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VARIATIONAL METHODS FOR THE TOTALLY MONOTONIC FUNCTIONS AND APPLICATIONS

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We derive new variational methods and formulas for the totally monotonic functions and apply them for finding the sharp estimates for the coefficients of the inverse functions of the examined ones. Two conjectures for these coefficients are stated.

Keywords: variational methods, totally monotonic functions, maximum and minimum of the coefficients of the inverse functions of totally monotonic functions

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1. INTRODUCTION

Let N denote the class of Nevanlinna analytic functions

$$w = f(z) = \int_0^1 \frac{\mu(t)}{z-t} = \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \quad z \notin [0,1], \tag{1}$$

where $\mu(t)$ is a probability measure on $[0,1]$, i.e. $\mu(t)$ is a nondecreasing function on $[0,1]$ with $\mu(0)=0$ and $\mu(1)=1$, and

$$a_n = \int_0^1 t^{n-1}\, d\mu(t), \quad n = 1, 2, \ldots, \quad a_1 = 1. \tag{2}$$

If we replace z by $1/z$ in (1), we obtain the class T of analytic functions

$$w = \varphi(z) \equiv f\left(\frac{1}{z}\right) = \int_0^1 \frac{z\,d\mu(t)}{1-tz} = \sum_{n=1}^{\infty} a_n z^n, \quad z \notin [1, +\infty], \tag{3}$$

with totally monotonic Taylor coefficients, which has been introduced by Hausdorff [1]. According to the Thale theorem [2, pp. 234–235, Theorem 2.3] (see also Goodman [3, pp. 183–184, Section 8]) the disk $\{z : |z-(1/2)| > (1/2)\}$ is the maximal domain of univalence for the class N. Hence the half-plane $\{z : \operatorname{Re} z < 1\}$ is the maximal domain of univalence for the class T. Wirths [4, p. 512, Corollary 2.3] has found the Koebe domain of the class T with respect to the unit disk $|z| < 1$. In [5] it is noted that the Koebe domains of the classes N and T with respect to the disks $|z| > 1$ and $|z| < 1$ are one and the same, respectively. Therefore we need to study only the class T in the unit disk $|z| < 1$. It follows from the Wirths result (see also [5, p. 345, Corollary 2]) that the largest common region of convergence of all Taylor series at the point $w = 0$ of the inverse functions $z = \psi(w)$ of the functions (3) in $|z| < 1$ is the disk $|w| < 1/2$. Let

$$z = \psi(w) = \sum_{n=1}^{\infty} b_n w^n, \quad |w| < \frac{1}{2}, \quad b_1 = 1, \tag{4}$$

be such series, where the coefficients b_n are determined by the coefficients a_n in (2) with the help of Theorem 3 below.

In this paper we derive variational methods which yield more precise information in comparison with the Wirths result [4, p. 513, Theorem 2.3] for the extremal functions of a given bounded real-valued continuous functional in the class T. As an application of these methods we find the minimum and the maximum of the coefficients b_2, b_3 and b_4 and state two conjectures for the extrema of all coefficients b_n, $n = 2, 3, 4, \ldots$, in (4).

2. VARIATIONAL FORMULAS FOR THE CLASS T

The variational methods and results represented by Theorems 1 and 2 below are new.

Theorem 1. *Let ε with $-1 < \varepsilon < 1$, $\varepsilon \neq 0$, be an arbitrary number and let the function $\varphi(z)$ belong to the class T. Then the varied function*

$$\varphi_*(z) = \int_0^1 \frac{z\,d\mu(t)}{1 - \dfrac{(1-\varepsilon)t}{1+\varepsilon-2\varepsilon t} z}, \qquad z \notin [1, +\infty], \tag{5}$$

also belongs to the class T and it has the asymptotic representation

$$\varphi_*(z) = \varphi(z) - 2\varepsilon z^2 \int_0^1 \frac{t(1-t)}{(1-tz)^2}\,d\mu(t) + O(\varepsilon^2), \quad |z| < 1, \tag{6}$$

where $O(\varepsilon^2)$ denotes a magnitude, the ratio of which to ε^2 is uniformly bounded for z lying in an arbitrary closed set of the disk $|z| < 1$.

