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EXTENSION OF THE DUHAMEL PRINCIPLE FOR THE HEAT EQUATION WITH DEZIN'S INITIAL CONDITION

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The classical Duhamel principle for the heat equation is extended to the case when the initial condition $u(x,0) = f(x)$ is replaced by the nonlocal A. Dezin's condition $\mu u(0) - u(T) = f(x)$, $\mu \neq 1$. To this end three types of operational calculi are developed: 1) operational calculus for $\frac{d}{dt}$ with the Dezin's functional, 2) operational calculus for $\frac{d^2}{dx^2}$ in a segment $[0,a]$ with boundary conditions $u(0) = 0$ and $u(a) = 0$, and 3) a combined operational calculus for functions $u(x,t)$ in $C(\Delta)$, $\Delta = [0,a] \times [0,T]$.

Keywords: convolution algebra, multiplier, operational calculus, divisor of zero, commutant, Duhamel principle

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1. INTRODUCTION

In [3] a general operational calculus for $\frac{d}{dt}$ with arbitrary boundary value functional Φ is developed. Following the pattern of Mikusinski's operational calculus [6] in the space $C[0,1]$, the convolution

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f(t+\tau-\sigma) g(\sigma)\, d\sigma \right\} \tag{1}$$

instead of Duhamel's convolution

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau \tag{2}$$

is used. For the details connected with the convolution (1) found in 1974 ([1]) by one of the authors, see [2] and [3].

In order Mikusinski's scheme to work, only the restriction $\Phi\{1\} \neq 0$ is needed. Without a loss of generality then we may assume

$$\Phi\{1\} = 1.$$

In the case $\Phi\{1\} = 0$ the Mikusinski's scheme is also applicable, but with some modifications.

In [4] and [5] A. A. Dezin considered non-local boundary value problems for the differentiation operator with boundary value condition of the form

$$\mu y(0) - y(T) = 0 \tag{3}$$

with $\mu \neq 1$. For unessential technical simplifications, in the sequel we assume that μ is real. The case of a complex μ can be treated in almost the same way. Instead of the functional $\mu f(0) - f(T)$ we may take the normed functional

$$\Phi\{f\} = \frac{1}{\mu - 1}\left[\mu f(0) - f(T)\right]$$

and then (1) takes the form

$$(f * g)(t) = \frac{1}{\mu - 1}\left[\mu \int_0^t f(t-\tau)g(\tau)\,d\tau + \int_t^T f(T+t-\tau)g(\tau)\,d\tau\right]. \tag{4}$$

According to [2] operation (4) is a convolution of the right inverse operator

$$lf(t) = \int_0^t f(\tau)\,d\tau + \frac{1}{\mu - 1}\int_0^T f(\tau)\,d\tau \tag{5}$$

of $\dfrac{d}{dt}$ in $C[0,T]$. This means that $f * g$ is a bilinear, commutative and associative operation, such that

$$l(f * g) = (lf) * g.$$

Since $lf(t)$ is determined as the solution of the boundary value problem

$$y' = f, \qquad \mu y(0) - y(T) = 0,$$

then from the representation

$$lf(t) = \{1\} * f,$$

where $\{1\}$ is the constant function 1, it follows that (4) is a convolution of l in $C[0,T]$.

2. CHARACTERIZATION OF THE MULTIPLIERS OF THE CONVOLUTION ALGEBRA $(C[0,T],*)$

According to Larsen [8], an operator $M : C[0,T] \to C[0,T]$ is a multiplier of the convolution algebra $(C[0,T],*)$ iff the relation

$$M(f*g) = (Mf)*g$$

holds for all $f, g \in C[0,T]$. In [8] it is shown that each multiplier is a continuous linear operator. From a general result of [2], p. 32, it follows that each multiplier operator M of the convolution algebra $(C[0,T],*)$ has the convolution representation

$$Mf = \frac{d}{dt}(m*f), \tag{6}$$

where $m(t) \stackrel{\text{def}}{=} M\{1\}$.

In order to characterize the multipliers it remains to specify the representation functions m in (6).

Theorem 1. *A linear operator $M : C[0,T] \to C[0,T]$ is a multiplier of the convolution algebra $(C[0,T],*)$ iff it has the representation*

$$Mf(t) = \frac{d}{dt}\left\{\frac{\mu}{\mu-1}\int_0^t m(t-\tau)f(\tau)\,d\tau + \frac{1}{\mu-1}\int_t^T m(T+t-\tau)f(\tau)\,d\tau\right\}, \tag{7}$$

where m is a continuous function with bounded variation in $[0,T]$.

Remark 1. If $m \in C \cap BV$, then (7) can be written in the form

$$Mf(t) = \frac{d}{dt}\frac{\mu}{\mu-1}\int_0^t m(t-\tau)\,dm(\tau) + \frac{1}{\mu-1}\int_t^T m(T+t-\tau)\,dm(\tau). \tag{8}$$

Proof. From a more general result in [2], p. 32, it follows that each multiplier of $(C[0,T],*)$ has the form (7), where $m(t) \stackrel{\text{def}}{=} M\{1\}$ is a continuous function. It remains only to prove that m is a function with bounded variation in $[0,T]$.

To this end let us fix t $(0 < t \le T)$ and consider $(Mf)(t)$ as a linear functional on $C[0,T]$. According to the F. Riesz representation theorem $(Mf)(t)$ has the form

$$(Mf)(t) = \int_0^T f(\tau)\,d\alpha_t(\tau), \tag{9}$$

where $\alpha_t(\tau)$ is a function with bounded variation in $[0,T]$ depending on t as a parameter. For the sake of uniqueness we may assume that $\alpha_t(\tau)$ is continuous from the left. It would be possible to accomplish the differentiation in (6) termwise provided $m \in C \cap BV$, but we can assume only $m \in C$, which is not enough to ensure the differentiability. Therefore we apply the operator l to (7) and obtain

$$lMf = m*f = \frac{\mu}{\mu-1}\int_0^t m(t-\tau)f(\tau)\,d\tau + \frac{1}{\mu-1}\int_t^T m(T+t-\tau)f(\tau)\,d\tau \tag{10}$$

since for $g \in C^1[0,T]$ we have

$$lg' = g - \frac{\mu g(0) - g(T)}{\mu - 1}$$

and the function $g = m * f$ satisfies the boundary value condition $\mu g(0) - g(T) = 0$.

