ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 94, 2000

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 94, 2000

FIBERED SURFACES*

AZNIV K. KASPARIAN

The fibered surfaces are shown to be finite branched coverings of products of algebraic curves. As a consequence, the fundamental group of a finite surface turns to be commensurable with a product of the fundamental groups of Riemann surfaces.

Keywords: fibered surfaces, fundamental groups

Mathematics Subject Classification 2000: 32J27

The compact Kähler surface S is said to be fibered if there is a surjective holomorphic map $S \to C_g$ with connected fibers onto a curve C_g of genus $g \geq 2$. The work focuses on some properties of fibered surfaces S. The first section exhibits S as a finite ramified covering $S \to C_g \times C_h$, $g+h = h^{1,0}(S)$ of products of curves. As a consequence, the second section shows the commensurability of the fundamental group $\pi_1(S)$ of a fibered surface with the product $\pi_1(C_m) \times \pi_1(C_n)$ of fundamental groups of appropriate Riemann surfaces.

1. STRUCTURE RESULT

Proposition 1. *Any fibered surface $f_1 : S \to C_g$, $g \geq 2$, with non-isotropic $H^{1,0}(S)$ is a finite ramified covering $f = (f_1, f_2) : S \to C_g \times C_h$, $g + h \leq h^{1,0}(S)$.*

According to the Theorem of Castelnuovo de Franchis (cf. [1]), for any fibered surface $f_1 : S \to C_g$ the subspace $f_1^* H^{1,0}(C_g) \subset H^{1,0}(S)$ is isotropic, which means

* The work is partially supported under Contract MM1003/2000 by the National Foundation for Scientific Research.

that the wedge product of any two forms from $f_1^* H^{1,0}(C_g)$ is zero. Let us start with the following

Lemma 2. *Let X be a compact complex manifold with functionally independent $\varphi_1, \varphi_2 \in H^{1,0}(X)$ and $\mathbb{C}$-linearly independent $\psi_1, \dots, \psi_m \in H^{1,0}(X)$. Suppose that with respect to some coordinate covering $X = \cup_{\alpha \in A} W^{(\alpha)}$ there hold $\psi_j = \sum_{i=1}^{k} \lambda_{ji}^{(\alpha)} \varphi_i$ for some local meromorphic functions $\lambda_{ji}^{(\alpha)} : W^{(\alpha)} \to \mathbf{P}_1$, $1 \leq j \leq m$, and $k = 1$ or 2. Then there exist global holomorphic functions $b_i, d_j, f_j, g_j : X \to \mathbb{C}$, such that $\omega_i := \frac{\varphi_i}{b_i}$, $i = 1, 2$, are global holomorphic $(1,0)$-forms, as well as $\psi_j = f_j \omega_1$ in the case of $k = 1$ and $d_j \psi_j = f_j \omega_1 + g_j \omega_2$ in the case of $k = 2$.*

Proof. The local rings $\mathcal{O}_{W^{(\alpha)}}$ of the holomorphic functions on $W^{(\alpha)}$ are factorial. Their fraction fields $\mathcal{M}_{W^{(\alpha)}}$ consist of the meromorphic functions on $W^{(\alpha)}$. That allows to represent uniquely $\lambda_{ji}^{(\alpha)} = \frac{a_{ji}^{(\alpha)}}{c^{(\alpha)}ji}$ as ratios of relatively prime $a_{ji}^{(\alpha)}, c_{ji}^{(\alpha)} \in \mathcal{O}_{W^{(\alpha)}}$. Since φ_i and ψ_j are globally defined, at $x \in W^{(\alpha)} \cap W^{(\beta)}$, one has $0 \equiv \psi_j^{(\alpha)}(x) - \psi_j^{(\beta)}(x) = \sum_{i=1}^{k} (\lambda_{ji}^{(\alpha)} - \lambda_{ji}^{(\beta)}) \varphi_i(x)$, which implies $\lambda_{ji}^{(\alpha)} = \lambda_{ji}^{(\beta)}$ due to the functional independence of φ_i.

