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LEAST ENUMERATIONS OF UNARY PARTIAL STRUCTURES

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In the present paper we consider structures with unary partial functions and partial predicates, called unary structures. Unary structures does not contain equality and inequality among the predicates of the structure. The main result obtained here is a characterization of the unary structures which have least enumerations, called degrees of the structures. As a corollary it is obtained a characterization of the unary structures which admit effective enumerations. There are some interesting results concerning the spectrum and the so-called quasi-degree of such structures.

Keywords: Enumeration, enumeration degree, enumeration operator, degree of a structure, type of a sequence of elements of a structure, Turing degree, universal set

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1. INTRODUCTION

There are a lot of attempts to find a measure of the complexity of a given structure. Richter [8] has defined a degree of a structure as the least T-degree (if it exists) of all bijective total enumerations of the structure. Then it has been introduced a spectrum of a structure according to T-degrees, using only bijective total enumerations, too. There are a lot of investigations that show some sufficient conditions for a structure to have a least enumeration [1] and [11], and another with complicated structures without degree [8, 7, 2, 6]. They use the equality among the predicates of the structure. Soskov [12, 11] has generalized the notion of spectrum of a structure, using not only bijective enumeration, but all total ones. In that definition enumeration degrees are considered. This gives a possibility to consider not only totally defined structures, but partially defined, as well. Soskov [11] has generalized a spectrum of a partial structure, defining a partial spectrum, using partial enumerations.

Here we consider partial structures with unary functions and predicates, calling them unary. Since the equality and the inequality are not unary, they are not among the predicates of the structure. We consider such structures because they are simple enough, and as we will see, they are rich enough. For unary structures we find necessary and sufficient conditions for possessing least enumerations w.r.t. to e-degrees. As a corollary we obtain similar conditions w.r.t. to T-degrees.

In Section 2 we introduce the main definitions and preliminary results. In Section 3 we introduce a type and ∃-type of an element of a unary partial structure. Roughly speaking, a type (∃-type) of such an element is the set of all codes of open (existential) formulas, which are true on that element in that structure. A characterization of all unary structures, which admits least enumerations in the terms of a universal set of all types (∃-types) is given. We show that a unary partial structure admits a least enumeration if and only if there exist sequence of finite elements such that the ∃-type of that sequence is the least upper bound of all ∃-types of the structure and there exists a computable sequence of enumeration operators, such that the sequence of these enumeration operators applied to the upper bound "describes" all types of the elements of the structure. As a corollary we characterize the structures which admit effective enumerations. In this section we show that it is not possible to have a spectrum of a unary partial structure with denumerably many minimal elements.

In Section 4 we prove that a partial spectrum of a unary partial structure is upward closed for all partial enumerations. We show that for every set of r-degrees there is a structure with a set of types which "almost" coincides with the set of r-degrees, $r \in \{e, T\}$. Here we show several interesting examples, some of them concerning the so called quasi-degrees [11]. For example, we show that there are structures which don't have degrees, but they have quasi-degrees.

2. PRELIMINARIES

In this paper we use ω to denote the set of all natural numbers; $Dom(f)$, $Ran(f)$ and G_f to denote respectively the domain, the range and the graph of the function f; $\langle f\rangle$ or $\langle G_f\rangle$ to denote the set $\{\langle x_1, \ldots, x_n, y\rangle | (x_1, \ldots, x_n, y) \in G_f\}$, where $\langle ., \ldots, .\rangle$ is some fixed coding function for all finite sequences of natural numbers.

We shall recall some definitions from [10, 3].

Let $\mathfrak{A} = \langle B; \theta_1, \ldots, \theta_n; R_1, \ldots, R_k\rangle$ be a partial structure, where B is an arbitrary denumerable set, $\theta_1, \ldots, \theta_n$ are partial unary functions in B and $R_1, \ldots, R_k$ are unary partial predicates on B. We call such structures *unary*. We identify the partial predicates with partial mapping taking values in $\{0, 1\}$, writing 0 for true and 1 for false.

Let $\mathfrak{B} = \langle \omega; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k \rangle$ be a partial structure over the set ω. By $\langle \mathfrak{B} \rangle$ we denote the set $\langle \varphi_1 \rangle \oplus \dots \oplus \langle \varphi_n \rangle \oplus \langle \sigma_1 \rangle \oplus \dots \oplus \langle \sigma_k \rangle$. Let W be a recursively enumerable set. For any set B let

$W(B) = \{x | \exists v(\langle v, x \rangle \in W \wedge E_v \subseteq B)\}$. In this case we say W is an enumeration operator. A sequence of enumeration operators $W_{z_0}, W_{z_1}, \dots$ is said to be computable if there exists a recursive function h such that $h(n) = z_n$ for any natural n.

Definition 1. *Let $\mathcal{A}$ be a family of subsets of ω. A set $U \subseteq \omega^2$ is said to be universal for the family $\mathcal{A}$ if the following conditions hold:*

a) For every fixed $e \in \omega$, $\{x_1 | (e, x_1) \in U\} \in \mathcal{A}$;

b) If $A \in \mathcal{A}$, then there exists e such that $A = \{x_1 | (e, x_1) \in U\}$.

Definition 2. *Let $\mathcal{F}$ be a family of unary partial functions. A binary partial function F is said to be universal for the family $\mathcal{F}$ if the following conditions hold:*

a) For every fixed $e \in \omega$, $\lambda x_1.F(e, x_1) \in \mathcal{F}$;

b) If $f \in \mathcal{F}$, then there exists e such that $f = \lambda x_1.F(e, x_1)$.

Definition 3. *An enumeration of a structure $\mathfrak{A}$ is any ordered pair $\langle \alpha, \mathfrak{B} \rangle$ where $\mathfrak{B} = \langle \omega; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k \rangle$ is a partial unary structure on ω and α is a partial surjective mapping of ω onto B such that the following conditions hold:*

(i) $Dom(\alpha) \leq_e \langle \mathfrak{B} \rangle$;

(ii) $\alpha(\varphi_i(x)) \cong \theta_i(\alpha(x))$ for every $x \in \omega$, $1 \leq i \leq n$;

(iii) $\sigma_j(x) \cong R_j(\alpha(x))$ for every $x \in \omega$, $1 \leq j \leq k$.

An enumeration $\langle \alpha, \mathfrak{B} \rangle$ is said to be *total* iff $Dom(\alpha) = \omega$.

An enumeration $\langle \alpha, \mathfrak{B} \rangle$ is said to be *effective* iff all functions and predicates in $\mathfrak{B}$ are computable.

Degree spectrum [12, 11] of the structure $\mathfrak{A}$ is the family

$$\mathbf{DS}(\mathfrak{A}) = \{d_e(\langle \mathfrak{B} \rangle) | \langle \alpha, \mathfrak{B} \rangle \text{ is a total enumeration of } \mathfrak{A}\}$$

Partial degree spectrum [11] of a structure $\mathfrak{A}$ is the family

$$\mathbf{PDS}(\mathfrak{A}) = \{d_e(\langle \mathfrak{B} \rangle) | \langle \alpha, \mathfrak{B} \rangle \text{ is an enumeration of } \mathfrak{A}\}.$$

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration of the structure $\mathfrak{A}$. We say that $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is a *least enumeration* of $\mathfrak{A}$ if and only if for every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$, $\langle \mathfrak{B}_0 \rangle \leq_e \langle \mathfrak{B} \rangle$.

Let $\mathcal{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i.e. $\mathcal{L}$ consists of n unary functional symbols $\mathbf{f_1}, \dots, \mathbf{f_n}$ and k unary predicate symbols $\mathbf{T_1}, \dots, \mathbf{T_k}$. We add a new unary predicate symbol $\mathbf{T_0}$ which will represent the unary total predicate R_0, where $R_0(s) = 0$ for all $s \in B$.