The operators M and l commute since they both are elements of the multipliers algebra and hence

$$lMf = Mlf = m * f. \tag{11}$$

From (9) we get

$$(Mlf)(t) = \int_0^T (lf)(\tau)\, d\alpha_t(\tau) = (lf)(T)\alpha_t(T) - (lf)(0)\alpha_t(0) - \int_0^T f(\tau)\alpha_t(\tau)\, d\tau. \tag{12}$$

Comparing (10), (11) and (12) we obtain the identity

$$\frac{\mu}{\mu-1}\int_0^t m(t-\tau)f(\tau)\, d\tau + \frac{1}{\mu-1}\int_t^T m(T+t-\tau)f(\tau)\, d\tau$$

$$= (lf)(T)\alpha_t(T) - (lf)(0)\alpha_t(0) - \int_0^T f(\tau)\alpha_t(\tau)\, d\tau.$$

Since

$$(lf)(T) = \mu(lf)(0) = -\frac{\mu}{\mu-1}\int_0^T f(\tau)\, d\tau,$$

then the right-hand side takes the form

$$-\int_0^T f(\tau)\left[\alpha_t(\tau) + \frac{\mu\alpha_t(T) - \alpha_t(0)}{\mu - 1}\right] d\tau.$$

If τ is a point of continuity of α_t, then the following two functional identities should hold:

$$-\alpha_t(\tau) - \frac{\mu\alpha_t(T) - \alpha_t(0)}{\mu - 1} = \frac{\mu}{\mu-1} m(t-\tau), \qquad 0 \le \tau \le t, \tag{13}$$

and

$$-\alpha_t(\tau) - \frac{\mu\alpha_t(T) - \alpha_t(0)}{\mu - 1} = \frac{1}{\mu-1} m(T+t-\tau), \qquad t \le \tau \le T. \tag{14}$$

From (13) and (14) it follows that $m \in BV$. Moreover, from (13) and (14) it follows that m satisfies the boundary value condition $\mu m(0) - m(T) = 0$. Indeed, if we take $\tau = t$ in (13) and (14), we get $\mu m(0) = m(T)$.

If $m \in C \cap BV$, then the derivative $\frac{d}{dt}(m * f)$ exists as a function from $C[0,T]$ and hence the linear operator

$$Mf = \frac{d}{dt}(m * f)$$

is well defined in $C[0,T]$. Obviously, it is a multiplier of the convolution algebra $(C[0,T],*)$.

Since the operator l has a cyclic element — the constant function $\{1\}$, then the multipliers ring of convolution (4) coincides with the commutant of l (see [2], p. 33).

Thus we obtained a complete characterization of the linear operators $M : C[0,T] \to C[0,T]$, which commute with the integration operator l. This explicit characterization can be considered as the "solution" of the non-local spectral problem considered.

In abstract setting the aim of any operational calculus for the operator l reduces to characterizing the class of operators commuting with l. But since the commutant of l coincides with the multipliers ring of the convolution algebra $\big(C[0,T],*\big)$, in this abstract setting the spectral problem obtains its solution by means of Theorem 1.

Usually, in the general spectral theory (see [9], pp. 287–296) only analytic functions of a given operator are considered.

Here we prefer to develop a direct algebraic operational calculus for the operator l, following the multiplier quotients scheme instead of Mikusinski's approach.

3. OPERATIONAL CALCULUS BY MULTIPLIERS QUOTIENTS

Here the basic elements of an operational calculus for the integration operator (5) will be developed. One can follow either the Mikusinski's scheme or the multipliers quotients scheme proposed in [2].

The basic multiplier is the Dezin's integration operator $lf = \{1\} * f$. This multiplier is the convolution operator $l = \{1\}*$. Let $\mathcal{M}$ be the ring of the multipliers of the convolution algebra $\big(C[0,T],*\big)$ and $\mathcal{N}$ be the multiplicative set of non-zero non-devisors of 0 of this algebra.

Let us denote by $\mathcal{R}$ the quotient ring of $\mathcal{M}$ with respect to $\mathcal{N}$, i.e. $\mathcal{R} = \mathcal{N}^{-1}\mathcal{M}$ (see [7], Ch. 2, Sec. 3). The elements of the ring $\mathcal{R}$ are quotients of the form

$$m = \frac{P}{Q}, \qquad \text{where } P \in \mathcal{M} \text{ and } Q \in \mathcal{N}.$$

We should always bear in mind the equivalence

$$\frac{P}{Q} = \frac{R}{S} \quad \Longleftrightarrow \quad PS = RQ.$$

If $c \in \mathbb{C}$, we will use the same letter for the numerical multiplier $c\{f(t)\} = \{cf(t)\}$. By 1 we denote the unit of $\mathcal{R}$, which is different from the convolution multiplier $\{1\}* = l$. Then the algebraic inverse element of l will be denoted by

$$s = \frac{1}{l}\,.$$

Theorem 2. *If $f \in C^1[0,T]$, then the relation*

$$y' = sy - \frac{\mu y(0) - y(T)}{\mu - 1} \tag{15}$$

holds, where the term $\dfrac{\mu y(0) - y(T)}{\mu - 1}$ *is viewed not as a constant function, but as a numerical multiplier, and y and y' are the convolution multipliers $\{y\}*$ and $\{y'\}*$, respectively.*

Proof. It is easy to verify the identity

$$l\{y'(t)\} = y(t) - y(0) + \frac{1}{\mu - 1}\{y(T) - y(0)\} = y(t) - \frac{\mu y(0) - y(T)}{\mu - 1}.$$

If we express this equality as an identity of multipliers, it takes the form

$$ly' = y - \frac{\mu y(0) - y(T)}{\mu - 1} l. \tag{16}$$

Multiplying both sides by s, we obtain (15).