One can represent the global meromorphic functions $\lambda_{ji} : X \to \mathbb{C}$ by global holomorphic numerators and denominators. Indeed, on $W^{(\alpha)} \cap W^{(\beta)}$ the relation $a_{ji}^{(\alpha)} c_{ji}^{(\beta)} = a_{ji}^{(\beta)} c_{ji}^{(\alpha)}$ requires a_{ji}^{β} to be divisible by $a_{ji}^{(\alpha)}$, according to $GCD(a_{ji}^{(\alpha)}, c_{ji}^{(\alpha)}) = 1$. Exchanging α with β, one obtains

$$a_{ji}^{(\alpha)}|_{W^{(\alpha)} \cap W^{(\beta)}} = u_{ji}^{(\alpha\beta)} a_{ji}^{(\beta)}|_{W^{(\alpha)} \cap W^{\beta}}, \quad c_{ji}^{(\alpha)}|_{W^{(\alpha)} \cap W^{\beta}} = u_{ji}^{(\alpha\beta)} c_{ji}^{(\beta)}|_{W^{(\alpha)} \cap W^{\beta}}$$

for some locally invertible $u_{ji}^{(\alpha\beta)}$. Due to the compactness of X, one can choose a finite coordinate covering and adjust all $u^{(\alpha\beta)}ji = 1$. After fixing some $a_{ji}^{(\alpha)}$, one puts $a_{ji}^{(\beta)}|_{W^{(\alpha)} \cap W^{(\beta)}} = a_{ji}^{(\alpha)}|_{W^{(\alpha)} \cap W^{(\beta)}}$ for all $\beta \in \{\beta_1, \dots, \beta_k\}$ with $W^{(\alpha)} \cap W^{(\beta)} \neq \varnothing$ and extends holomorphically $a_{ji}^{(\beta)}$ over the simply connected $W^{(\beta)}$. The same procedure is applied to all β with $W^{(\beta)} \cap W^{(\beta_i)} \neq \varnothing$, $1 \leq i \leq k$, etc.

In the case $k = 2$ let us consider the greatest common divisors $d_j := GCD(c_{j1}, c_{j2})$ and introduce $b_{ji} := \frac{c_{ji}}{d_j}$ for all $1 \leq j \leq m$, $i = 1, 2$. Then $\theta_j := d_j \psi_j = \sum_{i=1}^{2} \frac{a_{ji}}{b_{ji}} \varphi_i$. For future convenience let us put $b_{j1} := c_{j1}$, $\theta_j := \psi_j = \frac{a_{j1}}{b_{ji}} \varphi_1$ for $k = 1$.

Multiplying θ_j by $b_{j,3-i}$ for $i = 1, 2$ and bearing in mind that $GCD(a_{ji}, b_{ji}) = 1$, $GCD(b_{j,3-i}, b_{ji}) = 1$, one concludes that b_{ji} divide φ_i, i.e., $\frac{\varphi_i}{b_{ji}}$ are global holomorphic $(1,0)$-forms. The same holds if $k = 1$. Then the least common multiples

$b_i := LCM(b_{ji}|1 \leq j \leq m)$ divide φ_i and allow to define the global holomorphic $\omega_i := \frac{\varphi_i}{b_i}$. As a result, one obtains the representations $\theta_j = \sum_{i=1}^{k} a_{ji}\frac{b_i}{b_{ji}}\omega_i$ as $\mathcal{O}_X$-linear combinations of ω_i, Q.E.D.

Proof of Proposition 1. The subspace $U := f_1^* H^{1,0}(C_g) \subsetneq H^{1,0}(S)$ is maximal isotropic, according to the connectedness of the fibers of f_1. Therefore, any $\mathbb{C}$-basis $u_1, \ldots, u_g$ of U is of the form $u_i = \lambda_i^{(\alpha)} u_1$, $2 \leq i \leq g$, for some local meromorphic functions $\lambda_i^{(\alpha)} : W^{(\alpha)} \to \mathbf{P}_1$ on the coordinate charts $W^{(\alpha)} \subset S$. According to Lemma 2, there exist global holomorphic functions $\xi_1, \ldots, \xi_g$ and a global holomorphic $(1,0)$-form $\omega_1 \in U$ such that $u_i = \xi_i \omega_1$, $1 \leq i \leq g$.