Proof. The linear fractional function

$$\tau = \frac{(1-\varepsilon)t}{1+\varepsilon-2\varepsilon t}, \qquad 0 \le t \le 1, \quad -1 < \varepsilon < 1, \quad \varepsilon \ne 0, \tag{7}$$

for fixed ε, increases with t from 0 to 1. This property of (7) permits us to substitute $(1-\varepsilon)t/(1+\varepsilon-2\varepsilon t)$ for t in (3) to obtain (5). The function (5) belongs to the class T with the probability measure

$$\nu(\tau) := \mu\left(\frac{(1+\varepsilon)\tau}{1-\varepsilon+2\varepsilon\tau}\right), \qquad 0 \le \tau \le 1.$$

The difference between (5) and (3) is

$$\varphi_*(z) - \varphi(z) = -2\varepsilon z^2 \int_0^1 \frac{t(1-t)}{(1-tz)^2} \frac{d\mu(t)}{1-\varepsilon\dfrac{2t-1-tz}{1-tz}} \tag{8}$$

$$= -2\varepsilon z^2 \int_0^1 \frac{t(1-t)}{(1-tz)^2} \sum_{\nu=0}^{\infty} \varepsilon^\nu \left(\frac{2t-1-tz}{1-tz}\right)^\nu d\mu(t), \qquad |z| < 1,$$

since $|2t-1-tz| \le |1-tz|$ for $0 \le t \le 1$ and $|z| < 1$. Thus from (8) we obtain (6), which completes the proof of Theorem 1.

Theorem 2. *For given point z of the disk $|z| < 1$ and a given analytic function $\Phi(u_0, u_1, \ldots, u_n; z)$, $n \ge 0$, on the set $\bigcup_T \{\varphi(z), \varphi'(z), \ldots, \varphi^{(n)}(z); z\}$, the minimum (maximum) of the functional*

$$\operatorname{Re} \Phi\left(\varphi(z), \varphi'(z), \ldots, \varphi^{(n)}(z); z\right) \tag{9}$$

in the class T is attained only either in the subclass $T_1 \subset T$ of functions

$$\varphi(z) = cz + (1-c)\frac{z}{1-z} \in T_1, \quad 0 \le c \le 1, \tag{10}$$

or in the subclass $T_2 \subset T$ of functions

$$\varphi(z) = \sum_{k=1}^{p} \frac{c_k z}{1-t_k z} \in T_2 \tag{11}$$

with

$$1 \le p \le n+2, \quad 0 \le t_1 \le t_2 \le \cdots \le t_p \le 1, \quad 0 \le c_k \le 1, \quad \sum_{k=1}^{p} c_k = 1, \tag{12}$$

where $t_1, t_2, \ldots, t_p$ are among the numbers 0 and 1, and the roots in the interval $0 \le t \le 1$ of the equation

$$\operatorname{Re}\left\{ \left[\frac{\partial\Phi[\varphi(z)]}{\partial u_0} z^2(1-tz)^n \right.\right. \tag{13}$$

$$\left.\left. + \sum_{s=1}^{n} \frac{\partial\Phi[\varphi(z)]}{\partial u_s} s! t^{s-2}(s-1+2tz)(1-tz)^{n-s} \right] (1-t\bar{z})^{n+2} \right\} = 0,$$

where we assume that at the extremum of the functional (9) the equation (13) is not an identity for all t in the interval $0 \le t \le 1$,

$$\Phi[\varphi(z)] \equiv \Phi\left(\varphi(z), \varphi'(z), \ldots, \varphi^{(n)}(z); z\right),$$

and the empty sum for $n = 0$ is zero by convention.

Proof. The extremal functions $\varphi(z) \in T$ exist since the functional (9) is continuous and bounded on T and the class T is normal and compact in $|z| < 1$. If we set

$$u_s = \varphi^{(s)}(z), \quad u_s^* = \varphi_*^{(s)}(z) \quad (0 \le s \le n), \tag{14}$$

then the increments by the asymptotic formula (6) are

$$du_s = u_s^* - u_s = -2\varepsilon s! \int_0^1 t(1-t) I_s(t,z)\, d\mu(t) + O(\varepsilon^2) \quad (0 \le s \le n), \tag{15}$$

where

$$I_s(t,z) = \left(\frac{t}{1-tz}\right)^s \left[\binom{s+1}{1} \left(\frac{z}{1-tz}\right)^2 + 2\binom{s}{1} \frac{z}{(1-tz)t} + \binom{s-1}{1} \frac{1}{t^2} \right] \tag{16}$$