Let us fix some denumerable set $X_1, X_2, \dots$ of variables. We shall use capital letters X, Y, Z and the same letters by indexes to denote variables.

The definition of a term in the language $\mathcal{L}$ is the usual: every variable is a term; if τ is a term then $\mathbf{f_i}(\tau)$ is a term.

If τ is a term in the language $\mathcal{L}$, then we write $\tau(Y_1, \dots, Y_m)$ to denote that all variables which occur in the term τ are among $Y_1, \dots, Y_m$. If $a_1, \dots, a_m$ are elements of B and $\tau(Y_1, \dots, Y_m)$ is a term, then by $\tau_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)$ we denote the value of the term τ in $\mathfrak{A}$ over the elements $a_1, \dots, a_m$, if it exists.

Termal predicate in the language $\mathcal{L}$ is defined by the following inductive clauses:

If $\mathbf{T} \in \{\mathbf{T_0}, \dots, \mathbf{T_k}\}$ and τ is a term, then $\mathbf{T}(\tau)$ and $\neg\mathbf{T}(\tau)$ are termal predicates.

If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Let $\Pi(Y_1, \dots, Y_m)$ be a termal predicate whose variables are among $Y_1, \dots, Y_m$ and let $a_1, \dots, a_m$ be elements of B. The value $\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)$ of Π over $a_1, \dots, a_r$ in $\mathfrak{A}$ is defined by the following inductive clauses:

If $\Pi = \mathbf{T}_j(\tau)$, $0 \le j \le k$, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong R_j(\tau_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)) .$$

If $\Pi = \neg\Pi^1$, where Π^1 is a termal predicate, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong (\Pi^1_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0 \supset 1, 0).$$

If $\Pi = (\Pi^1 \& \Pi^2)$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong (\Pi^1_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0 \supset \Pi^2_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m), 1).$$

Formulae of the kind $\exists Y'_1 \dots \exists Y'_l(\Pi)$, where Π is a termal predicate are called *conditions*. Every variable which occurs in Π and is different from $Y'_1, \dots, Y'_l$ is called free in the condition $\exists Y'_1 \dots \exists Y'_l(\Pi)$.

Let $C = \exists Y'_1 \dots \exists Y'_l(\Pi)$ be a condition, all free variables in C be among $Y_1, \dots, Y_m$, and $a_1, \dots, a_m$ be elements of B. The value $C_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)$ is defined by the equivalence:

$$C_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0 \iff \exists t_1 \dots \exists t_l (\Pi_{\mathfrak{A}}(Y'_1/t_1, \dots, Y'_l/t_l, Y_1/a_1, \dots, Y_m/a_m) \cong 0).$$

We assume that some effective codding of all terms, termal predicates and conditions of the language $\mathcal{L}$ is fixed. We shall use τ^v, Π^v, C^v to denote the corespondent one with code v.

Let $A \subseteq \omega^r \times B^m$. The set A is said to be $\exists$-*definable* in the structure $\mathfrak{A}$ if and only if there exists a recursive function γ of $r+1$ variables such that for all $n, x_1, \dots, x_r$, $C^{\gamma(n,x_1,\dots,x_r)}$ is a condition with free variables among $Z_1, \dots, Z_l$, $Y_1, \dots, Y_m$ and for some fixed elements $t_1, \dots, t_l$ of B the following equivalence is true:

$$(x_1, \dots, x_r, a_1, \dots, a_m) \in A \iff$$
$$\exists n \in \omega (C^{\gamma(n,x_1,\dots,x_r)}_{\mathfrak{A}}(Z_1/t_1, \dots, Z_l/t_l, Y_1/a_1, \dots, Y_m/a_m) \cong 0).$$

Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; R_1, \dots, R_k \rangle$ be a partial structure. We shall give a generalized version of the normal enumerations [10] and call them normal *pseudo-enumerations.*

Define $f_i^*(p) = \langle i-1, p \rangle$, $i = 1, \dots, n$ and set $N_0 = \omega \backslash (Ran(f_1^*) \cup \dots \cup Ran(f_n^*))$. It is obvious that N_0 is a recursive set and let $\{\mathbf{p_0}, \mathbf{p_1}, \dots\} = N_0$, where $\mathbf{p_i} < \mathbf{p_j}$ if $i < j$.

Let $N_1 \subseteq N_0$. For every partial surjective mapping α^0 of N_1 onto B we define partial mapping α of ω onto B by the following inductive clauses:

(i) If $p \in N_1$, then $\alpha(p) \cong \alpha^0(p)$;

(ii) If $p = f_i^*(q)$, $\alpha(q) \cong a$ and $\theta_i(a) \cong b$, then $\alpha(p) \cong b$.

Let $D_1, \dots, D_n$ be partial predicates such that

$$D_i(x) \cong \begin{cases} 0, & \text{if } \theta_i(\alpha(x)) \text{ is defined,} \\ \text{undefined,} & \text{otherwise;} \end{cases}$$

and $f_1, \dots, f_n$ be partial functions such that

$$f_i(x) \cong \begin{cases} f_i^*(x), & \text{if } D_i(x) \cong 0, \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Let $\sigma_1, \dots, \sigma_k$ be the partial predicates defined by the equalities $\sigma_j(x) \cong R_j(\alpha(x))$, $j = 1, \dots, k$.

Let $\mathfrak{B}$ be the partial structure $\langle \omega; f_1, \dots, f_n; \sigma_1, \dots, \sigma_k \rangle$ and $\mathfrak{B}^*$ be the partial structure $\langle \omega; f_1^*, \dots, f_n^*; \sigma_1, \dots, \sigma_k \rangle$.

Every pair $\langle \alpha, \mathfrak{B} \rangle$ which is obtained by the method described above is called *normal pseudo-enumerations* of the structure $\mathfrak{A}$. The mapping α^0 again is called *basis* of the enumeration $\langle \alpha, \mathfrak{B} \rangle$. It is obvious that α^0 completely determines the normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$. Let us notice that in the general case a normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$ is not an enumeration at all. Nevertheless, we shall see that there are cases where they are enumerations and we shall use them.

In the case $N_1 = N_0$ normal pseudo-enumerations and normal enumerations coincide.

Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration. We shall reformulate all propositions for normal enumerations and shall formulate several new ones. The proofs are analogous to those for normal enumerations in [10] and we shall give proofs only of those which are different. Let us note that if $\theta_1, \dots, \theta_n$ are total, then the normal enumeration will be total.

Proposition 1. $Dom(\alpha) \leq_e N_1 \oplus D_1 \oplus \dots \oplus D_n$.

Proof. The result follows from

$$x \in Dom(\alpha) \iff x \in N_1 \vee \exists x_0 \exists x_1 \dots \exists x_l \exists i_1 \dots \exists i_l (1 \leq i_1, \dots, i_l \leq n \\ \& x_0 \in N_1 \& x_1 = \langle i_1 - 1, x_0 \rangle \& \dots \& x_l = \langle i_l - 1, x_{l-1} \rangle \\ \& D_{i_1}(x_0) \cong 0 \& \dots \& D_{i_l}(x_{l-1}) \cong 0 \& x_l = x).$$

□

Corollary 1. *If $N_1 \leq_e \langle \mathfrak{B} \rangle$, then $Dom(\alpha) \leq_e \langle \mathfrak{B} \rangle$.*

Proposition 2. *For every $1 \leq i \leq n$ and $y \in Dom(\alpha)$, $\alpha(f_i(y)) \cong \theta_i(\alpha(y))$.*

Corollary 2. *Let $\tau(Y)$ be a term, and $y \in Dom(\alpha)$. Then*

$$\alpha(\tau_{\mathfrak{B}}(Y/y)) \cong \tau_{\mathfrak{A}}(Y/\alpha(y)).$$

Corollary 3. *If $N_1 \leq_e \langle \mathfrak{B} \rangle$, then the normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$ is an enumeration of the structure $\mathfrak{A}$.*

Proposition 3. *There exists an effective way for every x in ω to define $y \in N_0$ and a term $\tau(Y)$ such that $x = \tau_{\mathfrak{B}^*}(Y/y)$.*

We call a term $\tau(X_i)$ *standard* for x if $x = \tau_{\mathfrak{B}^*}(X_i/\mathbf{p_i})$ for some $\mathbf{p_i} \in N$.