Remark 2. We will refer to (15) as the *basic formula of the Dezin's operational calculus.*

Theorem 3. *If $\lambda \in \mathbb{C}$ and $\lambda \notin \frac{1}{T}(\ln|\mu| + 2m\pi i)$, $m \in \mathbb{Z}$, then*

$$\frac{1}{s - \lambda} = \left\{\frac{e^{\lambda t}(\mu - 1)}{\mu - e^{\lambda T}}\right\}. \tag{17}$$

Proof. Using (15), we obtain

$$(s - \lambda)\{e^{\lambda t}\} = s\{e^{\lambda t}\} - \lambda\{e^{\lambda t}\} = \lambda\{e^{\lambda t}\} + \frac{\mu - e^{\lambda T}}{\mu - 1} - \lambda\{e^{\lambda t}\}$$

and (17) is obvious.

4. OPERATIONAL CALCULUS BY TRANSFORM APPROACH

An alternative approach is based on a finite integral transform associated with the Dezin's condition. (For a transform approach for a more general boundary value condition, see Dimovski [3].)

This finite integral transform can be defined, using the resolvent operator

$$\mathcal{R}_\lambda f = \left\{\frac{e^{\lambda t}(\mu - 1)}{\mu - e^{\lambda T}}\right\} * f. \tag{18}$$

Then the F. Riesz projector is defined as

$$\mathcal{P}_m\{f\} = -\frac{1}{2\pi i}\int\limits_{\Gamma_m} \mathcal{R}_\lambda f \, d\lambda, \tag{19}$$

where Γ_m is a small contour around the zero $\lambda_m = \frac{1}{T}(\ln|\mu| + 2m\pi i)$, $m \in \mathbb{Z}$, of

$$E(\lambda) = \frac{1}{\mu - 1}(\mu - e^{\lambda T}).$$

From (18) it follows that

$$\mathcal{P}_m\{f\} = \{e_m(t)\} * f,$$

where

$$e_m(t) = -\frac{1}{2\pi i}\int\limits_{\Gamma_m} \frac{e^{\lambda t}(\mu - 1)}{\mu - e^{\lambda T}}\, d\lambda = \frac{\mu - 1}{\mu T} e^{\lambda_m t}.$$

It is easy to verify the idempotency property

$$e_m^{*^2} = e_m * e_m = e_m.$$

It corresponds to the fact that $\mathcal{P}_m$ is a projector operator.

Using (4), it is easy to find that

$$\mathcal{P}_m\{f\} = \left(\frac{\mu}{\mu - 1}\int_0^T e^{-\lambda_m \tau} f(\tau)\, d\tau\right) e_m(t).$$

The coefficient of $e_m(t)$ is the corresponding finite Fourier transform

$$\mathcal{F}_m\{f\} = \frac{\mu}{\mu - 1}\int_0^T e^{-\lambda_m \tau} f(\tau)\, d\tau, \qquad m \in \mathbb{Z}. \tag{20}$$

This transform could be used as an alternative approach to the operational calculus we considered by a direct approach. In the following theorem we summarize the basic operational properties of the finite integral transform (20) for arbitrary $m \in \mathbb{Z}$.

Theorem 4. *For arbitrary $m \in \mathbb{Z}$ the following equalities hold:*

(i) $$\mathcal{F}_m\{1\} = \frac{1}{\lambda_m},$$

(ii) $$\mathcal{F}_m\{lf\} = \frac{1}{\lambda_m}\mathcal{F}_m\{f\},$$

(iii) $$\mathcal{F}_m\{f'\} = \lambda_m \mathcal{F}_m\{f\} - \frac{\mu f(0) - f(T)}{\mu - 1},$$

(iv) $$\mathcal{F}_m\{f * g\} = \mathcal{F}_m\{f\}\mathcal{F}_m\{g\},$$

and the inversion formula

(v) $$f(t) = \sum_{-\infty}^{\infty} \mathcal{F}_m\{f\}\frac{\mu-1}{\mu T}e^{\lambda_m t} = \frac{1}{T}\sum_{-\infty}^{\infty}\int_0^T e^{-\lambda_m \tau} f(\tau)\,d\tau e^{\lambda_m t},$$

when the series in the right-hand side converges uniformly.

Proof. (i) and (ii) are obvious.

(iii). From

$$lf' = f - \frac{\mu f(0) - f(T)}{\mu - 1}$$

and (i) it follows

$$\frac{1}{\lambda_m}\mathcal{F}_m\{lf'\} = \mathcal{F}_m\{f\} - \frac{\mu f(0) - f(T)}{\mu - 1}\frac{1}{\lambda_m}.$$

(iv) follows immediately from the representation

$$\mathcal{P}_m\{f\} = \mathcal{F}_m\{f\}e_m(t)$$

of the m-th Riesz projector (see Section 4).

(v) follows from the uniqueness theorem proven in [11], pp. 255–271.

Remark 3. Formula (iii) corresponds to the basic formula (15) and can be used in almost the same way.

5. OPERATIONAL CALCULUS FOR $\frac{D^2}{DX^2}$ WITH BOUNDARY VALUE CONDITIONS $U(0) = 0$, $U(A) = 0$

Following the multipliers quotients approach in [2], a survey of the operational calculus for the simplest boundary value problem for the second order differential operator $D = \dfrac{d^2}{dx^2}$ in $C[0,a]$ will be made. For more details one may consult our recent paper [10].