for $0 \le s \le n$, and $\binom{m}{1} = m$ for $m = 1, 2, \ldots$ and $\binom{m}{1} = 0$ for $m = 0, -1$. Further we introduce the abridged notations

$$\Phi \equiv \Phi(u_0, u_1, \ldots, u_n; z), \quad \Phi^* \equiv \Phi(u_0^*, u_1^*, \ldots, u_n^*; z), \tag{17}$$

where u_s and u_s^* $(0 \le s \le n)$ are given by (14). Then for sufficiently small $|\varepsilon|$ we have the Taylor series

$$\Phi^* = \Phi + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \left(\sum_{s=0}^{n} \frac{\partial}{\partial u_s} du_s \right)^{\nu} \Phi \tag{18}$$

for the functions (17). From (18) and (15)–(16) we obtain

$$\Phi^* = \Phi - 2\varepsilon \sum_{s=0}^{n} s! \frac{\partial\Phi}{\partial u_s} \int_0^1 t(1-t) I_s(t,z)\, d\mu(t) + O(\varepsilon^2). \tag{19}$$

It follows from (19) that

$$\operatorname{Re}\Phi^* = \operatorname{Re}\Phi - 2\varepsilon \int_0^1 t(1-t)\operatorname{Re}\left[\sum_{s=0}^{n} s!\frac{\partial\Phi}{\partial u_s} I_s(t,z)\right] d\mu(t) + O(\varepsilon^2). \tag{20}$$

The extremality of the function $\varphi(z)$ in the class T and the arbitrariness of ε imply that the coefficient of ε in (20) vanishes, i.e.

$$\int_0^1 t(1-t)\operatorname{Re}\left[\sum_{s=0}^{n} s!\frac{\partial\Phi}{\partial u_s} I_s(t,z)\right] d\mu(t) = 0. \tag{21}$$

If the equation

$$P(t) \equiv \operatorname{Re}\left[\sum_{s=0}^{n} s!\frac{\partial\Phi}{\partial u_s} I_s(t,z)\right] = 0 \tag{22}$$

is not an identity for all t in the interval $0 \leq t \leq 1$, i.e. if the conditions for the equation (13) hold, then the equation of the extremality (21) is fulfilled if and only if the measure $\mu(t)$ is a step function with points of discontinuity at 0, 1 and the roots of the equation (22) in t in the closed interval $[0,1]$, i.e. the roots of the equation (13) in $t \in [0,1]$, where the sum of the corresponding jumps equals to unit. In fact, this is evident if $\mu(t)$ is a corresponding step function. Conversely, it follows from the Goluzin variational formula applied to the class T (see, for example, [6, p. 93, formula (19)]) that $\mu(t)$ is a constant between any two adjacent roots of the equation (22) for the extremal function $\varphi(z)$ (see the comments for formulas (27)–(28) in [6, pp. 94–95]). Hence, the extremal functions $\varphi(z)$ belong to the subclasses $T_1 \subset T$ and $T_2 \subset T$ of functions (10) and (11)–(12), respectively, where the upper bound of the number p is determined in the following manner.

Let the real number ε be with a sufficiently small $|\varepsilon|$. If the extremal function $\varphi(z) \in T_2$ and in (11)–(12) we substitute $c_k + \varepsilon$ and $c_{k+1} - \varepsilon$ for c_k and c_{k+1}, respectively, then the varied function

$$\varphi_{**}(z) = \varphi(z) + \varepsilon\left[\frac{z}{1-t_k z} - \frac{z}{1-t_{k+1}z}\right] \tag{23}$$

also belongs to the subclass T_2. If we set analogously

$$u_s = \varphi^{(s)}(z), \quad u_s^{**} = \varphi_{**}^{(s)}(z) \quad (0 \leq s \leq n), \tag{24}$$

then by formula (23) the increments are

$$du_s = u_s^{**} - u_s = \varepsilon\left[\frac{\partial^s}{\partial z^s}\frac{z}{1-t_k z} - \frac{\partial^s}{\partial z^s}\frac{z}{1-t_{k+1}z}\right] \quad (0 \leq s \leq n). \tag{25}$$

