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AN EXPLICIT POLYNOMIAL SOLUTION OF THE REPRESENTATIVE PROBLEM OF THE MECHANICS OF FIBROUS COMPOSITES

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The paper is focused on an important aspect of the central problem of the mechanics of the unidirectionally reinforced fibrous composites. This problem concerns the features of the matrix-fibre load transfer phenomenon which, actually, provides by itself the very reinforcing effect of the fibres. The paper illustrates that the successful analysis of this problem definitely needs, first, the exact solution of a certain typical or, say, representative axisymmetric boundary value problem of the elasticity theory and, second, a representation of this solution in a form, which is convenient enough for additional mathematical manipulations. Such an explicit with respect to the problem variables polynomial representation is derived in the paper.

Keywords: fibrous composites, elasticity, special functions

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1. INTRODUCTION

A specific feature of the present contribution is that the purely mathematical result derived in it is, in practice, a useful and effective tool from the view point of the mechanics, i.e. of the study of the mechanical properties and the mechanical behaviour of an important for the engineering practice class of advanced structural materials, namely the unidirectionally reinforced fibrous composites. For this reason the analysis of the mathematical problem considered is preceded by a relatively extended preliminary section, the aim of which is to indicate at least the main problems associated with the mechano-mathematical modelling of the unidirectional fibrous composites and to illustrate, at the same time, the necessity and the practical value of the result obtained.

2. PRELIMINARIES

Composite materials consist, generally speaking, of a relatively weak and compliant continuous phase within which inclusions of different, most often stiff and strong materials, are discretely distributed and, as a rule, firmly bound to the surrounding continuous matrix phase. Depending on the material, the shape, the volume fraction and the geometry of the spatial distribution and mutual orientation of the inclusions, composite properties and, especially, their mechanical properties may differ considerably and even drastically from the respective properties of the matrix material. The strong effect that such inclusions produce on the properties of the matrix has been used since long in the fabrication of new structural materials with desired in advance unique combinations of mechanical properties. Basically, the goal of the practical use of this effect is to create materials with high strength and stiffness parameters, i.e. to reinforce the weak materials used as matrix phases. For this reason the effect is commonly referred to as a *reinforcing effect*.

This effect is especially pronounced in the so-called *fibre-reinforced composites*. The discrete reinforcing phase in these composites presents itself one or another network of rods, wires or whiskers, which are usually called *fibres*. An important for the practice class of such composites is that of the *unidirectionally reinforced fibrous composites*, i.e. of composites with straight parallelly aligned fibres. In most of the cases the fibres have the form of circular cylinders with a specific for each given composite radius-to-length ratio, which is often called fibre *slenderness ratio*.

The unidirectional composites are anisotropic or, speaking more precisely, transversely isotropic materials. They may have high strength and stiffness in the direction of the reinforcing fibres, but remain weak and compliant, as the matrix is, in the transverse direction. The practice proves that the degree of anisotropy or, which is the same, the strength of the very effect of the unidirectional fibrous reinforcement, depends on two basic structural parameters, namely the fibre volume fraction and the fibre slenderness ratio.

The strength and the stiffness of the unidirectional composites in their, say, strong direction, which is the direction of the reinforcing fibres, appear to be almost directly proportional to the fibre volume fraction. The simple "*rule of mixtures*"-type relations, used in the engineering practice, prove to be reliable quantitative approximations to the actual effect that the fibre volume fraction produces on the degree of strengthening and stiffenning of the matrix material.

The fibre slenderness ratio influences the strength and stiffness characteristics of the unidirectional composites in their strong direction in a more complicated way. In composites, which differ only in the lengths of the reinforcing fibres, but are otherwise identical in all remaining material and structural parameters, including fibre radii, those with longer fibres, i.e. with smaller slenderness ratios, are stronger and stiffer. A distinct feature of the practically observed dependencies of the strengthening effect on this ratio is that the decrease in the latter provides a respective increase in this effect only over certain restricted and specific for each composite structure ranges of increasing fibre lengths.

Briefly speaking, the unidirectional composites are characterized by specific slenderness ratios or, at fixed fibre radii, by specific fibre lengths. When the value of the specific for a given composite slenderness ratio is approached from above by means, for example, of a continuous increase of the fibre lengths at fixed fibre radii, the reinforcing effect continuously increases up to a specific for the composite maximum level and then, with further fibre lengths increase, this effect remains practically unchanged. The specific or, say, the *critical fibre length* upon which the reinforcing effect becomes insensitive to further fibre length increase, proves to be a distinct inherent characteristic of the unidirectional composites. Practical observations indicate that it is a complex and, as a matter of fact, still not quite well known function of the material parameters of the composites. The existing experimental data certify that the critical fibre lengths for different fibre-matrix systems vary from a few to tens and even hundreds fibre radii. This fact is a clear indication that the theoretical approaches to the problem of determining the critical fibre lengths should not be based on rough initial physical models, neither should involve, from a mathematical view point, rough simplifications and approximations.

