u_0,v_0) and satisfy the following conditions:*

1) $\quad (\nu_1 - \nu_2)\,\gamma_1\,(\nu_1)_v < 0, \quad (\nu_1 - \nu_2)\,\gamma_2\,(\nu_2)_u < 0,$

2.1) $\quad \left(\ln \dfrac{(\nu_1)_v}{\gamma_1}\right)_u = \dfrac{(\nu_1)_u}{\nu_1 - \nu_2}, \qquad \left(\ln \dfrac{(\nu_2)_u}{\gamma_2}\right)_v = -\dfrac{(\nu_2)_v}{\nu_1 - \nu_2},$

2.2) $\quad \dfrac{\nu_1 - \nu_2}{2}\left(\dfrac{(\gamma_2^2)_u}{(\nu_2)_u} + \dfrac{(\gamma_1^2)_v}{(\nu_1)_v}\right) - (\gamma_1^2 - \gamma_2^2) = \nu_1\nu_2.$

Let $z_0X_0Y_0l_0$ be an initial positive oriented orthonormal frame.

Then there exists a unique strongly regular time-like surface $\mathcal{M}:\ z=z(u,v)$, $(u,v)\in\mathcal{D}_0$ $((u_0,v_0)\in\mathcal{D}_0\subset\mathcal{D})$ with prescribed invariants ν_1, ν_2, γ_1, γ_2 such that

$$z(u_0,v_0)=z_0,\ X(u_0,v_0)=X_0,\ Y(u_0,v_0)=Y_0,\ l(u_0,v_0)=l_0.$$

Formulas (2.3) imply explicit expressions for the curvature and the torsion of any principal line on the time-like surface $\mathcal{M}$.

Let $c_1 : z=z(s)$, $\mathcal{M}\in J$ be a line from the family $\mathcal{F}_1$ ($v=\mathrm{const}$) parameterized by a natural parameter and κ_1, τ_1 be its curvature and torsion, respectively.

Since c_1 is an integral line of the unit time-like vector field X, then

$$z'=X,\quad z''=\nabla_X X=\gamma_1\,Y-\nu_1\,l,$$

$$z'''=\nabla_X\nabla_X X=-X(\nu_1)\,l+X(\gamma_1)\,Y+(\nu_1^2+\gamma_1^2)\,X,$$
$$\kappa_1^2=\nu_1^2+\gamma_1^2.$$

We use the formula
$$\tau=\frac{z'z''z'''}{z''^2}.$$

Since $\nu_1^2+\gamma_1^2>0$ along c_1, we find

$$\tau_1=\frac{\nu_1\,X(\gamma_1)-\gamma_1\,X(\nu_1)}{\nu_1^2+\gamma_1^2}=\frac{\nu_1^2}{\kappa_1^2}\,X\left(\frac{\gamma_1}{\nu_1}\right).$$

Denoting $\sin\theta_1=\frac{\gamma_1}{\kappa_1}$ and $\cos\theta_1=\frac{\nu_1}{\kappa_1}$, we obtain

$$\tau_1=X(\theta_1).$$

For the lines c_2 of the family $\mathcal{F}_2$ we obtain in a similar way the formulas

$$z'=Y,\quad z''=\nabla_Y Y=-\gamma_2\,X+\nu_2\,l,$$

$$z'''=\nabla_Y\nabla_Y Y=Y(\nu_2)\,l-Y(\gamma_2)\,X+(\gamma_2^2-\nu_2^2)\,Y,$$
$$\kappa_2^2=\varepsilon_2\,z''^2=\varepsilon_2\,(\nu_2^2-\gamma_2^2),\quad \varepsilon_2=\operatorname{sign} z''^2,$$

and in the case $z''^2\neq 0$,

$$\tau_2=\varepsilon_2\,\frac{\gamma_2\,Y(\nu_2)-\nu_2\,Y(\gamma_2)}{\kappa_2^2}=-\varepsilon_2\,\frac{\nu_2^2}{\kappa_2^2}\,Y\ \left(\frac{\gamma_2}{\nu_2}\right).$$

3. NATURAL PRINCIPAL PARAMETERS ON TIME-LIKE WEINGARTEN SURFACES

In this section we consider diagonalizable time-like Weingarten surfaces. For the sake of symmetry with respect to the principal curvatures ν_1 and ν_2 we use the following characterization of time-like Weingarten surfaces:

A diagonalizable time-like surface $\mathcal{M}: \ z = z(u,v), \ (u,v) \in \mathcal{D}$ *is Weingarten if there exist two real differentiable functions* $f(\nu), \ g(\nu), \ f(\nu) - g(\nu) \neq 0, \ f'(\nu)g'(\nu) \neq 0, \ \nu \in \mathcal{I} \subseteq \mathbb{R}$ *such that the principal curvatures of* $\mathcal{M}$ *at every point are given by* $\nu_1 = f(\nu), \ \nu_2 = g(\nu), \ \nu = \nu(u,v), \ (u,v) \in \mathcal{D}$.

The next statement gives a property of time-like Weingarten surfaces, which allows us to introduce special principal parameters on such surfaces.

