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SCATTERING OF ACOUSTOELECTRIC WAVES ON A CYLINDRICAL INHOMOGENEITY IN THE TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIUM

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Scattering of acoustoelectric waves on an inhomogeneity is studied. The scatterer is a circular continuous piezoelectric cylinder (fiber) embedded in the piezoelectric transversely isotropic medium. Expressions are found for the scattering amplitudes and total cross-sections of the three acoustoelectric waves propagating in the direction normal to the fiber axis. In the long-wave limit these expressions are obtained explicitly.

Keywords: piezoelectric medium, acoustoelectric waves, scattering, cylindrical inhomogeneity

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1. INTRODUCTION

The problem of the scattering of elastic waves on a single inhomogeneity in an elastic medium is of importance for several applications. First, such studies provide an information about the scatterer and therefore are relevant for the nondestructive evaluation of structural members. Another application is the investigation of the attenuation and velocity of elastic waves propagating through a medium, consisting of a set of noninteracting inhomogeneities. In recent years, significant progress has been achieved in solving this problem for ideally elastic materials [1–6].

In the present paper, we consider the scattering of acoustoelectric waves on a continuous cylindrical fiber embedded in a piezoelectric medium of hexagonal (transversely isotropic) symmetry. The expressions for scattering amplitudes of the acoustoelectric waves follow from the system of the integral equations for the electroelastic fields in the medium with inhomogeneity. This system is obtained in terms of Green's function of the coupled dynamic electroelastic problem (see Section 2). In Section 3, explicit expressions are obtained for the components of Green's function and scattering amplitudes for the quasiplane dynamic problem for the transversely isotropic piezoelectric medium. In Section 4, general formulae are derived for the total cross-section of acoustoelectric waves propagating in the direction normal to the fiber axis. Finally, explicit expressions are obtained for scattering amplitudes and total cross sections of three acoustoelectric waves in the long wave-length limit.

2. THE INTEGRAL EQUATIONS OF THE SCATTERING PROBLEM

We consider the piezoelectric medium obeying the following linear constitutive equations:

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \\ D_i &= e_{ikl}\varepsilon_{kl} + \eta_{ik}E_k, \end{aligned} \tag{2.1}$$

where σ and ε are the stress and strain tensors, E and D are the electric field intensity and electric displacement, respectively, $C = C^E$ is the tensor of elastic moduli at fixed E, $\eta = \eta^\varepsilon$ is the permittivity tensor at fixed strain ε, e is the piezoelectric constants tensor, and the superscript 'T' denotes the transposed tensor.

The substitution of Eqs. (2.1) into the equations of elastodynamics and Maxwell's equations leads to a coupled system of equations of linear electroelasticity. As usual, we disregard body sources of electrical nature. Hence, the equations of motion have the same form as in the theory of uncoupled elasticity

$$\partial_j\sigma_{ij} - \rho\ddot{u}_i = Q_i, \quad \partial_j = \partial/\partial x_j, \tag{2.2}$$

where u_i is the vector of elastic displacement, ρ is the material density, Q_i is the body force vector.

The solution of equation (2.2) together with Maxwell's equations describes the elastic-electromagnetic waves, i.e. elastic waves interacting with the electric field and the electromagnetic waves accompanying the deformation. If the characteristic velocity of the elastic waves is v, then the corresponding velocity of the electromagnetic waves has the order of $10^5\,v$. Therefore, we neglect the magnetic field generated by the elastic field propagating with the velocity v. It follows, then, that the magnetic effects can be neglected and the quasistatic approximation for the electric field can be used.

An additional field equation is the conservation of free electric charges:

$$\partial_i D_i = -q, \tag{2.3}$$

where q is the density of free electric charges and D_i is the dielectric displacement.

Since

$$E_i = -\partial_i\varphi, \quad \varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \tag{2.4}$$

where φ is the electric potential, the constitutive equations can be rewritten in the form

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\partial_l u_k + e_{ijk}\partial_k\varphi, \\ D_i &= e^T_{ikl}\partial_l u_k - \eta_{ik}\partial_k\varphi. \end{aligned} \tag{2.5}$$

Substituting them into (2.2) and (2.3) yields a coupled system of linear differential equations of electroelasticity for the piezoelectric medium:

$$\begin{aligned} &\partial_j C_{ijkl}\partial_l u_k + \partial_j e_{ijk}\partial_k\varphi - \rho\ddot{u}_i = -Q_i, \\ &\partial_i e_{ikl}\partial_l u_k - \partial_i\eta_{ik}\partial_k\varphi = -q. \end{aligned} \tag{2.6}$$

We consider now the harmonic oscillation of the medium with frequency ω. Since the dependence of quantities entering (2.6) on time is given by the multiplier $\exp(-i\omega t)$, the system (2.6) takes the form

$$\begin{aligned} &\partial_j C_{ijkl}\partial_k u_l + \rho\omega^2 u_i + \partial_j e_{ijk}\partial_k\varphi = -Q_i, \\ &\partial_i e^T_{ikl}\partial_k u_l - \partial_i\eta_{ik}\partial_k\varphi = -q. \end{aligned} \tag{2.7}$$

Let the body forces Q_i and electric charges q be distributed in some domain V. The solution of the system (2.7) that vanishes at infinity can be represented as

$$\begin{aligned} u_i(x) &= \int\limits_V G_{ik}(x-x')Q_k(x')\,\mathrm{d}x' + \int\limits_V \Gamma_i(x-x')q(x')\,\mathrm{d}x', \\ \varphi(x) &= \int\limits_V \gamma_k(x-x')Q_k(x')\,\mathrm{d}x' + \int\limits_V g(x-x')q(x')\,\mathrm{d}x' \end{aligned} \tag{2.8}$$

(the dependencies on frequency ω are omitted). The substitution of these expressions into the left-hand sides of (2.7) leads to a system of differential equations for the kernels $G_{ik}(x)$, $\Gamma_i(x)$, $\gamma_k(x)$ and $g(x)$ — the components of the electroelastic Green's function:

$$\begin{aligned} (\partial_j C_{ijkl}\partial_k + \rho\omega^2\delta_{il})G_{lm}(x) &+ \partial_j e_{ijk}\partial_k\gamma_m(x) = -\delta_{im}\delta(x), \\ (\partial_j C_{ijkl}\partial_k + \rho\omega^2\delta_{il})\Gamma_l(x) &+ \partial_j e_{ijk}\partial_k g(x) = 0, \\ \partial_i e^T_{ikl}\partial_k G_{lm}(x) &- \partial_i\eta_{ik}\partial_k\gamma_m(x) = 0, \\ \partial_i e^T_{ikl}\partial_k\Gamma_l(x) &- \partial_i\eta_{ik}\partial_k g(x) = -\delta(x), \end{aligned} \tag{2.9}$$

where $\delta(x)$ is the Dirac function. Fourier transformation of these equations yields

$$\begin{aligned} &\Lambda_{il}(k)G_{lj}(k) + h_i(k)\gamma_j(k) = \delta_{ij}, \\ &h^T_l(k)G_{lj}(k) - \lambda(k)\gamma_j(k) = 0, \\ &\Lambda_{il}(k)\Gamma_l(x) + h_i(k)g(k) = 0, \\ &h^T_l(k)\Gamma_l(k) - \lambda(k)g(k) = 1, \end{aligned} \tag{2.10}$$

where

$$\Lambda_{il} = k_j C_{ijkl} k_k - \rho\omega^2 \delta_{il}, \quad h_i(k) = e_{ikl} k_k k_l,$$
$$h_l^T = e_{ikl}^T k_i k_k, \quad \lambda(k) = \eta_{ik} k_i k_k. \tag{2.11}$$