From its purely qualitative side the existence of such a critical fibre length is easily understandable. The point is that in composites loads are not directly applied to the fibres, but to the matrix into which they are embedded. The role of the matrix is, besides to serve as a binding medium, also to transfer loads from composite surfaces, where the loads are applied, to the fibres. This *matrix-fibre load transfer* is, in reality, the essence of the very mechanism of realization of the reinforcing effect. Due to this mechanism the strong and stiff fibres take or absorb and, respectively, carry most of the loads applied, as mentioned, to the matrix. This actually explains the high load bearing capacities of the fibrous composites, i.e. their potential to carry loads that considerably exceed the restricted load bearing capacities of the weak matrix materials.

Obviously enough, the matrix transfers loads to the fibres through their end and cylindrical surfaces. Due to a number of reasons the end fibre surfaces have little effect on the overall load transfer pattern. One of these reasons is that the area of the end fibre surfaces is much smaller than that of the cylindrical lateral surfaces. Therefore, the contribution of fibre ends to the load transfer should not be expected to be comparable to that of the cylindrical surfaces. The increase in the area of fibre end surfaces, which is achievable by preparing fibres with oval instead of flat ends, proves practically not to affect the load transfer features. This is due to the fact that, mainly for technological reasons, the fibre-matrix bonds at fibre ends are not of the same necessarily high quality as those over the cylindrical fibre surfaces. Thus, irrespectively of whether fibre ends are flat or oval, the stresses developing over the weak fibre ends-matrix interfaces prove to be simply small enough to play a more or less decisive role in the load transfer processes. That is why in most of the studies of the load transfer phenomena fibre ends are viewed as stress free.

It would be probably instructive to note, before considering the load transfer through the cylindrical fibre surfaces, that fibrous composites are designed mainly to carry loads in the reinforcement direction, i.e. in the axial fibre direction. The transfer of such loads is evidently due to the axial interfacial shear stresses developing as a result of the mechanical fibre-matrix interaction. To illustrate this, it would be sufficient just to consider the equilibrium in the axial direction of an arbitrary fibre portion and immediately to notice that these are exactly the axial interfacial shear stresses that balance the axial loads acting at the ends of such a fibre portion.This fact demonstrates that the study of the load transfer should be first of all focused on the problem of determining the longitudinal (i.e. along the fibre length) distribution of the axial interfacial shear stresses.

Coming closer to this problem, let us remind once more that the increase of fibre lengths up to certain specific critical values results in higher load bearing capacities or that, briefly speaking, longer fibres absorb and, respectively, carry higher axial loads. In part, this effect is certainly due to the fact that since the fibre length increase is also increase in the area of the cylindrical fibre surfaces, then the axial interfacial shear stresses, when acting over increased areas, should introduce into the fibres and, respectively, balance increased axial loads. At the same time one should not exclude as a reason for this effect the eventual changes and, in particular, the possible increase of the intensities of the same shear stresses. Moreover, the very distribution along the fibre length of these stresses may change and thus have also effect on the level of the axial loads transferred from the matrix to the fibre.

It is hoped that the foregoing considerations reveal to a certain extent the complex multiaspect nature of the load transfer problem. In fact, this central for the mechanics of the unidirectional composites problem has been since long a subject of extended research. The result of this research is the number of *load transfer models* that the current composite materials literature offers. Without a discussion on their positive and negative sides it will be mentioned here that all these models are practically similar in a few basic aspects.

First, they are based on the experimentally observed fact that the mechanical behaviour, mainly in the reinforcement direction, of a typical unidirectional composite is very close to that of a representative composite element, say, *unit composite cell*, which, being composed of a single fibre with a firmly bound coaxial matrix coating, is stressed axisymmetrically and, in addition, symmetrically with respect to its middle cross-section.

Second, the models reflect, but rather in a qualitative than in a more or less satisfactory quantitative manner, the concentration of stresses close to the ends of the fibres where, as a result and a manifestation of the *stress concentration*, local failure phenomena, mainly in the form of fibre-matrix debonding (i.e. of interfacial cracks), often take place. In this regard the similarity of the models lies in the fact that they consider as a major reason for these phenomena mainly the concentration of the interfacial axial shear stresses. Accordingly, they deal, basically, with the problem of deriving the patterns of the shear stress concentration fields close to the fibre ends.