Lemma 3.1. *Let* $\mathcal{M}: \ z = z(u,v), \ (u,v) \in \mathcal{D}$ *be a diagonalizable time-like Weingarten surface parameterized with principal parameters. Then the function*

$$\lambda = \sqrt{-E} \exp \left(\int \frac{f' d\nu}{f - g} \right)$$

does not depend on v*, while the function*

$$\mu = \sqrt{G} \exp \left(\int \frac{g' d\nu}{g - f} \right)$$

does not depend on u*.*

Proof. Taking into account (2.4) and (2.2), we find

$$\gamma_1 = \frac{-f'(\nu)Y(\nu)}{f(\nu) - g(\nu)} = Y(\ln \sqrt{-E}), \qquad \gamma_2 = \frac{-g'(\nu)\,X(\nu)}{f(\nu) - g(\nu)} = -X(\ln \sqrt{G}),$$

which imply that

$$Y \left(\int \frac{f'(\nu)\,d\nu}{f(\nu) - g(\nu))} + \ln \sqrt{-E} \right) = 0, \qquad X \left(\int \frac{g'(\nu)\,d\nu}{g(\nu) - f(\nu)} + \ln \sqrt{G} \right) = 0.$$

The last equalities mean that $\lambda_v = 0$ and $\mu_u = 0$. □

We define special principal parameters on a time-like Weingarten surface as follows:

Definition 3.2. *Let* $\mathcal{M}: \ z = z(u,v), \ (u,v) \in \mathcal{D}$ *be a diagonalizable time-like Weingarten surface parameterized with principal parameters. The parameters* (u,v)

are said to be natural *principal, if the functions* $\lambda(u)$ *and* $\mu(v)$ *from Lemma 3.1 are constants.*

Proposition 3.3. *Any diagonalizable time-like Weingarten surface admits locally natural principal parameters.*

Proof. Let $\mathcal{M}: z = z(u,v),\ (u,v) \in \mathcal{D}$ be a time-like Weingarten surface in the Minkowski space $\mathbb{R}^3_1$, parameterized with principal parameters. Then $\nu_1 = f(\nu)$, $\nu_2 = g(\nu),\ \nu = \nu(u,v)$ for some differentiable functions f, g and ν satisfying the conditions $(f(\nu) - g(\nu))\, f'(\nu)\, g'(\nu) \neq 0,\ (u,v) \in \mathcal{D}$.

Let $\mathfrak{a} = \text{const} \neq 0,\ \mathfrak{b} = \text{const} \neq 0,\quad (u_0, v_0) \in \mathcal{D}$ and $\nu_0 = \nu(u_0, v_0)$. We change the parameters $(u,v) \in \mathcal{D}$ with $(\bar{u}, \bar{v}) \in \bar{\mathcal{D}}$ by the formulas

$$\bar{u} = \mathfrak{a} \int_{u_0}^{u} \sqrt{-E}\, \exp\left(\int_{\nu_0}^{\nu} \frac{f' d\nu}{f - g} \right) du + \overline{u}_0, \quad \bar{u}_0 = \text{const},$$
$$\bar{v} = \mathfrak{b} \int_{v_0}^{v} \sqrt{G}\, \exp\left(\int_{\nu_0}^{\nu} \frac{g' d\nu}{g - f} \right) dv + \overline{v}_0, \quad \bar{v}_0 = \text{const}.$$

According to Lemma 3.1 it follows that $(\bar{u}, \bar{v})$ are again principal parameters and

$$\bar{E} = -\frac{1}{\mathfrak{a}^2} \exp\left(-2 \int_{\nu_0}^{\nu} \frac{f' d\nu}{f - g} \right), \quad \bar{G} = \frac{1}{\mathfrak{b}^2} \exp\left(-2 \int_{\nu_0}^{\nu} \frac{g' d\nu}{g - f} \right). \tag{3.1}$$

Then for the functions from Lemma 3.1 we find

$$\lambda(\bar{u}) = |\mathfrak{a}|^{-1}, \quad \mu(\bar{v}) = |\mathfrak{b}|^{-1}.$$

Furthermore $\mathfrak{a}^2\, \bar{E}(u_0, v_0) = -1,\ \mathfrak{b}^2\, \bar{G}(u_0, v_0) = 1.$ □

We assume now that the considered time-like Weingarten surface $\mathcal{M}: z = z(u,v),\ (u,v) \in \mathcal{D}$ is parameterized with natural principal parameters (u,v). It follows from the above proposition that the coefficients E and G (consequently L and N) are expressed by the invariants of the surface.

As an immediate consequence from Proposition 3.3 we get

Corollary 3.4. *Let* $\mathcal{M}$ *be a time-like Weingarten surface parameterized by natural principal parameters* (u,v). *Then any natural principal parameters* $(\tilde{u}, \tilde{v})$ *on* $\mathcal{M}$ *are determined by* (u,v) *up to an affine transformation of the type*

$$\tilde{u} = a_{11}\, u + b_1, \quad \tilde{v} = a_{22}\, v + b_2, \quad a_{11} a_{22} \neq 0,$$

or of the type

$$\tilde{u} = a_{12}\, v + c_1, \quad \tilde{v} = a_{21}\, u + c_2, \quad a_{12} a_{21} \neq 0,$$

where a_{ij}, b_i, c_i; $i, j = 1, 2$ are constants.

Next we give a simple criterion principal parameters to be natural.