The solution of the system (2.10) can be written in the form

$$G_{ik} = \left(\Lambda_{ik} + \frac{1}{\lambda} h_i h_k^T\right)^{-1}, \quad g = -(\lambda + h_i^T \Lambda_{ij}^{-1} h_j)^{-1},$$
$$\gamma_i = \frac{1}{\lambda} h_k^T G_{ki}, \quad \Gamma_i = -\Lambda_{ik}^{-1} h_k g. \tag{2.12}$$

One can show that $\gamma_i = \Gamma_i$. Introducing the notation

$$\mathcal{G}(k,\omega) = \left\| \begin{matrix} G_{ik}(k,\omega) & \gamma_i(k,\omega) \\ \gamma_k^T(k,\omega) & g(k,\omega) \end{matrix} \right\|, \tag{2.13}$$

the x-presentation of Green's function can be obtained via the inverse Fourier transformation:

$$\mathcal{G}(x,\omega) = \frac{1}{(2\pi)^3} \int \mathcal{G}(k,\omega) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \, d\mathbf{k}. \tag{2.14}$$

The equations of motion (2.7) can be written in the following symbolic form:

$$\mathcal{L}(\nabla) f(x) = 0, \quad \mathcal{L}(\nabla,\omega) = \mathcal{T}(\nabla) + \omega^2 \rho \mathcal{J}, \tag{2.15}$$

where

$$\mathcal{T}(\nabla) = \left\| \begin{matrix} T_{ik}(\nabla) & t_i(\nabla) \\ t_k^T(\nabla) & -\tau(\nabla) \end{matrix} \right\|, \quad \mathcal{J} = \left\| \begin{matrix} \delta_{ik} & 0 \\ 0 & 0 \end{matrix} \right\|, \quad f(x) = \left\| \begin{matrix} u_k(x) \\ \varphi(x) \end{matrix} \right\|,$$
$$T_{ik}(\nabla) = \partial_j C_{ijkl} \partial_l, \quad t_i(\nabla) = \partial_j e_{ijk} \partial_k, \quad \tau(\nabla) = \partial_i \eta_{ik} \partial_k. \tag{2.16}$$

Consider now an unbounded medium with the electroelastic characteristics $\mathcal{L}^0$ and density ρ_0, where

$$\mathcal{L}^0 = \left\| \begin{matrix} C_{ijkl}^0 & e_{ijk}^0 \\ e_{ikl}^{0T} & -\eta_{ik}^0 \end{matrix} \right\|, \tag{2.17}$$

containing a region V with different electroelastic characteristics $\mathcal{L}$ and density ρ.

Let the harmonic vibrations of frequency ω propagate in the medium with the inhomogeneity. The electroelastic fields in such a medium satisfy equations (2.15) in which C, e, η and ρ are functions of coordinates. We represent these functions in the form

$$C(x) = C^0 + C^1 V(x), \quad e(x) = e^0 + e^1 V(x),$$
$$\eta(x) = \eta^0 + \eta^1 V(x), \quad \rho(x) = \rho_0 + \rho_1 V(x), \tag{2.18}$$

where $V(x)$ is the characteristic function of the region V, and the quantities with the superscript '1' denote the differences

$$C^1 = C - C^0, \quad e^1 = e - e^0, \quad \eta^1 = \eta - \eta^0, \quad \rho_1 = \rho - \rho_0. \tag{2.19}$$

The problem of electroelastic fields determination in the medium with an inclusion can be then reduced to the following system of integral equations:

$$f(x) = f^0(x) + \int\limits_V S(x - x')\mathcal{L}^1 F(x')\, \mathrm{d}x' + \omega^2 \rho_1 \int\limits_V \mathcal{G}(x - x')\mathcal{J} f(x')\, \mathrm{d}x', \tag{2.20}$$

with $f^0(x)$ denoting the "incident" field. The latter satisfies the equation

$$\left[\mathcal{T}^0(\nabla) + \omega^2 \rho_0 \mathcal{J}\right] f^0(x) = 0 \tag{2.21}$$

with the notations

$$\mathcal{L}^1 = \left\| \begin{matrix} C^1 & e^1 \\ e^{1T} & -\eta^1 \end{matrix} \right\|, \quad S(x) = \left\| \begin{matrix} G_{ik,l}(x) & \gamma_{i,k}(x) \\ \gamma^T_{k,l}(x) & -g_{,k}(x) \end{matrix} \right\|,$$
$$F(x) = \left\| \begin{matrix} \varepsilon(x) \\ -E(x) \end{matrix} \right\|. \tag{2.22}$$

Here $G(x)$, $\gamma(x)$ and $g(x)$ are the respective x-representations of the functions entering (2.13).

When $x \in V$, Eq. (2.20) describes the electroelastic fields inside the inhomogeneity on which the fields outside of it can be constructed uniquely.

3. ELECTROELASTIC FIELDS IN THE TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIUM CONTAINING A CONTINUOUS CYLINDRICAL FIBER

We consider an inhomogeneity having the shape of an infinite circular cylinder (continuous fiber) with the axis parallel to x_3-axis of the Cartesian coordinate system. Let the plane wave propagate in the direction normal to x_3-axis. Since $\mathcal{L}(x)$ and $\rho(x)$ are functions of x_1, x_2 only, the functions $f^0(x)$, $f(x)$, $F(x)$ are independent of x_3. Taking into account the relation

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-k_3 x'_3}\, \mathrm{d}x'_3 = \delta(k_3)\,,$$

Eq. (2.20) transforms into the following one:

$$f(y) = f^0(y) + \int\limits_S \mathcal{S}(y - y')\mathcal{L}^1 F(y')\, \mathrm{d}y' + \omega^2 \rho_1 \int\limits_S \mathcal{G}(y - y')\mathcal{J} f(y')\, \mathrm{d}y', \tag{3.1}$$

where S is the cylindrical cross-section, $y = (x_1, x_2)$ and

$$\mathcal{G}(y - y') = \frac{1}{(2\pi)^2} \int_0^\infty \overline{k}\, \mathrm{d}\overline{k} \int_0^{2\pi} \mathcal{G}(\overline{k}) \exp\big(-i\overline{\mathbf{k}} \cdot (\mathbf{y} - \mathbf{y}')\big)\, \mathrm{d}\phi, \quad \overline{k} = (k_1, k_2). \tag{3.2}$$

The expression for $\mathcal{G}(\overline{k})$ has to be obtained from $\mathcal{G}(k, \omega)$, given by (2.13) by setting $k_3 = 0$.