Proposition 4. *There exists an effective way for every x in ω to define an element $\mathbf{p_i}$ and a standard term $\tau(X_i)$ such that $x = \tau_{\mathfrak{B}^*}(X_i/\mathbf{p_i})$.*

Proposition 5. *Let $\tau(Y)$ be a term, $y \in \omega$, $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration and $\tau_{\mathfrak{B}^*}(Y/y) \in Dom(\alpha)$. Then $\tau_{\mathfrak{B}}(Y/y) \cong \tau_{\mathfrak{B}^*}(Y/y)$.*

If $\langle \alpha, \mathfrak{B} \rangle$ is a normal pseudo-enumeration, then we shall use the notation

$$R_\alpha := \cup_{j=1}^{k} \{\langle j, x, z \rangle | \sigma_j(x) = z\} \cup \cup_{j=1}^{n} \{\langle j + k, x, z \rangle | D_j(x) = z\}.$$

It is clear that for every $W \subseteq \omega$, $W \leq_e R_\alpha$ if and only if $W \leq_e \langle \mathfrak{B} \rangle$, i.e. $R_\alpha \equiv_e \langle \mathfrak{B} \rangle$.

Proposition 6. *There exists an effective way for every natural u to define elements $y_1, \dots, y_m \in N_0$ and a termal predicate $\Pi(Y_1, \dots, Y_m)$ such that for every normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$:*

$$u \in R_\alpha \iff \Pi_{\mathfrak{A}}(Y_1/\alpha(y_1), \dots, Y_m/\alpha(y_m)) \cong 0.$$

Proposition 7. *There exists an effective way for every code v of a finite set E_v to define elements $y_1^v, \dots, y_{m_v}^v \in N_0$ and a termal predicate $\Pi_v(Y_1, \dots, Y_{m_v})$ such that for every normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$:*

$$E_v \subseteq R_\alpha \iff \Pi_{v,\mathfrak{A}}(Y_1/\alpha(y_1^v), \dots, Y_{m_v}/\alpha(y_{m_v}^v)) \cong 0.$$

Therefore, there exists a recursive function γ such that

$$E_v \subseteq R_\alpha \iff \Pi_{\mathfrak{A}}^{\gamma(v)}(Y_1/\alpha(y_1^v), \ldots, Y_{m_v}/\alpha(y_{m_v}^v)) \cong 0.$$

We call a termal predicate $\Pi^{\gamma(v)}$ *standard* for v in the pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$, if

$$E_v \subseteq R_\alpha \iff \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \ldots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0.$$

Proposition 8. *There exists a recursive function γ such that for every code v of the finite set E_v to define elements $\mathbf{p_{j_1}}, \ldots, \mathbf{p_{j_{m_v}}}$ and a standard termal predicate $\Pi^{\gamma(v)}(X_{j_1}, \ldots, X_{j_{m_v}})$ such that for every normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$:*

$$E_v \subseteq R_\alpha \iff \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \ldots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0.$$

Lemmas 1 and 2 and Proposition 9 below have analogous proofs, therefore we shall give only the proof of Proposition 9.

Lemma 1. *Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration, $\tau(Y)$ be a term and $\varphi(y_1) \cong \tau_{\mathfrak{B}}(Y/y_1)$. Then $\langle G_\varphi \rangle \leq_e R_\alpha$.*

Lemma 2. *Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration, $\tau^v(X_r)$ be a term with code v and $r' \in \omega$. Set $\varphi(v, \langle x_1, \ldots, x_{r'} \rangle) \cong \tau_{\mathfrak{B}}^v(X_r/x_r)$, if $r \leq r'$, and $\varphi(v, \langle x_1, \ldots, x_{r'} \rangle) \cong \tau_{\mathfrak{B}}^v(X_r/x_{r'})$, if $r > r'$. Then $\langle G_\varphi \rangle \leq_e R_\alpha$.*

Proposition 9. *Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration, $\Pi^v(X_1, \ldots, X_r)$ be a termal predicate or a condition with a code v and $r' \in \omega$. Set*

$$\pi(v, \langle x_1, \ldots, x_{r'} \rangle) \cong \begin{cases} \Pi_{\mathfrak{B}}^v(X_1/x_1, \ldots, X_r/x_r), & \text{if } r \leq r', \\ \Pi_{\mathfrak{B}}^v(X_1/x_1, \ldots, X_{r'}/x_{r'}, \ldots, X_r/x_{r'}), & \text{if } r > r'. \end{cases}$$

Then $\langle G_\pi \rangle \leq_e R_\alpha$.

Proof. We shall consider only the case when π_1 is obtained from C^v by projection, i.e. $C^v \iff \exists X_j \Pi^{\gamma_1(v)}$, where $\Pi^{\gamma_1(v)}$ is a termal predicate and γ_1 is a recursive function. For the sake of simplicity let $j = 1$. Let us assume that for the corresponding function π of $\Pi^{\gamma_1(v)}$ we have $\langle G_\pi \rangle \leq_e R_\alpha$. Then

$$\begin{aligned} \langle v, \langle x_2, \ldots, x_{r'} \rangle, y \rangle \in \langle G_{\pi_1} \rangle &\iff \exists x_1(\langle \gamma_1(v), \langle x_1, x_2, \ldots, x_{r'} \rangle, y \rangle \in \langle G_\pi \rangle) \\ &\iff \exists v_1(\langle \langle \gamma_1(v), \langle x_2, \ldots, x_{r'} \rangle, y \rangle, v_1 \rangle \in W \& E_{v_1} \subseteq \langle G_\pi \rangle), \end{aligned}$$

where

$$W = \{\langle \gamma_1(v), \langle x_2, \ldots, x_{r'} \rangle, y \rangle, v_1 \rangle | E_{v_1} = \{\langle \gamma_1(v), \langle x_1, x_2, \ldots, x_{r'} \rangle, y \rangle\} \text{ for some } x_1\}.$$

Therefore, $\langle G_{\pi_1} \rangle \leq_e \langle G_\pi \rangle \leq_e R_\alpha$. □

3. THE MAIN RESULT

In this section we shall give necessary and sufficient conditions for a given unary partial structure to admit a least enumeration.

Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; R_1, \dots, R_k \rangle$ be a unary partial structure. Let Π^v be a termal predicate with variables among $X_1, \dots, X_m$.

Type of a sequence $b_1, \dots, b_m$ of elements of B is called the set

$$\{v | \Pi^v_{\mathfrak{A}}(X_1/b_1, \dots, X_m/b_m) \cong 0\}.$$

The type of the sequence $b_1, \dots, b_m$ will be denoted by $[b_1, \dots, b_m]_{\mathfrak{A}}$. The type of an element a of B is the type of the sequence a.

Let C^v be a condition with free variables among $X_1, \dots, X_m$. *∃-type* of a sequence $b_1, \dots, b_m$ of elements of B is called the set

$$\{v | C^v(X_1/b_1, \dots, X_m/b_m) \cong 0\}.$$

The ∃-type of the sequence $b_1, \dots, b_m$ is denoted by $\exists[b_1, \dots, b_m]_{\mathfrak{A}}$.