Proposition 3.5. *Let a time-like Weingarten surface $\mathcal{M}: \ z = z(u, v), \ (u, v) \in \mathcal{D}$ be parameterized with principal parameters. Then (u, v) are natural principal if and only if*

$$\sqrt{-EG}(\nu_1 - \nu_2) = \text{const} \neq 0. \tag{3.2}$$

Proof. The equality $\sqrt{-EG}\,(\nu_1 - \nu_2) = c\,\lambda\,\mu, \ c = \text{const} \neq 0$, and Lemma 3.1 imply the assertion. □

3.1. STRONGLY REGULAR TIME-LIKE W-SURFACES.

We consider strongly regular time-like W-surfaces, i.e. time-like W-surfaces, satisfying the condition

$$\nu_u(u, v)\nu_v(u, v) \neq 0, \qquad (u, v) \in \mathcal{D}.$$

Our main theorem for such surfaces is

Theorem 3.6. *Let $f(\nu)$, $g(\nu)$; $\nu \in \mathcal{I}$, be two differentiable functions satisfying $f(\nu) - g(\nu) \neq 0$, $f'(\nu)\,g'(\nu) \neq 0$, and let $\nu(u, v)$, $(u, v) \in \mathcal{D}$ be a differentiable function such that*

$$\nu_u\,\nu_v \neq 0, \quad \nu(u, v) \in \mathcal{I}.$$

Let $(u_0, v_0) \in \mathcal{D}$, $\nu_0 = \nu(u_0, v_0)$ and $\mathfrak{a} \neq 0$, $\mathfrak{b} \neq 0$ be two constants. If

$$\begin{aligned} &\mathfrak{a}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{f'd\nu}{f-g}\right)\left[g'\nu_{uu} + \left(g'' - \frac{2g'^2}{g-f}\right)\nu_u^2\right] \\ &\quad + \mathfrak{b}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{g'd\nu}{g-f}\right)\left[f'\nu_{vv} + \left(f'' - \frac{2f'^2}{f-g}\right)\nu_v^2\right] = fg(f-g), \end{aligned} \tag{3.3}$$

then there exists a unique (up to a motion) strongly regular time-like Weingarten surface $\mathcal{M}: \ z = z(u, v), \ \ (u, v) \in \mathcal{D}_0 \subset \mathcal{D}$ with invariants

$$\begin{gathered} \nu_1 = f(\nu), \quad \nu_2 = g(\nu), \\ \gamma_1 = \exp\left(\int_{\nu_0}^{\nu} \frac{g'd\nu}{g-f}\right)\frac{-\mathfrak{b}f'}{f-g}\,\nu_v, \quad \gamma_2 = \exp\left(\int_{\nu_0}^{\nu} \frac{f'd\nu}{f-g}\right)\frac{-\mathfrak{a}g'}{f-g}\,\nu_u. \end{gathered} \tag{3.4}$$

Furthermore, (u,v) are natural principal parameters for $\mathcal{M}$.

Proof. Using Proposition 3.3, we obtain that the integrability conditions 2.1) and 2.2) in Theorem 2.2 reduce to (3.3), which proves the assertion. □

Introducing the functions

$$I := \int_{\nu_0}^{\nu} \frac{f'(\nu)\,d\nu}{f(\nu) - g(\nu)}\,, \qquad J := \int_{\nu_0}^{\nu} \frac{g'(\nu)\,d\nu}{g(\nu) - f(\nu)}\,, \tag{3.5}$$

we can write the PDE (3.3) in the form

$$\mathfrak{a}^2\, e^{2I}\,\left(J_{uu} + I_u\, J_u - J_u^2\right) - \mathfrak{b}^2\, e^{2J}\,\left(I_{vv} + I_v\, J_v - I_v^2\right) = -f\, g, \tag{3.6}$$

and the principal geodetic curvatures (3.4) in the form

$$\gamma_1 = -\mathfrak{b}\, e^J\, I_v, \quad \gamma_2 = \mathfrak{a}\, e^I\, J_u. \tag{3.7}$$

Hence, with respect to natural principal parameters every strongly regular time-like Weingarten surface possesses a *natural PDE* (3.3) (or equivalently (3.6)).

3.2. TIME-LIKE W-SURFACES WITH $\gamma_1 = 0$.

In this subsection we consider time-like W-surfaces in Minkowski space with first principal geodesic curvature $\gamma_1 = 0$ and prove the fundamental theorem of Bonnet type for this class.

Let $\mathcal{M}:\ z = z(u,v),\ (u,v) \in \mathcal{D}$ be a time-like W-surface, parameterized by natural principal parameters. Then we can assume

$$\mathfrak{a}\,\sqrt{E} = e^I, \qquad \mathfrak{b}\,\sqrt{G} = e^J,$$

where I and J are the functions (3.5) and $\mathfrak{a}$, $\mathfrak{b}$ are some positive constants. We note that under the condition $\gamma_1 = 0$ it follows that the function $\nu = \nu(u)$ does not depend on v.

Considering the system (2.3), we obtain that the compatibility conditions for this system reduce to only one - the Gauss equation, which has the form:

$$X(\gamma_2) - \gamma_2^2 = -f(\nu)\, g(\nu).$$

Thus we obtain the following Bonnet type theorem for time-like W-surfaces satisfying the condition $\gamma_1 = 0$:

Theorem 3.7. *Let $f(\nu),\ g(\nu);\ \nu \in \mathcal{I}$, be two differentiable functions satisfying $f(\nu) - g(\nu) \neq 0$, $f'(\nu)\, g'(\nu) \neq 0$ and let $\nu(u,v) = \nu(u),\ (u,v) \in \mathcal{D}$ be a differentiable function such that*

$$\nu_u \neq 0, \quad \nu(u,v) \in \mathcal{I}.$$

Let $(u_0, v_0) \in \mathcal{D},\ \nu_0 = \nu(u_0, v_0)$ *and* $\mathfrak{a} > 0$ *be a constant. If*

$$\mathfrak{a}^2\, e^{2I}\,(J_{uu} + I_u\, J_u - J_u^2) = -f(\nu)\, g(\nu), \tag{3.8}$$

then there exists a unique (up to a motion) time-like W-surface $\mathcal{M}:\ z = z(u, v)$, $(u, v) \in \mathcal{D}_0 \subset \mathcal{D}$ *with invariants*

$$\begin{aligned} \nu_1 &= f(\nu), \quad \nu_2 = g(\nu), \\ \gamma_1 &= 0, \qquad \gamma_2 = \mathfrak{a}\, e^{I}\,(J)_u. \end{aligned} \tag{3.9}$$

Furthermore, (u, v) *are natural principal parameters on* $\mathcal{M}$.