The starting point of this operational calculus is the operator $L_{-\lambda^2}$. For $f \in C[0,a]$ the function $y = L_{-\lambda^2} f$ is defined as the solution of the boundary value problem

$$\begin{aligned} &y'' + \lambda^2 y = f(x), \\ &y(0) = y(a) = 0. \end{aligned} \tag{21}$$

It is easy to obtain the explicit expression

$$L_{-\lambda^2} f(x) = \frac{1}{\lambda}\int_0^x \sin\lambda(x-\xi)f(\xi)\,d\xi - \frac{\sin\lambda x}{\lambda\sin\lambda a}\int_0^x \sin\lambda(a-\xi)f(\xi)\,d\xi. \tag{22}$$

The operation

$$(f*g)(x) = -\frac{1}{2a}\int_0^a \left[\int_x^\xi f(\xi+x-\eta)g(\eta)\,d\eta - \int_{-x}^{\xi} f(|\xi-x-\eta|)g(|\eta|)\mathrm{sgn}\left((\xi-x-\eta)\eta\right)d\eta\right]d\xi \tag{23}$$

is a convolution of $L_{-\lambda^2}$ such that

$$L_{-\lambda^2}f(x) = \left\{\frac{a\sin\lambda x}{\sin\lambda a}\right\} * f(x). \tag{24}$$

The special case $\lambda = 0$ is used as the basic operator of the corresponding operational calculus. Denoting $L = L_0$, we have

$$Lf(x) = \{x\} * f(x), \tag{25}$$

i.e. L may be considered as the convolution operator $\{x\}*$. For simplicity we will write $L = \{x\}$. Also, if $f \in C[0,a]$, then by $\{f\}$ we will denote the convolution multiplier operator $\{f\}*$.

Let $\mathcal{M}_x$ be the ring of the multipliers of the convolution algebra $\left(C[0,T],*\right)$ and let by $\mathcal{N}_x$ we denote the multiplicative set of the non-divisors of 0 in $\mathcal{M}_x\backslash\{0\}$.

Further we consider the ring of the multipliers quotients $\mathcal{R}_x = \mathcal{N}_x^{-1}\mathcal{M}_x$ of the form P/Q with $P \in \mathcal{M}_x$, $Q \in \mathcal{N}_x$.

Basic is the role of the multipliers quotient

$$S = \frac{1}{L}, \tag{26}$$

where by 1 the identity operator in $\mathcal{M}_x$ is denoted.

The basic formula of the operational calculus under development can be obtained from the identity

$$Lf'' = f - \left(1-\frac{x}{a}\right)f(0) + \frac{x}{a}f(a).$$

Writing it as an identity of multipliers operators, it takes the form

$$Lf'' = f - \left\{1-\frac{x}{a}\right\}f(0) + \frac{1}{a}f(a)L. \tag{27}$$

Multiplying by S, we obtain

$$f'' = Sf - S\left\{1-\frac{x}{a}\right\}f(0) - \frac{1}{a}f(a), \tag{28}$$

where the numbers $f(0)$ and $\frac{1}{a}f(a)$ are to be considered as "numerical operators", i.e. as numerical multipliers in $\left(C[0,a],*\right)$. Using (28), we can find that

$$\frac{1}{S+\lambda^2} = \left\{\frac{a\sin\lambda x}{\sin\lambda a}\right\} \tag{29}$$

for $\lambda \neq \frac{n\pi}{a}$, $n \in \mathbb{N}$.

Proof. It is easy to see that

$$\frac{1}{S+\lambda^2} = L_{-\lambda^2}.$$

Indeed, if $y = L_{-\lambda^2} f$, then from (28) it follows $Sy = y'' = f - \lambda^2 y$. Now the assertion follows from (24).

The direct approach can be duplicated by the finite sine-transform (see [12]).

We will use a variant of the sine-transform having (23) as its convolution. It is slightly different from the finite Fourier sine-transform introduced by Churchill in [13], p. 349.

Definition 1. The transform $\mathcal{F}_n^s : C[0,a] \to \mathbb{C}^{\mathbb{N}}$ is defined by

$$\mathcal{F}_n^s\{f\} = \frac{(-1)^n}{n\pi} \int_0^a f(\xi) \sin \frac{n\pi}{a} \xi \, d\xi, \tag{30}$$

$n = 1, 2, \ldots$

Theorem 5. *The basic properties of the sine-transform are:*

(i)
$$\mathcal{F}_n^s\{x\} = -\left(\frac{a}{n\pi}\right)^2;$$

(ii)
$$\mathcal{F}_n^s\{Lf\} = -\left(\frac{a}{n\pi}\right)^2 \mathcal{F}_n^s\{f\};$$

(iii)
$$\mathcal{F}_n^s\{f''\} = -\left(\frac{n\pi}{a}\right)^2 \mathcal{F}_n^s\{f\} - \frac{1}{a}[f(a) - (-1)^n f(0)];$$

(iv)
$$\mathcal{F}_n^s\{f * g\} = \mathcal{F}_n^s\{f\}\mathcal{F}_n^s\{g\};$$

and the inversion formula

(v)
$$f(x) = \frac{2\pi}{a} \sum_{n=1}^{\infty} (-1)^n n \mathcal{F}_n^s\{f\} \sin \frac{n\pi}{a} x$$

holds when the right-hand side series converges uniformly.

Proof. (i) and (ii) can easily be obtained directly. (iii) follows from (27) using (i) and (ii):

$$\mathcal{F}_n^s\{Lf''\} = \mathcal{F}_n^s\{f\} - f(0)\mathcal{F}_n^s\{1 - \frac{x}{a}\} + \frac{1}{a} f(a)\mathcal{F}_n^s\{x\}$$

or

$$-\left(\frac{a}{n\pi}\right)^2 \mathcal{F}_n^s\{f''\} = \mathcal{F}_n^s\{f\} - f(0)\frac{(-1)^n a}{(n\pi)^2} + f(a)\frac{a}{(n\pi)^2}$$

since

$$\mathcal{F}_n^s\{1-\frac{x}{a}\} = \frac{(-1)^n a}{(n\pi)^2}.$$

(iv) It can easily be verified that

$$\left(\sin\frac{n\pi}{a}x\right) * \left(\sin\frac{n\pi}{a}x\right) = \frac{(-1)^n a}{2n\pi}\sin\frac{n\pi}{a}x.$$

Since

$$\mathcal{F}_n^s\{f\}\sin\frac{n\pi}{a}x = \left(\sin\frac{n\pi}{a}x\right) * f,$$

then

$$\begin{aligned}\mathcal{F}_n^s\{f*g\}\sin\frac{n\pi}{a}x &= \left(\sin\frac{n\pi}{a}x\right)*(f*g) = \mathcal{F}_n^s\{f\}\left(\sin\frac{n\pi}{a}x * g\right)\\ &= \mathcal{F}_n^s\{f\}\mathcal{F}_n^s\{g\}\sin\frac{n\pi}{a}x.\end{aligned}$$

(v) See [12], Ch. 3.