∃-type could be defined in any partial structure. In the case of unary structures we can characterize the ∃-types by types of the elements of B and a fixed set of natural numbers. A condition is said to be *simple* if it does not contain free variables and it is in the form $\exists X_1 \Pi$, where Π is a termal predicate. Let $V_0^{\mathfrak{A}} = \{v | C^v_{\mathfrak{A}} \cong 0 \ \& \ C^v \text{ is a simple condition }\}$. It is easy to see that the following proposition is true:

Proposition 10. *Let $\mathfrak{A}$ be a unary partial structure. Then for any elements $b_1, \dots, b_m$ of B, $\exists[b_1, \dots, b_m]_{\mathfrak{A}} \equiv_e [b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}}$.*

Lemma 3. *Let $\mathfrak{A}$ be a unary partial structure with degree* $\mathbf{a}$. *If there exists an universal set U for the family of all types of elements of B, then there exists an enumeration of $\mathfrak{A}$ which is normal pseudo-enumeration with e-degree* $\mathbf{a}$.

Proof. Let U be a universal set for the family of all types of elements of B with e-degree $\mathbf{a}$. By U_x we denote the set $\{v | (x, v) \in U\}$. In fact, for all x, U_x is a type of some element. We can assume that for every type $\mathbf{t}$ of an element of B there exist infinitely many x such that $\mathbf{t} = U_x$. Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration of $\mathfrak{A}$, defined by a basis α^0 satisfying: $\alpha^0(\mathbf{p}_x) = a \iff [a]_{\mathfrak{A}} = U_x$ and $Ran(\alpha^0) = B$.

Then $Dom(\alpha^0) = \{\mathbf{p}_x | \exists v((x, v) \in U)\} \le_e \langle U \rangle = \{\langle x, v \rangle | (x, v) \in U\}$. According to Proposition 9, $\langle U \rangle \le_e R_\alpha$. Therefore, $\langle \alpha, \mathfrak{B} \rangle$ is an enumeration. Furthermore, $\langle \mathfrak{B} \rangle \le_e \langle U \rangle \le_e R_\alpha$. Hence, $deg_e(R_\alpha) = \mathbf{a}$. □

Proposition 11. *Let $\mathfrak{A}$ be a unary partial structure. There exists a universal set U for the family of all types of elements of B with e-degree* $\mathbf{a}$ *iff there exists a universal set U_1 for the family of all ∃-types of sequences of elements of B with e-degree* $\mathbf{a}$.

Proof. Let us first assume that there exists a universal set U for the family of all types of elements of B with e-degree $\mathbf{a}$. According to Lemma 3, there exists a normal pseudo-enumeration $\langle\alpha, \mathfrak{B}\rangle$, which is an enumeration of $\mathfrak{A}$ such that $deg_e(R_\alpha) = \mathbf{a}$. Using Proposition 9, one can see in the enumeration $\langle\alpha, \mathfrak{B}\rangle$ that the family of all $\exists$-types of sequences of elements in the structure $\mathfrak{B}$ has a universal set with degree $\mathbf{a}$ and it is universal set for the family of all $\exists$-types of sequences of B in the structure $\mathfrak{A}$.

To prove the converse, let U_1 be a universal for the set of all $\exists$-types of sequences of elements of B with e-degree $\mathbf{a}$. Then the set $U = \{(x, v) | (x, v) \in U_1 \& \Pi^v$ is a termal predicate with variable $X_1\}$ is universal for the types $[a]_{\mathfrak{A}}$ of all elements a of B and $deg_e(U) \leq \mathbf{a}$. To ensure that there exists a universal set with degree $\mathbf{a}$, let us define the set U' as follows: $U' = U \oplus (A \times U_{x_0})$, where U_{x_0} is a fixed type of an element of B and A is a set of naturals such that $deg_e(A) = \mathbf{a}$. It is obvious that $deg_e(U') = \mathbf{a}$ and U' is a universal set for the set of all types of elements of B. □

Proposition 12. *If $\langle\alpha, \mathfrak{B}\rangle$ is an enumeration of the unary partial structure $\mathfrak{A}$ with e-degree $\mathbf{a}$, then there exists a universal set U for the family of all types of elements of B with e-degree $\mathbf{a}$.*

Proof. Let $\langle\alpha, \mathfrak{B}\rangle$ be enumeration of the unary partial structure $\mathfrak{A}$ with e-degree $\mathbf{a}$ and $\langle\mathfrak{B}\rangle = \cup_{j=1}^{n}\{\langle j, x, z\rangle | f_j(x) = z\} \cup \cup_{j=1}^{k}\{\langle n + j, x, z\rangle | \sigma_j(x) = z\}$. Define the set U as follows:

$$(x, v) \in U \iff \exists u(\langle\langle x, v\rangle, u\rangle \in W_a \& E_u \subseteq \langle\mathfrak{B}\rangle),$$

where the set W_a is defined as follows:

$$W_a = \{\langle\langle x, v\rangle, u\rangle | \Pi^v = \mathbf{T}_{n_k - n}(\mathbf{f}_{n_{k-1}}(\ldots \mathbf{f}_{n_0}(X_1)\ldots)) \& E_u = \{\langle n_0, x, y_0\rangle, \langle n_1, y_0, y_1\rangle, \ldots, \langle n_{k-1}, y_{k-1}, y_k\rangle, \langle n_k, y_k, 0\rangle\}\}$$

for some $n_0, \ldots, n_k, y_0, \ldots, y_k$.

It is obvious that $U \leq_e \langle\mathfrak{B}\rangle$ and it is easy to see that U is a universal set for the family of all types of all elements of the structure $\mathfrak{B}$. Therefore, it is a universal set for the family of all types of all elements of $\mathfrak{A}$. As in the previous proposition, we may assume that $deg_e(U) = \mathbf{a}$. □

One can easily prove also the following corollaries.

Corollary 4. *If $\langle\alpha, \mathfrak{B}\rangle$ is an enumeration of the unary partial structure $\mathfrak{A}$ with e-degree $\mathbf{a}$, then there exists an enumeration of $\mathfrak{A}$ which is normal pseudo-enumeration with e-degree $\mathbf{a}$.*

Corollary 5. *If the unary partial structure $\mathfrak{A}$ admits a least enumeration, then it admits a least enumeration which is normal pseudo-enumeration.*

Corollary 6. *Let $\langle \alpha, \mathfrak{B} \rangle$ be an enumeration of the unary partial structure $\mathfrak{A}$. Then $deg_e(\langle \mathfrak{B} \rangle)$ is an upper bound of the family of e-degrees of all types ($\exists$-types) of the elements of B.*

Proof. Let $deg_e(R_\alpha) = \mathbf{a}$. Then, according to Proposition 12, there exists a universal set $U(U_1)$ for the family of all types ($\exists$-types) of $\mathfrak{A}$ and $deg_e(U_1) = deg_e(U) = \mathbf{a}$. It is obvious that for all $b_1, \ldots, b_m \in B$, $deg_e([b_i]_{\mathfrak{A}}) \leq_e \mathbf{a}$, $i = 1, \ldots, m$ and $deg_e(\exists[b_1, \ldots, b_m]_{\mathfrak{A}}) \leq_e \mathbf{a}$. □

Theorem 1. *Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration of an arbitrary partial structure $\mathfrak{A}$ and there do not exist elements $b_1, \ldots, b_m$ of B such that $\langle \mathfrak{B}_0 \rangle \leq_e \exists[b_1, \ldots, b_m]_{\mathfrak{A}}$. Then there is a normal enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$ such that $\langle \mathfrak{B}_0 \rangle \not\leq_e R_\alpha$.*

Proof. Let us first mention that this theorem is valid for arbitrary partial structure and we will not use in the proof that it is unary. We shall define the normal enumeration $\langle \alpha, \mathfrak{B} \rangle$ constructing a basis α^0 of N_0 onto B. The construction is step by step. At each step s we define a partial mapping α_s of N_0 into B such that:

(i) $\alpha_s \subseteq \alpha_{s+1}$;

(ii) $Dom(\alpha_s)$ is a finite subset of N_0.