Hence, with respect to natural principal parameters every time-like Weingarten surface with $\gamma_1 = 0$ possesses a *natural ODE* (3.8).

4. PARALLEL TIME-LIKE SURFACES IN MINKOWSKI SPACE AND THEIR NATURAL PDE'S

Let $\mathcal{M}:\ \ z = z(u, v),\ (u, v) \in \mathcal{D}$ be a time-like surface, parameterized by principal parameters and $l(u, v),\ l^2 = 1$ be the unit normal vector field of $\mathcal{M}$. The parallel surfaces of $\mathcal{M}$ are given by

$$\overline{\mathcal{M}}(a):\ \bar{z}(u, v) = z(u, v) + a\, l(u, v), \quad a = \text{const} \neq 0, \quad (u, v) \in \mathcal{D}. \tag{4.1}$$

We call the family $\{\overline{\mathcal{M}}(a),\ a = \text{const} \neq 0\}$ the *parallel family* of $\mathcal{M}$.

Taking into account (4.1), we find

$$\bar{z}_u = (1 - a\,\nu_1)\, z_u, \quad \bar{z}_v = (1 - a\,\nu_2)\, z_v. \tag{4.2}$$

Excluding the points, where $(1 - a\,\nu_1)(1 - a\,\nu_2) = 0$, we obtain that the corresponding unit normal vector fields $\bar{l}$ to $\overline{\mathcal{M}}(a)$ and l to $\mathcal{M}$ satisfy the equality $\bar{l} = \varepsilon\, l$, where $\varepsilon := \operatorname{sign}(1 - a\,\nu_1)(1 - a\,\nu_2)$. In view of (4.2) it follows that $\bar{E} < 0$ and $\bar{G} > 0$. Hence, the parallel surfaces $\overline{\mathcal{M}}(a)$ of a time-like surface $\mathcal{M}$ are also time-like surfaces.

The relations between the principal curvatures $\nu_1(u, v)$, $\nu_2(u, v)$ of $\mathcal{M}$ and $\bar{\nu}_1(u, v)$, $\bar{\nu}_2(u, v)$ of its parallel time-like surface $\overline{\mathcal{M}}(a)$ are

$$\bar{\nu}_1 = \varepsilon\, \frac{\nu_1}{1 - a\,\nu_1}\,, \quad \bar{\nu}_2 = \varepsilon\, \frac{\nu_2}{1 - a\,\nu_2}; \quad \nu_1 = \frac{\varepsilon\,\bar{\nu}_1}{1 + a\,\varepsilon\,\bar{\nu}_1}\,, \quad \nu_2 = \frac{\varepsilon\,\bar{\nu}_2}{1 + a\,\varepsilon\,\bar{\nu}_2}. \tag{4.3}$$

Let $K = \nu_1\,\nu_2,\ H = \dfrac{1}{2}(\nu_2 + \nu_2),\ H' = \dfrac{1}{2}(\nu_2 - \nu_2)$ be the three invariants of the time-like surface $\mathcal{M}$. The equalities (4.3) imply the relations between the invariants $\bar{K}$, $\bar{H}$ and $\bar{H}'$ of $\overline{\mathcal{M}}(a)$ and the corresponding invariants of $\mathcal{M}$:

$$K = \frac{\bar{K}}{1 + 2a\,\varepsilon\bar{H} + a^2\bar{K}}\,, \quad H = \frac{\varepsilon\,\bar{H} + a\bar{K}}{1 + 2a\,\varepsilon\bar{H} + a^2\bar{K}}\,, \quad H' = \frac{\varepsilon\,\bar{H}'}{1 + 2a\,\varepsilon\,\bar{H} + a^2\bar{K}}\,. \tag{4.4}$$

Now let $\mathcal{M}: \ z = z(u,v), \ (u,v) \in \mathcal{D}$ be a time-like Weingarten surface with Weingarten functions $f(\nu)$ and $g(\nu)$. We suppose that (u,v) are natural principal parameters for $\mathcal{M}$. We show that (u,v) are also natural principal parameters for any parallel time-like surface $\overline{\mathcal{M}}(a)$.

Proposition 4.1. *The natural principal parameters (u,v) of a given time-like W-surface $\mathcal{M}$ are natural principal parameters for all parallel time-like surfaces $\overline{\mathcal{M}}(a), \ a = \mathrm{const} \neq 0$ of $\mathcal{M}$.*

Proof. Let $(u,v) \in \mathcal{D}$ be natural principal parameters for $\mathcal{M}$, (u_0, v_0) be a fixed point in $\mathcal{D}$ and $\nu_0 = \nu(u_0, v_0)$. The coefficients E and G of the first fundamental form of $\mathcal{M}$ are given by (3.1). The corresponding coefficients $\bar{E}$ and $\bar{G}$ of $\overline{\mathcal{M}}(a)$ in view of (4.2) are

$$\bar{E} = (1 - a\,\nu_1)^2\, E, \quad \bar{G} = (1 - a\,\nu_2)^2\, G. \tag{4.5}$$

Equalities (4.3) imply that $\overline{\mathcal{M}}(a)$ is again a Weingarten surface with Weingarten functions

$$\bar{\nu}_1(u,v) = \bar{f}(\nu) = \frac{\varepsilon f(\nu)}{1 - af(\nu)}\,, \quad \bar{\nu}_2(u,v) = \bar{g}(\nu) = \frac{\varepsilon g(\nu)}{1 - ag(\nu)}. \tag{4.6}$$

Using (4.6), we compute

$$\bar{f} - \bar{g} = \frac{\varepsilon(f-g)}{(1 - a\,f)(1 - a\,g)}\,,$$

which shows that $\mathrm{sign}\,(\bar{f} - \bar{g}) = \mathrm{sign}\,(f - g)$.