Let x_3 be the axis of transverse isotropy. The material is characterized by five independent elastic moduli $\mathbf{C}^0 = \left\{C_{11}^0, C_{12}^0, C_{13}^0, C_{33}^0, C_{44}^0, C_{66}^0 = \frac{1}{2}(C_{11}^0 - C_{12}^0)\right\}$, three piezoelectric constants $\mathbf{e}^0 = \left\{e_{31}^0, e_{15}^0, e_{33}^0\right\}$ and two permeability coefficients $\eta^0 = \left\{\eta_{11}^0, \eta_{33}^0\right\}$. To simplify the needed in the sequel tensorial operations (inversion, contractions, etc), the tensors $\mathbf{C}^0, \mathbf{e}^0$ and η^0 are expressed in the form

$$\begin{gathered}
\mathbf{C}^0 = \frac{1}{2}(C_{11}^0 + C_{12}^0)\mathbf{T}^2 + 2C_{66}^0 \left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) \\
+ C_{13}^0(\mathbf{T}^3 + \mathbf{T}^4) + 4C_{44}^0\mathbf{T}^5 + C_{33}^0\mathbf{T}^6, \\
\mathbf{e}^0 = e_{31}^0\mathbf{U}^1 + e_{15}^0\mathbf{U}^2 + e_{33}^0\mathbf{U}^3, \quad \eta^0 = \eta_{11}^0\mathbf{t}^1 + \eta_{33}^0\mathbf{t}^2.
\end{gathered} \tag{3.3}$$

The basic tensors $\mathbf{T}^1, \ldots, \mathbf{T}^6$, $\mathbf{U}^1, \mathbf{U}^2, \mathbf{U}^3$, $\mathbf{t}^1, \mathbf{t}^2$ are defined here by means of their components as follows:

$$\begin{gathered}
T_{ijkl}^1 = \theta_{i(k}\theta_{l)k}, \quad T_{ijkl}^2 = \theta_{ij}\theta_{kl}, \quad T_{ijkl}^3 = \theta_{ij}m_k m_l, \\
T_{ijkl}^4 = m_i m_j \theta_{kl}, \quad T_{ijkl}^5 = \theta_{i)(k}m_{l)}m_{(j}, \quad T_{ijkl}^6 = m_i m_j m_k m_l, \\
U_{ijk}^1 = \theta_{ij}m_k, U_{ijk}^2 = 2m_{(i}\theta_{j)k}, \quad U_{ijk}^3 = m_i m_j m_k, \\
t_{ij}^1 = \theta_{ij}, \quad t_{ij}^2 = m_i m_j,
\end{gathered} \tag{3.4}$$

in the Cartesian system whose x_3-axis is along the unit vector $\mathbf{m}$; the components of the tensor θ_{ij} are $\theta_{ij} = \delta_{ij} - m_i m_j$.

The appropriate formulae for the operations on these tensors are given in the Appendix.

The fiber material possesses the transverse isotropy aligned with the one of the matrix. The tensors of the elastic moduli, of the piezoelectric constants and of the dielectric coefficients of the fibers can be expressed in the same tensorial basis, similarly to (3.3) (without the superscript '0').

Using (3.3), one obtains

$$\begin{gathered}
\Lambda_{ik}(\overline{k}) = \Lambda_1\overline{n}_i\overline{n}_k + \Lambda_2(\theta_{ik} - \overline{n}_i\overline{n}_k) + \Lambda_3 m_i m_k, \\
h_i(\overline{k}) = h_i^T(\overline{k}) = \overline{k}^2 e_{15}^0 m_i, \quad \lambda(\overline{k}) = \overline{k}^2 \eta_{11}^0,
\end{gathered} \tag{3.5}$$

where

$$\Lambda_1 = \overline{k}^2 C_{11}^0 - \rho_0\omega^2, \quad \Lambda_2 = \overline{k}^2 C_{66}^0 - \rho_0\omega^2, \quad \Lambda_3 = \overline{k}^2 C_{44}^0 - \rho_0\omega^2. \tag{3.6}$$

These expressions and (2.12) imply that

$$G_{ik}(\overline{k},\omega) = \frac{1}{\Lambda_1}\overline{n}_i\overline{n}_k + \frac{1}{\Lambda_2}(\theta_{ik} - \overline{n}_i\overline{n}_k) + \frac{1}{\Lambda_3'}m_i m_k,$$

$$g(\overline{k},\omega) = -\frac{1}{\overline{k}^2\eta_{11}^0}\left[1 - \frac{\overline{k}^2(e_{15}^0)^2}{\eta_{11}^0\Lambda_3'},\right], \quad \gamma_i = \frac{e_{15}^0}{\eta_{11}^0\Lambda_3'}m_i, \tag{3.7}$$

$$\Lambda_3' = \overline{k}^2 C_{44}' - \rho_0\omega^2, \quad C_{44}' = C_{44}^0 + \frac{(e_{15}^0)^2}{\eta_{11}^0}.$$

Introducing the quantities

$$\alpha^2 = \frac{\rho_0\omega^2}{C_{11}^0}, \quad \beta^2 = \frac{\rho_0\omega^2}{C_{66}^0}, \quad \beta_\perp^2 = \frac{\rho_0\omega^2}{C_{44}'}, \tag{3.8}$$

the expressions (3.7) are recast as

$$\begin{aligned} G_{ik}(\overline{k},\omega) &= \frac{1}{\rho_0\omega^2}\left[\frac{\beta^2}{\overline{k}^2 - \beta^2}\theta_{ik}\right. \\ &\quad \left. + \frac{\overline{k}_i\overline{k}_k}{\overline{k}^2}\left(\frac{\alpha^2}{\overline{k}^2 - \alpha^2} - \frac{\beta^2}{\overline{k}^2 - \beta^2}\right) + m_i m_k\frac{\beta_\perp^2}{\overline{k}^2 - \beta_\perp^2}\right], \\ g(\overline{k},\omega) &= \frac{1}{\eta_{11}^0}\frac{1}{\overline{k}^2} - \frac{1}{\rho_0\omega^2}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)^2\frac{\beta_\perp^2}{\overline{k}^2 - \beta_\perp^2}, \\ \gamma_i(\overline{k},\omega) &= \frac{1}{\rho_0\omega^2}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)\frac{\beta_\perp^2}{\overline{k}^2 - \beta_\perp^2}m_i. \end{aligned} \tag{3.9}$$

To determine the x-representation of functions $G_{ik}(\overline{k},\omega)$, $\gamma_i(\overline{k},\omega)$ and $g(\overline{k},\omega)$, according to (2.14), we have to calculate the integral

$$I = \frac{1}{(2\pi)^2}\int_0^\infty \frac{\overline{k}\,\mathrm{d}\overline{k}}{\overline{k}^2 - \beta^2}\int_0^{2\pi} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{y}}\,\mathrm{d}\phi.$$

Since

$$\int_0^{2\pi} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{y}}\,\mathrm{d}\phi = \int_0^{2\pi} \mathrm{e}^{-i\overline{k}y\cos\phi}\,\mathrm{d}\phi = 2\int_0^{\pi}\cos(\overline{k}y\cos\phi)\,\mathrm{d}\phi = 2\pi J_0(\overline{k}y),$$

where $J_0(z)$ is the Bessel function, we have

$$I = \frac{1}{2\pi}\int_0^\infty \frac{J_0(\overline{k}y)\overline{k}\,\mathrm{d}\overline{k}}{\overline{k}^2 - \beta^2} = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_0^\infty \frac{J_0(\overline{k}y)\overline{k}\,\mathrm{d}\overline{k}}{\overline{k}^2 + (\varepsilon + i\beta)^2} = \frac{i}{4}H_0^{(1)}(\beta y).$$

Here, $H_0^{(1)}(z)$ is the Hankel function of the first kind. Hence, the x-representation of these functions has the form

$$G_{ik}(r,\omega) = \frac{i}{4\rho_0\omega^2}\left\{\theta_{ik}\beta^2 H_0^{(1)}(\beta r) - \frac{\partial^2}{\partial y_i \partial y_k}\left[H_0^{(1)}(qr)\right]_\beta^\alpha + m_i m_k \beta_\perp^2 H_0^{(1)}(\beta_\perp r)\right\},$$

$$\gamma_i(r,\omega) = \frac{i}{4\rho_0\omega^2}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)\beta_\perp^2 H_0^{(1)}(\beta_\perp r) m_i, \tag{3.10}$$

$$g(r,\omega) = \frac{1}{2\pi\eta_{11}^0}\ln r - \frac{i}{4\rho_0\omega^2}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)^2 \beta_\perp^2 H_0^{(1)}(\beta_\perp r),$$

where

$$[f(qr)]_\beta^\alpha \equiv f(\alpha r) - f(\beta r), \quad r = |y|. \tag{3.11}$$