6. A TWO-VARIATE OPERATIONAL CALCULUS FOR THE HEAT EQUATION WITH DEZIN'S BOUNDARY VALUE CONDITION

Our first aim is to develop an operational approach to the following non-local boundary value problem for the heat equation in the rectangle $\Delta = [0,a]\times[0,T]$:

$$\begin{aligned}&u_t = u_{xx} + F(x,t), \quad (x,t)\in\Delta,\\ &\mu u(x,0) - u(x,T) = f(x), \quad 0\le x\le a,\\ &u(0,t) = \varphi(t),\ u(a,t) = \psi(t), \quad 0\le t\le T,\end{aligned} \tag{31}$$

where $\mu \neq 1$ is a real parameter.

To this end we need a two-variate operational calculus for functions $u(x,t)$ of a space variable and a time variable. We will follow the pattern from [10].

We have to find an inner operation for functions $f\in C(\Delta)$, which is a convolution both for the operator

$$l\{u(x,t)\} = \int_0^t u(x,\tau)\,d\tau + \frac{1}{\mu-1}\int_0^T u(x,\tau)\,d\tau \tag{32}$$

and the operator

$$L\{u(x,t)\} = \int_0^x (x-\xi)u(\xi,t)\,d\xi - \frac{x}{a}\int_0^a (a-\xi)u(\xi,t)\,d\xi.$$

According to (23) the operation

$$\begin{aligned}(f \overset{x}{*} g)(x) \;=\; -\frac{1}{2a}\int_0^a\Bigg[&\int_x^\xi f(\xi+x-\eta)g(\eta)\,d\eta\\ &-\int_{-x}^\xi f(|\xi-x-\eta|)g(|\eta|)\mathrm{sgn}\left((\xi-x-\eta)\eta\right)d\eta\Bigg]d\xi\end{aligned} \tag{33}$$

is a convolution for L in $C([0,a])$.

Now we are to combine (33) with (4) in order to obtain a two-variate convolution, such that l and L to be multipliers of the corresponding convolution algebra.

Theorem 6. *Let $u, v \in C(\Delta)$. Then the operation*

$$
\begin{aligned}
(f \overset{(x,t)}{*} g)(x,t) = &-\frac{1}{2a(\mu-1)} \int_0^a \Bigg[\int_0^t \mu \int_x^\xi f(x+\xi-\eta, t-\tau) g(\eta,\tau)\, d\eta d\tau \\
&+ \int_t^T \int_x^\xi f(x+\xi-\eta, T+t-\tau) g(\eta,\tau)\, d\eta d\tau \\
&- \int_0^t \mu \int_{-x}^\xi f(|\xi-x-\eta|, t-\tau) g(|\eta|,\tau) \operatorname{sgn}\left[(\xi-x-\eta)\eta\right] d\eta d\tau \\
&- \int_t^T \int_{-x}^\xi f(|\xi-x-\eta|, T+t-\tau) g(|\eta|,\tau) \operatorname{sgn}\left[(\xi-x-\eta)\eta\right] d\eta d\tau \Bigg] d\xi
\end{aligned} \tag{34}
$$

is a bilinear, commutative and associative operation in $C(\Delta)$ such that the operators

$$l\{f\} = \{1\} \overset{t}{*} f$$

and

$$L\{f\} = \{x\} \overset{x}{*} f$$

are multipliers of the convolution algebra $\left(C(\Delta), \overset{(x,t)}{*}\right)$.

For a proof one can follow the lines of the corresponding proof in [10]. Since the bilinearity and the commutativity are almost obvious, only the associativity needs to be proved. We verify it for product functions $F(x,t) = f(x)\varphi(t)$ and $G(x,t) = g(x)\psi(t)$ using the identity

$$(F \overset{(x,t)}{*} G)(x,t) = (f \overset{x}{*} g)(x)(\varphi \overset{t}{*} \psi)(t),$$

followed by an approximation argument.

Let us denote by $\mathcal{R}$ the multipliers quotient ring of the convolution algebra $\left(C(\Delta), \overset{(x,t)}{*}\right)$. In $\mathcal{R}$ the one-variate identity

$$\{\varphi'(t)\} = s\{\varphi(t)\} - \frac{\mu\varphi(0) - \varphi(T)}{\mu - 1}$$

takes the form

$$\left\{\frac{\partial u(x,t)}{\partial t}\right\} = s\{u(x,t)\} - \left[\frac{\mu u(x,0) - u(x,T)}{\mu-1}\right]_t, \tag{35}$$

where by $[\cdot]_t$ it is denoted that the expression in the brackets is a numerical operator with respect to t, i.e. $[\cdot]_t = \{\cdot\} \overset{x}{*}$.

The corresponding identity with respect to x has the same form as in [10]:

$$\frac{\partial^2 u}{\partial x^2} = Su - S\left\{1 - \frac{x}{a}\right\}[u(0,t)]_x - \left[\frac{1}{a}u(a,t)\right]_x, \tag{36}$$

where $S = \frac{1}{L}$ and $[\cdot]_x$ is to be considered as a numerical multiplier with respect to x, i.e. $[\cdot]_x = \{\cdot\} \overset{t}{*}$.

For the proof of the uniqueness theorem in the next section we need to characterize the divisors of 0 of $\left(C(\Delta), \overset{(x,t)}{*}\right)$. This is done by means of the following two-variate finite integral transformation.