Further, we denote by $f_0 := f(\nu_0), \ g_0 := g(\nu_0)$ and taking into account (3.2) and (4.5), we compute

$$\sqrt{-\bar{E}\,\bar{G}}\,(\bar{f} - \bar{g}) = \sqrt{-E\,G}\,(f - g) = \mathrm{const} \neq 0,$$

which proves the assertion. □

Using the above statement, we prove the following theorem.

Theorem 4.2. T*he natural PDE of a given time-like W-surface $\mathcal{M}$ is the natural PDE of any parallel time-like surface $\overline{\mathcal{M}}(a), \ a = \mathrm{const} \neq 0$, of $\mathcal{M}$.*

Proof. We have to express equation (3.3) in terms of the Weingarten functions of the parallel time-like surface $\overline{\mathcal{M}}(a)$.

Putting

$$\bar{E}_0 = (1 - a\,\nu_1(u_0, v_0))^2\, E_0 = -\mathfrak{a}^{-2}\,(1 - a\,f_0)^2 =: -\bar{\mathfrak{a}}^{-2},$$

$$\bar{G}_0 = (1 - a\,\nu_2(u_0, v_0))^2\, G_0 = \mathfrak{b}^{-2}\,(1 - a\,g_0)^2 =: \bar{\mathfrak{b}}^{-2},$$

we obtain

$$
\begin{aligned}
&\bar{\mathfrak{a}}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{\bar f' d\nu}{\bar f - \bar g}\right)\left[\bar g' \nu_{uu} + \left(\bar g'' - \frac{2\bar g'^2}{\bar g - \bar f}\right)\nu_u^2\right] \\
&+ \bar{\mathfrak{b}}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{\bar g' d\nu}{\bar g - \bar f}\right)\left[\bar f' \nu_{vv} + \left(\bar f'' - \frac{2\bar f'^2}{\bar f - \bar g}\right)\nu_v^2\right] - \bar f\,\bar g(\bar f - \bar g) \\
&= \mathfrak{a}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{f' d\nu}{f - g}\right)\left[g' \nu_{uu} + \left(g'' - \frac{2g'^2}{g - f}\right)\nu_u^2\right] \\
&+ \mathfrak{b}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{g' d\nu}{g - f}\right)\left[f' \nu_{vv} + \left(f'' - \frac{2f'^2}{f - g}\right)\nu_v^2\right] - f\,g(f - g).
\end{aligned}
$$

Hence, the natural PDE of $\overline{\mathcal{M}}(a)$ in terms of the Weingarten functions $\bar f(\nu)$, $\bar g(\nu)$ coinsides with the natural PDE of $\mathcal{M}$ in terms of the Weingarten functions $f(\nu)$ and $g(\nu)$. □

5. TIME-LIKE SURFACES WHOSE CURVATURES SATISFY A LINEAR RELATION

We now consider time-like W-surfaces, whose three invariants K, H and H' satisfy a linear relation:

$$\delta K = \alpha\, H + \beta\, H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants}, \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0. \tag{5.1}$$

A time-like W-surface with principal curvatures ν_1 and ν_2 is said to be *linear fractional* if

$$\nu_1 = \frac{A\nu_2 + B}{C\nu_2 + D}, \quad BC - AD \neq 0. \tag{5.2}$$

We exclude the case $A = D$, $B = C = 0$, which characterizes the points with $H^2 - K = 0$, and show that the classes of surfaces with characterizing conditions (5.1) and (5.2), respectively, coincide.

Lemma 5.1. *Any surface whose invariants* $K = \nu_1\,\nu_2$, $H = \dfrac{1}{2}(\nu_1 + \nu_2)$ *and* $H' = \dfrac{1}{2}(\nu_1 - \nu_2)$ *satisfy the linear relation* (5.1) *is a linear fractional time-like Weingarten surface determined by* (5.2), *and vice versa.*

The relations between the constants $\alpha, \beta, \gamma, \delta$ in (5.1) and A, B, C, D in (5.2) are given by the equalities:

$$\alpha = A - D, \quad \beta = -(A + D), \quad \gamma = B, \quad \delta = C. \tag{5.3}$$

We denote by $\mathfrak{K}$ the class of all time-like surfaces with $H^2 - K > 0$, whose curvatures satisfy (5.1) or equivalently (5.2).

The aim of our study is to classify all natural PDE's of the surfaces from the class $\mathfrak{K}$.

The parallelism between two surfaces given by (4.1) is an equivalence relation. On the other hand, Theorem 4.2 shows that the surfaces from an equivalence class have one and the same natural PDE. Hence, it is sufficient to find the natural PDE's of the equivalence classes. For any equivalence class, we use a special representative, which we call *a basic class*. Thus the classification of the natural PDE's of the surfaces in the class $\mathfrak{K}$ reduces to the natural PDE's of the basic classes.

In view of Theorem 4.2, we prove the following classification theorem.