For a non-ruled fibered surface, $U := f_1^* H^{1,0}(C_g)$ is a proper subspace of $H^{1,0}(S)$. Any complement of U has a basis $v_1, \ldots, v_k$, $k = h^{1,0}(S) - g$ with $\omega_1 \wedge v_j \neq 0$ for all $1 \leq j \leq k$. The functionally independent ω_1, v_1 on the surface S generate $H^{1,0}(S)$ over the fields $\mathcal{M}_{W^{(\alpha)}}$ of local meromorphic functions. That allows to represent $v_i = \sigma_i^{(\alpha)}\omega_1 + \tau_i^{(\alpha)} v_1$ on $W^{(\alpha)}$, $\sigma_i^{(\alpha)}, \tau_i^{(\alpha)} \in \mathcal{M}_{W^{(\alpha)}}$. The application of Lemma 2 yields global holomorphic functions $b_1, b_2, d_j, \lambda_j, \mu_j$, $1 \leq j \leq k$, such that $\widetilde{\omega_1} := \frac{\omega_1}{b_1}$, $\omega_2 := \frac{v_1}{b_2}$ are global holomorphic $(1,0)$-forms and $d_j v_j = \lambda_j \widetilde{\omega_1} + \mu_j \omega_2$, $2 \leq j \leq k$. Let V_0 be the $\mathbb{C}$-span of $\varphi_1 = v_1 = b_2\omega_2 = \mu_1\omega_2$, $\varphi_j = d_j v_j - \lambda_j \widetilde{\omega_1} = \mu_2 \omega_2$, $2 \leq j \leq k$, and V be a maximal isotropic subspace of $H^{1,0}(S)$, containing V_0. Wedging by v_1 an arbitrary $v = \sum_{i=1}^{g} c_i \xi_i \omega_1 \in V \cap U$ and bearing in mind that $\omega_1 \wedge v_1 \neq 0$, one infers $\sum_{i=1}^{g} c_i \xi_i = 0$. As far as $\xi_1\omega_1, \ldots, \xi_g\omega_1$ are $\mathbb{C}$-linearly independent, there follows $c_i = 0$ for all $1 \leq i \leq g$. In other words, $U \cap V = 0$ and there exist maximal isotropic subspaces U, V with $U \oplus V \subsetneq H^{1,0}(S)$.

If $\dim_{\mathbb{C}} V \geq 2$, Castelnuovo-de Franchis' Theorem implies that there is a surjective holomorphic map $f_2 : S \to C_h$ with connected fibers, such that $f_2^* H^{1,0}(C_h) = V$. The holomorphic map $f = (f_1, f_2) : S \to C_g \times C_h$ is generically of $rank_{\mathbb{C}} df = 2$ since

$$f^* = f_1^* \oplus f_2^* : H^{1,0}(C_g \times C_h) = H^{1,0}(C_G) \oplus H^{1,0}(C_h) \to U \oplus V \subsetneq H^{1,0}(S)$$

and f_1^*, f_2^* are injective. According to Remmert's Proper Mapping Theorem, $f(S)$ is a 2-dimensional complex analytic subspace of $C_g \times C_h$. Therefore $f(S) = C_g \times C_h$. The generic fiber of f is a compact complex analytic 0-dimensional subspace of S, i.e., finite number of points.

In the case of $V = Span_{\mathbb{C}}(v_1)$, let us consider the dual $V^* \subset H_1(S, \mathbb{C})$ and its quotient $E := V^*/V^* \cap H_1(S, \mathbb{Z})_{free}$ modulo the free part of $H_1(S, \mathbb{Z})$. As a closed subtorus of the compact Albanese variety $Alb(S) = H^{1,0}(S)^*/H_1(S, \mathbb{Z})_{free}$, E is an elliptic curve. For any fixed $s_0 \in S$ the holomorphic map $f_2' : S \to E$, $f_2'(S) := \int_{s_0}^{s} v_1 modulo H_1(S, \mathbb{Z})_{free}$ is of $rank_{\mathbb{C}} df_2' = 1$, whereas surjective. Since the fibers of f_2' can be disconnected, we pass to Stein factorization $f_2 : S \to C_h$, $h \geq 1$. Then apply the rest of the proof for $\dim_{\mathbf{C}} V \geq 2$, Q.E.D.

Remark. *Generalization of Proposition* 1 *to higher dimensional compact Kähler manifolds.* Catanese has generalized in [3] the theorem of Castelnuovo-de Franchis. Let us say that the normal Kähler variety Y is of Albanese general type if the irregularity $h^{1,0}(Y) > \dim_{\mathbf{C}} Y$ and the image of Albanese map $\alpha : Y \to Alb(Y)$ is of $\dim_{\mathbf{C}} \alpha(Y) = \dim_{\mathbf{C}} Y$. The compact Kähler manifold X_n of $\dim_{\mathbf{C}} X_n = n$ is Albanese general type k-fibration if it admits a surjective holomorphic map $f_1 : X_n \to Y_k$ with connected fibers onto a normal k-dimensional Kähler variety of Albanese general type. Catanese has shown that a necessary and sufficient condition for the existence of an Albanese general type k-fibration $f_1 : X_n \to Y_k$ is the presence of a maximal subspace $U \subset H^{1,0}(X_n)$ with $\Lambda^{k+1} U = 0$, containing a subspace $U_0 \subseteq U$ of $\dim_{\mathbf{C}} U_0 \geq k+1$, whose k-wedge $\Lambda^k U_0$ is embedded in $H^{k,0}(X_n)$. A slight modification of the proof of Proposition 1 establishes that if a compact Kähler n-dimensional manifold X_n admits an Albanese general type $(n-1)$-fibration $f_1 : X_n \to X_{n-1}$, whose generic fibers are different from $\mathbf{P}_1$, then X_n is a finite ramified covering $f : X_n \to X_{n-1} \times X_1$ of the product of X_{n-1} and a Riemann surface X_1 of genus ≥ 1. The study of the complements of Albanese general k-fibrations $f_1 : X_n \to X_k$ with an arbitrary k is obstructed by the condition $\Lambda^{n-k} U_0 \hookrightarrow H^{n-k,0}(X_n)$, which is not easy to be understood.