At the end we take $\alpha^0 = \cup_{s=0}^{\infty} \alpha_s$.

With the even steps we ensure that α^0 is totally defined and that $Ran(\alpha^0) = B$, and with the odd steps we ensure that $\langle \mathfrak{B}_0 \rangle \not\leq_e R_\alpha$.

Let a_0, a_1,... be an arbitrary enumeration of the set B and let $W = \langle \mathfrak{B}_0 \rangle$. We remind that

$$W \leq_e R_\alpha \iff \exists e \forall x (x \in W \longleftrightarrow \exists v (\langle x, v \rangle \in W_e \& E_v \subseteq R_\alpha)) \iff$$

$$\exists e \forall x (x \in W \longleftrightarrow \exists v (\langle x, v \rangle \in W_e \& \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \ldots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0))$$

for some standard termal predicate $\Pi^{\gamma(v)}(X_{j_1}, \ldots, X_{j_{m_v}})$, some recursive function γ and some $\mathbf{p_{j_1}}, \ldots, \mathbf{p_{j_{m_v}}}$. Hence,

$$W \not\leq_e R_\alpha \iff$$

$$\forall e \exists x [(x \in W \& \forall v (\langle x, v \rangle \in W_e \to \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \ldots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \ncong 0))$$

$$\vee (x \notin W \& \exists v (\langle x, v \rangle \in W_e \& \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \ldots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0))].$$

In order that $W \not\leq_e R_\alpha$ we need to satisfy at least one of the two disjunctive members. In case we are able to satisfy the second member, we do it and the construction on that step will be completed. Otherwise we shall see that the first member will be satisfied automatically.

Step s=-1. $Dom(\alpha_{-1}) = Ran(\alpha_{-1}) = \emptyset$.

Step s=2e. Let x be the least element of N_0 such that $x \notin Dom(\alpha_{s-1})$ and a be the first element in the sequence $a_0, a_1, \ldots,$ such that $a \notin Ran(\alpha_{s-1})$. Set $\alpha_s(x) = a$ and $\alpha_s(y) = \alpha_{s-1}(y)$, for $y \in Dom(\alpha_{s-1})$.

Step s=2e+1. Let $Dom(\alpha_{s-1}) = \{x_0, \ldots, x_l\}$ and $c_i = \alpha_{s-1}(x_i)$, $i = 0, \ldots, l$. For every x we consider all v such that $\langle x, v\rangle \in W_e$.

There exists an effective way to find a standard termal predicate $\Pi^{\gamma(v)}(X_{j_1}, \ldots, X_{j_{m_v}})$ such that

$$E_v \subseteq R_\alpha \iff \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \ldots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0.$$

For the sake of simplicity, let us assume that $x_0 = \mathbf{p_0}, \ldots, x_l = \mathbf{p_l}$ and the list $X_0, \ldots, X_l, X_{l+1}, \ldots, X_{l+m}$ coincides with the list $X_{j_1}, \ldots, X_{j_{m_v}}$.

Then $E_v \subseteq R_\alpha \iff$

$$\Pi_{\mathfrak{A}}^{\gamma(v)}(X_0/\alpha(\mathbf{p_0}), \ldots, X_l/\alpha(\mathbf{p_l}), X_{l+1}/\alpha(\mathbf{p_{l+1}}), \ldots, X_{l+m}/\alpha(\mathbf{p_{l+m}}))) \cong 0.$$

Let $C_{\mathfrak{A}}^{v_1}(X_0, \ldots, X_l)$ be the condition $\exists X_{l+1} \ldots \exists X_{l+m}(\Pi^{\gamma(v)})$. We check whether there exist natural numbers $x \notin W$ and v, such that $\langle x, v\rangle \in W_e$ and $C_{\mathfrak{A}}^{v_1}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0$. If this is the case, we choose the least such v and find $b_1, \ldots, b_m$ such that

$$\Pi_{\mathfrak{A}}^{\gamma(v)}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l}), X_{l+1}/b_1, \ldots, X_{l+m}/b_m) \cong 0.$$

Set $\alpha_s(\mathbf{p_{l+j}}) = b_j$, $j = 1, \ldots, m$, $\alpha_s(y) = \alpha_{s-1}(y)$, for $y \in Dom(\alpha_{s-1})$. Otherwise, we do nothing, i.e. set $\alpha_s = \alpha_{s-1}$.

The construction is completed.

We continue proof of the theorem with a few auxiliary lemmas.

Lemma 4. *Let $C^{v_1}(X_0, \ldots, X_l)$ be the condition $\exists X_{l+1} \ldots \exists X_{l+m}(\Pi^{\gamma(v)})$ on step $s = 2e+1$ and there are no natural numbers $x \notin W$ and v, such that $\langle x, v\rangle \in W_e$ and $C_{\mathfrak{A}}^{v_1}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0$.*

Then there exists $x \in W$ such that for every v satisfying $\langle x, v\rangle \in W_e$ the conditional inequality

$$C_{\mathfrak{A}}^{v_1}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \not\cong 0$$

holds.

Proof. Let us mention that

$$\forall x(\exists v(\langle x, v\rangle \in W_e \& C_{\mathfrak{A}}^{v_1}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0) \longrightarrow x \in W).$$

If we assume that

$$\forall x(x \in W \longrightarrow \exists v(\langle x, v\rangle \in W_e \& C_{\mathfrak{A}}^{v_1}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0)),$$

then we would obtain that

$$\forall x(x \in W \longleftrightarrow \exists v(\langle x, v\rangle \in W_e \& C_{\mathfrak{A}}^{v_1}(X_0/\alpha_{s-1}(\mathbf{p_0}), \dots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0)).$$

Having in mind that we have obtained v_1 effectively from v, we conclude that $W \leq_e \exists[\alpha_{s-1}(\mathbf{p_0}), \dots, \alpha_{s-1}(\mathbf{p_l})]_{\mathfrak{A}}$ by index e, which contradicts the assumption of the theorem. □

The following lemma and corollary are obvious.

Lemma 5. *α^0 is a totally defined on N^0 surjective mapping.*

Corollary 7. *If all functions in the structure $\mathfrak{A}$ are total, then the normal enumeration $\langle \alpha, \mathfrak{B}\rangle$ is a totally defined surjective mapping.*

Let us assume now that $\langle \mathfrak{B}_0\rangle = W \leq_e R_\alpha$ by some index e. Then on step $s = 2e + 1$ we have satisfied first or second disjunctive member of the right part of the non-equivalence $W \not\leq_e R_\alpha$, which contradicts the assumption. Theorem 1 is proved. □

The following corollary is obvious.

Corollary 8. *Let $\mathfrak{A}$ be a unary partial structure. If $\mathfrak{A}$ admits a least enumeration $\langle \alpha_0, \mathfrak{B}_0\rangle$, then there exist elements $b_1, \dots, b_m$ of B such that $\langle \mathfrak{B}_0\rangle \leq_e \exists[b_1, \dots, b_m]_{\mathfrak{A}}$.*

Theorem 2. *Let $\mathfrak{A}$ be a unary partial structure. Then $\mathfrak{A}$ admits a least partial enumeration $\langle \alpha_0, \mathfrak{B}_0\rangle$ if and only if there exist elements $b_1, \dots, b_m$ of B such that $deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$ is the least upper bound of the e-degrees of all $\exists$-types of sequences of elements of B and there exists a universal set U of all types, such that $deg_e(U) = deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$.*

Proof. Let us assume first that $\mathfrak{A}$ admits a least enumeration $\langle \alpha_0, \mathfrak{B}_0\rangle$. According to Corollary 8, there exist $b_1, \dots, b_m$ in B such that $deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$ is the least upper bound of the e-degrees of all $\exists$-types of sequences of elements of B. By Proposition 12, there exists a universal set U of types, such that $deg_e(U) = deg_e(\langle \mathfrak{B}_0\rangle) = deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$.