Eqs. (3.1), when written in detail, have the form

$$u_i(y) = u_i^0(y) + \int_S \Big[\Psi_{imn}(R)\varepsilon_{mn}(y') - \psi_{im}(R)E_m(y') + \rho_1\omega^2 G_{ik}(R)u_k(y')\Big]\,\mathrm{d}y', \tag{3.12}$$

$$\varphi(y) = \varphi^0(y) + \int_S \Big[\Phi_{mn}(R)\varepsilon_{mn}(y') - \phi_m(R)E_m(y') + \rho_1\omega^2 \gamma_k(R)u_k(y')\Big]\,\mathrm{d}y', \tag{3.13}$$

where

$$\begin{aligned}
\Psi_{imn}(R) &= G_{ik,l}(R)C_{klmn}^1 + \gamma_{i,k}(R)e_{kmn}^{1T},\\
\psi_{im}(R) &= G_{ik,l}(R)e_{klm}^1 - \gamma_{i,k}(R)\eta_{km}^1,\\
\Phi_{mn}(R) &= \gamma_{k,l}(R)C_{klmn}^1 - g_{,k}(R)e_{kmn}^{1T},\\
\phi_m(R) &= \gamma_{k,l}(R)e_{klm}^1 + g_{,k}(R)\eta_{km}^1, \quad R = |y - y'|.
\end{aligned} \tag{3.14}$$

Eqs. (3.12) and (3.13) allow one to find the far-field asymptotics of the electroelastic fields. Taking into account the asymptotic formulas at $r \to \infty$

$$R^{-1} \sim r^{-1}, \quad R \sim r - (\mathbf{n}\cdot\mathbf{y}'), \quad n_i = \frac{y_i}{y}, \quad y = |\mathbf{y}|,$$

$$\frac{\partial}{\partial y_{k_1}} \cdots \frac{\partial}{\partial y_{k_m}} H_0^{(1)}(qR) \sim (iq)^m n_{k_1} \ldots n_{k_m} \sqrt{\frac{2}{\pi q y}}\, \mathrm{e}^{i(qy - \pi/4)} \mathrm{e}^{-iq(\mathbf{n}\cdot\mathbf{y}')},$$

one has

$$u_i(y) = u_i^0(y) + u_i^s(y), \quad \varphi(y) = \varphi^0(y) + \varphi^s(y), \tag{3.15}$$

and the "scattered" fields $u_i^s(y)$ and $\varphi^s(y)$ are determined by the expressions

$$u_i^s(y) = A_i(n)\frac{e^{i\alpha y}}{\sqrt{y}} + B_i(n)\frac{e^{i\beta y}}{\sqrt{y}} + C_i(n)\frac{e^{i\beta_\perp y}}{\sqrt{y}}, \qquad \varphi^s(y) = c(n)\frac{e^{i\beta_\perp y}}{\sqrt{y}}. \tag{3.16}$$

Here $A_i(n)$, $B_i(n)$, $C_i(n)$ and $c(n)$ are the amplitudes of the three cylindrical waves that can be represented in the form

$$A_i(n) = n_i n_k f_k(\alpha n), \quad B_i(n) = (\theta_{ik} - n_i n_k) f_k(\beta n),$$
$$C_i(n) = m_i m_k f_k(\beta_\perp n) + m_i f(\beta_\perp n), \tag{3.17}$$
$$c(n) = \frac{e_{15}^0}{\eta_{11}^0}\Big[m_k f_k(\beta_\perp n) + f(\beta_\perp n)\Big],$$

with the notations

$$f_k(qn) = \frac{i}{2\rho_0\omega^2}\sqrt{\frac{q^3}{2\pi}}e^{-i\pi/4}\Bigg\{iqn_l\int_S\Big[C_{klmn}^1\varepsilon_{mn}(y')$$
$$-e_{klm}^1 E_m(y')\Big]e^{-iq(\mathbf{n}\cdot\mathbf{y}')}\,dy' + \rho_1\omega^2\int_S u_k(R)e^{-iq(\mathbf{n}\cdot\mathbf{y}')}\,dy'\Bigg\}, \tag{3.18}$$
$$f(\beta_\perp n) = \frac{i}{2\rho_0\omega^2}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)\sqrt{\frac{\beta_\perp^3}{2\pi}}e^{-\frac{i\pi}{4}}i\beta_\perp n_k\int_S\Big[e_{kmn}^{1T}\varepsilon_{mn}(y')$$
$$+\eta_{km}^1 E_m(y')\Big]e^{-i\beta_\perp(\mathbf{n}\cdot\mathbf{y}')}\,dy', \quad q = \alpha, \beta, \beta_\perp.$$

4. SCATTERING CROSS-SECTION IN A PIEZOELECTRIC MEDIUM

We define the intensity vector I_i, associated with a stress field σ_{ij}, the electric potential φ and the velocities $\dot{u}_i$ and $\dot{D}_i$ by the relation

$$I_i = \sigma_{ij}\dot{u}_j + \varphi\dot{D}_i. \tag{4.1}$$

Similarly, we denote by I_i^s the intensity vector associated with the scattered fields, and by I_i^0 the intensity vector associated with the incident fields. The term "intensity" refers to the rate of energy transfer per unit area in the direction normal to the one of propagation, that is

$$I = I_i n_i, \tag{4.2}$$

where n_i is the unit vector in the direction of propagation. The power flux (the rate of energy transfer across the surface S with unit normal n_i) is

$$Q = \int_S I_i n_i \, \mathrm{d}S = \int_S (\sigma_{ij}\dot{u}_j + \varphi \dot{D}_i) n_i \, \mathrm{d}S. \tag{4.3}$$

For a given angular frequency corresponding to period T, the total cross section $Q(\omega)$ is the ratio of the average power flux over all directions to the average intensity of the incident fields:

$$Q(\omega) = \frac{\langle Q^s \rangle_t}{\langle I^0 \rangle_t}, \tag{4.4}$$

where $\langle \cdot \rangle_t$ denotes the time averaging over the period T.

Having found the far-field asymptotics of the scattered electroelastic fields we can now compute the total cross-section according to relation (4.4). Since the power flux is a real number,

$$\langle Q \rangle_t = \frac{1}{4} \int_S \Big\langle (\sigma_{ij} + \sigma_{ij}^*)(\dot{u}_j + \dot{u}_j^*) + (\varphi + \varphi^*)(\dot{D}_i + \dot{D}_i^*) \Big\rangle_t n_i \, \mathrm{d}S, \tag{4.5}$$

where '$*$' denotes the complex conjugate. Since we assume the vibrations to be harmonic,

$$\begin{aligned} \langle Q \rangle_t = \frac{i\omega}{4} \int_S \langle & -\sigma_{ij} u_i \mathrm{e}^{-2i\omega t} + \sigma_{ij}^* u_i^* \mathrm{e}^{2i\omega t} - \sigma_{ij}^* u_i + \sigma_{ij} u_i^* \\ & -\varphi D_j \mathrm{e}^{-2i\omega t} + \varphi^* D_j^* \mathrm{e}^{2i\omega t} - \varphi^* D_j + \varphi D_j^* \rangle_t n_j \, \mathrm{d}S. \end{aligned} \tag{4.6}$$

Computing the time average yields

$$\langle Q \rangle_t = -\frac{1}{2}\omega \, \mathrm{Im} \int_S (\sigma_{ij} u_i^* - D_j \varphi^*) n_j \, \mathrm{d}S. \tag{4.7}$$

Hence, the expression for the total cross-section takes the form

$$\begin{aligned} Q(\omega) &= -\frac{\omega}{2\langle I^0 \rangle_t} \, \mathrm{Im} \int_S (\sigma_{ij}^s u_i^{*s} - D_j^s \varphi^{*s}) n_j \, \mathrm{d}S, \\ \langle I^0 \rangle_t &= -\frac{1}{2}\omega \, \mathrm{Im} \, (\sigma_{ij}^0 u_i^{*0} - D_j^0 \varphi^{*0}) n_j^0, \end{aligned} \tag{4.8}$$

where n_j^0 is the normal to the front of the incident wave.