Definition 2. For $u \in C(\Delta)$ let

$$\mathcal{F}_{m,n}\{u\} = \frac{(-1)^n \mu}{(\mu - 1)n\pi} \int_0^a \int_0^T u(\xi, \tau) e^{-\lambda_m \tau} \sin \frac{n\pi}{a}\xi \, d\xi d\tau$$

be a two-variate finite integral transform corresponding to the two-variate convolution (34).

Theorem 7. *The following properties of $\mathcal{F}_{m,n}$ are satisfied:*

(i)
$$\mathcal{F}_{m,n}\{Lu\} = -\left(\frac{a}{n\pi}\right)^2 \mathcal{F}_{m,n}\{u\},$$

(ii)
$$\mathcal{F}_{m,n}\{lu\} = \lambda_m \mathcal{F}_{m,n}\{u\},$$

(iii)
$$\mathcal{F}_{m,n}\{u\} e^{\lambda_m t} \sin \frac{n\pi}{a} x = \left\{e^{\lambda_m t} \sin \frac{n\pi}{a} x\right\} \overset{(x,t)}{*} u,$$

(iv)
$$\mathcal{F}_{m,n}\{u \overset{(x,t)}{*} v\} = \mathcal{F}_{m,n}\{u\}\mathcal{F}_{m,n}\{v\},$$

(v)
$$\mathcal{F}_{m,n}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left(\frac{n\pi}{a}\right)^2 \mathcal{F}_{m,n}\{u\} - \frac{1}{a}\mathcal{F}_m\{u(a,t) - (-1)^n u(0,t)\},$$

(vi)
$$\mathcal{F}_{m,n}\left\{\frac{\partial u}{\partial t}\right\} = \lambda_m \mathcal{F}_{m,n}\{u\} - \mathcal{F}_n^s\left\{\frac{\mu u(x,0) - u(x,T)}{\mu - 1}\right\}.$$

Proof. Follows immediately from Theorems 4 and 5.

Lemma 1. *A function $u \in C(\Delta)$ is a divisor of 0 iff for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$ we have $\mathcal{F}_{m,n}\{u\} = 0$.*

Proof. Indeed, let for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$ we have $\mathcal{F}_{m,n}\{u\} = 0$. Then

$$\{e^{\lambda_m t} \sin \frac{n\pi}{a} x\} \overset{(x,t)}{*} \{u(x,t)\}$$
$$= \frac{(-1)^n \mu}{(\mu - 1)n\pi} \left[\int_0^a \int_0^T u(\xi, \tau) e^{-\lambda_m \tau} \sin \frac{n\pi}{a} \xi \, d\xi d\tau \right] e^{\lambda_m t} \sin \frac{n\pi}{a} x = 0.$$

Hence u is a divisor of 0.

Let us now conversely assume that u is a divisor of 0 of $\overset{(x,t)}{*}$, i.e. that $u \overset{(x,t)}{*} v = 0$ for some function $v \in C(\Delta)$, $v \neq 0$. If we assume that $\mathcal{F}_{m,n}\{u\} \neq 0$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, then

$$(u \overset{(x,t)}{*} v) \overset{(x,t)}{*} \left\{ e^{\lambda_m t} \sin \frac{n\pi}{a} x \right\} = \frac{(-1)^n \mu}{(\mu - 1)n\pi} \mathcal{F}_{m,n}\{u\} \mathcal{F}_{m,n}\{v\} = 0,$$

whence $\mathcal{F}_{m,n}\{v\} = 0$ $\forall m \in \mathbb{Z}$, $\forall n \in \mathbb{N}$. This is equivalent to

$$\left(\sin \frac{n\pi}{a} x \right) \overset{x}{*} \left[e^{\lambda_m t} \overset{t}{*} v(x,t) \right] = 0.$$

If we denote the functions in the brackets with $F_m(x,t)$, the last equality implies that

$$F_m(x,t) = 0, \qquad \forall m \in \mathbb{Z}.$$

Fixing now $x \in [0, a]$, we obtain

$$e^{\lambda_m t} \overset{t}{*} v(x,t) = 0,$$

whence $v(x,t) = 0$ $\forall t \in [0, T]$, contrary to the assumptions.

Theorem 8. *$L - l$ is a divisor of 0 of the multipliers ring of the convolution algebra $\left(C(\Delta), \overset{(x,t)}{*}\right)$ iff $\frac{a}{\pi} \sqrt{\frac{1}{T} \ln \frac{1}{\mu}} \in \mathbb{N}$.*

Proof. If $(L - l)M = 0$ for some $M \neq 0$, then there exists a function $v \in C(\Delta)$, $v \neq 0$, such that $Mv = u \neq 0$ and $(L - l)u = 0$. The fact that $u \neq 0$ implies that for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ we have

$$\mathcal{F}_{m,n}\{u\} \neq 0.$$

According to (i) and (ii) of Theorem 6 we have

$$\mathcal{F}_{m,n}\{Lu - lu\} = \left[-\left(\frac{a}{n\pi} \right)^2 - \lambda_m \right] \mathcal{F}_{m,n}\{u\} = 0.$$

This is possible only if

$$\left(\frac{a}{n\pi} \right)^2 + \lambda_m = 0.$$

Since $\lambda_m = \frac{1}{T}(\ln \mu + 2m\pi i)$, this is only possible for $m = 0$, i.e. when

$$\frac{1}{T} \ln \mu = -\left(\frac{a}{n\pi}\right)^2$$

for some $n \in \mathbb{N}$, whence the assertion follows. The converse is obvious.

7. GENERALIZED SOLUTIONS OF THE HEAT EQUATION WITH DEZIN'S INITIAL CONDITION

The common notion of a generalized solution in the sense of distribution theory is unpractical for boundary value problems. This is especially true for nonlocal boundary value problems in finite domains.