Third, the shear stress concentration patterns that the models provide are derived from either simple strength of materials-type analyses or from too much simplified axisymmetric boundary value problems of the elasticity (or plasticity) theory. The simplifications concern both the mathematical treatments of the equations governing these problems and the mechano-mathematical models of the actual physical boundary conditions, especially the conditions for the stresses at the fibre ends. These conditions will be discussed below.

Finally, the models generally assume that no interfacial shear stresses develop along most of the fibre lengths and suggest one or another specific monotonous increase of the intensities of these stresses along the remaining relatively short end fibre portions. But, strictly speaking, such simple distributions of stresses are just inconsistent with the elasticity theory since the equations of the latter are, as it is known, of elliptic type. Accordingly, the distributions of fibre stresses, including the interfacial shear stresses, might be either uniform along the entire fibre length or strictly non-uniform along each non-vanishingly short fibre portion. Thus, from a quantitative view point, such simple axial stress distributions are acceptable only in the sense of the Saint Venant's principle and as approximations to certain particular exact solutions of the general axisymmetric elasticity problem. Since these solutions, i.e. the exact solutions of the particular boundary value problems associated with the models, are not known, one could hardly derive estimates of the errors that such approximations involve and decide, respectively, about the reliability of the models.

Irrespectively of their roughness and simplicity, these models are widely used in the practice of composites design and application since they provide quantitative estimates of the load transfer parameters and, thus, of the load bearing potentials of the fibrous composites, which, even if not realistic enough in the details, are quite acceptable as mean or integral estimates.

Another positive side of the models is that they have revealed in part the specific nature of the above mentioned *nontrivial dependence of the reinforcing effect on the lengths of the fibres*. The models definitely indicate that the separation of a fibre into a central uniformly stressed portion, which only carries axial load, but does not take load directly from the surrounding matrix, and neighbouring this portion with relatively short end portions, along which the matrix transfers load to the fibre, are quite realistic. This separation of the fibre into a load bearing, or, effective and load transfer, or, ineffective portions is commonly adopted today. In view of these models the main problem of the load transfer analysis is to determine the current load transfer fibre length as a function of both the fibre-matrix system and the current load, applied to the composite, and, in addition, to specify the characteristic for the composite critical maximum load transfer length, now as a function of the fibre-matrix system only. If this critical length is specified, then, in order to gain maximum load bearing effect from a given unidirectional fibre-matrix system, one should use fibres of length which is greater than or at least equal to twice the critical length. Such fibres are usually referred to as *long fibres*. Fibres of lengths below the critical length are viewed as *short fibres*. The respective composites are usually called composites with long, or continuous and short, or discontinuous fibres.

It is not surprising that the models suggest estimates of the critical load transfer lengths which differ considerably. These estimates depend on the different in their roughness approximations involved in each particular model. There is a number of reasons for which these estimates should not be viewed as sufficiently precise though, as was mentioned above, they are acceptable in integral sense. In part, this is due to the fact that, as was also mentioned, the very concept of specifying effective and ineffective fibre portions is a matter of approximate interpretation of an exact solution of the axisymmetric elasticity problem. This interpretation implicitly assumes that the exactly determined interfacial shear stresses, acting along the central (effective) fibre portion, are small, i.e. practically negligible, and that they monotonously increase towards fibre ends along the load transfer (ineffective) fibre portions. But such an interpretation, being, let us mention again, acceptable on the whole, necessarily requires, first of all, to specify the level below which the stresses in question could be viewed as negligible. The models do not suggest such levels. Moreover, they can not specify the latter since such specifications require by themselves as a reference basis the unknown exact stress distributions.

The basic limitation of the existing models is that they take practically no account of the actual boundary conditions at the fibre ends. Generally, they assume that high interfacial shear stresses develop close to fibre ends, but entirely ignore the corresponding considerable drop that these stresses should necessarily undergo along certain end fibre portions, adjacent to fibre ends, due to the fact that these ends are practically almost free of shear stresses. The models simply ignore these end portions as elements of the load transfer pattern.