We apply now the general formula (4.8) to the scattering of the acoustoelectric waves on a continuous cylindrical surface of unit height and a radius r concentric with the fiber. Taking into account that the contribution to the energy flux through two cross-sections of this surface by the plane wave propagating normally to the fiber axis is zero, we have

$$Q(\omega) = -\frac{\omega}{2\left\langle I^0\right\rangle_t}\operatorname{Im}\int\limits_0^{2\pi}(\sigma_{ij}^s u_i^{*s} - D_j^s\varphi^{*s})n_j r\,\mathrm{d}\phi\,, \tag{4.9}$$

where ϕ is the angle between the wave normal n^0 and an arbitrary normal to the fiber surface.

To compute $Q(\omega)$, we have to find (utilizing (3.16))

$$\sigma_{ij}^s = in_k\left[C_{ijkl}^0\left(A_l(n)\alpha\frac{\mathrm{e}^{i\alpha y}}{\sqrt{y}} + B_l(n)\beta\frac{\mathrm{e}^{i\beta y}}{\sqrt{y}}\right.\right.$$
$$\left.\left.+C_l(n)\beta_\perp\frac{\mathrm{e}^{i\beta_\perp y}}{\sqrt{y}}\right) + \mathrm{e}_{ijk}^0 c(n)\beta_\perp\frac{\mathrm{e}^{i\beta_\perp y}}{\sqrt{y}}\right].$$

Substituting the expressions for transversely isotropic tensors C_{ijkl}^0 and e_{ijk}^0, we obtain

$$\sigma_{ij}^s n_j = i\left\{\left[\frac{1}{2}(C_{11}^0 + C_{12}^0)n_i n_k A_k(n) + C_{66}^0 A_i(n)\right]\alpha\frac{\mathrm{e}^{i\alpha y}}{\sqrt{y}}\right.$$
$$\left.+C_{66}^0 B_i(n)\beta\frac{\mathrm{e}^{i\beta y}}{\sqrt{y}} + C_{44}' m_i m_k C_k(n)\beta_\perp\frac{\mathrm{e}^{i\beta_\perp y}}{\sqrt{y}}\right\}, \tag{4.10}$$

where the relation

$$C_{44}^0 m_k C_k(n) + e_{15}^0 c(n) = \left(C_{44}^0 + \frac{(e_{15}^0)^2}{\eta_{11}^0}\right)(m_k f_k(\beta_\perp n) + f(\beta_\perp n)) = C_{44}' m_k C_k(n)$$

is taken into account with C_{44}' determined by (3.7). Similarly,

$$D_i^s = i\left[e_{31}^0 m_i n_k A_k(n)\alpha\frac{\mathrm{e}^{i\alpha y}}{\sqrt{y}} + n_i(e_{15}^0 m_k C_k(n) - \eta_{11}^0 c(n))\beta_\perp\frac{\mathrm{e}^{i\beta_\perp y}}{\sqrt{y}}\right].$$

This implies that

$$D_i^s n_i = i(e_{15}^0 m_k C_k(n) - \eta_{11}^0 c(n))\beta_\perp\frac{\mathrm{e}^{i\beta_\perp y}}{\sqrt{y}}\,. \tag{4.11}$$

Since

$$e_{15}^0 m_k C_k(n) - \eta_{11}^0 c(n) = \Big(m_k f_k(\beta_\perp n) + f(\beta_\perp n)\Big)(e_{15}^0 - e_{15}^0) = 0,$$

the second term in (4.9) does not contribute to the total scattering cross-section.

The substitution of these expressions, alongside with the relation

$$u_i^{*s} = A_i^*(n)\frac{\mathrm{e}^{-i\alpha y}}{\sqrt{y}} + B_i^*(n)\frac{\mathrm{e}^{-i\beta y}}{\sqrt{y}} + C_i^*(n)\frac{\mathrm{e}^{-i\beta_\perp y}}{\sqrt{y}}\,,$$

into (4.9) yields eventually:

$$Q(\omega) = -\frac{\omega}{2\langle I^0\rangle} \int_0^{2\pi} \left[C_{11}^0 \alpha |A_i|^2 + C_{66}^0 \beta |B_i|^2 + C_{44}' \beta_\perp |C_i|^2\right] \, d\phi, \tag{4.12}$$

where $|A_i|^2 = A_i A_i^*$.

We assume that the incident waves have the form

$$u_i^0(y,\omega) = U_i e^{ik\mathbf{n}^0\cdot\mathbf{y}}, \quad \varphi^0(y,\omega) = \Phi e^{ik\mathbf{n}^0\cdot\mathbf{y}}, \tag{4.13}$$

where k is the wave number, n_i^0 is the wave normal (perpendicular to the fiber axis), U_i is the polarization vector and Φ is the amplitude of the electric field. Since

$$\partial_j e_{jkl}^0 \partial_l u_k^0 - \partial_j \eta_{jk}^0 \partial_k \varphi^0 = 0,$$

it follows that

$$\Phi = \frac{e_{15}^0}{\eta_{11}^0} m_k U_k. \tag{4.14}$$

Hence, the expression for $\langle I^0\rangle_t$ can be represented in the form

$$\langle I^0\rangle_t = \frac{1}{2}\omega k \left[\frac{1}{2}(C_{11}^0 + C_{12}^0)(\mathbf{U}\cdot\mathbf{n}^0) + C_{66}^0(\mathbf{U}\cdot\mathbf{U}) + (C_{44}' - C_{66}^0)(\mathbf{U}\cdot\mathbf{m})\right]. \tag{4.15}$$

5. THE TOTAL SCATTERING CROSS-SECTION IN THE LONG-WAVE LIMIT

As it follows from expressions (3.17) for the amplitudes $A_i(n), B_i(n), C_i(n)$ and $c(n)$, the determination of vector $f_i(kn)$ and scalar $f(\beta_\perp n)$ plays a key role in the scattering problem. These quantities depend on the electroelastic fields u_i, φ (and the accompanying fields ε_{ij} and E_i), inside the region occupied by the scatterer. The mentioned fields have to be determined from the solution of the coupled electroelastic dynamic problem for the medium with the inhomogeneity. If these fields are found approximately, then the obtained formulae yield approximate expressions for the scattering cross-sections. Several approximations have been suggested (see the discussion of [4]), to mention only Born's approximation, quasistatic approximation and extended quasistatic approximation. We use the quasistatic (long-wave) approximation. The feature of this approximation is the replacement of the actual strain and electric fields inside of the inclusion by those of the static (infinite wavelength) problem. As it is well-known [7–11], if the external fields $F^0 = [\varepsilon^0, E^0]$ in the static limit ($\omega = 0$) are uniform in S, then the fields $F = [\varepsilon, E]$ inside this region are also uniform and have, after [11], the form