Conversely, assume that there exist elements $b_1, \dots, b_m$ of B such that $deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$ is the least upper bound of the e-degrees of all $\exists$-types of sequences of elements of B and there exists a universal set U of all types such that $deg_e(U) = deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$.

According to Lemma 3 there exists an enumeration $\langle \alpha_0, \mathfrak{B}_0\rangle$ of $\mathfrak{A}$ such that $deg_e(\langle \mathfrak{B}_0\rangle) = deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$ and $\langle \alpha_0, \mathfrak{B}_0\rangle$ is the least enumeration of the structure $\mathfrak{A}$. □

Let us assume that $\mathfrak{A}$ is a unary partial structure and there exist elements $b_1, \dots, b_m$ of B such that $deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$ is the least upper bound of the

e-degrees of all $\exists$-types of sequences of elements of B and there exists a universal set U of all types, such that $deg_e(U) = deg_e(\exists[b_1,\dots,b_m]_{\mathfrak{A}})$ and let us fix $A = \exists[b_1,\dots,b_m]_{\mathfrak{A}}$. Therefore, there exists an enumeration operator W_z such that $W_z(A) = \langle U\rangle$, i.e. for all natural x, u the following equivalence is true:

$$\langle x,u\rangle \in \langle U\rangle \iff \exists v(\langle\langle x,u\rangle\, v\rangle \in W_z \& E_v \subseteq A).$$

Using the S^m_n-theorem, we can find for a fixed z a recursive function h such that

$$\langle x,u\rangle \in \langle U\rangle \iff \exists v(\langle u,v\rangle \in W_{h(x)} \& E_v \subseteq A) \iff u \in W_{h(x)}(A),$$

i.e. the sequence $W_{h(0)}(\exists[b_1,\dots,b_m]_{\mathfrak{A}}), W_{h(1)}(\exists[b_1,\dots,b_m]_{\mathfrak{A}}),\dots$ is the sequence of all types of the elements of B. The converse is trivial. Thus we obtained the following

Corollary 9. *Let $\mathfrak{A}$ be a unary partial structure. Then $\mathfrak{A}$ admits a least partial enumeration $\langle\alpha_0,\mathfrak{B}_0\rangle$ if and only if there exist elements $b_1,\dots,b_m$ of B and computable sequence of enumeration operators $W_{z_0}, W_{z_1},\dots$ such that the family $\{W_{z_n}([b_1]_{\mathfrak{A}} \oplus \cdots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})\}_{n\in\omega}$ is the family of all types of elements of B.*

In order to formulate the corresponding corollaries for the case when the unary structures are total, we call *a type of some element a* the set $[a]_{\mathfrak{A}} \oplus (\omega \setminus [a]_{\mathfrak{A}})$, or equivalently $([a]_{\mathfrak{A}} \times \{0\}) \cup ((\omega \setminus [a]_{\mathfrak{A}}) \times \{1\})$, which is the graph of the characteristic function of the set $[a]_{\mathfrak{A}}$. Let us remind that a set A is total if and only if $A \equiv_e A \oplus (\omega \setminus A)$ and that an e-degree is total if it contains a total set. The following corollaries are obvious and we omit their proofs.

Corollary 10. *Let $\mathfrak{A}$ be a unary total structure. Then $\mathfrak{A}$ admits a least total enumeration $\langle\alpha_0,\mathfrak{B}_0\rangle$ if and only if there exist elements $b_1,\dots,b_m$ of B such that $deg_e([b_1]_{\mathfrak{A}} \oplus \cdots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}}))$ is a total e-degree which is the least upper bound of e-degrees of all $\exists$-types of sequences of elements of B and there exists universal function F for the characteristic functions of all types, such that $deg_e(F) = deg_e([b_1]_{\mathfrak{A}} \oplus \cdots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$.*

Corollary 11. *Let $\mathfrak{A}$ be a unary total structure. Then $\mathfrak{A}$ admits a least total enumeration $\langle\alpha_0,\mathfrak{B}_0\rangle$ if and only if there exist elements $b_1,\dots,b_m$ of B and computable sequence of recursive operators $W_{z_0}, W_{z_1},\dots$ such that $deg_e([b_1]_{\mathfrak{A}} \oplus \cdots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}}) = deg_e(A)$ for some total set A and the function $\lambda n\lambda u.W^A_{z_n}(u)$ is universal function for the family of the characteristic functions of all types of elements of $\mathfrak{A}$.*

Corollary 12. *Let $\mathfrak{A}$ be a unary partial structure. Then $\mathfrak{A}$ admits an effective enumeration $\langle\alpha_0,\mathfrak{B}_0\rangle$ if and only if all $\exists$-types of the elements of B are computably enumerable and there exists r.e. universal set U of all types of elements of $\mathfrak{A}$.*

The following corollaries are related to [4, 5].

Corollary 13. *Let $\mathfrak{A}$ be a unary total structure. Then $\mathfrak{A}$ admits an effective total enumeration $\langle\alpha_0, \mathfrak{B}_0\rangle$ if and only if all $\exists$-types of the elements of B are computably enumerable and there exists recursive universal function F of all types of elements of $\mathfrak{A}$.*

Corollary 14. *Let $\mathfrak{A}$ be a unary partial structure. Then $\mathfrak{A}$ admits an effective enumeration $\langle\alpha_0, \mathfrak{B}_0\rangle$ if and only if all $\exists$-types of the elements of B are r.e. and there is a computable sequence of enumeration operators $W_{z_0}, W_{z_1}, \dots$ such that the family $\{W_{z_n}(\omega)\}_{n\in\omega}$ is the family of all types of elements of B.*

Analogously to Theorem 1, one can prove the following

Theorem 3. *Let for every $i = 1, \dots, l$, $\langle\alpha_i, \mathfrak{B}_i\rangle$ be an enumeration of an arbitrary partial structure $\mathfrak{A}$, and for every $i = 1, \dots, l$ there do not exist elements $b_1, \dots, b_m$ of B such that $R_{\alpha_i} \le_e \exists[b_1, \dots, b_m]_{\mathfrak{A}}$. Then there is an enumeration $\langle\alpha, \mathfrak{B}\rangle$ of $\mathfrak{A}$ such that for all $i = 1, \dots, l$, $R_{\alpha_i} \not\le_e R_\alpha$.*

Theorem 4. *Let for every $i \in \omega$, $\langle\alpha_i, \mathfrak{B}_i\rangle$ be an enumeration of an arbitrary partial structure $\mathfrak{A}$, and for every $i \in \omega$ there do not exist elements $b_1, \dots, b_m$ of B such that $R_{\alpha_i} \le_e \exists[b_1, \dots, b_m]_{\mathfrak{A}}$. Then there is an enumeration $\langle\alpha, \mathfrak{B}\rangle$ of $\mathfrak{A}$ such that for all $i \in \omega$, $R_{\alpha_i} \not\le_e R_\alpha$.*

Proof. We only sketch the proof: At even steps we will ensure the enumeration $\langle\alpha, \mathfrak{B}\rangle$ to be total and surjective. At steps of the kind $2\langle e, i\rangle + 1$ we will ensure, as in Theorem 1, that $R_{\alpha_i} \not\le_e R_\alpha$ by index e. □

Corollary 15. *There doesn't exist a spectrum of a partial structure with denumerable minimal elements.*

Proof. Obvious. □

4. SOME CONSEQUENCES

As in [12], we can prove that for a unary partial structure $\mathfrak{A}$ the partial degree spectrum of $\mathfrak{A}$ is closed upward with respect to arbitrary e-degrees. As a special case we shall obtain that the degree spectrum is closed upward with respect to the total e-degrees, as well.