Two remarks are due with respect to this fact. First, from the view point of the pure load transfer, the ignorance of these portions is not an important disadvantage since, from most general considerations, the latter should be expected to be short enough and thus not to cause considerable corrections in the estimates of the load transfer lengths. Second and more important is that the models ignore the entire complex interfacial stress field developing along these end fibre portions. This specific stress field has been constructed in a recent author's paper [1], where some of the results of an extended author's study in progress are briefly reported. As it is shown in the paper, this field develops along end fibre portions, which are really short. Their lengths do not exceed a few fibre radii, which proves that these portions really could not contribute considerably to the overall load transfer. But at the same time it is definitely proved in the paper that this stress concentration field is the factor that actually governs the onset of the failure phenomena close to the fibre ends. In fact, the paper illustrates, first of all, that these phenomena are not governed, as the considered models suggest, by the interfacial shear stress concentration, but by the much higher concentration of the interfacial radial stresses. Moreover, as it is shown in the paper, the radial interfacial stress changes its sign along the short end fibre portions. In other words, this high by itself stress is always positive, i.e. tensile, along certain parts of these portions. Obviously, this high tensile interfacial radial stress is the factor dominating the critical conditions of onset of failure phenomena close to the fibre ends.

The load transfer model proposed in [1] could be referred to as a *full fibre length model* since, in contrast to the models considered, it involves the very end fibre portions which these models ignore. Being actually derived by means of sewing of two exact axisymmetric solutions of the elasticity theory, the model is not only free from the shortcomings of the existing simple models, but, in view of the considerations of the present section, is a real reference basis on which one could estimate the reliability of these approximate models.

It is hoped that this section clearly indicates that the general axisymmetric problem of the theory of elasticity is of immediate relevance to the mechanics of the unidirectional fibrous composites. The realistic modelling and prediction of the properties and the behaviour of these composites definitely require not only the derivation of exact solutions of certain particular, but similar on the whole axisymmetric boundary value elasticity problems. In fact, it is equally important to be easily able to manipulate further such solutions in order, for example, to construct superpositions, to derive approximations with desired accuracy, to estimate the accuracy of certain existing approximations or, as in [1], to sew such exact solutions.

It would be very advantageous, for these reasons, to have at one's disposal a convenient, i.e. an easy for mathematical manipulations, form of the exact solution of a certain, say, *representative problem* of the mechanics of the unidirectional fibrous composites. In what follows, this problem is first formulated and its solution is then shown to be representable indeed in a really convenient explicit analytical polynomial form.

3. THE REPRESENTATIVE AXISYMMETRIC PROBLEM

The commonly adopted representative model problem for a typically loaded long reinforcing fibre assumes that the state of stress of the latter is, as was pointed out above, axisymmetric and, in addition, symmetric with respect to its middle cross section. Basing upon the superposition principle of the elasticity theory and on the standard assumption that fibre ends are flat and free from stresses, one may further specify the fibre stress state as resulting from uniformly distributed normal (tensile or compressive) stresses of intensity, say σ_0, applied to its ends (or acting over its middle cross section), and from interfacial stresses, developing along the lateral cylindrical fibre surface as a result of the mechanical fibre-matrix interactions, caused by the mismatch of the mechanical properties of the fibre and matrix materials.

The fibre, when referred to cylindrical coordinates $\{r, \theta, z\}$, is assumed to occupy the domain $\{0 \le r \le r_f,\ |z| \le L,\ 0 \le \theta \le 2\pi\}$, where, obviously, r_f and L are fibre radius and half length, respectively. Then the boundary conditions of the representative problem could be specified as

$$\tau_{rz}(r, 0) = \tau_{rz}(r, \pm L) = 0, \tag{1}$$

$$\sigma_z(r, \pm L) = \sigma_0, \quad (\text{or} \quad \sigma_z(r, 0) = \sigma_0), \tag{2}$$

where $\tau_{rz}(r, z)$ and $\sigma_z(r, z)$ are the shear and the axial fibre stresses, respectively. The notations $\sigma_r(r, z)$ and $w(r, z)$ will be used below for the radial fibre stress and the axial displacement in the fibre.

Obviously, the remaining boundary conditions, concerning the stresses $\tau_{rz}(r_f, z)$ and $\sigma_r(r_f, z)$, acting over the cylindrical fibre surface $r = r_f$, could not be specified in advance. In fact, to determine these unknown stresses is, as was highlighted above, the sense of the central for the mechanics of the unidirectional composites problem of the matrix-fibre load transfer.

The approach to this problem is based on the understanding, developed in the mentioned author's study in progress and briefly described in [1], according to which the role of the weak and compliant matrix is rather to conduct the applied loads to the surface of the stiff and strong fibre than to influence considerably the specific and in much independent manner in which the fibre absorbs these loads through its surfaces, transforms them into internal stresses, and creates its own stress distribution pattern of the above discussed type. Such understanding of the dominant role of the fibre, or, of its more or less independent behaviour, suggests that the actual fibre state could be interpreted as an optimum or, say, a *natural* one, or, in other words, as a state corresponding to a solution of a certain variational problem of the elasticity theory to which the interfacial stresses in question serve as natural boundary conditions.