$$F = \mathcal{A}F^0, \quad \mathcal{A} = (\mathcal{I} + \mathcal{P}^0\mathcal{L}^1)^{-1}, \quad \mathcal{I} = \left\| \begin{matrix} I_{ijkl} & 0 \\ 0 & \delta_{ik} \end{matrix} \right\|, \quad I_{ijkl} = \delta_{i(k}\delta_{l)j}, \tag{5.1}$$

where $\mathcal{P}^0$ is an operator with constant components that can be represented in the tensor basis (3.4) as

$$\mathcal{P} = \left\| \begin{array}{cc} \mathbf{P}^0 & \mathbf{p}^0 \\ \mathbf{p}^0 & \pi^0 \end{array} \right\|,$$

$$\mathbf{P}^0 = -\left[\frac{1}{4C_{11}^0}\mathbf{T}^2 + \frac{1}{4}\left(\frac{1}{C_{11}^0} + \frac{1}{C_{66}^0}\right)\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + \frac{\eta_{11}^0}{2\Delta_0}\mathbf{T}^5\right], \tag{5.2}$$

$$\mathbf{p}^0 = -\frac{e_{15}^0}{4\Delta_0}\mathbf{U}^2, \quad \pi^0 = \frac{C_{44}^0}{2\Delta_0}\mathbf{t}^1, \quad \Delta_0 = \eta_{11}^0 C_{44}^0 + (e_{15}^0)^2.$$

Note that only the product

$$\mathcal{L}^1 F = \mathcal{L}^1 \mathcal{A} F^0 = \mathcal{L}^{\mathcal{A}} F^0, \quad \mathcal{L}^{\mathcal{A}} = \mathcal{L}^1 \mathcal{A} \tag{5.3}$$

enters the right-hand sides of Eqs. (3.18). The components of the constant operator $\mathcal{L}^{\mathcal{A}}$ can be obtained by using tensorial operations in the basis (3.4), see the Appendix. Then

$$\mathcal{L}^{\mathcal{A}} = \left\| \begin{array}{cc} \mathbf{C}^{\mathcal{A}} & \mathbf{e}^{\mathcal{A}} \\ \mathbf{e}^{T\mathcal{A}} & -\eta^{\mathcal{A}} \end{array} \right\|,$$

$$\begin{aligned} \mathbf{C}^{\mathcal{A}} &= \frac{1}{2}(C_{11}^{\mathcal{A}} + C_{12}^{\mathcal{A}})\mathbf{T}^2 + 2C_{66}^{\mathcal{A}}\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) \\ &+ C_{13}^{\mathcal{A}}(\mathbf{T}^3 + \mathbf{T}^4) + 4C_{44}^{\mathcal{A}}\mathbf{T}^5 + C_{33}^{\mathcal{A}}\mathbf{T}^6, \end{aligned} \tag{5.4}$$

$$\mathbf{e}^{\mathcal{A}} = e_{31}^{\mathcal{A}}\mathbf{U}^1 + e_{15}^{\mathcal{A}}\mathbf{U}^2 + e_{33}^{\mathcal{A}}\mathbf{U}^3, \quad \eta^{\mathcal{A}} = \eta_{11}^{\mathcal{A}}\mathbf{t}^1 + \eta_{33}^{\mathcal{A}}\mathbf{t}^2,$$

with the notations

$$\frac{1}{2}\left(C_{11}^{\mathcal{A}} + C_{12}^{\mathcal{A}}\right) = \frac{1}{2}(C_{11}^1 + C_{12}^1)\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1},$$

$$C_{66}^{\mathcal{A}} = C_{66}^1\left[1 + \frac{C_{66}^1}{2}\left(\frac{1}{C_{11}^0} + \frac{1}{C_{66}^0}\right)\right]^{-1},$$

$$C_{13}^{\mathcal{A}} = C_{13}^1\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1}, \quad C_{44}^{\mathcal{A}} = \frac{1}{\Delta_f}\left[C_{44}^1 + \frac{C_{44}^0}{2\Delta_0}\left(C_{44}^1\eta_{11}^1 + (e_{15}^1)^2\right)\right],$$

$$C_{33}^{\mathcal{A}} = C_{33}^1 - \frac{(C_{13}^1)^2}{C_{11}^0}\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1}, \tag{5.5}$$

$$e_{31}^{\mathcal{A}} = e_{31}^1\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1}, \quad e_{15}^{\mathcal{A}} = \frac{1}{\Delta_f}\left[e_{15}^1 + \frac{e_{15}^0}{2\Delta_0}\left(C_{44}^1\eta_{11}^1 + (e_{15}^1)^2\right)\right],$$

$$e_{33}^{\mathcal{A}} = e_{33}^1 - \frac{C_{13}^1 e_{31}^1}{C_{11}^0}\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1},$$

$$\eta_{11}^{\mathcal{A}} = \frac{1}{\Delta_f}\left[\eta_{11}^1 + \frac{\eta_{11}^0}{2\Delta_0}\left(C_{44}^1\eta_{11}^1 + (e_{15}^1)^2\right)\right],$$

$$\eta_{33}^{\mathcal{A}} = \left[\eta_{33}^1 + \frac{(e_{31}^1)^2}{C_{11}^0}\left(1 + \frac{C_{11}^1 + C_{12}^1}{2C_{11}^0}\right)^{-1}\right],$$

$$\Delta_f = \left[1 + \frac{1}{2\Delta_0}\left(e_{15}^0 e_{15}^1 + C_{44}^0\eta_{11}^1\right)\right]\left[1 + \frac{1}{2\Delta_0}\left(e_{15}^0 e_{15}^1 + C_{44}^1\eta_{11}^0\right)\right]$$

$$-\frac{1}{4\Delta_0^2}\left(C_{44}^1 e_{15}^0 - C_{44}^0 e_{15}^1\right)\left(\eta_{11}^0 e_{15}^1 - \eta_{11}^1 e_{15}^0\right).$$

In the foregoing expressions the quantities with superscript '1' refer to the difference between the inclusion and the matrix electroelastic constants.

The determination of amplitudes $A_i(n)$, $B_i(n)$ and $C_i(n)$ utilizes the following relations that hold in the long-wave approximation:

$$\begin{gathered} e^{-iq\mathbf{n}\cdot\mathbf{y}'} \simeq 1, \quad u_k^0 = U_k^0, \quad \varepsilon_{kl}^0 = ikU_{(k}^0 n_{l)}^0 \quad (k = \alpha, \beta, \beta_\perp), \\ \varphi^0 = \frac{e_{15}^0}{\eta_{11}^0}(U_k^0 m_k), \quad E_k^0 = -i\beta_\perp n_k^0 \frac{e_{15}^0}{\eta_{11}^0}(U_k^0 m_k), \end{gathered} \tag{5.6}$$

where U_k^0 is the polarization vector, k is the wave number and n^0 is the normal to the wave front.

In accordance with (3.18), we have

$$\begin{aligned} f_k(qn) = {} & \frac{i\pi a^2}{2\rho_0\omega^2}\sqrt{\frac{q^3}{2\pi}}e^{-i\pi/4}\Bigg[\rho_1\omega^2 U_k^0 \\ & - qkC_{klpq}^{\mathcal{A}} n_l n_q^0 U_p^0 - q\beta_\perp \frac{e_{15}^0}{\eta_{11}^0}(U_k^0 m_k) e_{klp}^{\mathcal{A}} n_l n_p^0\Bigg], \\ f(\beta_\perp n) = {} & -\frac{i\pi a^2}{2\rho_0\omega^2}\sqrt{\frac{q^3}{2\pi}}e^{-i\pi/4}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)\beta_\perp \\ & \times \left[k e_{kpq}^{\mathcal{A}T} n_k n_p^0 U_q^0 + \beta_\perp \frac{e_{15}^0}{\eta_{11}^0}(U_k^0 m_k)\eta_{kp}^{\mathcal{A}} n_k n_p^0\right], \end{aligned} \tag{5.7}$$

where $C^{\mathcal{A}}$, $e^{\mathcal{A}}$ and $\eta^{\mathcal{A}}$ are defined in (5.4) and (5.5), respectively, and a is the fiber radius.