2. THE FUNDAMENTAL GROUP

Corollary 3. *If the surface S is a finite ramified covering $f = (f_1, f_2) : S \to C_g \times C_h$, $g \geq 2$, $h \geq 2$, then its fundamental group $\pi_1(S)$ is commensurable with $\pi_1(C_m) \times \pi_1(C_n)$ for some $m \geq g$, $n \geq h$.*

Proof. Campana has shown in [2] that for any surjective holomorphic map $X \to C$ of a compact Kähler manifold X onto a Riemann surface C there is a finite etale cover $r : \widetilde{X} \to X$ such that the Stein factorization $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ of $fr : \widetilde{X} \to C$ has no multiple fibers and there is a finite map $\rho : \widetilde{C} \to C$ with $\rho \widetilde{f} = fr$. The application of this result to $f_1 : S \to C_g$ yields a finite etale cover $r_1 : S_1 \to S$, a surjective holomorphic map $f_1' : S_1 \to C_m$, $m \geq g$, without multiple fibers, and a finite map $\rho_1 : C_m \to C_g$ such that $f_1 r_1 = \rho_1 f_1'$. The subsequent application of the aforementioned result to $f_2 r_1 : S_1 \to C_h$ provides a finite etale cover $r_2 : Z \to S_1$, a holomorphic surjection $\varphi_2 : Z \to C_n$, $n \geq h$, without multiple fibers, and a finite map $\rho_2 : C_n \to C_h$ with $f_2 r_1 r_2 = \rho_2 \varphi_2$. Consequently, the composition $\varphi_1 := f_1' r_2 : Z \to C_m$ of the unramified r_2 and f_1' has no multiple fibers. The Cartesian product $\varphi = (\varphi_1, \varphi_2) : Z \to C_m \times C_n$ is a finite covering, as far as $r_1 r_2 : Z \to S$ is a finite etale, $f = (f_1, f_2) : S \to C_g \times C_h$ is finite and there is a projection $(\rho_1, \rho_2) : C_m \times C_n \to C_g \times C_h$. We claim that φ is unramified since the generic fibers of $\varphi_1 : Z \to C_m$ and $\varphi_2 : Z \to C_n$ have no self-intersections. Indeed, for apropriate ramified coverings $p_1 : C_m \to \mathbf{P}_1$ and $p_2 : C_n \to \mathbf{P}_1$ one obtains linear pencils of divisors $p_1 \varphi_1 : Z \to \mathbf{P}_1$ and $p_2 \varphi_2 : Z \to \mathbf{P}_1$. According to Bertini's theorem, the generic fibers $(p_i\varphi_i)^{-1}(x) = \varphi_i^{-1}(p_i^{-1}(x))$, $i = 1, 2$, have no singularities outside the base locus. Thus, $Z \to C_m \times C_n$ is a finite unramified covering and $\pi_1(Z)$ is a finite index subgroup of $\pi_1(C_m) \times \pi_1(C_n)$. On the other hand, $r_1 r_2 : Z \to S$ is finite and unramified, so that $\pi_1(Z)$ is a finite index subgroup of $\pi_1(S)$. That justifies the commensurability of $\pi_1(S)$ and $\pi_1(C_m)\times\pi_1(C_n)$, Q.E.D.

Acknowledgements. The author is extremely indebted to Prof. V. Tsanov for the useful advices and conversations.

REFERENCES

1. Amor'os, J., M. Burger, K. Corlette, D. Kotschick, D. Toledo. Fundamental Groups of Compact Kähler Manifolds. *Math. Surv. Mon.*, **44**, Providence, 1996.
2. Campana, F. Negativity of compact curves in infinite covers of projective surfaces. *J. Alg. Geom.*, **7**, 1998, 673–693.
3. Catanese, F. Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations. *Inv. Math.*, **104**, 1991, 263–289.

Received July 16, 2000

Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5 James Bourchier Blvd.
BG-1164 Sofia, Bulgaria
E-mail: kasparia@fmi.uni-sofia.bg