This understanding is further combined with the concept of the stress function (or, stress potential) as a function which, once introduced as a solution of the *general variational problem* of the elasticity theory, provides full exact solutions of each particular boundary value problem when subject to the respective boundary conditions. Use is made of the known representations of the stresses and displacements by means of the stress function for the general axisymmetric elasticity problem. For the quantities of interest in the present study these representations read, cf. [2],

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\Delta\varphi - \frac{\partial^2\varphi}{\partial z^2}\right], \tag{3}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\Delta\varphi - \frac{\partial^2\varphi}{\partial z^2}\right], \tag{4}$$

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\Delta\varphi - \frac{\partial^2\varphi}{\partial r^2}\right), \tag{5}$$

$$w = \frac{1}{2G}\left[2(1-\nu)\Delta\varphi - \frac{\partial^2\varphi}{\partial z^2}\right], \tag{6}$$

where $\varphi(r, z)$ is the stress function ($\Delta^2\varphi(r, z) = 0$), Δ is the Laplace operator, ν and G are, respectively, the Poisson's ratio and the shear modulus of the material considered or, in our case, of the fibre material.

Finally, use is also made of the productive Timoshenko's idea (cf.[2]) of representing the particular solutions for the stress function in terms of Legendre polynomials of the first kind $P_n(x)$, $n = 1, 2, 3, \ldots$, $x = z/R$, $R = \sqrt{r^2 + z^2}$. According to this representation each stress function of the form

$$\varphi_{2p+1}(r, z) = A_{2p+1} R^{2p-2} P_{2p-2}(x) + B_{2p+1} R^{2p-2} P_{2p-4}(x), \tag{7}$$

where $p = 1, 2, 3, \ldots$, A_{2p+1} and B_{2p+1} are constants to be determined from the boundary conditions, provides an axisymmetric stress state with stresses $\sigma_z(r, z; p)$ and $\tau_{rz}(r, z; p)$, which, as the representative problem requires, besides being axisymmetric, are, respectively, even and odd functions of z, i.e. symmetric and antisymmetric with respect to the middle cross-section of the fibre.

The practical realization of this approach to the representative model problem considered implies the following forms of the quantities of interest:

$$\sigma_z(r, z; p) = \sigma_0 \left(\frac{R}{L}\right)^{2p-2} x P_{2p-3}(x), \tag{8}$$

$$\begin{aligned} \tau_{rz}(r, z; p) = & -\sigma_0 \left(\frac{R}{L}\right)^{2p-2} \left\{ \frac{p(2p-3)}{(p-1)(2p-1)} \frac{x}{\sqrt{1-x^2}} \right. \\ & \left. \times \Big[P_{2p-4}(x) - x P_{2p-3}(x) \Big] + \frac{1}{2p-1} \sqrt{1-x^2} P_{2p-3}(x) \right\}, \end{aligned} \tag{9}$$

$$\begin{aligned} \sigma_r(r, z; p) = & -\sigma_0 \left(\frac{R}{L}\right)^{2p-2} \left\{ \frac{2(p+1-\nu)}{(2p-1)(2p-2)} \frac{1}{1-x^2} \Big(P_{2p}(x) - x P_{2p-1}(x) \Big) \right. \\ & \left. + \frac{1}{4p-3} \left[(2p-3) P_{2p-4}(x) + 2x P_{2p-3}(x) \right] + \frac{4p^2+4p-5}{(4p-3)(2p-2)} P_{2p-2}(x) \right\}, \end{aligned} \tag{10}$$

$$\begin{aligned} w(r, z; p) = & \frac{\sigma_0 L}{2G} \left(\frac{R}{L}\right)^{2p-1} \frac{1}{(2p-1)(2p-2)(4p+1)} \\ & \times \left[\left(4p^2 - 8p + 8p\nu + 2\nu + \frac{4(2p-1)^2}{4p-3} \right) P_{2p-1}(x) \right. \\ & \left. + \frac{(2p-1)(2p-2)(4p+1)}{4p-3} P_{2p-3}(x) \right]. \end{aligned} \tag{11}$$

In fact, the model proposed in [1] is a result of a procedure of sewing of two solutions of the type presented by Eqs. (8) — (11) with their own appropriately chosen p-indices. Each of these solutions satisfies a pair of boundary conditions of the type of Eqs. (1) and (2) and is thus a solution of the representative axisymmetric problem. One of the solutions concerns the central fibre portion, say $|z| < l$, $l < L$, while the other one governs the end fibre portions $l \leq |z| \leq L$, where, obviously, $L - l$ is the length of the latter fibre portions. These two solutions are sewed over the cross sections $|z| = l$ of the fibre.