Proposition 13. *Let $\langle\alpha, \mathfrak{B}\rangle$ be an enumeration of the unary partial structure $\mathfrak{A}$ and $deg_e(R_\alpha) \le_e A$. Then there exists an enumeration $\langle\alpha_0, \mathfrak{B}_0\rangle$ of $\mathfrak{A}$ such that $deg_e(\langle\mathfrak{B}_0\rangle) = deg_e(A)$.*

Proof. Let a be an element of B such that at least one function of $\mathfrak{A}$ is defined on A. Define an enumeration $\langle\alpha_0, \mathfrak{B}_0\rangle$ which is a normal pseudo-enumeration as follows:

$$\alpha_0^0(\mathbf{p_i}) \cong \begin{cases} a, & \text{if } i \text{ is even } \& \frac{i}{2} \in A, \\ \alpha(\frac{i-1}{2}), & \text{if } i \text{ is odd}, \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

It is easy to see that $deg_e(\langle\mathfrak{B}_0\rangle) = deg_e(R_\alpha \oplus (A \oplus [a]_{\mathfrak{A}})) = deg_e(A)$. □

Analogously one can prove

Proposition 14. *Let $\langle\alpha, \mathfrak{B}\rangle$ be an enumeration of the unary partial structure $\mathfrak{A}$ such that $\theta_1, \ldots, \theta_n$ are total and $deg_e(R_\alpha) \leq_e A$, where A is a total set. Then there exists a total enumeration $\langle\alpha_0, \mathfrak{B}_0\rangle$ of $\mathfrak{A}$ such that $deg_e(\langle\mathfrak{B}_0\rangle) = deg_e(A)$.*

Proposition 15. *Let $\mathbf{a}$ be an arbitrary e-degree. Then there exists a unary partial structure $\mathfrak{A} = \langle B; \theta_1; R_1, R_2\rangle$ with total function θ_1, such that $\mathfrak{A}$ has a least enumeration with e-degree $\mathbf{a}$.*

Proof. Let A be an arbitrary set of natural numbers, such that $deg_e(A) = \mathbf{a}$. The idea of constructing such structure is the following. We take infinite disjoint copies of natural numbers with successor functions on all of them. Then on one of them we take copy of the set A and on the remaining infinite copies we ensure the codes of all existential formulas, which are true in the structure $\mathfrak{A}$ will be recursive and all types of those elements in that copies will be finite, hence recursive. Take $B = \{a_0, a_1, \ldots, b_0^0, b_1^0, \ldots, b_0^1, b_1^1, \ldots\}$, where all $a_0, a_1, \ldots, b_0^0, b_1^0, \ldots, b_0^1, b_1^1, \ldots$ are different. Set $\theta_1(a_n) = a_{n+1}$, $\theta_1(b_n^i) = b_{n+1}^i$ for all natural i, n; set $R_1(a_0) = R_1(b_0^i) = 0$ for all natural i while $R_1(a_n)$ and $R_1(b_n^i)$ are undefined for all natural i and positive n. Further,

$$R_2(a_n) \cong \begin{cases} 0, & \text{if } n \in A, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

and $\Pi^v_{\mathfrak{A}}(X_j/b_0^v) \cong 0$ for all $v \in \omega$ such that the only predicate symbols and variables which occur in Π^v is a termal are $\mathbf{T}_2$ and X_j. Moreover, let R_2 be defined on the smallest finite subset of $\{b_0^v, b_1^v, \ldots\}$ which guarantee that $\Pi^v_{\mathfrak{A}}(X_1/b_0^v) \cong 0$. Thus, the types $[b_i^j]_{\mathfrak{A}}$ will be finite sets and will ensure that the set of all $\exists$-types is recursive. Indeed, a closed condition of the type $\exists X_j \Pi^v$ is true on the structure $\mathfrak{A}$ if and only if $\Pi^v = \mathbf{T}_1(X_j)\&\Pi^{v'}$, where $\Pi^{v'}$ is an arbitrary termal predicate with predicate symbol $\mathbf{T}_2$ and variable X_j.

Since $deg(\mathfrak{A}) = \mathbf{a}$ it is easy to see that for all positive n
$[a_n]_{\mathfrak{A}} \equiv_e [a_0]_{\mathfrak{A}} \equiv_e \{m | R_1(a_0) = 0 \ \& \ R_2(\theta_1^m(a_0)) = 0\} \equiv_e A$. □

Proposition 16. *Let $\mathbf{a}$ be an arbitrary T-degree. Then there exists unary total structure $\mathfrak{A} = \langle B; \theta_1; R_1, R_2\rangle$, such that $\mathfrak{A}$ has a least total enumeration with T-degree $\mathbf{a}$.*

Proof. Let A be an arbitrary set of natural numbers, such that $deg_T(A) = \mathbf{a}$. The idea is the same as in the previous proposition: we take B and θ_1 to be the same as in the previous proposition; take $R_i(a) = 0$ whenever $R_i(a) = 0$ in the previous proposition and $R_i(a) = 1$ whenever in the previous proposition $R_i(a)$ is undefined, $i = 1, 2$. □

Analogously one can prove the following

Proposition 17. *Let* $\mathbf{A}$ *be a denumerable set of e-(T-)degrees. Then there exists a unary partial(total) structure* $\mathfrak{A} = \langle B; \theta_1; R_1, R_2\rangle$ *with totally defined function* θ_1*, such that the set of the e-(T-)degrees of all types of* $\mathfrak{A}$ *coincides with the set* $\mathbf{A} \cup \{0\}$*.*

Proof. We consider only the case of T-degrees. Let $\mathbf{A} = \{a_i\}_{i\in I}$ for some countable index set I and A_i be an arbitrary set of natural numbers, such that $deg_e(A_i) = \mathbf{a_i}$ for any $i \in I$. Take $\mathfrak{A}_i = \langle B_i; \theta_1^i; R_1^i, R_2^i\rangle$ such that $deg_e(A_i) = \mathbf{a_i}$ for any $i \in I$ and all types of elements of B_i are finite or a_i. Assume that $B_i \cap B_j \neq \emptyset$ for all $i, j \in I$, $i \neq j$ and let $B = \cup_{i\in I} B_i$. Then $\theta_1(a) = \theta_1^i(a)$ and $R_j(a) = R_j^i(a)$ if $a \in B_i, i \in I$ and $j = 1, 2$. Then it is obvious that all type of B form the set $\mathbf{A} \cup \{0\}$. □

This proposition shows how to construct a various structures with or without degree. At the same time it shows that we can construct structures which contain different independent structures.

Proposition 18. *Let us consider the family of all recursive sets. There exists a unary total structure* $\mathfrak{A}_\mathbf{0} = \langle B; \theta_1; R_1, R_2\rangle$*, such that the family of all types of elements of* B *coincides with the family of copies of all recursive sets (or with the characteristic functions of copies of all recursive sets).*

Proof. Let $A_0, A_1, \ldots$ be a sequence of all recursive sets. As above, for any recursive set A_i we take an independent copy $B_i = \{a_0^i, a_1^i, \ldots\}$ of the set of natural numbers and a total function successor θ_1^i such that $\theta_1^i(a_n^i) = a_{n+1}^i$ for all $i, n \in \omega$. Then we take $R_1^i(a_0^i) = 0$ and $R_1^i(a_n^i) = 1$ for all positive n;

$$R_2^i(a_n^i) \cong \begin{cases} 0, & \text{if } n \in A_i, \\ 1, & \text{otherwise.} \end{cases}$$

Here R_1^i defines "zeros" and R_2^i defines a copy of the set A_i. Take the structure $\mathfrak{A}_i = \langle B_i; \theta_1^i; R_1^i, R_2^i\rangle$ for all i and assume that $B_i \cap B_j = \emptyset$ for all $i, j \in \omega$, $i \neq j$ and let $B = \cup_{i\in\omega} B_i$. Then $\theta_1(a) = \theta_1^i(a)$ and $R_j(a) = R_j^i(a)$ if $a \in B_i, i \in \omega$ and $j = 1, 2$. It is obvious that all types of elements of B of the structure $\mathfrak{A}_\mathbf{0} = \langle B; \theta_1; R_1, R_2\rangle$ are recursive sets and are copies of all recursive sets. Moreover, the set $V_0^{\mathfrak{A}}$ is recursive. Therefore, the least upper bound of all degrees of $\exists$-types is 0. If we assume that the structure $\mathfrak{A}$ admits least enumeration, then we would obtain that the family of all recursive set has a universal recursive set. This is a contradiction, which shows that we cannot omit the condition with universal set (function) in Theorem 3. □

Question 1. What is $\mathbf{DS}(\mathfrak{A}_\mathbf{0})$?