Theorem 5.2. *Up to similarity, the time-like surfaces in Minkowski space, whose curvatures K, H and H' satisfy the linear relation*

$$\delta K = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants}; \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0,$$

are described by the natural PDE's of the following basic surfaces:

(1) $H = 0: \quad \nu = e^{\lambda}, \quad \bar{\Delta}\lambda = e^{\lambda};$

(2) $H = \dfrac{1}{2}: \quad \nu = \dfrac{1}{2}(1 - e^{\lambda}), \quad \bar{\Delta}\lambda = \sinh\lambda;$

(3) $H' = 1: \quad \bar{\Delta}^*(e^{\nu}) = 2\,\nu\,(\nu + 2);$

(4) $H = \beta\, H' \; (\beta^2 > 1): \qquad \bar{\Delta}^*(\nu^{\beta}) = 2\,\dfrac{\beta\,(\beta+1)}{(\beta-1)^2}\,\nu;$

(5) $H = \beta\, H' \; (\beta^2 < 1): \qquad \Delta^*(\nu^{\beta}) = 2\,\dfrac{\beta\,(\beta+1)}{(\beta-1)^2}\,\nu;$

(6) $\left|\begin{array}{l} H = \beta H' + 1 \\ \beta^2 > 1 \end{array}\right. : \ \nu = \dfrac{(\beta-1)\lambda+2}{2}, \ \bar{\Delta}^*(\lambda^{\beta}) = \dfrac{\beta((\beta-1)\lambda+2)((\beta+1)\lambda+2)}{2(\beta-1)\lambda};$

(7) $\left|\begin{array}{l} H = \beta\, H' + 1 \\ \beta^2 < 1 \end{array}\right. : \ \nu = \dfrac{(\beta-1)\,\lambda+2}{2}, \ \Delta^*(\lambda^{\beta}) = \dfrac{\beta((\beta-1)\lambda+2)((\beta+1)\lambda+2)}{2(\beta-1)\,\lambda};$

(8) $K = -1: \quad \nu = \tan\lambda, \quad \Delta\lambda = -\sin\lambda;$

(9) $K = 2\,H': \quad \nu = \dfrac{\lambda-4}{\lambda-2}, \quad \bar{\Delta}^*(e^{\lambda}) = 2;$

$$(10)\ K=\beta H'+\gamma\ (\beta\neq 0,\ \gamma<0): \quad \left| \begin{array}{l} \nu=\lambda+\dfrac{\beta}{2}, \quad \mathcal{I}=\dfrac{1}{\sqrt{-\gamma}}\arctan\dfrac{\lambda}{\sqrt{-\gamma}}, \\ \bar{\Delta}^*(e^{\beta\mathcal{I}})=-\dfrac{\beta\gamma}{2}\,\dfrac{\lambda\,(\beta\lambda+2\gamma)}{\lambda^2-\gamma}. \end{array} \right.$$

Proof. According to the constant C in (5.2), the linear fractional time-like W-surfaces are divided into two classes: linear fractional time-like W-surfaces, determined by the condition $C=0$ and linear fractional time-like W-surfaces, determined by the condition $C\neq 0$.

I. Linear fractional time-like Weingarten surfaces with $C=0$.

This class is determined by the equality

$$\alpha H+\beta H'+\gamma=0, \quad (\alpha,\gamma)\neq(0,0), \quad \alpha^2-\beta^2\neq 0. \tag{5.4}$$

For the invariants of the time-like parallel surface $\overline{\mathcal{M}}(a)$ of $\mathcal{M}$, because of (4.4), we get the relation

$$\varepsilon\,(\alpha+2\,a\,\gamma)\,\bar{H}+\varepsilon\,\beta\,\bar{H}'+\gamma=-a\,(\alpha+a\,\gamma)\,\bar{K}. \tag{5.5}$$

Let $\eta:=\operatorname{sign}(\alpha^2-\beta^2)$. Each time choosing appropriate values for the constants $\mathfrak{a}$, $\mathfrak{b}$ and ν_0 in (3.3), we consider the following subclasses and their natural PDE's:

1) $\alpha=0,\ \beta\neq 0,\ \gamma\neq 0$. Assuming that $\gamma=1$, the relation (5.4) becomes

$$\beta H'+1=0.$$

The natural PDE for these W-surfaces is

$$(e^{-\beta\nu})_{uu}-(e^{\beta\nu})_{vv}=\frac{2}{\beta}\,\nu\,(\beta\nu-2). \tag{5.6}$$

Up to similarities these time-like W-surfaces are generated by the basic class $H'=1$ with the natural PDE

$$(e^{\nu})_{uu}-(e^{-\nu})_{vv}=2\,\nu\,(\nu+2), \tag{5.6*}$$

which is the case (3) in the statement of the theorem.

2) $\alpha\neq 0,\ \gamma=0$. Assuming that $\alpha=1$, the relation (5.4) becomes

$$H+\beta H'=0.$$

2.1) $\beta\neq 0,\ \eta=-1\ (\beta^2-1>0)$. Choosing $\mathfrak{b}^2\,\dfrac{\beta-1}{\beta+1}\,\nu_0^{-(\beta+1)}=1,\ \mathfrak{a}^2\,\nu_0^{\beta-1}=1$, the natural PDE becomes

$$\left(\nu^{-\beta}\right)_{uu}-\left(\nu^{\beta}\right)_{vv}=2\,\frac{\beta(\beta-1)}{(\beta+1)^2}\,\nu, \tag{5.7}$$

which is the case (4) in the statement of the theorem.

2.2) $\beta \neq 0,\ \eta = 1\ (\beta^2 - 1 < 0)$. Choosing $\mathfrak{b}^2\,\dfrac{\beta-1}{\beta+1}\nu_0^{-(\beta+1)} = -1,\ \mathfrak{a}^2\,\nu_0^{\beta-1} = 1$, the natural PDE becomes

$$\left(\nu^{-\beta}\right)_{uu} + \left(\nu^{\beta}\right)_{vv} = 2\,\frac{\beta(\beta-1)}{(\beta+1)^2}\,\nu, \tag{5.8}$$

which is the case (5) in the statement of the theorem.