The closed analytical form of the solution of the general representative problem, Eqs. (8) — (11), has its definite advantages, but, at the same time, this form is easily seen to be not convenient for further mathematical manipulations which one is necessarily forced to perform for one or another reason. Such manipulations are, for example, unavoidable part of the further elastic analysis of the complete problem of determining the stress-strain state of the entire unit composite cell (of which the fibre is only an element). The difficulties, arising when this form is eventually subject to further manipulations, result, on the one hand, from the fact that Eqs. (8) — (11) are implicit with respect of the problem variables r and z. On the other hand, the known analytical forms of the Legendre polynomials are by themselves complicated enough. To reduce to some extent these difficulties, most of which the author met during his work on [1], was the author's motive to try to derive a simpler representation of the latter equations, namely an explicit with respect to the variables r and z polynomial representation.

It should be recognized that the work on [1] and, in particular, the analysis of the axial fibre stress, Eq. (8), gave the author the hint for the form of the simple representation derived below. Unexpectedly, it appeared that to prove the generalization of this form was not a trivial combinatorics problem. As the reminder part of this contribution illustrates, the use of some familiar special functions provides a short and effective way to this generalization.

4. AXIAL FIBRE STRESS

The desired simplification of the axial fibre stress representation concerns, obviously, only the term $\sigma_z(r, z\,;\,p)/\sigma_0$ in Eq. (8). Upon introducing dimensionless coordinates ρ, ζ this term takes, in accordance with one of the standard representations of the Legendre polynomials $P_n(x)$, cf. [3], the form

$$\frac{\sigma_z(\rho, \zeta; p)}{\sigma_0} = \zeta^{2p-2} \sum_{k=0}^{p-2} \frac{(-1)^k (4p-2k-6)!}{2^{2p-3} k! (2p-k-3)! (2p-2k-3)!} \left(1 + \frac{\rho^2}{\zeta^2}\right)^k, \qquad (12)$$

where $(\ldots)!$ stays, as usual, for the factorials of the numbers in brackets.

The sum in Eq. (12) is a polynomial of ρ^2/ζ^2. Let this sum be denoted by Σ_z and written in the form

$$\Sigma_z = \sum_{m=0}^{p-2} \lambda_m \left(\frac{\rho^2}{\zeta^2}\right)^m. \qquad (13)$$

Provided a more or less convenient representation of the coefficients λ_m, $m = 1, 2, \ldots, p-2$, is found, the axial fibre stress will take the explicit polynomial form

$$\sigma_z(\rho, \zeta; p) = \sigma_0 \sum_{m=0}^{p-2} \lambda_m \rho^{2m} \zeta^{2p-2m-2}, \qquad (14)$$

which is definitely more informative and easier to deal with than the form suggested by Eq. (12) suggests.

It will be mentioned, before deriving the λ_m-presentation in question, that the solutions of the representative problem, which are of interest both from a mechanical view point and in the context of the present contribution, are those with relatively large p-values. Solutions with small p-values are physically unacceptable for the load bearing central fibre portion. As it is shown in [1], superpositions of such solutions govern, in actual fact, the state of stress of the very end fibre portions, so that they have their definite mechanical meaning. But, at the same time, as Eq. (12) proves, their explicit polynomial presentation is a matter of trivial transforms.

According to Eq. (8), the term $\sigma_z(\rho,\zeta;p)/\sigma_0$ here considered is

$$\frac{\sigma_z(\rho,\zeta;p)}{\sigma_0} = \zeta^{2p-2}\left(1+\frac{\rho^2}{\zeta^2}\right)^{\frac{2p-3}{2}} P_{2p-3}\left[\left(1+\frac{\rho^2}{\zeta^2}\right)\right]. \tag{15}$$

The general representation of the Legendre polynomials by means of the hypergeometric function $F(a,b;c;d)$ will be used below. According to this representation, cf. [3],

$$P_{2p-3}\left[\left(1+\frac{\rho^2}{\zeta^2}\right)^{-\frac{1}{2}}\right] = \frac{(4p-7)!!}{(2p-3)!!}\left(1+\frac{\rho^2}{\zeta^2}\right)^{-\frac{2p-3}{2}} F(a,b;c;d), \tag{16}$$

where, in our case, $a=-p+\frac{3}{2}$, $b=-p+2$, $c=-2p+\frac{7}{2}$, $d=1+\frac{\rho^2}{\zeta^2}$, and $(\ldots)!!$ are double factorials.