The next definition belongs to Soskov [11]. Let W be a set of natural numbers. It is said that $d_e(W)$ is a quasi-degree of the structure $\mathfrak{A}$ if for all sets $A \subseteq \omega^m$ the following equivalence is true:

$$A \text{ is } \exists\text{−definable in } \mathfrak{A} \iff A \leq_e W.$$

Let us mention that this definition is not the original, but it is equivalent to the original one.

Proposition 19. *There exists a class of unary partial structures and sets of natural numbers W such that for all sets $A \subseteq \omega^m$ the following equivalence is true:*

$$A \text{ is } \exists\text{−definable in } \mathfrak{A} \iff A \leq_e W.$$

Proof. Let $\mathfrak{A}$ be a unary partial structure such that there exist elements $b_1, \dots, b_m$ of B such that $deg_e([b_1]_\mathfrak{A} \oplus \dots \oplus [b_m]_\mathfrak{A})$ is the least upper bound of e-degrees of all types of elements of B and $W = [b_1]_\mathfrak{A} \oplus \dots \oplus [b_m]_\mathfrak{A}$. As in the previous propositions, take enough copies of natural numbers such that all types of the new elements to be finite and all (or recursive set of all) simple conditions to be true on those new elements and denote the new structure by $\mathfrak{A}'$. For the sake of simplicity let assume that $\mathfrak{A}' = \mathfrak{A}$. It is easy to see that $\mathfrak{A}'$ satisfies the required condition. Indeed, let A be $\exists$-definable in $\mathfrak{A}$, i.e. there exists recursive function γ of $m+1$ variables, having values in the set of all codes of conditions with free variables among $X_1, \dots, X_l$ such that for some elements $b'_1, \dots, b'_l$ the following equivalence is true:

$$(x_1, \dots, x_m) \in A \iff \exists n \in \omega(C^{\gamma(n,x_1,\dots,x_m)}(X_1/b'_1, \dots, X_l/b'_l) \cong 0).$$

Let us represent the condition $C^{\gamma(n,x_1,\dots,x_m)}(X_1, \dots, X_l) = C^{\gamma(n,\overline{x})}(X_1, \dots, X_l)$ in the form $\Pi^{\gamma_1(n,\overline{x})}(X_1)\& \dots \&\Pi^{\gamma_l(n,\overline{x})}(X_l)\&C^{\gamma_{l+1}(n,\overline{x})}$, where $C^{\gamma_{l+1}(n,\overline{x})}$ is a simple condition and all $\gamma_1, \dots, \gamma_{l+1}$ are recursive. Then, $[b'_i]_\mathfrak{A} \leq_e W$ and for some W_{z_i}, the following equivalence holds:

$$z \in [b'_i]_\mathfrak{A} \iff \exists v_i(\langle z, v_i\rangle \in W_{z_i}\&E_{v_i} \subseteq W), i = 1, \dots, m.$$

Therefore,

$$\overline{x} \in A \iff \exists n(\gamma_1(n, \overline{x}) \in [b'_1]_\mathfrak{A}\& \dots \&\gamma_l(n, \overline{x}) \in [b'_l]_\mathfrak{A}\&\gamma_{l+1}(n, \overline{x}) \in V_0^\mathfrak{A})$$

$$\iff \exists n(\gamma_1(n, \overline{x}) \in [b'_1]_\mathfrak{A}\& \dots \&\gamma_l(n, \overline{x}) \in [b'_l]_\mathfrak{A}) \iff$$

$$\exists n(\exists v_1(\langle\gamma_1(n, \overline{x}), v_1\rangle \in W_{z_1}\&E_{v_1} \subseteq W)\& \dots \&\exists v_l(\langle\gamma_l(n, \overline{x}), v_l\rangle \in W_{z_l}\&E_{v_l} \subseteq W))$$

$$\iff \exists v(\langle\langle\overline{x}\rangle, v\rangle \in W_z\&E_v \subseteq W),$$

where $\langle\langle\overline{x}\rangle, v\rangle \in W_z \iff$

$$\exists n(\exists v_1(\langle\gamma_1(n, \overline{x}), v_1\rangle \in W_{z_1})\& \dots \&\exists v_l(\langle\gamma_l(n, \overline{x}), v_l\rangle \in W_{z_1}\&E_v = E_{v_1} \cup \dots \cup E_{v_l}).$$

The converse, i.e. if $A \leq_e W$, then A is $\exists$-definable in $\mathfrak{A}$ is obvious. □

Proposition 20. *Let A be an arbitrary set of natural numbers with $deg_T(A) = \mathbf{a}$ and let us consider the family of all recursive in A sets. There exists a unary total structure $\mathfrak{A}_{\mathbf{a}} = \langle B; \theta_1; R_1, R_2 \rangle$, such that the family of all types of elements of B coincides with the family of copies of all recursive in A sets (or with the characteristic functions of copies of all recursive in A sets).*

Proof. Let $A_0, A_1, \dots$ be the sequence of all recursive in A sets. As above, for any recursive in A set A_i we take independent copy of set of natural numbers $B_i = \{a_0^i, a_1^i, \dots\}$ and a total function successor θ_1^i such that $\theta_1^i(a_n^i) = a_{n+1}^i$ for all $i, n \in \omega$. Then take $R_1^i(a_0^i) = 0$ and $R_1^i(a_n^i) = 1$, for all positive n;

$$R_2^i(a_n^i) \cong \begin{cases} 0, & \text{if } n \in A_i, \\ 1, & \text{otherwise.} \end{cases}$$

Here again R_1^i defines zero and R_2^i defines a copy of the set A_i. Take the structure $\mathfrak{A}_i = \langle B_i; \theta_1^i; R_1^i, R_2^i \rangle$ for all i and assume that $B_i \cap B_j = \emptyset$ for all $i, j \in \omega$, $i \neq j$ and let $B = \cup_{i \in \omega} B_i$. Then $\theta_1(a) = \theta_1^i(a)$ and $R_j(a) = R_j^i(a)$ if $a \in B_i, i \in \omega$ and $j = 1, 2$. Then it is obvious that all types of elements of B of the structure $\mathfrak{A}_{\mathbf{a}} = \langle B; \theta_1; R_1, R_2 \rangle$ are recursive in A sets and are copies of all recursive in A sets. Moreover, the set $V_0^{\mathfrak{A}}$ is recursive. Therefore the least upper bound of all degrees of $\exists$-types is $\mathbf{a}$. If we assume that the structure $\mathfrak{A}$ admits a least enumeration, then we would obtain that the family of all recursive sets in A has a universal recursive in A set. This is a contradiction, which shows that this structure $\mathfrak{A}_{\mathbf{a}}$ does not admit a least enumeration. At the same time it satisfies the condition of the previous proposition. Therefore, $\mathfrak{A}_{\mathbf{a}}$ has quasi-degree $\mathbf{a}$. □

Thus, we proved also the following

Corollary 16. *There exists a unary total structures $\mathfrak{A}_{\mathbf{a}} = \langle B; \theta_1; R_1, R_2 \rangle$, such that $\mathfrak{A}_{\mathbf{a}}$ has a quasi-degree but does not have a least enumeration.*

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