2.3) $\beta = 0$. Putting $\nu = e^{\lambda}$, we get the natural PDE for time-like surfaces with $H = 0$:

$$\lambda_{uu} - \lambda_{vv} = e^{\lambda}, \tag{5.9}$$

which is the case (1) in the statement of the theorem.

3) $\alpha \neq 0,\ \beta = 0,\ \gamma \neq 0$. Assuming that $\alpha = 1$, the relation (5.4) becomes

$$H + \gamma = 0.$$

Putting $|H|\,e^{\lambda} := H - \nu = H' > 0$, we get the one-parameter system of natural PDE's for CMC time-like surfaces with $H = -\gamma$:

$$\lambda_{uu} - \lambda_{vv} = 2\,|H|\,\sinh\lambda. \tag{5.10}$$

Up to similarities these time-like W-surfaces are generated by the basic class $|H| = \frac{1}{2}$ with the natural PDE

$$\lambda_{uu} - \lambda_{vv} = \sinh\lambda, \tag{5.10*}$$

which is the case (2) in the statement of the theorem.

4) $\alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0$. Assuming that $\alpha = 1$ we have

$$H + \beta\,H' + \gamma = 0, \quad \beta^2 - 1 \neq 0.$$

Let $\lambda := 2\,H' = \dfrac{-2}{\beta+1}\,(\nu+\gamma) > 0$.

4.1) If $\eta = -1\ (\beta^2 - 1 > 0)$ and choosing

$$\mathfrak{b}^2 = \frac{\beta+1}{\beta-1}\left(\frac{-2}{\beta+1}(\nu_0+\gamma)\right)^{\beta+1},\ \mathfrak{a}^2 = \left(\frac{-2}{\beta+1}(\nu_0+\gamma)\right)^{-(\beta-1)},$$

the natural PDE becomes

$$\left(\lambda^{-\beta}\right)_{uu} - \left(\lambda^{\beta}\right)_{vv} = \frac{\beta}{2\,(\beta+1)}\,\frac{((\beta+1)\lambda + 2\,\gamma)((\beta-1)\lambda + 2\,\gamma)}{\lambda}. \tag{5.11}$$

Up to similarities these time-like W-surfaces are generated by the basic class $H = \beta\,H' + 1,\ \beta^2 > 1$ with the natural PDE

$$\left(\lambda^{\beta}\right)_{uu} - \left(\lambda^{-\beta}\right)_{vv} = \frac{\beta}{2\,(\beta-1)}\,\frac{((\beta+1)\lambda + 2)((\beta-1)\lambda + 2)}{\lambda}, \tag{5.11*}$$

which is the case (6) in the statement of the theorem.

4.2) If $\eta = 1$ ($\beta^2 - 1 < 0$) and choosing

$$\mathfrak{b}^2 = -\frac{\beta+1}{\beta-1}\left(\frac{-2}{\beta+1}(\nu_0+\gamma)\right)^{\beta+1}, \ \mathfrak{a}^2 = \left(\frac{-2}{\beta+1}(\nu_0+\gamma)\right)^{-(\beta-1)},$$

the natural PDE becomes

$$\left(\lambda^{-\beta}\right)_{uu} + \left(\lambda^{\beta}\right)_{vv} = \frac{\beta}{2\,(\beta+1)}\,\frac{((\beta+1)\lambda + 2\,\gamma)((\beta-1)\lambda + 2\,\gamma)}{\lambda}. \quad (5.12)$$

Up to similarities these time-like W-surfaces are generated by the basic class $H = \beta\, H' + 1$, $\beta^2 < 1$ with the natural PDE

$$\left(\lambda^{\beta}\right)_{uu} + \left(\lambda^{-\beta}\right)_{vv} = \frac{\beta}{2\,(\beta-1)}\,\frac{((\beta+1)\lambda + 2)((\beta-1)\lambda + 2)}{\lambda}, \quad (5.12^*)$$

which is the case (7) in the statement of the theorem.

II. Linear fractional time-like Weingarten surfaces with $C \neq 0$.

Let $C = 1$. The equality (5.1) gets the form

$$K = \alpha\, H + \beta\, H' + \gamma. \quad (5.13)$$

The corresponding relation for the parallel surface $\overline{\mathcal{M}}(a)$ is

$$\varepsilon(\alpha + 2\,a\,\gamma)\,\bar{H} + \varepsilon\,\beta\,\bar{H}' + \gamma = (1 - a\,\alpha - a^2\,\gamma)\,\bar{K}. \quad (5.14)$$

Each time choosing appropriate values for the constants $\mathfrak{a}$, $\mathfrak{b}$ and ν_0 in (3.3), we consider the following subclasses and their natural PDE's:

5) $\alpha = \gamma = 0$, $\beta \neq 0$. The relation (5.13) becomes

$$K = \beta H' \quad \Longleftrightarrow \quad \rho_1 - \rho_2 = -\frac{2}{\beta},$$

where $\rho_1 = \dfrac{1}{\nu_1}$, $\rho_2 = \dfrac{1}{\nu_2}$ are the principal radii of curvature of $\mathcal{M}$.