We now consider several special cases.

5.1. LONGITUDINAL WAVES

In this case,

$$k = \alpha, \quad U_k^0 = n_k^0, \quad U_k^0 m_k = 0 \tag{5.8}$$

and the expressions for $f_k(\alpha n)$ and $f_k(\beta n)$ read

$$\begin{aligned} f_k(\alpha n) &= \frac{i\pi a^2}{2\rho_0\omega^2}\sqrt{\frac{\alpha^3}{2\pi}}\mathrm{e}^{-i\pi/4}\Big\{\rho_1\omega^2 n_k^0 \\ &\quad - \alpha^2\left[\frac{1}{2}(C_{11}^{\mathcal{A}}+C_{12}^{\mathcal{A}}) + +C_{66}^{\mathcal{A}}(2\cos\phi-1)\right]n_k\Big\}, \\ f_k(\beta n) &= \frac{i\pi a^2}{2\rho_0\omega^2}\sqrt{\frac{\beta^3}{2\pi}}\mathrm{e}^{-i\pi/4}\Big\{\rho_1\omega^2 n_k^0 \\ &\quad - \alpha\beta\left[\frac{1}{2}(C_{11}^{\mathcal{A}}+C_{12}^{\mathcal{A}})n_k + C_{66}^{\mathcal{A}}(2n_k^0\cos\phi - n_k)\right]\Big\}. \end{aligned} \tag{5.9}$$

According to (3.17), we find now

$$A_i(n) = \frac{i\pi a^2}{2}\sqrt{\frac{\alpha^3}{2\pi}}\mathrm{e}^{-i\pi/4}\left\{\frac{\rho_1}{\rho_0}\cos\phi - \frac{1}{C_{11}^0}\left[\frac{1}{2}\Big(C_{11}^{A}+C_{12}^{A}\Big) + C_{66}^{A}\cos 2\phi\right]\right\}n_i,$$

$$B_i(\beta n) = \frac{i\pi a^2}{2}\sqrt{\frac{\beta^3}{2\pi}}\mathrm{e}^{-i\pi/4}\left[\frac{\rho_1}{\rho_0} - 2\zeta\frac{C_{66}^{A}}{C_{66}^0}\cos\phi\right](n_i^0 - n_i\cos\phi), \tag{5.10}$$

where

$$\zeta = \alpha/\beta. \tag{5.11}$$

Obviously, $f(\beta_\perp n) = 0$ and the vector $f_k(\beta_\perp n)$ lies in the x_1x_2-plane. Therefore $f_k(\beta_\perp n)m_k = 0$ and $C_i(n) = c(n) = 0$.

Taking into account the relation

$$\langle I^0\rangle_t = -\frac{1}{2}\omega\alpha C_{11}^0 \tag{5.12}$$

and substituting (5.10) into the right-hand side of (4.12), we obtain, after integration with respect to φ,

$$\begin{aligned} Q_L(\omega) &= \frac{\pi^2}{8}a(\alpha a)^3\left\{\frac{1}{(\rho_0 v_L^2)^2}\left[\frac{1}{2}(C_{11}^{\mathcal{A}}+C_{12}^{\mathcal{A}})^2\right.\right. \\ &\quad \left.\left. + (C_{66}^{\mathcal{A}})^2\left(1+\frac{1}{\zeta^4}\right)\right] + \left(\frac{\rho_1}{\rho_0}\right)^2\left(1+\frac{1}{\zeta^2}\right)\right\}, \quad v_L^2 = C_{11}^0/\rho_0. \end{aligned} \tag{5.13}$$

5.2. SHEAR WAVES POLARIZED IN THE x_1x_2-PLANE

In this case,

$$\begin{aligned} U_k^0 &= e_k^0 \quad (e_k^0 n_k^0 = e_k^0 m_k^0 = 0), \quad k = \beta, \\ \langle I^0\rangle_t &= -\frac{1}{2}\omega\beta C_{66}^0. \end{aligned} \tag{5.14}$$

The scalar $f(\beta_\perp n)$ vanishes (as in the previous case) and

$$\begin{aligned} f_k(\alpha n) &= \frac{i\pi a^2}{2\rho_0\omega^2}\sqrt{\frac{\alpha^3}{2\pi}}\mathrm{e}^{-i\pi/4}\left[\rho_1\omega^2 e_k^0 - \alpha\beta C_{66}^{\mathcal{A}}(e_k^0\cos\phi + n_k^0\sin\phi)\right], \\ f_k(\beta n) &= \frac{i\pi a^2}{2\rho_0\omega^2}\sqrt{\frac{\beta^3}{2\pi}}\mathrm{e}^{-i\pi/4}\left[\rho_1\omega^2 e_k^0 - \beta^2 C_{66}^{\mathcal{A}}(e_k^0\cos\phi + n_k^0\sin\phi)\right]. \end{aligned} \tag{5.15}$$

The amplitudes $A_i(n)$ and $B_i(n)$ of the scattered waves take the form

$$\begin{aligned} A_i(n) &= \frac{i\pi a^2}{2}\sqrt{\frac{\alpha^3}{2\pi}}\mathrm{e}^{-i\pi/4}\left(\frac{\rho_1}{\rho_0}\sin\phi - \zeta\frac{C_{66}^{\mathcal{A}}}{C_{66}^0}\sin 2\phi\right)n_i, \\ B_i(n) &= \frac{i\pi a^2}{2}\sqrt{\frac{\beta^3}{2\pi}}\mathrm{e}^{-i\pi/4}\left[\frac{\rho_1}{\rho_0}\left(e_i^0 - n_i\sin\phi\right)\right. \\ &\left. -\beta^2\frac{C_{66}^{\mathcal{A}}}{C_{66}^0}\left(e_i^0\cos\varphi + n_i^0\sin\phi - n_i\sin 2\phi\right)\right]. \end{aligned} \tag{5.16}$$

Hence, the total scattering cross-section of the waves, according to the general expression (4.12), is

$$\begin{aligned} Q_T(\omega) &= \frac{\pi^2}{8}a(\beta a)^3\left[\frac{1}{(\rho_0 v_T^2)^2}(C_{66}^{\mathcal{A}})^2(1+\zeta^4)\right. \\ &\left.+\left(\frac{\rho_1}{\rho_0}\right)^2(1+\zeta^2)\right], \quad v_T^2 = C_{66}^0/\rho_0. \end{aligned} \tag{5.17}$$

As it follows from (5.13) and (5.17), the total scattering cross-sections of the longitudinal and shear waves polarized in the x_1x_2-plane do not contain any dielectric or piezoelectric constants. This was to be expected, since the x_1x_2-plane is the plane of isotropy, so that the piezoelectric behaviour does not manifest itself. The situation, however, is quite different when shear waves, polarized in the x_3-direction, are considered.