For brevity, we again denote

$$\Phi \equiv \Phi(u_0, u_1, \ldots, u_n; z), \qquad \Phi^{**} \equiv \Phi(u_0^{**}, u_1^{**}, \ldots, u_n^{**}; z), \tag{26}$$

where u_s and u_s^{**} $(0 \leq s \leq n)$ are given by (24). Then the corresponding Taylor series (18) for the functions (26) and (25) yield

$$\operatorname{Re}\Phi^{**} = \operatorname{Re}\Phi + \varepsilon\operatorname{Re}\left\{\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\left[\frac{\partial^s}{\partial z^s}\frac{z}{1-t_k z} - \frac{\partial^s}{\partial z^s}\frac{z}{1-t_{k+1}z}\right]\right\} + O(\varepsilon^2). \tag{27}$$

In addition, from the conditions for the equation (13) (or (22)) it follows that we have the inequality

$$\frac{\partial\Phi[\varphi(z)]}{\partial u_s} \equiv \frac{\partial}{\partial u_s}\Phi\left(\varphi(z),\varphi'(z),\ldots,\varphi^{(n)}(z);z\right) \neq 0 \tag{28}$$

at least for one $s \in \{0,1,\ldots,n\}$. Then the extremality of $\varphi(z)$ in (27), the arbitrariness of ε and the inequality (28) imply the condition

$$\operatorname{Re}\left\{\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\left[\frac{\partial^s}{\partial z^s}\frac{z}{1-t_k z} - \frac{\partial^s}{\partial z^s}\frac{z}{1-t_{k+1}z}\right]\right\} = 0. \tag{29}$$

The condition (29) shows that the function

$$Q(t) \equiv \operatorname{Re}\left[\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\frac{\partial^s}{\partial z^s}\frac{z}{1-tz}\right], \quad 0 \le t \le 1, \tag{30}$$

has equal values at any two adjacent points of discontinuity t_k and t_{k+1} of the measure $\mu(t)$ for the subclass T_2, i.e. $Q(t)$ has equal values at all the points of discontinuity of the measure $\mu(t)$ for the subclass T_2. Hence, the derivative $Q'(t)$ vanishes at least at one point inside the intervals between any two adjacent points of discontinuity of $\mu(t)$ in $0 \le t \le 1$. But from (30) and (22), having in mind (16), we conclude that

$$Q'(t) = \operatorname{Re}\left[\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\frac{\partial^s}{\partial z^s}\frac{z^2}{(1-tz)^2}\right] = \operatorname{Re}\left[\sum_{s=0}^{n}s!\frac{\partial\Phi}{\partial u_s}I_s(t,z)\right] = P(t). \tag{31}$$

The equation (22) or the equivalent algebraic equation (13) have no more than $2n+2$ roots in t. Taking into account the endpoints 0 and 1, we conclude that the step measure $\mu(t)$ has no more than $2n+4$ points of discontinuity in the interval $0 \le t \le 1$. It follows from (31) that if the points of discontinuity of $\mu(t)$ in $0 \le t \le 1$ are more than $n+2$, then the equation (22) (or (13)) will have more than $2n+2$ roots in $0 \le t \le 1$, which is impossible. Hence, the number p satisfies the inequalities in (12). If the extremal function $\varphi(z) \in T_1$, the corresponding assertions are established in the same way.

This completes the proof of Theorem 2.

3. APPLICATION TO THE COEFFICIENT PROBLEM OF THE INVERSE FUNCTIONS IN THE CLASS T

We need the following

Theorem 3. *In terms of the coefficients a_n in (2), the coefficients b_n in (4) have the following simplest explicit form:*

$$b_n = \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k\binom{n+k-1}{k}D_{n-1,k}(a_2,a_3,\ldots,a_{n-k+1}), \quad n \ge 2, \tag{32}$$

where

$$D_{n-1,k}(a_2, a_3, \ldots, a_{n-k+1}) \equiv \sum \frac{k!(a_2)^{\nu_1}(a_3)^{\nu_2}\ldots(a_{n-k+1})^{\nu_{n-k}}}{\nu_1!\nu_2!\ldots\nu_{n-k}!} \tag{33}$$

for $1 \le k \le n-1$, $n \ge 2$, *are the ordinary Bell polynomials in* $a_2, a_3, \ldots, a_{n-k+1}$, *and the sum is taken over all nonnegative integers* $\nu_1, \nu_2, \ldots, \nu_{n-k}$ *satisfying*

$$\begin{aligned} &\nu_1 + \nu_2 + \cdots + \nu_{n-k} = k, \\ &\nu_1 + 2\nu_2 + \cdots + (n-k)\nu_{n-k} = n-1, \quad 1 \le k \le n-1, \ n \ge 2. \end{aligned} \tag{34}$$

Proof. We use the method in [6, pp. 91–93, Theorem 1], which is applicable to each analytic function $F(z)$ in $|z| < 1$ normalized by the requirements $F(0) = F'(0) - 1 = 0$ (see in [6] a recurrence relation for the polynomials (33) and tables for the polynomials (33) and the coefficients (32)).