Putting $\lambda := 4\,\dfrac{\nu - \beta}{2\,\nu - \beta}$, the natural PDE of these time-like surfaces gets the form

$$\left(e^{\lambda}\right)_{uu} - \left(e^{-\lambda}\right)_{vv} - \frac{\beta^4}{8} = 0. \quad (5.15)$$

Up to similarities these time-like W-surfaces are generated by the basic class $K = 2\,H'$ with the natural PDE

$$\left(e^{\lambda}\right)_{uu} - \left(e^{-\lambda}\right)_{vv} - 2 = 0, \quad (5.15^*)$$

which is the case (9) in the statement of the theorem.

6) $(\alpha, \gamma) \neq (0, 0)$, $\alpha^2 + 4\gamma \geq 0$. The relation (5.14) implies that there exists a time-like surface $\overline{\mathcal{M}}(a)$, parallel to $\mathcal{M}$, which satisfies the relation (5.4). Hence the natural PDE of $\mathcal{M}$ is one of the PDE's (5.6) - (5.12).

7) $\alpha^2 + 4\,\gamma < 0$. It follows that $\gamma < 0$. The relation (5.14) implies that there exists a time-like surface $\overline{\mathcal{M}}(a)$ parallel to $\mathcal{M}$, which satisfies the relation

$$K = \beta H' + \gamma. \tag{5.16}$$

7.1) $\beta = 0$. The relation (5.16) becomes $K = \gamma < 0$, i.e. $\overline{\mathcal{M}}$ is of constant negative sectional curvature γ. Putting $\lambda := 2 \arctan \dfrac{\nu}{\sqrt{-\gamma}}$, we get the natural PDE of this surface

$$\lambda_{uu} + \lambda_{vv} = -K^2 \sin \lambda. \tag{5.17}$$

Up to similarities these time-like W-surfaces are generated by the basic class $K = -1$ with the natural PDE

$$\lambda_{uu} + \lambda_{vv} = -\sin \lambda, \tag{5.17*}$$

which is the case (8) in the statement of the theorem.

7.2) $\beta \neq 0$, $\gamma < 0$. Choosing $\nu_0 = \dfrac{\beta}{2}$, the natural PDE of $\overline{\mathcal{M}}$ becomes

$$(\exp{(\beta\, \mathcal{I})})_{uu} - (\exp{(-\beta\, \mathcal{I})})_{vv} = -\frac{\beta\,\gamma}{2}\, \frac{\lambda\; (\beta\, \lambda + 2\,\gamma)}{\lambda^2 - \gamma}\,, \tag{5.18}$$

where

$$\mathcal{I} = \frac{1}{\sqrt{-\gamma}} \arctan \frac{\lambda}{\sqrt{-\gamma}}, \quad \lambda := \nu - \frac{\beta}{2},$$

which is the case (10) in the statement of the theorem.

The proof of Theorem 5.2 is complete. □

6. SUMMARY

Summarizing the results in [6, 7] and in the present paper, we obtain the following parallel between the natural PDE's describing linear fractional W-surfaces in $\mathbb{R}^3$, linear fractional space-like and time-like W-surfaces in $\mathbb{R}^3_1$, respectively.

(i) The natural PDE for a Weingarten surface in Euclidean space is of the type:

$$\begin{aligned} &\mathfrak{a}^2 \exp\left(2 \int_{\nu_0}^{\nu} \frac{f' d\nu}{f-g}\right) \left[g' \nu_{uu} + \left(g'' - \frac{2g'^2}{g-f}\right) \nu_u^2\right] \\ -\;\; &\mathfrak{b}^2 \exp\left(2 \int_{\nu_0}^{\nu} \frac{g' d\nu}{g-f}\right) \left[f' \nu_{vv} + \left(f'' - \frac{2f'^2}{f-g}\right) \nu_v^2\right] = -fg(f-g), \end{aligned}$$

or, equivalently,

$$\mathfrak{a}^2\, e^{2I}\,(J_{uu} + I_u\, J_u - J_u^2) + \mathfrak{b}^2\, e^{2J}(I_{vv} + I_v\, J_v - I_v^2) = f(\nu)\, g(\nu).$$

(ii) The natural PDE for a space-like Weingarten surface in Minkowski space is of the type:

$$a^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{f' d\nu}{f-g}\right)\left[g'\nu_{uu} + \left(g'' - \frac{2g'^2}{g-f}\right)\nu_u^2\right]$$
$$-b^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{g' d\nu}{g-f}\right)\left[f'\nu_{vv} + \left(f'' - \frac{2f'^2}{f-g}\right)\nu_v^2\right] = fg(f-g),$$

or, equivalently,

$$\mathfrak{a}^2\, e^{2I}\,(J_{uu} + I_u\, J_u - J_u^2) + \mathfrak{b}^2\, e^{2J}(I_{vv} + I_v\, J_v - I_v^2) = -f(\nu)\, g(\nu).$$

(iii) The natural PDE for a time-like Weingarten surface with real principal curvatures in Minkowski space is of the type:

$$\mathfrak{a}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{f' d\nu}{f-g}\right)\left[g'\nu_{uu} + \left(g'' - \frac{2g'^2}{g-f}\right)\nu_u^2\right]$$
$$+\mathfrak{b}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{g' d\nu}{g-f}\right)\left[f'\nu_{vv} + \left(f'' - \frac{2f'^2}{f-g}\right)\nu_v^2\right] = fg(f-g),$$

or, equivalently,

$$\mathfrak{a}^2\, e^{2I}\,\left(J_{uu} + I_u\, J_u - J_u^2\right) - \mathfrak{b}^2\, e^{2J}\,\left(I_{vv} + I_v\, J_v - I_v^2\right) = -f(\nu)\, g(\nu).$$

Therefore for the corresponding basic linear fractional surfaces in $\mathbb{R}^3$ and $\mathbb{R}^3_1$ we obtain the correspondence between their natural PDE's.

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