In the case of the boundary value problem (31) it is very useful to introduce the notion of a generalized solution in the framework of the algebraic analysis of D. Przeworska-Rolevicz [14].

Let $C^{2,1}(\Delta)$ be the space of functions that are twice continuously differentiable with respect to x and continuously differentiable with respect to t. Let us assume that (31) has a classical solution $u \in C^{2,1}(\Delta)$. Applying the operator Ll to the equation

$$u_t = u_{xx} + F(x,t),$$

we obtain

$$L(lu_t) = l(Lu_{xx}) + LlF(x,t).$$

From (16) we get

$$lu_t = u - \frac{1}{\mu - 1} f(x)$$

and from (27) we have

$$Lu_{xx} = u - \left(1 - \frac{x}{a}\right)\varphi(t) + \frac{x}{a}\psi(t).$$

Hence

$$L\left[u(x,t) - \frac{1}{\mu-1} f(x)\right] = l\left[u(x,t) - \left(1 - \frac{x}{a}\right)\varphi(t) + \frac{x}{a}\psi(t)\right] + LlF(x,t)$$

or

$$(L - l)u = \frac{1}{\mu - 1} Lf(x) - \left(1 - \frac{x}{a}\right) l\varphi(t) + \frac{x}{a} l\psi(t) + LlF(x,t). \qquad (37)$$

Conversely, if a function $u \in C^{2,1}(\Delta)$ satisfies (37), then it is a solution of (31). Indeed, if we apply the differential operator $\frac{\partial^3}{\partial x^2 \partial t}$ to (37), we obtain

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u = F(x,t)$$

and hence $u(x,t)$ is a solution of the heat equation.

In order to verify that the initial conditions are satisfied, let us apply $\frac{\partial^2}{\partial x^2}$ only. We obtain

$$u - lu_{xx} = \frac{1}{\mu - 1} f(x) + lF(x,t).$$

Applying to this equality the functional

$$\Psi\{g(t)\} = \mu g(0) - g(T), \tag{38}$$

we get

$$\Psi\{u\} = f(x),$$

i.e. the Dezin's initial conditions since $\Psi\{lg\} = 0$.

In a similar way applying $\frac{\partial}{\partial t}$ to (37) we get

$$Lu_t - u = -\left(1 - \frac{x}{a}\right)\varphi(t) + \frac{x}{a}\psi(t) + LF(x,t).$$

Since $(Lg)(x)$ satisfies the boundary conditions $(Lg)(0) = (Lg)(a) = 0$, then the above equality for $x = 0$ and for $x = a$ gives $u(0,t) = \varphi(t)$ and $u(a,t) = \psi(t)$.

The above considerations justify the next definition.

Definition 3. If $u \in C(\Delta)$ satisfies the integral relation (37), it is said to be a generalized solution of the boundary value problem (31).

Lemma 2. *If* $u \in C(\Delta)$ *satisfies* (37), *then* $u(0,t) = \varphi(t)$, $u(a,t) = \psi(t)$ *and* $\mu u(x,0) - u(x,T) = f(x)$.

Proof. From (37) for $x = 0$ we get

$$-lu(0,t) = -l\varphi(t),$$

whence $u(0,t) = \varphi(t)$. In a similar way, substituting $x = a$ in (37), we obtain $u(a,t) = \psi(t)$.

Applying the functional (38) to (37), we get

$$L\Psi_t\{u(x,t)\} = \frac{1}{\mu - 1} Lf(x)\Psi\{1\}$$

(since $L\Psi_t = \Psi_t L$) or

$$L\{\mu u(x,0) - u(x,T)\} = Lf(x).$$

Now Dezin's initial condition follows applying the operator $\frac{\partial^2}{\partial x^2}$ to both sides of the above equality.

The two-variate operational calculus developed above allows to algebraize completely the boundary value problem (31). We use formulas (35) and (36) to obtain

$$su - \frac{[f(x)]_t}{\mu - 1} = Su - S\left\{1 - \frac{x}{a}\right\}[\varphi(t)]_x - \frac{1}{a}[\psi(t)]_x + \{F(x,t)\}.$$

The solution u exists as an element of the ring $\mathcal{R}$, i.e. as a multipliers quotient and it can be represented in the algebraic form

$$u = \frac{\{F(x,t)\}}{s-S} + \frac{[f(x)]_t}{(\mu-1)(s-S)} - \frac{S}{s-S}\left\{1 - \frac{x}{a}\right\}[\varphi(t)]_x - \frac{1}{s-S}\frac{1}{a}[\psi(t)]_x. \quad (39)$$

It is valid provided $s - S$ is a non-divisor of 0 in $\mathcal{R}$.

Theorem 9 (for uniqueness). *The element $s - S$ of $\mathcal{R}$ is a non-divisor of 0 provided $\frac{a}{\pi}\sqrt{\frac{1}{T}\ln\frac{1}{\mu}} \notin \mathbb{N}$.*

Proof. Let us assume the contrary. As in [10] it is easy to show that this assumption reduces to the existence of a non-zero function $u \in C(\Delta)$ such that

$$(L - l)u = 0.$$

Then the proof follows from Theorem 7.

Now the formal solution (39) exists provided $\frac{a}{\pi}\sqrt{\frac{1}{T}\ln\frac{1}{\mu}} \notin \mathbb{N}$. In order to interpret it, we introduce

$$\Omega = \frac{1}{sS(s-S)} = \frac{Ll}{s-S}.$$

Since $Ll = \{x\}$, then Ω can be interpreted as a solution to the boundary value problem (31) with $F(x,t) \equiv x$ and $f(x) = 0$, $\varphi(t) = 0$, $\psi(t) = 0$.