5.3. SHEAR WAVES POLARIZED IN x_3-DIRECTION

Indeed, we have in this case,

$$U_k^0 = m_k, \quad k = \beta_\perp, \quad \langle I^0\rangle_t = -\frac{1}{2}\omega\beta_\perp C_{44}'. \tag{5.18}$$

Obviously, $A_i(n) = B_i(n) = 0$ and the vector $f_k(\beta_\perp)$ and the scalar $f(\beta_\perp)$ become

$$\begin{aligned} f_k(\beta_\perp n) &= \frac{i\pi a^2}{2}\sqrt{\frac{\beta_\perp^3}{2\pi}}\mathrm{e}^{-i\pi/4}\left[\frac{\rho_1}{\rho_0} - \frac{1}{C_{44}'}\left(C_{44}^{\mathcal{A}} + \frac{e_{15}^{\mathcal{A}}e_{15}^0}{\eta_{11}^0}\right)\right]\cos\phi, \\ f(\beta_\perp n) &= -\frac{i\pi a^2}{2}\sqrt{\frac{\beta_\perp^3}{2\pi}}\mathrm{e}^{-i\pi/4}\frac{e_{15}^0}{\eta_{11}^0 C_{44}'}\left(e_{15}^{\mathcal{A}} + \frac{e_{15}^0}{\eta_{11}^0}\eta_{11}^{\mathcal{A}}\right)\cos\phi. \end{aligned} \tag{5.19}$$

It follows, then, that

$$C_i(n) = \frac{i\pi a^2}{2}\sqrt{\frac{\beta_\perp^3}{2\pi}}\mathrm{e}^{-\frac{i\pi}{4}}\left\{\frac{\rho_1}{\rho_0} - \frac{1}{C'_{44}}\left[C_{44}^A + \left(\frac{e_{15}^0}{\eta_{11}^0}\right)^2 \eta_{11}^A\right]^2 \cos\phi\right\} m_i. \quad (5.20)$$

The total scattering cross-section for these waves is

$$Q_{T\perp}(\omega) = \frac{\pi^2}{8}(\beta_\perp a)^3 a \left\{\frac{1}{(\rho_0 v_{T\perp}^2)^2}\left[C_{44}^A + \left(\frac{e_{15}^0}{\eta_{11}^0}\right)^2 \eta_{11}^A\right]^2 + 2\left(\frac{\rho_1}{\rho_0}\right)^2\right\}, \quad (5.21)$$

$$v_{T\perp}^2 = C'_{44}/\rho_0.$$

For the purely elastic behaviour ($e_{15}^0 = 0$) this expression coincides with the one obtained in [12].

6. CONCLUSIONS

The obtained results for the scattering amplitudes and cross-sections of a circular scatterer in a piezoelectric medium of hexagonal (transversely isotropic) symmetry may be useful for many future applications, e.g., for the determination of the symmetry of the scatterer by measuring its scattering cross-section. The Green's function method, presented in Section 3, can be extended to scatterers of arbitrary symmetry. Here a similar amplitude equation as (3.16) occurs wherein the scattering amplitudes reflect the symmetry of the scatterer. Thus the presented method will hopefully stimulate further work in the treatment of the scattering of acoustoelectric waves at inhomogeneities.

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APPENDIX

The needed formulae, concerning the tensorial basis (3.4) used in the paper, are collected here. Their application, as already demonstrated, allows one to substantially simplify and standardize the appropriate tensorial operations in the problem under consideration.

If a certain tensor **A** is expressed in the **T**-basis as

$$\mathbf{A} = A_1\mathbf{T}^2 + A_2\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) + A_3\mathbf{T}^3 + A_4\mathbf{T}^4 + A_5\mathbf{T}^5 + A_6\mathbf{T}^6, \tag{A.1}$$

then the inverse tensor $\mathbf{A}^{-1}$ is given by the expression

$$\begin{aligned}\mathbf{A}^{-1} = \frac{A_6}{2\Delta}\mathbf{T}^2 + \frac{1}{A_2}\left(\mathbf{T}^1 - \frac{1}{2}\mathbf{T}^2\right) - \frac{A_3}{\Delta}\mathbf{T}^3 - \frac{A_4}{\Delta}\mathbf{T}^4 \\ + \frac{4}{A_5}\mathbf{T}^5 + \frac{2A_1}{\Delta}\mathbf{T}^6, \quad \Delta = 2(A_1A_6 - A_3A_4).\end{aligned} \tag{A.2}$$

If two tensors **A** and **B** are given in the **T**-basis, the contraction of these tensors with respect to two pairs of indices reads

$$\begin{aligned}A_{ijkl}B_{klmn} = (2A_1B_1 + A_3B_4)T^2_{ijmn} + A_2B_2\left(T^1_{ijmn} - \frac{1}{2}T^2_{ijmn}\right) \\ + (2A_1B_3 + A_3B_6)T^3_{ijmn} + (2A_4B_1 + A_6B_4)T^4_{ijmn} \\ + \frac{1}{2}A_5B_5T^5_{ijmn} + (A_6B_6 + 2A_4B_3)T^6_{ijmn}.\end{aligned} \tag{A.3}$$

Consider now two tensors $\mathbf{C}$ and $\mathbf{D}$, presented in the $\mathbf{U}$-basis,

$$C_{ijk} = \sum_{r=1}^{3} C_r U_{ijk}^r, \quad D_{ijk} = \sum_{s=1}^{3} D_s U_{ijk}^s. \tag{A.4}$$

The contraction of these tensors with respect to one index gives the tensor in $\mathbf{T}$-basis:

$$C_{ijm} D_{mkl}^T = C_1 D_1 T_{ijkl}^2 + C_1 D_3 T_{ijkl}^3 + C_3 D_1 T_{ijkl}^4 + 4C_2 D_2 T_{ijkl}^5 + C_3 D_3 T_{ijkl}^6. \tag{A.5}$$

The contraction of the tensors $\mathbf{C}$ and $\mathbf{D}$ with respect to two pairs of indices gives a tensor, which is presented in the $\mathbf{t}$-basis as

$$C_{ikl}^T D_{klj} = 2C_2 D_2 t_{ij}^1 + (2C_1 D_1 + C_3 D_3) t_{ij}^2. \tag{A.6}$$

It can be shown that the $\mathbf{t}$-basis is orthogonal in the sense that if

$$\alpha_{ij} = \alpha_1 t_{ij}^1 + \alpha_2 t_{ij}^2, \quad \beta_{ij} = \beta_1 t_{ij}^1 + \beta_2 t_{ij}^2, \tag{A.7}$$

then

$$\alpha_{ik} \beta_{kj} = \alpha_1 \beta_1 t_{ij}^1 + \alpha_2 \beta_2 t_{ij}^2 \tag{A.8}$$

and

$$\alpha_{ij}^{-1} = \frac{1}{\alpha_1} t_{ij}^1 + \frac{1}{\alpha_2} t_{ij}^2. \tag{A.9}$$

The following formulae are also useful:

$$A_{ijmn} C_{mnk} = (2A_1 C_1 + A_3 C_3) U_{ijk}^1 + \frac{1}{2} A_5 C_2 U_{ijk}^2 + (2A_4 C_1 + A_6 C_3) U_{ijk}^3,$$

$$C_{imn}^T A_{mnkl} = (2C_1 A_1 + C_3 A_4) U_{ijk}^{1T} + \frac{1}{2} C_2 A_5 U_{ijk}^{2T} + (2C_1 A_3 + C_3 A_6) U_{ijk}^{3T},$$

$$\alpha_{im} C_{mkl}^T = \alpha_2 C_1 U_{ikl}^{1T} + \alpha_1 C_2 U_{ikl}^{2T} + \alpha_2 C_3 U_{ikl}^{3T}, \tag{A.10}$$

$$C_{ijm} \alpha_{mk} = C_1 \alpha_2 U_{ijk}^1 + C_2 \alpha_1 U_{ijk}^2 + C_3 \alpha_2 U_{ijk}^3.$$

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