Theorem 4. *The minimum (maximum) of the coefficients* b_n, $n \ge 2$, *from* (32) *in the class* T *is attained only either in the subclass* $T_1 \subset T$ *of functions* (10) *or in the subclass* $T_2 \subset T$ *of functions* (11)–(12) *with:*

(i) $1 \le p \le m$ *if* $n = 2m$, $m = 1, 2 \ldots$,

(ii) $1 \le p \le m+1$ *if* $n = 2m+1$, $m = 1, 2 \ldots$,

where in (12) *the points* $t_1, t_2, \ldots, t_p$ *are among the numbers* 0 *and* 1 *and the roots in the interval* $0 \le t \le 1$ *of the equation*

$$P(t) = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s}(s-1)t^{s-2} = 0, \quad n \ge 2 \tag{35}$$

(for $n = 2$ *this equation is impossible — see below Corollary* 1), *and the function*

$$Q(t) = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s} t^{s-1}, \quad Q'(t) = P(t), \quad n \ge 2, \tag{36}$$

has equal values at any two adjacent points of the sequence $t_1, t_2, \ldots, t_p$.

Proof. We apply Theorem 2 for $z = 0$ and the function

$$\begin{aligned} b_n &\equiv \Phi(u_0, u_1, \ldots, u_n; 0) \\ &\equiv \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k \binom{n+k-1}{k} D_{n-1,k}\left(\frac{u_2}{2!}, \frac{u_3}{3!}, \ldots, \frac{u_{n-k+1}}{(n-k+1)!}\right) \end{aligned} \tag{37}$$

on the set $\bigcup_T \{\varphi(0), \varphi'(0), \ldots, \varphi^{(n)}(0); 0\}$, where $n \ge 2$, having in mind (32)–(34). For the function (37), the equation (22) (or (13)) and the function (30) for the condition (29) are reduced to (35) and (36), respectively, where

$$\frac{\partial b_n}{\partial a_s} = s!\frac{\partial \Phi}{\partial u_s}, \quad a_s = \frac{\varphi^{(s)}(0)}{s!} \equiv \frac{u_s}{s!}, \quad 2 \le s \le n. \tag{38}$$

It is clear from (38) and (32)–(34) that for the function (37) the equation (35) is not an identity in t in the interval $0 \le t \le 1$ since, for example, $\partial b_n/\partial a_n = -1 \ne 0$, $n \ge 2$. Further:

(i) For $n = 2m$, $m = 1, 2, \ldots$, the function $\mu(t)$ in (3) for the extremum of (37) has not more than $2m$ points of discontinuity among the roots of the equation (35) and the points 0 and 1. For all the points of discontinuity of the extremal step function $\mu(t)$, if they are more than one, the function (36) has equal values. If $\mu(t)$ has more than m ($m > 1$) points of discontinuity in $0 \le t \le 1$, then the equation (35) will have more than $2m - 2$ roots in $0 \le t \le 1$, which is impossible. Hence, the points of discontinuity of $\mu(t)$ in $0 \le t \le 1$ are not more than m ($m \ge 1$). Therefore, the interval of the integer p in (12) is contracted to $1 \le p \le m$.

(ii) For $n = 2m + 1$, $m = 1, 2, \ldots$, the function $\mu(t)$ in (3) for the extremum of (37) has not more than $2m + 1$ points of discontinuity among the roots of the equation (35) and the points 0 and 1. For all the points of discontinuity of the extremal step function $\mu(t)$, if they are more than one, the function (36) has equal values. If $\mu(t)$ has more than $m + 1$ points of discontinuity in $0 \le t \le 1$, then the equation (35) will have more than $2m - 1$ roots in $0 \le t \le 1$, which is impossible. Hence, the points of discontinuity of $\mu(t)$ in $0 \le t \le 1$ are not more than $m + 1$. Therefore, the interval of the integer p in (12) is contracted to $1 \le p \le m + 1$.