If such a solution $u(x,t)$ exists, we can find it by means of the finite sine transform. If $u(x,t)$ is the solution, then

$$\mathcal{F}_n^s\{u_t\} = \mathcal{F}_n^s\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}_n^s\{x\}.$$

Using (i) and (iii) of Theorem 5 and the boundary conditions for

$$u_n(t) = \mathcal{F}_n^s\{u(x,t)\},$$

we obtain the equations

$$\begin{aligned} \frac{du_n}{dt} &= -\left(\frac{n\pi}{a}\right)^2 u_n(t) - \left(\frac{a}{n\pi}\right)^2, \\ \mu u_n(0) - u_n(T) &= 0 \end{aligned} \quad (40)$$

for $n = 1, 2, \ldots$ For the functions $u_n(t)$ we find the explicit expressions

$$u_n(t) = \frac{(\mu-1)a^4}{(n\pi)^4\left(\mu - e^{-\left(\frac{n\pi}{a}\right)^2 T}\right)} e^{-\left(\frac{n\pi}{a}\right)^2 t} - \frac{a^4}{(n\pi)^4}.$$

Taking into account the inversion formula (v) in Theorem 5, we now define

$$\begin{aligned} \Omega(x,t) &= \frac{2\pi}{a}\sum_{n=1}^{\infty}(-1)^n n u_n(t)\sin\frac{n\pi}{a}x = -2\left(\frac{a}{\pi}\right)^3\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi}{a}x \\ &+ 2\frac{(\mu-1)a^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3\left(\mu - e^{-\left(\frac{n\pi}{a}\right)^2 T}\right)} e^{-\left(\frac{n\pi}{a}\right)^2 t}\sin\frac{n\pi}{a}x \qquad (41) \\ = -\frac{1}{6}(x^3 - a^2 x) &+ 2\frac{(\mu-1)a^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3\left(\mu - e^{-\left(\frac{n\pi}{a}\right)^2 T}\right)} e^{-\left(\frac{n\pi}{a}\right)^2 t}\sin\frac{n\pi}{a}x. \end{aligned}$$

It is a matter of a simple verification that $\Omega(x,t)$ satisfies the integral relation

$$L\Omega - l\Omega = Ll\{x\}. \qquad (42)$$

Theorem 10. *Let $F(x,t) \in C^{2,1}(\Delta)$ and let it satisfy the zero initial-boundary conditions of (31). Then the problem (31) has a generalized solution $u(x,t)$, which has the Duhamel type representation*

$$u(x,t) = \Omega \overset{(x,t)}{*} \frac{\partial^3 F}{\partial x^2 \partial t}. \qquad (43)$$

Proof. In order to prove that u is a generalized solution of (31), we are to show that it satisfies the integral relation

$$(L-l)u = LlF.$$

From the assumptions made on f we have

$$Ll\frac{\partial^3 F}{\partial x^2 \partial t} = F.$$

Then

$$Ll(l-l)u = L^2 lu - Ll^2 u = L\left(\Omega \overset{(x,t)}{*} Ll\frac{\partial^3 F}{\partial x^2 \partial t}\right) - l\left(\Omega \overset{(x,t)}{*} Ll\frac{\partial^3 F}{\partial x^2 \partial t}\right)$$

$$= L(\Omega \overset{(x,t)}{*} F) - l(\Omega \overset{(x,t)}{*} F) = (L\Omega - l\Omega) \overset{(x,t)}{*} F.$$

But $L\Omega - l\Omega = Ll\{x\}$ (see (42)) and then

$$Ll(l-l)u = Ll\{x\} \overset{(x,t)}{*} F.$$

Since $\{x\} \overset{(x,t)}{*} F = LlF$, then

$$Ll(L-l)u = L^2l^2F.$$

Now "canceling" the term Ll, we get $Lu - lu = LlF$.

From (43) it follows that $u(x,t)$ is continuous on Δ and it satisfies the initial-boundary value conditions $u(0,t) = u(a,t) = 0$ and $\mu u(x,0) - u(x,T) = 0$ according to Lemma 2.

Theorem 11. *Let $f \in C^4[0,a]$ and let $f(0) = f''(0) = f(a) = f''(a) = 0$. Then the function $v \in C^{2,1}(\Delta)$ defined by*

$$v(x,t) = -\frac{1}{\mu-1}\left(\Omega \overset{x}{*} f^{(4)}(x)\right) - \frac{1}{\mu-1}f(x) \tag{44}$$

is a generalized solution of the equation $v_t = v_{xx}$ with $v(0,t) = v(a,t) = 0$, $\mu v(x,0) - v(x,T) = f(x)$.

Proof. It is not difficult to obtain this Duhamel-type representation from

$$v = -\frac{1}{\mu-1}\frac{[f(x)]_t}{s-S},$$

but it is easier to verify directly that (44) satisfies the equation $(L-l)v = Lf$.

Theorem 12. *Let $\psi \in C^2[0,T]$ satisfy $\mu\psi(0) - \psi(T) = 0$ and $\mu\psi'(0) - \psi'(T) = 0$. Then the function $w \in C^{2,1}(\Delta)$, given by*

$$w(x,t) = -\frac{1}{a}\Omega \overset{t}{*} \psi'' + \frac{x}{a}\psi(t), \tag{45}$$

is a generalized solution of the equation $w_t = w_{xx}$ with $w(0,t) = 0$, $w(a,t) = \psi(t)$ and $\mu w(x,0) - w(x,T) = 0$.

Proof. We have to show that w satisfies the integral relation

$$(L-l)w = -\frac{1}{a}x\psi(t).$$

This is a matter of simple calculations.

Remark 4. The case $F \equiv 0$, $\psi(t) \equiv 0$, $f(x) \equiv 0$ is not essentially different from the just considered case. Although the corresponding expression

$$\frac{-S\left\{1 - \frac{x}{a}\right\}[\varphi(t)]_x}{s-S}$$

looks more involved than the expression

$$\frac{-[\psi(t)]_x}{(\mu-1)(s-S)}$$

in the previous case, it can be simplified by introducing the new independent variable $z = a - x$.

Remark 5. In order the generalized solution of problem (31) to be a classical solution, only a slight increase of the smoothness assumptions on F, φ, ψ and f is necessary. It is sufficient to require the corresponding derivatives of the highest order to be not only continuous, but absolutely continuous.

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