Now, in view of Eqs. (12) — (16) and of the explicit form of $F(a,b;c;d;)$ with negative integer variable $b=-p+2<0$, namely, cf. [4],

$$\begin{aligned} &F\left(-p+\frac{3}{2},-p+2;-2p+\frac{7}{2};1+\frac{\rho^2}{\zeta^2}\right) \\ &=\sum_{k=0}^{p-2}\frac{\left(-p+\frac{3}{2}\right)_k(-p+2)_k}{\left(-2p+\frac{7}{2}\right)_k k!}\left(1+\frac{\rho^2}{\zeta^2}\right)^k, \end{aligned} \tag{17}$$

the coefficients λ_m, $m=0,1,\ldots,p-2$, take the form

$$\lambda_m = \frac{(4p-7)!!}{(2p-3)!!}\sum_{k=m}^{p-2}\frac{\left(-p+\dfrac{3}{2}\right)_k(-p+2)_k}{\left(-2p+\dfrac{7}{2}\right)_k k!}\binom{k}{m}, \tag{18}$$

where the Pochhammer symbol $(a)_i = a(a+1)(a+2)\ldots(a+i-1)$, $i=1,2,\ldots$, is used along with the binomial coefficients representation

$$\binom{k}{m} = \frac{k!}{m!(k-m)!} = \frac{(k-m+1)_m}{m!}. \tag{19}$$

Upon setting $s = k - m$ Eq. (18) takes, due to the obvious relation $(a)_{i+j} = (a)_i(a+i)_j$, the form

$$\lambda_m = \frac{(4p-7)!!}{(2p-3)!!} \frac{\left(-p+\frac{3}{2}\right)_m (-p+2)_m}{\left(-2p+\frac{7}{2}\right)_m m!} \times \sum_{s=0}^{p-m-2} \frac{\left(-p+m+\frac{3}{2}\right)_s (-p+m+2)_s}{\left(-2p+m+\frac{7}{2}\right)_s s!}. \tag{20}$$

The sum in Eq. (20) is easily seen to be, by definition (cf. Eq. (17)), equal to $F(-p+m+\frac{3}{2}, -p+m+2; -2p+m+\frac{7}{2}; 1)$ and to be thus, due to the relation, cf. [4], $F(a,b;c;d) = \Gamma(d)\Gamma(d-a-b)/\Gamma(d-a)\Gamma(d-b)$, representable as $\Gamma(-2p+m+7/2)\Gamma(-m)/\Gamma(-p+2)\Gamma(-p+3/2)$, where Γ denotes the well-known Gamma-function.

With the latter form and the known representations of the Γ-function Eq. (20) implies upon due transforms and manipulations the following compact expressions for the λ_m-coefficients, $m = 0, 1, 2, \ldots, p-2$:

$$\lambda_m = (-1)^m \frac{(2m-1)!!}{(2m)!!} \binom{2p-3}{2m}. \tag{21}$$

Other equivalent representations of the λ_m-coefficients become now available from Eq. (21) as, for example,

$$\lambda_m = (-1)^m \frac{1}{2^{2m}} \binom{2p-3}{2m} \binom{2p-m-3}{m}. \tag{22}$$

With Eq. (21) the final desired explicit in the $\{\rho, \zeta\}$-variables polynomial representation of the axial fibre stress becomes

$$\sigma_z(\rho,\zeta;p) = \sigma_0 \sum_{m=0}^{p-2} (-1)^m \frac{(2m-1)!!}{(2m)!!} \binom{2p-3}{2m} \rho^{2m} \zeta^{2p-2m-2}. \tag{23}$$

5. SHEAR FIBRE STRESS

The explicit polynomial representation of the shear fibre stress is, of course, derivable in a way, similar to the one just used for the axial stress. But it would be much simpler just to introduce Eq. (23) for the axial stress into the equilibrium equation

$$\frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = 0 \tag{24}$$

and to get almost immediately, upon satisfying the axial symmetry condition $\tau_{rz}(0,\zeta;p) = 0$, the form

$$\tau_{rz}(\rho,\zeta;p) = -\sigma_0 \sum_{m=0}^{p-2} \lambda_m \frac{p-m-1}{m+1} \rho^{2m+1} \zeta^{2p-2m-3}, \tag{25}$$

where λ_m are the same coefficients as in Eq. (21) (or Eq. (22)).