This completes the proof of Theorem 4.

Corollary 1. *The coefficient b_2 from* (32) *satisfies the sharp inequalities*

$$-1 \le b_2 \le 0, \tag{39}$$

where the equalities hold only for the following extremal functions:

— on the left-hand side of (39), *for the function*

$$\psi(w) = \frac{w}{1+w} = \sum_{n=1}^{\infty} (-1)^{n-1} w^n, \tag{40}$$

inverse of the function

$$\varphi(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \in T_1; \tag{41}$$

— on the right-hand side of (39), *for the function*

$$\psi(w) = w, \tag{42}$$

inverse of the function

$$\varphi(z) = z \in T_1. \tag{43}$$

Proof. For $n = 2$, Theorem 4(i) yields $p = 1$. For $n = 2$, from (32)–(34) and (35) we obtain

$$b_2 = -a_2, \quad \frac{\partial b_2}{\partial a_2} = -1, \tag{44}$$

and

$$P(t) = -1 \ne 0, \tag{45}$$

respectively. It follows from (45) that the point of discontinuity of $\mu(t)$ can be either $t_1 = 0$ or $t_1 = 1$ with the corresponding jumps $c_1 = 1$ and $c_1 = 1$. Therefore, we obtain the two extremal functions (41) and (43) of the form (10), the inverse ones of which (40) and (42) supply the equalities in (39), respectively.

Remark 1. The inequalities (39) and the extremal functions (41) (or (40)) and (43) (or (42)) follow from (44) and (2) for $n = 2$ as well.

Corollary 2. *The coefficient b_3 from (32) satisfies the sharp inequalities*

$$-\frac{1}{8} \leq b_3 \leq 1, \tag{46}$$

where the equalities hold only for the following extremal functions:
— on the left-hand side of (46), for the function

$$\begin{aligned}\psi(w) &= \frac{2}{3}\left(1 + w - \sqrt{1 - w + w^2}\right)\\ &= w + \frac{2}{3}\sum_{n=2}^{\infty}(-1)^{n-1}w^n \sum_{n/2\leq\nu\leq n}\binom{1/2}{\nu}\binom{\nu}{n-\nu},\end{aligned} \tag{47}$$

where $\sqrt{1} = 1$ and the inner sum is taken over all integers ν satisfying $n/2 \leq \nu \leq n$, inverse of the function

$$\varphi(z) = \frac{3}{4}z + \frac{1}{4}\frac{z}{1-z} = z + \frac{1}{4}\sum_{n=2}^{\infty} z^n \in T_1; \tag{48}$$

— on the right-hand side of (46), for the function (40), inverse of the function (41), respectively.

Proof. For $n = 3$, Theorem 4(ii) yields $p = 1, 2$. For $n = 3$, from (32)–(34), (35), (36) and (38) we obtain

$$b_3 = -a_3 + 2a_2^2, \quad a_2 = \frac{\varphi''(0)}{2}, \quad a_3 = \frac{\varphi'''(0)}{6}, \tag{49}$$

$$\frac{1}{2}P(t) = 2a_2 - t = 0, \tag{50}$$

and

$$Q(t) = 4a_2t - t^2, \quad Q'(t) = P(t), \tag{51}$$

respectively.

If $p = 1$, then (11)–(12) are reduced to

$$\varphi(z) = \frac{z}{1 - tz} \in T_2, \tag{52}$$

where t can be either the root $t = 2a_2$ of (50) or any of the points 0 and 1. From (52) we obtain $\varphi''(0) = 2t$. On the other hand, $\varphi''(0) = 2a_2 = t$, and hence $t = 0$. Then (52) takes the form $\varphi(z) = z$, the inverse one of which is $\psi(w) = w$. It is clear that the identity is not an extremal function. If $t = 1$ in (52), then we obtain that $\varphi''(0) = 2$, $\varphi'''(0) = 6$ and the equations (49) yield $b_3 = 1$. Thus for the function (40), inverse of the extremal function (41), the bound 1 in (46) is attained.