6. RADIAL FIBRE STRESS

As the structure of Eqs. (5), (6) suggests, the derivation of the explicit polynomial forms for the radial fibre stress σ_r and the axial displacement w is only a matter of further, mainly technical manipulations and transforms of the type already considered for the axial fibre stress. Omitting the details they involve, these manipulations reduce Eq. (10) to the form

$$\sigma_r(\rho, \zeta; p) = -\sigma_0 \sum_{m=0}^{p} \mu_m \rho^{2m-2} \zeta^{2p-2m}, \tag{26}$$

where the coefficients μ_m, $m = 1, 2, \ldots, p$, read

$$\begin{aligned} \mu_m = (-1)^m \frac{(2m-1)!!}{(2m)!!} \frac{p+1-\nu}{(2p-1)(p-1)} \frac{m}{p} \binom{2p}{2m} \\ - (-1)^m \frac{(2m-3)!!}{(2m-2)!!} \frac{2p-2m+3}{2p-2} \binom{2p-2}{2m-2}. \end{aligned} \tag{27}$$

Note that due to the standard convention $\binom{a}{b} = 0$, when $b < 0$, the term involving the multiplier $\binom{2p-2}{2m-2}$ in μ_0 is zero, so that the coefficient μ_0 is itself zero and the summation in Eq. (26) starts, practically, with $m = 1$.

7. AXIAL DISPLACEMENT

In a similar manner the axial displacement could be reduced to the form

$$w(\rho, \zeta; p) = \frac{\sigma_0 L}{2G(4p-3)} \sum_{m=0}^{p-1} \omega_m \rho^{2m} \zeta^{2p-2m-1}, \tag{28}$$

where

$$\omega_m = \frac{(2m-1)!!}{(2m)!!} \left[b \binom{2p-1}{2m} + \binom{2p-3}{2m} \right] - \frac{(2m-3)!!}{(2m-2)!!} \binom{2p-3}{2m-2} \tag{29}$$

and

$$b = \frac{16p^3 + 4(8\nu - 7)p^2 + 8(1-2\nu)p - 6\nu + 4}{(2p-1)(2p-2)(4p+1)}. \tag{30}$$

The terms $\binom{2p-3}{2m-2}$ in ω_0 and $\binom{2p-3}{2m}$ in ω_{p-1} are zeros due to the above mentioned convention for the binomial coefficients and the convention $\binom{a}{b} = 0$ for $b < a$, respectively.

8. CONCLUDING REMARKS

The σ_z, τ_{rz}, and σ_r representations, derived above, when introduced into the equilibrium equation

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{31}$$

and the Hooke's law relation

$$u = \frac{r}{2(1+\nu)G}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] \tag{32}$$

imply almost directly similar polynomial representations for the remaining basic quantities of the considered problem, namely the circumferential stress $\sigma_\theta(\rho, \zeta; p)$ and the radial displacement $u(\rho, \zeta; p)$. The coefficients in these representations are linear combinations of the coefficients λ_m and μ_m.

The following remark is due with respect to the forms of the coefficients λ_m, μ_m and ω_m. Obviously, Eqs. (21) (or (22)), (27) and (29) are only particular and certainly not the optimum forms of the otherwise large varieties of equivalent and maybe even simpler and more compact forms in which these coefficients are representable. Each of the particular forms derived above should be actually viewed as a basis for deriving other, eventually more convenient in one or another sense, equivalent forms of the same coefficients.

It should be probably mentioned in addition that the stresses and displacements in the fibre are not, as the first impression might be, independent of the mechanical properties of the matrix, the geometry of the unit composite cell (i.e. of the thickness of the matrix coating, or, which is the same, of the fibre volume fraction), and of the current loading parameter. In fact, these parameters enter the above derived expressions for the fibre stresses and displacements through the multiplier σ_0. The latter specifies the boundary conditions for the representative problem (cf. Eq. (2)) and presents itself the axial fibre stress in the trivial case of uniformly stressed fibre, i.e. the case which corresponds, formally, to the solution of the representative problem with $p = 1$. The coupling of this trivial fibre state with that of the surrounding matrix implies the so-called *plane cross sections-type* problems for the entire unit composite cell. The determination of the σ_0-stress is a basic element of the solution of these problems. References [5, 6] provide the σ_0-values for two particular but typical problems of thermal and mechanical loading of a unit composite cell, namely the problems of uniform cooling (heating) of the matrix phase and of longitudinal extension (compression) of a unit composite cell.

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