If $p = 2$, then (11) has two terms, corresponding to the condition $Q(t_1) = Q(t_2)$, where $Q(t)$ is determined by (51) and t_1 and t_2 are among the numbers 0, $t = 2a_2$ and 1. According to this condition and the Rolle theorem, the equation (50) has an odd number of roots between t_1 and t_2. This is possible only if $t_1 = 0$ and $t_2 = 1$. Hence, the extremal function $\varphi(z)$ is

$$\varphi(z) = cz + (1-c)\frac{z}{1-z} \in T_2, \quad 0 < c < 1. \tag{53}$$

Further, the condition $Q(0) = Q(1)$, where $Q(t)$ is given by (51), yields $a_2 = \varphi''(0)/2 = 1/4$. On the other hand, from (53) we obtain $\varphi''(0) = 2(1-c)$, and hence $c = 3/4$. For $c = 3/4$, from (53) we obtain the extremal function (48) and its inverse function (47) for which the bound $-1/8$ in (46) is attained.

Remark 2. The second sharp inequality in (46) and the extremal function (41) (or (40)) can be obtained in another way. With the help of the Cauchy inequality and (2) we obtain that

$$a_2^2 = \left(\int_0^1 1.t\, d\mu(t)\right)^2 \leq \int_0^1 1^2\, d\mu(t). \int_0^1 t^2\, d\mu(t) = a_3. \tag{54}$$

Now from (54) and the first equation in (49) we obtain the sharp inequalities

$$b_3 \leq a_3 \leq 1$$

with the unique extremal function (41) (or (40)).

Corollary 3. *The coefficient* b_4 *from* (32) *satisfies the sharp inequalities*

$$-1 \leq b_4 \leq \frac{5+4\sqrt{10}}{135} = 0.13073415\ldots, \tag{55}$$

where the equalities hold only for the following extremal functions:

— *on the left-hand side of* (55), *for the function* (40), *inverse of the function* (41);

— *on the right-hand side of* (55), *for the inverse function of the function* (53) *for*

$$c = \frac{10-\sqrt{10}}{15} = 0.45584816\ldots\,. \tag{56}$$

Proof. For $n = 4$, Theorem 4(i) yields $p = 1, 2$. For $n = 4$, from (32)–(34), (35), (36) and (38) we obtain

$$b_4 = -a_4 + 5a_2a_3 - 5a_2^3, \tag{57}$$

$$P(t) = 5a_3 - 15a_2^2 + 10a_2t - 3t^2 = 0, \tag{58}$$

$$Q(t) = \left(5a_3 - 15a_2^2\right)t + 5a_2t^2 - t^3, \quad Q'(t) = P(t), \tag{59}$$

where $a_{2,3,4}$ are the coefficients of the extremal functions $\varphi(z) \in T_2$, i.e.

$$a_2 = \frac{\varphi''(0)}{2}, \quad a_3 = \frac{\varphi'''(0)}{6}, \quad a_4 = \frac{\varphi^{(IV)}(0)}{24}, \tag{60}$$

respectively.

If $p = 1$, then (11)–(12) are reduced to (52), where t can be either any root of (58) in $0 \le t \le 1$ or any of the points 0 and 1. Converting (52) or by means of (60), (52) and (57), we obtain

$$b_4 = -t^3, \quad 0 \le t \le 1. \tag{61}$$

If $p = 2$, then (11), having in mind (12), can have the following forms:

$$\varphi(z) = cz + (1-c)\frac{z}{1-tz} \in T_2, \quad 0 < c < 1, \quad 0 < t < 1, \tag{62}$$

$$\varphi(z) = c\frac{z}{1-tz} + (1-c)\frac{z}{1-z} \in T_2, \quad 0 < c < 1, \quad 0 < t < 1, \tag{63}$$

and (53), where t (in general different for each function) is a real root of (58) and the other root of (58) has to lie in the open intervals $(0, t)$, $(t, 1)$ and $(0, 1)$ in accordance to (59) and the conditions

$$Q(0) = Q(t), \quad Q(t) = Q(1), \quad Q(0) = Q(1), \tag{64}$$

respectively.

(a) From the first equation of (64) and (59) we obtain the corresponding equation for (62), namely,

$$5a_3 - 15a_2^2 + 5a_2 t - t^2 = 0. \tag{65}$$

It follows from (58) and (65) that

$$a_2 = \frac{2t}{5}, \quad a_3 = \frac{7t^2}{25}. \tag{66}$$

On the other hand, from (60) and (62) we get

$$a_2 = (1-c)t, \quad a_3 = (1-c)t^2, \quad a_4 = (1-c)t^3. \tag{67}$$

The equations (66) and (67) yield the different values $c = 3/5$ and $c = 18/25$, respectively, i.e. the extremal function $\varphi(z)$ is not of the form (62).

(b) From the second equation of (64) and (59) we obtain the corresponding equation for (63), namely,

$$5a_3 - 15a_2^2 + 5a_2(t+1) - t^2 - t - 1 = 0. \tag{68}$$

It follows from (58) and (68) that

$$a_2 = \frac{2t+1}{5}, \quad a_3 = \frac{7t^2 + 2t + 3}{25}. \tag{69}$$

On the other hand, from (60) and (63) we get

$$a_2 = ct + 1 - c, \quad a_3 = ct^2 + 1 - c, \quad a_4 = ct^3 + 1 - c. \tag{70}$$

The equations (69) and (70) yield the equations

$$c = \frac{2(t-2)}{5(t-1)}, \quad 3t^2 - 12t + 2 = 0, \quad t = \frac{6 - \sqrt{30}}{3} = 0.1742581\ldots \tag{71}$$

(the other root of the second equation is not in the open interval $(0,1)$). From (71), (69) and (70) we obtain

$$\begin{aligned} c &= \frac{2\left(10+\sqrt{30}\right)}{35}, \quad a_2 = \frac{15-2\sqrt{30}}{15}, \\ a_3 &= \frac{35-6\sqrt{30}}{15}, \quad a_4 = \frac{345-62\sqrt{30}}{45}. \end{aligned} \tag{72}$$

By the values of $a_{2,3}$ from (72) the equation (58) becomes

$$9t^2 - 2\left(15-2\sqrt{30}\right)t + 2\left(17-3\sqrt{30}\right) = 0. \tag{73}$$

Really, for the roots of (73) we have

$$0 < \frac{6-\sqrt{30}}{3} < \frac{12-\sqrt{30}}{9} < 1.$$

Finally, by the values of $a_{2,3,4}$ from (72) and (57) we obtain

$$b_4 = -\frac{45-8\sqrt{30}}{45} = -0.026271\ldots . \tag{74}$$

(c) From the third equation of (64) and (59) we obtain the corresponding equation for (53), namely,

$$5a_3 - 15a_2^2 + 5a_2 - 1 = 0. \tag{75}$$

It follows from (60) and (53) that

$$a_2 = 1-c, \quad a_3 = 1-c, \quad a_4 = 1-c. \tag{76}$$

From (75)–(76) we get the values

$$c_{1,2} = \frac{10 \pm \sqrt{10}}{15}. \tag{77}$$

The equations (76) and (77) yield

$$a_2 = \frac{5 \mp \sqrt{10}}{15}, \quad a_3 = \frac{5 \mp \sqrt{10}}{15}, \quad a_4 = \frac{5 \mp \sqrt{10}}{15}, \tag{78}$$

respectively. By the values of $a_{2,3}$ from (78) the equation (58) becomes

$$9t^2 - 2\left(5 \mp \sqrt{10}\right)t + 2 \mp \sqrt{10} = 0. \tag{79}$$

Really, each equation of (79) has one root in $0 < t < 1$, respectively. Finally, from (78) and (57) we find

$$b_4 = \frac{5-4\sqrt{10}}{135} = -0.05666\ldots, \quad b_4 = \frac{5+4\sqrt{10}}{135} = 0.13073415\ldots, \tag{80}$$

respectively. Now the comparison of (61), (74) and (80) leads us to (55) and (56), which completes the proof of Corollary 3.

Remark 3. The first inequality in (55) and the extremal function (41) (or (40)) follow also from (57), (54) and (2), namely,

$$b_4 = -a_4 + 5a_2\left(a_3 - a_2^2\right) \geq -a_4 \geq -1.$$

For the coefficients $b_5, b_6, \ldots$, we can proceed in the same way.

Conjecture 1. *In the class T each coefficient b_n, $n = 2, 3, \ldots$, from* (32) *attains its minimum (maximum) only for the rational functions of the form* (10).

Conjecture 2. *In the class T each coefficient b_n, $n = 2, 3, \ldots$, from* (32) *satisfies the sharp inequalities*

$$b_{2m} \geq -1, \quad m = 1, 2, \ldots,$$

and

$$b_{2m+1} \leq 1, \quad m = 1, 2, \ldots,$$

where the equalities hold only for the extremal function (40), *inverse of the function* (41).

For $n = 2, 3, 4$ these conjectures are proved in the above corollaries 1–3.

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